MFML - Special Session

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ a_1 & a_2 & & a_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$a_i^T a_j = 0 \quad \forall i \neq j$

orthogonal

$a_i^T a_j = 0 \quad \forall i \neq j$

$a_i^T a_i = 1 \quad \forall i$

orthonormal

$$\begin{array}{cc} a_1 & a_2 \end{array}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{orthogonal}$$

$a_1 \cdot a_2 = a_1^T a_2$

$$\begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

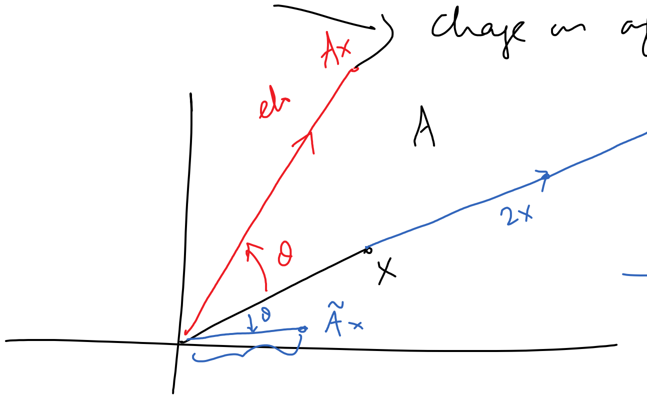
0

$$\rightarrow \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \quad \text{orthonormal.}$$

1 1

$A_{n \times n} x \longrightarrow$ Extension / compression

$\longrightarrow$ change in angle / rotation



$x \neq 0$, λ

$\underline{A}\,\underline{x} = \lambda \underline{x}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdot & a_{nn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$a_{i1} x_1 + a_{i2} x_2 + \underset{\wedge}{} a_{ii} x_i + a_{in} x_n = \lambda x_i \quad \forall i$

$a_{i1} x_1 + a_{i2} x_2 + (a_{ii} - \lambda) x_i + \cdot + a_{in} x_n = 0$

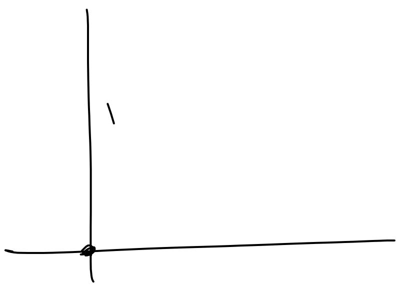
$\forall i$

$a_{i1}x_1 + a_{i2}x_2 + \dots + (a_{ii}-\lambda)x_i + \dots + a_{in}x_n = 0$

$$(A-\lambda I)\begin{pmatrix}x_1\\x_2\\ \vdots \\ x_n\end{pmatrix} = 0 \; ; \; (A-\lambda I)x = 0$$

$$\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \lambda\begin{pmatrix}x_1\\x_2\end{pmatrix}$$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 &= \lambda x_2 \end{aligned} \Rightarrow \begin{bmatrix}(a_{11}-\lambda)x_1 + a_{12}x_2 = 0\\ a_{21}x_1 + (a_{22}-\lambda)x_2 = 0\end{bmatrix}$$



$$\begin{pmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$(A-\lambda I)x = 0 \quad \text{—(1)}$$

Suppose $(A-\lambda I)$ is non-singular $\det(A-\lambda I) \neq 0$

$$(1)\; \underbrace{(A-\lambda I)^{-1}(A-\lambda I)}\, x = (A-\lambda I)^{-1}\, 0$$

$$Ix = 0 \Rightarrow x = 0$$

$\det(A-\lambda I) = 0$ yield eigenvalues of A

$$A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \qquad (A-\lambda I) = \begin{bmatrix}1-\lambda & 2\\ 3 & 4-\lambda\end{bmatrix}$$

$$\begin{aligned}\det(A-\lambda I) &= (1-\lambda)(4-\lambda) - 6\\ &= \lambda^2 - 5\lambda + 4 - 6 = \lambda^2 - 5\lambda - 2 = 0\end{aligned}$$

$$\lambda = \frac{5 \pm \sqrt{25+8}}{2} = \frac{5 \pm \sqrt{33}}{2}$$

$$\lambda_1 = \frac{5+\sqrt{33}}{2}, \quad \lambda_2 = \frac{5-\sqrt{33}}{2}$$

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Result. An $n \times n$ matrix A has n eigenvalues

λ_1. $\quad \underbrace{(A - \lambda_1 I)x = 0}$ $\quad$ x is called the eigenvector corr to λ_1

$Ax_2 = \lambda_2 x_2$

$$A_{n\times n} \longrightarrow \begin{matrix} \lambda_1, & \lambda_2 & \cdots & \lambda_n \\ \downarrow & \downarrow & & \downarrow \\ x_1 & x_2 & \cdot & x_n \end{matrix}$$

$\underbrace{Ax_1 = \lambda_1 x_1}$, $\quad \underbrace{Ax_n = \lambda_n x_n}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{bmatrix}$$

$$\boxed{Ax = \lambda x}$$

$$A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}, \qquad (\lambda+1)(\lambda+4) - 4 = 0$$

$$\lambda^2 + 5\lambda = 0$$

$$\lambda = 0 \text{ or } \lambda = -5$$

$$\begin{pmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{pmatrix}$$

$\lambda = -5$ $\qquad Ax = -5x$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(A + 5I)x = 0$$

$$(A+5I)x = 0$$

$$\begin{pmatrix} -4+5 & 2 \\ 2 & -1+5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$Ax = \lambda x$$
$$(A-\lambda I)x = 0$$
$$(A+5I)x = 0$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 = 0$$
$2\times$
$$2x_1 + 4x_2 = 0$$

$$x_1 + 2x_2 = 0$$
$$x_1 = -2x_2$$

$$\begin{pmatrix} -2\alpha \\ \alpha \end{pmatrix} = \alpha\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

eigen vector is generated by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}}_{A}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \end{pmatrix} = -5\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Properties : 1. Sum of eigenvalues = Sum of diagonal elements of a matrix

$$-4-1 = -5+0$$

2 Product of eigenvalues = determinant

$$-5 \cdot 0 = 0$$

3 If $\lambda = 0$ is an eigenvalue, det $= 0$ the matrix is singular

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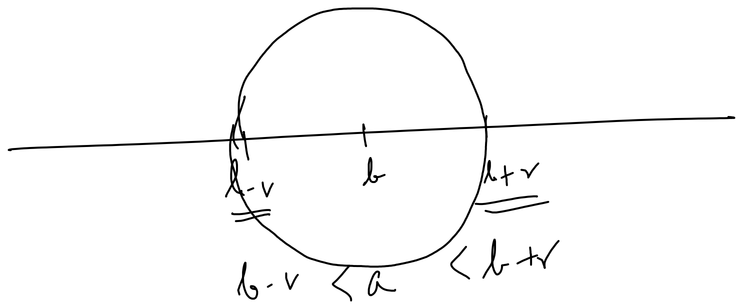
$A = \Bigg(\qquad \Bigg)$

(11)

$$A_{10\times10} = \begin{pmatrix} & \\ & \end{pmatrix}$$

$|a - b| < r$

Sphere with radius r & centre b.



$|Vah - Lah| < 50$

$L - 50$ $\quad$ $L + 50$

Gerschgorin

$$A = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}$$

λ

$$|\lambda - (-4)| < |2|$$

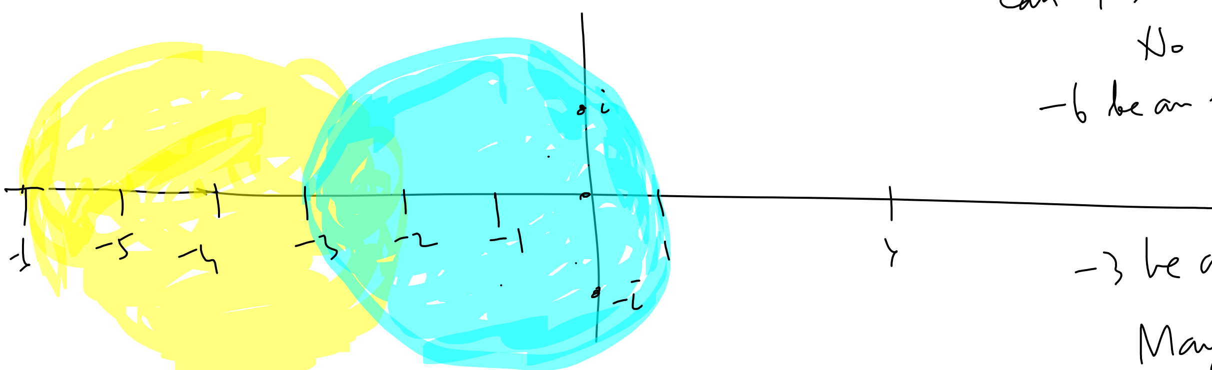
$$|\lambda - a_{ii}| < \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

$$|\lambda - (-1)| < |2|$$

$$|\lambda + 4| < 2, \quad |\lambda + 1| < 2$$

Can 4 be an eigenvalue? No

-6 be an eigenvalue? No



-3 be an eigenvalue? May be.

$$A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow (A - \lambda I) = \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \big((-\lambda)(-\lambda) + 1\big) = 0$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$$

$\lambda^2+1=0 \Rightarrow \lambda^2=-1$

$\lambda = \pm i$ Complex no

$0+i$, $0-i$

$\det \begin{bmatrix} 0-\lambda & 2 \\ 2 & 0-\lambda \end{bmatrix} \Rightarrow \det \begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix}$ $\lambda^2-4=0$

$\lambda = \pm 2$

$\cancel{A} = \begin{bmatrix} -\lambda & -2 \\ 2 & 4-\lambda \end{bmatrix}$ $A = \begin{bmatrix} 0 & -2 \\ 2 & 4 \end{bmatrix}$

$\det \; -\lambda(4-\lambda)+4=0$

$\lambda^2-4\lambda+4=0$

$(\lambda-2)^2=0$; $\underline{\lambda_1=2 , \lambda_2=2}$

If $a+ib$ is a root of ch. eq, $a-ib$ will also be a root

<u>Theorem.</u> If $\bar{A}^T = A$ (<u>Hermitian</u>) (<u>Symmetric</u>) $A = \begin{bmatrix} 1+i & 2+i \\ 3-i & 4-i \end{bmatrix}$

the <u>eigenvalues</u> <u>are real</u>. $\bar{A} \begin{bmatrix} 1-i & 2-i \\ 3+i & 4+i \end{bmatrix}$

Sy $\begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ch pn. det $\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21}$$

$$a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21}$$

$$\underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{C_0} \underbrace{-(a_{11}+a_{22})}_{C_1}\lambda + \underbrace{(-1)^2}_{C_2}\lambda^2$$

$ax^2 + bx + c = 0$

Sum of the roots $= -\frac{b}{a}$

Product of the roots $= c/a$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$C_0 + C_1\lambda + C_2\lambda^2 = 0$ ch poly

Sum of the roots $= -\frac{C_1}{C_2} = \frac{a_{11}+a_{22}}{1}$

$(\lambda_1+\lambda_2) \rightarrow = a_{11}+a_{22}$

Product of the roots $= \frac{C_0}{C_2} = \frac{(a_{11}a_{22}-a_{21}a_{12})}{1}$

$\lambda_1\lambda_2 \rightarrow = \det(A)$

$\underline{(Ax=\lambda x)}$, $\underbrace{A \text{ is orthogonal}}$

$AA^T = I$

$(Ax)^T = (\lambda x)^T \Rightarrow x^T A^T = \lambda x^T$

$Ax = \lambda x$

$x^T \cancel{A^T A} x = \lambda x^T \lambda x$

$\underbrace{x^T x} = \lambda^2 \underbrace{x^T x}$, $x \neq 0 \Rightarrow x^T x \neq 0$

$1 = \lambda^2 \Rightarrow \lambda = \pm 1$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ eigenvalues $\det\begin{pmatrix} (1-\lambda) & 0 \\ 0 & (1-\lambda) \end{pmatrix} = 0$

$(\lambda-1)^2 = 0 \Rightarrow \lambda = 1, 1$

$\lambda_1 = 1, \lambda_2 = 1$

$\lambda = 1$ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cdot \\ x_2 \end{bmatrix} = \begin{bmatrix} \cdot \\ 0 \end{bmatrix}$$

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$$

A^T . $\det(A^T - \lambda I) = 0$

$= \underbrace{\det(A - \lambda I)^T} = 0$

$= \det(A - \lambda I) = 0$

λ is an eigenvalue of $A \Leftrightarrow \lambda$ is an eigenvalue of A^T

$A = \lambda_1, \lambda_2, \ldots \lambda_n$ and <u>all are different</u>

$\underbrace{x_1 \quad x_2 \quad \ldots \quad x_n}$ are <u>LI</u>

<u>Proof</u>. Let $x_1, x_2 \ldots x_n$ be not LI

Sp $\underline{x_1, x_2 \ldots x_r}$ are only LI $\quad r < n$

r is the max index upto which $x_1 \ldots x_r$ are LI

$$\boxed{x_{r+1} = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r} \quad -(1)$$

$$A x_{r+1} = \alpha_1 A(x_1) + \alpha_2 A(x_2) + \ldots \alpha_r A(x_r)$$

(1) x A $\quad (2) \quad - \lambda_{r+1} x_{r+1} = \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 + \ldots + \alpha_r \lambda_r x_r$

(1) x λ_{r+1} $\quad (3) \quad - \lambda_{r+1} x_{r+1} = \alpha_1 \lambda_{r+1} x_1 + \alpha_2 \lambda_{r+1} x_2 + \ldots + \alpha_r \lambda_{r+1} x_r$

$$0 = \alpha_1(\lambda_1 - \lambda_{r+1}) x_1 + \ldots + \alpha_r(\lambda_r - \lambda_{r+1}) x_r$$

$\alpha_1 \cancel{(\lambda_1 - \lambda_{r+1})} = 0$, $\ldots$ $\alpha_r(\lambda_r - \lambda_{r+1}) = 0$

$\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$

$\Rightarrow x_{r+1} = 0$ a contradiction as x_{r+1} is

eigenvector and hence $\neq 0$

$\Rightarrow X_{r+1} = 0$ a contradiction as X_{r+1} is an eigenvector and hence $\neq 0$

$$A \longrightarrow \underset{\substack{\downarrow \\ X_1}}{11.2}, \underset{\substack{\downarrow \\ X_2}}{26}, \underset{\substack{\downarrow \\ X_3}}{41}, \underset{\substack{\downarrow \\ X_4}}{-2.8}$$

$\therefore X_1 \ldots X_4$ are LI

Spectral Thm: A is a sym matrix in $\mathbb{R}^{n \times n}$

a) Eigenvalues are real

b) eigenvectors are orthonormal & they form a basis of $\mathbb{R}^n$

$\lambda, x \qquad Ax = \lambda x.$

$\lambda, \mu x \qquad A\underline{\mu x} = \mu Ax = \mu \lambda x = \lambda \underline{\mu x}$

$\lambda, x, y \qquad Ax = \lambda x$

$\qquad\qquad Ay = \lambda y$

$\Rightarrow A(x+y) = \lambda(x+y)$

$(x+y)$ is an eigenvector corr to λ.

$$x = \begin{bmatrix} 4+i \\ 3-i \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} 4-i \\ 3+i \end{bmatrix}, \quad \bar{x}^T = \begin{bmatrix} 4-i & 3+i \end{bmatrix}$$

Given any x, be it real or complex

$\bar{x}^T x$ to be real?

$$x = \begin{bmatrix} a_1+ib_1 \\ a_2+ib_2 \\ a_n+ib_n \end{bmatrix}, \quad \bar{x}^T = [a_1-ib_1 \quad a_2-ib_2 \quad a_n-ib_n]$$

$$\therefore \bar{x}^T x = \underbrace{a_1^2+b_1^2}_{\text{real}} + \underbrace{a_2^2+b_2^2}_{\text{real}} \dots + \underbrace{a_n^2+b_n^2}_{\text{real}}$$

is real

$x \neq 0$, $\bar{x}^T x$ or $x^T x > 0$

$Ax = \lambda x$; A is Sym or Hermitian.

To prove λ is real

$$\bar{x}^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

$$\therefore \lambda = \frac{\bar{x}^T A x}{\bar{x}^T x} \quad \text{real}$$

To prove λ is real, we need to prove $\bar{x}^T A x$ is real whenever A is Sym

$$\mu = \bar{x}^T A x$$

$$\bar{\mu} = \mu^T = (\bar{x}^T A x)^T = x^T A^T (\bar{x}^T)^T = \overline{x^T A^T \bar{x}}$$

$$= \overline{\bar{x}^T \bar{A}^T x} = \bar{x}^T A x = \bar{\mu}$$

Let $\mu = \mu_1 + i\mu_2$

$$\cancel{\mu_1} + i\mu_2 = \cancel{\mu_1} - i\mu_2$$

$$2i\mu_2 = 0 \Rightarrow \mu_2 = 0$$

$\therefore$ μ is real

$\therefore$ λ is real.

Corr A is called Skew Hermitian if $\bar{A}^T = -A$
Skew Sym $A^T = -A$

Corr A is called Skew Her... Skew Sy $A^T = -A$

$$\mu = -\bar{\mu}$$

$$\mu_1 + i\mu_2 = -(\mu_1 - i\mu_2)$$

$$\Rightarrow \mu_1 + \cancel{i\mu_2} = -\mu_1 + \cancel{i\mu_2}$$

$$2\mu_1 = 0 \Rightarrow \mu_1 = 0$$

$\therefore$ For Skew Hermitian eigenvalues are either 0 or purely imaginary