1) Recent studies show that air traffic controllers correctly identify 95 out of 100 signals. 25 signals arrive in a 30 minute period.

a) **(0.5 points)** Find the probability that exactly 24 signals are identified correctly.

b) **(1 point)** Find the probability that at least 20 signals are identified correctly.

c) **(1 point)** Let X denote the number of correctly identified signals. Find the probability distribution function of X. What type of distribution is it?

d) **(1 point)** Consider an hour and a half in which 25 signals arrive every half hour. Find the probability that at least 20 signals are identified correctly in every half hour.

a) 95 correct out of 100 $\Rightarrow p = 0.95 \Rightarrow X \sim \text{Binomial}(0.95)$

$$P(X=24) = C_{25}^{24} \cdot 0.95^{24} \cdot 0.05 = 25 \cdot 0.95^{24} \cdot 0.05$$

b) A: at least 20 sign. $\Rightarrow \bar{A}$: less than 20 sign.

$$\Rightarrow P(A) = 1 - P(\bar{A}) = 1 - \sum_{i=0}^{19} C_{25}^{i} \cdot (0.95)^{i} \cdot (0.05)^{25-i}$$

B: less than 6 are wrong

$$\Rightarrow P(B) = \sum_{i=0}^{5} C_{25}^{i} \cdot (0.05)^{i} \cdot (0.95)^{25-i}$$

C: $\sum_{i=20}^{25} C_{25}^{i} \cdot (0.95)^{i} \cdot (0.05)^{25-i}$

1) Recent studies show that air traffic controllers correctly identify 95 out of 100 signals. 25 signals arrive in a 30 minute period.

a) **(0.5 points)** Find the probability that exactly 24 signals are identified correctly.

b) **(1 point)** Find the probability that at least 20 signals are identified correctly.

c) **(1 point)** Let X denote the number of correctly identified signals. Find the probability distribution function of X. What type of distribution is it?

d) **(1 point)** Consider an hour and a half in which 25 signals arrive every half hour. Find the probability that at least 20 signals are identified correctly in every half hour.

c) $X \sim Bin(0.95)$

$$\Rightarrow \quad X \begin{pmatrix} k \\ C_{25}^{k} \cdot (0.95)^{k} \cdot (0.05)^{25-k} \end{pmatrix} \quad k = \overline{0,25}$$

d) $(P(A))^3$

$\hookrightarrow$ se at b)

2) (1.5 points) Let X be a random variable with pdf $\begin{pmatrix} -2 & -1 & 0 & 2 \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}$. Find the pdf of $Y = 2X^2$.

$X = -2 \Rightarrow Y = 8$

$X = -1 \Rightarrow Y = 2$

$X = 0 \Rightarrow Y = 0$

$X = 2 \Rightarrow Y = 8$

$\Rightarrow P(Y = 8) = P((X = -2) \cup (X = 2)) = \frac{1}{4}$

$P(Y = 2) = P(X = -1) = \frac{1}{4}$

$P(Y = 0) = P(X = 0) = \frac{1}{2}$

$\Rightarrow$ pdf of $Y \Rightarrow Y \begin{pmatrix} 0 & 2 & 8 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from distribution with pdf $f(x;\theta) = \frac{\theta^x}{x!}e^{-\theta}$, $x = 0, 1, ...,$ $E(X) = V(X) = \theta$, with $\theta > 0$, unknown.

a) (1.5 points) Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

b) (0.5 points) Is it an absolutely correct estimator? Explain.

c) (2 points) Find the efficiency of $\overline{\theta}$, $e(\overline{\theta})$.

13

a) $\frac{\partial \ln L(x_1, x_2, \dots, x_n; \theta)}{\partial \theta} = 0$

$L(x_1, x_2, \dots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) =$

$= e^{-n\theta} \cdot \underbrace{\prod_{i=1}^{n} \frac{1}{x_i!}}_{k} \cdot \theta^{\sum_{i=1}^{n} x_i} = e^{-n\theta} \cdot k \cdot \theta^{n\overline{x}}$

$\Rightarrow \ln L = -n\theta + \ln k + n\overline{x} \ln \theta$

$\Rightarrow \frac{\partial \ln L}{\partial \theta} = -n + \frac{n\overline{x}}{\theta} = 0 \Leftrightarrow \frac{n\overline{x}}{\theta} = n \Leftrightarrow \theta = \frac{n\overline{x}}{n} \Leftrightarrow \theta = \overline{x}$

$\Rightarrow \overline{\theta} = \overline{x}$ estimator obtained using the method of maximum likelihood

3) Let $X_1, X_2, \dots, X_n$ be a random sample drawn from distribution with pdf $f(x;\theta) = \frac{\theta^x}{x!}e^{-\theta}$, $x = 0, 1, \dots,$
$E(X) = V(X) = \theta$, with $\theta > 0$, unknown.

a) (1.5 points) Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

b) (0.5 points) Is it an absolutely correct estimator? Explain.

c) (2 points) Find the efficiency of $\overline{\theta}$, $e(\overline{\theta})$.

13

b) $\overline{\theta} = \overline{X}$

i) $E(\overline{\theta}) \overset{?}{=} \theta \quad E(\overline{\theta}) = E(\overline{x}) = E(X) = \theta$

ii) $\lim_{n\to\infty} V(\overline{\theta}) \overset{?}{=} 0 \quad V(\overline{\theta}) = \frac{V(\theta)}{n} = \frac{\theta}{n} \Rightarrow \lim_{n\to\infty} \frac{\theta}{n} = 0$

c) $e(\overline{\theta}) = \frac{1}{I_n(\theta)\, V(\overline{\theta})}, \quad I_n(\theta) = E\left(\left(\frac{\partial \ln L}{\partial \theta}\right)^2\right)$

$I_n(\theta) = E\left(\left(\frac{n\overline{X}}{\theta} - n\right)^2\right) = E\left(\frac{n^2\overline{X}^2}{\theta^2} - \frac{2n^2\overline{X}}{\theta} + n^2\right) = E\left(\frac{n^2\overline{X}^2}{\theta^2}\right) +$

$E\left(-\frac{2n^2\overline{X}}{\theta}\right) + E(n^2) = \frac{n^2}{\theta^2}E(\overline{X}^2) - \frac{2n^2}{\theta}E(\overline{x}) + n^2$

$$E(\bar{X}) = E(X) = \theta$$

$$V(\bar{X}) = E(\bar{X}^2) - (E(\bar{X}))^2 \Rightarrow E(\bar{X}^2) = V(\bar{X}) + E(\bar{X})^2 =$$

$$= \frac{V(X)}{n} + E(X)^2 = \frac{\theta}{n} + \theta^2$$

$$\Rightarrow I_n(\theta) = \frac{n^2}{\theta^2} \cdot \left(\frac{\theta}{n} + \theta^2\right) - \frac{2n^2}{\theta} \cdot \theta + n^2 =$$

$$= \frac{n}{\theta} + n^2 - 2n^2 + n^2 = \frac{n}{\theta}$$

$$V(\hat{\theta}) = V(\bar{X}) = \frac{V(X)}{n} = \frac{\theta}{n}$$

$$\Rightarrow e(\hat{\theta}) = \frac{1}{\frac{n}{\theta} \cdot V(\hat{\theta})} = 1$$

$\Rightarrow$ efficient