1) The battery of a particular car brand starts with probability 0.95. Find the probability of the following events:

a) **(1 point)** A : the battery only starts on the 5^{th} attempt;

b) **(2 points)** B : the battery starts on the first at least 20 consecutive attempts.

Pascal (Negative Binomial) Model: The probability of the n^{th} success occurring after k failures in a sequence of Bernoulli trials with probability of success p $(q = 1 - p)$, is $P(n;k) = C^{n-1}_{n+k-1} p^n q^k = C^k_{n+k-1} p^n q^k$.

Geometric Model: The probability of the 1^{st} success occurring after k failures in a sequence of Bernoulli trials with probability of success p $(q = 1 - p)$, is $p_k = pq^k$.

a) $k = 4$

$q = 0.05$

$p = 0.95$

$P_4 = 0.95 \cdot 0.05^4$ ✓

b) $P(X \geq 20) = 1 - P(X < 20) = 1 - \sum_{k=0}^{19} 0.05 \cdot 0.95^k$

2) (2 points) Let $X \in Exp(\mu)$. Find the pdf of $Y = \sqrt{X}$.

$\cancel{X = e^{\mu}}$ $\quad f_X(x) = \lambda e^{-\lambda x}$

$Y = \sqrt{e^{\mu}}$

$\sqrt{x} = y \Rightarrow x = y^2$

$g(x) = \sqrt{x}$

$g'(x) = \frac{1}{2\sqrt{x}} > 0 \Rightarrow g(x)$ strictly increasing

$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}$ $y \in g(\mathbb{R})$

$f(x) = y \Rightarrow g(y) = x$

$g^{-1}(y) = y^2 \Rightarrow$ $f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|} =$

$= \frac{f_X(y^2)}{|g'(y^2)|} = \frac{\lambda e^{-\lambda y^2}}{\frac{1}{2y}} = \boxed{2\lambda y e^{-\lambda y^2}}$

2) (2 points) Let $X \in Exp(\mu)$. Find the pdf of $Y = \sqrt{X}$.

3) Let $X_1, X_2, \dots, X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \frac{2}{\theta^2}x$, for $0 < x < \theta$, with $\theta > 0$ unknown.

a) (2 points) Find the method of moments estimator, $\hat{\theta}$, for θ.

b) (2 points) Is $\hat{\theta}$ an absolutely correct estimator? Explain.

a) $f(x;\theta) = \frac{2}{\theta^2}x$

$$\nu_1 = \overline{\nu_1}$$

$$\nu_1 = E(X) = \int_{-\infty}^{\infty} x f(x,\theta)\,dx =$$

$$= \int_0^\theta \frac{2x^2}{\theta^2} = \frac{2x^3}{3\theta^2}\Big|_0^\theta = \frac{2\theta^3}{3\theta^2} = \frac{2}{3}\theta$$

$$\overline{\nu_1} = \overline{X}$$

$$\Rightarrow \frac{2}{3}\theta = \overline{X} \Rightarrow \theta = \frac{3}{2}\overline{X} \Rightarrow \overline{\theta} = \frac{3}{2}\overline{X}$$

estimator obtained using method of moments

2) **(2 points)** Let $X \in Exp(\mu)$. Find the pdf of $Y = \sqrt{X}$.

3) Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \frac{2}{\theta^2}x$, for $0 < x < \theta$, with $\theta > 0$ unknown.

a) **(2 points)** Find the method of moments estimator, $\tilde{\theta}$, for θ.

b) **(2 points)** Is $\tilde{\theta}$ an absolutely correct estimator? **Explain.**

b) $\tilde{\theta} = \frac{3}{2}\bar{X}$

(i) $E(\tilde{\theta}) \overset{?}{=} \theta$

$$E(\tilde{\theta}) = E\left(\frac{3}{2}\bar{X}\right) = \frac{3}{2}E(\bar{X}) = \frac{3}{2}E(X) = \frac{3}{2}\cdot\frac{2}{3}\theta = \theta$$

(ii) $\lim_{n\to\infty} V(\tilde{\theta}) \overset{?}{=} 0$

$$V(\tilde{\theta}) = \frac{V(\theta)}{n} = \frac{E(X^2) - (E(X))^2}{n} =$$

$$E(X^2) = \int_0^\theta x^2 \frac{2}{\theta^2} x \, dx = \frac{2x^4}{4\theta^2}\Big|_0^\theta = \frac{1}{2}\theta^2 \Rightarrow V(\tilde{\theta}) = \frac{\frac{1}{2}\theta^2 - \frac{4}{9}\theta^2}{n} = \frac{\theta^2}{18n} \to 0$$