BITS Pilani
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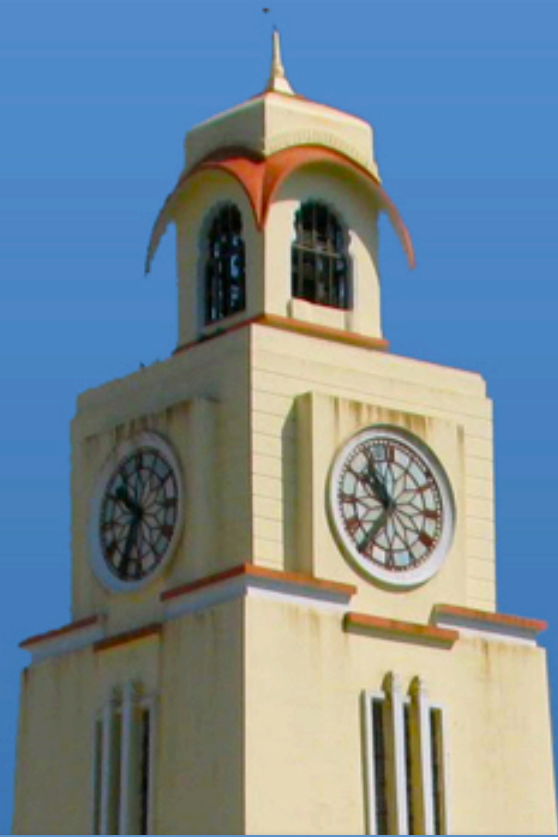
Mathematical Foundations

MFDS Team



Mathematical Foundations

Webinar 2

Agenda

- Gram Schmidt Orthogonalization
- Cholesky decomposition
- Eigenvalues and Eigen vectors
- Eigen Decomposition
- Spectral Theorem
- Singular Value Decomposition

Gram Schmidt Process

- The Gram-Schmidt process is a method for orthonormalizing a set of vectors in an inner product space

- The Gram-Schmidt process takes a finite, linearly independent set of vectors

$$S = \{v_1, \ldots, v_k\} \quad \text{for } k \leq n$$

and generates an orthogonal set

$$S' = \{u_1, \ldots, u_k\}$$

- The Gram-Schmidt process works as follows:

$$\begin{aligned} u_1 &= v_1 \\ u_2 &= v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 \\ u_3 &= v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 \end{aligned}$$

Gram Schmidt process (contd.)

$$u_4 = v_4 - \frac{\langle v_4, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_4, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2 - \frac{\langle v_4, u_3 \rangle}{\langle u_3, u_3 \rangle} u_3$$

$$\vdots$$

$$\vdots$$

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{\langle v_k, u_j \rangle}{\langle u_j, u_j \rangle} u_j$$

- The sequence $u_1, \ldots, u_k$ is the required system of orthogonal vectors.
- The normalized vectors

$$e_1 = \frac{u_1}{||u_1||}, e_2 = \frac{u_2}{||u_2||} \ldots, e_k = \frac{u_k}{||u_k||}$$

form an orthonormal set.

Example of Gram Schmidt process

Qus: Obtain an orthogonal basis for the subspace of $\mathbb{R}^4$ spanned by

$$x_1 = (1,0,1,0), x_2 = (1,1,1,1), x_3 = (-1,2,0,1)$$

Sol: Following the Gram-Schmidt process, we set

$$v_1 = x_1 = (1,0,1,0).$$

Next, we have

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) = (0,1,0,1)$$

and

Example of Gram Schmidt process (contd.)

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle x_3, v_2 \rangle}{||v_2||^2} v_2$$

$$= (-1,2,0,1) + \frac{1}{2}(1,0,1,0) - \frac{3}{2}(0,1,0,1)$$

$$= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) .$$

The orthogonal basis so obtained is

$$\{(1,0,1,0), (0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

We can normalise the vectors to get

$$\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

Cholesky decomposition

For a symmetric positive definite matrix A, a decomposition of the form A = LL^T is possible. For example

$$A = \begin{bmatrix} 42 & 32 & 37 \\ 32 & 34 & 31 \\ 37 & 31 & 35 \end{bmatrix}$$

This is symmetric and positive definite

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

Cholesky decomposition

$$a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}} = \sqrt{42} = 6.4807$$

$$a_{21} = l_{21} l_{11} \Rightarrow l_{21} = \frac{32}{6.4807} = 4.9377$$

$$a_{13} = l_{11} l_{31} \Rightarrow l_{31} = \frac{37}{6.4807} = 5.7092$$

$$a_{22} = (l_{21})^2 + (l_{22})^2$$

$$34 = (4.9377)^2 + (l_{22})^2$$

$$\Rightarrow l_{22} = 3.1015$$

$$a_{32} = l_{31} l_{21} + l_{32} l_{22} \Rightarrow l_{32} = .906$$

Cholesky decomposition

$$a_{33} = (l_{31})^2 + (l_{32})^2 + (l_{33})^2$$
$$l_{33} = 1 \cdot 2586$$

$$L = \begin{bmatrix} 6.4807 & 0 & 0 \\ 4.9377 & 3.1015 & 0 \\ 5.7092 & 0.9059 & 1.2586 \end{bmatrix}$$

Eigenvalues and Eigenvectors

A matrix eigenvalue problem considers the vector equation

(1) $$\mathbf{A}\mathbf{x} = \lambda \mathbf{x},$$

where **A** is a given square matrix, λ an unknown scalar(real or complex), and **x** an unknown vector.
The task is to determine λ's and **x**'s (dependent on λ's)that satisfy (1).
Since $\mathbf{x} = \mathbf{0}$ is always a solution for any λ,we only admit solutions with $\mathbf{x} \neq \mathbf{0}$.
The solutions to (1) are given the following names: The λ's that satisfy (1) are called **eigenvalues of A** and the corresponding nonzero **x**'s that also satisfy (1) are called **eigenvectors of A**.

Multiple Eigenvalues

Example 2: Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}.$$

Solution.

For our matrix, the characteristic determinant gives the characteristic equation

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0.$$

The roots (eigenvalues of **A**) are $\lambda_1 = 5,\ \lambda_2 = \lambda_3 = -3.$

Multiple Eigenvalues

Solution. ***(continued 1)***

To find eigenvectors, we apply the Gauss elimination to the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$, first with $\lambda = 5$ and then with $\lambda = -3$. For $\lambda = 5$ the characteristic matrix is

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}.$$

It row-reduces to $$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -24/7 & -48/7 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiple Eigenvalues

Solution. *(continued 2)*

Hence it has rank 2. Choosing $x_3 = -1$ we have $x_2 = 2$ from $-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$ and then $x_1 = 1$ from $-7x_1 + 2x_2 - 3x_3 = 0$. Hence an eigenvector of **A** corresponding to $\lambda = 5$ is $\mathbf{x}_1 = [1 \quad 2 \quad -1]^T$.

For $\lambda = -3$ the characteristic matrix

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

row-reduces to

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Multiple Eigenvalues

Solution. *(continued 3)*

Hence it has rank 1.

From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$. Choosing $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$, we obtain two linearly independent eigenvectors of $\mathbf{A}$ corresponding to $\lambda = -3$

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Exam question

A professor teaching linear algebra wanted to test students understanding of eigenvalues and eigenvectors. Professor told class that β is some real number. By using this β, professor constructed a 3 by 3 matrix C given by

$$C = \begin{bmatrix} 2 & \beta & 0 \\ \beta & 2 & \beta \\ 0 & \beta & 2 \end{bmatrix}$$

i) If it is given to you that C has three non-zero eigenvalues, then can you tell me all the possible values of β?

ii) If it is given to you that C has 3 positive eigenvalues, then can you tell me all the possible values of β?

Exam question

i) If matrix has only nonzero eigenvalues, then determinant is not zero
This means $2 - \beta^2 \neq 0$. So $\beta \neq \sqrt{2}$

ii) Similarly, one necessary condition for all positive eigenvalues is
$2 - \beta^2 > 0$

In summary $-\sqrt{2} \leq \beta \leq \sqrt{2}$

Similarity of Matrices

Similar Matrices. Similarity Transformation

An $n \times n$ matrix $\hat{\mathbf{A}}$ is called **similar** to an $n \times n$ matrix $\mathbf{A}$ if

$$\hat{\mathbf{A}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{4}$$

for some (nonsingular!) $n \times n$ matrix $\mathbf{P}$. This transformation, which gives $\hat{\mathbf{A}}$ from $\mathbf{A}$, is called a **similarity transformation**.

Eigenvalues and Eigenvectors of Similar Matrices

If $\hat{\mathbf{A}}$ is similar to $\mathbf{A}$, then $\hat{\mathbf{A}}$ has the same eigenvalues as $\mathbf{A}$. Furthermore, if $\mathbf{x}$ is an eigenvector of $\mathbf{A}$, then $\mathbf{y} = \mathbf{P}^{-1}\mathbf{x}$ is an eigenvector of $\hat{\mathbf{A}}$ corresponding to the same eigenvalue.

Diagonalization of a Matrix

Diagonalization of a Matrix

If an $n \times n$ matrix **A** *has a basis of eigenvectors, then*

(5) $$\mathbf{D} = \mathbf{X}^{-1}\mathbf{A}\mathbf{X}$$

is diagonal, with the eigenvalues of **A** *as the entries on the main diagonal. Here* **X** *is the matrix with these eigenvectors as column vectors. Also,*

(5*) $$\mathbf{D}^{m} = \mathbf{X}^{-1}\mathbf{A}^{m}\mathbf{X} \qquad (m = 2, 3, \dots).$$

Diagonalization of a Matrix

Diagonalize

$$\mathbf{A} = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}.$$

Solution.

The characteristic determinant gives the characteristic equation $-\lambda^3 - \lambda^2 + 12\lambda = 0$. The roots (eigenvalues of $\mathbf{A}$) are $\lambda_1 = 3$, $\lambda_2 = -4$, $\lambda_3 = 0$. By the Gauss elimination applied to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ with $\lambda = \lambda_1, \lambda_2, \lambda_3$ we find eigenvectors and then $\mathbf{X}^{-1}$ by the Gauss–Jordan elimination

Diagonalization of a Matrix

Solution. (continued 1)

The results are

$$\begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix},$$

$$\mathbf{X}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Diagonalization of a Matrix

Solution. *(continued 2)*
Calculating $\mathbf{AX}$ and multiplying by $\mathbf{X}^{-1}$ from the left, we thus obtain

$$\mathbf{D} = \mathbf{X}^{-1}\mathbf{AX} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} -3 & -4 & 0 \\ 9 & 4 & 0 \\ -3 & -12 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Spectral theorem

- An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that
$$A = PDP^T = PDP^{-1} \qquad (1)$$
- Such a diagonalization requires n linearly independent and orthonormal eigenvectors.
- When is this possible?
- If A is orthogonally diagonalizable as in (1), then
$$A^T = (PDP^T)^T = P^{TT} D^T P^T = PDP^T = A$$

Spectral theorem

- **Theorem 2:** An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric matrix.
- **Theorem 3:** The Spectral Theorem for Symmetric Matrices
- An $n \times n$ symmetric matrix A has the following properties:
  a. A has n real eigenvalues, counting multiplicities.
  b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

Spectral Theorem

c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

d. A is orthogonally diagonalizable.

- **Example** : Orthogonally diagonalize the matrix

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix},$$ whose characteristic equation is

$$0 = -\lambda^3 + 12\lambda^2 - 21\lambda - 98 = -(\lambda - 7)^2(\lambda + 2)$$

Spectral Theorem

- **Solution:** The usual calculations produce bases for the eigenspaces:

$$\lambda = 7: \mathrm{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathrm{v}_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}; \lambda = -2: \mathrm{v}_3 = \begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix}$$

- Although $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, they are not orthogonal.
- The projection of $\mathbf{v}_2$ onto $\mathbf{v}_1$ is $\frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1$.
- The component of $\mathbf{v}_2$ orthogonal to $\mathbf{v}_1$ is

$$\mathrm{z}_2 = \mathrm{v}_2 - \frac{v_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} - \frac{-1/2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 1/4 \end{bmatrix}$$

Spectral Theorem

- Then $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an orthogonal set in the eigenspace for $\lambda = 7$.
- (Note that $\mathbf{z}_2$ is linear combination of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$, so $\mathbf{z}_2$ is in the eigenspace).
- Since the eigenspace is two-dimensional (with basis $\mathbf{v}_1, \mathbf{v}_2$), the orthogonal set $\{\mathbf{v}_1, \mathbf{z}_2\}$ is an *orthogonal basis* for the eigenspace, by the Basis Theorem.
- Normalize $\mathbf{v}_1$ and $\mathbf{z}_2$ to obtain the following orthonormal basis for the eigenspace for $\lambda = 7$:

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, u_2 = \begin{bmatrix} -1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$$

Spectral Theorem

- An orthonormal basis for the eigenspace for $\lambda = -2$ is

$$\mathbf{u}_3 = \frac{1}{\|2\mathbf{v}_3\|} 2\mathbf{v}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

- By Theorem 1, $\mathbf{u}_3$ is orthogonal to the other eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$.

- Hence $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

Spectral Theorem

- Let

$$P = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix}, D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

Exam question

Let P be a real square matrix satisfying $P = P^T$ and $P^2 = P$.

1. Can the matrix P have complex eigenvalues? If so, construct an example, else, justify your answer.
2. What are the eigenvalues of P ?

$P = P^T \implies P$ is symmetric.

Spectral theorem: If $A \in \mathbb{R}^{n \times n}$ is symmetric there exists an orthonormal basis of the corresponding vector space V consisting of the eigenvectors of A, and each eigenvalue is real.

Therefore, by the Spectral theorem, P cannot have complex eigenvalues.

Exam question

Suppose λ is an eigenvalue of P and let v be the corresponding eigenvector.

Then
$$Pv = \lambda v$$

$$\Longrightarrow \; P^2 v = \lambda P v = \lambda^2 v$$

$$\Longrightarrow \; Pv = \lambda^2 v$$

$$\Longrightarrow \; \lambda v = \lambda^2 v \qquad (\because P^2 = P)$$

$$\Longrightarrow \; (\lambda^2 - \lambda) v = \mathbf{0}$$

Since $v \neq \mathbf{0}$, we have

$$\lambda^2 - \lambda = 0 \;\Longrightarrow\; \lambda(\lambda - 1) = 0 \quad \Longrightarrow\; \lambda = 0 \text{ or } \lambda = 1$$

Hence, the eigenvalues of P are 0 and 1.

Singular Value Decomposition

Singular Value Decomposition(SVD) is a factorization of an m x n matrix into

- U is an m x m orthogonal matrix (Its columns are Left Singular Vectors)
- Σ is an m x n diagonal matrix with **singular values** on the diagonal

$$\Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & & 0 \end{pmatrix} \qquad \text{Convention: } \sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant 0$$

- V^T is an n x n orthogonal matrix (V's columns are called Right Singular Vectors) such that $\mathbf{A = U\Sigma V^T}$

Evaluation of U and V

1. Find an orthogonal diagonalization of A^TA
   - Find the eigenvalues of A^TA and corresponding orthonormal set of eigenvectors
2. Set up V and Σ
   - Arrange the eigenvalues of A^TA in decreasing order and compute the square roots of the eigen values. Σ will be same size as A with D(diagonal entries are non zero singular values) in upper left corner and with 0's elsewhere
3. Derive U for $\mathbf{A = U\Sigma V^T}$

Singular Value Decomposition Example

Find Singular value decomposition of the matrix

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$$

<u>Step 1</u>: Compute $A^T A$

$$A^T A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

<u>Step 2</u>: Find the eigenvalues of $A^T A$.

The characteristic polynomial of $A^T A$ is:

$$\lambda^2 - trace(A^T A)\lambda + det(A^T A) = 0$$

$$\implies \lambda^2 - 90\lambda = 0 \qquad (\because det(A^T A) = 0)$$

SVD

$$\implies \lambda(\lambda - 90) = 0$$

$\implies \lambda_1 = 90$ and $\lambda_2 = 0$ are the two eigenvalues of $A^T A$.

Step 3: Find the singular values of A.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{90} \text{ and } \sigma_2 = \sqrt{\lambda_2} = 0.$$

Step 4: Construct the diagonal matrix of same size as A with singular values as its diagonal entries in decreasing order.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

SVD

<u>Step 5</u>: Find the eigenvectors (called right singular vectors) corresponding to each eigenvalue of $A^T A$.

Suppose $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue λ_2. Then

$$A^T A z = \lambda_2 z \quad \Longrightarrow \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 81z_1 - 27z_2 = 0 \text{ and } -27z_1 + 9z_2 = 0$$

$$\Longrightarrow z_2 = 3z_1$$

$$\Longrightarrow z = z_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Choose $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ as an eigenvector corresponding to λ_2.

SVD

Similarly, we can find $v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ as an eigenvector corresponding to λ_1.

Step 6: Write the orthogonal matrix consisting of the normalised eigenvectors of $A^T A$.

$$V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Step 7: Find the left eigenvectors (called left singular vectors) $\hat{u}_i$ using

$$\hat{u}_i = \frac{Av_i}{\sigma_i}.$$

For $i = 1$, we have.

$$\hat{u}_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{900}} \begin{bmatrix} 10 \\ -20 \\ -20 \end{bmatrix}.$$

SVD

Since $\sigma_2 = 0$, to find the other vector we need to use the orthogonality condition

Let $w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the next vector. Then

$$w \cdot u_1 = 0 \implies 10x - 20y - 20z = 0 \implies x = 2y + 2z$$

$$\implies w = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let

$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ And $w_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

SVD

Hence the vectors orthogonal to u_1 are

$$u_2 = w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \implies \hat{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and}$$

$$u_3 = w_2 - \frac{\langle w_2, u_2 \rangle}{||u_2||^2} u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \implies \hat{u}_3 = \frac{1}{\sqrt{45}} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

$$\text{Hence, } U = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix}.$$

SVD

<u>**Step 8**</u>: The singular value decomposition of A is

$$A = U\Sigma V^T = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Thank You