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HW4

1. Determine the values $\int_1^2 e^x \sin(4x)dx$ with $h = 0.1$ by

a. Use the composite trapezoidal rule

b. Use the composite Simpsons' method

c. Use the composite midpoint rule

a. Composite Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2}[f(a) + 2\sum_{i=1}^{2n-1} f(x_i) + f(b)] - \frac{b-a}{12}h^2 f''(\xi)$$

b. Composite Midpoint rule

$$\int_a^b f(x)dx = 2h[f(x_1) + f(x_3) + \ldots + f(x_{2n-1})] + \frac{b-a}{6}h^2 f''(\xi)$$

c. Composite Simpson's rule

$$\int_a^b f(x)dx = \int_{x_0}^{x_{2n}} f(x)dx = \sum_{i=1}^{n} \int_{x_{2(i-1)}}^{x_{2i}} f(x)dx = \sum_{i=1}^{n} \frac{h}{3}[f(x_{2i-2}) + 4f(x_{2i-1}) + f(x_{2i})]$$

$$= \frac{h}{3}[f(x_0) + 2\sum_{i=2}^{n} f(x_{2i-2}) + 4\sum_{i=1}^{n} f(x_{2i-1}) + f(x_{2n})] - \frac{b-a}{180}h^4 f^{iv}(\xi)$$

a. Trapezoidal rule

$I = \int_1^2 e^x \sin(4x)dx$

$a=1,\ b=2,\ h=0.1=\frac{2-1}{2n},\ n=10,\ x_i = x_0 + h\cdot i = 1+0.1i$

$x_0 = 1,\ x_{10} = 2,\ f(x) = e^x \sin(4x)$

$f(x) = e^x \sin(4x)$

$f'(x) = e^x \sin(4x) + e^x 4\cos(4x)$

$f''(x) = e^x 4\cos(4x) + e^x \sin(4x) + e^x 4\cos(4x) - e^x 16\sin(4x)$

- $\int_a^b f(x)dx \approx \frac{h}{2}[f(x_0) + 2\sum_{i=1}^{2n-1} f(x_i) + f(x_n)]$

$\approx \frac{h}{2}[f(x_0) + 2f(x_1) + \ldots + 2f(x_9) + f(x_{10})]$

$\approx \frac{0.1}{2}[f(1) + 2f(1.1) + 2f(1.2) + \ldots + f(2)]$

$\approx \frac{0.1}{2}[-2.057 + 2(-2.8588 - 3.3074 - 3.2417 - 2.5599 - 1.2523 + 0.5773 + 2.7048 + 4.8014 + 6.4714) + 7.3104]$

$\approx 0.05[-2.057 + 2\times 1.3348 + 7.3104] = 0.39615$ #

b. Simpsons' method

- $\int_a^b f(x)dx \approx \frac{h}{3}[f(x_0) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + f(x_n)$

$\approx \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots f(x_{10})]$

$\approx \frac{0.1}{3}[-2.057 + 4(-2.8588 + -3.2417 + -1.2523 + 2.7048 + 6.4714)$
$+ 2(-3.3074 - 2.5599 + 0.5773 + 4.8014) + 7.3104]$

$\approx \frac{0.1}{3}[-2.057 + 4\times 1.8234 + 2\times(-0.4886) + 7.3104] = 0.38566$ #

c. composite Midpoint

$\int_a^b f(x)dx \approx 2h[f(x_1) + f(x_3) + \ldots + f(x_{n-1})]$

$\approx 0.2[-2.8588 - 3.2417 - 1.2523 + 2.7048 + 6.4714]$

$\approx 0.2\times 1.8234 = 0.36468$ #

2. Approximate $\int_1^{1.5} x^2 \ln x dx$ using Gaussian Quadrature with $n=3$ and $n=4$. Then compare the result to the exact value of the integral.

G.Q $[1, 1.5] \to [-1, 1]$

$x = \frac{b-a}{2} t + \frac{b+a}{2}$ $a=1$, $b=1.5$

$x = 0.25t + 1.25$ $dx = 0.25dt$

$\int_1^{1.5} x^2 \cdot \ln(x) dx = \int_{-1}^{1} (0.25t+1.25)^2 \ln(0.25t+1.25) \cdot 0.25 dt$

• $\int_a^b f(x)dx \approx \sum_{i=1}^{n} w_i \cdot f(x_i)$

$f(x) = x^2 \cdot \ln(x)$, $x_i = 0.25t_i + 1.25$, $(b-a)/2 = 0.25$

$\Rightarrow \int_1^{1.5} f(x)dx \approx \frac{b-a}{2} \sum_{i=1}^{n} w_i f(x_i) = 0.25 \sum w_i \cdot f(0.25t_i + 1.25)$

① $n=3$

t_i	w_i
-0.7746	0.5556
0	0.8889
0.7746	0.5556

$x_1 \approx 0.25 \times (-0.7746) + 1.25 = 1.05635$

$x_2 = 1.25$

$x_3 \approx 0.25(0.7746) + 1.25 = 1.44365$

$f(1.05635) \approx (1.05635)^2 \cdot \ln(1.05635) \approx 0.0612$

$f(1.25) = (1.25)^2 \ln(1.25) \approx 0.34866$

$f(1.44365) \approx (1.44365)^2 \ln(1.44365) \approx 0.76524$

$I \approx 0.25(0.5556 \times 0.0612 + 0.8889 \times 0.34866 + 0.5556 \times 0.76524)$

$= 0.25(0.034 + 0.3099 + 0.4252) = 0.25 \times 0.7691 \approx 0.192275$ #

② $n=4$

t_i	w_i
-0.8611	0.3479
-0.3399	0.6521
0.3399	0.6521
0.8611	0.3479

$x_1 = 0.25(-0.8611) + 1.25 = 1.0347$

$x_2 = 0.25(-0.3399) + 1.25 = 1.1650$

$x_3 = 0.25(0.3399) + 1.25 = 1.3350$

$x_4 = 0.25(0.8611) + 1.25 = 1.4653$

$f(1.0347) \approx (1.0347)^2 \cdot \ln(1.0347) = 0.0365$

$f(1.1650) \approx (1.1650)^2 \cdot \ln(1.1650) = 0.2073$

$f(1.3350) \approx (1.3350)^2 \cdot \ln(1.3350) = 0.5149$

$f(1.4653) \approx (1.4653)^2 \cdot \ln(1.4653) = 0.8203$

$I \approx 0.25(0.3479 \times 0.0365 + 0.6521 \cdot 0.2073 + 0.6521 \times 0.5149 + 0.3479 \times 0.8203)$

$= 0.25(0.01269 + 0.13518 + 0.33576 + 0.28538) = 0.25 \cdot 0.76901 \approx 0.1922525$ #

③

exact value $\to$ 0.19225938

method	手算	exact	誤差
$n=3$	0.192275	0.19225938	0.00001562
$n=4$	0.1922525	0.19225938	0.00000688

#

3. Approximate $\int_0^{\pi/4}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx$ using

a. Simpson's rule for $n=4$ and $m=4$

b. Gaussian Quadrature, $n=3$ and $m=3$

c. Compare these results with the exact value.

a. $\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx$

$x_i=a+ih_x$, $h_x=(\frac{\pi}{4}-0)/4=\frac{\pi}{16}$ $x_0=0$, $x_1=\frac{\pi}{16}$, $x_2=\frac{\pi}{8}$, $x_3=\frac{3\pi}{16}$, $x_4=\frac{\pi}{4}$,

$y_i=\sin x_i+jh_y(x_i)$, $h_y(x_i)=\dfrac{\cos x_i-\sin x_i}{4}$

i	$x_i=i\cdot h_x$	$\sin x_i$	$\cos x_i$	$h_y=\frac{\cos x_i-\sin x_i}{4}$
0	0	0	1	0.250
1	0.19635	0.1951	0.9808	0.1964
2	0.39270	0.3827	0.9239	0.1353
3	0.58905	0.5556	0.8315	0.06897
4	0.78540	0.7071	0.7071	0

$f(x_i,y_i)=2y_i\sin x_i+\cos^2 x_i$

① 以 $x_0=0$ 爲例

$x=0$, $\sin x=0$, $\cos x=1$, $h_y=0.25$

y節點 : $y_0=0$ $y_1=0.25$ $y_2=0.5$ $y_3=0.75$ $y_4=1$

$f(x,y)=\cancel{2y\sin x}+\cos x^2=\cos x^2=1$

Simpson's rule ⇒

$\int_0^1 f(0,y)\,dy$

$\approx\frac{h_y}{3}[f(0,0)+4f(0,0.25)+2f(0,0.5)+4f(0,0.75)+f(0,1)]$

$=\frac{0.25}{3}(1+4\cdot1+2\cdot1+4\cdot1+1\cdot1)=\frac{0.25}{3}\cdot16=\frac{4}{3}\approx1.333$

$\to I(x_0)=1.333$

② 以 $x_1\approx0.19635$

$\sin x_1\approx0.1951$ $\cos x_1\approx0.9808$ $h_y=0.1964$

$y_0=0.1951$ $y_1=0.3915$ $y_2=0.5879$ $y_3=0.7843$ $y_4=0.9808$

$f(x,y)=2y\sin x+\cos^2 x$ $\sin x_1\approx0.1951$ $\cos^2 x_1\approx0.9619$

$f(0.19635,0.1951)=2\cdot0.1951\cdot0.1951+0.9619\approx1.0371$

$f(0.19635,0.3915)=2\cdot0.3915\cdot0.1951+0.9619\approx1.1147$

$f(0.19635,0.5879)=2\cdot0.5879\cdot0.1951+0.9619\approx1.1912$

$f(0.19635,0.7843)=2\cdot0.7843\cdot0.1951+0.9619\approx1.2679$

$f(0.19635,0.9808)=2\cdot0.9808\cdot0.1951+0.9619\approx1.3446$

$\int f(x_1,y)\,dy$

$\approx\frac{0.1964}{3}(1.0371+4\cdot1.1147+2\cdot1.1912+4\cdot1.2679+1.3446)$

$\approx\frac{0.1964}{3}\times14.2945=0.9358$

$\to I(x_1)\approx0.9358$

③ 以 $x_2=0.3927$

$\sin x_2\approx0.3827$ $\cos x_2\approx0.9239$ $h_y\approx0.1353$

$\cos^2 x_2\approx0.8536$

y_i	$f(x_2,y_j)$
$y_0=0.3827$	$2\cdot0.3827\cdot0.3827+0.8536\approx1.1465$
$y_1=0.5180$	$2\cdot0.5810\cdot0.3827+0.8536\approx1.2501$
$y_2=0.6533$	$2\cdot0.6533\cdot0.3827+0.8536\approx1.3536$
$y_3=0.7886$	$2\cdot0.7886\cdot0.3827+0.8536\approx1.4572$
$y_4=0.9239$	$2\cdot0.9239\cdot0.3827+0.8536\approx1.5608$

$\int(x_2,y)\,dy$

$\approx\frac{0.1353}{3}(1.1465+4\cdot1.2501+2\cdot1.3536+4\cdot1.4572+1.5608)$

$\approx0.0451\times16.2437=0.7326$

$\to I(x_2)\approx0.7326$

④ 以 $x_3=0.58905$

$\sin x_3\approx0.5556$ $\cos x_3\approx0.8315$ $h_y\approx0.06897$ $\cos^2 x_3\approx0.6914$

y_i	$f(x_3,y_j)$
$y_0=0.5556$	$2\cdot0.5556\cdot0.5556+0.6914\approx1.3088$
$y_1=0.6246$	$2\cdot0.6246\cdot0.5556+0.6914\approx1.3854$
$y_2=0.6936$	$2\cdot0.6936\cdot0.5556+0.6914\approx1.4621$
$y_3=0.7626$	$2\cdot0.7626\cdot0.5556+0.6914\approx1.5388$
$y_4=0.8315$	$2\cdot0.8315\cdot0.5556+0.6914\approx1.6154$

$\int f(x_3,y)\,dy$

$\approx\frac{0.06897}{3}[1.3088+4\times1.3854+2\times1.4621+4\times1.5388+1.6154]$

$\approx0.02299\times17.5452=0.4034$

$I(x_3)\approx0.4034$

⑤ $x_4=0.7854$

$\sin x_4=\cos x_4$

$\to I(x_4)=0$

$\int_0^{\pi/4}I(x)\,dx\approx\frac{h_x}{3}[I(x_0)+4I(x_1)+2I(x_2)+4I(x_3)+I(x_4)]$

$\approx\frac{\pi/16}{3}[1.333+4\times0.9358+2\times0.7326+4\times0.4034+0]$

$\approx\frac{\pi}{48}\cdot8.155=0.5337$ #

b. G.Q $n=m=3$

$x_i = \frac{b-a}{2}\xi_i + \frac{b+a}{2}$

$\xi_0 = -0.7746$, $w_0 = 0.5556$

$\xi_1 = 0$, $w_1 = 0.8889$

$\xi_2 = 0.7746$, $w_2 = 0.5556$

① $x_0 = \frac{\pi}{8}(-0.7746+1) \approx \frac{\pi}{8} 0.2254 \approx 0.0885$, $w_0 = \frac{5}{9}$

$\sin x_0 \approx 0.0884$. $\cos x_0 \approx 0.9961$

$y \in [0.0884, 0.9961]$

$y_i = \frac{0.9961-0.0884}{2}\xi_j + \frac{0.9961+0.0884}{2} = 0.4539\cdot\xi_j + 0.5422$

ξ_j	y_j
-0.7746	0.1906
0	0.5422
0.7746	0.8938

$f(x,y) = 2y\sin x + \cos^2 x$ $\quad \cos^2 x \approx 0.9922$

y_j	$f(x_0, y_j)$
$y_0 = 0.1906$	$2\cdot 0.1906\cdot 0.0884 + 0.9922 \approx 1.0259$
$y_1 = 0.5422$	$2\cdot 0.5422\cdot 0.0884 + 0.9922 \approx 1.0881$
$y_2 = 0.8938$	$2\cdot 0.8950\cdot 0.0884 + 0.9922 \approx 1.1504$

$\int_{\sin x_0}^{\cos x_0} f(x_0,y)dy \approx \frac{0.9961-0.0884}{2}\left[\frac{5}{9}\cdot 1.0259 + \frac{8}{9}1.0881 + \frac{5}{9}1.1504\right] \approx 0.4539 \cdot 2.1763 \approx 0.9878$

外層加權 $\frac{\pi}{8}\cdot\frac{5}{9}\cdot 0.9878 \approx 0.2155$

② $x_1 = \frac{\pi}{8} = 0.3927$

$\sin x_1 \approx 0.3827$, $\cos x_1 \approx 0.9239$

$y_i = \frac{0.9239-0.3827}{2}\xi_j + \frac{0.9239+0.3827}{2} = 0.2706\,\xi_j + 0.6533$

ξ_j	y_j
-0.7746	0.4437
0	0.6533
0.7746	0.8629

$f(x,y) = 2y\sin x + \cos^2 x$ $\quad \cos^2 x \approx 0.8536$

y_j	$f(x_1, y_j)$
$y_0 = 0.4437$	$2\cdot 0.4437\cdot 0.3827 + 0.8536 \approx 1.1932$
$y_1 = 0.6533$	$2\cdot 0.6533\cdot 0.3827 + 0.8536 \approx 1.3536$
$y_2 = 0.8629$	$2\cdot 0.8629\cdot 0.3827 + 0.8536 \approx 1.5141$

$\int f(x_1,y)dy \approx \frac{0.9239-0.3827}{2}\left[\frac{5}{9}\cdot 1.1932 + \frac{8}{9}1.3536 + \frac{5}{9}1.5141\right] \approx 0.2706 \times 2.7073 \approx 0.7326$

外層加權 $w = \frac{8}{9}$, $\frac{\frac{\pi}{4}-0}{2} = \frac{\pi}{8} \Rightarrow \frac{\pi}{8}\cdot\frac{8}{9}\cdot 0.7326 \approx 0.2557$

③ $x_2 = \frac{\pi}{8}(1+0.7746) = 0.6969$

$\sin x_2 \approx 0.6418$ $\quad \cos x_2 \approx 0.7668$

$y_j = \frac{0.7668-0.6418}{2}\xi_j + \frac{0.7668+0.6418}{2} = 0.0625\,\xi_j + 0.7043$

ξ_j	y_j
-0.7746	0.6559
0	0.7043
0.7746	0.7527

$f(x,y) = 2y\sin x + \cos^2 x$ $\quad \cos^2 x \approx 0.5880$

y_j	$f(x_2, y_j)$
$y_0 = 0.6559$	$2\cdot 0.6559\cdot 0.6418 + 0.5880 \approx 1.4299$
$y_1 = 0.7043$	$2\cdot 0.7043\cdot 0.6418 + 0.5880 \approx 1.4920$
$y_2 = 0.7527$	$2\cdot 0.7527\cdot 0.6418 + 0.5880 \approx 1.5542$

$\int f(x_2,y)dy \approx \frac{0.7668-0.6418}{2}\left[\frac{5}{9}\cdot 1.4299 + \frac{8}{9}1.4920 + \frac{5}{9}1.5542\right] \approx 0.0625 \times 2.98405 \approx 0.1865$

外層加權 $w_2 = \frac{5}{9}$, $\frac{\frac{\pi}{4}-0}{2} = \frac{\pi}{8} \Rightarrow \frac{\pi}{8}\cdot\frac{5}{9}\cdot 0.1865 \approx 0.0407$

總和 $0.2155 + 0.2557 + 0.0407 = 0.5119$ #

c. exact value $\Rightarrow I_{exact} = 0.5236$

method	result	exact	error
Simpson's Rule (a)	0.5337	0.5236	0.0101
GQ (b)	0.5119	0.5236	0.0117

→ Simpson's Rule 更加精確 #

4. Use the composite Simpson's rule and $n=4$ to approximate the improper integral a) $\int_0^1 x^{-1/4}\sin x dx$, b) $\int_1^\infty x^{-4}\sin x dx$ by use the transform $t = x^{-1}$

(a)

$\int_0^1 x^{-1/4} \sin(x) dx$

$n=4 \Rightarrow h = \frac{1-0}{4} = 0.25$

$x_0=0,\ x_1=0.25,\ x_2=0.5,\ x_3=0.75\ x_4=1$

$f(0)=0 \quad f(0.25)=0.3499 \quad f(0.5)=0.5701 \quad f(0.75)=0.7325 \quad f(1)=0.8415$

$\int_a^b f(x)dx \approx \frac{h}{3}[0+4\cdot 0.3499 + 2\times 0.5701 + 4\times 0.7325 + 0.8415]$

$\approx \frac{0.25}{3} \times 6.3113 = 0.5259$ #

(b) $\int_1^\infty x^{-4} \sin(x) dx$

$x=\frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$\int_1^\infty x^{-4}\sin(x)dx = \int_1^0 ((\frac{1}{t})^{-4}\sin(\frac{1}{t})(-\frac{1}{t^2}))dt = \int_0^1 t^2 \sin(\frac{1}{t})dt$

$f(0)=0,\ f(0.25)=-0.0473,\ f(0.5)=0.2273,\ f(0.75)=0.5467,\ f(1)=0.8415$

$\int_0^1 f(x)dx \approx \frac{0.25}{3}[0+4\times(-0.0473)+2\times 0.2273 + 4\times 0.5467 + 0.8415]$

$\approx \frac{0.25}{3} \times 3.2937 = 0.27448$ #