1) The probability of a certain basketball player making a free throw is known to be 0.6. Find the probability of the following events:

a) **(1 point)** A : the player makes his first free throw only on the 5^{th} shot;

b) **(2 points)** B : the player makes the first at least 7 consecutive free throws.

a) Geometric model, success: "makes free throw", failure: "no free throw"

$p = 0.6$ $q = 0.4$, $K = 4$, trial: "makes a shot"

$P(X=5) = 0.6 \cdot 0.4^4$

b) success: "doesn't make a free throw"

failure: "makes a free throw"

$A =$ "1st success after 7 or more failures".

$\bar{A} =$ "1st success after less than 7 failures"

$P(A) = 1 - P(\bar{A}) = 1 - \sum_{k=0}^{6} 0.4 \cdot 0.6^k$

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}$, for $0 < x < 1$, with $\theta > 0$, unknown.

a) **(2 points)** Find the method of moments estimator, $\hat{\theta}$, for θ.

b) **(2 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

a) $\vartheta_k = \overline{\vartheta}_k$

$\vartheta_1 = E(x)$, $\overline{\vartheta}_1 = \overline{x}$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \theta \int_0^1 x^\theta\,dx = \theta \left.\frac{x^{\theta+1}}{\theta+1}\right|_0^1 = \theta \cdot \frac{1}{\theta+1}$$

$$= \frac{\theta}{\theta+1} \Rightarrow \frac{\theta}{\theta+1} = \overline{x} \Leftrightarrow \theta = \theta\overline{x} + \overline{x} \Rightarrow \theta(1-\overline{x}) = \overline{x}$$

$$\Rightarrow \theta = \frac{\overline{x}}{1-\overline{x}} \Rightarrow \overline{\theta} = \frac{\overline{x}}{1-\overline{x}}$$ estimator obtained using method of moments

3) Let $X_1, X_2, ..., X_n$ be a random sample drawn from a distribution with pdf $f(x;\theta) = \theta x^{\theta-1}$, for $0 < x < 1$, with $\theta > 0$, unknown.

a) **(2 points)** Find the method of moments estimator, $\hat{\theta}$, for θ.

b) **(2 points)** Find the maximum likelihood estimator, $\overline{\theta}$, for θ.

b) $\frac{\partial \ln L(x_1, ..., x_n; \theta)}{\partial \theta} = 0$

$$L(x_1, ..., x_n; \theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \Big(\underbrace{\prod_{i=1}^{n} x_i}_{K}\Big)^{\theta-1} =$$

$$= \theta^n \cdot K^{\theta-1} \Rightarrow \ln L = n \ln \theta + (\theta-1)\ln K$$

$$\Rightarrow \frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \ln K = 0 \Rightarrow \theta = \frac{-\ln K}{n}$$

$$\Rightarrow \overline{\theta} = -\frac{\ln\left(\prod_{i=1}^{n} x_i\right)}{n}$$

estimator obtained using maximum likelihood method