HW5

1. The initial-value problem

$y' = 1 + (y/t) + (y/t)^2$, $1 \le t \le 2$, $y(1) = 0$ has the exact solution $y(t) = t\tan(\ln t)$.

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use Taylor's method of order 2 with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

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Euler's method

1. $h = t_{i+1} - t_i = 0.1$, $y(t_{i+1}) = y(t_i) + f[t_i, y(t_i)]h = y(t_i) + \frac{y(t_{i+1}) - y(t_i)}{t_{i+1} - t_i} \cdot h$

$t_0 = 1$, $y_0 = 0$, $f_0 = f(1, 0) = 1$

$\Rightarrow t_i = 0.1i + 1$

$\frac{dy}{dt} = 1 + \frac{y}{t} + (\frac{y}{t})^2 \Rightarrow f(t_i, y_i) = 1 + \frac{y_i}{1+0.1i} + (\frac{y_i}{1+0.1i})^2$

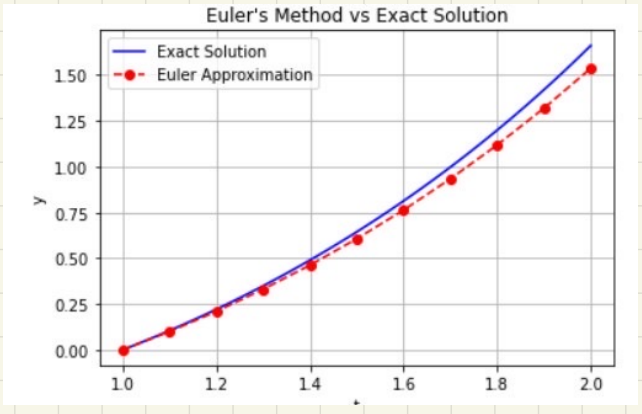
$\Rightarrow y_{i+1} = y_i + 0.1 \times [1 + \frac{y_i}{1+0.1i} + (\frac{y_i}{1+0.1i})^2]$, $i = 0, 1, 2, \ldots 10$

$\Delta y = y - f(y)$

$\varepsilon = |\frac{\Delta y}{y}|$

i	t_i	y_i (approximate)	y_i (exact)	Δy (絕對誤差)	ε (相對誤差)
0	1.0	0	0	0	0
1	1.1	0.1	0.105160	0.005160	0.0491
2	1.2	0.209917	0.221243	0.011325	0.05118
3	1.3	0.330471	0.349121	0.018651	0.05342
4	1.4	0.462354	0.489682	0.027328	0.05581
5	1.5	0.606285	0.643875	0.037590	0.05838
6	1.6	0.763041	0.812753	0.049711	0.06116
7	1.7	0.933475	0.997494	0.064019	0.06418
8	1.8	1.118537	1.199439	0.080902	0.06745
9	1.9	1.319293	1.420116	0.100823	0.07100
10	2.0	1.536943	1.661282	0.124338	0.07484

t	Euler y	Exact y	Error
1.0	0.000000	0.000000	0.000000
1.1	0.100000	0.105160	0.005160
1.2	0.209917	0.221243	0.011325
1.3	0.330471	0.349121	0.018651
1.4	0.462354	0.489682	0.027328
1.5	0.606285	0.643875	0.037590
1.6	0.763041	0.812753	0.049711
1.7	0.933475	0.997494	0.064019
1.8	1.118537	1.199439	0.080902
1.9	1.319293	1.420116	0.100823
2.0	1.536943	1.661282	0.124338



Taylor's method

b. Taylor $\rightarrow y(t_{i+1}) = y(t_i) + hT^{(n)}[t_i, y(t_i)]$

Order 2 $\rightarrow T^{(2)} = f + \frac{h}{2}\frac{df}{dt} = f + \frac{h}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right]$

$\frac{dy}{dt} = f(t,y) = 1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2$

$\frac{d^2}{dt^2}y = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\cdot f(t,y) = \frac{-y}{t^2} - \frac{2y^2}{t^3} + \left[1 + \frac{y}{t} + \left(\frac{y}{t}\right)^2\right]\left[\frac{1}{t} + \frac{2y}{t^2}\right]$

$= \frac{-y}{t^2} - \frac{2y^2}{t^3} + \frac{1}{t} + \frac{2y}{t^2} + \frac{y}{t^2} + \frac{2y^2}{t^3} + \frac{y^2}{t^3} + \frac{2y^3}{t^4} = \frac{1}{t} + \frac{2y}{t^2} + \frac{y^2}{t^3} + \frac{2y^3}{t^4}$

$y_{i+1} = y_i + 0.1\left[\left(1 + \frac{y_i}{t_i} + \left(\frac{y_i}{t_i}\right)^2\right) + \frac{0.1}{2}\left(\frac{1}{t_i} + \frac{2y_i}{t_i^2} + \frac{y_i^2}{t_i^3} + \frac{2y_i^3}{t_i^4}\right)\right]$

$\Rightarrow$ $t_i = 1 + 0.1i$

i	t_i	y_i (approximate)	y_i (exact)	Δy (絕對誤差)	ε (相對誤差)
0	1.0	0	0	0	0
1	1.1	0.105000	0.105160	0.000160	0.001521
2	1.2	0.220919	0.221243	0.000324	0.001464
3	1.3	0.348612	0.349121	0.000509	0.001458
4	1.4	0.488954	0.489682	0.000728	0.001487
5	1.5	0.642883	0.643875	0.000993	0.001519
6	1.6	0.811438	0.812753	0.001315	0.001618
7	1.7	0.995787	0.997494	0.001707	0.001711
8	1.8	1.197252	1.199439	0.002187	0.001823
9	1.9	1.417344	1.420116	0.002772	0.001952
10	2.0	1.657795	1.661282	0.003487	0.002099

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t = 1.0, Approx = 0.000000, Exact = 0.000000, Error = 0.000000
t = 1.1, Approx = 0.105000, Exact = 0.105160, Error = 0.000160
t = 1.2, Approx = 0.220919, Exact = 0.221243, Error = 0.000324
t = 1.3, Approx = 0.348612, Exact = 0.349121, Error = 0.000509
t = 1.4, Approx = 0.488954, Exact = 0.489682, Error = 0.000728
t = 1.5, Approx = 0.642883, Exact = 0.643875, Error = 0.000993
t = 1.6, Approx = 0.811438, Exact = 0.812753, Error = 0.001315
t = 1.7, Approx = 0.995787, Exact = 0.997494, Error = 0.001707
t = 1.8, Approx = 1.197252, Exact = 1.199439, Error = 0.002187
t = 1.9, Approx = 1.417344, Exact = 1.420116, Error = 0.002772
t = 2.0, Approx = 1.657795, Exact = 1.661282, Error = 0.003487
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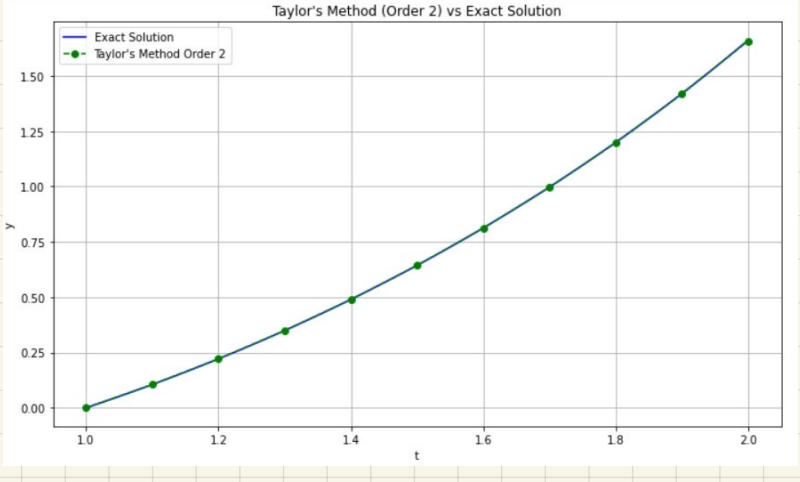

2. The system of initial-value problems

$$u_1' = 9u_1 + 24u_2 + 5\cos t - \frac{1}{3}\sin t, \quad u_1(0) = \frac{4}{3},$$

$$u_2' = -24u_1 - 52u_2 - 9\cos t + \frac{1}{3}\sin t, \quad u_2(0) = \frac{2}{3},$$

has the unique solution

$$u_1 = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t, \quad u_2 = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t.$$

Try $h = 0.05$ and $h = 0.1$ in Runge-Kutta method, and compare their results with the exact value.

$u_1'(t) = f_1(t, u_1, u_2)$

$u_2'(t) = f_2(t, u_1, u_2)$

$u_j \Rightarrow j = 1$

$u_1 : K_{1,1} = h f_1(t_i, u_{i,1}, u_{i,2})$

$K_{2,1} = h f_1(t_i + \frac{h}{2}, u_{i,1} + \frac{K_{1,1}}{2}, u_{i,2} + \frac{K_{1,2}}{2})$

$K_{3,1} = h f_1[t_i + \frac{h}{2}, u_{i,1} + \frac{K_{2,1}}{2}, u_{i,2} + \frac{K_{2,2}}{2}]$

$K_{4,1} = h f_1[t_{i+1}, u_{i,1} + K_{3,1}, u_{i,2} + K_{3,2}]$

$u_{i+1,1} = u_{i,1} + \frac{1}{6}(K_{1,1} + 2K_{2,1} + 2K_{3,1} + K_{4,1})$

$t=0 \quad t_i = 0.05i$

① $h = 0.05$

$u_1 : K_{1,1} = h[9u_{i,1} + 24u_{i,2} + 5\cos t_i - \frac{1}{3}\sin t_i]$

$= 0.05[9u_{i,1} + 24u_{i,2} + 5\cos(0.05i) - \frac{1}{3}\sin(0.05i)]$

$K_{2,1} = h[9(u_{i,1} + \frac{K_{1,1}}{2}) + 24(u_{i,2} + \frac{K_{1,2}}{2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= 0.05[9(u_{i,1} + \frac{K_{1,1}}{2}) + 24(u_{i,2} + \frac{K_{1,2}}{2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$K_{3,1} = h[9(u_{i,1} + \frac{K_{2,1}}{2}) + 24(u_{i,2} + \frac{K_{2,2}}{2}) + 5\cos(t_i + \frac{h}{2}) - \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= 0.05[9(u_{i,1} + \frac{K_{2,1}}{2}) + 24(u_{i,2} + \frac{K_{2,2}}{2}) + 5\cos(0.05i + 0.025) - \frac{1}{3}\sin(0.05i + 0.025)]$

$K_{4,1} = h[9(u_{i,1} + K_{3,1}) + 24(u_{i,2} + K_{3,2}) + 5\cos(t_i + h) - \frac{1}{3}\sin(t_i + h)]$

$= 0.05[9(u_{i,1} + K_{3,1}) + 24(u_{i,2} + K_{3,2}) + 5\cos(0.05i + 0.05) - \frac{1}{3}\sin(0.05i + 0.05)]$

$u_{i+1,1} = u_{i,1} + \frac{1}{6}(K_{1,1} + 2K_{2,1} + 2K_{3,1} + K_{4,1})$

$u_2 : K_{1,2} = h[-24u_{i,1} - 52u_{i,2} - 9\cos t_i + \frac{1}{3}\sin t_i]$

$= 0.05[-24u_{i,1} - 52u_{i,2} - 9\cos(0.05i) + \frac{1}{3}\sin(0.05i)]$

$K_{2,2} = h[-24(u_{i,1} + \frac{K_{1,1}}{2}) - 52(u_{i,2} + \frac{K_{1,2}}{2}) - 9\cos(t_i + \frac{h}{2}) + \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= 0.05[-24(u_{i,1} + \frac{K_{1,1}}{2}) - 52(u_{i,2} + \frac{K_{1,2}}{2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$K_{3,2} = h[-24(u_{i,1} + \frac{K_{2,1}}{2}) - 52(u_{i,2} + \frac{K_{2,2}}{2}) - 9\cos(t_i + \frac{h}{2}) + \frac{1}{3}\sin(t_i + \frac{h}{2})]$

$= 0.05[-24(u_{i,1} + \frac{K_{2,1}}{2}) - 52(u_{i,2} + \frac{K_{2,2}}{2}) - 9\cos(0.05i + 0.025) + \frac{1}{3}\sin(0.05i + 0.025)]$

$K_{4,2} = h[-24(u_{i,1} + K_{3,1}) - 52(u_{i,2} + K_{3,2}) - 9\cos(t_i + h) + \frac{1}{3}\sin(t_i + h)]$

$= 0.05[-24(u_{i,1} + K_{3,1}) - 52(u_{i,2} + K_{3,2}) - 9\cos(0.05i + 0.05) + \frac{1}{3}\sin(0.05i + 0.05)]$

$u_{i+1,2} = u_{i,2} + \frac{1}{6}(K_{1,2} + 2K_{2,2} + 2K_{3,2} + K_{4,2})$

t_i	j	$K_{1,j}$	$K_{2,j}$	$K_{3,j}$	$K_{4,j}$
0	1	1.65	-0.24924	1.68076	-2.18175
	2	-3.78333	0.14556	-3.82245	4.13953
0.05	1	0.42418	-0.21073	0.44279	-0.85810
	2	-1.21590	0.11138	-1.23314	1.46144
0.10	1	0.02508	-0.18257	0.03852	-0.39550
	2	-0.35363	0.09272	-0.36295	0.54744
0.15	1	-0.09602	-0.15990	-0.08526	-0.22650
	2	-0.06808	0.08027	-0.07426	0.23204
0.20	1	-0.12479	-0.14081	-0.11574	-0.15858
	2	0.02297	0.07076	0.01824	0.12025

0.25	1	-0.12388	-0.12444	-0.11611	-0.12632
	2	0.04889	0.06298	0.045	0.07813
0.30	1	-0.11432	-0.1103	-0.10761	-0.10737
	2	0.05343	0.05642	0.05013	0.06020
0.35	1	-0.10307	-0.09807	-0.09725	-0.09401
	2	0.05133	0.05082	0.04848	0.05100
0.40	1	-0.09232	-0.08748	-0.08727	-0.08347
	2	0.04749	0.04604	0.04502	0.04519
0.45	1	-0.08267	-0.07833	-0.07829	-0.07471
	2	0.04350	0.04194	0.04137	0.04092
0.50	1	-0.07421	-0.07042	-0.07042	-0.06726
	2	0.03987	0.03845	0.03802	0.03752

Compare

i	ti	(approx) $U_{i,1}$	exact $u_1(t_i)$	Δy	ε	(approx) $U_{i,2}$	exact $u_2(t_i)$	Δy	ε
0	0	1.333333	1.333333	0	0	0.666667	0.666667	0	0
1	0.05	1.721880	1.912059	0.190179	0.0995	-0.499599	-0.909077	0.409	0.4499
2	0.10	1.726915	1.793063	0.066148	0.0369	-0.832598	-1.032002	0.199	0.1928
3	0.15	1.617161	1.601967	0.015194	0.0095	-0.890373	-0.961459	0.0711	0.0740
4	0.20	1.481687	1.423902	0.05779	0.0406	-0.861042	-0.874681	0.0136	0.01554
5	0.25	1.348945	1.267646	0.081299	0.0641	-0.807505	-0.795221	0.0123	0.01546
6	0.30	1.227063	1.131577	0.095486	0.0844	-0.750341	-0.724999	0.0253	0.0349
7	0.35	1.117478	1.012999	0.104479	0.1031	-0.695886	-0.663060	0.0328	0.0495
8	0.40	1.019525	0.909409	0.110116	0.1211	-0.645732	-0.608214	0.0375	0.0617
9	0.45	0.931977	0.818630	0.113347	0.1385	-0.599934	-0.559389	0.0405	0.0724
10	0.50	0.853541	0.738788	0.114753	0.1553	-0.558092	-0.515658	0.0424	0.0822

步長 h = 0.05:

t	u1(approx)	u1(exact)	error_u1	u2(approx)	u2(exact)	error_u2
0.00	1.333333	1.333333	0.00e+00	0.666667	0.666667	1.11e-16
0.05	1.721880	1.912059	1.90e-01	-0.499599	-0.909077	4.09e-01
0.10	1.726915	1.793063	6.61e-02	-0.832598	-1.032002	1.99e-01
0.15	1.617161	1.601967	1.52e-02	-0.890373	-0.961459	7.11e-02
0.20	1.481687	1.423902	5.78e-02	-0.861042	-0.874681	1.36e-02
0.25	1.348945	1.267646	8.13e-02	-0.807505	-0.795221	1.23e-02
0.30	1.227063	1.131577	9.55e-02	-0.750341	-0.724999	2.53e-02
0.35	1.117478	1.012999	1.04e-01	-0.695886	-0.663060	3.28e-02
0.40	1.019525	0.909409	1.10e-01	-0.645732	-0.608214	3.75e-02
0.45	0.931977	0.818630	1.13e-01	-0.599934	-0.559389	4.05e-02
0.50	0.853541	0.738788	1.15e-01	-0.558092	-0.515658	4.24e-02
0.55	0.783017	0.668275	1.15e-01	-0.519706	-0.476225	4.35e-02
0.60	0.719337	0.605710	1.14e-01	-0.484290	-0.440411	4.39e-02
0.65	0.661560	0.549909	1.12e-01	-0.451407	-0.407635	4.38e-02
0.70	0.608868	0.499860	1.09e-01	-0.420673	-0.377404	4.33e-02

② $h = 0.1 \Rightarrow t_i = 0.1i$

u_1 : $K_{1,1} = 0.1[9u_{i,1} + 24u_{i,2} + 5\cos(0.1i) - \frac{1}{3}\sin(0.1i)]$

$K_{2,1} = 0.1[9(u_{i,1} + \frac{K_{1,1}}{2}) + 24(u_{i,2} + \frac{K_{1,2}}{2}) + 5\cos(0.1i+0.05) - \frac{1}{3}\sin(0.1i+0.05)]$

$K_{3,1} = 0.1[9(u_{i,1} + \frac{K_{2,1}}{2}) + 24(u_{i,2} + \frac{K_{2,2}}{2}) + 5\cos(0.1i+0.05) - \frac{1}{3}\sin(0.1i+0.05)]$

$K_{4,1} = 0.1[9(u_{i,1} + K_{3,1}) + 24(u_{i,2} + K_{3,2}) + 5\cos(0.1i+0.1) - \frac{1}{3}\sin(0.1i+0.1)]$

$u_{i+1,1} = u_{i,1} + \frac{1}{6}(K_{1,1} + 2K_{2,1} + 2K_{3,1} + K_{4,1})$

u_2 : $K_{1,2} = 0.1[-24u_{i,1} - 52u_{i,2} - 9\cos(0.1i) + \frac{1}{3}\sin(0.1i)]$

$K_{2,2} = 0.1[-24(u_{i,1} + \frac{K_{1,1}}{2}) - 52(u_{i,2} + \frac{K_{1,2}}{2}) - 9\cos(0.1i+0.05) + \frac{1}{3}\sin(0.1i+0.05)]$

$K_{3,2} = 0.1[-24(u_{i,1} + \frac{K_{2,1}}{2}) - 52(u_{i,2} + \frac{K_{2,2}}{2}) - 9\cos(0.1i+0.05) + \frac{1}{3}\sin(0.1i+0.05)]$

$K_{4,2} = 0.1[-24(u_{i,1} + K_{3,1}) - 52(u_{i,2} + K_{3,2}) - 9\cos(0.1i+0.1) + \frac{1}{3}\sin(0.1i+0.1)]$

$u_{i+1,2} = u_{i,2} + \frac{1}{6}(K_{1,2} + 2K_{2,2} + 2K_{3,2} + K_{4,2})$

t_i	j	$K_{1,j}$	$K_{2,j}$	$K_{3,j}$	$K_{4,j}$
0	1	3.3	-4.29729	11.14328	-43.30659
	2	-7.56667	8.14946	-23.59572	88.39502
0.1	1	19.32131	-20.3617	59.73636	-222.8475
	2	-40.3107	41.3188	-123.2983	457.4901
0.2	1	101.8829	-104.088	311.2367	-1154.084
	2	-209.843	213.501	-640.0268	2371.354
0.3	1	529.5652	-538.524	1614.879	-5982.614
	2	-1088.65	1106.38	-3319.004	12294.49
0.4	1	2746.724	-2791.23	8373.754	-31018.17
	2	-5644.98	5735.91	-17208.85	63744.09
0.5	1	14242.03	-14471.3	43416.99	-160822.42
	2	-29268.5	29739.2	-89224.88	330500.2

Compare

i	t_i	(approx) $u_{i,1}$	exact $u_1(t_i)$	Δy	ε	(approx) $u_{i,2}$	exact $u_2(t_i)$	Δy	ε
0	0	1.333333	1.333333	0	0	0.666667	0.666667	0	0
1	0.10	-3.052437	1.793063	4.8455	2.7024	8.989305	-1.032002	-10.0213	9.7105
2	0.20	-23.847795	1.423902	25.2716	17.7481	51.192704	-0.874681	-52.0674	59.5273
3	0.30	-130.165202	1.131577	131.2968	116.03	269.269193	-0.724999	-269.9942	372.4063
4	0.40	-680.231485	0.909409	681.1409	748.993	1399.368584	-0.608214	-1399.977	2301.78
5	0.50	-3531.299585	0.738788	3532.0384	4780.855	7258.241839	-0.515658	-7258.757	14076.69

步長 h = 0.1:

t	u1(approx)	u1(exact)	error_u1	u2(approx)	u2(exact)	error_u2
0.00	1.333333	1.333333	0.00e+00	0.666667	0.666667	1.11e-16
0.10	-3.052437	1.793063	4.85e+00	8.989305	-1.032002	1.00e+01
0.20	-23.847795	1.423902	2.53e+01	51.192704	-0.874681	5.21e+01
0.30	-130.165202	1.131577	1.31e+02	269.269193	-0.724999	2.70e+02
0.40	-680.231485	0.909409	6.81e+02	1399.368584	-0.608214	1.40e+03
0.50	-3531.299585	0.738788	3.53e+03	7258.241839	-0.515658	7.26e+03