Mathematical Foundations

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Mathematical Foundations

Webinar#1

Agenda

- Particular Solution and Solution to Homogenous system

- Gauss Jordan method for finding inverse of the matrix

- Linearly independent and dependent Vectors.

- Basis and Dimension of the Vector Spaces

- Examples of inner product spaces

- Gram Schmidt Orthogonalization process

Q1: Modelling of electrical / traffic networks would lead to a linear system Ax=b.

Refer to the text book / other resources and construct a network which

has the following properties

a) the number of equations is 6

b) A has rank 5

c) the system is consistent.

Solution:

Consider the following traffic flow diagram and construct the linear system AX= b.
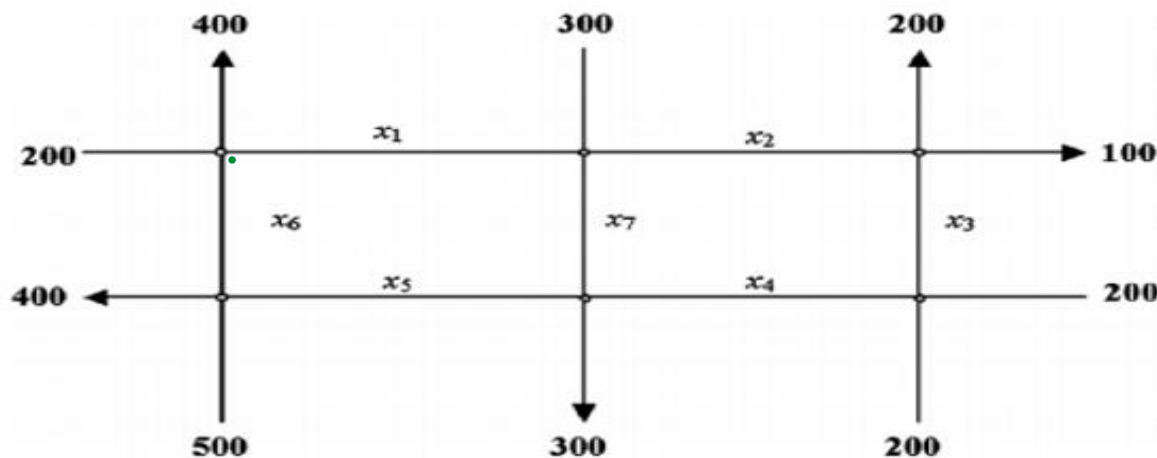
On modelling, using the fact that, the out-flow traffic= inflow traffic we get,

$$200 + x_6 = 400 + x_1$$
$$x_1 + 300 = x_2 + x_7$$
$$x_2 + x_3 = 200 + 100$$
$$200 + 200 = x_3 + x_4$$
$$x_7 + x_4 = x_5 + 300$$
$$500 + x_5 = 400 + x_6$$

on rewriting the equations

$$x_1 - x_6 = -200$$
$$x_1 - x_2 - x_7 = -300$$
$$x_2 + x_3 = 300$$
$$x_3 + x_4 = 400$$
$$x_4 - x_5 + x_7 = 300$$
$$x_5 - x_6 = -100$$

The Augmented matrix for the above system is

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & : & -300 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & : & 300 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix}$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & : & 300 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_2 = -R_2 + R_1$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_3 = R_3 - R_2$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_4 = R_4 - R_3$$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_5 = -R_5 + R_4 \quad [\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} R'_6 = R_6 - R_7$$

$Rank\ of\ [A:B] = Rank\ of\ [A] = 5 \neq 7\ (Number\ of\ unknows)$

So, 7-5 =2 arbitrary value can be assumed to solve the above system

If $\qquad x_6 = k,$ $\qquad\qquad\qquad\qquad\qquad$ **Again if** $\qquad x_7 = l$

$x_5 = k - 100$ $\qquad x_2 + x_7 = k + 100$ $\qquad\qquad\qquad\qquad x_2 = k - l + 100$

$x_1 = k - 200$ $\qquad x_3 - x_7 = -k + 200$ $\qquad\qquad\qquad\quad x_3 = -k + l + 200$

$\qquad\qquad\qquad\qquad x_4 + x_7 = k + 200$ $\qquad\qquad\qquad\qquad x_4 = k - l + 200$

is the required solution.

The above system will have infinitely many solution , Since rank of [A:B] and A =5 and which is less than no of Unknowns i.e equal to 7. You get particular solution by giving values for k and l.

1.Show that the following homogeneous system has non-trivial solutions:

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 \quad\quad\quad + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

Solution: The reduction of the augmented matrix to reduced row-echelon form is

$$\begin{bmatrix} 1 & -1 & 2 & -1 & | & 0 \\ 2 & 2 & 0 & 1 & | & 0 \\ 3 & 1 & 2 & -1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 2 & -1 & | & 0 \\ 0 & 4 & -4 & 3 & | & 0 \\ 0 & 4 & -4 & 2 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & | & 0 \\ 0 & 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The leading variables are x1,x2,and x4,so x3 is assigned as a parameter say x3= t. Then the general solution is x1=−t, x2= t, x3= t, x4=0.

Hence, taking t =1(say),we get a nontrivial solution: x1=−1,x2=1,x3=1,x4=0.

Facts:

- If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution(infact, infinitely many).

- The gaussian algorithm systematically produces solutions to any homogeneous linear system, called basic solutions, one for every parameter.

- Any nonzero scalar multiple of a basic solution will still be called a basic solution.

- Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:
 1. The system has exactly n−r basic solutions, one for each parameter.
 2. Every solution is a linear combination of these basic solutions.

2. Find basic solutions of the homogeneous system with coefficient matrix A, and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution: The reduction of the augmented matrix to reduced row-echelon form is

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 2 & | & 0 \\ -2 & 6 & 1 & 2 & -5 & | & 0 \\ 3 & -9 & -1 & 0 & 7 & | & 0 \\ -3 & 9 & 2 & 6 & -8 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 2 & 2 & | & 0 \\ 0 & 0 & 1 & 6 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So the general solution is x1=3r−2s−2t, x2=r,x3=−6s+t, x4=s, and x5=t where r,s, and t are parameters.

In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

EXAMPLE : Find the Inverse of a Matrix by Gauss–Jordan Elimination

Determine the inverse of the matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

Solution.

We apply the Gauss elimination to the following
$n \times 2n = 3 \times 6$ matrix,

$$\begin{bmatrix} \mathbf{A} & \mathbf{I} \end{bmatrix} = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \quad \begin{array}{l} \text{Row 2 + 3 Row 1} \\ \\ \text{Row 3 − Row 1} \end{array}$$

Solution. *(continued)*

$$= \begin{bmatrix} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{bmatrix}$$

Row 3 – Row 2

Solution.

This is [**U** **H**] as produced by the Gauss elimination. Now follow the additional Gauss–Jordan steps, reducing **U** to **I**, that is, to diagonal form with entries 1 on the main diagonal.

$$= \begin{bmatrix} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix} \quad \begin{array}{l} \text{– Row 1} \\ 0.5 \text{ Row 2} \\ -0.2 \text{ Row 3} \end{array}$$

Solution. *(continued)*

$$= \begin{bmatrix} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix}$$
Row 1 + 2 Row 3
Row 2 − 3.5 Row 3

$$= \begin{bmatrix} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{bmatrix}$$
Row 1 + Row 2

Solution. *(continued 4)*

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}\begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}.$ Similarly $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$

Linear Dependent and Independent vectors

Definition :

Suppose that V is a vector space and that $x_1, x_2, \ldots\ldots x_k$ are vectors in V.

The set of vectors $\{x_1, x_2, \ldots\ldots x_k\}$ is linearly independent if

$c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots + c_k x_k = 0$ for some scalars $c_1, c_2, \ldots\ldots c_k \in$ R

where all $c_1, c_2, \ldots\ldots c_k$ are zero.

Example:

the set of vectors $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ are linearly independent.

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The set of vectors $\{x_1, x_2, \ldots\ldots x_k\}$ is linearly dependent if

$\qquad c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots + c_k x_k = 0$ for some scalars $c_1, c_2, \ldots\ldots c_k \in \mathrm{R}$

where at least one of $c_1, c_2, \ldots\ldots c_k$ is non–zero.

Example :

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\5\\7\end{bmatrix} - \begin{bmatrix}5\\9\\13\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

So the set of vectors $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}, \begin{bmatrix}5\\9\\13\end{bmatrix} \right\}$ are linearly dependent.

Example:

Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $y = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$ and $z = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. Is $\{x, y, z\}$ linearly dependent?

Solution: We have to determine whether or not we can find real numbers c_1, c_2, c_3 which are not all zero, such that $c_1 x + c_1 y + c_2 z = 0$

We have to solve the augmented matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 9 & -1 & 0 \end{array}\right] \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & -16 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 := -\frac{1}{16} R_3]{R_2 := -\frac{1}{4} R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So this set of equations has a zero solution. Therefore, $\{x, y, z\}$ is a linearly independent set of vectors.

Linear Dependent and Independent vectors

Example : Let x = $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, y = $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ and z = $\begin{bmatrix} 0 \\ 4 \\ 8 \end{bmatrix}$. Is{x, y, z } linearly dependent?

Solution: We have to determine whether or not we can find real numbers c_1, c_2, c_3 which are not all zero, such that $c_1 x + c_1 y + c_2 z = 0$

We have to solve the augmented matrix equation:

$$\begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 2 & 2 & 4 & | & 0 \\ 3 & 1 & 8 & | & 0 \end{bmatrix} \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 0 & -4 & 4 & | & 0 \\ 0 & -8 & 8 & | & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 := R_3 - 2R_2} \begin{bmatrix} 1 & 3 & 0 & | & 0 \\ 0 & -4 & 4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

So this set of equations has a non–zero solution. Therefore,{x, y,z} is a linearly dependent set of vectors.

Example: Consider the polynomials $p(x) = 1 + 3x + 2x^2, q(x) = 3 + x + 2x^2$ and $r(x) = 2x + x^2$ in P_2. Is $\{p(x), q(x), r(x)\}$ linearly dependent?

Solution :

Let $c_1 p(x) + c_2 q(x) + c_3 r(x)$=0

That is $c_1(1 + 3x + 2x^2) + c_2(3 + x + 2x^2) + c_3(2x + x^2) = 0$

Which gives the matrix form

$$\begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \xrightarrow[R_3:=R_3-2R_1]{R_2:=R_2-3R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -8 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2:=R_2-R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

Hence $\{p(x), q(x), r(x)\}$ is linearly dependent.

Linear Spans

Linear Spans:

Suppose $u_1, u_2, \ldots u_n$ are any vectors in a vector space V.

Then any vector of the form

$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where a_i are scalers, is called linear combination of $u_1, u_2, \ldots, u_n$.

The collection of all such linear combinations, denoted by

$$span(u_1, u_2, \ldots, u_n)$$

is called Linear span of $u_1, u_2, \ldots u_n$.

Basis and Dimensions

Basis: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a basis of V if
it has the following two properties.

- S is linearly independent
- S spans V.

Dimension of Vector Space:

A vector space V is said to be of dimension n if V has a basis with n elements.

Example

1. Determine whether or not each of the following form a
basis of $\mathbb{R}^3$:

a)   (1,1,1), (1,0,1)   b) (1,2,3), (1,3,5), (1,0,1), (2,3,0)

c) (1,1,1), (1,2,3)(2, −1,1)   d) (1,1,2), (1,2,5), (5,3,4)

Solution: a and b are not basis of $\mathbb{R}^3$ because basis of $\mathbb{R}^3$ must contain exactly three elements

c) The three vectors of $\mathbb{R}^3$ form basis if they are linearly independent.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows; hence the vectors are LID.

d) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

The echelon matrix has zero row hence the vectors are not LID so they do not form the basis of $\mathbb{R}^3$.

Example

2. Find the basis and dimension of the subspace W of $\mathbb{R}^3$

 where

a) $W = \{(a, b, c): a + b + c = 0\}$

b) $W = \{(a, b, c): a = b = c\}$

Solution:

a)   Note that number of free variables=3-1=2 , so dimension is 2 and any two independent vectors which satisfying equation $a + b + c = 0$ is the basis set of V.

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.

b) Any vector $w \in W$ has the form $w = (k, k, k), k \in \mathbb{R}$. Hence

$w = k(1,1,1) = ku$, where $u = (1,1,1)$. Since spanning set contain one element, therefor dimension is 1.

Example 3. Let W be subspace of $\mathbb{R}^4$ spanned by the

vectors $u_1 = (1, -2,5, -3), u_2 = (2,3,1, -4), u_3 = (3,8, -3, -5)$.

a)   Find basis and dimension of W.

b)   Extend the basis of W to basis of $\mathbb{R}^4$.

$$\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows $(1, -2, 5, -3)$ and $(0, 7, -9, 2)$ form the basis of W.
   Therefore, dimension of W is 2.

b) To extend the basis of W to a basis of $\mathbb{R}^4$ we have to

find four linearly independent vectors which include the

above two vectors. The four vectors are

$$(1, -2, 5, -3), (0, 7, -9, 2), (0, 0, 1, 0), (0, 0, 0, 1).$$

They form the basis of $\mathbb{R}^4$.

4. Let V be the vector space of 2×2 matrices over $\mathbb{R}$. Let W be the subspace of symmetric matrices. Find the dimension and basis of W.

Solution: For $w \in W$, we have $w = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We shoe $S = \{e_1, e_2, e_3\}$ is a basis of W i.e. we have to prove

a) S spans W. b) S is linearly independent

Solution:

a)Any matrix $w = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = ae_1 + be_2 + ce_3$. Thus S spans W.

b) Suppose $xe_1 + ye_2 + ze_3 = 0$

$$x \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $x = y = z = 0$. Hence e_1, e_2, e_3 are linearly independent.

Therefore S is a basis of W.

Inner Product spaces:-

The vector space V over F is said to be an inner product space if there is defined for any two vectors $u, v \in V$ an element in F such that

i. $\langle u, v \rangle = \langle \overline{v, u} \rangle$
ii. $\langle u, v \rangle \geq 0 \; and \; \langle u, v \rangle = 0 \; if \; and \; only \; if \; u = 0$
iii. $\langle \alpha u + \beta v, \; w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \; for \; any \; u, v, w \in V \; and \; \alpha, \beta \in F$

Remarks:-

i.   In this definition F is either the field of real or complex number
ii. $\langle \overline{v, u} \rangle$ denote the conjugate of $\langle v, w \rangle$.

Example:-

If $z = x + iy$ then $\bar{z} = x - iy$ we have $\bar{\bar{z}} = z, z\bar{z} = |z|^2 = x^2 + y^2 \geq 0$ and also $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$

i.   Using properties 1 and 3 in the definition,
      We have
      $\langle u, \; \alpha v + \beta w \rangle = \langle \overline{\alpha v + \beta w, \; u} \rangle$ (by i)
      $= \overline{\alpha \langle v, u \rangle + \beta \langle w, u \rangle}$ (by iii)
      $= \bar{\alpha} \langle \overline{v, u} \rangle + \bar{\beta} \langle \overline{w, u} \rangle$
      $= \bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, w \rangle$

1. In $F^{(n)}$, let $u = (\alpha_1, \alpha_2, \ldots \alpha_n)$ and $v = (\beta_1, \beta_2, \ldots \beta_n)$ defined $\langle u, v \rangle = \alpha_1 \overline{\beta_1} + \alpha_2 \overline{\beta_2} + \cdots + \alpha_n \overline{\beta_n}$ show that this defines an inner product on $F^{(n)}$.

Solution:-

i. We have $\langle v, u \rangle = \beta_1 \overline{\alpha_1} + \beta_2 \overline{\alpha_2} + \cdots + \beta_n \overline{\alpha_n}$

Now $\langle \overline{v, u} \rangle = \overline{\beta_1 \overline{\alpha_1} + \beta_2 \overline{\alpha_2} + \cdots + \beta_n \overline{\alpha_n}}$

$$= \overline{\beta_1 \overline{\alpha_1}} + \overline{\beta_2 \overline{\alpha_2}} + \cdots + \overline{\beta_n \overline{\alpha_n}}$$

$$= \overline{\beta_1} \, \overline{\overline{\alpha_1}} + \overline{\beta_2} \, \overline{\overline{\alpha_2}} + \cdots + \overline{\beta_n} \, \overline{\overline{\alpha_n}}$$

$$= \overline{\beta_1} \, \alpha_1 + \overline{\beta_2} \, \alpha_2 + \cdots + \overline{\beta_n} \, \alpha_n$$

$$= \alpha_1 \overline{\beta_1} + \alpha_2 \overline{\beta_2} + \cdots + \alpha_n \overline{\beta_n}$$

$$\langle \overline{v, u} \rangle = \langle u, v \rangle$$

ii. We have $\langle u, u \rangle = \alpha_1 \overline{\alpha_1} + \alpha_2 \overline{\alpha_2} + \cdots + \alpha_n \overline{\alpha_n}$

$\quad = |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 \quad (because\ z\bar{z} = |z|^2)$

$\langle u, u \rangle \geq 0$

Next, $\langle u, u \rangle = 0$

$$\Leftrightarrow |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 0$$

$$\Leftrightarrow |\alpha_i|^2 = 0\ for\ each\ i = 1, 2, \ldots, n$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$$\Leftrightarrow u = 0$$

iii. Let $w = (\gamma_1, \gamma_1, \dots \gamma_n)$ and $\propto, \beta \in F$

Now $\propto u + \beta v = \alpha(\alpha_1, \alpha_2, \dots \alpha_n) + \beta(\beta_1, \beta_2, \dots \beta_n)$

$\qquad = (\alpha\alpha_1, \alpha\alpha_2, \dots \alpha\alpha_n) + (\beta\beta_1, \beta\beta_2, \dots \beta\beta_n)$

$\propto u + \beta v = (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2, \dots \alpha\alpha_n + \beta\beta_n)$

Then

$\langle \alpha u + \beta v, \qquad w \rangle = \langle \alpha\alpha_1 + \beta\beta_1 \rangle \overline{\gamma_1} + \langle \alpha\alpha_2 + \beta\beta_2 \rangle \overline{\gamma_2} + \dots + \langle \alpha\alpha_n + \beta\beta_n \rangle \overline{\gamma_n}$

$\qquad = (\alpha\alpha_1\gamma_1, \alpha\alpha_2\gamma_2, \dots, \alpha\alpha_n\overline{\gamma_n}) + (\beta\beta_1\gamma_1, \beta\beta_2\gamma_2, \dots, \beta\beta_n\overline{\gamma_n})$

$\qquad = \alpha(\alpha_1\gamma_1, \alpha_2\gamma_2, \dots, \alpha_n\overline{\gamma_n}) + \beta(\beta_1\gamma_1, \beta_2\gamma_2, \dots, \beta_n\overline{\gamma_n})$

$\qquad \langle \alpha u + \beta v, \qquad w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

Thus all the three properties of the inner product satisfied.

Hence $\langle u, v \rangle$ defines an inner on $F^{(n)}$

2. In $F^{(2)}$, let $u = (\alpha_1, \alpha_2)$ and $v = (\beta_1, \beta_2)$ defined $\langle u, v \rangle = 2\alpha_1\overline{\beta_1} + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_2\beta_2$ show that this defines an inner product on $F^{(2)}$.

Solution:-

i. We have $\langle v, u \rangle = 2\beta_1\overline{\alpha_1} + \beta_1\overline{\alpha_2} + \beta_2\overline{\alpha_1} + \beta_2\overline{\alpha_2}$

   Now $\overline{\langle v, u \rangle} = \overline{2\beta_1\overline{\alpha_1} + \beta_1\overline{\alpha_2} + \beta_2\overline{\alpha_1} + \beta_2\overline{\alpha_2}}$

   $= 2\overline{\beta_1\overline{\alpha_1}} + \overline{\beta_1\overline{\alpha_2}} + \overline{\beta_2\overline{\alpha_1}} + \overline{\beta_2\overline{\alpha_2}}$

   $\qquad\qquad\qquad becuase\ \overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

   $\qquad\qquad = 2\overline{\beta_1}\ \overline{\overline{\alpha_1}} + \overline{\beta_1}\ \overline{\overline{\alpha_2}} + \overline{\beta_2}\ \overline{\overline{\alpha_1}} + \overline{\beta_2}\ \overline{\overline{\alpha_2}}$

   $\qquad\qquad = 2\overline{\beta_1}\ \alpha_1 + \overline{\beta_1}\ \alpha_2 + \overline{\beta_2}\ \alpha_1 + \overline{\beta_2}\ \alpha_2$

   $\qquad\qquad = 2\alpha_1\overline{\beta_1} + \overline{\beta_1}\alpha_2 + \overline{\beta_2}\alpha_1 + \overline{\beta_2}\alpha_2$

   $\qquad\qquad\qquad \overline{\langle v, u \rangle} = \langle u, v \rangle$

ii. We have $\langle u, u \rangle = 2\alpha_1\overline{\alpha_1} + \alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1} + \alpha_2\overline{\alpha_2}$

   $\qquad\qquad = \alpha_1\overline{\alpha_1} + \alpha_1\overline{\alpha_1} + \alpha_1\overline{\alpha_2} + \alpha_2\overline{\alpha_1} + \alpha_2\overline{\alpha_2}$

   $= |\alpha_1|^2 + \alpha_1(\overline{\alpha_1} + \overline{\alpha_2}) + \alpha_2(\overline{\alpha_1} + \overline{\alpha_2})\quad (because\ z\bar{z} = |z|^2)$

   $\qquad\qquad = |\alpha_1|^2 + (\alpha_1 + \alpha_2)(\overline{\alpha_1} + \overline{\alpha_2})$

   $\qquad\qquad = |\alpha_1|^2 + (\alpha_1 + \alpha_2)(\overline{\alpha_1 + \alpha_2})$

   $= |\alpha_1|^2 + |\alpha_1 + \alpha_2|^2\ (because\ z\bar{z} = |z|^2)$

   $\langle u, u \rangle \geq 0\quad (because |\alpha_1|^2 \geq 0, |\alpha_1 + \alpha_2|^2 \geq 0\ )$

Next, $\langle u, u \rangle = 0$

$$\Leftrightarrow |\alpha_1|^2 + |\alpha_1 + \alpha_2|^2 = 0$$
$$\Leftrightarrow |\alpha_1|^2 = 0 \text{ and } |\alpha_1 + \alpha_2|^2 = 0$$
$$\Leftrightarrow \alpha_1 = 0 \text{ and } \alpha_2 = 0$$
$$\Leftrightarrow u = 0$$

iii. Let $w = (\gamma_1, \gamma_2)$ and let $\propto, \beta \in F$

Now $\propto u + \beta v = \alpha(\alpha_1, \alpha_2) + \beta(\beta_1, \beta_2)$
$$= (\alpha\alpha_1, \alpha\alpha_2) + (\beta\beta_1, \beta\beta_2)$$
$$\propto u + \beta v = (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2)$$

Then

$$\langle \alpha u + \beta v, \qquad w \rangle$$
$$= 2\langle \alpha\alpha_1 + \beta\beta_1 \rangle \overline{\gamma_1} + \langle \alpha\alpha_1 + \beta\beta_1 \rangle \overline{\gamma_2} + \langle \alpha\alpha_2 + \beta\beta_2 \rangle \overline{\gamma_1} + \langle \alpha\alpha_2 + \beta\beta_2 \rangle \overline{\gamma_2}$$
$$= \alpha(2\alpha_1\overline{\gamma_1} + \alpha_1\overline{\gamma_2} + \alpha_2\overline{\gamma_1} + \alpha_2\overline{\gamma_2}) + \beta(2\beta_1\overline{\gamma_1} + \beta_1\overline{\gamma_2} + \beta_2\overline{\gamma_1} + \beta_2\overline{\gamma_2})$$
$$\langle \alpha u + \beta v, \qquad w \rangle = \alpha\langle u, w \rangle + \beta\langle v, w \rangle$$

Thus all the three properties of an inner product satisfied.

Hence $\langle u, v \rangle$ defines an inner on $F^{(n)}$

Examples of inner product spaces

3. Compute the length of vector $v = (1, \; -2, \; 2, \; 0)$ and also find an unit vectors and length of unit vector.

Solution: Given that $v = (1, \; -2, \; 2, \; 0)$

Length of vector $= \lVert v \rVert$

$$= \sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2 + v_4{}^2}$$

$$= \sqrt{(1)^2 + (-2)^2 + (2)^2 + (0)^2}$$

$$= \sqrt{1 + 4 + 4 + 0} = \sqrt{9} = 3$$

$$\lVert v \rVert = 3$$

Unit vector $(\hat{n}) = \dfrac{v}{\lVert v \rVert} = \dfrac{(1,-2,2,0)}{3}$

u = Unit vector $(\hat{n}) = \left(\dfrac{1}{3}, \dfrac{-2}{3}, \dfrac{2}{3}, \; 0 \right)$

Length of an unit vector $\lVert u \rVert = \sqrt{\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{-2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^2 + (0)^2}$

$$= \sqrt{\dfrac{1}{9} + \dfrac{4}{9} + \dfrac{4}{9}} = \sqrt{\dfrac{9}{9}} = \sqrt{1}$$

$$\lVert u \rVert = 1$$

Examples of inner product spaces

4. Calculate the length of vector $v = (2, \ -1, \ 1, \ 2)$ and also find an unit vector.

Solution: Given that $v = (2, \ -1, \ 1, \ 2)$

Length of vector $= \|v\|$

$$= \sqrt{v_1{}^2 + v_2{}^2 + v_3{}^2 + v_4{}^2}$$
$$= \sqrt{(2)^2 + (-1)^2 + (1)^2 + (2)^2}$$
$$= \sqrt{4 + 1 + 1 + 4} = \sqrt{10}$$
$$\|v\| = \sqrt{10}$$

Unit vector $(\hat{n}) = \dfrac{v}{\|v\|} = \dfrac{(2, -1, \ 1, \ 2)}{\sqrt{10}}$

$\mathrm{u} =$ Unit vector $(\hat{n}) = \left(\dfrac{2}{\sqrt{10}}, \dfrac{-1}{\sqrt{10}}, \dfrac{1}{\sqrt{10}}, \ \dfrac{2}{\sqrt{10}} \right)$

Examples of inner product spaces

5. Compute the distance between the vectors $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and v= $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Solution: Given that $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and v= $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$\text{Distance between u and v} = ||u - v||$$
$$= \sqrt{(u-v)(u-v)}$$
$$= \sqrt{(||u-v||)^2}$$
$$= ||u - v||$$

Let $u - v = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$||u - v|| = \sqrt{(4)^2 + (-1)^2}$$
$$= \sqrt{16 + 1}$$
$$||u - v|| = \sqrt{17}$$

Gram Schmidt Process

- The Gram–Schmidt process is a method for orthonormalizing a set of vectors in an inner product space

- The Gram–Schmidt process takes a finite, linearly independent set of vectors

$$S = \{v_1, \ldots, v_k\} \quad \text{for } k \leq n$$

and generates an orthogonal set

$$S' = \{u_1, \ldots, u_k\}$$

- The Gram–Schmidt process works as follows:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

Gram Schmidt process (contd.)

$$u_4 = v_4 - \frac{\langle v_4, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_4, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_4, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$\vdots$$

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

➡ The sequence $u_1, \ldots, u_k$ is the required system of orthogonal vectors.

➡ The normalized vectors

$$e_1 = \frac{u_1}{||u_1||}, e_2 = \frac{u_2}{||u_2||} \ldots, e_k = \frac{u_k}{||u_k||}$$

form an orthonormal set.

Qus: Obtain an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$x_1 = (1,0,1,0), x_2 = (1,1,1,1), x_3 = (-1,2,0,1)$$

Sol: Following the Gram-Schmidt process, we set

$$v_1 = x_1 = (1,0,1,0).$$

Next, we have

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1)$$

and

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle x_3, v_2 \rangle}{||v_2||^2} v_2$$

$$= (-1, 2, 0, 1) + \frac{1}{2}(1, 0, 1, 0) - \frac{3}{2}(0, 1, 0, 1)$$

$$= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

The orthogonal basis so obtained is

$$\{(1, 0, 1, 0), (0, 1, 0, 1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

We can normalise the vectors to get

$$\{\frac{1}{\sqrt{2}}(1, 0, 1, 0), \frac{1}{\sqrt{2}}(0, 1, 0, 1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

Q. Apply Gram-Schmidt orthogonalization to the following

sequence of vectors in R^3 : $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Solution:

Apply the process with $x1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad x2 = \begin{pmatrix} 8 \\ 1 \\ -6 \end{pmatrix}, \quad x3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Step 1 produces an orthogonal basis:

$v1 = x1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

Example of Gram Schmidt process

$$v2 = x2 - \frac{(x2, v1)}{(v1, v1)} v1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{\left(\begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{pmatrix} 6 \\ -3 \\ 6 \end{pmatrix}$$

$$v3 = x3 - \frac{(x3, v1)}{(v1, v1)} v1 - \frac{(x3, v2)}{(v2, v2)} v2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \right)}{\left(\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \right)} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{-6}{81} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$$

Step 2 produces an orthonormal basis by replacing each vector with a vector of norm 1:

$$\text{Replace } v_1 \text{ with } \frac{v_1}{|v_1|} = \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

$$\text{Replace } v_2 \text{ with } \frac{v_2}{|v_2|} = \frac{\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}}{\left\| \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \right\|} = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

$$\text{Replace } v_3 \text{ with } \frac{v_3}{|v_3|} = \frac{\frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}}{\left\| \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} \right\|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$$

So the final solution is $v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, v_3 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$

THANK YOU