BITS Pilani
Pilani Campus

Mathematical Foundations

MFDS Team

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Webinar#1

Agenda

- Particular Solution and Solution to Homogenous system
- Gauss Jordan method for finding inverse of the matrix
- Linearly independent and dependent Vectors.
- Basis and Dimension of the Vector Spaces
- Examples of inner product spaces
- Gram Schmidt Orthogonalization process

Finding Particular Solution

Q1: Modelling of electrical / traffic networks would lead to a linear system Ax=b. Refer to the text book / other resources and construct a network which has the following properties

a) the number of equations is 6

b) A has rank 5

c) the system is consistent.

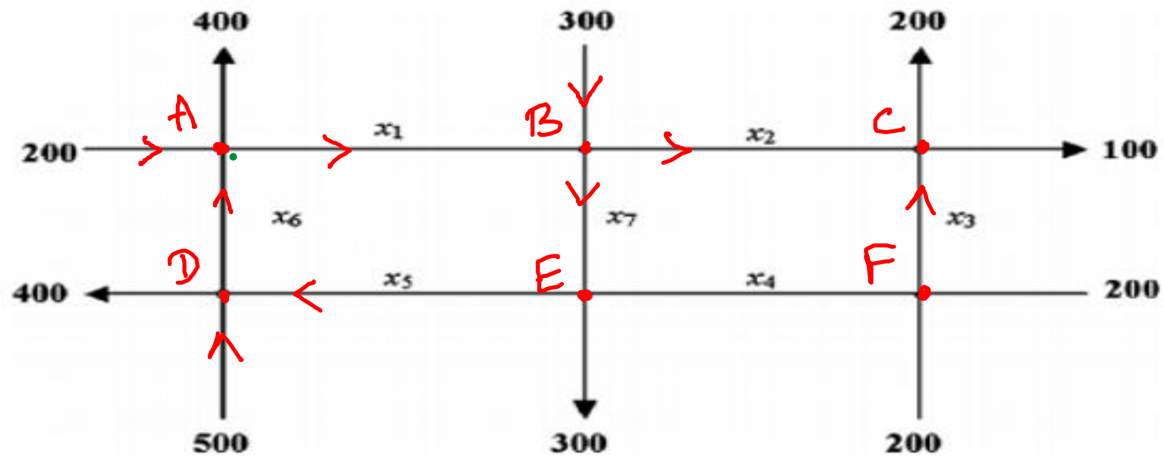
Solution:

Consider the following traffic flow diagram and construct the linear system AX= b.

unique

infinitely many

rank(A)=5



On modelling, using the fact that, the out-flow traffic= inflow traffic we get,

Inflow outflow

$$
\begin{aligned}
200 + x_6 &= 400 + x_1 \\
x_1 + 300 &= x_2 + x_7 \\
x_2 + x_3 &= 200 + 100 \\
200 + 200 &= x_3 + x_4 \\
x_7 + x_4 &= x_5 + 300 \\
500 + x_5 &= 400 + x_6
\end{aligned}
$$

on rewriting the equations

$$
\begin{aligned}
x_1 - x_6 &= -200 \\
x_1 - x_2 - x_7 &= -300 \\
x_2 + x_3 &= 300 \\
x_3 + x_4 &= 400 \\
x_4 - x_5 + x_7 &= 300 \\
x_5 - x_6 &= -100
\end{aligned}
$$

$x_1 = -200 + K$

6

$x_5 = -100 + K$

The Augmented matrix for the above system is

$$[\mathbf{A}:\mathbf{B}] = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & & & x_7 & & \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & : & -300 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & : & 300 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} \end{array}$$

$$[\mathbf{A}:\mathbf{B}] = \begin{array}{r} \\ R_2' \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & \underline{-1} & 1 & : & 100 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & : & 300 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_2 = -R_2 + R_1$$

$0 - (-1)$

$R_1 - R_2$

$$[\mathbf{A}:\mathbf{B}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & : & 400 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_3 = R_3 - R_2$$

$$[\mathbf{A}:\mathbf{B}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & : & 300 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_4 = R_4 - R_3$$

$a+b=2$

$a-b=3$

$a=1 \qquad b=1$

$$[\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \end{bmatrix} R'_5=-R_5+R_4$$

$$\begin{array}{c} \begin{array}{ccccccccc} \checkmark & \checkmark & \checkmark & \checkmark & \checkmark & x_6 & x_7 & & \end{array} \\ [\mathbf{A}:\mathbf{B}]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 & : & -200 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1 & : & 100 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & : & 200 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 & : & 200 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & : & -100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & : & 0 \end{bmatrix} R'_6=R_6-R_5 \end{array}$$

$Rank\ of\,[A:B] = Rank\ of\,[A] = 5 \neq 7\,(Number\ of\ unknowns)$

$\neq 7$ Infinitely many solution

So, 7-5 =2 arbitrary value can be assumed to solve the above system

$\alpha, \beta \in \mathbb{R}$

If $x_6 = k,$

$x_5 = k-100$ ✓ $\qquad x_2 + x_7 = k+100$

$x_1 = k-200$ ✓ $\qquad x_3 - x_7 = -k+200$

$x_4 + x_7 = k+200$

Again if $x_7 = l$

$x_2 = k-l+100$

$x_3 = -k+l+200$

$x_4 = k-l+200$

is the required solution.

The above system will have infinitely many solution , Since rank of [A:B] and A =5 and which is less than no of Unknowns i.e equal to 7. You get particular solution by giving values for k and l.

Solution to Homogenous system

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$Ax=0$

1.Show that the following homogeneous system has non-trivial solutions:

$[A\,;\,0]$

$A_{3\times4}$

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 + 2x_2 + x_4 &= 0 \\ 3x_1 + x_2 + 2x_3 - x_4 &= 0 \end{aligned}$$

4 variables
3 equations

Solution: The reduction of the augmented matrix to reduced row-echelon form is

$$\begin{array}{c} \quad A \qquad\qquad 0 \\ \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 2 & 2 & 0 & 1 & 0 \\ 3 & 1 & 2 & -1 & 0 \end{array}\right] \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & 0 \\ 0 & 4 & -4 & 3 & 0 \\ 0 & 4 & -4 & 2 & 0 \end{array}\right] \rightarrow \begin{array}{c} x_1 \checkmark \ \ x_2 \checkmark \ \ x_3 \ \ x_4 \checkmark \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The leading variables are x1,x2,and x4,so x3 is assigned as a parameter say x3= t. Then the general solution is x1=−t, x2= t, x3= t, x4=0.
Hence, taking t =1(say),we get a nontrivial solution: x1=−1,x2=1,x3=1,x4=0.

Let $\boxed{x_3 = t}$

Substituting x_3 values in the equations:

$$\boxed{x_2 = x_3}$$

$\checkmark\ x_1 + x_3 = 0$

$\checkmark\ x_2 - x_3 = 0$

$\checkmark\ x_4 = 0$

$\therefore x_1 = -t$

$x_2 = t$

$x_4 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$Ax = 0$

$$2 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

Facts:

- If a homogeneous system of linear equations has more variables than equations, then it has a nontrivial solution(infact, infinitely many).
- The gaussian algorithm systematically produces solutions to any homogeneous linear system, called basic solutions, one for every parameter.
- Any nonzero scalar multiple of a basic solution will still be called a basic solution.
- Let A be an $m \times n$ matrix of rank r, and consider the homogeneous system in n variables with A as coefficient matrix. Then:

1. The system has exactly $n-r$ basic solutions, one for each parameter.
2. Every solution is a linear combination of these basic solutions.

$n=4$, $r=3$ basic solutions $=4-3=1$

2. Find basic solutions of the homogeneous system with coefficient matrix A, and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} 1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

Solution: The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is x1=3r−2s−2t, x2=r,x3=−6s+t, x4=s, and x5=t where r,s, and t are parameters.

In matrix form this is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix} = r\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Hence basic solutions are

$$\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Determination of the Inverse by the Gauss–Jordan Method

EXAMPLE : Find the Inverse of a Matrix by Gauss–Jordan Elimination

Determine the inverse of the matrix:

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}_{3\times 3}.$$

$$[A \,\vdots\, I] \xrightarrow{\text{Gauss Jordan}} [I \,\vdots\, A^{-1}]$$

Solution.

We apply the Gauss elimination to the following $n \times 2n = 3 \times 6$ matrix,

I → Identity matrix

$$[\mathbf{A} \quad \mathbf{I}] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ \text{Row 2} + 3\,\text{Row 1} \\ \text{Row 3} - \text{Row 1} \end{array}$$

Solution. *(continued)*

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \quad \text{Row 3} - \text{Row 2}$$

Solution.

This is [**U H**] as produced by the Gauss elimination. Now follow the additional Gauss–Jordan steps, reducing **U** to **I**, that is, to diagonal form with entries 1 on the main diagonal.

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right] \quad \begin{array}{l} -\text{Row } 1 \\ 0.5 \text{ Row } 2 \\ -0.2 \text{ Row } 3 \end{array}$$

Solution. *(continued)*

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Row 1 + 2 Row 3
Row 2 − 3.5 Row 3

$I \qquad\qquad A^{-1}$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array}\right]$$

Row 1 + Row 2

Solution. ***(continued 4)***

The last three columns constitute $\mathbf{A}^{-1}$. Check:

$$\begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$. Similarly $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Linear Dependent and Independent vectors

Definition :

Suppose that V is a vector space and that $x_1, x_2, \ldots\ldots. x_k$ are vectors in V.

The set of vectors $\{x_1, x_2, \ldots\ldots. x_k\}$ is linearly independent if

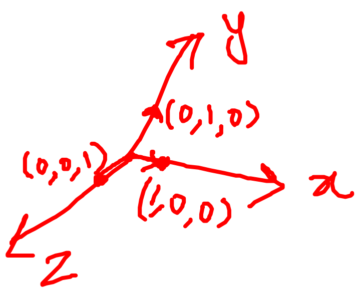
$c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_kx_k = 0$ for some scalars $c_1, c_2, \ldots\ldots. c_k \in$R

where all $c_1, c_2, \ldots\ldots. c_k$ are zero.

$\mathbb{R}^3$

Example:

the set of vectors $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ are linearly independent.



$$\begin{bmatrix}2\\5\\3\end{bmatrix} = \alpha_1\begin{bmatrix}1\\0\\0\end{bmatrix} + \alpha_2\begin{bmatrix}0\\1\\0\end{bmatrix} + \alpha_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Linear Dependent and Independent vectors

Definition :

Suppose that V is a vector space and that $x_1, x_2, \ldots\ldots. x_k$ are vectors in V.

The set of vectors $\{x_1, x_2, \ldots\ldots. x_k\}$ is linearly dependent if

$c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_kx_k = 0$ for some scalars $c_1, c_2, \ldots\ldots. c_k \in R$

where at least one of $c_1, c_2, \ldots\ldots. c_k$ is non–zero.

Example :

$$2\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\5\\7\end{bmatrix} - \begin{bmatrix}5\\9\\13\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$2(1) + 3 - 5 = 0$

So the set of vectors $\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}3\\5\\7\end{bmatrix}, \begin{bmatrix}5\\9\\13\end{bmatrix}\right\}$ are linearly dependent.

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Linear Dependent and Independent vectors

$c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + c_2\begin{bmatrix}3\\2\\9\end{bmatrix} + c_3\begin{bmatrix}5\\2\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Example:

Let x = $\begin{bmatrix}1\\2\\3\end{bmatrix}$, y = $\begin{bmatrix}3\\2\\9\end{bmatrix}$ and z = $\begin{bmatrix}5\\2\\-1\end{bmatrix}$. Is{x, y, z } linearly dependent?

Solution: We have to determine whether or not we can find real numbers $c_{1,} c_2, c_3$ which are not all zero, such that $c_1 x + c_1 y + c_2 z = 0$ 2 3

We have to solve the augmented matrix equation:

$c_1 + 3c_2 + 5c_3 = 0$

$2c_1 + 2c_2 + 2c_3 = 0$

$3c_1 + 9c_2 - c_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 9 & -1 & 0 \end{array}\right] \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 0 & -16 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 := -\frac{1}{16}R_3]{R_2 := -\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So this set of equations has a zero solution. Therefore,{x, y,z} is a linearly independent set of vectors.

Linear Dependent and Independent vectors

Example : Let $x = \begin{bmatrix}1\\2\\3\end{bmatrix}$, $y = \begin{bmatrix}3\\2\\1\end{bmatrix}$ and $z = \begin{bmatrix}0\\4\\8\end{bmatrix}$. Is {x, y, z } linearly dependent?

Solution: We have to determine whether or not we can find real numbers c_1, c_2, c_3 which are not all zero, such that $c_1 x + c_2 y + c_3 z = 0$

We have to solve the augmented matrix equation:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 2 & 2 & 4 & 0 \\ 3 & 1 & 8 & 0 \end{array}\right] \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -8 & 8 & 0 \end{array}\right]$$

$2-2(1)$

$2-2(3)=-4$

$$\xrightarrow{R_3 := R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or}$$

So this set of equations has a non–zero solution. Therefore,{x, y,z} is a linearly dependent set of vectors.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Linear Dependent and Independent vectors

$P(x) = a_0 x + a_1 x^2 + a_2 x^3 + \dots a_{n-1} x^n + a_n$

Example: Consider the polynomials $p(x) = 1 + 3x + 2x^2, q(x) = 3 + x + 2x^2$ and $r(x) = 2x + x^2$ in P_2 . Is $\{p(x), q(x), r(x)\}$linearly dependent?

degree 2

$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0$?

$[A : 0]$

Solution :

$Let\ \ c_1 p(x) + c_2 q(x) + c_3 r(x)$=0

That is $c_1(1 + 3x + 2x^2) + c_2(3 + x + 2x^2) + c_3(2x + x^2) = 0$

$c_1 + 3c_1 x + 2c_1 x^2 + 3c_2 + c_2 x + 2c_2 x^2 + 2c_3 x + c_3 x^2 = 0$

Which gives the ~~matrix~~ coefficient matrix form

$$\begin{array}{c c} & \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \begin{array}{c} \text{constant} \\ x \\ x^2 \end{array} & \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \end{array} \xrightarrow[R_3 := R_3 - 2R_1]{R_2 := R_2 - 3R_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -8 & 2 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 := R_2 - R_3} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$c_1 + 3c_2 = 0$

$3c_1 x + c_2 x + 2c_3 x = 0$

$2c_1 x^2 + 2c_2 x^2 + c_3 x^2 = 0$

Hence $\{p(x), q(x), r(x)\}$is linearly dependent.

Linear Spans

Linear Spans:

Suppose $u_1, u_2, \dots u_n$ are any vectors in a vector space V.

Then any vector of the form

$a_1 u_1 + a_2 u_2 + \cdots + a_n u_n$, where a_i are scalars, is called linear combination of $u_1, u_2, \dots, u_n$.

The collection of all such linear combinations, denoted by

$$span(u_1, u_2, \dots, u_n)$$

is called Linear span of $u_1, u_2, \dots u_n$.

Basis and Dimensions

Basis: A set $S = \{u_1, u_2, \ldots, u_n\}$ of vectors is a basis of V if it has the following two properties.

- S is linearly independent
- S spans V.

Dimension of Vector Space:

A vector space V is said to be of dimension n if V has a basis with n elements.

Example

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1. Determine whether or not each of the following form a basis of $\mathbb{R}^3$:

(x, y, z)
$(1, 1, 1)$

a) $(1,1,1), (1,0,1)$ b) $(1,2,3), (1,3,5), (1,0,1), (2,3,0)$

c) $(1,1,1), (1,2,3)(2,-1,1)$ d) $(1,1,2), (1,2,5), (5,3,4)$

$x+y+z$
$x+2y+3z$
$2x-y+z$

Solution: a and b are not basis of $\mathbb{R}^3$ because basis of $\mathbb{R}^3$ must contain exactly three elements

c) The three vectors of $\mathbb{R}^3$ form basis if they are linearly independent.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

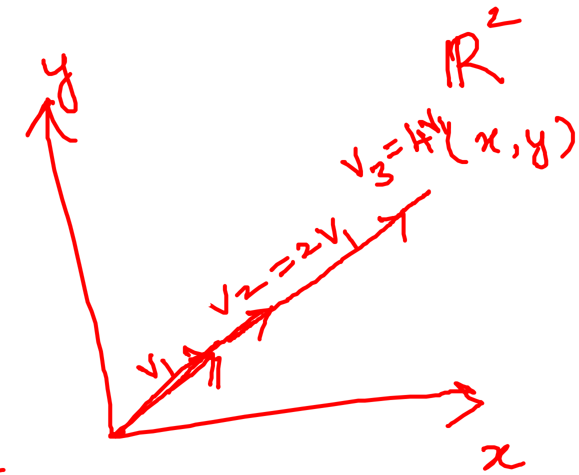
The echelon matrix has no zero rows; hence the vectors are LID.

Example

d) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 5 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & -2 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

The echelon matrix has zero row hence the vectors are not LID so they do not form the basis of $\mathbb{R}^3$.

Span $\{v_1\}$



$$\begin{array}{cc} v_1 & v_2 \end{array}$$
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$x_1 + y_1 = 0$

$2x_1 + 2y_1 = 0$

Example

$b = -c - a$

2. Find the basis and dimension of the subspace W of $\mathbb{R}^3$
 where

a) $W = \{(a, b, c) : a + b + c = 0\}$

b) $W = \{(a, b, c) : a = b = c\}$

$\boxed{a = -b - c}$

$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

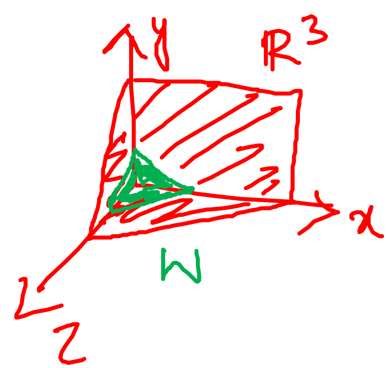
No. of unknowns

a) Note that number of free variables=3-1=2 , so dimension is 2 and any two independent vectors which satisfying equation $a + b + c = 0$ is the basis set of V.

$a + b + c = 0$

$a = -b - c$

e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$.



Example

b) Any vector $w \in W$ has the form $w = (k, k, k), k \in \mathbb{R}$. Hence $w = k(1,1,1) = ku$, where $u = (1,1,1)$. Since spanning set contain one element, therefore dimension is 1.

$\mathbb{R}^3$ (x, y, z)

$\mathbb{R}^4$ (w, x, y, z)

Example 3.Let W be subspace of $\mathbb{R}^4$ spanned by the vectors $u_1 = (1, -2,5, -3), u_2 = (2,3,1, -4), u_3 = (3,8, -3, -5)$.

a) Find basis and dimension of W.

b) Extend the basis of W to basis of $\mathbb{R}^4$.

$(1,0,0)$ $(0,1,0)$

$(0,0,1)$

Example

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero rows $(1,-2,5,-3)$ and $(0,7,-9,2)$ form the basis of W.

Therefore, dimension of W is 2.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) To extend the basis of W to a basis of $\mathbb{R}^4$ we have to find four linearly independent vectors which include the above two vectors. The four vectors are

$$(1,-2,5,-3), (0,7,-9,2), (0,0,1,0), (0,0,0,1).$$

They form the basis of $\mathbb{R}^4$.

Example

4. Let V be the vector space of 2×2 matrices over $\mathbb{R}$. Let W be the subspace of symmetric matrices. Find the dimension and basis of W.

Solution: For $w \in W$, we have $w = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$.

Let $e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

We show $S = \{e_1, e_2, e_3\}$ is a basis of W i.e. we have to prove

a) S spans W.

b) S is linearly independent.

Example

Solution:

a)Any matrix $w = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$=$ae_1 + be_2 + ce_3$. Thus S spans W.

b) Suppose $xe_1 + ye_2 + ze_3 = 0$

$$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This gives

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So $x = y = z = 0$. Hence e_1, e_2, e_3 are linearly independent.

Therefore S is a basis of W.

Examples of inner product spaces

Inner Product spaces:-

The vector space V over F is said to be an inner product space if there is defined for any two vectors $u, v \in V$ an element in F such that

i. $\langle u, v\rangle = \langle \overline{v, u}\rangle$

ii. $\langle u, v\rangle \geq 0$ and $\langle u, v\rangle = 0$ $if\ and\ only\ if$ $u = 0$

iii. $\langle \alpha u + \beta v,\ \ w\rangle = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$ $for\ any$ $u, v, w\ \in V$ and $\alpha, \beta\ \in\ F$

Remarks:-

i. In this definition F is either the field of real or complex number

ii. $\langle \overline{v, u}\rangle$ denote the conjugate of $\langle v, u\rangle$.

Example:-

If $z = x + iy$ then $\bar{z} = x - iy$ we have $\bar{\bar{z}} = z, z\bar{z} = |z|^2 = x^2 + y^2 \geq 0$ and also $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$, $\overline{z_1 . z_2} = \overline{z_1} . \overline{z_2}$

i. Using properties 1 and 3 in the definition,

We have

$\langle u,\ \ \alpha v + \beta w\rangle = \langle \overline{\alpha v + \beta w,\ \ u}\rangle$ (by i)

$= \overline{\alpha\langle v, u\rangle + \beta\langle w, u\rangle}$ (by iii)

$= \bar{\propto} \langle \overline{v, u}\rangle + \bar{\beta}\langle \overline{w, u}\rangle$

$= \bar{\propto} \langle u, v\rangle + \bar{\beta}\langle u, w\rangle$

Examples of inner product spaces

1. In $F^{(n)}$, let $u = (\alpha_1, \alpha_2, \ldots \alpha_n)\ and\ v = (\beta_1, \beta_2, \ldots \beta_n)$ defined $\langle u, v\rangle = \alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots + \alpha_n\,\overline{\beta_n}$ show that this defines an inner product on $F^{(n)}$.

Solution:-

i. We have $\langle v, u\rangle = \beta_1\overline{\alpha_1} + \beta_2\overline{\alpha_2} + \cdots + \beta_n\,\overline{\alpha_n}$

Now $\langle \overline{v, u}\rangle = \overline{\beta_1\overline{\alpha_1} + \beta_2\overline{\alpha_2} + \cdots + \beta_n\,\overline{\alpha_n}}$

$$= \overline{\beta_1\overline{\alpha_1}} + \overline{\beta_2\overline{\alpha_2}} + \cdots + \overline{\beta_n\overline{\alpha_n}}$$

$$= \overline{\beta_1}\ \overline{\overline{\alpha_1}} + \overline{\beta_2}\ \overline{\overline{\alpha_2}} + \cdots + \overline{\beta_n}\ \overline{\overline{\alpha_n}}$$

$$= \overline{\beta_1}\ \alpha_1 + \overline{\beta_2}\ \alpha_2 + \cdots + \overline{\beta_n}\ \alpha_n$$

$$= \alpha_1\overline{\beta_1}\ + \alpha_2\overline{\beta_2}\ + \cdots + \alpha_n\overline{\beta_n}$$

$$\langle \overline{v, u}\rangle = \langle u, v\rangle$$

ii. We have $\langle u, u\rangle = \alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} + \cdots + \alpha_n\,\overline{\alpha_n}$

$= |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 \quad (because\ z\bar{z} = |z|^2)$

$\langle u, u\rangle \geq 0$

Next, $\langle u, u\rangle = 0$

$$\Leftrightarrow |\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_n|^2 = 0$$

$$\Leftrightarrow |\alpha_i|^2 = 0\ for\ each\ i = 1,2, \ldots, n$$

$$\Leftrightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$$\Leftrightarrow u = 0$$

Examples of inner product spaces

iii. Let $w = (\gamma_1, \gamma_1, \dots \gamma_n)$ and $\propto, \beta \in F$

Now $\propto u + \beta v = \alpha(\alpha_1, \alpha_2, \dots \alpha_n) + \beta(\beta_1, \beta_2, \dots \beta_n)$

$$= (\alpha\alpha_1, \alpha\alpha_2, \dots \alpha\alpha_n) + (\beta\beta_1, \beta\beta_2, \dots \beta\beta_n)$$

$$\propto u + \beta v = (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2, \dots \alpha\alpha_n + \beta\beta_n)$$

Then

$$\langle \alpha u + \beta v, \qquad w \rangle = \langle \alpha\alpha_1 + \beta\beta_1 \rangle \overline{\gamma_1} + \langle \alpha\alpha_2 + \beta\beta_2 \rangle \overline{\gamma_2} + \cdots + \langle \alpha\alpha_n + \beta\beta_n \rangle \overline{\gamma_n}$$

$$= (\alpha\alpha_1\gamma_1, \alpha\alpha_2\gamma_2, \dots, \alpha\alpha_n\overline{\gamma_n}) + (\beta\beta_1\gamma_1, \beta\beta_2\gamma_2, \dots, \beta\beta_n\overline{\gamma_n})$$

$$= \alpha(\alpha_1\gamma_1, \alpha_2\gamma_2, \dots, \alpha_n\overline{\gamma_n}) + \beta(\beta_1\gamma_1, \beta_2\gamma_2, \dots, \beta_n\overline{\gamma_n})$$

$$\langle \alpha u + \beta v, \qquad w \rangle = \alpha\langle u, w \rangle + \beta\langle v, w \rangle$$

Thus all the three properties of the inner product satisfied.

Hence $\langle u, v \rangle$ defines an inner product on $F^{(n)}$

Examples of inner product spaces

2. In $F^{(2)}$, let $u=(\alpha_1,\alpha_2)$ and $v=(\beta_1,\beta_2)$ defined $\langle u,v\rangle = 2\alpha_1\overline{\beta_1}+\alpha_1\overline{\beta_2}+\alpha_2\overline{\beta_1}+\alpha_2\overline{\beta_2}$ show that this defines an inner product on $F^{(2)}$.

Solution:-

i. We have $\langle v,u\rangle = 2\beta_1\overline{\alpha_1}+\beta_1\overline{\alpha_2}+\beta_2\,\overline{\alpha_1}+\beta_2\,\overline{\alpha_2}$

Now $\langle \overline{v,u}\rangle = \overline{2\beta_1\overline{\alpha_1}+\beta_1\overline{\alpha_2}+\beta_2\,\overline{\alpha_1}+\beta_2\,\overline{\alpha_2}}$

$= 2\overline{\beta_1\overline{\alpha_1}}+\overline{\beta_1\overline{\alpha_2}}+\overline{\beta_2\overline{\alpha_1}}+\overline{\beta_2\overline{\alpha_2}}$

$becuase\ \overline{z_1+z_2}=\overline{z_1}+\overline{z_2}$

$= 2\overline{\beta_1}\,\overline{\overline{\alpha_1}}+\overline{\beta_1}\,\overline{\overline{\alpha_2}}+\overline{\beta_2}\,\overline{\overline{\alpha_1}}+\overline{\beta_2}\,\overline{\overline{\alpha_2}}$

$= 2\overline{\beta_1}\,\alpha_1+\overline{\beta_1}\,\alpha_2+\overline{\beta_2}\,\alpha_1+\overline{\beta_2}\,\alpha_2$

$= 2\alpha_1\overline{\beta_1}+\overline{\beta_1}\alpha_2+\overline{\beta_2}\alpha_1+\overline{\beta_2}\alpha_2$

$\langle \overline{v,u}\rangle = \langle u,v\rangle$

ii. We have $\langle u,u\rangle = 2\alpha_1\overline{\alpha_1}+\alpha_1\overline{\alpha_2}+\alpha_2\overline{\alpha_1}+\alpha_2\,\overline{\alpha_2}$

$= \alpha_1\overline{\alpha_1}+\alpha_1\overline{\alpha_1}+\alpha_1\overline{\alpha_2}+\alpha_2\overline{\alpha_1}+\alpha_2\,\overline{\alpha_2}$

$= |\alpha_1|^2+\alpha_1(\overline{\alpha_1}+\overline{\alpha_2})+\alpha_2(\overline{\alpha_1}+\overline{\alpha_2})\quad (because\ z\bar{z}=|z|^2)$

$= |\alpha_1|^2+(\alpha_1+\alpha_2)(\overline{\alpha_1}+\overline{\alpha_2})$

$= |\alpha_1|^2+(\alpha_1+\alpha_2)(\overline{\alpha_1+\alpha_2})$

$= |\alpha_1|^2+|\alpha_1+\alpha_2|^2\ \ (because\ z\bar{z}=|z|^2)$

$\langle u,u\rangle \geq 0\quad (because|\alpha_1|^2\geq 0, |\alpha_1+\alpha_2|^2\geq 0\ )$

Next, $\langle u,u\rangle = 0$

$$\Leftrightarrow |\alpha_1|^2 + |\alpha_1+\alpha_2|^2 = 0$$
$$\Leftrightarrow |\alpha_1|^2 = 0 \; and \; |\alpha_1+\alpha_2|^2 = 0$$
$$\Leftrightarrow \alpha_1 = 0 \;\; and \; \alpha_2 = 0$$
$$\Leftrightarrow u = 0$$

iii. Let $w = (\gamma_1, \gamma_2) \; and \; let \;\; \propto, \beta \in F$

Now $\propto u + \beta v = \alpha(\alpha_1,\alpha_2) + \beta(\beta_1,\beta_2)$

$$= (\alpha\alpha_1, \alpha\alpha_2) + (\beta\beta_1, \beta\beta_2)$$
$$\propto u + \beta v = (\alpha\alpha_1 + \beta\beta_1, \alpha\alpha_2 + \beta\beta_2)$$

Then

$\langle \alpha u + \beta v, \qquad w\rangle$

$= 2\langle \alpha\alpha_1 + \beta\beta_1\rangle\overline{\gamma_1} + \langle \alpha\alpha_1 + \beta\beta_1\rangle\overline{\gamma_2} + \langle \alpha\alpha_2 + \beta\beta_2\rangle\overline{\gamma_1} + \langle \alpha\alpha_2 + \beta\beta_2\rangle\overline{\gamma_2}$

$= \alpha(2\alpha_1\overline{\gamma_1} + \alpha_1\overline{\gamma_2} + \alpha_2\overline{\gamma_1} + \alpha_2\overline{\gamma_2}) + \; \beta(2\beta_1\overline{\gamma_1} + \beta_1\overline{\gamma_2} + \beta_2\overline{\gamma_1} + \beta_2\overline{\gamma_2})$

$$\langle \alpha u + \beta v, \qquad w\rangle = \; \alpha\langle u,w\rangle + \beta\langle v,w\rangle$$

Thus all the three properties of an inner product satisfied.

Hence $\langle u,v\rangle$ defines an inner on $F^{(n)}$

Examples of inner product spaces

3. Compute the length of vector $v = (1,\ -2,\ 2,\ 0)$ and also find an unit vectors and length of unit vector.

Solution: Given that $v = (1,\ -2,\ 2,\ 0)$

Length of vector $= ||v||$

$$= \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2 + {v_4}^2}$$

$$= \sqrt{(1)^2 + (-2)^2 + (2)^2 + (0)^2}$$

$$= \sqrt{1+4+4+0} = \sqrt{9} = 3$$

$$||v|| = 3$$

Unit vector $(\hat{n}) = \frac{v}{||v||} = \frac{(1,-2,2,0)}{3}$

u = Unit vector $(\hat{n}) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{2}{3},\ 0\right)$

Length of an unit vector $||u|| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + (0)^2}$

$$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = \sqrt{1}$$

$$||\mathrm{u}|| = 1$$

Examples of inner product spaces

4. Calculate the length of vector $v = (2,\ -1,\ 1,\ 2)$ and also find an unit vector.

Solution: Given that $v = (2,\ -1,\ 1,\ 2)$

Length of vector $= \left||v|\right|$

$$= \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2 + {v_4}^2}$$

$$= \sqrt{(2)^2 + (-1)^2 + (1)^2 + (2)^2}$$

$$= \sqrt{4 + 1 + 1 + 4} = \sqrt{10}$$

$$\left||v|\right| = \sqrt{10}$$

Unit vector $(\hat{n}) = \frac{v}{||v||} = \frac{(2, -1, 1, 2)}{\sqrt{10}}$

u = Unit vector $(\hat{n}) = \left(\frac{2}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

Examples of inner product spaces

5. Compute the distance between the vectors $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and v= $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Solution: Given that $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ and v= $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Distance between u and v $= ||u - v||$

$$= \sqrt{(u-v)(u-v)}$$

$$= \sqrt{(u-v)^2}$$

Let $u - v = \begin{bmatrix} 7 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$||u - v|| = \sqrt{(4)^2 + (-1)^2}$$

$$= \sqrt{16 + 1}$$

$$||u - v|| = \sqrt{17}$$

Gram Schmidt Process

- The Gram-Schmidt process is a method for orthonormalizing a set of vectors in an inner product space

- The Gram-Schmidt process takes a finite, linearly independent set of vectors

$$S = \{v_1, \dots, v_k\} \quad \text{for } k \leq n$$

and generates an orthogonal set

$$S' = \{u_1, \dots, u_k\}$$

- The Gram-Schmidt process works as follows:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

Gram Schmidt process (contd.)

$$u_4 = v_4 - \frac{\langle v_4, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_4, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_4, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$\vdots$$

$$\vdots$$

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

- The sequence $u_1, \ldots, u_k$ is the required system of orthogonal vectors.
- The normalized vectors

$$e_1 = \frac{u_1}{||u_1||}, e_2 = \frac{u_2}{||u_2||} \ldots, e_k = \frac{u_k}{||u_k||}$$

form an orthonormal set.

Example of Gram Schmidt process

Qus: Obtain an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$x_1 = (1,0,1,0), x_2 = (1,1,1,1), x_3 = (-1,2,0,1)$$

Sol: Following the Gram-Schmidt process, we set

$$v_1 = x_1 = (1,0,1,0) .$$

Next, we have

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1)$$

and

Example of Gram Schmidt process (contd.)

innovate achieve lead

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle x_3, v_2 \rangle}{||v_2||^2} v_2$$

$$= (-1,2,0,1) + \frac{1}{2}(1,0,1,0) - \frac{3}{2}(0,1,0,1)$$

$$= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

The orthogonal basis so obtained is

$$\{(1,0,1,0), (0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

We can normalise the vectors to get

$$\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

Example of Gram Schmidt process

Q. Apply Gram-Schmidt orthogonalization to the following sequence of vectors in R^3 : $\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}8\\1\\-6\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$

Solution:

Apply the process with x1= $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $x2 = \begin{pmatrix}8\\1\\-6\end{pmatrix}$, $x3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Step 1 produces an orthogonal basis:

v1= x1= $\begin{pmatrix}1\\2\\0\end{pmatrix}$

Example of Gram Schmidt process

$$v2 = x2 - \frac{(x2,v1)}{(v1,v1)} v1 = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{\left(\begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ -6 \end{bmatrix} - \frac{10}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{pmatrix} 6 \\ -3 \\ -6 \end{pmatrix}$$

$$v3 = x3 - \frac{(x3,v1)}{(v1,v1)} v1 - \frac{(x3,v2)}{(v2,v2)} v2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)}{\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)}{\left(\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}\right)} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} =$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{5} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \frac{-6}{81} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$$

Example of Gram Schmidt process

Step 2 produces an orthonormal basis by replacing each vector with a vector of norm 1:

$$\text{Replace } v_1 \text{ with } \frac{v_1}{|v_1|} = \frac{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\|} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

$$\text{Replace } v_2 \text{ with } \frac{v_2}{|v_2|} = \frac{\begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix}}{\left\| \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} \right\|} = \frac{1}{9} \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}.$$

$$\text{Replace } v_3 \text{ with } \frac{v_3}{|v_3|} = \frac{\frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}}{\left\| \frac{1}{9}\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} \right\|} = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$$

Example of Gram Schmidt process

So the final solution is $v_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_2 = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, v_3 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix}.$

THANK YOU