1) A contestant participates in a game show where three important prizes are offered. His chances of winning the three prizes are $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$, respectively.

a) **(1 point)** Find the probability that the contestant wins exactly one prize.

b) **(1.5 points)** Find the probability that the contestant loses at least two prizes.

c) **(1.5 points)** Let X denote the number of prizes won by the contestant. Find the probability distribution function of X.

d) **(1 point)** How many prizes can the contestant expect to win?

a) $P(X=1)$ $\quad p_1 = \frac{1}{6}$; $p_2 = \frac{1}{3}$; $p_3 = \frac{1}{2}$

<u>Poisson Model</u>: The probability of k successes $(0 \le k \le n)$ in n trials, with probability of success p_i in the i^{th} trial $(q_i = 1 - p_i)$, $i = \overline{1,n}$, is $P(n;k) = \sum_{1 \le i_1 < \ldots < i_k \le n} p_{i_1} \ldots p_{i_k} q_{i_{k+1}} \ldots q_{i_n}$, $i_{k+1}, \ldots, i_n \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ = the coefficient of x^k in the expansion $(p_1 x + q_1)(p_2 x + q_2) \ldots (p_n x + q_n)$.

$K = 1$

$$\left(\frac{1}{6}x + \frac{5}{6}\right)\cdot\left(\frac{1}{3}x + \frac{2}{3}\right)\cdot\left(\frac{1}{2}x + \frac{1}{2}\right) = \left(\frac{1}{6}x + \frac{5}{6}\right)\left(\frac{1}{6}x^2 + \frac{1}{3}x + \frac{1}{6}x + \frac{1}{3}\right) = \left(\frac{1}{6}x + \frac{5}{6}\right)\cdot\left(\frac{1}{6}x^2 + \frac{1}{2}x + \frac{1}{3}\right) =$$

$$= \frac{1}{36}x^3 + \frac{1}{12}x^2 + \frac{1}{18}x + \frac{5}{36}x^2 + \frac{5}{12}x + \frac{5}{18} =$$

$$= \frac{1}{36}x^3 + \frac{8}{36}x^2 + \frac{17}{36}x + \frac{5}{18}x^0$$

$$P(X=1) = \frac{17}{36}$$

1) A contestant participates in a game show where three important prizes are offered. His chances of winning the three prizes are $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$, respectively.

a) **(1 point)** Find the probability that the contestant wins exactly one prize.

b) **(1.5 points)** Find the probability that the contestant loses at least two prizes.

c) **(1.5 points)** Let X denote the number of prizes won by the contestant. Find the probability distribution function of X.

d) **(1 point)** How many prizes can the contestant expect to win?

b) lose at least 2 prizes $\Leftrightarrow$ wins at most 1 prize

$$P(B) = P(0) + P(1) = \frac{17}{36} + \frac{5}{18} = \frac{27}{36} = \frac{3}{4}$$

c)
$$X \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{5}{18} & \frac{17}{36} & \frac{8}{36} & \frac{1}{36} \end{pmatrix}$$

d)
$$E(X) = \frac{5}{18} \cdot 0 + \frac{17}{36} \cdot 1 + \frac{8}{36} \cdot 2 + 3 \cdot \frac{1}{36} = \frac{17}{36} + \frac{16}{36} + \frac{3}{36} = 1$$

2) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2, 3\theta)$ distribution, with $\theta > 0$ unknown. (for $X \in Gamma(a, b)$, the pdf is $f(x; a, b) = \frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}$, $x > 0$, $E(X) = ab$, $V(X) = ab^2$)

a) **(1.5 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

b) **(0.5 points)** Is it an absolutely correct estimator? Explain.

c) **(2 points)** Find the efficiency of $\overline{\theta}$, $e(\overline{\theta})$.

$\Gamma(n+1) = n!$

$\Gamma(2) = \Gamma(1+1) = 1!$

a) Since 2 is a constant, we'll rewrite the function f as $f(x;3\theta) = \frac{1}{(3\theta)^2\Gamma(2)} \cdot x e^{-\frac{x}{3\theta}}$

$\Gamma(2) \to 1!$

$$f(x;3\theta) = \frac{1}{9\theta^2} x e^{-\frac{x}{3\theta}}$$

$$L(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} \frac{1}{9\theta^2} x_i e^{-\frac{x_i}{3\theta}} = k \cdot \left(\frac{1}{\theta^2}\right)^n \cdot e^{-\frac{1}{3\theta}\sum_{i=1}^{n}(x_i)} = k \cdot \frac{1}{\theta^{2n}} \cdot e^{-\frac{1}{3\theta}n\overline{x}}$$

$$k = \prod_{i=1}^{n} \frac{1}{9} x_i$$

$$\ln(L) = \ln k + \ln\frac{1}{\theta^{2n}} - \frac{1}{3\theta}n\overline{x} = \cancel{\ln k} - 2n\ln\theta - \frac{n\overline{x}}{3\theta}$$

$$\frac{\partial L}{\partial\theta} = -\frac{2n}{\theta} + \frac{n\overline{x}}{3\theta^2} = 0 \Leftrightarrow \frac{n\overline{x}}{3\theta^2} = \frac{2n}{\theta} \quad \Big| \cdot \frac{\theta}{n}$$

$$\frac{\overline{x}}{3\theta} = 2 \Rightarrow \theta = \frac{\overline{x}}{6} \Rightarrow \overline{\theta} = \frac{\overline{x}}{6}$$

2) Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a $Gamma(2, 3\theta)$ distribution, with $\theta > 0$ unknown.
(for $X \in Gamma(a, b)$, the pdf is $f(x; a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$, $x > 0$, $E(X) = ab$, $V(X) = ab^2$)

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c) **(2 points)** Find the efficiency of $\overline{\theta}$, $e(\overline{\theta})$.

b) E $\quad \overline{\theta} = \frac{\overline{X}}{6}$

i) $E(\overline{\theta}) \overset{?}{=} \theta \qquad E(\overline{\theta}) = E\left(\frac{\overline{X}}{6}\right) = \frac{1}{6} E(\overline{X}) = \frac{1}{6} E(X) \Rightarrow$

$E(\overline{\theta}) = 6\theta \cdot \frac{1}{6} \Rightarrow E(\overline{\theta}) = \theta$

ii) $\lim_{n \to \infty} V(\overline{\theta}) \overset{?}{=} 0 \qquad V(\overline{\theta}) = V\left(\frac{\overline{X}}{6}\right) = \frac{1}{36} V(\overline{X}) = \frac{1}{36} \cdot \frac{V(X)}{n}$

$= \frac{1 \cdot 18\theta^2}{36n} = \frac{\theta^2}{2n} \xrightarrow{n \to \infty} 0 \qquad$ i, ii) $\Rightarrow$ absolutely correct estimator

$\frac{\theta^2}{2n}$

2) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2, 3\theta)$ distribution, with $\theta > 0$ unknown. (for $X \in Gamma(a, b)$, the pdf is $f(x; a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$, $x > 0$, $E(X) = ab$, $V(X) = ab^2$)

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Definition 5.11. *Let $\overline{\theta} = \overline{\theta}(X_1, \ldots, X_n)$ be an absolutely correct estimator for θ. The **efficiency** of $\overline{\theta}$ is the quantity*

$$e(\overline{\theta}) = \frac{I_n^{-1}(\theta)}{V(\overline{\theta})} = \frac{1}{I_n(\theta) V(\overline{\theta})}. \quad (5.9)$$

*The estimator $\overline{\theta}$ is said to be **efficient** for θ, if $e(\overline{\theta}) = 1$.*

Fisher's Information: $I_n(\theta) = E\left[\left(\frac{\partial \ln L(X_1, ..., X_n|\theta)}{\partial \theta}\right)^2\right]$;

- if the range of X does not depend on θ, then $I_n(\theta) = -E\left[\frac{\partial^2 \ln L(X_1, ..., X_n|\theta)}{\partial^2 \theta}\right]$ and $I_n(\theta) = nI_1(\theta)$.

c) $e(\overline{\theta}) =$

$$I_n(\theta) = E\left[\left(-\frac{2n}{\theta} + \frac{n\overline{x}}{3\theta^2}\right)^2\right] = E\left[\left(\frac{n}{\theta}\right)^2\left(-2 + \frac{\overline{x}}{3\theta}\right)^2\right] =$$

$$= \left(\frac{n}{\theta}\right)^2 E\left(4 - 4\frac{\overline{x}}{3\theta} + \left(\frac{\overline{x}}{3\theta}\right)^2\right) = \left(\frac{n}{\theta}\right)^2\left(4 - \frac{4}{3}E\left(\frac{\overline{x}}{\theta}\right) + \frac{1}{9}E\left(\left(\frac{\overline{x}}{\theta}\right)^2\right)\right)$$

$$= \left(\frac{n}{\theta}\right)^2\left(\frac{36 - 12E\left(\frac{\overline{x}}{\theta}\right) + E\left(\left(\frac{\overline{x}}{\theta}\right)^2\right)}{9}\right) = \frac{1}{9}\left(\frac{n}{\theta}\right)^2\left(36 - 12E\left(\frac{\overline{x}}{\theta}\right) + E\left(\left(\frac{\overline{x}}{\theta}\right)^2\right)\right.$$

$$= \frac{1}{9}\left(\frac{n}{\theta}\right)^2\left(36 - 12\frac{1}{\theta}E(\bar{x}) + \frac{1}{\theta^2}E(\bar{x}^2)\right)$$

$$= \frac{1}{9}\left(\frac{n}{\theta}\right)^2\left(36 - \cancel{7}2 + \frac{1}{\theta^2}\cdot\left(\frac{18\theta^2}{n} + 36\theta^2\right)\right)$$

$$E(x^2) = V(x) + (E(x))^2 = V(\bar{x}) + (E(\bar{x}))^2 = \frac{V(x)}{n} + (E(x))^2$$

$$= \frac{2\cdot 9\theta^2}{n} + (6\theta)^2 = \frac{18\theta^2}{n} + 36\theta^2$$

$$\to = \frac{1}{9}\left(\frac{n}{\theta}\right)^2\left(\cancel{-36} + \frac{18}{n} + \cancel{36}\right) = \frac{2}{n}\cdot\left(\frac{n}{\theta}\right)^2 = \frac{2n}{\theta^2}$$

$e(\theta)$

$$e(\theta) = \frac{\mathcal{I}_n^{-1}(\theta)}{V(\bar{\theta})} = \frac{1}{\frac{2n}{\theta^2} \cdot \frac{\theta^2}{2n}} = 1 \Rightarrow \text{is efficient!}$$