1) The probability that the battery of a particular car brand does not start is 0.03. Find the probability of the following events:

a) **(1 point)** A : the battery only starts on the 3^{rd} attempt;

b) **(2 points)** B : the battery starts on the first at least 25 consecutive attempts.

a) $X \sim$ Geometric model, success: starts, $p = 0.97$, $q = 0.03$

trial: starting battery, failure: doesn't start

$$P(X=3) = 0.97 \cdot 0.03^2$$

b) success: doesn't start ($p = 0.03$)

failure: starts ($q = 0.97$)

A: "1st success after 25 or more failures"

$\bar{A}$: "1st success after less than 25 failures"

$$\Rightarrow P(A) = 1 - P(\bar{A}) = 1 - \sum_{k=0}^{24} 0.03 \cdot 0.97^k$$

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = (1+\theta)x^{\theta}$, for $0 < x < 1$, with $\theta > -1$, unknown.

a) **(2 points)** Find the method of moments estimator, $\hat{\theta}$, for θ.

b) **(2 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

a) $\nu_k = \overline{\nu}_k$ (solve the system)

$k=1$ (1 unknown)

$\nu_1 = E(X) \qquad \overline{\nu}_1 = \overline{X}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = (1+\theta)\int_0^1 x^{\theta+1}\,dx = (1+\theta)\cdot \frac{x^{\theta+2}}{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}$$

$$\Rightarrow \frac{\theta+1}{\theta+2} = \overline{X} \Leftrightarrow \theta + 1 = \theta\overline{X} + 2\overline{X} \Leftrightarrow \theta(\overline{X}-1) = 1-2\overline{X} \Leftrightarrow$$

$$\Rightarrow \theta = \frac{1-2\overline{X}}{\overline{X}-1} \Rightarrow \hat{\theta} = \frac{1-2\overline{X}}{\overline{X}-1},$$ estimator obtained using method of moments

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = (1+\theta)x^\theta$, for $0 < x < 1$, with $\theta > -1$, unknown.

a) **(2 points)** Find the method of moments estimator, $\hat{\theta}$, for θ.

b) **(2 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

b) $\dfrac{\partial \ln L(x_1, x_2, \dots x_n ; \theta)}{\partial \theta} = 0$ (solve the system)

$$L(x_1, x_2, \dots, x_n ; \theta) = \prod_{i=1}^{n} f(x_i ; \theta) = \prod_{i=1}^{n} (1+\theta) x_i^\theta = (1+\theta)^n \underbrace{\left(\prod_{i=1}^{n} x_i\right)}_{k}^\theta =$$

$$= (1+\theta)^n \cdot k^\theta \Rightarrow \ln L = n \ln(1+\theta) + \theta \ln k$$

$$\Rightarrow \frac{\partial \ln L}{\partial \theta} = \frac{n}{1+\theta} + \ln k = 0 \Rightarrow \theta = \frac{-n}{\ln k} - 1 \Rightarrow$$

$$\Rightarrow \overline{\theta} = \frac{-n}{\ln \prod_{i=1}^{n} x_i} - 1$$, estimator obtained using method of maximum likelihood