1) A student has to take two exams. His chances of passing the two tests are $\frac{4}{5}$ and $\frac{3}{4}$, respectively.

a) **(1 point)** Find the probability that the student passes exactly one exam.

b) **(1 point)** Let X denote the number of exams the student passes. Find the probability distribution function of X.

c) **(0.5 points)** How many tests is the student expected to pass?

1) We will use Poisson Model

$k = 1$

$n = 2$

$p_1 = \frac{4}{5}$ $q_1 = \frac{1}{5}$

$p_2 = \frac{3}{4}$ $q_2 = \frac{1}{4}$

$$P(2,1) = \left(\frac{4}{5}x + \frac{1}{5}\right)\left(\frac{3}{4}x + \frac{1}{4}\right)$$

$$= \frac{12x^2}{20} + \frac{4x}{20} + \frac{3x}{20} + \frac{1}{20}$$

$$= \frac{12x^2 + 7x + 1}{20}$$

To pass exactly one exam we need the coff. of x^1 which is $\frac{7}{20}$

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b) X = nr of exams student passes

pdf(x) = ?

$$\begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{20} & \frac{7}{20} & \frac{12}{20} \end{pmatrix}$$

Since we use the Poisson Model we will use the coeff of x for every possible chance.

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c) $E(x) = \sum_{i=0}^{2} x_i p_i = 0 \cdot \frac{1}{20} + \frac{7}{20} + \frac{12}{10} = \frac{31}{20} = 1.55$

2) **(2 points)** The pdf of the random variable X is given by $f_X(x) = 2e^{-2x}$, $x > 0$. Find the pdf of $Y = \frac{1}{2}X - 1$.

$g(x) = \frac{1}{2}x - 1$

$g'(x) = \frac{1}{2} \Rightarrow g$ strictly increasing

$\frac{1}{2}x - 1 = y \Rightarrow \frac{1}{2}x = y + 1 \Rightarrow x = 2y + 2$

$x = 2(y+1)$

$\Rightarrow g^{-1}(y) = 2(y+1)$

2) **(2 points)**. The pdf of the random variable X is given by $f_X(x) = 2e^{-2x}$, $x > 0$. Find the pdf of $Y = \frac{1}{2}X - 1$.

$$f_Y(y) = \frac{f_X(g^{-1}(y))}{|g'(g^{-1}(y))|}, \quad y \in g(R)$$

$$f_Y(y) = \frac{2e^{-4(y+1)}}{\frac{1}{2}} = \boxed{4e^{-4(y+1)}} \text{ pdf of } Y$$

$$f_X(2(y+1)) = 2e^{-2(2(y+1))}$$

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2,\theta)$ distribution, with $\theta > 0$ unknown. (for $X \in Gamma(a,b)$, the pdf is $f(x;a,b) = \frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}$, $x > 0$, $E(X) = ab$, $V(X) = ab^2$)

a) **(1.5 points)** Find the method of moments estimator, $\hat{\theta}$, for θ;

b) **(2 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

c) **(1 point)** Is $\overline{\theta}$ an absolutely correct estimator? Explain.

$\Gamma(2) = 1! = 1$

a) $\nu_k = \overline{\nu_k}$ (solve the system), $k = 1$ (nr of unknowns)

$\nu_1 = E(X) = a \cdot b = 2\theta$

$\overline{\nu_1} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$ (mean)

$2\theta = \overline{X} \Rightarrow \theta = \frac{\overline{X}}{2} \Rightarrow \hat{\theta} = \frac{\overline{X}}{2}$ estimator obtained using method of moments

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2, \theta)$ distribution, with $\theta > 0$ unknown. (for $X \in Gamma(a, b)$, the pdf is $f(x; a, b) = \frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}, \quad x > 0, \ E(X) = ab, \ V(X) = ab^2$)

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b) $\frac{\partial \ln L(x_1, ..., x_n)}{\partial \theta} = 0$

$L(x_1, ..., x_n) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} \frac{1}{\theta^2 \cdot 1} x_i^1 e^{-x_i/\theta} =$

$= \left(\frac{1}{\theta^2}\right)^n \prod_{i=1}^{n} x_i \cdot e^{-\frac{x_i}{\theta}} = \left(\frac{1}{\theta^2}\right)^n \underbrace{\prod_{i=1}^{n} x_i}_{K} \cdot e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} =$

$= \frac{1}{\theta^{2n}} \cdot K \cdot e^{-\frac{1}{\theta} \cdot n \cdot \overline{X}} \Rightarrow \ln L = -2n \ln \theta + \ln k - \frac{n\overline{X}}{\theta} \cdot 1$

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2, \theta)$ distribution, with $\theta > 0$ unknown. (for $X \in Gamma(a, b)$, the pdf is $f(x; a, b) = \frac{1}{b^a \Gamma(a)} x^{a-1} e^{-x/b}$, $x > 0$, $E(X) = ab$, $V(X) = ab^2$)

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$$\frac{\partial \ln L}{\partial \theta} = -\frac{2n}{\theta} + \frac{n\bar{x}}{\theta^2} = 0 \Rightarrow$$

$$\Rightarrow -2n + \frac{n\bar{x}}{\theta} = 0 \Rightarrow \frac{n\bar{x}}{\theta} = 2n \Rightarrow \frac{\cancel{n}\bar{x}}{2\cancel{n}} = \theta$$

$$\Rightarrow \theta = \frac{\bar{x}}{2}$$

$$\Rightarrow \overline{\theta} = \frac{\bar{x}}{2}$$

estimator obtained using maximum likelihood

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a $Gamma(2, \theta)$ distribution, with $\theta > 0$ unknown.
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a) (1.5 points) Find the method of moments estimator, $\hat{\theta}$, for θ;

b) (2 points) Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

c) (1 point) Is $\overline{\theta}$ an absolutely correct estimator? Explain.

i) $E(\bar{\theta}) = \theta$ and ii) $\lim_{n \to \infty} V(\bar{\theta}) = 0$

$$E(\bar{\theta}) = E\left(\frac{\bar{X}}{2}\right) = \frac{1}{2} E(\bar{X}) = \frac{1}{2} E(X) = \frac{1}{2} \cdot 2 \cdot \theta = \theta$$

$$\lim_{n \to \infty} V(\bar{\theta}) = \lim_{n \to \infty} \frac{\theta^2}{2n} = 0$$

$$V(\bar{\theta}) = V\left(\frac{\bar{X}}{2}\right) = \frac{1}{4} V(\bar{X}) = \frac{1}{4} \frac{V(X)}{n} = \frac{2\theta^2}{4n} = \frac{\theta^2}{2n}$$

i, ii) $\rightarrow$ $\bar{\theta}$ an absolutely correct estimator