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Quantum Assisted Secure Multiparty Computation

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Abstract

Quantum cryptography is a field of study that utilizes the properties of quantum physics to develop cryptographic primitives that are beyond the reach of classical cryptography. Its main objective is to improve existing classical implementations and to introduce new cryptographic methods that can withstand the power of quantum computers. While much of the research in this field has focused on quantum key distribution (QKD), there have been important advances in the understanding and development of other two-party primitives such as quantum oblivious transfer (QOT). QOT protocols, similar to QKD protocols, allow for quantum-safe computation, however, the conditions under which QOT is fully quantum-safe are still under intense scrutiny. The thesis begins by surveying the work done on the concept of oblivious transfer within theoretical quantum cryptography, highlighting proposed protocols and their security requirements, discussing impossibility results, and examining quantum security models in which QOT security can be proven.

The most significant application of oblivious transfer (OT) is in the realm of secure multiparty computation (SMC). This technology has the potential to revolutionize fields such as data analysis and computation by enabling multiple parties to compute virtually any function while maintaining the privacy of their inputs. However, the security and efficiency of SMC protocols are heavily dependent on the security and efficiency of OT. To address this, the thesis conducts a detailed comparison of the complexity of quantum oblivious transfer based on oblivious keys and two of the fastest classical OT protocols. This comparison provides insight into the potential benefits and limitations of using quantum techniques in SMC.

Building on the theoretical comparison of quantum and classical approaches to oblivious transfer, the thesis integrates and compares both within an SMC system for genomic analysis. The proposed system utilizes quantum cryptographic protocols to compute a phylogenetic tree from a set of private genome sequences. This system significantly improves the privacy and security of the computation by incorporating three quantum cryptographic protocols that provide enhanced security against quantum computer attacks. The system adapts several distance-based methods, such as the Unweighted Pair Group Method with Arithmetic mean, Neighbour-Joining, and Fitch-Margoliash, into a

private setting where the sequences owned by each party are not disclosed to other members. The performance and privacy guarantees of the system are evaluated theoretically through complexity analysis and security proof. Additionally, the thesis provides an extensive explanation of the implementation details and cryptographic protocols used. The implementation of quantum-assisted secure phylogenetic tree computation is based on the Libscapi implementation of the Yao protocol, the PHYLIP library, and simulated keys of two quantum systems: quantum oblivious key distribution and quantum key distribution. The implementation is benchmarked against a classical-only solution, and the results indicate that both approaches have similar execution times, with the only difference being the time overhead taken by the oblivious key management system of the quantum-assisted approach.

Finally, the thesis presents the first quantum protocol for oblivious linear evaluation. Oblivious linear evaluation is a generalization of oblivious transfer, where two distrustful parties, Alice and Bob, obliviously compute a linear function, $f(x) = ax + b$, without revealing their inputs to each other. Alice inputs the function coefficients, a and b, and Bob inputs the function input, x. The output, $f(x)$, is only delivered by Bob. This primitive is essential for arithmetic-based secure multiparty computation protocols from both a structural and security point of view. In the classical setting, it is known that oblivious linear evaluation can be generated based on oblivious transfer, and quantum counterparts of these protocols can, in principle, be constructed as straightforward extensions based on quantum oblivious transfer. However, the thesis presents a novel quantum protocol for oblivious linear evaluation that does not rely on quantum oblivious transfer. The protocol is first presented for the semi-honest setting and then extended to the dishonest setting using a commit-and-open strategy. The protocol uses high-dimensional quantum states to obliviously compute the linear function, $f(x)$, on Galois fields of prime dimension, $GF(d) \cong \mathbb{Z}_d$, or prime-power dimension, $GF(d^M)$. The protocol utilizes a complete set of mutually unbiased bases in prime-power dimension Hilbert spaces and their linear behavior upon the Heisenberg-Weyl operators. The protocol is also generalized to achieve vector oblivious linear evaluation, which increases efficiency by generating several instances of oblivious linear evaluation. The security of the protocol is proven in the framework of quantum universal composability.

Key-words: quantum cryptography, quantum oblivious transfer, quantum oblivious linear evaluation, secure multiparty computation.

Resumo

A criptografia quântica é o campo da criptografia que explora as propriedades quânticas da matéria. Geralmente, visa desenvolver primitivas fora do alcance da criptografia clássica e melhorar as implementações clássicas existentes. Embora grande parte do trabalho neste campo se foque na distribuição de chaves quânticas (*quantum key distribution*, QKD), também têm existido desenvolvimentos cruciais para a compreensão e desenvolvimento de outras primitivas criptográficas, como a transferência oblívia quântica (*quantum oblivious transfer*, QOT). Pode-se mostrar a semelhança entre a estrutura de aplicação das primitivas QKD e QOT. Assim como os protocolos QKD permitem comunicação com segurança quântica, os protocolos QOT permitem computação com segurança quântica. No entanto, as condições sob as quais o QOT é totalmente seguro quântico têm sido sujeitas a um intenso estudo. Nesta tese, começamos por fazer um levantamento do trabalho desenvolvido em torno do conceito de OT dentro da criptografia quântica teórica. Aqui concentramo-nos em alguns protocolos propostos e nos seus requisitos de segurança. Revisitamos os resultados de impossibilidade desta primitiva e discutimos vários modelos quânticos de segurança sob os quais é possível provar a segurança do QOT.

A aplicação mais famosa do OT está no domínio da computação multipartidária segura (*secure multiparty computation*, SMC). Esta tecnologia tem o potencial de ser disruptiva nas áreas de análise e computação de dados. Esta permite que vários participantes calculem um certa função, preservando a privacidade dos seus dados. No entanto, a maior parte da segurança e eficiência do protocolos SMD dependem da segurança e eficiência do OT. Por esta razão, fazemos uma comparação detalhada entre a complexidade da QOT baseada em chaves oblívias e dois dos protocolos OT clássicos mais rápidos.

Seguindo a comparação teórica entre OT quântico e clássico, integramos e comparamos ambas as abordagens dentro de um sistema SMC baseado na análise de sequências genéticas. Em resumo, propomos um sistema SMC auxiliado por protocolos criptográficos quânticos com o objectivo de computar uma árvore filogenética a partir de um conjunto de sequências genéticas privadas. Este sistema melhora significativamente a privacidade e a segurança da computação graças a três protocolos criptográficos quânticos que fornecem segurança aprimorada contra ataques de computadores quânticos. Este sistema adapta

vários métodos baseados em distância (Unweighted Pair Group Method with Arithmetic mean, Neighbour-Joining, Fitch-Margoliash) num ambiente privado onde as sequências de cada participante não são divulgadas aos demais membros presentes no protocolo. Avaliamos teoricamente as garantias de desempenho e privacidade do sistema através de uma análise de complexidade e prova de segurança, e fornecemos uma extensa explicação dos detalhes de implementação e protocolos criptográficos. Implementamos este sistema com base na implementação Libscapi do protocolo de Yao, na biblioteca PHYLIP e em chaves simuladas de dois sistemas quânticos: distribuição de chaves oblívias quânticas e distribuição de chaves quânticas. Comparamos esta implementação com uma solução somente clássica e concluímos que ambas as abordagens apresentam tempos de execução semelhantes. A única diferença entre os dois sistemas é a sobrecarga de tempo tomada pelo sistema de gestão de chaves oblívias da abordagem quântica.

Finalmente, apresentamos o primeiro protocolo quântico de avaliação linear oblívia (*oblivious linear evaluation*, OLE). O OLE é uma generalização do OT, em que dois participantes calculam de forma oblívia uma função linear, $f(x) = ax + b$. Ou seja, cada participante fornece os seus dados de forma privada, a fim de calcular o resultado $f(x)$ que se torna conhecido por apenas um deles. Do ponto de vista estrutural e de segurança, o OLE é fundamental para protocolos SMC baseados em circuitos aritméticos. No caso clássico, sabe-se que o OLE pode ser gerado com base no OT, e as contrapartes quânticas desses protocolos podem, em princípio, ser construídas como extensões directas baseadas em QOT. Aqui, apresentamos o primeiro, protocolo quântico OLE que, além disso, não depende de QOT. Começamos apresentando um protocolo semi-honesto e depois estendemo-lo para o cenário desonesto através de uma estratégia *commit-and-open*. O nosso protocolo usa estados quânticos para calcular a função linear, $f(x)$, em campos de Galois de dimensão prima, $GF(d) \cong \mathbb{Z}_d$, ou dimensão de potência prima, $GF(d^M)$. Estas construções utilizam a existência de um conjunto completo de *mutually unbiased bases* em espaços de Hilbert de dimensão de potência prima e o seu comportamento linear sobre os operadores de Heisenberg-Weyl. Também generalizamos o nosso protocolo para obter uma versão vectorial do OLE, onde são geradas várias instâncias de OLE, tornando o protocolo mais eficiente. Provamos que os protocolos têm segurança estática no âmbito da composição universal quântica.

Palavras-chave: criptografia quântica, transferência oblívia quântica, avaliação linear oblívia quântica, computação multipartidária segura.

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I dedicate this thesis to my loving wife Teresinha and my two children Henrique and Helena who came to life during this journey to help me finish it.

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List of Abbreviations

AES – Advanced encryption standard.

ALSZ13 – OT extension protocol developed by Gilad Asharov, Yehuda Lindell, Thomas Schneider, and Michael Zohner [3].

API – Application programming interface.

BB84 – Quantum key distribution protocol developed by Charles Bennet and Gilles Brassard in 1984 [5].

BBCS – Quantum oblivious transfer protocol developed by Bennet, Brassard, Crépeau and Skubiszewska [6].

BCJL – Quantum bit commitment protocol developed by Brassard, Crépeau, Jozsa and Langlois [7].

BGW – Secure multiparty computation protocol developed by Ben-Or, Goldwasser and Wigderson [8].

BM – Oblivious transfer protocol developed by Bellare and Micali [2].

BMR – Secure multiparty computation protocol developed by Beaver, Micali and Rogaway [9].

BQS – Bounded-quantum-storage model.

BQS-UC – Bounded-quantum-storage universal composability model.

CBMC-GC – C Bounded model checker - Garbled Circuit.

CCD – Secure multiparty computation protocol developed by Chaum, Crépeau and Damgárd [10].

COM – Commitment.

CP – Completely positive map.

CPTP – Completely positive trace preserving map.

CPU – Central processing unit.

CRS – Common reference string.

CSRNG – Cryptographically secure pseudorandom number generator.

DI – Device independent.

EGL – Even, Goldreich, Lempel.

F84 – Evolutionary distance developed by Felsenstein.

FSA – Faked-state attacks.

GDPR – General data protection regulation.

GISAID – Global initiative on sharing avian influenza data.

GMW – Secure multiparty computation protocol developed by Goldreich, Micali and Wigderson [11].

GWAS – Genome-wide association studies.

HyCC – Compilation of hybrid protocols developed in [12].

JC – Jukes-Cantor.

K2P – Kimura 2-parameter.

KOS15 – OT extension protocol developed by Keller, Orsini and Scholl [4].

LAN – Local area network.

LD – LogDet.

LWE – Learning with errors.

M-LWE – Module learning with errors.

MDI – Measurement device independent.

MMH – Multi-linear modular hashing.

MUB – Mutually unbiased bases.

NP – Oblivious transfer protocol developed by Naor and Pinkas [13].

NQS – Noisy quantum storage model.

NTRU – Number theory research unit.

OKM – Oblivious key management system.

OLE – Oblivious linear evaluation.

OT – Oblivious transfer.

PDQ – Private database queries.

PET – Privacy-enhancing technologies.

PHYLIP – Phylogeny inference package.

POVM – Positive operator-valued measure.

PRG – Pseudorandom generator.

QBC – Quantum bit commitment.

QKD – Quantum key distribution.

QOKD – Quantum oblivious key distribution.

QOLE – Quantum oblivious linear evaluation.

QOT – Quantum oblivious transfer.

QRNG – Quantum random number generator.

quantum-UC – Quantum universal composability model.

RNG – Random number generator.

RSA – Public-key cryptosystem developed by Rivest, Shamir and Adleman.

RWOLE – Random weak oblivious linear evaluation.

SARS-CoV-2 – Severe acute respiratory syndrome coronavirus 2.

SHA – Secure hash algorithm.

SMC – Secure multiparty computation.

THA – Trojan-horse attack.

UC – Universal composability model.

UPGMA – Unweighted pair group method with arithmetic mean.

VM – Virtual machine.

VOLE – Vector oblivious linear evaluation.

WOLE – Weak oblivious linear evaluation.

WSE – Weak string erasure.

Notation

General

$\mathbb{Z}_q$	Set of integers $a \mod q$.
$\gcd(a, b)$	Greatest common divisor between integers a and b.
$\mathbb{Z}_q^*$	Set of integers $a \in \mathbb{Z}_q$ that are coprime with q, i.e. $\gcd(a, q) = 1$.
$\lvert I \rvert$	Size of a set I.
$\bar{T}$	Complement of set T.
$s \leftarrow_\$ I$	s is drawn uniformly at random from the set I.
$\boldsymbol{v}_{\lvert J}$	Subvector of $\boldsymbol{v}$ restricted to the indices $i \in J$.
$[m]$	The ordered set $\{1, 2, \ldots, m\}$ for $m \in \mathbb{Z}_q$.
$[m, n]$	The ordered set $\{m, m+1, \ldots, n-1, n\}$ for $m, n \in \mathbb{Z}_q$ such that $m < n$.
$d_H(\boldsymbol{x}, \boldsymbol{y})$	Hamming distance given by $\lvert\{i : x_i \neq y_i\}\rvert$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_q^n$.
$r_H(\boldsymbol{x}, \boldsymbol{y})$	The relative Hamming distance given by $d_H(\boldsymbol{x}, \boldsymbol{y})/n$.
$\mu(n)$	Negligible function.
log	Natural logarithm.
$\log_d$	Logarithm with base d.
$\mathbb{E}_x[f(x)]$	Expectation of $f(x)$ over random choices of x.
dim	Dimension.

Classical information

μ_X	Average value of elements in set X.
$I(A)$	Amount of information some event A.
$H(X)$	Shannon entropy of random variable X.
$h_d(x)$	d-ary entropy function.
$H_{\min}(X)$	Min-entropy of classical random variable X.
$\mathfrak{F}$	Two-universal hash family.

Quantum information

$\mathcal{H}_A\,(\mathcal{H}_A^*)$	Hilbert space (its dual) of quantum system A.
$\langle\phi\|\psi\rangle$	Scalar product of the vectors $\|\phi\rangle$ and $\|\psi\rangle$.
$\|\phi\rangle\langle\phi\|$	Projector onto the vector $\|\phi\rangle$.
ρ, σ	Density operators.
$\|B_{a,b}\rangle$	Generalised Bell states.
$\tau_{\mathcal{X}}$	Uniform distribution over $\mathcal{X}$.
$\mathcal{S}_1(\mathcal{H})$	Set of normalized vectors on $\mathcal{H}$.
$\mathrm{Herm}(\mathcal{H})$	Sets of hermitian operators on $\mathcal{H}$.
$\mathrm{Pos}(\mathcal{H})$	Sets of positive-semi definite on $\mathcal{H}$.
$\mathcal{P}(\mathcal{H})$	The sets density operators on $\mathcal{H}$.
$\delta(\rho,\sigma)$	Trace distance between ρ and σ.
$\mathbb{1}$	Identity operator.
$\mathrm{tr}(\rho)$	Trace of the hermitian operator ρ.
$\mathrm{rank}(\rho)$	Rank of the hermitian operator ρ.
$\mathcal{E}$, $\mathcal{T}$, $\mathcal{C}$, $\mathcal{O}^{()}$, $\mathcal{M}$	CPTP maps.

$H_{\min}(X\|B)_\rho$	Conditional min-entropy of a cq-state ρ_{XB}.
$H_{\min}(A\|B)_{\rho\|\sigma}$	Min-entropy of ρ_{AB} relative to σ_B.

Secure multiparty computation

P_i	Party i.
P_{TTP}	Third trusted party.
Enc_k	Symmetric encryption method with key k.
$\perp$	Empty string/element.
$\mathcal{F}$	Ideal functionality.
$\mathcal{F}_{\mathbf{OT}}$	OT functionality.
$\mathcal{F}_{\mathbf{OLE}}$	OLE functionality.
$\mathcal{F}_{\mathbf{VOLE}}$	VOLE functionality.
$\mathcal{F}_{\mathbf{COM}}$	COM functionality.
$\mathcal{F}_{\mathbf{ext}}$	External functionality.
$\mathcal{F}_{\mathbf{FakeCOM}}$	Fake commitment functionality.
Π	Protocol.
$\mathcal{Z}$	Environment.
Adv	Adversary.
$\mathcal{S}$	Simulator.
$\mathcal{S}_A$	Simulator against dishonest Alice.
$\mathcal{S}_B$	Simulator against dishonest Bob.
$\text{EXEC}_{\pi^C,Adv,\mathcal{Z}}$	The output of the environment $\mathcal{Z}$ at the end of the real execution, and by C the corrupted party.
$\text{EXEC}_{\rho^C,\mathcal{S},\mathcal{Z}}$	The output of the environment $\mathcal{Z}$ at the end of the ideal execution, and by C the corrupted party.
$\mathtt{DM}_d$	Distance matrix functionality.

$\mathtt{D}_d$	Two-party distance functionality.
$\mathtt{A}$	Phylogenetic tree functionality.

Protocols

m_i	Alice's messages.
b	Bob's bit choice.
l	Length of OT protocol messages.
κ	Symmetric security parameter.
$\boldsymbol{\theta}^{\mathsf{A}}$, $\boldsymbol{\theta}^{\mathsf{B}}$	Alice and Bob's bases vectors.
ok	Oblivious key.
ok^{A}	Alice's oblivious key.
ok^{A}	Bob's oblivious key.
e^{B}	Bob's signal string.
pk_i	Public key.
T	Test set.
Δt	Waiting time.
ν	Fraction of the transmitted qubits kept by the adversary.
$c_{\mathcal{N}}$	Classical capacity of quantum channel $\mathcal{N}$.
$\mathcal{A}_{\mathbf{qok}}(J)$	Alice's quantum hacking procedure.
$\mathcal{B}_{\mathbf{qok}}(J)$	Bob's quantum hacking procedure.
$\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$	BBCS protocol in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model.
$\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$	BBCS protocol in the BQS model.
$\Pi^{\mathbf{BBCS}}_{\mathbf{nqs}}$	BBCS protocol in the NQS model.
$\Pi^{\mathbf{BBCS}}_{\mathbf{O}}$	Optimised BBCS protocol.
$\Pi^{\mathbf{QOKD}}_{\mathbf{bqs}}$	Quantum oblivious key distribution protocol in the BQS model.

$\Pi^{\mathsf{A}}_{\mathbf{FSA}}$	Alice's faked-state attack on qubits indexed by J.
$\Pi^{\mathsf{A}}_{\mathbf{THA}}$	Alice's trojan-horse attack on qubits indexed by J.
$\Pi^{\mathsf{B}}_{\mathbf{THA}}$	Bob's trojan-horse attack on qubits indexed by J.
$B_{\text{op}}^{\text{ALSZ13}}$	Number of binary operations executed by ALSZ13 protocol.
$B_{\text{op}}^{\text{BBCS}}$	Number of binary operations executed by $\Pi^{\mathbf{BBCS}}_{\mathbf{O}}$ protocol.
$B_{\text{op}}^{\text{KOS15}}$	Number of binary operations executed by KOS15 protocol.
A_d^a	Private phylogenetic tree algorithm for distance d and algorithm a.
M	Number of genomic sequences.
s	Length of genomic sequences.
L_{ok}^j	Size of total oblivious key.
L_{bok}^j	Total size of oblivious key for base OT.
L_{QRNG}^j	Random bits generated by QRNG.
L_{qkd}^j	Total size of QKD keys.
N_{Yao}^j	Number of Yao protocol executions.
N_{OT}^j	Number of OT executions.
N_{bOT}^j	Number of base OT executions.
N_{int}^j	Number of internal computations.
V_a^b	Heisenberg-Weyl operators.
$\|e_r^x\rangle$	$r-$th eigenket of the V_1^x operator.
$\mathbf{F}_0$	Random variable for Alice's functions in the RWOLE protocol.
$\mathbf{F}$	Random variable for Alice's functions in the WOLE protocol.
A'	Dishonest Alice quantum system.
B'	Dishonest Bob quantum system.
$\Pi^n_{\mathbf{RWOLE}}$	RWOLE protocol.
$\Pi^n_{\mathbf{WOLE}}$	WOLE protocol.
$\Pi_{\mathbf{EXT}}$	Extraction protocol.

$\Pi_{\mathbf{VEXT}}$	Vector extraction protocol.
$\Pi_{\mathbf{QOLE}}$	Quantum OLE protocol.
$\Pi_{\mathbf{QVOLE}}$	Quantum vector OLE protocol.

Chapter 1

Introduction

The field of data mining and data analysis has seen significant advancements with the increasing power of computers [14]. However, the need for large-scale data collection can lead to the compromise of sensitive and private information, particularly in fields such as genomics [15–18]. As a result, the aggregation of data from different sources is often restricted by laws and regulations such as the General Data Protection Regulation (GDPR) [19]. While these regulations aim to protect individuals' privacy, they also limit the ability of honest parties to access the data needed to address important societal issues.

Secure multiparty computation

To address the privacy concerns outlined above, various privacy-enhancing technologies have been proposed, such as secure multiparty computation (SMC) [20–22]. SMC enables a group of n parties, denoted as P_i, to jointly compute a function $f(x_1, ..., x_n) = (y_1, ..., y_n)$ without revealing their inputs to the other parties. The security requirements of SMC are equivalent to an ideal scenario, where each party P_i sends their inputs to a separate, trusted third party, who then computes $f()$ and returns the corresponding output to each party.

The research area of SMC has been greatly advanced with the development of various protocols and frameworks [23–25]. These protocols can be broadly classified into two categories: those based on boolean logic and those based on arithmetic logic. The efficiency and security of SMC protocols rely heavily on the efficiency and security of important cryptographic primitives, such as oblivious transfer (OT) for boolean-based protocols and oblivious linear evaluation (OLE) for arithmetic-based protocols [22, 26]. However, a fundamental limitation of these primitives is their reliance on public-key cryptography, as proven by Impagliazzo and Rudich [27]. This dependence on public-key cryptography is a significant drawback in terms of both performance and security, as it requires more

computational resources and is vulnerable to attacks from quantum computers, as demonstrated by Shor's algorithm [28]. To address these issues and ensure the safe deployment of SMC methods in the face of quantum computing, it is crucial to develop SMC methods that are secure against quantum attacks while maintaining state-of-the-art performance levels.

A quantum era

The second quantum revolution is upon us and quantum technology has advanced to a point where we can integrate its unique features into complex engineering systems. Quantum cryptography, in particular, has been a major area of focus, with research aiming to develop protocols that offer advantages over their classical counterparts. As outlined in [29, 30], these advantages can take two forms:

1. Improving security requirements and achieving information-theoretically secure protocols or those that require fewer computational assumptions;

2. Developing new cryptographic primitives that were previously unattainable using classical techniques.

While quantum key distribution (QKD) is the most well-known application of quantum cryptography, other cryptographic tasks such as bit commitment [31], coin flipping [32], delegated quantum computation [33], position verification [34], and password-based identification [35, 36] among others, also play important roles in this field.

Also, the intrinsic randomness provided by quantum phenomena can be leveraged to develop quantum communication protocols for oblivious transfer (OT) [6]. Importantly, there is a significant difference between classical and quantum OT from a security standpoint, as the latter can be achieved with only the assumption of the existence of quantum-hard one-way functions [37, 38]. This means quantum OT requires less security assumptions than classical OT, as the latter cannot be based on one-way functions alone [39, 40]. Furthermore, these quantum protocols often possess the desirable property of everlasting security, which guarantees information-theoretic security after the execution of the protocol [41]. This greatly improves the security of SMC protocols, allowing them to rely on symmetric cryptography and one-way functions, and to possess the important feature of everlasting security. With regards to the oblivious linear evaluation (OLE) primitive, it is known that it can be reduced to OT [26] through classical methods that do not require additional assumptions. Therefore, it seems natural to use quantum OT to generate quantum-secure OLE instances.

Contributions

Despite the many advances, the adoption of quantum cryptography in secure multiparty computation (SMC) systems has been limited due to the efficiency challenges posed by quantum technology and the need for high throughput of both OT and OLE primitives in boolean- and arithmetic-based SMC.

The aim of this thesis is to further the adoption of quantum cryptography in SMC systems through three key contributions. The first contribution involves a study comparing the efficiency of classical and quantum protocols. The second contribution is the implementation of a specialized SMC system for genomics analysis utilizing quantum OT. And the final contribution is the development of the first quantum OLE protocol that does not rely on OT. Additionally, we have also created a comprehensive review dedicated solely to quantum OT protocols, which is often overlooked in broader surveys on the topic of "quantum cryptography". We describe the contributions in a bit more detail.

Efficiency of classical and quantum OT protocols. As far as we are aware, there is no comparative study on the efficiency of quantum and classical approaches. This is largely due to two factors. From a theoretical perspective, the use of different types of information (quantum and classical) makes it challenging to establish a fair comparison based on the complexity of the protocols. From a practical perspective, there is also a significant gap in the technological maturity between quantum and classical techniques. Quantum technology is still in its early stages, whereas classical processors and communication have undergone decades of development.

We compare the complexity and efficiency of classical and quantum protocols, despite their constraints. Both types of protocols can be broken down into two phases: precomputation and transfer. The precomputation phase is independent of the parties' inputs and is used to generate the resources needed in the transfer phase, which takes into account the parties' inputs. This phase may not be as efficient as the transfer phase, so for comparison purposes, we focus on the transfer phase. Notably, the transfer phase of quantum OT only involves classical communication, making it possible and fair to compare it to the transfer phase of classical protocols.

We compare the complexity of the transfer phase of two classical OT extension protocols [3, 4] and an optimized quantum OT protocol in detail. Our conclusion is that the transfer phase of quantum OT is on par with its classical counterparts and has the potential to be more efficient.

Quantum assisted secure multiparty computation. In light of individuals' privacy concerns and legal regulations, it is crucial to handle and study genomic data using

highly secure privacy-preserving techniques. We propose a practical secure multiparty computation (SMC) system that utilizes quantum cryptographic protocols to compute a phylogenetic tree from a set of private genome sequences. This system applies several distance-based methods, such as Unweighted Pair Group Method with Arithmetic mean, Neighbour-Joining, and Fitch-Margoliash, in a private setting where the sequences owned by each party are not revealed to other members during the protocol. Instead of using a generic SMC implementation for phylogenetic trees, we develop a specialized private protocol that improves efficiency for this specific use case.

We conduct a theoretical evaluation of the performance and privacy guarantees of our proposed system, providing a complexity analysis and security proof. We also provide a detailed explanation of the implementation details and cryptographic protocols used. We demonstrate the effectiveness and practicality of our quantum-assisted secure phylogenetic tree computation by implementing it using the Libscapi implementation of the Yao protocol, the PHYLIP library and simulated keys of two quantum systems: quantum oblivious key distribution and quantum key distribution. [1] We compare our implementation with a classical-only solution and find that both approaches have similar execution times. The only difference between the quantum and classical systems is the time overhead taken by the quantum-assisted approach for oblivious key management.

Quantum oblivious linear evaluation protocol. Our second contribution is a quantum protocol for OLE that provides quantum-UC security in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model, which assumes the availability of a commitment functionality, $\mathcal{F}_{\mathbf{COM}}$. To ensure security, we leverage the properties of Mutually Unbiased Bases in high-dimensional Hilbert spaces with prime and prime-power dimensions. This approach is motivated by recent theoretical and experimental advancements in quantum cryptography [42–46] that have opened the door for new solutions in the field.

As far as we are aware, our protocol is the first to propose a quantum-UC secure quantum OLE. Furthermore, it does not rely on any quantum OT implementation, which is a common approach. We design the protocol to handle static corruption adversarial model for both semi-honest and malicious adversaries. Additionally, we introduce a weaker version of OLE, which has potential independent value. We also modify the proposed protocol to generate quantum-UC secure vector OLE (VOLE) and provide bounds on the size of VOLE based on the security parameters.

[1] The code can be accessed at the following repo: github.com/manel1874/private-phylogenetic-analysis.

Outline

The results are presented as follows. We start presenting the technical elements required throughout the thesis in Chapter 2. Chapter 3 is devoted to quantum oblivious transfer protocols. Then, in Chapter 4 we compare classical and quantum approaches for OT. In Chapter 5, we presented our implementation of quantum-assisted SMC system applied to phylogeny analysis. In Chapter 6, we present our quantum OLE protocol along with its security proof. Finally, in Chapter 7 we present an overall conclusion of the thesis and propose some future work.

Published research

This thesis draws on research published in various journals and presents the results of my PhD work, which involved the following projects.

- [47] Manuel B. Santos, Paulo Mateus, and Chrysoula Vlachou. "Quantum Universally Composable Oblivious Linear Evaluation". 2022. DOI: 10.48550/ARXIV.2204.14171. Poster at QCrypt2022.
- [48] Manuel B. Santos, Paulo Mateus, and Armando N. Pinto. "Quantum Oblivious Transfer: A Short Review". In: Entropy 24.7 (2022), p. 945.
- [49] Manuel B. Santos et al. "Private Computation of Phylogenetic Trees Based on Quantum Technologies". In: IEEE Access 10 (2022), pp. 38065–38088.
- [50] Manuel B. Santos, Armando N. Pinto, and Paulo Mateus. "Quantum and classical oblivious transfer: A comparative analysis". In: IET Quantum Communication 2.2 (2021), pp. 42–53.
- [51] Manuel B. Santos et al. "Quantum Secure Multiparty Computation of Phylogenetic Trees of SARS-CoV-2 Genome". In: 2021 Telecoms Conference (ConfTELE). IEEE, Feb. 2021.
- [52] Armando N. Pinto et al. "Quantum Enabled Private Recognition of Composite Signals in Genome and Proteins". In: 2020 22nd International Conference on Transparent Optical Networks (ICTON). IEEE, July 2020.

Chapter 3 is based on [48] and [49]. Chapter 4 is based on the work developed on both [50] and [49]. Chapter 5 is the combination of [49, 51, 52]. Finally, Chapter 6 presents all the results from [47].

Chapter 2

Technical overview

In this chapter, we present the fundamental mathematical and information-theoretic concepts that are utilized throughout the thesis.

2.1 Mathematical preliminaries

In this section, we define and explain the mathematical notation and concepts used throughout the thesis.

First, we use $\gcd(a, b)$ to denote the greatest common divisor between integers a and b, with $a, b \in \mathbb{Z}$. We use $\mathbb{Z}_q$ to denote the set of integers $a \mod q$, and $\mathbb{Z}_q^*$ to denote the set of integers $a \in \mathbb{Z}_q$ that are coprime with q, i.e., $\gcd(a, q) = 1$. When q is prime, $\mathbb{Z}_q^*$ forms a multiplicative group of order $q - 1$ and $\mathbb{Z}_q$ forms a finite field of order q. A generator g of a multiplicative group $\mathbb{G}$ is an element in $\mathbb{G}$ such that for all $a \in \mathbb{G}$, there exists an integer r such that $g^r = a$. The discrete logarithm base g of an element $a \in \mathbb{G}$, denoted by $\log_g a$, is the power r of g such that $g^r = a$.

We use $|I|$ to denote the size of a set I and use the notation $s \leftarrow_\$ I$ to describe a situation where an element s is drawn uniformly at random from the set I. Vectors $\boldsymbol{v} = (v_1, \ldots, v_n)$ are denoted in bold. Given a set J, $\boldsymbol{v}_{|J}$ denotes the subvector of $\boldsymbol{v}$ restricted to the indices $i \in J$. For $m \in \mathbb{Z}_q$, $[m]$ is the ordered set $1, 2, \ldots, m$, and for $m, n \in \mathbb{Z}_q$ such that $m < n$, $[m, n] = m, m + 1, \ldots, n - 1, n$. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{Z}_q^n$, $r_H(\boldsymbol{x}, \boldsymbol{y}) = d_H(\boldsymbol{x}, \boldsymbol{y})/n$ is the relative Hamming distance, where the Hamming distance is given by $d_H(\boldsymbol{x}, \boldsymbol{y}) = |i : x_i \neq y_i|$.

Finally, we use the big-$\mathcal{O}$ notation to denote the fastest-growing term of the number of operations with respect to some security parameter n. A negligible function $\mu(n)$ is a function such that $\mu(n) < 1/p(n)$ for some polynomial $p(n)$ and sufficiently large n.

2.2 Secure multiparty computation

Secure multiparty computation (SMC) allows multiple parties, denoted as P_i with $i \in 1, \ldots, n$, to jointly compute a function, $f(x_1, \ldots, x_n) = (y_1, \ldots, y_n)$, without revealing their individual inputs, x_i, to each other. The only information received by each party P_i is their corresponding output y_i of the function $f()$, which may reveal some information about the parties' inputs depending on the function being computed. This functionality is designed to be equivalent to a scenario where each party P_i sends their input x_i to an independent and trusted third party P_{TTP}, who computes $f(x_1, \ldots, x_n)$ and sends the output y_i to each party.

It is important to note that SMC may not completely hide the inputs of the parties, even with a perfectly secure protocol. This is due to the security guarantees of the ideal scenario, where it is possible for a perfectly legitimate SMC protocol (such as using a trusted third party) to leak all the inputs of the parties. This can happen when one of the parties can use their inputs and outputs to invert the function $f()$. For example, if two parties want to compute the average of their weight, it is straightforward for both parties to use their weight and the average value to compute the other party's weight, as the function is bijective with the adversaries' inputs fixed. In this scenario, SMC does not improve the privacy of the computation.

The following are some informal descriptions of properties of SMC:

1. Correctness: If all the parties abide by the protocol, the protocol will evaluate the correct output according to $f()$ and the parties' inputs $x_1, \ldots, x_n$.

2. Passive security: If the adversaries do not deviate from the protocol, they do not learn the inputs of the honest parties. In this thesis, we refer to adversaries who do not deviate from the protocol as semi-honest parties, also known as honest-but-curious adversaries in the literature.

3. Active security: If the adversaries deviate arbitrarily from the protocol (dishonest parties), they do not learn the inputs of the honest parties. In active security, there are two types of protocols that react differently to adversarial behavior. They can be robust against the adversaries, meaning the honest parties will still receive the correct answer, or the honest parties can abort the protocol when there is malicious activity.

Regarding the corruption strategy of the adversaries, they can be of two types: static or adaptive. Static security guarantees that the protocol is secure against an adversary who only corrupts parties before the execution of the protocol. Adaptive security is a more challenging property to attain, as it assumes that the adversary can choose which party

to corrupt throughout the protocol. It is also worth noting that there is a fundamental difference between the adversarial structure of encryption methods and SMC methods. In encryption methods, the adversary is considered an external party (usually referred to as Eve) that interferes with the communication between the protocol parties. In the case of SMC methods, the adversaries are a subset of the protocol parties.

Next, we present two common approaches used for SMC protocols: the garbled circuit approach and the secret sharing approach. The garbled circuit approach is based on boolean circuits and follows from the techniques developed by Yao [22]. The secret sharing approach is commonly based on arithmetic circuits (although it can also be used with boolean circuits) and follows from the properties of secret sharing [8, 10]. It should be noted that throughout this thesis, we will focus on two-party protocols. For this reason, we name these parties Alice and Bob, and follow the convention that Alice plays the role of the protocol's sender and Bob plays the role of the receiver.

2.2.1 Garbled circuit approach

The garbled circuit approach, which is based on Yao's seminal work [22], proposes a technique to "encrypt" boolean circuits in such a way that preserves the security requirements of both parties. This "encrypted" version is called a garbled circuit and is presented in this section along with the Yao protocol description. This approach is typically best suited for scenarios with higher latency, as it typically requires a fixed number of communication rounds, regardless of the complexity of the function being evaluated. However, for large circuits, high bandwidth is required [53].

Before delving into the details of the Yao protocol, it is important to introduce a crucial primitive: oblivious transfer (OT).

Oblivious transfer

The study of oblivious transfer (OT) has been active since its first proposal by Rabin in 1981 [54]. The importance of OT comes from its wide range of applications. In particular, it can be proven that OT is equivalent to the secure two-party computation of general functions [55, 56], meaning that a secure two-party computation can be implemented using OT as its building block. Additionally, this primitive can also be used for secure multiparty computation (SMC) [26], private information retrieval [57], private set intersection [58], and privacy-preserving location-based services [59].

The OT functionality can be presented in many flavours. In this thesis, when we refer to OT, we mean the 1-out-of-2 OT that is specified in Figure 2.1. Consequently, we have that OT must satisfy the following security requirements:

- Concealing: Alices knows nothing about Bob's bit choice b.

- Obliviousness: Bob knows nothing about the message $m_{b\oplus 1}$.

OT can be generalized to the case of k-out-of-N OT, where Alice owns N messages, and Bob can choose k of them. For $k = 1$, this is commonly called private database query (PDQ). Also, we call random OT when both parties' inputs are random.

Yao protocol

A solution for SMC was first proposed by Yao [22], where he developed the concept of garbled circuits, which is one of the key elements for secure computation. The Yao's garbled circuit protocol is originally designed for only two parties, but its generalization to multiple parties was later achieved by GMW [23] and BMR [9]. Additionally, various implementation optimizations have been developed to improve the performance of the Yao protocol, such as point-and-permute [9], row reduction [60, 61], FreeXOR [62] and half gates [63].

As mentioned before, the main idea of the Yao protocol is to represent the desired function $f()$ as a boolean circuit C, i.e. a sequence of logical gates interconnected with wires. After the generation of the circuit C, each party will have two distinct roles. Generally speaking, Alice (also known as the garbler) randomly generates keys for each input bit, encrypts each circuit's gate, and sends both elements to Bob (also known as the evaluator). This procedure masks Alice's inputs from Bob. Then, through the oblivious transfer (OT) functionality, Bob receives the keys corresponding to his input bits. This allows to mask Bob's inputs from Alice. Finally, since the evaluator has all the input keys, he can decrypt every gate, and evaluate the circuit. To better understand how the protocol works, let us consider a four-input boolean circuit description of the Millionaires'

$\mathcal{F}_{\mathbf{OT}}$ functionality

- **Input phase:** Alice sends $(m_0, m_1) \in \{0,1\}^l$ (two messages) to $\mathcal{F}_{\mathbf{OT}}$ and Bob sends $b \in \{0,1\}$ (bit choice) to $\mathcal{F}_{\mathbf{OT}}$.
- **Output phase:** Alice receives nothing $\perp$ from the functionality and Bob receives m_b.

Figure 2.1: OT functionality.

problem. This problem can be described by the following expression:

$$f(a,b) = \begin{cases} 1 & \text{if } a > b, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for $a, b \in \{0,1\}^2$. In summary, it allows two parties to discover who has the largest value without revealing them.

The protocol goes as follows:

1. *Circuit generation:* The garbler Alice generates a boolean circuit of function (2.1):

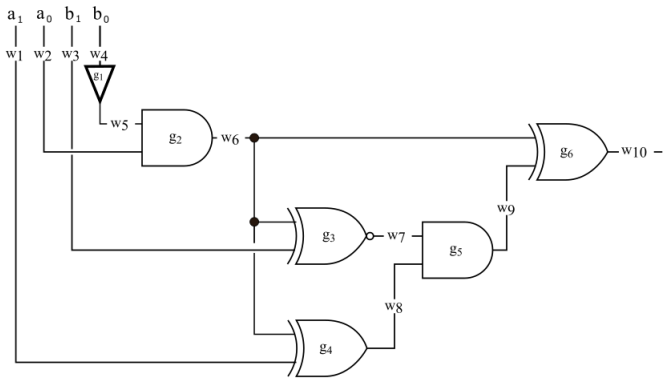


Figure 2.2: Boolean circuit of the Millionaires' Problem. Optimised circuit according to the construction in [1].

   In this case, the circuit contains one NOT gate (g_1), two AND gates (g_2, and g_5), two XOR gate (g_4 and g_6), one XNOR gate (g_3) and four input wires (w_1 and w_2 belonging to Alice and w_3 and w_4 to Bob).

2. *Wire encryption:* Alice uses a random number generator to generate two keys k_i^0 and k_i^1 for each wire w_i, $i \in \{1, ..., 10\}$. These keys correspond to the possible values (0 or 1) on the wire. Note that this is done to prevent Bob from knowing the true value of the wires during the evaluation process.

3. *Gate encryption:* For every gate g_l in the circuit with corresponding input wires w_i and w_j and output wire w_s, Alice creates the following table:

$\mathsf{Enc}_{k_i^0}\left(\mathsf{Enc}_{k_j^0}\left(k_s^{g_l(0,0)}\right)\right)$
$\mathsf{Enc}_{k_i^0}\left(\mathsf{Enc}_{k_j^1}\left(k_s^{g_l(0,1)}\right)\right)$
$\mathsf{Enc}_{k_i^1}\left(\mathsf{Enc}_{k_j^0}\left(k_s^{g_l(1,0)}\right)\right)$
$\mathsf{Enc}_{k_i^1}\left(\mathsf{Enc}_{k_j^1}\left(k_s^{g_l(1,1)}\right)\right)$

where $g_l(a, b)$ is the output of gate g_l for inputs $a, b \in \{0, 1\}$. So, we could think of each row as a locked box that requires two keys to be opened. If the two correct keys are used, it outputs the key corresponding to the desired output value given by g_l. After encrypting each gate, Alice permutes the rows of the corresponding table, otherwise, it would be easy to know the real value of the input keys. Then, she sends to Bob the garbled tables along with Alice's input keys.

As an example, we can easily see that if we use input keys k_i^0 and k_j^1 (corresponding to real values 0 and 1), we would only be able to decipher the second row of the table, $\mathsf{Enc}_{k_i^0}(\mathsf{Enc}_{k_j^1}(k_s^{g_l(0,1)}))$, and get $k_s^{g_l(0,1)}$.

4. *Oblivious Transfer:* At this stage of the protocol, the evaluator Bob knows the garbled circuit and Alice's input keys but he does not know the keys corresponding to his real inputs. However, since Bob wants to keep his input value private he cannot directly ask for those keys. At this point, the OT functionality enables the evaluator to receive his input keys without compromising neither the evaluator's nor garbler's security. In fact, for every input wire, both parties perform an OT where Alice plays the role of the sender and Bob plays the role of the receiver.

   Let us assume Alice's input keys to be k_1^0 and k_2^1 (corresponding to the real value 01) and Bob's input bits to be 11. This means that Bob must use the respective input keys (k_3^1 and k_4^1) in order to correctly evaluate the circuit. So, they will execute two OT protocols where:

   - Alice inputs: (k_3^0, k_3^1) and (k_4^0, k_4^1);
   - Bob inputs: $b_1 = 1$ and $b_2 = 1$.

5. *Evaluation:* Once the evaluator has all the necessary elements, he can proceed with the circuit evaluation. In this step, he simply has to decipher the correct rows of the garbled tables sent by Alice with the corresponding keys. Since the rows of the tables are shuffled, the evaluator does not know which row is the correct one. This small issue can be solved by simple techniques (Point-and-Permute or encryption with a certain number of 0 padded) which, for the sake of brevity, we will not explore here. At the end of the evaluation, the evaluator receives the key that corresponds to the result. Finally, the evaluator sends the resulting key to the garbler and the garbler tells him the final bit.

   According to our Millionaires' problem, the evaluation yields the following results for $a = 01$ and $b = 11$: $g_1(k_4^1) = k_5^0$, $g_2(k_5^0, k_2^1) = k_6^0$, $g_3(k_6^0, k_3^1) = k_7^0$, $g_4(k_6^0, k_1^0) = k_8^1$, $g_5(k_7^0, k_8^1) = k_9^0$, $g_6(k_6^0, k_9^0) = k_{10}^0$. Actually, the desired result is 0.

The Yao protocol has its security based on two main building blocks: garbled circuits and oblivious transfer. Although garbled circuits can be generated with symmetric encryption (i.e. using double AES encryption), OT protocols cannot be classically achieved with symmetric cryptography alone [39]. Thus, it is crucial to find efficient protocols for a quantum-resistant OT.

2.2.2 Secret sharing approach

The secret sharing approach, first introduced by BGW [8] and CCD [10], does not involve encrypting the circuit. Instead, parties use a secret sharing scheme to evaluate the circuit. This approach involves simple operations such as addition and multiplication, but the number of communication rounds needed will depend on the size of the circuit being evaluated. An important primitive for secret sharing based protocols is oblivious linear evaluation (OLE).

Oblivious linear evaluation

Oblivious linear evaluation (OLE) can be thought of as a generalization of oblivious transfer (OT) [54]. It has been shown to be a building block for securely evaluating arithmetic circuits, such as in [64–67]. Specifically, OLE can be used to generate multiplication triples, which are essential for securely computing multiplication gates [67]. OLE also has applications in tasks such as two-party secure computation [68–72] and private set intersection [73].

$\mathcal{F}_{\mathbf{OLE}}$ functionality

- **Input phase:** Alice sends $(a, b) \in \mathbb{Z}_d^2$ (two field elements) to $\mathcal{F}_{\mathbf{OLE}}$ and Bob sends $x \in \mathbb{Z}_d$ to $\mathcal{F}_{\mathbf{OLE}}$.
- **Output phase:** Alice receives nothing $\perp$ from the functionality and Bob receives $f(x) := ax + b$.

Figure 2.3: OLE functionality.

The OLE functionality specification is presented in Figure 2.3. Similarly, we have that OLE must satisfy the following security requirements:

- Concealing: Alices knows nothing about Bob's field element x.
- Obliviousness: Bob knows nothing about the function $f()$ other than its evaluation at x, i.e. $f(x)$.

We can also generalize the OLE functionality to a vectorized version. The vector OLE (VOLE) functionality is presented in Figure 2.4. Note that Bob only inputs one field element x and Alice inputs two vectors.

$\mathcal{F}_{\mathbf{VOLE}}$ functionality

- **Input phase:** Alice sends $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}_d^{2n}$ (two vectors of field elements) to $\mathcal{F}_{\mathbf{VOLE}}$ and Bob sends only $x \in \mathbb{Z}_d$ to $\mathcal{F}_{\mathbf{VOLE}}$.
- **Output phase:** Alice receives nothing $\perp$ from the functionality and Bob receives $\boldsymbol{f}(x) := \boldsymbol{a}x + \boldsymbol{b}$.

Figure 2.4: VOLE functionality.

Basic operations

To highlight the importance of OLE in secret sharing based SMC protocols, we go through a passively secure protocol [74]. We consider the two party case (Alice and Bob) where the parties own additive shares of the secret. So, for some secret value x, where $x = x_A + x_B$, Alice owns x_A and Bob owns x_B. Depending on the circuit, the operations used in the protocol are as follows:

- **Input**. For Alice to secret share her input value x, she randomly chooses x_B and sends it to Bob. Alice defines x_A as $x_A = x - x_B$;
- **Addition**. There are two scenarios to consider:
  - **Scalar**. For Alice and Bob to add a scalar to a secret x ($z = a + x$), Alice computes $z_A = a + x_A$ and Bob sets $z_B = x_B$.
  - **Shares**. For Alice and Bob to add secrets x and y ($z = x+y$), they individually add their corresponding shares, i.e. $z_A = x_A + y_A$ and $z_B = x_B + y_B$.
- **Multiplication**. There are two scenarios to consider:
  - **Scalar**. For Alice and Bob to multiply a secret x by a scalar a ($z = a \cdot x$), Alice computes $z_A = a \cdot x_A$ and Bob computes $z_B = a \cdot x_B$.
  - **Shares**. Observe that, for Alice and Bob to multiply secrets x and y ($z = x \cdot y$), they require some sort of communication to compute cross terms:

$$\begin{aligned} x \cdot y &= (x_A + x_B) \cdot (y_A + y_B) && (2.2) \\ &= x_A \cdot y_A + x_A \cdot y_B + x_B \cdot y_A + x_B \cdot y_B && (2.3) \end{aligned}$$

At this point, Alice and Bob can execute two OLEs to secret share the cross terms $x_A \cdot y_B$ and $x_B \cdot y_A$. Indeed, if Alice inputs $(x_A, -s_A)$ and $(y_A, -s'_A)$ for random values s_A, s'_A and Bob inputs y_B and x_B, Bob will output $s_B = x_A \cdot y_B - s_A$ and $s'_B = y_A \cdot x_B - s'_A$. Thus, we have that $s_A + s_B = x_A \cdot y_B$ and $s'_A + s'_B = y_A \cdot x_B$. So, Alice share is $z_A = x_A \cdot y_A + s_A + s'_A$ and Bob share is $z_B = s_B + s'_B + x_B \cdot y_B$.

- **Output**. For Alice to receive the output value x of some output wire, Bob simply sends x_B to Alice. Alice outputs $x = x_A + x_B$.

2.3 Quantum information

Quantum information theory is a field that studies the implications of using quantum systems as the medium of information. The information carriers in quantum systems are governed by the laws of quantum mechanics, allowing for properties not present in classical methods to be exploited. In this section, we present the basic elements of quantum information that will be used in the quantum protocols presented and their security proofs.

In quantum information theory, a quantum system is described by a Hilbert space $\mathcal{H}_A$. In this thesis, we will consider only finite-dimensional Hilbert spaces, where $\dim \mathcal{H}_A = d < \infty$. The space $\mathcal{H}_A$ can be identified with the complex vector space $\mathbb{C}^d$, as well as its corresponding dual space $\mathcal{H}_A^*$. We use the Dirac bra-ket notation to describe the states of a quantum system. A pure state is described by a normalized vector $|\psi\rangle_A \in \mathcal{H}_A$ and its dual vector $\langle\psi|_A \in \mathcal{H}_A^*$. To simplify notation, we may omit specifying the Hilbert space to which a state belongs if it is clear from context. The standard basis of $\mathbb{C}^d$ can be identified with the computational basis of $\mathcal{H}_A$, denoted as $\{|i\rangle\}_{i=0}^{d-1}$. The joint system of multiple subsystems $\mathcal{H}_1, \ldots, \mathcal{H}_n$ can be described by their tensor product, denoted as $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. The vectors in this joint system are represented as $|\boldsymbol{x}\rangle = |x_1\rangle \otimes \ldots \otimes |x_n\rangle$, where $\boldsymbol{x} \in \mathbb{Z}_d^n$.

We can generate quantum pure states, denoted as $|\psi_i\rangle \in \mathcal{H}$, according to a probability distribution p_i. This situation is described by a density operator, denoted as $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, which is commonly referred to as a mixed state. Density operators are positive semi-definite hermitian operators with unitary trace, that is, $\rho \geq 0$ and $\operatorname{tr} \rho = 1$. The set of hermitian operators, positive semi-definite operators and density operators on a Hilbert space $\mathcal{H}$ are denoted as $\mathrm{Herm}(\mathcal{H})$, $\mathrm{Pos}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ respectively.

A mixed state is considered classical if it is of the form $\rho_{\mathcal{X}} = \sum_{x \in \mathcal{X}} P_X(x) |x\rangle\langle x|$, where $\mathcal{X}$ is a finite set and P_X is a probability distribution over $\mathcal{X}$. The uniform distribution over $\mathcal{X}$ is denoted as $\tau_{\mathcal{X}} = \frac{1}{|\mathcal{X}|} \sum_{x \in \mathcal{X}} |x\rangle\langle x|$, where $|\mathcal{X}|$ is the size of $\mathcal{X}$. The identity operator is denoted by $\mathbb{1}$. Additionally, for a bipartite quantum state ρ_{XB}, it is said to be a classical-

quantum state (cq-state for short) if it is of the form $\rho_{XB} = \sum_{x\in\mathcal{X}} P_X(x)\,|x\rangle\langle x| \otimes \rho_B^x$, where P_X is a probability distribution over the finite set $\mathcal{X}$.

2.3.1 Trace distance

Proving the security of quantum protocols requires a method for distinguishing quantum states. Fortunately, there is a useful metric, known as the trace distance, that measures the distinguishability of two quantum states, $\sigma, \rho \in \mathcal{P}(\mathcal{H})$, by any procedure, regardless of efficiency. The trace distance is defined as [75]

$$\delta(\rho, \sigma) := \frac{1}{2}||\rho - \sigma||_1,$$

where $||\cdot||_1$ is the $1-$Schatten norm in the space of bounded operators acting on a Hilbert space. Its name comes from the fact that we can write it using the trace operator as follows

$$||\rho - \sigma||_1 = \mathrm{Tr}\left\{\sqrt{(\rho-\sigma)^\dagger(\rho-\sigma)}\right\}.$$

In this work, we will utilize completely positive trace preserving (CPTP) maps. These maps are defined as preserving the normalization of input states and mapping positive operators to positive operators. As a result, they ensure that density operators are mapped to density operators, making them useful in describing all physically possible operations. They will be a key focus in Chapter 6, which deals with the quantum oblivious linear evaluation protocol. However, it is important to note that CPTP maps do not increase the distinguishability between quantum states, as proven in Lemma 1. In other words, the trace distance between two quantum states remains unchanged after being transformed by a CPTP map.

Lemma 1 (Lemma 7, [75])**.** *The trace distance has the following properties:*

1. *For any CPTP map $\mathcal{E}$ and any $\sigma, \rho \in \mathcal{P}(\mathcal{H})$ we have that*

$$\delta(\mathcal{E}(\sigma), \mathcal{E}(\rho)) \leq \delta(\sigma, \rho).$$

2. *Let $\sigma, \sigma' \in \mathcal{P}(\mathcal{H})$ and $\rho \in \mathcal{P}(\mathcal{H}')$. Then,*

$$\delta(\sigma \otimes \rho, \sigma' \otimes \rho) = \delta(\sigma, \sigma').$$

Although the following lemma is not directly related to the trace distance, it will be used in Chapter 6 to bound the trace distance between two states.

Lemma 2 (Corollary 4, [76]). *Let $X := \{x_1, \ldots, x_n\}$ be a list of (not necessarily distinct) values in $[0,1]$ with the average $\mu_X := \frac{1}{n}\sum_{i=1^n} x_i$. Let T of size t be a random subset of X with the average $\mu_T := \frac{1}{t}\sum_{i\in T} x_i$. Then, for any $\epsilon > 0$, the set $\bar{T} = X \setminus T$ with average $\mu_{\bar{T}} = \frac{1}{n-t}\sum_{i\in K} x_i$ satisfies*

$$P\left[\mu_{\bar{T}} - \mu_T \geq \sqrt{\frac{n(t+1)}{2(n-t)t^2}\log\frac{1}{\epsilon}}\right] \leq \epsilon.$$

2.3.2 Entropy

Entropy measures are used to quantify the unpredictability of a random variable and the amount of information gained by observing a system. One of the earliest and most widely used classical entropy measures was developed by Shannon in 1948 [77]. Shannon's entropy measure captures the idea that more predictable events convey less information. The more surprising and unpredictable an event is, the more informative it is. Shannon started by proposing the following function to describe the amount of information an event A has:

$$I(A) = -\log\left(P(A)\right),$$

where $P(A)$ is the probability of event A. Shannon's entropy is defined as the average amount of information of all possible events, and, for a discrete random variable X, is calculated as follows:

$$H(X) = \mathbb{E}\left[I(X)\right].$$

In the case of a distribution $P(X)$ over the set $\{0,1\}$ that selects 1 with probability p and 0 with probability $1-p$, the binary entropy is given by the following equation:

$$H(X) = -p\log_2 p - (1-p)\log_2(1-p).$$

Throughout our analysis, we will also frequently use the $d-$ary entropy function, which is a generalization of the standard binary entropy function. However, it should be noted that the $d-$ary entropy does not possess the same operational meaning as the binary entropy measure.

Definition 1. *For $d \geq 2$, the d-ary entropy function $h_d : [0,1] \to \mathbb{R}$ is given by*

$$h_d(x) = x\log_d(d-1) - x\log_d x - (1-x)\log_d(1-x).$$

The $d-$ary entropy is specially useful to bound the size of an important object in coding theory, the Hamming ball. The Hamming ball with radius μ centered at some point $\boldsymbol{r}$ is defined to be the set of vectors $\boldsymbol{z}$ at a distance μ from $\boldsymbol{r}$, when the distance is given by the Hamming distance d_H. So, we have the following Lemma.

Lemma 3 (Lemma 5, [78]). *For an integer $d \geq 2$ and $\mu \in [0, 1 - \frac{1}{d}]$,*

$$|\{\boldsymbol{z} \in \mathbb{Z}_d^n : d_H(\boldsymbol{z}, \boldsymbol{r}) \leq \mu n\}| \leq d^{h_d(\mu)n}.$$

When proving the security of protocols, it is crucial to understand the worst-case scenario rather than the average behavior. Therefore, the standard binary entropy definitions are not sufficient for securing protocols, and a new measure is needed. This is achieved through the use of min-entropy. Classically, for a finite random variable X, where $P(X = x) = p_x$, the min-entropy is defined as:

$$H_{\min}(X) = -\log \max p_x.$$

Operationally, this gives the probability of correctly guessing the element drawn from X, when choosing the element x with maximum probability. That is, $P_{\text{guess}} = \max p_x$. This definition can also be extended to cq-states $\rho_{XB} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B)$ as defined in Definition 2.

Definition 2. *Let $\rho_{XB} \in \mathcal{P}(\mathcal{H}_X \otimes \mathcal{H}_B)$ be a cq-state. The conditional min-entropy is given by*

$$H_{\min}(X|B)_\rho = -\log P_{guess}(X|B),$$

where $P_{guess}(X|B)$ is given by

$$P_{guess}(X|B) = \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[M_x \rho_x^B\right],$$

where the maximization is taken over all positive operator-valued measures (POVM), i.e. $\{M_x \geq 0 : \sum_x M_x = \mathbb{1}\}$.

In Definition 2, $P_{\text{guess}}(X|B)$ represents the probability of correctly guessing x given access to system B. Additionally, the maximization is taken over the most general type of measurements allowed in quantum mechanics. The following lemma states how min-entropy changes when a fixed bijective function is applied to the classical subsystem of a cq-state. This lemma will be important in the proof of security for the quantum oblivious linear evaluation protocol presented in Chapter 6.

Lemma 4. *Let $\rho_{XB} \in \mathcal{P}(\mathcal{H}_X \otimes \mathcal{H}_B)$ be a cq-state and let $f : \mathcal{X} \to \mathcal{X}$ be a fixed bijective function. Then,*

$$H_{\min}(X|B)_\rho \leq H_{\min}(f(X)|B)_\rho.$$

Proof. Consider the unitary operator,

$$U = \sum_x |f(x)\rangle\langle x| .$$

We check that U is indeed unitary:

$$UU^\dagger = \left(\sum_x |f(x)\rangle\langle x| \right) \left(\sum_{x'} |x'\rangle\langle f(x')| \right) = \sum_x |f(x)\rangle\langle f(x)| = I,$$

where in the last step we used the fact that the function f is a bijection. The same holds for $U^\dagger U = I$.

Now, observe the following,

$$\begin{aligned} H_{\min}(f(X)|B) &= -\log \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[M_x \rho^B_{f(x)}\right] \\ &= -\log \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[M_x U \rho^B_x U^\dagger\right] \\ &= -\log \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[U^\dagger M_x U \rho^B_x\right] . \end{aligned}$$

It is important to note that $\{N_x\}_x = \left\{U^\dagger M_x U\right\}_x$ is also a POVM, as they are all positive semidefinite operators and they sum up to unity. Therefore, we have that $\left\{U^\dagger M_x U\right\}_x$ can only decrease the space of possible POVMs, which is why we have:

$$\max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[U^\dagger M_x U \rho x^B\right] \leq \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[M_x \rho x^B\right] .$$

This means that,

$$H_{\min}(f(X)|B) \geq -\log \max_{\{M_x\}_x} \sum_x p_x \operatorname{tr}\left[M_x \rho^B_x\right] = H_{\min}(X|B).$$

□

The conditional min-entropy can be generalized to the fully quantum case where both systems are quantum (Definition 3).

Definition 3. *Let $\rho_{AB'} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}'_B)$ and $\sigma_{B'} \in \mathcal{P}(\mathcal{H}'_B)$. The min-entropy of $\rho_{AB'}$ relative to $\sigma_{B'}$ is given by*

$$H_{min}(A|B')_{\rho|\sigma} = -\log \min\{\lambda : \lambda \cdot id_A \otimes \sigma_{B'} \geq \rho_{AB'}\},$$

and

$$H_{min}(A|B')_\rho = \sup_{\sigma_{B'}} H_{min}(A|B')_{\rho|\sigma_{B'}}.$$

Furthermore, consider the superposition state $|\phi\rangle_{AB'} = \sum z \in B\alpha_z |z\rangle |\psi^z\rangle$ for some set $\mathcal{B}$ and arbitrary coefficients α_z. We define $\rho_{AB'} = |\phi\rangle\langle\phi|_{AB'}$ and the mixture $\tilde{\rho}_{AB'} = \sum_{z\in\mathcal{B}} |\alpha_z|^2 |z\rangle\langle z| \otimes |\psi^z\rangle\langle\psi^z|$. The following lemma gives a lower bound on the min-entropy of $\rho_{AB'}$ in terms of the min-entropy of $\tilde{\rho}_{AB'}$.

Lemma 5 (Lemma 3.1.13, [79])**.** *Let $\rho_{AB'}$ and $\tilde{\rho}_{AB'}$ be defined as above. Then,*

$$H_{min}(A|B')_\rho \geq H_{min}(A|B')_{\tilde{\rho}} - \log|\mathcal{B}|.$$

It is important to understand the changes in min-entropy that occur when a completely positive (CP) map is applied, as this is a crucial aspect of the security proof for the quantum oblivious linear evaluation protocol outlined in Chapter 6. It is known that, for a unital CP map $\mathcal{M}$ (i.e. $\mathcal{M}(\mathbb{1}) = \mathbb{1}$), the conditional min-entropy does not decrease, i.e. $H_{\min}(\mathcal{M}(A)|B) \geq H_{\min}(A|B)$. However, this result alone is insufficient for deriving practical min-entropy bounds. To obtain meaningful bounds for specific operators $\mathcal{M}$, it is necessary to utilize Theorem 6, in conjunction with Lemma 8 and Lemma 9. It is important to note that for clarity, the theorem employs the notation outlined in Chapter 6.

Lemma 6 (Theorem 1, [80])**.** *Let $\mathbf{X}$ denote a system with n qudits, and $\mathcal{M}_{\mathbf{X}\to\mathbf{FY}}$ be a CP map such that $((\mathcal{M}^\dagger \circ \mathcal{M})_{\mathbf{X}} \otimes id_{\bar{\mathbf{X}}})(\Phi_{\mathbf{X}\bar{\mathbf{X}}}) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}_d^{2n}} \lambda_{(\boldsymbol{a},\boldsymbol{b})}\Phi_{(\boldsymbol{a},\boldsymbol{b})}$. Then, for any partition of $\mathbb{Z}_d^{2n} = \mathfrak{S}_+ \cup \mathfrak{S}_-$ into subsets $\mathfrak{S}_+$ and $\mathfrak{S}_-$, and $\mathcal{M}(\sigma_{\mathbf{X}E}) = \sigma_{\mathbf{FY}E}$ we have*

$$2^{-H_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}|\sigma_{\mathbf{X}E}}} \leq \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_+} \lambda_{(\boldsymbol{a},\boldsymbol{b})} 2^{-H_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} + \left(\max_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_-} \lambda_{(\boldsymbol{a},\boldsymbol{b})}\right) d^n, \tag{2.4}$$

where, in general, for a (not necessarily normalized) quantum state $\rho_{AB} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B)$, $H_2(A|B)$ is the so-called collision entropy [79], given as

$$H_2(A|B)_{\rho_{AB}} = -\log\left(\mathrm{Tr}\left\{\left(\rho_B^{-1/4}\rho_{AB}\rho_B^{-1/4}\right)^2\right\}\right).$$

If we further condition on a general quantum state $\sigma_B \in \mathcal{P}(\mathcal{H}_B)$, we have

$$H_2(A|B)_{\rho_{AB}|\sigma_B} = -\log\left(\mathrm{Tr}\left\{\left(\sigma_B^{-1/4}\rho_{AB}\sigma_B^{-1/4}\right)^2\right\}\right).$$

It is interesting to note that when $\mathcal{M}$ is trace preserving, we have,

$$2^{-\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}|\sigma_{\mathbf{X}E}}} = 2^{-\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}}}.$$

This follows from the definition of the collision entropy and the fact that $\mathrm{Tr}_{\boldsymbol{FY}}\left[\mathcal{M}(\sigma_{\boldsymbol{X}E})\right] = \sigma_E$ [80].

Next, we present a chain rule for the collision entropy.

Lemma 7 (Proposition 8, [81]). *For any $\rho_{ABC} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$, it holds that*

$$H_2(A|BC)_\rho \geq H_2(AC|B) - \log d_C,$$

where d_C is the rank of ρ_C.

Now, we need a way to relate min-entropy and collision entropy to have useful bounds for min-entropy. This is done through the following two Lemmas.

Lemma 8 (Lemma 17, [80]). *Let $\rho_{AB'} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_{B'})$ and $d_A = \dim \mathcal{H}_A$. Then*

$$H_{\min}(A|B')_\rho \leq H_2(A|B')_\rho \leq 2H_{\min}(A|B')_\rho + \log d_A.$$

Lemma 9 (Lemma 18, [80]). *Let $\rho_{XB'} \in \mathcal{P}(\mathcal{H}_X \otimes \mathcal{H}_{B'})$ be a cq-state. Then*

$$H_{\min}(X|B')_\rho \leq H_2(X|B')_\rho \leq 2H_{\min}(X|B')_\rho.$$

Finally, we present a data-processing inequality, which reflects the intuitive idea that the min-entropy of a system A, given side information B, does not decrease under local physical operations applied to B.

Lemma 10 (Data processing inequality, Theorem 6.19, [82]). *Let $\rho_{AB} \in \mathcal{P}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Moreover, let $\mathcal{E}$ be a sub-unital CPTP map from system A to A' (i.e. $\mathcal{E}(\mathbb{1}_A) \leq \mathbb{1}_{A'}$) and $\mathcal{T}$ be a CPTP map from system B to B'. Then, the state $\sigma_{A'B'} = (\mathcal{E} \otimes \mathcal{T})\,\rho_{AB}$ satisfies*

$$H_{\min}(A|B)_\rho \leq H_{\min}(A'|B')_\sigma.$$

2.3.3 Two-universal functions

We start by defining a particular set of functions that are usually used to amplify the privacy of the parties' input and output elements.

Definition 4 ($\delta-$almost two-universal hash family; two-universal hash family). *A family, $\mathfrak{F}$, of functions, g, with domain D and range R is called a $\delta-$almost two-universal hash family if for any two distinct elements $w, w' \in D$ and for g chosen at random from $\mathfrak{F}$, the probability of a collision $g(w) = g(w')$ is at most δ. In the special case that $\delta = 1/|R|$, where $|R|$ is the size of the range R, the family is called two-universal.*

Now, we present a particular two-universal hash family, known as Multi-linear Modular Hashing (MMH), that preserves the structure of the OLE input and output while maintaining its privacy amplification guarantees. This family is based on the modular inner product of vectors [83].

Definition 5 (Definition 2, [83]). *Let d be a prime and let n be an integer $n > 0$. Define a family MMH* (Multi-linear Modular Hashing) of functions from $\mathbb{Z}_d^n$ to $\mathbb{Z}_d$ as follows*

$$MMH^* := \{g_x : \mathbb{Z}_d^n \to \mathbb{Z}_d \,|\, x \in \mathbb{Z}_d^n\},$$

where the functions g_x are defined for any $x = (x_1, \ldots, x_n)$, $m = (m_1, \ldots, m_n) \in \mathbb{Z}_d^n$

$$g_x(m) = x \cdot m \mod d = \sum x_i \, m_i \mod d.$$

Theorem 1 (Theorem 3, [83]). *The family MMH* is two-universal.*

Halevi and Krawczyk [83] actually prove a stronger result, namely that the MMH* family is $\Delta-$*universal*, which is more general than two-universal. For the sake of simplicity, we only present the simpler version of this theorem here.

The Generalized Leftover Hash Lemma, presented below, is a crucial component in the security proof of Chapter 6. It ensures that, after applying a known function g from a two-universal family to a random variable X, the resulting random variable $Z = g(X)$ is close to uniform, given some (possibly quantum) side information E. This is a high-dimensional version of the Leftover Hash Lemma, which can be easily derived by using Lemma 4 from [84] with $d_A = d^l$. Note that this is a special version, as Tomamichel et al. in [84] prove it in the more general case for $\delta-$almost two-universal hash families.

Lemma 11 (Generalized Leftover Hash Lemma [84]). *Let X be a random variable, E a quantum system, and $\mathfrak{F}$ a two-universal family of hash functions from X to $\mathbb{Z}_d^l$. Then, on average over the choices of g from $\mathfrak{F}$, the output $Z := g(X)$ is ξ-close to uniform conditioned on E, where*

$$\xi = \frac{1}{2}\sqrt{2^{l \log d - H_{min}(X|E)}}. \tag{2.5}$$

2.4 Universal composability

The universal composability (UC) framework, first introduced by Canetti in the classical setting [85], was extended to the quantum setting by Unruh, Ben-Or, and Mayers [86, 87]

(see also [88, 89]). It provides strong composability guarantees by ensuring the security of a protocol is independent of any external execution of the same or other protocols. Both the classical and quantum frameworks use the same ideal-real world comparison structure and consider similar interactions between machines. However, the quantum-UC framework allows for the manipulation of quantum states in addition to classical operations.

Specifically, the quantum-UC security of a protocol Π is determined by comparing its execution in a real scenario, where Π is executed, to an ideal scenario, where an ideal functionality $\mathcal{F}$ that carries out the same task is executed. The comparison is performed by a special machine called the environment, $\mathcal{Z}$, which supervises the execution of both scenarios and has access to any external information, such as concurrent executions of the same or any other protocol. In the two-party case, the structure of the machines in both scenarios is as follows: in the real scenario, there is the environment $\mathcal{Z}$, the adversary Adv, and the two parties, Alice and Bob. In the ideal scenario, there is the environment $\mathcal{Z}$, the simulator $\mathcal{S}$, the two parties Alice and Bob, and the ideal functionality $\mathcal{F}$. Informally, a protocol Π is considered quantum-UC secure if the environment $\mathcal{Z}$ cannot distinguish between the execution of Π in the real scenario and the execution of the functionality $\mathcal{F}$ in the ideal scenario. Any possible attack of the adversary Adv in the execution of Π can be simulated by the simulator $\mathcal{S}$ in the ideal-world execution of $\mathcal{F}$, without any noticeable difference from the point-of-view of the environment $\mathcal{Z}$. As the ideal functionality $\mathcal{F}$ is secure by definition, the real-world adversary is not able to extract any more information than what is allowed by the functionality $\mathcal{F}$.

The formal definition of quantum-UC security can be stated as follows. Let Π and ρ represent the real and ideal two-party protocols, respectively. Let $\text{EXEC}\Pi^C, Adv, \mathcal{Z}$ denote the output of the environment $\mathcal{Z}$ at the end of the real execution, where C denotes the corrupted party and Adv denotes the adversary. Similarly, let $\text{EXEC}\rho^C, \mathcal{S}, \mathcal{Z}$ denote the output of the environment $\mathcal{Z}$ at the end of the ideal execution, where $\mathcal{S}$ is the simulator.

Definition 6 (Statistical quantum-UC security, Computational quantum-UC security [88])**.** *Let protocols π and ρ be given. We say that π statistically quantum-UC emulates ρ if and only if for every party, C, and for every adversary, Adv, there exists a simulator, $\mathcal{S}$, such that for every environment $\mathcal{Z}$, and every $z \in \{0,1\}^*$, $n \in \mathbb{N}$,*

$$\left|P[EXEC_{\Pi^C,Adv,\mathcal{Z}}(n,z)=1] - P[EXEC_{\rho^C,\mathcal{S},\mathcal{Z}}(n,z)=1]\right| \leq \mu(n),$$

where $\mu(n)$ is a negligible function and n is the security parameter. We furthermore require that if Adv is quantum-polynomial-time, so is $\mathcal{S}$. Finally, if we consider quantum-polynomial-time Adv and $\mathcal{Z}$ we have computational quantum-UC security.

$\mathcal{F}_{\mathbf{COM}}$ functionality

- **Commitment phase**. Upon receiving $(\texttt{commit}, M)$ from Bob, the functionality sends `commit` to Alice.
- **Opening phase**. Upon receiving `open` from Bob, the functionality sends $(\texttt{open}, M)$ to Alice.

Figure 2.5: Commitment functionality.

The role of the simulator, $\mathcal{S}$, in the quantum-UC framework is to simulate the execution of the protocol Π in such a way that the environment $\mathcal{Z}$ is not able to distinguish between the real execution and the ideal execution. To accomplish this, $\mathcal{S}$ runs a simulated honest party that interacts with the environment, which is acting as the adversary. Additionally, $\mathcal{S}$ controls the dishonest party and their inputs to the ideal functionality $\mathcal{F}$, as well as the external functionality $\mathcal{F}_{\mathbf{ext}}$ if used in the real execution.

In order to generate a simulated execution that cannot be distinguished by the environment, $\mathcal{S}$ relies on its ability to extract the inputs provided to the dishonest party by the environment and uses them along with the ideal functionality outputs. Furthermore, $\mathcal{S}$ can reprogram $\mathcal{F}_{\mathbf{ext}}$ in the ideal world as needed to produce an indistinguishable simulation of the real world.

In summary, the simulator $\mathcal{S}$ plays a crucial role in the quantum-UC framework by simulating the execution of the real protocol Π in the ideal scenario, in order to ensure that the environment $\mathcal{Z}$ is not able to distinguish between the real execution and the ideal execution, thereby providing strong composability guarantees for the protocol Π.

Ideal functionalities

Whenever a protocol Π utilizes an external functionality $\mathcal{F}_{\mathbf{ext}}$, we say that Π is in the $\mathcal{F}_{\mathbf{ext}}-$hybrid model. The quantum OLE protocol $\Pi_{\mathbf{QOLE}}$ presented in Chapter 6 employs the ideal commitment functionality, $\mathcal{F}_{\mathbf{COM}}$, defined in Figure 2.5. Note that the protocol makes multiple calls to $\mathcal{F}_{\mathbf{COM}}$ and only opens a subset of the committed elements. To specify different instance calls, we use an index element i. In the commitment phase, Bob sends $(\texttt{commit}, i, M)$ to the functionality, which in turn sends $(\texttt{commit}, i)$ to Alice. In the opening phase, Bob sends $(\texttt{open}, i)$, and the functionality sends $(\texttt{open}, i, M)$ to Alice.

The $\mathcal{F}_{\mathbf{COM}}$ functionality can be replaced by the commitment protocol $\Pi_{\mathbf{COM}}$ presented in [90], which is computationally UC-secure in the Common Reference String (CRS) model. As analyzed in [91] (Theorem 3.), the protocol $\Pi_{\mathbf{COM}}$ computationally

quantum-UC realizes $\mathcal{F}_{\mathbf{COM}}$ in the CRS model. Therefore, since $\Pi_{\mathbf{QOLE}}$ is proved to be quantum-UC secure, the resulting protocol $\Pi_{\mathbf{QOLE}}^{\Pi_{\mathbf{COM}}}$ is quantum-UC secure by the composition theorem [88].

2.5 Conclusion

Throughout this chapter, we introduce the essential concepts that underpin the rest of the thesis. The chapter is divided into four main sections. First, we provide a succinct overview of the mathematical notation. Next, we give an informal description of secure multiparty computation. Finally, we introduce the basic formalism of quantum information and the universal composability framework in the quantum setting.

Chapter 3

Quantum oblivious transfer

In the classical setting, it is not possible to develop information-theoretic secure OT or reduce it to one-way functions, and therefore public-key computational assumptions are required. Impaggliazzo and Rudich [39] demonstrated that one-way functions alone cannot imply key agreement, which is an example of asymmetric cryptography. Additionally, Gertner et al. [40] noted that since OT implies key agreement, this establishes a separation between symmetric cryptography and OT. Therefore, it is not possible to generate OT using only symmetric cryptography, as one could potentially use one-way functions to implement key agreement through the OT construction. This poses a threat to all classical OT protocols [13, 92, 93] that rely on mathematical assumptions that can be provably broken by a quantum computer [28].

Aside from the security issue, asymmetric cryptography tends to be computationally more complex than symmetric cryptography, creating problems with speed when a large number of OTs are required. The classical post-quantum approach aims to find protocols that are resistant to quantum computer attacks, but these protocols are still based on computational complexity problems and are not necessarily less computationally expensive than the previously mentioned protocols.

In parallel to the classical post-quantum approach, the field of quantum cryptography has attempted to address this security issue by developing OT protocols based on quantum technologies. Interestingly, Wiesner proposed a similar concept more than a decade before Rabin's classical OT was published in 1981 [54]. At the time, Wiesner's proposal was rejected for publication due to a lack of acceptance in the research community. However, the first published quantum OT (QOT) protocol, known as the BBCS (Bennett-Brassard-Crépeau-Skubiszewska) protocol [6], was not presented until 1992. There is a key difference between classical and quantum OT from a security perspective: quantum OT has been proved to be possible under the assumption of only the existence of quantum-hard one-way functions [37, 38], meaning it requires weaker security assumptions than

classical OT.

In this chapter, we delve into the subject of quantum oblivious transfer (OT). Our focus is on several key OT protocols, their security models, and underlying assumptions. To the best of our knowledge, there is no existing survey that is solely dedicated to quantum OT protocols. Typically, these protocols are analysed as part of broader surveys on "quantum cryptography", resulting in a less in-depth examination of the topic. For further reference, we have included some comprehensive reviews on the broader subject of quantum cryptography in [29, 94–100].

The structure of this chapter is as follows: first, we provide a brief overview of the impossibility results related to quantum OT. Then, we delve into the exposition of some of the most well-known quantum OT protocols. Finally, we offer a brief overview of OT protocols that are not covered in detail in this thesis.

3.1 Impossibility results

The emergence of quantum oblivious transfer (QOT) was closely linked to the development of quantum bit commitment (QBC). In fact, the first proposed QOT protocol (BBCS [6]) reduces QOT to QBC, which sets a clear distinction between classical and quantum protocols. While classical bit commitment (BC) can be reduced to classical oblivious transfer (OT) [56], the reverse is not true using only classical communication [101]. As a result, Yao's proof [102] of the BBCS protocol [6] highlights an enhanced characteristic of quantum communications: the equivalence between QOT and QBC - they can be reduced to each other - a relationship that is not present in the classical realm.

At the time of the BBCS protocol, researchers aimed to achieve unconditionally secure quantum oblivious transfer (QOT). This was based on the potential for unconditionally secure quantum bit commitment (QBC). One year after the proposal of BBCS, Brassard et al. introduced the BCJL protocol [7]. However, it was later discovered that the protocol had a flawed proof of its unconditional security [103]. This was followed by independent proof from Lo and Chau [104] and Mayers [105] that unconditionally secure QBC is actually impossible. Despite this, the question of unconditionally secure QOT not based on QBC remained open [94], even after the no-go theorems [104, 105]. Lo eventually proved that unconditionally secure QOT is also impossible [106]. Lo's conclusion was drawn as a corollary of a more general result, which stated that secure two-party computations where only one party learns the result cannot be unconditionally secure. Lo's results sparked a line of research on the possibility of two-sided secure two-party computation, which was later proven impossible by Colbeck [107] and further extended in subsequent works [108–110]. For further reading on the impossibility results presented by Lo, Chau and

Mayers, the reader can refer to the works [101, 111].

Although the impossibility results in quantum cryptography have been widely accepted, some criticism has been raised regarding their generality [112–115]. Yuen [112] argues that since there is no known characterization of all possible QBC protocols, it is logically impossible to have a general impossibility proof. Nevertheless, subsequent analyses [116–118] have confirmed the general belief of the impossibility of unconditionally secure QBC. The criticism was finally put to rest with the proof by Ariano et al. [119] in 2007, which covered all conceivable protocols based on classical and quantum information theory. Subsequent works have tried to simplify the proof [120–122] and translate it into categorical quantum mechanics language [123–125].

In response to these impossibility results, the community has taken two main approaches:

1. Development of OT protocols under certain assumptions, such as limiting adversary's technology (e.g. noisy-storage model, relativistic protocols, isolated-qubit model) or assuming the security of additional functionalities (e.g. bit commitment);

2. Development of OT protocols with a relaxed security definition, where the adversary is allowed to extract some information about the honest party's input/output with a given probability. This approach gives rise to the concepts of weak OT and weak private database query.

The next section will examine OT protocols that generate a special primitive known as *oblivious keys* as an intermediate step.

3.2 BBCS-based protocols

This section investigates protocols that overcome the no-go theorems [104, 105] through certain assumptions. These protocols are either based on one-way functions, considered to be quantum-hard [37, 38, 126], or on the adversary's technological or physical limitations [127–132]. The latter differs qualitatively from the complexity-based assumptions that post-quantum protocols rely on. Moreover, the security of these protocols is only dependent on the validity of the assumptions during the protocol execution and remains intact even if the assumptions are no longer valid later on. This property, known as *everlasting* security [133], is a defining feature of quantum protocols compared to classical cryptography.

We begin by introducing the first quantum oblivious transfer (QOT) protocol and then examine how it led to the development of two assumption models: $\mathcal{F}_{\text{COM}}-$hybrid and the limited-quantum-storage models.

3.2.1 BBCS protocol

The concept of "quantum conjugate coding" was introduced by Wiesner in 1983 [134]. It's a key element in many quantum cryptographic protocols, including quantum oblivious transfer [6]. The technique is also known as "quantum multiplexing" [135], "quantum coding" [136], and "BB84 coding" [101]. It involves encoding classical information in two non-orthogonal (conjugate) bases, leading to a property where measuring in one basis erases information encoded in the other. So, when bit 0 and 1 are encoded by these two bases, no measurement is able to perfectly distinguish the states. We will be using the following bases in the two-dimensional Hilbert space $\mathcal{H}_2$:

- Computational basis: $+ := \{|0\rangle_+, |1\rangle_+\}$;
- Hadamard basis: $\times := \{|0\rangle_\times, |1\rangle_\times\} = \left\{\frac{1}{\sqrt{2}}(|0\rangle_+ + |1\rangle_+), \frac{1}{\sqrt{2}}(|0\rangle_+ - |1\rangle_+)\right\}$.

Throughout this chapter, we simplify the notation by associating the basis set $\{+, \times\}$ with the binary set $\{0, 1\}$: $+$ is associated with 0 and $\times$ with 1. This allows us to easily compare strings of bases between parties, i.e. the XOR operation ($\oplus$) between two vectors $\boldsymbol{\theta}^{\mathsf{A}}, \boldsymbol{\theta}^{\mathsf{B}} \in \{+, \times\}^n$ is defined as the XOR operation between the corresponding binary vectors $\boldsymbol{\theta}^{\mathsf{A}}, \boldsymbol{\theta}^{\mathsf{B}} \in \{0, 1\}^n$.

Protocol [6]. The first proposal for a quantum oblivious transfer protocol, known as Bennett-Brassard-Crépeau-Skubiszewska (BBCS), is shown in Figure 3.1. It utilizes the quantum conjugate coding technique. The first phase of the BBCS QOT protocol, referred to as the *BB84 phase*, consists of Alice encoding a set of qubits that are randomly measured by Bob. The parties then use Bob's output bits and Alice's random elements to generate a special type of key, known as *oblivious key*. Alice reveals her bases $\boldsymbol{\theta}^{\mathsf{A}}$ to Bob to achieve this. Using the oblivious key as a resource and a two-universal family of hash functions $\mathfrak{F}$, Alice can then obliviously send one of the messages m_0, m_1 to Bob, ensuring he is only able to know one of the messages. Recall, we use the notation $s \leftarrow_{\$} S$ to describe a situation where an element s is drawn uniformly at random from the set S.

Oblivious keys. The term "oblivious key" was first used by Fehr and Schaffner [89] in reference to random OT. The notion was refined by Jakobi et al. [137] for implementing private database queries (PDQ). In the BBCS protocol, oblivious keys are used as a resource to perform OT. Like standard encryption keys, they enable the performance of OT. In other words, encryption methods consume standard keys and OT methods consume oblivious keys. The concept of oblivious keys applied to OT protocols was recently presented by Lemus et al. [138]. We can define it as follows.

$\Pi^{\mathbf{BBCS}}$ protocol

Parameters: n, security parameter; two-universal family of hash functions $\mathfrak{F}$ from $\{0,1\}^{n/2}$ to $\{0,1\}^l$.
Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

BB84 phase:

1. Alice generates random bits $\boldsymbol{x}^{\mathsf{A}} \leftarrow_{\$} \{0,1\}^n$ and random bases $\boldsymbol{\theta}^{\mathsf{A}} \leftarrow_{\$} \{+,\times\}^n$. Sends the state $\left|\boldsymbol{x}^{\mathsf{A}}\right\rangle_{\boldsymbol{\theta}^{\mathsf{A}}}$ to Bob.

2. Bob randomly chooses bases $\boldsymbol{\theta}^{\mathsf{B}} \leftarrow_{\$} \{+,\times\}^n$ to measure the received qubits. We denote by $\boldsymbol{x}^{\mathsf{B}}$ his output bits.

Oblivious key phase:

3. Alice reveals to Bob the bases $\boldsymbol{\theta}^{\mathsf{A}}$ used during the *BB84 phase* and sets his oblivious key to $\mathsf{ok}^{\mathsf{A}} := \boldsymbol{x}^{\mathsf{A}}$.

4. Bob computes $\mathsf{e}^{\mathsf{B}} = \boldsymbol{\theta}^{\mathsf{B}} \oplus \boldsymbol{\theta}^{\mathsf{A}}$ and sets $\mathsf{ok}^{\mathsf{B}} := \boldsymbol{x}^{\mathsf{B}}$.

Transfer phase:

5. Bob defines $I_0 = \{i : \mathsf{e}_i^{\mathsf{B}} = 0\}$ and $I_1 = \{i : \mathsf{e}_i^{\mathsf{B}} = 1\}$ and sends the $(I_b, I_{b\oplus 1})$ to Alice.

6. Alice picks two uniformly random hash functions $f_0, f_1 \in \mathfrak{F}$, computes the pair of strings (s_0, s_1) as $s_i = m_i \oplus f_i(\mathsf{ok}^{\mathsf{A}}_{I_{b\oplus i}})$ and sends the pairs (f_0, f_1) and (s_0, s_1) to Bob.

7. Bob computes $m_b = s_b \oplus f_i(\mathsf{ok}^{\mathsf{B}}_{I_0})$.

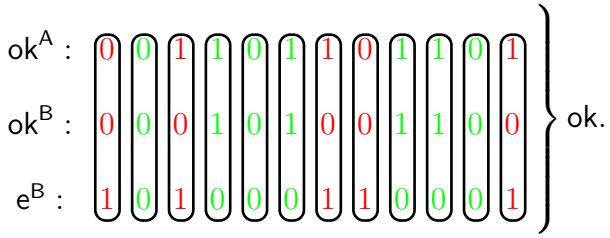
Alice's output: $\perp$.
Bob's output: m_b.

Figure 3.1: BBCS OT protocol.

Definition 7 (Oblivious key)**.** *An oblivious key shared between two parties, Alice and Bob, is a tuple* $\mathsf{ok} := \left(\mathsf{ok}^{\mathsf{A}}, (\mathsf{ok}^{\mathsf{B}}, \mathsf{e}^{\mathsf{B}})\right)$ *where* ok^{A} *is Alice's key,* ok^{B} *is Bob's key and* e^{B} *is Bob's signal string.* e^{B} *indicates which indexes of* ok^{A} *and* ok^{B} *are correlated and which indexes are uncorrelated, i.e.* $\mathsf{e}_i^{\mathsf{B}} = 0$ *when the corresponding indexes are correlated and* $\mathsf{e}_i^{\mathsf{B}} = 1$ *when they are not.*

Note that, for some index i, when two index elements $\mathsf{ok}_i^{\mathsf{A}}$ and $\mathsf{ok}_i^{\mathsf{B}}$ are correlated,

$\mathsf{ok}_i^{\mathsf{A}} = \mathsf{ok}_i^{\mathsf{B}}$. However, when they are uncorrelated, they are drawn independently. This means that both index elements may either be equal or different. Consider the following oblivious key $\mathsf{ok} = (001101101101, (000101001100, 101000110001))$ as an example. We can check it is a well-structured oblivious key:



It is worth stressing that oblivious keys are independent of the sender's messages m_0, m_1 and are not the same as random OT. In fact, as Alice does not know the groups of indexes I_0 and I_1 computed by Bob after the basis revelation, Alice does not have her messages fully defined. A similar concept was defined by König et al. [129] under the name of *weak string erasure*.

Security. Regarding security, the BBCS protocol provides unconditional security against dishonest Alice, as she only receives some set of indexes I_0 from Bob. However, it is insecure against dishonest Bob, who can carry out a memory attack and obtain complete knowledge of both messages m_0 and m_1 undetected [6]. In the memory attack, Bob delays his measurements in step 2 until after step 3. This requires quantum memory, which is why it's called the memory attack. To mitigate this issue, the authors suggest forcing Bob to measure the received states at step 2. In subsequent sections, two common approaches to tackle this security flaw are presented: either by assuming the existence of commitments or by setting physical constraints that prevent Bob from delaying his measurements.

3.2.2 BBCS in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model

As previously discussed, the BBCS protocol needs Bob to measure his qubits in step 2 for security. A solution to this issue, as proposed in [6], is the use of a commitment scheme. This approach results in a $\mathcal{F}_{\mathbf{COM}}$-hybrid model, where $\mathcal{F}_{\mathbf{COM}}$ refers to any commitment scheme (including ideal functionalities)[1].

Protocol. The modified BBCS (Figure 3.2) adds a *cut and choose* step using commitment scheme **COM**. In this step, Bob commits to the measurement bases and output bits from the *BB84 phase*. Alice then selects a subset of qubits to verify, and Bob reveals the corresponding commitments. If no inconsistencies are found, the protocol continues. To

[1]The notation $\mathcal{F}_{\mathbf{COM}}$ is commonly used for ideal functionalities. However, for simplicity, we use $\mathcal{F}_{\mathbf{COM}}$ to refer to any commitment scheme (including the ideal commitment functionality).

$\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocol

Parameters: n, security parameter; $\mathfrak{F}$ two-universal family of hash functions.
Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

BB84 phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Cut and choose phase:

3. Bob commits to the bases used and the measured bits, i.e. $\mathbf{COM}(\boldsymbol{\theta}^{\mathsf{B}}, \boldsymbol{x}^{\mathsf{B}})$, and sends to Alice.
4. Alice asks Bob to open a subset T of commitments (e.g. $n/2$ elements) and receives $\{\theta_i^{\mathsf{B}}, x_i^{\mathsf{B}}\}_{i \in T}$.
5. In case any opening is not correct or $x_i^{\mathsf{B}} \neq x_i^{\mathsf{A}}$ for $\theta_i^{\mathsf{B}} = \theta_i^{\mathsf{A}}$, abort. Otherwise, proceed.

Oblivious key phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Transfer phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Alice's output: $\perp$.
Bob's output: m_b.

Figure 3.2: BBCS OT protocol in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model.

ensure security, the size of the tested subset must be proportional to n to guarantee Bob's measurement with high probability in n.

Security. The security of the modified BBCS protocol has been extensively studied in a long line of research [6, 36–38, 41, 89, 102, 139–143]. Early studies from the 1990s focused on analysing security against limited adversaries who only made individual measurements [140]. Later, Yao [102] showed its security against general adversaries capable of fully coherent measurements. However, these initial works [102, 140, 141] lacked a comprehensive security definition and relied on weak security measures like Collision Entropy and Mutual Information [144, 145]. In modern quantum cryptography, the protocol's security is established in quantum simulation-paradigm frameworks [36, 41, 89, 129] using a simulator to show that a real execution is indistinguishable from an ideal, secure execution.

Desirable worst-case security measures for quantum oblivious transfer (QOT) were applied a decade later [79, 146]. These were based on the concept of *min-entropy* [144,

145], $H_{\min}$, which, intuitively, reflects the maximum probability of an event to happen. To prove security against dishonest Bob, one is interested in measuring Bob's min-entropy on Alice's oblivious key ok^{A} given any quantum side information E he may have, i.e. $H_{\min}(\mathsf{ok}^{\mathsf{A}}|E)$. Informally, for a bipartite classical-quantum state ρ_{XE} the conditional min-entropy $H_{\min}(X|E)$ is given by

$$H_{\min}(X|E)_{\rho_{XE}} := -\log P_{guess}(X|E),$$

where $P_{guess}(X|E)$ is the maximum probability the adversary can guess x given all possible measurements. Damgård et al. [36] proved the stand-alone QOT security using this min-entropy measure and Fehr and Schaffner's [89] quantum simulation-paradigm framework. Their argument to prove the security of the protocol against dishonest Bob can be summarized as follows. The cut and choose phase ensures that Bob's conditional min-entropy on the elements of ok^{A} belonging to I_1 (indexes with uncorrelated elements) is lower-bounded by some value that is proportional to the security parameter, i.e. $H_{\min}(\mathsf{ok}^{\mathsf{A}}_{I_1}|E) \geq n\lambda$ for some $\lambda > 0$. Note that this is equivalent to derive an upper bound on the guessing probability $P_{guess}(\mathsf{ok}^{\mathsf{A}}_{I_1}|E) \leq 2^{-n\lambda}$. Having deduced an expression for λ, they proceed by applying a random hash function f from a two-universal family $\mathfrak{F}$, $f \leftarrow_{\$} \mathfrak{F}$. This final step ensures that $f(\mathsf{ok}^{\mathsf{A}}_{I_1})$ is statistically indistinguishable from uniform (privacy amplification theorem [146–148]). The proof provided by Damgård et al. [36] was extended by Unruh [41] to the quantum Universal Composable (UC) model with ideal commitments. Now, a natural question arises:

Which commitment schemes can be used to render simulation-based security?

Commitment scheme. The work by Aaronson [126] provides non-constructive evidence "that collision-resistant hashing might still be possible in a quantum setting", which supports the use of commitment schemes based on quantum-hard one-way functions in $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$. It has been demonstrated that any one-way function, including quantum-hard ones, can be used to construct commitment schemes [149–151]. However, using a commitment scheme based on a quantum-hard one-way function in $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ does not necessarily lead to a simulation-secure protocol. This is due to the difficulty or impossibility of simulation-based proof due to the nature of the commitment scheme. For more information, see [37].

For a commitment scheme to render simulation-based security, the simulator in its proof must have access to two intriguing properties: *extractability* and *equivocality*. Extractability allows the simulator to obtain the committed value from a malicious committer, while equivocality enables the simulator to modify the committed value later.

Despite the counter-intuitive nature of using a commitment scheme where we can violate both security properties (hiding and biding properties), these properties are crucial for the scheme's security. The extractability is used to prove security against the dishonest sender, and equivocality is used to prove security against the dishonest receiver. There are proposals in the literature for commitment schemes COM with these properties based on:

- Quantum-hard one-way functions [37, 38];
- Common Reference String (CRS) model [41, 90];
- Bounded-quantum-storage model [152];
- Quantum hardness of the Learning With Errors assumption [36].

Composability. The integration of secure OT executions in secure multiparty protocols [55] is critical to ensuring security, as improper integration could lead to security breaches. Although it seems intuitive to assume that a secure OT protocol can be integrated within more complex protocols, proving this is highly non-trivial as it is not clear a priori under which circumstances protocols can be composed [153].

To address this challenge, the development of simulation-based security was introduced. However, simulation-based security alone is not sufficient for composability, as a composability framework is also necessary [153]. Several composability frameworks have been proposed in the literature. Fehr and Schaffner [89] proposed a framework for sequential composition of quantum protocols in a classical environment. Ben-Or and Mayers [154] and Unruh [41, 155] extended the classical Universal Composability model [85] to a quantum setting, allowing concurrent composability. Maurer and Renner [156] developed a more general composability framework that accommodates various models of computation, communication, and adversary behavior. Recently, Broadbent and Karvonen [125] introduced an abstract model of composable security in terms of category theory. As far as we are aware, the composable security of the protocol $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ has been proven in the Fehr and Schaffner model [89] by Damgård et al. [36] and in the quantum-UC model by Unruh [41].

3.2.3 BBCS in the limited-quantum-storage model

In this section, we review protocols based on the limited-quantum-storage model. These protocols sidestep the no-go theorems by relying on reasonable assumptions about the storage capacities of both parties. There are two main approaches within this model.

The first, the bounded-storage model, was introduced by Damgård et al. [127], who assumed that parties can store only a limited number of qubits. The second approach, the noisy-storage model, was initiated by Wehner et al. [128], who assumed that parties can store all qubits but they are unstable, i.e., they have imperfect, noisy storage that results in some decoherence. Both models force adversaries to use their quantum memories as both parties have to wait a predetermined time (Δt) during the protocol.

3.2.4 Bounded-quantum-storage model

In the bounded-quantum-storage (BQS) model, we assume that adversaries can only store a fraction $0 < \gamma < 1$ of the transmitted qubits during the waiting time Δt. The adversary's storage capacity is limited to $q = n\gamma$ qubits. γ is referred to as the storage rate.

Protocol. The protocol in the BQS model, $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$, is very similar to the BBCS protocol $\Pi^{\mathbf{BBCS}}$ presented in Figure 3.1. The difference is that both parties have to wait a predetermined time (Δt) after step 2. This protocol is presented in Figure 3.3.

$\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ protocol

Parameters: n, security parameter; $\mathfrak{F}$ two-universal family of hash functions.
Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

BB84 phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Waiting time phase:

3. Both parties wait time Δt.

Oblivious key phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Transfer phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Alice's output: $\perp$.
Bob's output: m_b.

Figure 3.3: BBCS OT protocol in the bounded-quantum-storage model.

Security. In this section, we focus on security against dishonest Bob in the BQS model. The justification for security against dishonest Alice follows from the original BBCS pro-

tocol, as detailed in Section 3.2.1.

In the BQS model, the waiting time (Δt) prevents Bob from holding a significant portion of qubits until Alice reveals the bases choices $\boldsymbol{\theta}^{\mathsf{A}}$ used during the BB84 phase. This is because Bob is forced to measure a fraction of the qubits, causing him to lose information about Alice's bases $\boldsymbol{\theta}^{\mathsf{A}}$.

Damgård et al. [146] showed that, with overwhelming probability, the loss of information about Alice's oblivious key ($\mathsf{ok}^{\mathsf{A}}_{I_1}$) is described by the min-entropy lower bound:

$$H_{\min}(\mathsf{ok}^{\mathsf{A}}_{I_1}|E) \geq \frac{1}{4}n - \gamma n - l - 1.$$

Similar to the $\mathcal{F}_{\mathbf{COM}}-$hybrid model, the min-entropy value must be proportional to the security parameter n. To ensure security, an upper bound on the fraction of qubits that can be stored by the receiver must be set, i.e. $\gamma < \frac{1}{4}$.

König et al. [129] improved the upper bound to $\gamma < \frac{1}{2}$ and showed that the BQS model is a special case of the noisy-quantum-storage model. Further, Mandayam and Wehner [157] presented a protocol that remains secure even when an adversary cannot store any fraction of the transmitted pulses, using higher-dimensional mutually unbiased bases. In this latter work, the storage rate γ approaches 1 with increasing dimension.

Composability. The security of the protocol $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ was initially proven by Damgård et al. [127, 146] under the stand-alone security model [158]. However, this model does not guarantee the composability of the protocol. The sequential composability was later established by Wehner and Wullschleger [158] in a simulation-based framework and by Fehr and Schaffner in a sequential composability framework.

Unruh [152] extended the quantum-UC framework and combined it with the BQS model to develop BQS-UC security. In this framework, composability is ensured while keeping track of the quantum memory bound used by the machines. Under this framework, Unruh follows a different approach as he does not use the protocol $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ (Figure 3.3). Unruh presented a BQS-UC secure commitment protocol and combined it with the $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocol to obtain a constant-round protocol that emulates any two-party functionality in a BQS-UC secure manner.

3.2.5 Noisy-quantum-storage model

The noisy-quantum-storage (NQS) model is a generalization of the bounded-quantum-storage (BQS) model. In the NQS model, the adversary is allowed to retain any fraction ν of the transmitted qubits, including the case where $\nu = 1$. However, the adversary's quantum memory is assumed to be noisy, meaning that qubits cannot be stored for a

certain amount of time (Δt) without undergoing decoherence [129].

Mora formally, the decoherence of qubits in the noisy storage can be described by a completely positive trace-preserving (CPTP) map, also known as a channel, $\mathcal{C} : \mathcal{P}(\mathcal{H}_{\text{in}}) \to \mathcal{P}(\mathcal{H}_{\text{out}})$. The Hilbert space of the stored qubits before and after the storage period (Δt) is represented by $\mathcal{H}_{\text{in}}$ and $\mathcal{H}_{\text{out}}$, respectively. $\mathcal{P}(\mathcal{H})$ represents the set of positive semi-definite operators with unitary trace acting on a Hilbert space $\mathcal{H}$. The channel $\mathcal{C}$ takes a quantum state $\rho \in \mathcal{H}_{\text{in}}$ at time t and returns a quantum state $\rho' \in \mathcal{H}_{\text{out}}$ at time $t + \Delta t$.

With this formulation, it's clear that the BQS model is a subset of the NQS. In BQS, the channel takes the form $\mathcal{C} = \mathbb{1}^{\otimes \nu n}$, where the storage rate ν is the fraction of transmitted qubits stored in the quantum memory. The most commonly studied scenario is restricted to $n-$fold quantum channels ($\mathcal{C} = \mathcal{N}^{\otimes \nu n}$), where the channel $\mathcal{N}$ is applied independently to each stored qubit [128, 129, 159]. In this scenario, specific security parameters can be derived.

Protocols. The BQS model protocol $\Pi_{\mathbf{bqs}}^{\mathbf{BBCS}}$ is considered secure in the NQS model [159]. However, the first proposed protocol analysed in this general NQS model was developed by König et al. [129] and draws inspiration from classical OT in the bounded-classical-storage model [160–162]. König et al.'s protocol uses two key techniques in its classical post-processing phase: encoding of sets and interactive hashing. The encoding of sets involves an injective function $\mathsf{Enc} : 0, 1^t \to T$, where T is a set of all subsets of $[n]$ with size $n/4$. The interactive hashing is a two-party protocol between Alice and Bob where Bob inputs a message W^t and both parties receive messages W_0^t and W_1^t, with one of them equal to W^t, but with the index unknown to Alice and the choice randomly determined.

In this section, we present the naïve version of the protocol introduced by König et al. in [129]. Although both $\Pi_{\mathbf{bqs}}^{\mathbf{BBCS}}$ and $\Pi_{\mathbf{nqs}}^{\mathbf{BBCS}}$ are distinct, we maintain a similar notation for ease of comparison. The protocol $\Pi_{\mathbf{nqs}}^{\mathbf{BBCS}}$ is outlined in Figure 3.4. The first two phases (*BB84* and *Waiting time*) are the same as in $\Pi_{\mathbf{bqs}}^{\mathbf{BBCS}}$ (Figure 3.3).

After the *Waiting time* phase, both parties generate a resource similar to oblivious keys, known as weak string erasure (WSE). Alice holds the entire key ok^{A}, while Bob holds one-fourth of it, represented by the tuple $(I, \mathsf{ok}^{\mathsf{B}} := \mathsf{ok}_I^{\mathsf{A}})$, where I is the set of indices they measured in the same basis and has size $|I| = \frac{n}{4}$. Using a method to encode sets into binary strings and interactive hashing, both parties generate two index subsets, I_0 and I_1. These subsets, along with two 2-universal hash functions, allow Alice to compute her output messages (m_0, m_1) and Bob to obtain his bit choice b and the corresponding message m_b. For more details on encoding sets and interactive hashing, refer to [161] and [162].

Naïve $\Pi^{\mathbf{BBCS}}_{\mathbf{nqs}}$ **protocol**

Parameters: n, security parameter; $\mathfrak{F}$ two-universal family of hash functions.
Alice's input: $\perp$.
Bob's input: $\perp$.

BB84 phase: Same as in $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

Waiting time phase: Same as in $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ (Figure 3.3).

Weak String Erasure phase: Similar to *Oblivious key phase* of $\Pi^{\mathbf{BBCS}}$ (Figure 3.1).

4. Alice reveals to Bob the bases $\boldsymbol{\theta}^{\mathsf{A}}$ used during the *BB84 phase* and sets her oblivious key to $\mathsf{ok}^{\mathsf{A}} := \boldsymbol{x}^{\mathsf{A}}$.
5. Bob computes $\mathsf{e}^{\mathsf{B}} = \boldsymbol{\theta}^{\mathsf{B}} \oplus \boldsymbol{\theta}^{\mathsf{A}}$. Then, he defines $I = \{i : \mathsf{e}^{\mathsf{B}}_i = 0\}$ and sets $\mathsf{ok}^{\mathsf{B}} := \boldsymbol{x}^{\mathsf{B}}_I$.
6. If $|I| < n/4$, Bob randomly adds elements to I and pads the corresponding positions in ok^{B} with 0s. Otherwise, he randomly truncates I to size $n/4$, and deletes the corresponding values in ok^{B}.

Interactive hashing phase:

7. Alice and Bob execute interactive hashing with Bob's input W to be equal to a description of $I = \mathsf{Enc}(W)$. They interpret the outputs W_0 and W_1 as descriptions of subsets I_0 and I_1 of $[n]$.

Transfer phase:

5. Alice generates random $f_0, f_1 \leftarrow_{\$} \mathfrak{F}$ and sends them to Bob.
6. Alice computes the pair of messages (m_0, m_1) as $m_i = f_i(\mathsf{ok}^{\mathsf{A}}_{I_i})$.
7. Bob computes $b \in \{0,1\}$ by comparing $I = I_b$ and computes $m_b = f_b(\mathsf{ok}^{\mathsf{B}}_I)$.

S **output:** $(m_0, m_1) \in \{0,1\}^l$ (two messages).
R **output:** (b, m_b) where $b \in \{0,1\}$ (bit choice).

Figure 3.4: BBCS OT protocol in the noisy-quantum-storage model.

Security. The first proofs in the NQS model for the original BQS protocol (Figure 3.3) were developed by Schaffner, Wehner, and Terhal [128, 163]. However, these initial works only considered individual-storage attacks, in which the adversary treats all incoming qubits equally. Schaffner later proved [159] the security of $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ against arbitrary attacks

in the more general NQS model defined by König et al. [129].

In the general NQS model, the security of both protocols $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathbf{nqs}}$ (Figures 3.3 and 3.4) against a dishonest receiver relies on setting a lower-bound for the min-entropy of the "unknown" key $\mathsf{ok}^{\mathsf{A}}_{I_{1-b}}$ given the receiver's quantum side information, represented by the output of the quantum channel $\mathcal{C}$ applied to the received states. More formally, one has to lower-bound the expression $H_{\min}\left(\mathsf{ok}^{\mathsf{A}}_{I_{1-b}}|\mathcal{C}\left(Q_{\text{in}}\right)\right)$, where Q_{in} denotes the subsystem of the received states before undergoing decoherence. This lower-bound, proven in [129], depends on the receiver's maximal success probability $P^{\mathcal{C}}_{\text{succ}}(n)$ of correctly decoding a randomly chosen n-bit string $x \in \{0,1\}^n$ sent over the quantum channel $\mathcal{C}$.

For particular channels $\mathcal{C} = \mathcal{N}^{\otimes\nu}$, König et al. [129] concluded that security in the NQS model can be obtained in case

$$c_{\mathcal{N}} \cdot \nu < \frac{1}{2},$$

where $c_{\mathcal{N}}$ is the classical capacity of quantum channels $\mathcal{N}$ satisfying a particular property (strong-converse property).

3.2.6 Experimental attacks

While QKD and QOT protocols are proven to be secure in theory, their experimental implementations may contain loopholes that undermine their security. The mismatch between theory and practice stems from the fact that theoretical proofs often assume the honest parties' physical apparatus is invulnerable to hacking. However, flaws in both the generation and measurement of qubits can be used to carry out various quantum attacks. For a comprehensive overview of QKD attacks and countermeasures, we refer the reader to the review articles by Lo et al. [164] and Pirandola et al. [165]. In this context, we briefly examine the impact of such attacks on QOT protocols based on BBCS.

QOT attacks

It is important to stress that there is a fundamental difference between QKD and QOT protocols. In QKD, the parties have a mutual trust and can work together to identify an external attack, while in QOT, the parties are inherently distrustful of each other. External attacks in QKD presume that the attacker has physical access to the quantum channel and can launch a man-in-the-middle attack. On the other hand, QOT protocols are inherently linked through a quantum channel, thus QOT attacks may require less effort to launch as the attacker is already utilizing the channel.

According to the security requirements of QOT protocols, it is crucial for both Alice and Bob to maintain their respective privacy. In particular, Alice should not know Bob's bit b,

and Bob should not have knowledge of m_{1-b}. The security of BBCS-based QOT protocols depends on the security of oblivious keys, which requires Alice not to have information about the set of indexes known to Bob (i.e. e^{B}), and Bob having limited knowledge of Alice's key (i.e. ok^{A}). These two pieces of information can be easily obtained by the adversary if they have access to the quantum bases used by either party (i.e. $\boldsymbol{\theta}^{\mathsf{A}}$ or $\boldsymbol{\theta}^{\mathsf{B}}$).

For instance, Alice can compute e^{B} by taking the XOR of $\boldsymbol{\theta}^{\mathsf{B}}$ and $\boldsymbol{\theta}^{\mathsf{A}}$, while Bob can obtain ok^{A} by measuring all the qubits with Alice's bases $\boldsymbol{\theta}^{\mathsf{A}}$. Therefore, the objective of the adversary is to gain information (or control) about the set of bases used by the other party through their quantum channel.

Two common attacks in quantum systems are faked-state attacks (FSA) and trojan-horse attacks (THA) [166, 167]. FSA targets measurement apparatus only, while THA can target both preparation and measurement apparatus. In a prepare-and-measure setting, FSA can only be executed by Alice, while THA can be executed by either party. In this context, let's examine how these two attacks can be applied to both $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocols. The attacks on $\Pi^{\mathbf{BBCS}}_{\mathbf{nqs}}$ follow the same reasoning but the notation vary slightly.

We denote by $\tilde{\boldsymbol{\theta}}^{\mathsf{B}}_J \leftarrow \mathcal{A}_{\mathbf{qok}}(J)$ Alice's quantum hacking procedure ($\mathcal{A}_{\mathbf{qok}}(J)$) that breaks the security requirements of oblivious keys and provides her with Bob's bases ($\tilde{\boldsymbol{\theta}}^{\mathsf{B}}_J$) from index set J. Similarly for Bob, i.e. $\tilde{\boldsymbol{\theta}}^{\mathsf{A}}_J \leftarrow \mathcal{B}_{\mathbf{qok}}(J)$.

$\Pi^{\mathsf{A}}_{\mathbf{FSA}}$ attack

Alice's input: set of indexes J of size q.

1. Alice performs some faked-state attack $\left\{\tilde{\theta}^{\mathsf{B}}_j\right\}_{j\in J} \leftarrow \mathcal{A}_{\mathbf{qok}}(J)$ where $\tilde{\theta}^{\mathsf{B}}_j \in \{+,\times\}$ or $\tilde{\theta}^{\mathsf{B}}_j = \perp$.
2. If $\exists j \in J$ such that $\tilde{\theta}^{\mathsf{B}}_j \neq \perp$:
   (a) $b = 0$ if $j \in I_b$, ;
   (b) $b = 1$ if $j \notin I_b$.
3. Otherwise, sets $b = \perp$.

Alice's output: b.

Figure 3.5: Alice faked-state attack to $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocols.

FSA attacks. The attacks described by Jain et al. [168] involve well-crafted optical sig-

nals that allow Alice to manipulate Bob's measurement outcomes. When Alice and Bob's bases coincide, Bob's detector clicks. Conversely, when their bases are orthogonal, Bob does not detect an event ($\perp$). Alice exploits this by forcing Bob to only use measurements where their bases coincide, thus gaining full knowledge of Bob's bases. By discarding the indexes corresponding to no detection events and using the others in the protocol, Alice can easily distinguish between I_0 and I_1.

It's worth noting that Alice only needs to successfully manipulate one measurement round in order to guess one basis, which occurs with high probability after a sufficient number of attacks q. The probability of success for Alice in q rounds is calculated as follows:

$$P[\text{Successful attack by Alice in } q \text{ rounds}] = 1 - \left(\frac{1}{2}\right)^q.$$

With knowledge of Bob's basis, Alice can determine to which set (I_0 or I_1) a corresponding index (j) belongs. As Bob computes his message m_b using the set where their basis coincides and Alice computes both messages m_0 and m_1 using both sets, Alice can determine Bob's message m_b by identifying which message came from the set to which j belongs. The attack $\Pi^{\mathsf{A}}_{\mathbf{FSA}}$ against both $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ is summarized in Figure 3.5.

$\Pi^{\mathsf{A}}_{\mathbf{THA}}$ attack

Alice's input: one index element, j.

1. Alice performs some trojan-horse attack $\left\{\tilde{\theta}^{\mathsf{B}}_j\right\} \leftarrow \mathcal{A}_{\mathbf{qok}}(i)$ where $\tilde{\theta}^{\mathsf{B}}_j \in \{+, \times\}$.
2. Alice compares the received basis $\tilde{\theta}^{\mathsf{B}}_j$ with her corresponding base θ^{A}_j. Denote by $\tilde{\mathsf{e}}^{\mathsf{B}}_j := \tilde{\theta}^{\mathsf{B}}_j \oplus \theta^{\mathsf{A}}_j$.
3. Upon receiving I_b from R:
   (a) $b = \tilde{\mathsf{e}}^{\mathsf{B}}_j$ if $j \in I_b$;
   (b) $b = 1 - \tilde{\mathsf{e}}^{\mathsf{B}}_j$ if $j \notin I_b$.

Alice's output: b.

Figure 3.6: Alice trojan-horse attack to $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocols.

THA attacks. These attacks are executed by sending bright optical signals into the target equipment and analysing the different reflections to determine the bases in use.

Similarly to the FSA, Alice only needs to identify one of Bob's bases to compromise the system. By comparing her basis with Bob's basis for that particular round, she can determine Bob's bit b. This attack is depicted in Figure 3.6 as $\Pi^{\mathsf{A}}_{\mathbf{THA}}$.

Bob's THA attack, on the other hand, is more challenging. He must not only accurately guess all of Alice's bases but also properly measure the corresponding qubits after revealing the sender's bases. This task is much more difficult without the assistance of quantum memories. Bob's attack is illustrated in Figure 3.7 as $\Pi^{\mathsf{B}}_{\mathbf{THA}}$.

$\Pi^{\mathsf{B}}_{\mathbf{THA}}$ attack

Parameters: n, security parameter..

1. Bob performs some trojan-horse attack to all qubits sent by Alice, i.e. $\left\{\tilde{\theta}_i^{\mathsf{A}}\right\}_{i\in[n]} \leftarrow \mathcal{B}_{\mathbf{qok}}([n])$ where $\tilde{\theta}_i^{\mathsf{A}} \in \{+,\times\}$.
2. Bob measures the received states $\left|\boldsymbol{x}^{\mathrm{A}}\right\rangle_{\boldsymbol{\theta}^{\mathrm{A}}}$ with the correct bases, $\left\{\tilde{\theta}_i^{\mathsf{A}}\right\}_{i\in[n]}$.

Bob's output: ok^{A}.

Figure 3.7: Bob trojan-horse attack to $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ and $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ protocols.

Countermeasures

We have seen how two well-known quantum hacking techniques can undermine the security of oblivious keys and, consequently, the security of oblivious transfer. Fortunately, there are countermeasures available to prevent these quantum hacking techniques from compromising the security of oblivious keys and, in turn, oblivious transfer. These countermeasures can be classified into two categories: security patches designed to address specific vulnerabilities and novel protocols that allow for the use of faulty devices.

Regarding the two presented possible attacks, it is commonly possible to implement security patches that prevent them. The FSA attack can be mitigated by placing an additional detector (watchdog) at the entrance of the receiver's measurement device to monitor for malicious radiation that blinds the detector. On the other hand, the THA attack can be prevented by using an isolator at the entrance of both parties' devices. However, as noted by Jain et al. [168], these countermeasures may not be foolproof in practice as they only offer perfect protection if the isolators and watchdogs are effective at all relevant frequencies.

The study of security patches for each technological vulnerability is an active area of research [169]. However, this approach requires a challenging task of making experimental implementations match ideal protocols, leading to potential security vulnerabilities. Instead, it is more advantageous to develop protocols that already account for faulty devices and are robust against quantum hacking attacks. This is the objective of device-independent (DI) cryptography, where quantum devices are treated as black-boxes and the assumption that they cannot be controlled by the adversary is dropped [170, 171]. This section provides a brief overview of the current advancements in DI protocols. For a more comprehensive understanding, refer to the original works.

Kaniewski-Wehner DI protocol [172]. The first DI protocol for QOT was proposed by Kaniewski and Wehner [172] and further improved by Ribeiro et al. in [173]. The protocol was proved to be secure in the noisy-quantum-storage (NQS) model as it uses the original NQS protocol $\Pi_{\mathbf{nqs}}^{\mathbf{BBCS}}$ (Figure 3.4) for trusted devices.

The protocol considers two scenarios. The first assumes that the devices have a memoryless behavior every time they are used, which enables the devices to be tested independently from the actual protocol, resulting in a DI protocol with two phases: a device-testing phase and a protocol phase. Under this memoryless assumption, the protocol is proven to be secure against general attacks using proof techniques from [129]. The second scenario removes the memoryless assumption, making it unrealistic to test the devices in advance, as their behavior can change. As a result, the structure of the initial DI protocol must be altered, interweaving the rounds of the device-testing phase with the rounds of the protocol phase.

As is typical in DI protocols, the DI property arises from a violation of Bell inequalities [174] that ensures a certain level of entanglement. In the protocol phase, the entanglement-based variant of $\Pi_{\mathbf{nqs}}^{\mathbf{BBCS}}$ must be used, with Alice preparing maximally entangled states $|\Phi^+\rangle\langle\Phi^+|$, where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The Bell inequality used in this case is based on the Clauser-Holt-Shimony-Horne (CHSH) inequality [175].

Broadbent-Yuen DI protocol [176]. Recently, Broadbent and Yuen [176] proposed a DI protocol in the BQS (bounded-quantum-storage) model using $\Pi_{\mathbf{bqs}}^{\mathbf{BBCS}}$ (Figure 3.3). Like the work of Kaniewski and Wehner [172], the protocol is secure under the memoryless assumption, but with a difference. Unlike Kaniewski and Wehner's work, which relies on the violation of Bell inequalities [174] for security, Broadbent and Yuen's protocol does not require non-communication assumptions. Instead, it employs a recent self-testing protocol [177, 178] that is based on the hardness of the Learning with Errors (LWE) problem [179]. This approach allows the protocol to be secure without relying on the violation of Bell

inequalities.

Ribeiro-Wehner MDI protocol [180]. In the work by Ribeiro and Wehner [180], an OT protocol in the measurement-device-independent (MDI) regime [181] was developed to tackle the technological difficulties in implementing DI protocols [182]. The MDI regime assumes that the measurement devices are untrusted while the sources are trusted. This protocol was motivated by the lack of security proof in the DI setting and the vulnerability of non-device-independent protocols to attacks on the measurement devices [183]. The presented protocol follows the research line of König et al. [129] and is also proved to be secure in the NQS model.

The protocol starts with a weak string erasure (MDI-WSE) phase, similar to the approach taken by König et al. [129]. In this phase, Alice and Bob send random states $\left|\boldsymbol{x}^{\mathsf{A}}\right\rangle_{\boldsymbol{\theta}^{\mathsf{A}}}$ and $\left|\boldsymbol{x}^{\mathsf{B}}\right\rangle_{\boldsymbol{\theta}^{\mathsf{B}}}$ to an external agent who performs a Bell measurement and announces the result. Bob adjusts his bit to match Alice's based on the announcement. Subsequently, both parties proceed with the $\Pi^{\mathbf{BBCS}}_{\mathbf{nqs}}$ protocol (Figure 3.4) from the waiting time phase onward. A similar protocol was proposed by Zhou et al. [184] which includes error estimation to enhance security.

3.3 Conclusion

Since the proposal of quantum OT 40 years ago, active and fruitful research on this topic has deepened our understanding of the limits and advantages of quantum cryptography. It was first proved that two fundamental primitives, bit commitment and oblivious transfer, are equivalent in the quantum setting, a relation that does not hold classically. Unfortunately, it was also proven that both primitives cannot be unconditionally secure in the quantum setting, matching the impossibility results in the classical setting. However, this equivalence in the quantum setting implies that quantum OT requires weaker security assumptions than classical OT. Quantum OT can be implemented solely with quantum-hard one-way functions, whereas classical OT requires at least one-way functions with trapdoors. This makes classical OT potentially more vulnerable to quantum computer attacks and tendentiously less computationally efficient. Additionally, some quantum OT implementations benefit from an important feature known as everlasting security, which does not have a classical counterpart. It states that even if the security assumptions lose validity after the protocol execution, the security of the protocol is not compromised. In other words, quantum OT implementations are considered unconditionally secure after the protocol execution.

In this chapter, we have discussed some of the most common assumptions used to

implement secure quantum OT. Hybrid approaches are based on both quantum physical laws and computational complexity assumptions. Limited-quantum-storage approaches offer secure solutions as long as the technological limitations are met during protocol execution. In the next chapter, we will compare the security and efficiency of these quantum OT protocols with their classical counterparts.

Chapter 4

Classical and quantum oblivious transfer

Secure multiparty computation (SMC) has the potential to revolutionize data analysis and computation by enabling multiple parties to compute any function while preserving the privacy of their inputs. The security and efficiency of SMC protocols rely heavily on the security and efficiency of oblivious transfer (OT). Thus, it is crucial to understand the advantages and drawbacks of both classical and quantum OT protocols.

In this chapter, we begin by examining the security and efficiency of classical OT protocols. Then, we compare these classical protocols with their quantum counterparts. However, it is important to note that classical and quantum approaches utilize different information medium and that classical technology is more established than quantum technology. These factors raise questions about the validity of comparing the two approaches.

In Chapter 3, we reviewed various quantum OT protocols and focused on BBCS-based QOT protocols. These protocols offer a practical solution for performing OT within SMC while being resistant to quantum computer attacks. The protocols are divided into two separate phases: the oblivious key (precomputation) phase and the transfer phase. The oblivious key phase uses quantum technologies and is independent of the parties' input elements (m_0, m_1 and b), while the transfer phase only requires classical communication and is based on the precomputed elements (oblivious keys). It can be argued that the precomputation phase is not so hungry-efficient as the transfer phase, as it is independent of the parties' inputs and can be performed well ahead of an SMC execution. The classical OT protocols can also be divided into these two phases, allowing for a comparison of the transfer phase between quantum and classical approaches. Additionally, no concurrent use of quantum equipment is necessary during the SMC execution.

4.1 Classical oblivious transfer

Let us start by presenting the Bellare-Micali (BM) OT protocol [2] based on public key Diffie-Hellman. This exposition aims to shed some light on the issues related to classical OT implementations. The security and efficiency issues explored in this section also apply to most of the major classical protocols [13, 92, 93].

We consider $\mathbb{G}_q$ to be a subgroup of $\mathbb{Z}_p^*$ with generator g and order q, where p is prime and $p = 2q+1$. Also, we assume public knowledge on the value of some constant $C \in \mathbb{G}_q$. This constant guarantees that Bob follows the protocol. Also, for simplicity, we assume the protocol uses a random oracle described as a function H. For comparison purposes with quantum OT version presented in Chapter 3, we split the BM OT protocol into two phases: precomputation phase and transfer phase. The first phase sets the necessary resources to execute the oblivious transfer in the second phase. The BM OT protocol Π_{BM} is shown in Fig. 4.1.

Π_{BM} **protocol**

Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

(Precomputation phase)

1. Bob randomly generates $k \in \mathbb{Z}_q$ and computes g^k.
2. Alice randomly generates $r_0, r_1 \in \mathbb{Z}_q$ and computes g^{r_0} and g^{r_1}.

(Transfer phase)

3. Bob sets $\mathsf{pk}_b := g^k$. Also, he computes $\mathsf{pk}_{b\oplus 1} = C \cdot \mathsf{pk}_b^{-1}$.
4. Bob sends both public keys $(\mathsf{pk}_0, \mathsf{pk}_1)$ to Alice.
5. Alice checks if $(\mathsf{pk}_0, \mathsf{pk}_1)$ were correctly generated by computing their product: $C = \mathsf{pk}_0 \times \mathsf{pk}_1$.
6. Alice computes and sends to Bob the two tuples: $E_0 = (g^{r_0}, H(\mathsf{pk}_0^{r_0}) \oplus m_0)$ and $E_1 = (g^{r_1}, H(\mathsf{pk}_1^{r_1}) \oplus m_1)$ for some hash function H.
7. Bob is now able to compute $H(\mathsf{pk}_b^{r_b})$ and recover m_b.

Alice's output: $\perp$.
Bob's output: m_b.

Figure 4.1: Bellare-Micali classical OT protocol divided into two phases [2].

4.1.1 Security issues

The security of the Bellare-Micali (BM) OT protocol depends on both concealing and obliviousness properties. The concealing property is maintained as Bob does not send to Alice any information that reveals his input bit choice b. The obliviousness property, on the other hand, relies on Alice's ability to keep the randomly generated elements r_0 and r_1 confidential. This property is compromised if Bob is able to compute the discrete logarithm of g^{r_i} $(i = 0, 1)$ (discrete logarithm problem).

The hardness of the discrete logarithm problem on cyclic groups is fundamental to several other protocols, making it imperative to understand its limitations. However, it is still unproven whether there exists a polynomial-time algorithm that can compute r from g^r $(r \in \mathbb{Z}_q)$ in a general cyclic group $\mathbb{G}_q$ with generator g and order q. The security of the BM OT protocol assumes that Bob has limited computational power and is unable to calculate the discrete logarithm of a generic number.

Although the generic discrete logarithm problem is not known to be tractable in polynomial-time, there are specific cases where it is possible to compute it efficiently. Indeed, the security of the discrete logarithm problem in cyclic groups can be compromised if the structure of the group is not robust enough. For instance, if a prime p is randomly generated without ensuring that $p - 1$ contains a big prime p_b in its decomposition, it is possible to use a divide-and-conquer technique [185] along with some other methods (Shank's method [186], Pollard's rho [187], Pollard's lambda [187]) to solve the discrete logarithm problem. The efficiency of the algorithms depends on the size of p_b; the smaller p_b, the faster the algorithm can solve the discrete logarithm problem. To avoid these attacks, it is recommended to use safe primes, i.e., primes of the form $p = 2q + 1$ where q is also prime. However, finding safe primes is computationally more expensive compared to finding regular primes.

The size of the prime numbers is also an important consideration. In [188], it is reported that the number field sieve algorithm can compute the discrete logarithm in a 512-bit group after a week-long precomputation and in just one minute. So, by following this method, after a week-long computation, Bob would be able to find both messages m_0 and m_1 of the BM OT protocol in one minute. In an SMC scenario based on the Yao approach [22], where each OT performed corresponds to one input bit of Alice and the chosen group parameters are fixed, Bob would be able to get the keys corresponding to both 0 and 1 bit and, consequently, he would be able to discover all Alice's inputs. Hence, at the expense of efficiency, it is necessary to use large prime numbers (2048-bit or larger) that are resistant to these classical attacks.

We have just seen specific examples where it is possible to break the security of the OT protocol using classical techniques. However, the larger threat to the security of OT and

many other asymmetric cryptographic protocols such as RSA, elliptic-curve cryptography, and Diffie-Hellman key exchange is posed by quantum computers, which can efficiently solve the general discrete logarithm problem. This was first demonstrated by Peter Shor in his 1995 publication of a quantum algorithm that can solve both the prime factorization and discrete logarithm problems in polynomial time [28]. Therefore, in the BM OT protocol Bob would be able to perform two attacks with the help of a quantum computer:

Quantum attack 1:

1. Bob computes the discrete logarithm of $g^{r_{b\oplus 1}}$ received from Alice using Shor's algorithm, i.e. $r_{b\oplus 1} = \log_g g^{r_{b\oplus 1}}$.
2. Bob is then able to compute $H\big((g^{r_b})^k\big) = H(\mathsf{pk}_b^{r_b})$ and $H(\mathsf{pk}_{b\oplus 1}^{r_{b\oplus 1}})$ and get both messages m_b and m_{b-1}.

Quantum attack 2:

1. Bob computes the discrete logarithm of $\mathsf{pk}_{b\oplus 1}$ with the Shor's algorithm, i.e. $s = \log_g \mathsf{pk}_{b\oplus 1}$.
2. Bob is then able to compute $H\big((g^{r_b})^k\big) = H(\mathsf{pk}_b^{r_b})$ and $H\big((g^{r_{b\oplus 1}})^s\big) = H(\mathsf{pk}_{b\oplus 1}^{r_{b\oplus 1}})$ and get both messages m_b and $m_{b\oplus 1}$.

The research literature mainly presents two approaches to address this issue: developing protocols with assumptions about the computational power of quantum computers, or developing protocols that utilize quantum technology. The former approach, known as post-quantum cryptography [189], often requires more demanding public-key cryptography protocols due to the computational assumptions employed. Notably, these assumptions are yet unproven and have only been subject to a few years of scrutiny, making them vulnerable to attack in the near future. The latter approach, referred to as quantum cryptography [165], provides solutions without relying on asymmetric cryptography, but significantly increases the cost of the necessary technological equipment. It's crucial to note that quantum protocols do not experience the *intercept now, decipher later* attack (everlasting security) because their security is based on quantum theory. Conversely, this type of attack is always a possibility in protocols that rely on computational assumptions.

4.1.2 Efficiency issues

In the previous section, we highlighted that increasing security through mitigation processes always has an impact on efficiency. This is because generating secure primes is more demanding, larger exponents and prime modules result in heavier computations,

and post-quantum solutions generally require stronger computational assumptions, leading to increased computational complexity.

Now, let's examine the efficiency limitations of the BM OT protocol. To start, we will consider the operations used in the protocol, including random number generation, modular multiplication, modular inversion, modular exponentiation, hash function evaluation, and XOR operation. Of all these operations, modular exponentiation is the most demanding, meaning that the complexity of BM OT is heavily influenced by the complexity of modular exponentiation. The number of modular exponentiations performed in each phase is summarized in Table 4.1.

	Alice	Bob
Precomputation phase	2	1
Transfer phase	2	1

Table 4.1: Number of modular exponentiations in the BM protocol for each phase.

One of the most efficient ways to perform general modular exponentiation with n-bit numbers is to use a combination of square-and-multiply algorithm and Karatsuba multiplication. The former has a complexity of $\mathcal{O}(n)$ multiplications, while the latter has a complexity of $\mathcal{O}(n^{1.58})$. The overall complexity of this method is $\mathcal{O}(n^{2.58})$ n-bit operations [190]. To overestimate the rate of OT generation, we'll consider the time (in CPU cycles) required to perform all modular exponentiation operations. The rate can be calculated using the following expression:

$$\left(\frac{C_{mexp}}{C_{cycles}} \times N_{mexp}\right)^{-1} \tag{4.1}$$

where C_{mexp} is the number of CPU cycles required to perform one modular exponentiation, C_{cycles} is the CPU frequency (number of cycles per second), and N_{mexp} is the number of modular exponentiations performed in the OT implementation. It's important to note that this expression only provides an overestimation, as it depends on both the implementation of the modular exponentiation operation and the CPU frequency being used.

Given a standard CPU operating at 2.5 GHz ($C_{cycles} = 2.5 \times 10^9$ cycles per second) and an efficient implementation of modular exponentiation ($C_{mexp} \sim 400,000$ CPU cycles) [191], the BM OT protocol could perform at most ~ 1041 BM OTs per second, as shown in Fig. 4.2. This, however, is a loose overestimation of the number of OTs per second, as it only takes into account the computational complexity of modular exponentiation and

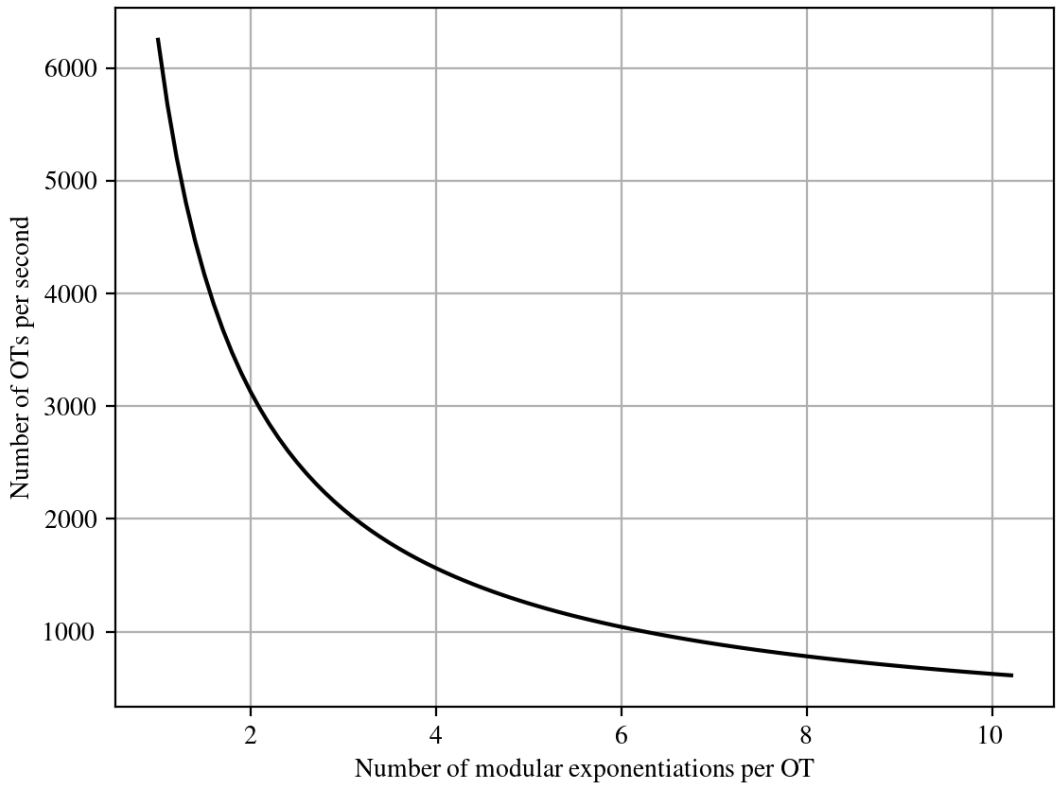


Figure 4.2: Plot of expression (4.1) on the overestimation of OT rate against the number of modular exponentiation operations required per OT.

assumes that other operations have minimal impact on computation time. Therefore, the actual OT rate must be lower.

For comparison, a study in [3] reported that it takes around 18 ms to generate a Naor-Pinkas OT [13] which requires 5 modular exponentiations, yielding a rate of 56 OTs per second. These OT rates pose a serious challenge for SMC protocols that rely on OT, such as the Yao SMC protocol [22]. The Yao protocol uses boolean circuits to compute a desired functionality privately and requires half the number of input wires as the number of required OTs. Using the rough OT rate estimation, the OT phase of the Yao protocol with a $32,000$ input boolean circuit would take at least 16 seconds, and around 2 minutes and 23 seconds using Naor-Pinkas OT rate. These execution times can become impractical in deployment environments that require several rounds of circuit evaluation and require higher OT rates.

4.1.3 OT extension protocols

To improve the efficiency of OT, one potential solution is to replace the computation-intensive asymmetric cryptography with more efficient symmetric cryptography. Symmetric cryptography has the advantage of being faster than asymmetric cryptography. In addition, all known quantum attacks to symmetric cryptography based on the Grover's algorithm only provide a quadratic advantage over classical approaches, which can be mitigated by doubling the size of the symmetric keys [189]. However, despite its efficiency, symmetric cryptography is not enough for OT because it does not meet the asymmetric cryptographic assumptions required by Impagliazzo and Rudich's result [27]. Hence, OT cannot be performed solely with symmetric cryptography methods.

To overcome the limitations imposed by Impagliazzo and Rudich's result [27] on the use of solely symmetric cryptography for OT, researchers have developed hybrid protocols that combine both symmetric and asymmetric cryptography. Beaver [192] introduced the idea of extending the number of OTs by using symmetric cryptography, once a small number of base OTs are established using asymmetric cryptography. Although Beaver's original protocol was inefficient, it paved the way for more efficient implementations [3, 193–196]. Currently, one of the most efficient protocols can generate about 10 million OTs in 2.62 seconds [3]. The security of these protocols mainly relies on the security of the base OT protocol and the use of quantum secure symmetric tools. However, it's important to note that the protocol analysed in Section 4.2.3 [3] is not secure against malicious parties and should only be used in a semi-honest environment. To ensure security against malicious parties, extra consistency check phases are necessary, increasing the complexity of the protocol [4, 196], as discussed in Section 4.2.3.

4.2 Oblivious transfer complexity analysis

In this section, we compare the complexity of the transfer phase of an optimized version of the BBCS-based QOT protocols ($\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ and $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$) presented before and several well known classical protocols. We start by explaining the optimized version..

4.2.1 Optimization

Recall that both $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ and $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ can be divided into two phases: the oblivious key distribution phase (we also call it a *precomputation* phase) and the transfer phase. It is interesting to note that both protocols follow the same steps in the transfer phase. We present the transfer phase of both protocols in Figure 4.3. We slightly rewrite the protocol by using only one hash function (H describes a random oracle) instead of two random hash functions f_0 and f_1. This is done for comparison purposes and because, in practice, H is implemented as a specific hash function, such as SHA.

$\Pi^{\mathbf{BBCS}}$ protocol

Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

Precomputation phase: Alice and Bob generate an oblivious key $(\mathsf{ok}^{\mathsf{A}}, (\mathsf{ok}^{\mathsf{B}}, \mathsf{e}^{\mathsf{B}}))$ according to the corresponding procedure. $\Pi^{\mathbf{BBCS}}_{\mathcal{F}_{\mathbf{COM}}}$ as in Figure 3.2 and $\Pi^{\mathbf{BBCS}}_{\mathbf{bqs}}$ as in Figure 3.3.

Transfer phase:

5. Bob defines $I_0 = \{i : \mathsf{e}^{\mathsf{B}}_i = 0\}$ and $I_1 = \{i : \mathsf{e}^{\mathsf{B}}_i = 1\}$ and sends the pair $(I_b, I_{b\oplus 1})$ to Alice.
6. Alice computes the pair of strings (s_0, s_1) as $s_i = m_i \oplus H(\mathsf{ok}^{\mathsf{A}}_{I_{b\oplus i}})$ and sends to Bob.
7. Bob computes $m_b = s_b \oplus H(\mathsf{ok}^{\mathsf{B}}_{I_0})$.

Alice's output: $\perp$.
Bob's output: m_b.

Figure 4.3: Transfer phase of BBCS-based QOT protocols in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model and bounded-quantum-storage model.

In the first communication round of the protocol in Figure 4.3, Bob sends two sets $(I_b, I_{b\oplus 1})$ to Alice (Step 5). This can be optimized by only sending one set (I_b), as Alice

can determine its complement ($\overline{I_b} = I_{b\oplus 1}$) with just one set. This leads to the optimized protocol ($\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$) shown in Figure 4.4. This optimization results in lower bandwidth requirements compared to the original transfer phase.

The size of the sets can be identified by a symmetric security parameter κ, as they define the keys (ok_{I_i}, $i = 0, 1$) used in the hash function H. For comparison, we consider $\kappa = 128$. Also, the messages m_0 and m_1 can be viewed as garbled circuit keys, with a size of $l = 128, 192$ or 256. If we assume $l \sim \kappa$, the same number of bits are required. This means that in Step 5, Bob only needs to send l bits to Alice, resulting in a reduction of one fourth in the number of bits sent during the transfer phase.

$\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol

Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

Precomputation phase: Alice and Bob generate an oblivious key $(\mathsf{ok}^{\mathsf{A}}, (\mathsf{ok}^{\mathsf{B}}, \mathsf{e}^{\mathsf{B}}))$ according to the corresponding procedure. $\Pi_{\mathcal{F}_{\mathbf{COM}}}^{\mathbf{BBCS}}$ as in Figure 3.2 and $\Pi_{\mathbf{bqs}}^{\mathbf{BBCS}}$ as in Figure 3.3.

Transfer phase:

5. Bob defines $I_0 = \{i : \mathsf{e}_i^{\mathsf{B}} = 0\}$ and $I_1 = \{i : \mathsf{e}_i^{\mathsf{B}} = 1\}$ and **sends only** I_b to Alice.
6. Alice computes the pair of strings (s_0, s_1) as $s_i = m_i \oplus H(\mathsf{ok}^{\mathsf{A}}_{I_{b\oplus i}})$ and sends to Bob.
7. Bob computes $m_b = s_b \oplus H(\mathsf{ok}^{\mathsf{B}}_{I_0})$.

Alice's output: $\perp$.
Bob's output: m_b.

Figure 4.4: Transfer phase of BBCS-based QOT protocols in the $\mathcal{F}_{\mathbf{COM}}$−hybrid model and bounded-quantum-storage model.

To fairly compare the transfer phase of the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol with other classical protocols, we divide classical protocols into precomputation and transfer phases. All steps that are independent of the messages (m_0 and m_1) and the bit choice (b) are considered part of the precomputation phase, while others are included in the transfer phase. The transfer phase is more important to optimize as it is executed during the Yao GC protocol, while the precomputation phase can be performed beforehand.

We stress we will only compare the complexity of different protocols' transfer phase

because their precomputation phase rely on different technologies. Since quantum technologies are still in their infancy and constantly evolving, it is difficult to compare the efficiency with classical approaches. However, the oblivious key phase of the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol has a linear time complexity in all its security parameters, as shown by Lemus et al. [138]. The time complexity of $\Pi_{\mathcal{F}_{\mathbf{COM}}}^{\mathbf{BBCS}}$ is $\mathcal{O}(\kappa(2l+t))$, where κ is the security parameter of the hash-based commitments, $2l$ is the number of qubits used to generate the oblivious keys, and t is the number of testing qubits.

4.2.2 Classical OT

In section 4.1, we divided the well known Bellare-Micali protocol in these two phases and we observed that it uses three exponentiations during the transfer phase. In Table 4.2, we present the number of required modular exponentiations and communication rounds during the transfer phase of four well known classical protocols that have their security based on the computational hardness of the Discrete Logarithm problem.

Protocol	Exponentiation	Comm. rounds
EGL [92]	3	2
BM [2]	3	2
NP [13]	2	2
SimpleOT [93]	1	2

Table 4.2: Number of modular exponentiation operations and communication rounds executed during the transfer phase of four classical protocols.

From Table 4.2, we see that the most efficient protocol (SimpleOT [93]) still requires one exponentiation operation and 2 communication rounds. From the above formula (4.1) and setting $C_{mcyles} = 2.5 \times 10^9$, $C_{mexp} = 400\,000$ and $N_{mexp} = 1$, we get an overestimation of around 6000 OT per second. Comparing with the rate achieved by OT extension protocols (10 million OT in 2.62 s), it is still very inefficient.

This means current classical OT protocols have a computational complexity limited by $\mathcal{O}(n^{2.58})$ bit operations due to modular exponentiation. The $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol only depends on simple bit operations (XOR, truncation and comparison), meaning its computational complexity is linear in the length of the messages $\mathcal{O}(n)$.

Despite their security guarantees, none of the classical OT protocols discussed are secure against quantum computer attacks. To achieve this level of security, post-quantum approaches must be adopted, which may result in higher computational demands [197].

For example, the use of Kyber key encapsulation based on the module learning with errors (M-LWE) problem [198] in a LAN network results in a rate of only 41 OT per second (24 ms per OT), as reported in [199]. This rate is even lower than the rate of 56 OT per second achieved by the Naor-Pinkas protocol [13] reported in [3]. The NTRU post-quantum encryption system [200] was used in [201, 202] to develop a 1-out-of-n OT, which was compared with the SimpleOT protocol [93]. Although the individual sides are more efficient in NTRU OT, the overall protocol is still less efficient, with a rate of 728 OT per second (1.372 ms per OT) for the highest security level compared to 1375 OT per second (0.727 ms per OT) using SimpleOT. It is important to note that these protocols are still vulnerable to *intercept now, decipher later* attacks as they rely on computational assumptions that are only believed to be secure against quantum computer attacks and not proven.

4.2.3 OT extension

As we explained in section 4.1.3, several techniques based on an hybrid symmetric-asymmetric approach were developed as a way to increase the OT execution rate. These techniques use a small number κ (= 128) of base OT protocols (e.g. EGL, BM, NP, SimpleOT) and extend this resource to m (= 10 000 000) OT executions, where $m >> \kappa$.

Again, to fairly compare $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ with OT extension protocols, we divide them into precomputation and transfer phases. In this section, we compare the communication and computational complexity of m executions of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ to one execution of an OT extension protocol, as the latter generates a predetermined number (m) of OTs. We compare $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ with the semi-honest ALSZ13 protocol and then with the maliciously secure KOS15 protocol.

ALSZ13 comparison

Let's consider the OT extension protocol proposed in [3] (ALSZ13), as illustrated in Figure 4.5. At the time of writing, this protocol reports the fastest implementation with 10 million OTs generated in just 2.68 seconds. The ALSZ13 protocol is divided into two phases: an initial OT phase and an OT extension phase. For comparison purposes, we will focus solely on the second phase, which aligns with our distinction between the precomputation and transfer phases.

In Tables 4.3 and 4.4, we present a comparison of the computational and communication complexity of the OT extension protocol (ALSZ13) and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$. In Table 4.3, PRG refers to a pseudorandom generator, κ is the number of base OTs executed during the precomputation phase of the OT extension, m represents the total number of OTs, and l

ALSZ13 OT extensions protocol [3]

Alice's input: m pairs (x_j^0, x_j^1), $\forall j \in [m]$ of l-bit strings.
Bob's input: m selection bits $\boldsymbol{r} = (r_1, ..., r_m)$.

Initial OT phase (Precomputation phase)

1. Alice randomly generates a string $\boldsymbol{s} = (s_1, ..., s_\kappa)$.
2. Bob randomly chooses κ pairs of κ-bit strings $\{(\boldsymbol{k}_i^0, \boldsymbol{k}_i^1)\}_{i=1}^{\kappa}$.
3. Bob and Alice execute κ base OTs, where Alice plays the role of the receiver with input $\boldsymbol{k}$ and Bob plays the role of the sender with messages $(\boldsymbol{k}_i^0, \boldsymbol{k}_i^1)$, $\forall i \in [\kappa]$.

OT extension phase (Transfer phase)

4. Bob applies a pseudorandom number generator G to $\boldsymbol{k}_i^0$, i.e. $\boldsymbol{t}^i = G(\boldsymbol{k}_i^0)$. Computes $\boldsymbol{u}^i = \boldsymbol{t}^i \oplus G(\boldsymbol{k}_i^1) \oplus \boldsymbol{r}$ and sends $\boldsymbol{u}^i$ to Alice for every $i \in [\kappa]$.
5. Alice computes $\boldsymbol{q}^i = (s_i \cdot \boldsymbol{u}^i) \oplus G(\boldsymbol{k}_i^{s_i})$.
6. Alice sends (y_j^0, y_j^1) for every $j \in [m]$, where $y_j^0 = x_j^0 \oplus H(j, \boldsymbol{q}_j)$, $y_j^1 = x_j^1 \oplus H(j, \boldsymbol{q}_j \oplus \boldsymbol{s})$ and $\boldsymbol{q}_j$ is the j-th row of the matrix $Q = [\boldsymbol{q}^1|...|\boldsymbol{q}^\kappa]$. Note that, in practice, it is required to transpose Q to access its j-th row.
7. Bob computes $x_j^{r_j} = y_j^{r_j} \oplus H(j, \boldsymbol{t}_j)$.

Alice's output: $\perp$.
Bob's output: $(x_1^{r_1}, ..., x_m^{r_m})$.

Figure 4.5: Precomputation and transfer phases of OT extensions protocol presented in [3].

is the length of the OT strings. It is assumed that $l \sim \kappa$ have a similar magnitude, as the key length used in the garbled circuits is $l = 128, 192$, or 256, while $\kappa = 128$ [3]. Now, we justify the analysis presented in Tables 4.3 and 4.4.

Regarding the ALSZ13 protocol, for every $i \in [\kappa]$, Bob computes two PRGs in step 4 and Alice computes one PRG in step 5. This accounts for 3κ PRG executions. For every $j \in [m]$, Alice computes two hash functions in step 6 and Bob computes one hash function. This accounts for $3m$ hash functions. For every $i \in [\kappa]$, Bob computes two $m-$bit XOR operations in step 4 and Alice computes one $m-$bit XOR operation. For every $j \in [m]$, Alice computes two $l-$bit XOR operations in step 6 and Bob computes one $l-$bit XOR operation in step 7. Also, for every $j \in [m]$, Alice computes one $\kappa-$bit XOR operation in step 6. This accounts for $3m\kappa + 3ml + m\kappa$ bitwise XOR operations. For every $i \in [\kappa]$, Alice computes one $m-$bit AND operation in step 5. Finally, Alice

has to perform a matrix inversion which accounts for around $m \log m$ bit operations. The communication complexity is given by the following elements: Bob sends an $m-$bit vector for every $i \in [\kappa]$ and Alice sends two $l-$bit messages for every $j \in [m]$. This accounts for $2ml + m\kappa$ bits sent.

Regarding the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol, for every execution of the protocol, Alice computes two hash functions in step 6 and Bob computes one hash function in step 7. This accounts for $3m$ hash functions. Also, Alice computes two $l-$bit XOR operations in step 6 and Bob computes one $l-$bit XOR operation in step 7. This accounts for $3ml$ bitwise XOR operations. For every execution of the protocol, Alice performs 2κ bitwise comparisons in step 5. Also, Alice computes two $\kappa-$bit truncation in step 6 and Bob computes one $\kappa-$bit truncation in step 7. The communication complexity is given by the following elements: Bob sends a $\kappa-$bit vector and Alice sends two $l-$bit messages, for every execution of the protocol. This accounts for $2ml + m\kappa$ bits sent.

Operation	ALSZ13	$\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$
PRG (AES)	3κ	-
Hash (SHA-1)	$3m$	$3m$
Bitwise XOR	$3m\kappa + 3ml + m\kappa$	$3ml$
Bitwise AND	$m\kappa$	-
Matrix transposition	$m \log m$	-
Bitwise comparison	-	$2m\kappa$
Bitwise truncation	-	$3m\kappa$

Table 4.3: Computational complexity comparison between ALSZ13 [3] OT extension protocol and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol from section 4.2.1.

	ALSZ13	$\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$
Bits sent	$2ml + m\kappa$	$2ml + m\kappa$

Table 4.4: Communication complexity comparison between ALSZ13 [3] OT extension protocol and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol from section 4.2.1.

The communication complexity is exactly the same in both protocols: $\sim 3ml$. So, the OT extension protocol does not have any advantage over $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ during the communication phase. Regarding their computational complexity, we have to compare the binary operations executed by each protocol.

Firstly, we can see that $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ transfer phase is asymptotically more efficient than ALSZ13 OT extension transfer phase. The computational complexity of OT extension is not linear in the number of OT executions, $\mathcal{O}(m \log m)$, whereas it is linear in the case of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$, $\mathcal{O}(m)$. Now, let us compare the binary operations between each protocol. Denote by $B_{\text{op}}^{\text{ALSZ13}}$ and $B_{\text{op}}^{\text{BBCS}}$ the number of binary operations executed by ALSZ13 and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$, respectively. As both protocols execute $3m$ hash functions, we do not take into account their execution. Also, assuming that $\kappa \sim l$, $B_{\text{op}}^{\text{ALSZ13}}$ is roughly given by,

$$\begin{aligned} B_{\text{op}}^{\text{ALSZ13}} &= 3\kappa + 3m\kappa + 3ml + m\kappa + m\kappa + m \log m \\ &= 8m\kappa + 3\kappa + m \log m \end{aligned}$$

and $B_{\text{op}}^{\text{BBCS}} = 8m\kappa$. Here, we simplify and assume that 3κ PRGs executions consume only 3κ bit operations. Therefore, ALSZ13 has more $B_{\text{op}}^{\text{ALSZ13}} - B_{\text{op}}^{\text{BBCS}} \geq m \log m$ binary operations than the transfer phase of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol.

From the results of our comparison, we can conclude that the transfer phase of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is competitive with the corresponding phase of the semi-honest ALSZ13 protocol, and has the potential to be even more efficient. Furthermore, the performance of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ transfer phase is achieved while providing stronger security guarantees. Unlike the ALSZ13 protocol, which relies on computational assumptions of the base OT, $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ has been proven secure against quantum computers. Moreover, while ALSZ13 is a semi-honest protocol (assumes well-behaved parties that follow the protocol), $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol is secure against any corrupted party. To obtain a fair comparison, it is appropriate to consider OT extension protocols that are secure against malicious parties. The work developed in [193] presented the first protocol in the malicious scenario, which was latter optimised by KOS15 [4] and ALSZ15 [196]. Both optimizations carry out one run of the semi-honest OT extension presented in ALSZ13 plus some consistency checks. The protocol presented in [4] adds to ALSZ13 a *check correlation* phase after the transfer phase and the protocol presented in [196] adds a *consistency check* phase during the transfer phase. This means that both malicious protocols' transfer phases have greater computational and communication complexity when compared with ALSZ13. Therefore, we can infer that the transfer phase of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ has lower computational and communication complexity than its malicious classical equivalents. In the next step, we compare the KOS15 protocol [4] with $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$.

KOS15 comparison

KOS15 protocol is very similar to ALSZ13, but it includes an additional phase called *check correlation* phase. This phase ensures that the receiver is well behaved and does

Operation	KOS15	$\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$
Hash (SHA-1)	$3m$	$3m$
Bitwise XOR	$3m\kappa + 3ml + m\kappa$	$3ml$
Bitwise AND	$m\kappa$	-
Matrix transposition	$m \log m$	-
Bitwise comparison	-	$2ml$
Bitwise truncation	-	$3ml$
κ-bit additon	$3(m + (\kappa + w))\kappa$	-
κ-bit mult	$2(m + (\kappa + w))\kappa^{1.58}$	-

Table 4.5: Computational complexity comparison between KOS15 [4] OT extension protocol and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol from section 4.2.1.

not cheat. In Figure 4.6, it is presented the KOS15 protocol that generates m l-bit string OT out of κ base OT, with computational security given by κ and statistical security given by w. Note that, in Figure 4.6, we join all the subprotocols presented in the original paper: $\prod_{\text{COT}_\text{E}}^{\kappa,m'}$, $\prod_{\text{ROT}}^{\kappa,m}$ and $\prod_{\text{DEROT}}^{\kappa,m}$. Also, they identify $\mathbb{Z}_2^\kappa$ with the finite field $\mathbb{Z}_{2^\kappa}$ and use "$\cdot$" for multiplication in $\mathbb{Z}_{2^\kappa}$. For example, the element $\boldsymbol{t}_j$ in $\sum_{j=1}^{m'} \boldsymbol{t}_j \cdot \chi_j$ (Figure 4.6, step 10) should be considered in $\mathbb{Z}_{2^\kappa}$.

	KOS15	$\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$
Bits sent	$2ml + m\kappa + \kappa$	$2ml + m\kappa$

Table 4.6: Communication complexity comparison between KOS15 [4] OT extension protocol and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol from section 4.2.1.

The KOS15 protocol, like $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ and ALSZ13, begins with a precomputation phase that can be performed prior to the actual OT computation. However, the KOS15 paper [4] originally carried out the computation of PRGs G during the OT extension phase. These 3κ computations of G can actually be done during the precomputation phase as they are independent of the input elements. The main difference between KOS15 and ALSZ13 lies in steps $9 - 11$, the check correlation phase. In this phase, both parties utilize a random oracle functionality $\mathcal{F}_{\text{Rand}}(\mathbb{F}_{2^\kappa}^{m'})$ to obtain equal random values. Bob then performs twice m' κ-bit sums, m' κ-bit multiplications and sends 2κ bits (x and t) to Alice, who in turn performs m' κ-bit sums and m' κ-bit multiplications. For the purpose of simplicity, we assume that each κ-bit sum takes κ bit operations, and multiplication takes $\kappa^{1.585}$,

KOS15 OT extensions protocol [4]

Alice's input: m pairs (x_j^0, x_j^1), $\forall j \in [m]$ of l-bit strings.
Bob's input: m selection bits $\boldsymbol{r} = (r_1, ..., r_m)$.

Initial OT phase (Precomputation phase)

1. Alice randomly generates a string $\boldsymbol{s} = (s_1, ..., s_\kappa)$ and Bob randomly chooses κ pairs of κ-bit strings $\{(\boldsymbol{k}_i^0, \boldsymbol{k}_i^1)\}_{i=1}^{\kappa}$.
2. Bob and Alice execute κ base OTs. Alice plays the role of the receiver with input $\boldsymbol{s}$ and Bob plays the role of the sender with messages $(\boldsymbol{k}_i^0, \boldsymbol{k}_i^1)$, $i \in [\kappa]$.
3. Bob applies a pseudorandom number generator G to $\boldsymbol{k}_i^0$ and $\boldsymbol{k}_i^1$: $\boldsymbol{t}^i = G(\boldsymbol{k}_i^0)$ and $\boldsymbol{t}_1^i = G(\boldsymbol{k}_i^1)$. Also, set $\boldsymbol{T}^i = \boldsymbol{t}^i \oplus \boldsymbol{t}_1^i$.
4. Alice applies G to $\boldsymbol{k}_i^{s_i}$ and sets $\boldsymbol{g}_i^{s_i} = G(\boldsymbol{k}_i^{s_i})$.

OT extension phase (Transfer phase)
Extend

5. Bob generates random elements r_j, for $r \in [m+1, m']$ and resize $\boldsymbol{r} = (r_1, ..., r_m, r_{m+1}, ..., r_{m'})$, where $m' = m + (\kappa + w)$.
6. Bob computes $\boldsymbol{u}^i = \boldsymbol{T}^i \oplus \boldsymbol{r}$ and sends $\boldsymbol{u}^i$ to Alice for every $i \in [\kappa]$.
7. Alice computes $\boldsymbol{q}^i = (s_i \times \boldsymbol{u}^i) \oplus \boldsymbol{g}_i^{s_i}$ for every $i \in [\kappa]$.

Check correlation

8. Sample $(\chi_1, ..., \chi_{m'}) \leftarrow \mathcal{F}_{\text{Rand}}(\mathbb{F}_{2^\kappa}^{m'})$.
9. Bob computes $x = \sum_{j=1}^{m'} r_j \cdot \chi_j$ and $t = \sum_{j=1}^{m'} \boldsymbol{t}_j \cdot \chi_j$, where $\boldsymbol{t}_j$ is the j-th row of the matrix $[\boldsymbol{t}^1|...|\boldsymbol{t}^\kappa]$ and sends these to Alice.
10. Alice computes $q = \sum_{j=1}^{m'} \boldsymbol{q}_j \cdot \chi_j$, where $\boldsymbol{q}_j$ is the j-th row of the matrix $Q = [\boldsymbol{q}^1|...|\boldsymbol{q}^\kappa]$, and checks that $t = q + r \cdot \boldsymbol{s}$. If the check fails, output ABORT, otherwise continue.

Randomize and encrypt

11. Alice sends (y_j^0, y_j^1) for every $j \in [m]$, where $y_j^0 = x_j^0 \oplus H(j, \boldsymbol{q}_j)$, $y_j^1 = x_j^1 \oplus H(j, \boldsymbol{q}_j \oplus \boldsymbol{s})$.
12. Bob computes $x_j^{r_j} = y_j^{r_j} \oplus H(j, \boldsymbol{t}_j)$.

Alice's output: $\perp$.
Bob's output: $(x_1^{r_1}, ..., x_m^{r_m})$.

Figure 4.6: Precomputation and transfer phases of OT extensions protocol presented in [4].

using the Karatsuba method for multiplication with $O(\kappa^{1.585})$ complexity and schoolbook addition with $O(\kappa)$ complexity.

Let us compare the binary operations between KOS15 and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ as we did with the ALSZ13 protocol. Denote by $B_{\text{op}}^{\text{KOS15}}$ and $B_{\text{op}}^{\text{BBCS}}$ the number of binary operations executed by KOS15 and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$, respectively. Again, without taking into account the execution of $3m$ hash functions and assuming that $\kappa \sim l$, $B_{\text{op}}^{\text{KOS15}}$ is roughly given by,

$$\begin{aligned} B_{\text{op}}^{\text{KOS15}} &= 3m\kappa + 3ml + m\kappa \\ &+ m\kappa + m\log m \\ &+ 3(m + (\kappa + w))\kappa \\ &+ 2(m + (\kappa + w))\kappa^{1.58} \\ &= 11m\kappa + m\log m \\ &+ 3\kappa^2 + 3w\kappa \\ &+ 2m\kappa^{1.58} + 2\kappa^{2.58} + 2w\kappa^{1.58} \end{aligned}$$

and $B_{\text{op}}^{\text{BBCS}} = 8m\kappa$. Therefore, KOS15 has more $B_{\text{op}}^{\text{KOS15}} - B_{\text{op}}^{\text{BBCS}} \geq 5m\kappa + m\log m$ binary operations than $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ transfer phase. For this estimation, note that we are considering the lower bound $2m\kappa$ instead of $2m\kappa^{1.58}$ and we are not taking into account the implementation of the random oracle $\mathcal{F}_{\text{Rand}}(\mathbb{F}_{2^\kappa}^{m'})$, which would add an extra cost linear in the number of OT executions.

Regarding the communication complexity, the number of bits sent during both KOS15 and $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is almost the same. KOS15 only adds κ bits to the communication during the check correlation phase. However, since this overhead is independent of m (number of OTs executed) its effect is amortized for big m.

4.3 Conclusion

The security and efficiency of OT implementations is crucial for secure computations, especially in the context of Yao's garbled circuit protocol. While classical OT protocols rely on asymmetric cryptographic primitives, which are known to be vulnerable to quantum attacks or have security based on conjectures, several works [37, 38, 41, 138] have used the laws of physics to prove the security of BBCS-based QOT protocols against malicious adversaries with access to quantum computers. Additionally, using oblivious keys can separate the quantum technological burden from the execution of OT and enable efficient implementation.

In this chapter, we compared the transfer phase of an optimized version ($\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$) of the BBCS-based QOT protocol with the transfer phase of the currently fastest imple-

mentation of OT (ALSZ13). Our results showed that the transfer phase of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ has the potential to be faster than the ALSZ13 protocol while offering higher security. In addition to being secure against quantum computer attacks, BBCS-based QOT protocols are also secure in the malicious setting, whereas ALSZ13 is only secure in the semi-honest model. Furthermore, our analysis revealed that the transfer phases of current maliciously secure implementations (ALSZ15 and KOS15) have a higher computation and communication complexity than $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$. In the next chapter, we compare the performance of a secure multiparty computation system based on both classical OT and BBCS-based QOT protocols.

Chapter 5

Private phylogenetic trees

Several privacy-enhancing technologies (PETs), such as differential privacy [20], homomorphic encryption [21], and secure multiparty computation, have been applied in the field of biomedical data analysis, including genomic data analysis [203–207]. To advance the state of the art, there have been competitions [208] aimed at developing faster and more secure solutions in genomic analysis. Recent surveys [209, 210] have discussed the role of PETs in various computational domains within the genomic field, including genomic aggregation, GWASs and statistical analysis, sequence comparison, and genetic testing. However, these surveys do not cover the application of privacy-preserving methods to phylogeny inference.

In contrast to classical technologies, the use of quantum cryptographic technologies in private computation has been limited. Only a few works have explored their integration, such as Chan et al.'s development of quantum-assisted real-world private database queries [211] and the suggestion by Ito et al. [212] that quantum OT is suitable for secure multiparty computation. However, quantum cryptographic technologies have matured to a level where their integration with privacy-enhancing technologies is possible. Technologies such as quantum key distribution (QKD) and quantum random number generators (QRNG) are being commercialized for critical applications, such as governmental data storage and communications, and have seen in-field deployment (e.g., OpenQKD, https://openqkd.eu/). The quantum oblivious key distribution (QOKD) protocol, which leverages the same technology as QKD and QRNG, benefits from its development and provides the necessary resources to perform OT [129, 137, 138].

In this chapter, we present a feasible modular private phylogenetic tree protocol that leverages quantum communications. It provides enhanced security against quantum computer attacks and decreases the complexity of the computation phase when compared to a state-of-the-art classical-only system. The system is built on top of Libscapi [213] implementation of Yao protocol and PHYLIP phylogeny package [185]. It integrates three

crucial quantum primitives: quantum oblivious transfer, quantum key distribution and quantum random number generator.

This chapter follows a top-down approach. In Section 5.1, we start by explaining the concept of phylogenetic trees and the distance-based algorithms used to generate these trees. In Section 5.2, we set down the security definitions that will be used to analyse and prove the system's security. In Section 5.3 and 5.4, we describe the quantum cryptographic tools and the software tools that are integrated into the protocol, respectively. In Section 5.5, we describe the proposed SMC system for phylogenetic trees. In Section 5.6 we explain how the quantum cryptographic tools are integrated into the system. Section 5.7 is devoted to the theoretical security analysis of the protocol and in Section 5.8 we perform a complexity analysis. In the last Section we present a performance comparison of the system between a classical-only and a quantum-assisted implementation.

5.1 Phylogenetic trees

Phylogenetic trees are diagrams that depict the evolutionary ties between groups of organisms [214] and are composed of several nodes and branches. The nodes represent genome sequences and each branch connects two nodes. It is important to note that the terminal nodes (also called leaves) represent known data sequences, whether internal nodes are ancestral sequences inferred from the known sequences [215, 216]. The length of the branches connecting two nodes represents the number of substitutions that have occurred between them. However, this quantity must be estimated because it cannot be computed directly using the sequences. In fact, by simply counting the number of sites where two nodes have different base elements (Hamming distance), we underestimate the number of substitutions that have occurred between them.

The best way to compute a correct phylogenetic tree depends on the type of species and sequences under analysis and the assumptions made by the sequences substitution model. By a correct tree, we mean a tree that depicts as approximate as possible the real phylogeny of the sequences, i.e. the real ties between known sequences and inferred ancestors. These assumptions lead to different algorithms which can be divided into two categories:

1. Distance-based methods: they base their analysis on the evolutionary distance matrix which contains the evolutionary distances between every pair of sequences. The evolutionary distance used also depends on the substitution model considered. These methods are computationally less expensive when compared to character-based methods;

2. Character-based methods: they base their analysis on comparing every site (character) of the known data sequences and do not reduce the comparison of sequences to a single value (evolutionary distance).

We only consider the distance-based algorithms that are part of the PHYLIP [217] distance matrix models, namely: Fitch-Margoliash (`fitch` and `kitsch`), Neighbour Joining (`neighbor`) and UPGMA (`neighbor`). Also, we only consider the evolutionary distances developed in PHYLIP `dnadist` program: Jukes-Cantor (JC) [218], Kimura 2-parameter (K2P) [219], F84 [220] and LogDet [221]. For readers interested in learning more about phylogenetic analysis, we recommend the textbooks by Ziheng [215] and Felsenstein [216].

Next, we give an overview of these distance-based methods to build some intuition on how to tailor them to a private setting. We start by looking at the different evolutionary distances and then at the distance-based algorithms.

5.1.1 Evolutionary distances

The evolutionary distance depends on the number of estimated substitutions between two sequences, which is governed by the substitution model used. So, before defining a suitable distance, it is important to have a model that describes the substitution probability of each nucleotide across the sequence at a given time.

The distances considered in this work can be divided into two groups by their assumptions. JC, K2P and F84 assume that the substitution probabilities remain constant throughout the tree, (i.e. stationary probabilities), whether the LogDet distance assumes that the probabilities are not stationary.

Also, the first three evolutionary distances (JC, K2P and F84) assume an evolutionary model that can be described by a *time-homogeneous stationary Markov* process. This Markov process is based on a probability matrix $\mathbf{P}(t)$ that defines the transition probabilities from one state to the other after a certain time period t. It can be shown [222] that this probability is given by

$$\mathbf{P}(t) = e^{\mathbf{Q}t} \tag{5.1}$$

where the rate matrix $\mathbf{Q}$ is of the form given by (5.2).

$$\mathbf{Q} = \begin{pmatrix} -\mu(a\pi_C + b\pi_G + c\pi_T) & a\mu\pi_C & b\mu\pi_G & c\mu\pi_T \\ g\mu\pi_A & -\mu(g\pi_A + d\pi_G + c\pi_T) & d\mu\pi_G & e\mu\pi_T \\ h\mu\pi_A & i\mu\pi_C & -\mu(h\pi_A + j\pi_C + f\pi_T) & f\mu\pi_T \\ j\mu\pi_A & k\mu\pi_C & l\mu\pi_G & -\mu(i\pi_A + k\pi_C + l\pi_G) \end{pmatrix} \tag{5.2}$$

In $\mathbf{Q}$, each entry $\mathbf{Q}_{ij}$ represents the substitution rate from nucleotide i to j and both its columns and rows follow the order A, C, G, T. μ is the total number of substitutions per unit time and we can define the evolutionary distance, d, to be given by $d = \mu t$. The parameters $a, b, c, ..., l$ represent the relative rate of each nucleotide substitution to any other. Finally, $\pi_A, \pi_C, \pi_G, \pi_T$ describe the frequency of each nucleotide in the sequences.

From Expression (5.1), it is possible to define a likelihood function on the distance d and use the maximum likelihood approach to get an estimation of the evolutionary distance. The likelihood function defines the probability of observing two particular sequences, x and y, given the distance d:

$$L(d) = \prod_{i=1}^{n} \pi_{x_i} P_{x_i,y_i}\Big(\frac{d}{\mu}\Big).$$

The parameters of $\mathbf{Q}$ are defined differently depending on the evolutionary model used and the maximum likelihood solution leads to different evolutionary distances.

Jukes-Cantor

The Jukes-Cantor model [218] is the simplest possible model based on $\mathbf{Q}$ as given in (5.2). It assumes the frequencies of the nucleotide to be the same, i.e. $\pi_A = \pi_C = \pi_G = \pi_T = \frac{1}{4}$ and sets the relative rates $a = b = ... = l = 1$. This model renders an evolutionary distance between two sequences x and y given by:

$$d_{xy} = -\frac{3}{4} \ln\left(1 - \frac{4}{3}\frac{h_{xy}}{n}\right) \tag{5.3}$$

where h_{xy} is the uncorrected hamming distance and n the length of the sequences.

Kimura 2-parameter

This model [219] distinguishes between two different nucleotide mutations:

1. Type I (transition): $A \leftrightarrow G$, i.e. from purine to purine, or $C \leftrightarrow T$, i.e. from pyrimidine to pyrimidine.

2. Type II (transversion): from purine to pyrimidine or vice versa.

These two different types of transformation lead to different probability distributions denoted by P and Q, where P is the probability of homologous sites showing a type I difference, while Q is that of these sites showing a type II difference. So, the Kimura [219] metric between x and y is given by the following:

$$d_{xy} = -\frac{1}{2}\ln\left((1-2P-Q)\sqrt{1-2Q}\right) \tag{5.4}$$

where $P = \frac{n_1}{n}$, $Q = \frac{n_2}{n}$ and n_1 and n_2 are respectively the number of sites for which two sequences differ from each other with respect to type I ("transition" type) and type II ("transversion" type) substitutions.

F84

This model [220] also distinguishes different nucleotide transitions but do not assume the nucleotide frequencies to be the same. This leads to a more general distance which can be estimated in closed form:

$$d_{xy} = -2A\ln\left(1-\frac{P}{2A}-\frac{(A-B)Q}{2AC}\right) + 2(A-B-C)\ln\left(1-\frac{Q}{2C}\right) \tag{5.5}$$

where $A = \frac{\pi_C\pi_T}{\pi_Y} + \frac{\pi_A\pi_G}{\pi_R}$, $B = \pi_C\pi_T + \pi_A\pi_G$ and $C = \pi_R\pi_Y$ for $\pi_Y = \pi_C + \pi_T$ and $\pi_R = \pi_A + \pi_G$, and P and Q are defined as in the Kimura 2-parameter model above.

Although more complex models can be considered with different combinations of parameters in $\mathbf{Q}$, not all of them produce a distance function that can be estimated in closed form.

LogDet

As mentioned before, the models based on matrix $\mathbf{Q}$ assume that the probability matrix $\mathbf{P}(t)$ is stationary, i.e. remains constant throughout the tree. However, there are evolutionary scenarios where this assumption does not give a correct description of reality. The LogDet evolutionary distance [221] suits a wider set of models and considers the case where $\mathbf{P}(t)$ is different at each branch in the tree. This is given by

$$d_{xy} = -\frac{1}{4}\ln\left(\frac{\det F_{xy}}{\sqrt{\det\prod_x\prod_y}}\right) \tag{5.6}$$

where the divergence matrix F_{xy} is a 4×4 matrix such that the $ij-$th entry gives the proportion of sites with nucleotide i in sequence x and j in sequence y. Also, $\prod_x$ and

$\prod_y$ are diagonal matrices where its $i-$th component correspond to the proportion of i nucleotide in the sequence x and y, respectively.

5.1.2 Distance-based algorithms

All distance-based methods make use of evolutionary distances to compare different genomic sequences. Although it may lead to less accurate phylogenetic trees, these methods are highly popular among researchers who have to handle large number of sequences. All methods assume the following:

1. The evolutionary distance computed between each pair is independent of all other sequences;

2. The estimated distance between each pair of sequences is given by the sum of the size of the branches that connect both of them.

These algorithms are thus divided into two phase:

1. Distance computation phase: all the pairwise evolutionary distances are computed according to the selected model. This step is common to all distance-based methods;

2. Iterative clustering: aggregate the sequences in clusters iteratively. This step is specific to each method.

Let us briefly describe three of the most common distance-based methods [215].

UPGMA

The Unweighted Pair Group Method with Arithmetic mean (UPGMA) method produces a rooted phylogenetic tree and assumes the data to be ultrametric, i.e. assumes that

$$d_{xy} \leq \max(d_{xz}, d_{yz})$$

for sequences x, y and z. These two assumptions imply that all the sequences are equidistant to the inferred root sequence.

It starts by considering every sequence as a single-valued cluster. Then, it goes on merging the clusters according to the smallest difference between them and recomputes the distance matrix through a simple average of distances. In summary, we have the following steps:

1. Merge clusters, $C_i = \{c_i\}$ and $C_j = \{c_j\}$ for sets c_i and c_j, with the smallest distance present in the distance matrix, i.e. $d_{i,j} \leq d_{k,l} \, \forall k, l$. Create a new cluster $C_{i/j} = \{\{c_i, c_j\}\}$. This new cluster represents a branch between clusters C_i and C_j;

2. Recompute the distance matrix according to the following formula:

$$d_{i/j,l} = \frac{d_{i,l} + d_{j,l}}{2}$$

for all other clusters l;

3. Eliminate clusters C_i and C_j from the distance matrix and add cluster $C_{i/j}$ with the distances computed as in the previous step;

4. Repeat steps $1 - 3$ until there is only one cluster left.

Neighbour-Joining

As we have seen, the UPGMA joins the clusters with the minimum distance between them. Now, the Neighbour-Joining method considers not only how close two clusters are, but it also considers how far these two clusters are from the others. Thus, the clusters to be merged should minimize the following quantity:

$$q(C_i, C_j) = (r-2)d(C_i, C_j) - u(C_i) - u(C_j)$$

where r is the number of clusters in the current iteration and $u(C_i) = \sum_j d(C_i, C_j)$.

As opposed to the UPGMA algorithm, this method produces an unrooted tree and it can be summarised in the following steps:

1. Consider every sequence as a single-valued cluster and connect it to a central point;

2. Compute a matrix $\mathcal{Q}$ where its entries are given by the quantity above, i.e. $\mathcal{Q}_{ij} = q(C_i, C_j)$;

3. Identify clusters C_i and C_j with the smallest value in the matrix $\mathcal{Q}$. Create a new node $C_{i,j}$ and join both clusters C_i and C_j to it;

4. Assign to the branch $C_i C_{i/j}$ a distance given by:

$$\frac{1}{2}d(C_i, C_j) - \frac{1}{2}\frac{(u_i - u_j)}{r-2}$$

and to the branch $C_j C_{i/j}$ a distance given by:

$$\frac{1}{2}d(C_i, C_j) - \frac{1}{2}\frac{(u_j - u_i)}{r-2};$$

5. Eliminate clusters C_i and C_j from the distance matrix and add cluster $C_{i/j}$ with the distances to the other clusters computed as follows:

$$d(C_l, C_{i/j}) = \frac{1}{2}(d(C_l, C_i) + d(C_l, C_j) - d(C_i, C_j))$$

for all other nodes C_l;

6. Repeat steps $2 - 5$ until there is only one cluster left.

Fitch-Margoliash

This method renders an unrooted tree and also assumes that the distances are additive. It analyses iteratively three-leaf trees and computes the distance between three known nodes and one created internal node. This is based on the following observation. Given three clusters C_i, C_j and C_l, and one internal node a that is connected to all these three clusters, the distances between the clusters are given by:

$$\begin{aligned} d(C_i, C_j) &= d(C_i, a) + d(a, C_j) \\ d(C_i, C_l) &= d(C_i, a) + d(a, C_l) \\ d(C_l, C_j) &= d(C_l, a) + d(a, C_j), \end{aligned}$$

from which we can easily see that

$$\begin{aligned} d(a, C_i) &= \tfrac{1}{2}\Big(d(C_i, C_j) + d(C_i, C_l) - d(C_l, C_j)\Big) \\ d(a, C_j) &= \tfrac{1}{2}\Big(d(C_i, C_j) + d(C_l, C_j) - d(C_i, C_l)\Big) \\ d(a, C_l) &= \tfrac{1}{2}\Big(d(C_i, C_l) + d(C_l, C_j) - d(C_i, C_j)\Big). \end{aligned} \tag{5.7}$$

Thus, we can estimate the distances from the known clusters to the new internal node using the distances between the clusters as given in (5.7). Based on this, the Fitch-Margoliash algorithm goes as follows:

1. Consider every sequence as a single-valued cluster;

2. Identify the two clusters, C_i and C_j, with the smallest distance in the distance matrix;

3. Consider all the other clusters as a single cluster C_l and recompute the distance matrix with just three clusters. The distances between the identified clusters and the new cluster is given by an average value of the distances between the identified clusters and the elements inside the cluster C_l, i.e.

$$d(C_i, C_l) = \frac{1}{|C_l|} \sum_{c \in C_l} d(C_i, c)$$

and similarly for C_j;

4. Using expressions (5.7), we compute the distances from the three clusters and the central node;

5. Merge clusters, C_i and C_j, into a new one $C_{i/j}$ and recompute the distance matrix between $C_{i/j}$ and all the other clusters $c \in C_l$ by a simple average expression:

$$d(c, C_{i/j}) = \frac{d(c, C_i) + d(c, C_j)}{2};$$

6. Repeat steps $2 - 4$ until there is only one cluster left.

All these methods output a tree with some topology, $\mathcal{T}$ along with the distances between the branches.

5.2 Security definition

In this chapter, we consider a multiparty computation scenario that is secure against *semi-honest* parties. This means that all the parties strictly follow the protocol but can use their inputs, received messages and outputs to deduce any additional information. As such, these are also commonly called *honest-but-curious* parties. Nevertheless, we can extend the protocol to the malicious setting, by simply implementing a two-party secure computation protocol that is secure against malicious adversaries [74]. Our security will follow the simulation paradigm and we start with the definition of security in a multi-party setting. The formal definition is taken from [74] and it requires the following elements:

- $\mathcal{F}$ denotes the ideal functionality to be computed in the SMC session, i.e. $\mathcal{F} : \mathcal{X}^n \to \mathcal{Y}^n$ where n is the number of parties participating in the SMC and $\mathcal{X}$ and $\mathcal{Y}$ are the input and output space of each party, respectively. $X^i \in \mathcal{X}$ and $Y^i \in \mathcal{Y}$ denote the sets of input and output of party P^i, respectively. Also, for short, $X = (X^1, ..., X^n)$ and $Y = (Y^1, ..., Y^n)$;

- π denotes the protocol that implements the ideal functionality $\mathcal{F}$;
- C is the set of corrupted parties;
- $\texttt{view}_\pi^i(X) := (X^i, r^i; m_1^i, ..., m_t^i)$. This tuple is called the view of party P^i and it contains its inputs (X^i), its random-tape value (r^i) and the messages m_j^i received during the SMC execution;
- $\texttt{output}_\pi(X) = (\texttt{output}_\pi^1(X), ..., \texttt{output}_\pi^n(X))$, where $\texttt{output}_\pi^i(X)$ is the output of party i computed from its view $\texttt{view}_\pi^i(X)$;
- S is a probabilistic polynomial-time simulator in the ideal-world;
- The distribution on inputs X given by a real-world execution of the protocol π:

$$\texttt{Real}_\pi(C; X) := \big\{\{\texttt{view}_\pi^i(X) : i \in C\}, \texttt{output}_\pi(X)\big\}_X.$$

- The distribution on inputs X given by the ideal-world simulation of the parties' view:

$$\texttt{Ideal}_{S,\mathcal{F}}(C; X) := \big\{S\big(\{(X^i, \mathcal{F}(X^i)) : i \in C\}\big), \mathcal{F}(X)\big\}_X.$$

Definition 8 (Semi-honest security)**.** *A protocol securely realizes $\mathcal{F}$ in the presence of semi-honest adversaries if there exists a simulator S such that, for every subset of corrupted parties C and all inputs X, we have*

$$\texttt{Real}_\pi(C; X) \stackrel{c}{\equiv} \texttt{Ideal}_{S,\mathcal{F}}(C; X), \tag{5.8}$$

where $\stackrel{c}{\equiv}$ denotes computational indistinguishability.

This definition conveys the notion that whatever can be computed by a party during the execution of the protocol is only based on his inputs and outputs, i.e. the execution of the protocol do not provide any further information. This is equivalent to expression (5.8), which states that the distribution of the view and outputs in a real-world execution is computationally indistinguishable from the distribution generated by a simulator and the functionality output. It is also worth noting that, as it is proved in [223], for deterministic $\mathcal{F}$ we have that definition III.1 is equivalent to the simpler case where the `Real` and `Ideal` distributions do not take into account the output of the real protocol execution and the output of the functionality, respectively, i.e.

$$\texttt{Real}_\pi(C; X) = \{\texttt{view}_\pi^i(X) : i \in C\}_X$$

and

$$\texttt{Ideal}_{S,\mathcal{F}}(C;X) = \big\{S\big(\{(X^i,\mathcal{F}(X^i)) : i \in C\}\big)\big\}_X.$$

Therefore, we just need to build a simulator that satisfies expression (5.8) for the $\texttt{Real}_\pi(C;X)$ and $\texttt{Ideal}_{S,\mathcal{F}}(C;X)$ given as above in order to prove security.

5.2.1 Distance matrix functionality

For our private phylogenetic tree problem, the ideal functionality $\mathcal{F}$ outputs the distance matrix according to the selected evolution model (Jukes-Cantor, Kimura 2-parameter, F84 or LogDet). We denote by $\texttt{DM}_d$, $d \in \{\text{JC}, \text{K2P}, \text{F84}, \text{LD}\}$ such a functionality. Note that this functionality is deterministic and, as we pointed before, we just have to prove expression (5.8) to hold for the simpler definition of `Real` and `Ideal`.

The protocol that privately computes the distance matrix $\texttt{DM}_d$ is built up by many invocations of a two-party distance functionality, denoted by $\texttt{D}_d$ for $d \in \{\text{JC}, \text{K2P}, \text{F84}, \text{LD}\}$. Consequently, we can reduce the the security of $\texttt{DM}_d$ to that of $\texttt{D}_d$ and use the composition theorem proved in [224] to prove $\texttt{DM}_d$ security.

Before presenting the composition theorem, we provide some informal definitions. We have that an *oracle-aided* protocol using the *oracle-functionality* f is a protocol where the parties can interact with an oracle which outputs to each party according to f. Also, when an oracle-aided protocol privately computes some g in the sense of (5.8) using the oracle-functionality f, we say that it *privately reduces* g to f. For a more detailed discussion on this topic, we refer the interested reader to [224]. The composition theorem for the semi-honest model can therefore be stated as follows:

Theorem 2. *(Composition theorem) Suppose that g is privately reducible to f and that there exists a protocol for privately computing f. Then, there exists a protocol for privately computing g.*

In other words, there exists a private protocol of g when the oracle-functionality f is substituted by its real private protocol in the corresponding oracle-aided protocol g.

5.3 Quantum tools

In this section, we present three quantum primitives used in the private computation of phylogenetic trees, rendering a full quantum-proof solution.

5.3.1 Quantum oblivious key distribution

We explored the concept of oblivious keys in chapter 3. To generate these oblivious keys, we saw that we can follow the prepare-and-measure quantum approach developed by Bennet [6] along with some commitment functionality. As an example, Lemus et al. [138] proposed to use the Halevi and Micali classical bit commitments based on universal and cryptographic hashing [225]. Thus, its security is based on the laws of physics and on the fact that there is no significant quantum speed-up in finding collisions on the hash-based bit commitments [138, 226, 227]. Also, as discussed in [50, 138], this protocol has an important security feature: it is resistant against *intercept now, decipher later* attacks. It this chapter, we call quantum oblivious key distribution (QOKD) the subprotocol of BBCS-based QOT protocols that comprises all the phases excluding the transfer phase. For illustration, the QOKD protocol in the bounded-quantum-storage model, $\Pi_{\mathbf{bqs}}^{\mathbf{QOKD}}$, is summarised in Figure 5.1.

5.3.2 Quantum random number generator

A random number generator (RNG) is another very important tool in the realm of secure multiparty computation. The SMC security can be compromised and the parties' privacy can be broken if the RNG used is predictable. An attack of this kind was reported in [3] where the authors exploited the Java weak random number generator used in v0.1.1 FastGC [228]. This attack allowed them to disclose the inputs of both parties in an SMC scenario. It also highlights the fact that it is not possible to use any kind of RNG for cryptographic purposes.

In the case of cryptographically secure pseudorandom number generators (CSRNG), it is crucial that it provides both forward and backward security. The former means that an attacker should not be able to predict the next generated number even when he knows all the generated sequence. The latter means that an attacker should not be able to predict all the generated sequence from a small set of generated elements. These two properties are not present in common RNGs. For example, linear congruential generators do not fit for cryptographic tasks since they can be easily predicted as reported in [229]. Also, Krawczk found that a large class of general congruential generators do not provide forward security even for obscured parameters [230]. So, in order to produce some CSRNG, instead of using linear operations, the research community decided to rely on the computational intractability of computing the discrete logarithm. Both [231] and [232] use modular exponentiation as an intermediate step in order to generate some pseudorandom bit. As mentioned above, all the cryptographic protocols with their security based on the discrete logarithm problem are threatened by quantum computers and these CSRNG

$\Pi^{\mathbf{QOKD}}_{\mathbf{bqs}}$ protocol

Parameters: n, security parameter.
Alice's input: $(m_0, m_1) \in \{0,1\}^l$ (two messages).
Bob's input: $b \in \{0,1\}$ (bit choice).

BB84 phase:

1. Alice generates random bits $\boldsymbol{x}^{\mathsf{A}} \leftarrow_{\$} \{0,1\}^n$ and random bases $\boldsymbol{\theta}^{\mathsf{A}} \leftarrow_{\$} \{+,\times\}^n$. Sends the state $\left|\boldsymbol{x}^{\mathsf{A}}\right\rangle_{\boldsymbol{\theta}^{\mathsf{A}}}$ to Bob.
2. Bob randomly chooses bases $\boldsymbol{\theta}^{\mathsf{B}} \leftarrow_{\$} \{+,\times\}^n$ to measure the received qubits. We denote by $\boldsymbol{x}^{\mathsf{B}}$ his output bits.

Waiting time phase:

3. Both parties wait time Δt.

Oblivious key phase:

4. Alice reveals to Bob the bases $\boldsymbol{\theta}^{\mathsf{A}}$ used during the *BB84 phase* and sets his oblivious key to $\mathsf{ok}^{\mathsf{A}} := \boldsymbol{x}^{\mathsf{A}}$.
5. Bob computes $\mathsf{e}^{\mathsf{B}} = \boldsymbol{\theta}^{\mathsf{B}} \oplus \boldsymbol{\theta}^{\mathsf{A}}$ and sets $\mathsf{ok}^{\mathsf{B}} := \boldsymbol{x}^{\mathsf{B}}$.

Alice's output: ok^{A}.
Bob's output: $(\mathsf{ok}^{\mathsf{B}}, \mathsf{e}^{\mathsf{B}})$.

Figure 5.1: QOKD protocol in the bounded-quantum-storage model.

protocols are not an exception. Besides this technique, one could use either AES or DES as cryptographically random generator.

Although these techniques are used to provide unpredictability and backward secrecy, all the randomness relies on some initial seed. This seed is used because all the process is based on deterministic algorithms. So, a pseudo RNG can be viewed as a randomness extractor from some initial random value. For this reason, it is crucial to use an initial random value that is as close as possible to a truly random value. This can be generated from different sources and usually, the best randomness comes from physical devices (e.g. atomic decay [233] or thermal noise [234]). So, a potentially good source of true RNG comes from natural phenomena where some part of the system is used as the source of entropy. In the case of classical natural phenomena, the entropy is frequently taken from some unknown or chaotic subsystem which can ultimately be described by a deterministic

theory. In this case, the unpredictability drawn from the system's entropy comes from our lack of knowledge and inability to fully grasp the underlying complex natural mechanisms. Also, some classical phenomena (e.g. mouse pointers) may not have enough entropy to generate good quality random numbers. However, quantum natural phenomena have their roots in quantum mechanics which is intrinsically related to probability theory. For this reason, quantum systems can be potential sources of entropy even assuming complete knowledge of the system. This comes from the fact that, in quantum mechanics, we only have access to the probability distribution of the system's state and we can only know it after measuring it [235].

Within the scope of SMC, the generation of the circuit's wire keys must be guaranteed to be unpredictable and efficient. All these features can be achieved with a quantum RNG (QRNG) [236].

5.3.3 Quantum key distribution

As we will explain in the last section, part of the communication between the parties should be kept encrypted. Message encryption is commonly achieved with symmetric cryptographic tools, such as AES (Advanced Encryption Scheme) or the perfect cypher one-time pad. These symmetric tools are used to encrypt the communication content through a common key assumed to be only known by both communicating parties. However, the techniques used to distribute a common key cannot be realized using just symmetric cryptography and one needs asymmetric cryptography. Unfortunately, most of the commonly used techniques in asymmetric cryptography (RSA, Elliptic Curves or Diffie-Hellman) rely on computational assumptions that can be broken by a quantum computer through the already mentioned Shor's algorithm [28].

So, to render a quantum-resistant privacy-preserving solution, we make use of quantum key distribution (QKD) protocol to share symmetric keys to be used along with symmetric cryptography [165, 237–239]. Its security relies on the laws of quantum physics and it is proven to be resistant against computationally unbounded adversaries [79, 240]. This level of security comes from one very important quantum property known as the No-Cloning theorem. This property ensures that it is not possible to measure a quantum state without introducing a measurable perturbation in the system. Thus, both parties enrolling in the QKD protocol will be able to detect a potential eavesdropper in case some adversary tries to intercept and read the quantum signals.

5.4 Software tools

Next, we present the open-source tools used to implement the system presented in the subsequent sections.

5.4.1 CBMC-GC

The CBMC-GC compiler [241] is used in step 1) of Yao protocol to generate the boolean circuit representation of the desired function. It translates C-like code into boolean circuits based on a model checking tool called CBMC and it optimizes circuits for size and depth [242, 243]. HyCC [12] is also a potential candidate for this step as it builds upon CBMC-GC. However, it aims to build circuits for hybrid SMC protocols in which our system is not based.

5.4.2 Libscapi

The Libscapi library [213] implements several important cryptographic primitives for two-party and multi-party protocols. It is extensively used to implement steps $2-5$ of the Yao protocol in the repository MPC-Benchmark [244]. This implementation has integrated one of the most efficient OT extension protocols [245] along with the base OTs proposed by Chou and Orlandi [246].

5.4.3 PHYLIP

The PHYLIP package [217] is a C++ open-source project that provides a set of programs to infer phylogenies. Among other programs, it implements distance-based methods (UPGMA, Neighbour-Joining, Fitch-Margoliash) and computes the evolutionary distances described previously in Section 5.1.1 (JK, K2P, F84, LD). Due to its modularity, we integrate PHYLIP distance methods with Yao protocol for evolutionary distances assisted with quantum technologies.

5.5 Secure multiparty computation of phylogenetic trees

The proposed system allows to securely compute a suite of algorithms that perform phylogeny analysis through the computation of phylogenetic trees. Based on the modular nature of distance-based algorithms, the system combines different evolution models with

different phylogenetic algorithms. In this section, we describe how to integrate the tools presented in previews sections 5.3-5.4 to develop this modular private system.

5.5.1 Functionality definition

As already mentioned in Section 5.1, all distance-based methods are divided into two phases: distance matrix computation and distance matrix processing. Apart from the metric used, the first phase is similar among all methods whereas the second phase is specific to each one while depending only on the distance matrix. Therefore, each phase corresponds to a particular functionality that can be formalized as follows:

- Functionality `DM`: receives some distance metric $d \in \{\text{JC}, \text{K2P}, \text{F84}, \text{LD}\}$ and all input sequences, and outputs a matrix with the pairwise distances between every sequence, i.e.

$$\texttt{DM}(d; s_1, ..., s_m) = \begin{pmatrix} 0 & d_{1,2} & \cdots & d_{1,m} \\ d_{2,1} & 0 & & d_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m,1} & d_{m,2} & \cdots & 0 \end{pmatrix}$$

  where $d_{i,j} = d(s_i, s_j)$ for short.

- Functionality `A`: receives a distance matrix M and an algorithm type

$$a \in \{\text{UPGMA}, \text{NJ}, \text{FM}\},$$

  and outputs the structure of the tree in newick tree format, i.e.

$$\texttt{A}(M, a) = (\text{subtree}_1 : l_1, \text{subtree}_2 : l_2),$$

  where each l_1 and l_2 denotes the distance to its parent node, subtree is built up by other subtrees and the leaves are given by $(\text{subtree}_{k-1} : l_{k-1}, s_{i_k} : l_k)$. For consistency, leaves are also considered as a subtrees. Note that this representation is not unique, e.g. $(s_1 : 0.7, (s_2 : 0.3, s_3 : 0.5) : 0.5)$ and $((s_3 : 0.5, s_2 : 0.3) : 0.5, s_1 : 0.7)$ represent the same rooted tree depicted in Figure 5.2.

  Therefore, if we consider the equivalence relation, $\sim$, given by

$$(\text{subtree}_1 : l_1, \text{subtree}_2 : l_2) \sim (\text{subtree}_2 : l_2, \text{subtree}_1 : l_1),$$

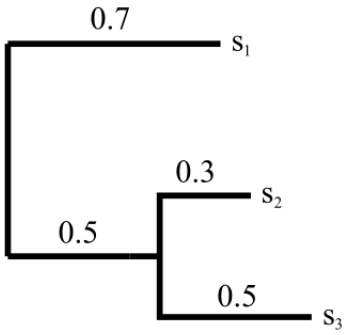


Figure 5.2: Example of rooted phylogenetic tree.

we have that the quotient set of the trees by $\sim$ satisfy the uniqueness property from an evolutionary point of view.

For simplicity, denote by A_d^a the private protocol that implements sequentially both functionalities described above, i.e. $A_d^a(s_1, ..., s_m) = \texttt{A}(\texttt{DM}(d; s_1, ..., s_m), a)$. This leads to twelve possible combinations of algorithms A_d^a for $d \in \{\text{JC}, \text{K2P}, \text{F84}, \text{LD}\}$ and $a \in \{\text{UPGMA}, \text{NJ}, \text{FM}\}$.

5.5.2 Private protocol

During the distance matrix computation phase (`DM`) of the private A_d^a, each party has to compute the distance between his sequences and the other parties' sequences privately, i.e. without revealing his sequences to the other parties. Since this corresponds to several instances of a two-party secure computation, we make use of the Yao protocol described in Section 2.2.1. This means that each party has to generate the boolean circuit representation of the elected distance d, which is accomplished by the CBMC-GC software tool before the beginning of the protocol. In Section 5.7.1, we analyse how to generate these circuits.

Now, since the Yao protocol is executed only between two different parties P^i and P^j for $i, j \in [n]$, the other participating parties P^t, $t \in [n] \setminus \{i, j\}$, do not have access to the distances computed between theses two parties. For this reason, P^t has to receive the result of the Yao protocol execution from both P^j and P^i. After this, each party outputs the distance matrix that is used as the input of PHYLIP programs: `fitch`, `kitsch` or `neighbor`.

In the second phase of the protocol (`A`), the parties do not need to communicate because this phase only depends on the quantities computed during the first phase. For this reason, this phase is executed internally by each party, who then compute the phylogenetic tree. This phase is carried out by the PHYLIP programs mentioned in the previous paragraph.

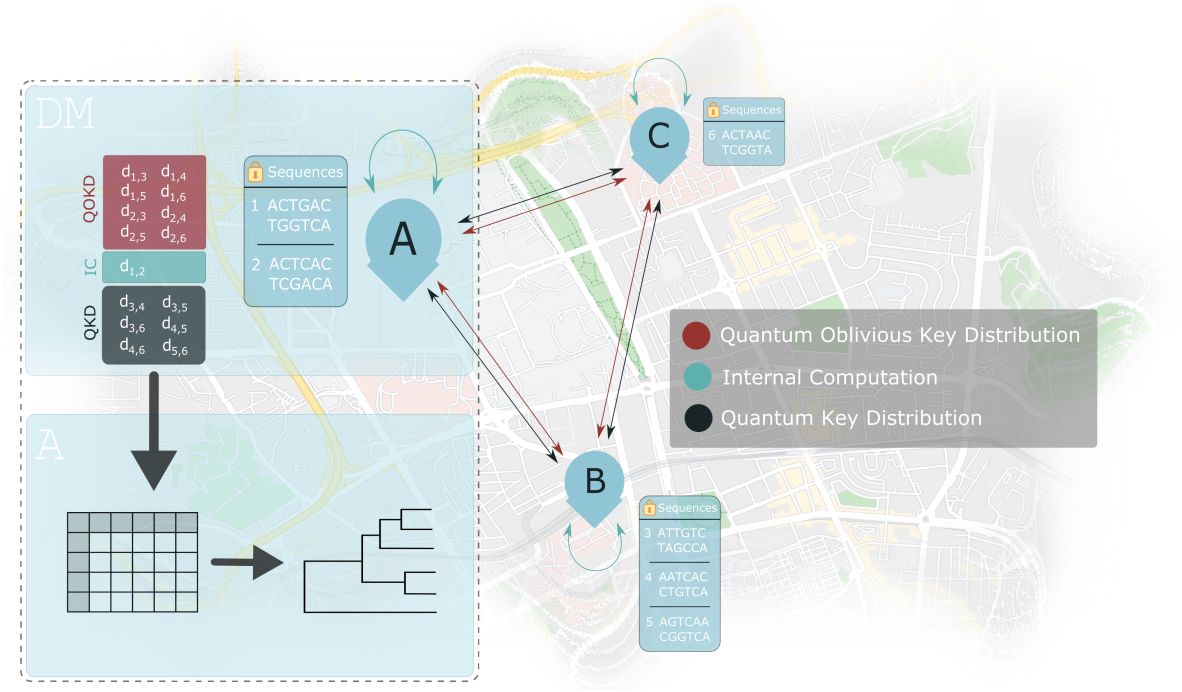


Figure 5.3: Overview of the A_d^a network structure.

These two phases are shown in Figure 5.3 and we give more details about the protocol assisted with quantum technologies in the next section.

5.5.3 Quantum private protocol

Let us specify the private A_d^a protocol with the quantum cryptographic tools. Following the scenario depicted in Figure 5.3, we define $S_i = \{s_{i,1}, ..., s_{i,l}\}$ to be the set of sequences owned by party P^i. Also, we denote by $d_{(i,l),(j,k)}$ the distance between the l-th sequence of party P^i and the k-th sequence of party P^j, i.e. $d_{(i,l),(j,k)} = d(s_{i,l}, s_{j,k})$.

As briefly described before, the private A_d^a protocol has two phases. The first phase requires different types of interactions between the parties to compute the desired distance matrix and the second phase is computed internally. Since the second phase is carried out internally, there is no need for communication between the parties. Therefore, the quantum cryptographic tools will only be used during the first private phase. In summary, each pair of parties require two quantum channels as depicted in Figure 5.3: one to generate oblivious keys for oblivious transfer and the other to generate symmetric keys for encryption.

Consider the case where P_t has to compute the distance matrix entry corresponding to distance $d_{(i,l),(j,k)}$. Depending on whether P_t owns both sequences, one of the sequences or none of the sequences $(s_{(i,l)}, s_{(j,k)})$, P_t proceed as follows:

1. If $i = j = t$ (i.e. both sequences are owned by P_t), $d_{(i,l),(j,k)}$ is computed internally

by P_t (blue arrow in Figure 5.3);

2. If $i = t$ and $j \neq t$ (i.e. one of the sequences is owned by P_t), $d_{(i,l),(j,k)}$ is computed privately with Yao protocol assisted with QOKD system (red arrow in Figure 5.3);

3. If $i \neq t$ and $j \neq t$ (i.e. none of the sequences is owned by P_t), both parties P_i and P_j (or just party P_i in case $i = j$) must send to P_t the distance $d_{(i,j),(k,l)}$ encrypted with the symmetric key generated through the QKD system (black arrow in Figure 5.3).

5.6 Quantum technologies integration

Now, let us see the role of quantum technologies in this private system and its integration with quantum networks.

5.6.1 Quantum oblivious transfer

Libscapi implementation of Yao protocol combines a very efficient base OT protocol with one of the fastest OT extension protocols. It uses the base OT (SimpleOT) proposed by Chou and Orlandi [93] integrated with the OT Extension (KOS15 [4]) presented in chapter 4. In this setting, the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol can be implemented in two different ways depending on the number of oblivious keys generated between the two parties: as a base OT protocol integrated within OT extension protocol or as a stand-alone method substituting all Libscapi OT implementation. If the number of oblivious keys generated is scarce compared to the number of OT required, then one should integrate $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ in the OT extension. Otherwise, one could directly use the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol. A scheme of the integration of the quantum oblivious key distribution (QOKD) system is depicted in Figure 5.4.

It is important to note that the base OTs executed during the pre-computation phase of the OT extension have the parties' roles reversed. This means that the OT extension sender is the base OT receiver and vice-versa. This should be taken into consideration in case the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is integrated within OT extension because $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is not symmetric in the sense that the apparatus used by the sender is different from that of the receiver. However, since it is known that OT is symmetric, we can use the reduction proposed in [247] without having to swap the quantum technological material.

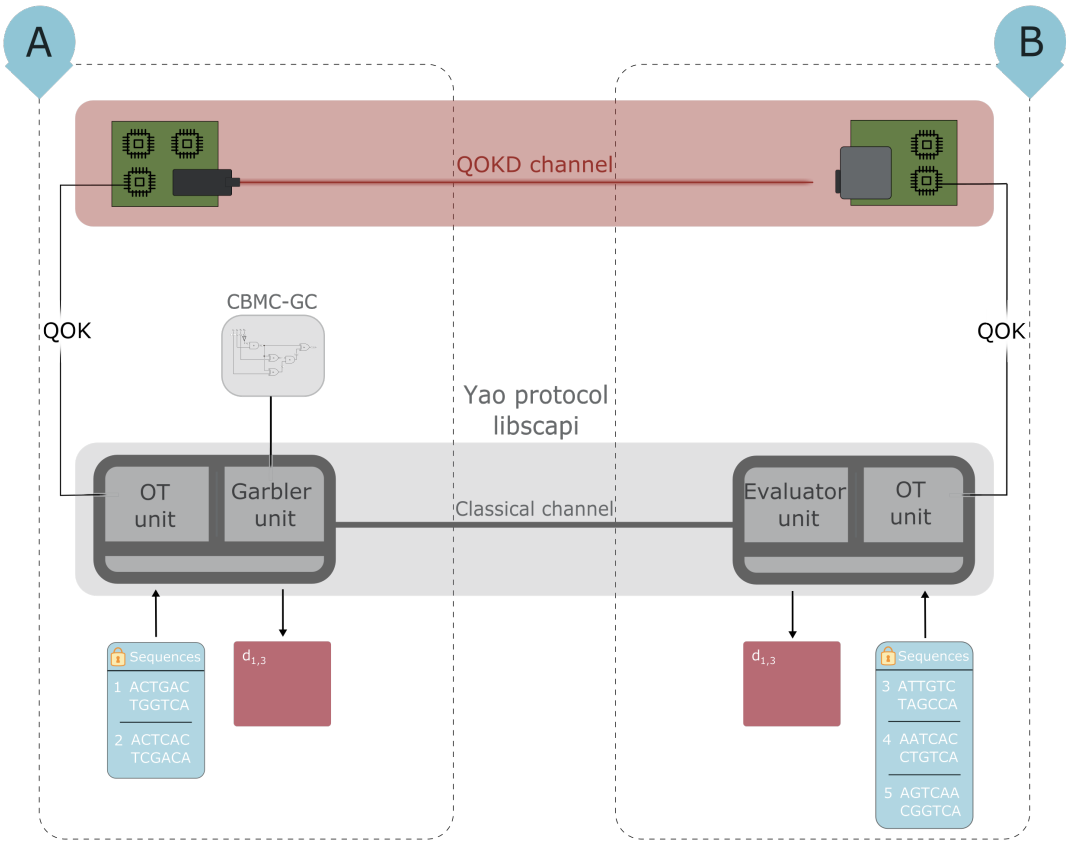


Figure 5.4: Overview of the integration of the QOKD service and the CBMC-GC tool in the Yao protocol.

5.6.2 Quantum random number generation

As previously described, the Yao protocol needs to generate random numbers for the keys in the *Wire encryption* step. This is crucial for the security of the protocol because its predictability allows deducing the parties' input as reported in [3].

Libscapi implementation makes use of OpenSSL library function `RAND_bytes` to randomly generate a seed from which it computes new numbers. In this private system, we substitute this function to a call of QRNG.

5.6.3 Quantum key distribution

The QKD system allows the parties to receive the distance elements of the sequences they do not own, while preserving the security of the system. We use the keys generated by the QKD system along with the perfect cipher: one-time pad.

5.6.4 Quantum network integration

Technological equipment

Both QKD and QOKD protocols rely on the same physical processes. They can both be realized either with continuous or discrete variables [138, 165, 238, 248]. Also, the technological equipment used by the receiver (Bob) and transmitter (Alice) is the same in both quantum services (QKD and QOKD). As for the case of the prepare-and-measure setting, the first quantum step is the same in both protocols: Alice randomly sends quantum states in two different bases and Bob measures these states on random bases. The difference relies on the classical post-processing phase. So, we can conclude that both services share the same technological equipment (fibre, receiver and transmitter). Moreover, as proposed by Pinto et al. [249] in a similar setting, both QKD and QOKD services can coexist with classical signals in the same fibre.

Network topology

The quantum private protocol explained above in Section 5.5.3 assumes that every two parties have a direct quantum channel between them that is used to generate oblivious keys and symmetric keys, i.e. a fully connected quantum network. This approach follows from the fact that the first QKD and quantum OT (QOT) protocols were based on prepare-and-measure techniques [5, 6]. However, as discussed in chapter 3, there are also protocols that implement device-independent QOT (DI-QOT) [172, 180] (under some constraints) and DI-QKD [165]. In addition to the advantages from a security point of view, these DI protocols can also be implemented within a star-structured quantum network having an untrusted party as the middle point. This increases the implementation flexibility of the proposed quantum private protocol of phylogenetic trees (Section 5.5.3).

As analysed by Joshi et al. [250], existing networks fall into three possible types: trusted node networks, actively switched and fully connected quantum networks based on entanglement sharing and wavelength multiplexing. Using the two types of protocols just mentioned (prepare-and-measure and device-independent), it is possible to implement our proposed system in all three existing quantum network implementation types.

Moreover, Kumaresann et al. [251] analyses possible SMC infrastructure topologies that can be created based on a set of OT channels shared between some pairs of parties in the network. They developed "secure protocols that allow additional pairs of parties to establish secure OT correlations using the help of other parties in the network in the presence of a dishonest majority" (Abstract, [251]). Since they work in the information-theoretical setting, there is no security loss in combining Kumaresann protocol with quantum approaches. This integration increases the range of configurations allowed. However,

further efficiency analysis has to be done to understand the impact of this approach in practice.

5.7 System security

In this section, we analyse the security of the proposed system. We start by describing the methods used to privately compute the distance between two sequences and then we prove the security of the private protocol proposed in Section 5.5.3 which implements the functionality described in Section 5.5.1.

5.7.1 Private computation of distances

The private computation of the distance between sequences is an important building block in the security of the system. We have that the privacy of the sequences directly relies on this step. Here, we go through the methods used to compute the distances used by the PHYLIP program: Jukes-Cantor, Kimura 2-parameter, F84 and LogDet.

A common building block to all these four distance metrics is the computation of the Hamming distance between two sequences x and y, h_{xy}. We start by looking at an adapted divide-and-conquer way to compute the Hamming distance between two sequences and then we see how to apply it to the private computation of distance metrics.

Hamming distance

We are interested in the boolean representation of the Hamming distance and, as mentioned above, we use the CBMC-GC tool to translate ANSI-C code into this representation. Usually, to compute the Hamming distance between two binary strings, x and y, we start by applying the XOR operation, $z = x \oplus y$. Then, we just have to count the number of 1's in z. This operation is commonly known as population count or popcount(z) for short. So, the binary Hamming distance is given by $h_{xy} = \text{popcount}(x \oplus y)$.

We use an adapted divide-and-conquer technique for the computation of popcount(z) [252]. Originally, this divide-and-conquer technique starts by dividing the sequence into 2-bit blocks and then counts the number of 1's inside each 2-bit block. After that, it allocates the result of each block in a new 2-bit block. Then, we can sum the values inside these 2-bit blocks iteratively.

We follow the approach described above but we have to tailor it for the computation of the Hamming distance between two four-based sequences (A, C, G, T). Since we are using a boolean circuit representation, the nucleotide sequences must be represented in binary. So, by convention, we use the following 2-bit encoding: $A = 00$, $C = 01$, $G = 10$

and $T = 11$. If we follow directly the approach described above, we would have that the Hamming distance between the single-valued sequences "A" and "C" is smaller than the single-valued sequence between "A" and "T":

$$\begin{aligned} d_H(A, C) &= \text{popcount}(00 \oplus 01) \\ &= \text{popcount}(01) = 1, \end{aligned}$$

$$\begin{aligned} d_H(A, T) &= \text{popcount}(00 \oplus 11) \\ &= \text{popcount}(11) = 2. \end{aligned}$$

This issue comes from the fact that we are counting the number of 1's inside every 2-bit blocks. Instead, we are just interested in knowing if there is at least one element 1 inside each 2-bit block because it indicates that the bases at that site are different. Therefore, before counting the number of 1's in the `XOR`ed sequence, we apply an `OR` operation to the bits inside every 2-bit blocks. We call this operation $\text{popcount}^t(z)$. For simplicity, hereafter we denote by h_{xy} the tailored Hamming distance between sequences x and y. Now, we have that the tailored Hamming distance between "A" and "T" gives the desired result:

$$\begin{aligned} d_H(A, T) &= \text{popcount}^t(00 \oplus 11) \\ &= \text{popcount}^t(11) \\ &= \text{popcount}\big(\texttt{OR}(1, 1)\big) = 1. \end{aligned}$$

In Figure 5.5, we show an example on how to compute the Hamming distance between two-valued sequences "AG" and "GC".

Jukes-Cantor

As described in Section 5.1.1, the Jukes-Cantor distance between two sequences is given by:

$$d_{xy} = -\frac{3}{4} \ln \Big(1 - \frac{4}{3} \frac{h_{xy}}{N}\Big),$$

where h_{xy} is the hamming distance between sequence x and sequence y.

Now, note that the function $f(x) = -\frac{3}{4} \ln \left(1 - \frac{4}{3} \frac{x}{N}\right)$ is one-to-one. This means that, from a privacy point of view, $f(x)$ carries the same amount of information than x. Therefore, we could simply proceed as follows:

1. Privately compute the Hamming distance, h_{xy}, using the tailored Hamming distance method described above and the Yao protocol assisted with quantum oblivious keys;

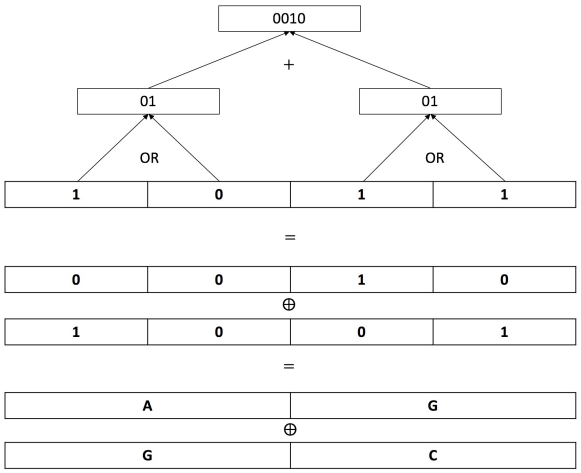


Figure 5.5: Overview of the tailored divide-and-conquer technique. This corresponds to lines `12-19` in Figure A.1 in Appendix A.

2. Internally compute $d_{xy} = f(h_{xy})$ (no need of quantum SMC).

This way, we just have to generate the boolean circuit for h_{xy} rather than generating for the full expression d_{xy}.

Kimura

In Section 5.1.1, we saw that the Kimura 2-parameter model leads to the following distance:

$$d_{xy} = -\frac{1}{2}\ln\left(\left(1-2P-Q\right)\sqrt{1-2Q}\right),$$

where $P = \frac{n_1}{N}$, $Q = \frac{n_2}{N}$ and n_1 and n_2 are respectively the number of sites for which two sequences differ from each other with respect to type I ("transition" type) and type II ("transversion" type) substitutions.

Similar to the case of Jukes-Cantor metric, note that $h(x) = -\frac{1}{2}\ln\left(\sqrt{\frac{x}{N^3}}\right)$ is one-to-one and only defined for $x > 0$. Thus, we can proceed as follows:

1. Privately compute the expression $c = (N - 2n_1 - n_2)^2(N - 2n_2)$ using the tailored Hamming distance method described above and the Yao protocol assisted with quantum oblivious keys;

2. Internally computes $d_{xy} = h(c)$ (no need of quantum SMC).

More precisely, the ANSI-C code that privately computes expression $c = (N - 2n_1 - n_2)^2(N - 2n_2)$ proceeds as follows. It uses the function $\text{popcount}_t(z)$ described above to compute the quantities n_1 and n_2. Observe that a transition type ($A \leftrightarrow G$ or $C \leftrightarrow T$)

renders the same XOR value:

$$A \oplus G = 00 \oplus 10 = 10$$
$$T \oplus C = 11 \oplus 01 = 10.$$

Therefore, using a four-sized sequence, the quantities n_1 and n_2 are given by:

$$\begin{aligned} n_1 &= 4 - \text{popcount}_t(x \oplus y \oplus 10101010) \\ n_2 &= \text{popcount}_t(x \oplus y) - n_1. \end{aligned}$$

F84 and LogDet

Recall from Section 5.1.1, that the F84 (F_{xy}) and LogDet (L_{xy}) distances are given, respectively, by:

$$F_{xy} = -2A \ln\left(1 - \frac{P}{2A} - \frac{(A-B)Q}{2AC}\right) + 2(A - B - C)\ln\left(1 - \frac{Q}{2C}\right), \tag{5.9}$$

$$L_{xy} = -\frac{1}{4}\ln\left(\frac{\det F_{xy}}{\sqrt{\det \prod_x \prod_y}}\right), \tag{5.10}$$

where $A = \frac{\pi_C\pi_T}{\pi_Y} + \frac{\pi_A\pi_G}{\pi_R}$, $B = \pi_C\pi_T + \pi_A\pi_G$ and $C = \pi_R\pi_Y$ for $\pi_Y = \pi_C + \pi_T$ and $\pi_R = \pi_A + \pi_G$, and P and Q are defined as in the Kimura 2-parameter mode above. Also, the divergence matrix F_{xy} is a 4×4 matrix such that the $ij-$th entry gives the proportion of sites in sequence x and y with nucleotide i and j, respectively. Also, $\prod_x$ and $\prod_y$ are diagonal matrices where its $i-$th component correspond to the proportion of i nucleotide in the sequence x and y, respectively.

As before, we want to split the private computation of both F_{xy} and L_{xy} in two steps. Note that, in this case, there is no clear way to define two bijective functions, $g()$ and $q()$, on some simple parameters, d and e, such that $F_{xy} = g(d)$ and $L_{xy} = p(e)$. By simple parameters, we mean parameters that do not depend on complex operations such as logarithm or square root. Instead, one can use the CORDIC algorithm [253, 254] for square-roots and logarithm functions and translate an approximation of both F_{xy} and L_{xy} into boolean circuits.

5.7.2 Private computation of phylogenetic trees

In this section, we prove that the protocol A_d^a described in Section 5.5.3 securely implements functionality $\mathtt{A} \circ \mathtt{DM}$ described in Section 5.5.1 according to the security definition 8. So, we want to prove the following theorem:

Theorem 3. *The protocol A_d^a securely realizes* $\mathtt{A} \circ \mathtt{DM}$ *in the presence of semi-honest adversaries.*

We start by noting that the ideal functionality outputs the distance matrix to the parties and that during A computation there is no interaction between the parties. Therefore, the security of the system is independent of the distance-based algorithm used (UPGMA, Neighbour-Joining or Fitch-Margoliash) and we can only focus on the computation of DM functionality.

As already mentioned, the protocol that implements the functionality DM is built up by many invocations of a two-party distance functionality, denoted by $\mathtt{D}_d$ for $d \in \{\mathrm{JC}, \mathrm{K2P}, \mathrm{F84}, \mathrm{LD}\}$. So, in order to prove the above theorem, we will need to following two lemmas:

Lemma 12. *A_d^a privately reduces* DM *to* $\mathtt{D}_d$*, i.e. an oracle-aided A_d^a protocol privately computes* DM *using the oracle-functionality* $\mathtt{D}_d$*.*

Proof. In order to prove this lemma, we have to develop a simulator S that simulates the view of a set of corrupted parties C. S starts from receiving all the input sequences from the corrupted parties. It then proceeds as follows:

1. Generates random sequences of the honest parties, H.
2. Invokes the oracle-functionality $\mathtt{D}_d$ on these sequences.
3. Sends to all corrupted parties C the results of distances computed from honest parties sequences.
4. Invokes the oracle-functionality $\mathtt{D}_d$ on the sequences owned by the corrupted parties.
5. Invokes the oracle-functionality $\mathtt{D}_d(s_i, s_j)$ for $s_i \in H$ and $s_j \in C$.

In a real execution, the corrupted parties will only receive the distances computed by $\mathtt{D}_d$ on the honest parties sequences (as in step 2.), on their sequences (as in step 4.) and between corrupt and honest parties. Therefore, we have that the oracle-aided A_d^a protocol privately computes DM using the oracle-functionality $\mathtt{D}_d$.

□

Lemma 13. *Yao protocol with the OT primitive instantiated by $\Pi_{\boldsymbol{O}}^{\boldsymbol{BBCS}}$ protocol (Figure 4.3) privately computes* $\mathtt{D}_d$*.*

Proof. In [89] it was developed a framework that allows quantum protocols to be composed in a classical environment. They also mention that a general secure function evaluation

remains secure when instantiating the OT primitive by a secure quantum version. In [255], it was proved that $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol is secure according to the security definition given in [89]. Therefore, we can compose the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol with a Yao protocol [256] while preserving the overall security.

□

So, from Lemma 12 and 13 we can use the composition theorem 2 and conclude that the protocol A_d^a is secure.

We have proved that our system is well designed and secure against quantum computer attacks under the semi-honest model. In order to extend the protocol to the malicious setting, we just have to implement a two-party secure computation protocol that is secure in the malicious adversary model [74].

5.8 Complexity analysis

In this section, we start by analysing the complexity of the protocol A_d^a presented before. We assume there are n parties, $P_1, \ldots, P_n$, with $M_1, \ldots, M_n$ sequences, respectively. Also, we assume that the sequences are aligned and that they have the same number of nucleotides, s.

5.8.1 Protocol complexity analysis

Now, let us analyse the complexity of the protocol presented in Section 5.5.3.

Yao protocol executions

Regarding the number of Yao protocol executions, we have that each party P_j owning M_j sequences has to perform $N_{\text{Yao}}^j = M_j \sum_{i \neq j} M_i$ secure distance computations. So, the total number of Yao protocol executions is given by

$$N_{\text{Yao}} = \sum_j N_{\text{Yao}}^j = \sum_{j,i \neq j} M_j M_i.$$

If we assume the number of sequences per party to be the same, i.e. $M_j = M \, \forall j \in [n]$, then we can simplify the expression above and conclude that $N_{\text{Yao}} = M^2 n(n-1)$. This means that the number of Yao protocol executions is quadratic in the number of sequences per party ($\mathcal{O}(n^2)$) and also in the number of parties ($\mathcal{O}(M^2)$).

OT executions

From N_{Yao} we can deduce the number of OT executions. In the Yao protocol, we need to execute one OT for each of the evaluator's input wires. For a sequence with s nucleotides and using a two-bit representation of each nucleotide, the boolean circuit that computes the distance between two sequences will have $2s$ input wires for each party input. Therefore, each party executes the following number of OT executions ($\forall j$):

$$\begin{aligned} N^j_{\text{OT}} &= N^j_{\text{Yao}} \cdot 2s \\ &= 2sM^2(n-1). \end{aligned}$$

It is important to note that N^j_{OT} is independent of the size of the boolean circuit used, i.e. it is independent of the distance metric d used in the protocol. This is a consequence of using the Yao protocol where the number of OT only depends on the input size. In case we were using GMW [23] protocol, the number of OT per party would depend on the size of the circuit.

As mentioned in Section 5.6.1, in case the number of oblivious keys generated is scarce compared to the number of OT required, we can use the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol to generate the base OT used within OT extension protocol. In this case, we just have to generate κ $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocols per Yao execution: $L^j_{\text{bOT}} = N^j_{\text{Yao}} \cdot \kappa = \kappa M^2(n-1)$.

Oblivious keys

At this point, we can easily deduce the size of oblivious keys that each pair of parties have to generate when using messages of size l.

In case we use $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol to generate the final OT:

$$\begin{aligned} L^j_{\text{ok}} &= N^j_{\text{OT}} \cdot 2l \\ &= 4slM^2(n-1). \end{aligned}$$

Also, we can use the number of OT executions per party and the analysis from Table 4.5 and [50] to compute the computational and communication complexity (in bits) of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$:

$$\begin{aligned} \mathcal{C}^j_{\text{comp}} &= N^j_{\text{OT}} \cdot 8l \\ &= 16slM^2(n-1), \end{aligned}$$

$$\begin{aligned} \mathcal{C}^j_{\text{comm}} &= N^j_{\text{OT}} \cdot 3l \\ &= 6slM^2(n-1). \end{aligned}$$

In case we use $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol to generate the base OT, the total size of oblivious key required is:

$$\begin{aligned} L_{\text{bok}}^{j} &= N_{\text{bOT}}^{j} \cdot 2l \\ &= 2\kappa l M^2(n-1). \end{aligned}$$

QRNG

The QRNG has to generate twice the total length of oblivious keys, i.e. $L_{\text{QRNG}} = 2L_{\text{ok}}$.

Internal computation

Number of internal computations per party:

$$N_{\text{int}}^{j} = \binom{M}{2} = \frac{M!}{2!(M-2)!}.$$

Encryption keys

As discussed before, for every party P^j, P^t $(t \neq j)$ has to receive from P_j the distances known by P_j that P_t does not have access. So, P_j has to send $M^2(n-2) + N_{\text{int}}^{j}$ distance values to P_t. Consequently, the length of the QKD key used to send these distances to P_t is:

$$32(M^2(n-2) + N_{\text{int}}^{j}),$$

for a 32−bit number representation. Therefore, the total size of key shared between two parties P_j and P_t must be:

$$L_{qkd}^{jt} = 64(M^2(n-2) + N_{\text{int}}).$$

Also, each party must have an overall shared key of $L_{qkd}^{j} = \sum_{i \neq j} L_{qkd}^{i} = 64(n-1)(M^2(n-2) + N_{\text{int}}^{j})$.

5.8.2 Use case

We now present the scenario used to test and compare both quantum-assisted and classical-only approaches. We start by exploring the complexity analysis and the OT comparison carried out in previous sections. We extend this analysis in the next section with a testbed implementation.

We consider a scenario where three parties $n = 3$ have M SARS-CoV-2 genome sequences (with length $s = 32\,000$) and want to privately compute a phylogenetic tree from them. In the next section we consider a varying number of sequences, but, for now, we set

Parameter	Formula	Amount	Generation Time
L^j_{ok}	$4slM^2(n-1)$	3.3×10^9 bit	5m30s
L^j_{bok}	$2\kappa lM^2(n-1)$	6.6×10^6 bit	0.64s
L^j_{QRNG}	$8slM^2(n-1)$	6.6×10^9 bit	28s
L^j_{qkd}	$64(n-1)(M^2(n-2)+\binom{M}{2})$	18.6×10^3 bit	1.9×10^{-3}s
N^j_{Yao}	$M^2(n-1)$	200	
N^j_{OT}	$2sM^2(n-1)$	12.8×10^6	
N^j_{bOT}	$\kappa M^2(n-1)$	25.6×10^3	
N^j_{int}	$\binom{M}{2}$	45	

Table 5.1: Complexity analysis where $n = 3$, $M = 10$, $s = 32\,000$ and $l, \kappa = 128$. L^j_{ok}: size of total oblivious key. L^j_{bok}: total size of oblivious key for base OT. L^j_{QRNG}: random bits generated by QRNG. L^j_{qkd}: total size of QKD keys. N^j_{Yao}: number of Yao protocol executions. N^j_{OT}: number of OT executions. N^j_{bOT}: number of base OT executions. N^j_{int}: number of internal computations.

$M = 10$. Following a standard choice [3], we consider garbled circuit keys with $l = 128$ bits, computational security parameter with $\kappa = 128$ bits and statistical security parameter with $w = 64$ bits. For these parameter values, we can instantiate the expressions deduced in the complexity analysis (Section 5.8.1). This information is summarised in Table 5.1. As expected, the total size of oblivious keys (L^j_{ok}) required for a scenario where $\Pi^{\mathbf{BBCS}}_{\mathbf{O}}$ is the main OT protocol is three orders of magnitude higher than the case where $\Pi^{\mathbf{BBCS}}_{\mathbf{O}}$ serves as a base OT protocol in KOS15 (L^j_{bok}). Also, we note that the total size of symmetric keys required in the protocol (L^j_{qkd}) is much smaller than that of oblivious keys (L^j_{ok} and L^j_{bok}), pointing to the fact that its management should be less expensive than the oblivious keys management system. This will be discussed further in the next section.

We can also estimate the time required to generate the keys based on their size. If we consider state-of-the-art rates of 10 Mbit/s for both QKD and QOKD systems [257] and a rate of 240 Mbit/s for QRNG (ID Quantique QRNG PCIe cards [258]), we would need around 5 minutes for L^j_{ok}, 0.64s for L^j_{bok}, 28s for L^j_{QRNG} and 1.9×10^{-3}s for L^j_{qkd}. Note that we can significantly reduce the time of the precomputation phase in case we integrate $\Pi^{\mathbf{BBCS}}_{\mathbf{O}}$ with KOS15 OT extension protocol.

5.9 Performance evaluation

In this section, we set out to explore and compare the performance of two implementations of the proposed secure phylogenetic tree computation (A_d^a): classical-only and quantum-assisted. The quantum-assisted system replaces Libscapi base OT (SimpleOT [246]) implementation with the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol presented before (Figure 4.4). It also uses symmetric keys along with one-time pad to encrypt distance values as described in Section 5.5. More specifically, we benchmark our implementation for the duration of its main components: circuit generation, communication, (internal) computation and SMC operation.

Here, we do not assess the generation performance of both symmetric keys and oblivious keys. We precompute these keys using a simulator that mimics the structure of the quantum generated keys and we do not include their generation time in the performance analysis. The reason for this is twofold: performance in quantum cryptography is an active field of research with no clear way on how to be compared with classical approaches; quantum generation of both keys (symmetric and oblivious) can be precomputed without depending on the parties' inputs and used later as a resource in the execution of the system.

5.9.1 Setup

We leverage a testbed on a virtual environment composed of three Ubuntu (64-bit) 16.04.3 Virtual Machines (VM) with 3GB of RAM. The virtual environment was created using VirtualBox and the VMs were running on a 2.6 GHz Intel Core i7 processor.

The performance of the implementation was measured on the VMs with the clock type `CLOCK_REALTIME` from the C++ library `time`. Although the values might differ for different host machines, this method is certainly adequate to use as a comparison between a classical-only and a quantum-assisted system.

We follow the scenario presented in Section 5.8.2, where we have three parties ($n = 3$) owning at most ten sequences ($M \leq 10$) with 32 000 nucleotides. For the sake of comparison, we use the Jukes-Cantor phylogenetic distance along with PHYLIP implementation of UPGMA algorithm, i.e. $(d, a) = (\mathrm{JC}, \mathrm{UPGMA})$.

Sequences preprocessing

The 30 sequences used in this testbed were taken from GISAID database [259] which collects SARS-CoV-2 genome sequences. These sequences were then aligned using the Clustal Omega API [260]. After alignment, the sequences (4-based) were translated to bits according to the following rule: $A \to 00$, $C \to 01$, $G \to 10$ and $T \to 11$. Note

Min. Time	Time	Nº of gates	Depth
0s	1m42.7s	2 489 218	29 771
100s	3m30.7s	2 205 372	21 711
200s	5m9.3s	2 205 372	21 711

Table 5.2: Generation of Jukes-Cantor boolean circuit. Min. Time: Minimization Time.

that this alignment procedure is not privacy-preserving and was only used for testing purposes. A privacy-preserving alignment can be easily executed if all parties agree on a public reference sequence and align locally their sequences against this reference.

5.9.2 Circuit generation

As mentioned above, the CBMC-GC tool can generate a boolean circuit description of the phylogenetic distance from its corresponding ANSI-C code. In Table 5.2, we present the generation time of the Jukes-Cantor boolean circuit for three different minimization time values (CBMG-GC parameter). The minimization time is a parameter of the CBMC-GC tool that regulates the time spent to minimize the size of the boolean circuit. We note that the generation of the circuit only has to be carried out once. From Table 5.2, we can see that the minimization time for values above $100s$ does not have a great impact on the minimization of both the number of gates and circuit depth. The C code describing the Jukes-Cantor distance is shown in Appendix A.

5.9.3 System execution time

We start by recalling that the proposed secure algorithm is divided into the following parts:

1. Distance Matrix, `DM`:
   (a) Pairwise SMC computation of distances, `SMC`;
   (b) Pairwise internal computation of distances, `IC`;
   (c) Sending/Receiving other sequences, `Com`;
2. Phylogenetic computation, `A`.

We join the internal computation of sequences and PHYLIP phylogenetic computation into the same category and assess three different components for both classical and quantum runs: Communication (`Com`), SMC (`SMC`) and Computation (`IC`, `A`). In Tables 5.3

Nº of Seq.	2	4	6	8	10
Comm.	3,95%	0,98%	0,44%	0,25%	0,16%
SMC	95,95%	98,94%	99,48%	99,68%	99,77%
Comp.	0,10%	0,08%	0,07%	0,07%	0,07%

Table 5.3: Percentage weight of each component in the classical-only system.

Nº of Seq.	2	4	6	8	10
Comm.	3,75%	0,93%	0,39%	0,22%	0,14%
SMC	96,15%	98,99%	99,55%	99,72%	99,81%
Comp.	0,10%	0,07%	0,06%	0,06%	0,05%

Table 5.4: Percentage weight of each component in the quantum-assisted system.

and 5.4, we show the proportion of each component. As expected, in both systems the pairwise SMC computation of distances represents the greatest portion, accounting for more than 95% of the time for all different numbers of sequences. However, the weight of SMC in the quantum-assisted system is consistently higher than the classical-only system for all cases. This can be explained by the fact that the quantum-assisted SMC takes longer than the classical-only SMC.

Figure 5.6 present us with the average duration of both systems with standard deviation as error bars. Here, we see that the quantum-assisted approach has a higher cost than the classical-only implementation. As discussed in Section 5.6.1, we can either use the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ protocol as the main OT in the Libscapi implementation or we can use it as a base OT in the KOS15 OT Extension used by Libscapi. Since we have implemented the latter, our $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is competing against the SimpleOT [246] base OT implementation. As analysed by the authors (Section 4 [50]), the $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ transfer phase is expected to outperform base OT implementations and to have comparable performance to OT Extension protocols. However, these analyses only compared cryptographic and computational operations and did not take into account implementation constraints and memory complexity.

In the quantum-assisted implementation, we separate the precomputation phase (generation of symmetric and oblivious keys) from the secure computation phase of the proposed protocol, A_d^a. For this reason, it is necessary to develop a key management system to save and keep key synchronization between parties. Consequently, the key management system becomes the bottleneck as the number of sequences increases. In particular, the key management system of oblivious keys is responsible for most of the overhead (Fig-

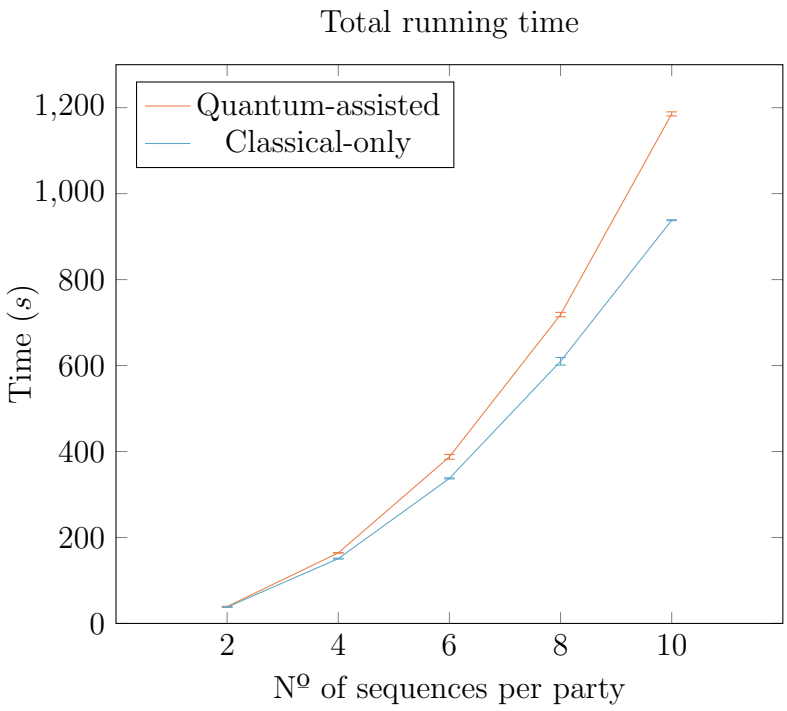


Figure 5.6: Total running time of both quantum-assisted and classical-only systems.

ure 5.7). We refer to the difference between the quantum-assisted and the classical-only system as the overhead of the quantum-assisted system.

The reason for oblivious keys management to be more expensive than symmetric management and to be the main cause of overhead is twofold: the total size of oblivious keys used is three orders of magnitude higher than that of symmetric keys (compare L^j_{qkd} and L^j_{bok} from Table 4.5); oblivious keys are saved in files (slower access) whereas symmetric keys are loaded into RAM memory (faster access). The main reason for oblivious keys to be managed from a file system is that it allows to use Libscapi implementation of Yao protocol in a modular way, i.e. we only have to change the type of base OT used by Libscapi implementation without tailoring any other module.

As the management of files is time-sensitive to their size, the proportion of time of the system's overhead due to the oblivious key management system (OKMS) increases with the number of shared keys per party. This can be confirmed by Figure 5.8 which shows the proportion of time spent by the oblivious key management system during the overhead of the quantum-assisted system.

Future work is required to develop more efficient oblivious key management systems. Despite this difference, we stress that the quantum-assisted system has a significantly higher degree of security against quantum computer attacks.

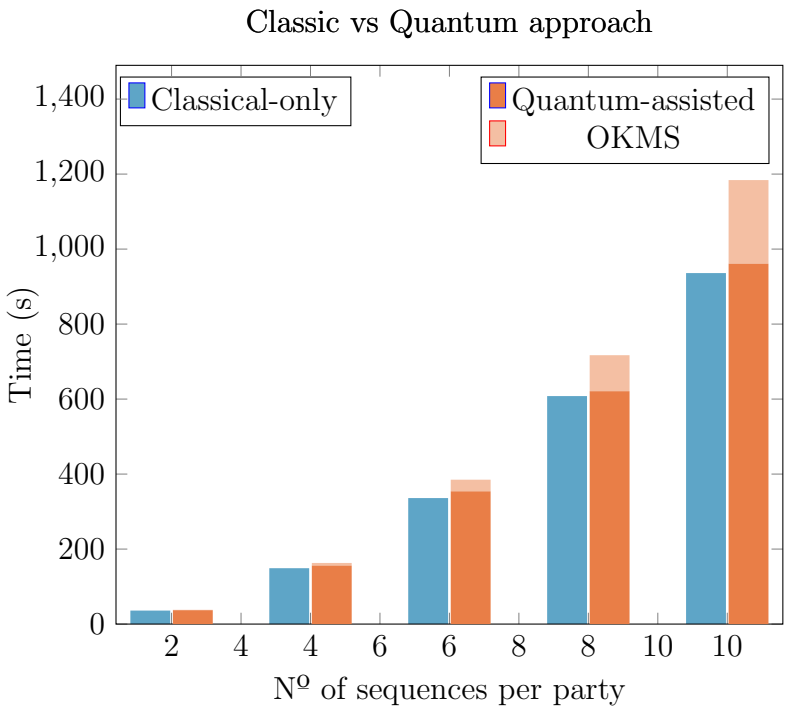


Figure 5.7: Total running time of the pairwise SMC computation of distances for both quantum-assisted and classical-only systems.

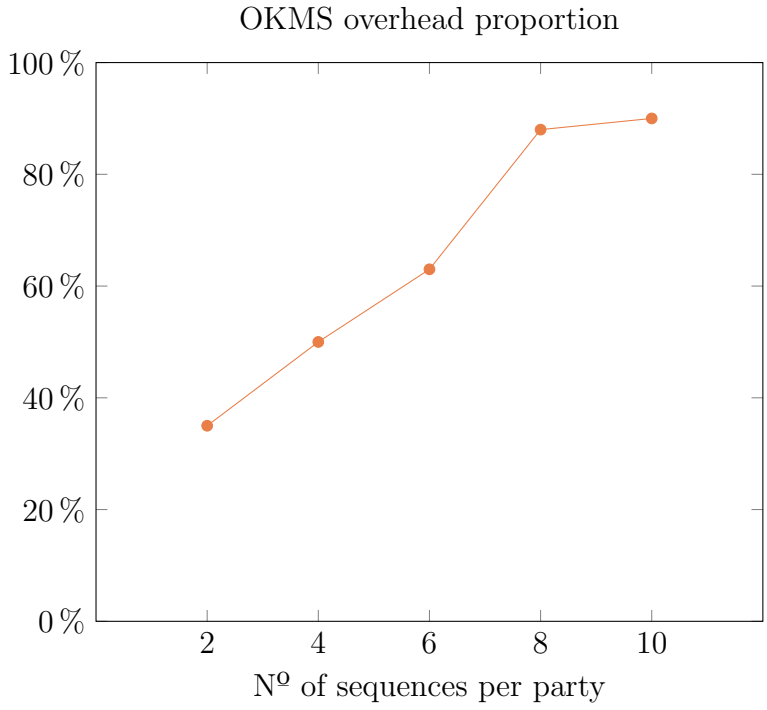


Figure 5.8: The proportion of the quantum-assisted system's overhead that is attributable to the Oblivious Key Management System (OKMS).

5.10 Conclusion

In this chapter, we presented an SMC protocol assisted with quantum technologies tailored for distance-based algorithms of phylogenetic trees. It is a modular protocol that uses one distance metric taken from four possible evolutionary models (Jukes-Cantor, Kimura 2-parameter, F84 and LogDet) and three different protocols (UPGMA, Neighbour-Joining and Fitch-Margoliash). In total, we can implement twelve different combinations of protocols.

The proposed system is based on ready to use libraries (CBMC-GC, Libscapi and PHYLIP) that are integrated with quantum technologies to provide a full quantum-proof solution. We use the quantum version of primitives that play a central role in the security of the system: oblivious transfer, encryption and random number generation.

We compare the performance of a classical-only and a quantum-assisted system based on simulated symmetric and oblivious keys. Previous analyses on the computation and communication complexity point to a scenario where the quantum-assisted version does not add an extra efficiency cost. This is confirmed by comparing the running times of both approaches without considering the overhead created by the oblivious key management system that increases with the number of shared keys. Further work is required to develop more efficient key management systems. Despite this extra cost, the quantum-assisted version significantly improves the system security when compared with the classical-only as it renders a protocol with enhanced security against quantum computers.

Chapter 6

Quantum oblivious linear evaluation

Oblivious Linear Evaluation (OLE) is a cryptographic task that permits two distrustful parties, say Alice and Bob, to jointly compute the output of a linear function $f(x) = ax+b$ in some finite field, $\mathbb{F}$. Alice provides inputs $a, b \in \mathbb{F}$ and Bob provides $x \in \mathbb{F}$, while the output, $f(x)$, becomes available only to Bob. As the parties are distrustful, a secure OLE protocol should not permit Alice to learn anything about Bob's input, while also Alice's inputs should remain unknown to Bob. OLE can be seen as a generalization of oblivious transfer (OT) [54], a basic primitive for secure two-party computation, which is a special case of secure multi-party computation [10, 261, 262]. OT has been shown to be complete for secure multi-party computation, i.e., any such task, including OLE, can be achieved given an OT implementation.

Impagliazzo and Rudich proved that OT protocols require public-key cryptography and cannot just rely on symmetric cryptography [39]. Consequently, OLE cannot rely on symmetric cryptography either, and we need to resort to public-key cryptography. However, Shor's quantum algorithm [28] poses a threat to the currently deployed public-key systems, motivating the search for protocols secure against quantum attacks. Bennet et al. [6] and Crépeau [263] proposed the first protocols for quantum OT (QOT). As far as quantum OLE (QOLE) is concerned, to the best of our knowledge, no protocol has been proposed as of now. Analogously to the classical case, it is expected that one can implement QOLE based on QOT protocols. That said, in this work we propose a protocol for QOLE that, additionally, does not rely on any QOT implementation.

OLE is commonly generalised to vector OLE (VOLE). In this setting, Alice defines a set of k linear functions $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}^k \times \mathbb{F}^k$ and Bob receives the evaluation of all these functions on a specified element $x \in \mathbb{F}$, i.e. $\boldsymbol{f} := \boldsymbol{a}x + \boldsymbol{b}$. One can think of VOLE as the arithmetic analog of string OT and show how it can be used in certain Secure Arithmetic Computation and Non-Interactive Zero Knowledge proofs [70]. Ghosh et. al put further in evidence the usefulness of VOLE by showing that it serves as the building block of

Oblivious Polynomial Evaluation [66], a primitive which allows more sophisticated applications, such as password authentication, secure list intersection, anonymous complaint boxes [264], anonymous initialization for secure metering of client visits in servers [265], secure Taylor approximation of relevant functions (e.g. logarithm) [266], secure set intersection [267] and distributed generation of RSA keys [268]. We also show how our QOLE protocol can be adapted to achieve secure VOLE.

6.1 Contributions overview

We present a quantum protocol for OLE with universally composable security (quantum-UC security, see Definition 6) in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model, i.e. when assuming the existence of a commitment functionality, $\mathcal{F}_{\mathbf{COM}}$ (see Figure 2.5). To obtain a secure protocol, we take advantage of the properties of mutually unbiased bases (MUBs) in high-dimensional Hilbert spaces with prime and prime-power dimension. Such a choice is motivated by recent theoretical and experimental advances that pave the way for the development and realization of new solutions for quantum cryptography [42–46, 269–274]. To the best of our knowledge, our protocol is the first proposal of a QOLE protocol proved to be quantum-UC secure. Moreover, it is not based on any QOT implementation which would be the standard approach. To prove its security, the only assumption we make is the existence of a commitment functionality. We consider the static corruption adversarial model with both semi-honest and dishonest adversaries. Finally, we modify the proposed protocol to generate quantum-UC secure VOLE.

Main tool. The proposed protocol $\Pi_{\mathbf{QOLE}}$ (see Figure 6.5) is based on the fact that in a Hilbert space of dimension d (isomorphic to $\mathbb{Z}_d$) there exists a set of MUBs $\{|e_r^x\rangle\}_{x,r\in\mathbb{Z}_d}$, such that, upon the action of a certain operator V_a^b, each basis element r is shifted by some linear factor $ax-b$ inside the same basis x:

$$V_a^b \left|e_r^x\right\rangle = c_{a,b,x,r} \left|e_{ax-b+r}^x\right\rangle, \tag{6.1}$$

where $a, b, x, r \in \mathbb{Z}_d = \{0, 1, \ldots, d-1\}$. If Alice controls the operator V_a^b and Bob controls the quantum state $|e_r^x\rangle$, they are able to compute a linear function $f(x) = ax - b$ where effectively Alice controls the function $f = (a, b)$ and Bob controls its input x. Moreover, since Bob controls x and r, he can receive $f(x)$ by measuring the output element.

Protocol overview. In a nutshell, the QOLE protocol (see Figure 6.5) with inputs $f = (a, b)$ from Alice and x from Bob is divided into two main phases. In the first *quantum phase*, Alice and Bob use high-dimensional quantum states to generate n random weak

OLE (RWOLE) instances, where n is the security parameter. In this phase, Alice outputs n random elements $f_i^0 = (a_i^0, b_i^0)$, and Bob outputs n elements $(x_i^0, y^0 = f_i^0(x_i^0))$. These instances are considered to be weaker because Bob is allowed to have some amount of information about the n outputs of Alice (a_i^0, b_i^0). In the second *post-processing phase*, Alice and Bob use classical tools to extract one secure OLE from the aforementioned n instances.

More specifically, in the quantum phase, Bob randomly generates $m = (1+t)n$ quantum states $\left|e_{r_i}^{x_i^0}\right\rangle$ and sends them to Alice. Then, Bob commits to his choice (x_i^0, r_i), $\forall i \in [m]$, where for any $l \in \mathbb{N}$, $[l]$ denotes the set $\{1, \ldots, l\}$, using an ideal commitment functionality, $\mathcal{F}_{\mathbf{COM}}$, and Alice asks to verify a subset T of size tn of these commitments. This intermediate *commit-and-open* step allows Alice to test Bob's behaviour and ensure that he does not deviate *too much* from the protocol, and it is a common method used in security proofs of QOT protocols [36, 88]. If Bob passes all the tests, Alice randomly generates (a_i^0, b_i^0) and applies $V_{a_i^0}^{b_i^0}$ to the remaining n received states $\left|e_{r_i}^{x_i^0}\right\rangle$, for $i \in [m] \setminus T$. For the rest of this section we relabel and denote $[n] = [m] \setminus T$. According to the expression (6.1), the output states are given by $\left|e_{a_i^0 x_i^0 - b_i^0 + r_i}^{x_i^0}\right\rangle$ and she sends them to Bob, who outputs $y_i^0 = a_i^0 x_i^0 - b_i^0$ by measuring the received states in the corresponding basis x_i^0 and subtracting r_i, $\forall i \in [n]$.

The post-processing phase uses two subprotocols: a derandomization step (see Figure 6.3) and an extraction step (see Figure 6.4). The derandomization step is based on the protocol Π_{OLE}^n from [275] and transforms the n RWOLE instances into n weak OLE (WOLE) instances with inputs $(a_i, b_i)_{i\in[n]}$ chosen by Alice and inputs x_i for $i \in [n]$ chosen by Bob. The extraction protocol uses the so-called *Multi-linear Modular Hashing* family, MMH*, of two-universal hash functions [83] to render Bob's information on Alice's system useless and to extract one secure OLE out of n instances of WOLE. In the extraction phase, Alice samples a two-universal hash function $g_{\boldsymbol{\kappa}}$ from MMH* and sends it to Bob. Then, with adequately-crafted vectors $(\boldsymbol{a}, \boldsymbol{b}) = \big((a_1, \ldots, a_n), (b_1, \ldots, b_n)\big)$, Alice has $a = g_{\boldsymbol{\kappa}}(\boldsymbol{a})$ and $b = g_{\boldsymbol{\kappa}}(\boldsymbol{b})$, and Bob outputs $y = g_{\boldsymbol{\kappa}}(\boldsymbol{y})$, where $\boldsymbol{y} = \boldsymbol{a}\boldsymbol{x} + \boldsymbol{b}$ after point-wise vector multiplication with the constant vector $\boldsymbol{x} = (x, \ldots, x)$.

quantum-UC security. Due to the quantum nature of the states $\left|e_{r_i}^{x_i^0}\right\rangle_{i\in[n]}$, a dishonest Alice is not able to distinguish which bases $x_i^0, i \in [n]$ are used by Bob. From her point of view, Bob's states are maximally mixed and therefore completely hide x_i^0. This is enough to ensure that, in the derandomization step, Alice does not receive any information about Bob's final input x. For a dishonest Bob, to correctly pass all Alice's tests, it means he did not cheat at all rounds with overwhelming probability. This ensures that he has some *bounded* information on Alice's random elements $(a_i^0, b_i^0)_{i\in[n]}$, and using privacy

amplification techniques in the extraction step, Alice can guarantee that Bob's information about her final input (a, b) is the same as in the case of an ideal OLE functionality, i.e. the probability distribution of a is close to uniform.

Turning this intuition into a quantum-UC security proof requires some additional insights. First, we need a way to quantify Bob's information on Alice's elements (a_i^0, b_i^0) after the testing phase and the application of the corresponding $V_{a_i^0}^{b_i^0}$ operators, for $i \in [n]$; for this purpose we use the quantum *min-entropy* (see Definition 3). We follow the approach of [36] to guarantee that Bob does not significantly deviate from the protocol in all the rounds, and we use Theorem 1 from [80] to compute a concrete lower bound of Bob's min-entropy on Alice elements $(a_i^0, b_i^0)_{i\in[n]}$. Along with Lemma 11, we have that $a = g_{\boldsymbol{\kappa}}(\boldsymbol{a})$ is close to uniform, which is sufficient to prove that Bob does not know more about (a, b) than what the output $y = ax + b$ reveals.

In order to show that the protocol $\Pi_{\mathbf{QOLE}}$ is quantum-UC secure, we need to show that an ideal execution of $\Pi_{\mathbf{QOLE}}$ with access to $\mathcal{F}_{\mathbf{OLE}}$ (Figure 2.3) is indistinguishable from a real execution of the protocol from the point of view of an external entity called the *environment*. To prove this indistinguishability, we have to build a simulator that simulates the execution of the protocol in the ideal setting and generates messages on behalf of the honest simulated parties, while trying to extract the dishonest party's inputs and feed them in $\mathcal{F}_{\mathbf{OLE}}$. In particular, for a dishonest Alice, we have to demonstrate the existence of a simulator, $\mathcal{S}_A$, that generates messages on behalf of honest Bob and extracts Alice's input (a, b) which, in turn, feeds into $\mathcal{F}_{\mathbf{OLE}}$. To this end, we consider that $\mathcal{S}_A$ simulates an attack by Bob at all rounds, i, of the protocol which allows to extract the m values of Alice (a_i^0, b_i^0). However, the commit-and-open scheme described above is designed to catch such an attack, and to work around this issue we substitute the ideal commitment functionality, $\mathcal{F}_{\mathbf{COM}}$, with a fake commitment functionality, $\mathcal{F}_{\mathbf{FakeCOM}}$, that allows $\mathcal{S}_A$ to open the commitments later [88]. From the remaining n values (a_i^0, b_i^0), $\mathcal{S}_A$ computes Alice's input (a, b) and feeds it to $\mathcal{F}_{\mathbf{OLE}}$.

For a dishonest Bob, we have to show the existence of a simulator, $\mathcal{S}_B$, that generates messages on behalf of honest Alice and extracts Bob's input x. We assume that $\mathcal{S}_B$ has full control over $\mathcal{F}_{\mathbf{COM}}$, which means that it has access to Bob's m committed values (x_i^0, r_i); the input x can be easily extracted from these values.

Protocol generalization. We start by generalizing the main relation (6.4) to Galois Fields of prime-power dimension, $GF(d^M)$ for $M > 1$. Then, we show how we can obtain a protocol for quantum VOLE. In particular, from n WOLE instances, we are able to generate a VOLE with size proportional to n, and we bound this proportion by the min-entropy value on the WOLE instances.

6.1.1 Organization

In Section 6.2, we introduce the main tool used in the QOLE protocol. In Section 6.3, in order to build some intuition, we present a QOLE protocol that is secure only if we consider Bob to be semi-honest; in case Bob is dishonest, its security is compromised. In Section 6.4, we construct a secure protocol that comprises the first part of our main QOLE protocol presented in Section 6.4.2. Next, in Section 6.5, we prove the security of the QOLE protocol in the quantum-UC framework. Then, in Section 6.6, we show how to generalise the presented QOLE protocol to Galois Fields of prime-power dimensions and we also present a quantum-UC secure protocol achieving VOLE.

6.2 Mutually unbiased bases

In this section, we present the basics and some properties of mutually unbiased bases (MUBs) in some high-dimensional Hilbert space $\mathcal{H}^d$. This is the main tool that is used in our protocol. For more details about MUBs see [43].

Definition 9. *Let $\mathcal{B}_0 = \{|\psi_1\rangle, \ldots, |\psi_d\rangle\}$ and $\mathcal{B}_1 = \{|\phi_1\rangle, \ldots, |\phi_d\rangle\}$ be orthonormal bases in the d-dimensional Hilbert space $\mathcal{H}^d$. They are said to be mutually unbiased if $|\langle\psi_i|\phi_j\rangle| = \frac{1}{\sqrt{d}}$ for all $i, j \in \{1, \ldots, d\}$. Furthermore, a set $\{\mathcal{B}_0, \ldots, \mathcal{B}_m\}$ of orthonormal bases on $\mathcal{H}^d$ is said to be a set of MUBs if, for every $i \neq j$, $\mathcal{B}_i$ is mutually unbiased with $\mathcal{B}_j$.*

MUBs are extensively used in quantum cryptography because, in some sense, these bases are as far as possible from each other and the overlap between two elements from different bases is constant. Let $\{|0\rangle, \ldots, |d-1\rangle\}$ be the computational basis of $\mathcal{H}^d$, where d is a prime number, and $\{|\tilde{0}\rangle, \ldots, |\widetilde{d-1}\rangle\}$ be the dual basis which is given by the Fourier transform on the computational basis:

$$|\tilde{j}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{-ij} |i\rangle ,$$

where $\omega = e^{\frac{2\Pi i}{d}}$. We can easily verify that the computational basis and its dual basis are mutually unbiased, and we will make use of the following two operators, V_a^0 and V_0^b, to encode Alice's functions during the first (quantum) phase of the protocol.

Definition 10 (Shift operators)**.** *The shift operator V_a^0 shifts the computational basis by a elements, i.e.*

$$V_a^0 |i\rangle = |i + a\rangle .$$

Similarly, the dual shift operator V_0^b shifts the dual basis by b elements, i.e.

$$V_0^b \left|\tilde{j}\right\rangle = \left|\widetilde{j-b}\right\rangle.$$

The operators V_a^0 and V_0^b are diagonal in the dual and computational basis, respectively[1], i.e.

$$V_a^0 = \sum_{j=0}^{d-1} \omega^{aj} \left|\tilde{j}\right\rangle\!\left\langle\tilde{j}\right| \text{ and } V_0^b = \sum_{i=0}^{d-1} \omega^{bi} \left|i\right\rangle\!\left\langle i\right|.$$

Furthermore, following the convention from [43], we can define

$$V_a^b \;:=\; V_0^b V_a^0 = \sum_{l=0}^{d-1} \omega^{(l+a)b} \left|l+a\right\rangle\!\left\langle l\right|,$$

obtaining the so-called *Heisenberg-Weyl operators*. These operators form a group of unitary transformations with d^2 elements; the group has $d+1$ commuting abelian subgroups of d elements, and for each abelian subgroup, there exists a basis of joint eigenstates of all V_a^b in the subgroup. These $d+1$ bases are pairwise mutually unbiased. Let $x \in \mathbb{Z}_{d+1}$ label the abelian subgroups, let $l \in \mathbb{Z}_d$ label the elements of each subgroup, and let U_l^x denote the corresponding subgroup operators. Finally, let the $i-$th basis element associated with the $x-$th subgroup be denoted by $|e_i^x\rangle$. Then, it can be seen that [43],

$$U_l^x = \sum_{i=0}^{d-1} \omega^{il} \left|e_i^x\right\rangle\!\left\langle e_i^x\right| \text{ and } \left|e_i^x\right\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{-il+\frac{l(l-1)}{2}x} \left|l\right\rangle,$$

where

$$U_l^x = \alpha_l^x V_l^{xl} \text{ with } \alpha_l^x = \omega^{-xl(l+1)/2}.$$

One can show that

$$V_a^b \left|e_0^x\right\rangle = c_{x,a,b} \left|e_{ax-b}^x\right\rangle,\ x \in \mathbb{Z}_d \text{ and } V_a^b \left|e_0^d\right\rangle = c_{d,a,b} \left|e_a^d\right\rangle \text{ for } x = d,$$

or more generally

$$V_a^b \left|e_r^x\right\rangle = c_{a,b,x,r} \left|e_{ax-b+r}^x\right\rangle, \text{ with } c_{a,b,x,r} = \omega^{ar+\frac{a(a+1)}{2}x}. \tag{6.2}$$

[1]Note that V_a^0 and V_0^b can be seen as a generalization of the Pauli X and Z operators, respectively.

Proof. By definition, we have that

$$\begin{aligned}
V_a^b \left|e_r^x\right\rangle &= \frac{1}{\sqrt{d}} \sum_{k,l=0}^{d-1} \omega^{(k+a)b} \left|k+a\right\rangle\!\left\langle k\right| \omega^{-rl+\frac{l(l-1)}{2}x} \left|l\right\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{(l+a)b} \omega^{-rl+\frac{l(l-1)}{2}x} \left|l+a\right\rangle \\
&= \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{lb} \omega^{-r(l-a)+\frac{(l-a)(l-a-1)}{2}x} \left|l\right\rangle \\
&= \frac{\omega^{ar}}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{-l(-b+r)} \omega^{\frac{l(l-1)}{2}x+lax+\frac{a(a+1)}{2}x} \left|l\right\rangle \\
&= \frac{\omega^{ar}}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{-l(-b+r)} \omega^{\frac{l(l-1)}{2}x-lax+\frac{a(a+1)}{2}x} \left|l\right\rangle \\
&= \frac{\omega^{ar+\frac{a(a+1)}{2}x}}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{-l(ax-b+r)+\frac{l(l-1)}{2}x} \\
&= \omega^{ar+\frac{a(a+1)}{2}x} \left|e_{ax-b+r}^x\right\rangle .
\end{aligned}$$

□

This last property is the main ingredient for the construction of our protocol as it encodes a linear evaluation based on values a, b and $x \in \mathbb{Z}_d$[2]. In our protocol, we take a, b – that determine the operators V_a^b – to be Alice's inputs and x to be Bob's input.

Finally, let us see how the operators V_a^b act on the so-called *generalised Bell states*, since Bob's attack to the protocol is based on that. We start with the definition of the *seed* Bell state

$$\left|B_{0,0}\right\rangle = \frac{1}{\sqrt{d}} \sum_i \left|i^*, i\right\rangle ,$$

where the map $\left|\psi\right\rangle \to \left|\psi^*\right\rangle$ is defined by taking the complex conjugate of the coefficients:

$$\left|\psi\right\rangle = \sum_i \beta_i \left|i\right\rangle \to \left|\psi^*\right\rangle = \sum_i \beta_i^* \left|i\right\rangle .$$

Using the properties of the operators V_a^b, we can derive the rest of the generalised Bell states from the seed state, as

$$\left|B_{a,b}\right\rangle = (\mathbb{1} \otimes V_a^b) \left|B_{0,0}\right\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} \omega^{(i+a)b} \left|i^*, i+a\right\rangle , \tag{6.3}$$

[2]While $x \in \mathbb{Z}_{d+1}$, henceforth we consider $x \in \mathbb{Z}_d$, since we only use d out of the $d+1$ MUBs.

and one can prove that the set $\{|B_{a,b}\rangle\}_{(a,b)\in\mathbb{Z}_d^2}$ constitutes an orthonormal maximally entangled basis in the Hilbert space of two-qudit states [43].

6.3 Semi-honest QOLE protocol

In order to build some intuition on the proposed protocol for QOLE, we start by presenting a simpler protocol that is only secure under the semi-honest adversarial model. This semi-honest version leverages the properties of MUBs explored in Section 6.2 and, in particular, the one presented in expression (6.2). As we saw, given the set of MUBs $\{|e_r^x\rangle\}_{r\in\mathbb{Z}_d}$, $\forall x \in \mathbb{Z}_d$, the operators V_a^b simply permute the elements inside the basis x, according to a linear combination of the elements a, b, x and r:

$$V_a^b \, |e_r^x\rangle = c_{a,b,x,r} \left|e^x_{ax-b+r}\right\rangle . \tag{6.4}$$

Alice and Bob can use the above property to compute together a linear function $f(x) = ax - b$, where Alice chooses the parameters a and b, and Bob chooses the input element x. The protocol summarized in Figure 6.1. Bob starts by choosing a basis x and an element r therein, and prepares the state $|e_r^x\rangle$: the basis choice x plays the role of the input element x, and the basis element r is used to enhance Bob's security against a potentially dishonest Alice. Then, he sends the state $|e_r^x\rangle$ to Alice, who, in turn, applies on it the operator V_a^b and sends back to Bob the resulting state. According to (6.4), Bob receives $\left|e^x_{ax-b+r}\right\rangle$, measures it in the x basis, and outputs the linear function evaluation $f(x) = ax - b$ by subtracting r. Thus, the correctness of the protocol is ensured by expression (6.4).

As far as the security of this protocol is concerned, we can easily see that it is secure against a dishonest Alice. From her point of view, all the density matrices describing the several possible cases for $x = 0, \ldots, d-1$ are maximally mixed states. Therefore, she cannot know anything about the value of x.

If, moreover, Bob is semi-honest the protocol remains secure. On the other hand, if Bob is dishonest and deviates from the protocol, he is able to find out Alice's inputs a and b with certainty. In Section 6.2 equation (6.3), we saw that the generalised Bell basis is generated by Alice's operators, V_a^b, i.e. $|B_{a,b}\rangle = (1 \otimes V_a^b)\,|B_{0,0}\rangle$, and Bob can make use of this property in order to extract her inputs a and b. His attack can be described as follows:

1. Bob prepares the state $|B_{0,0}\rangle$ and sends the second qudit to Alice.

2. Alice applies her chosen operator V_a^b.

3. Bob measures both qudits in the generalised Bell basis and outputs a, b.

Semi-honest QOLE

Alice's input: $(a, b) \in \mathbb{Z}_d^2$
Bob's input: $x \in \mathbb{Z}_d$

1. Bob randomly generates $r \in \mathbb{Z}_d$. He prepares and sends the state $|e_r^x\rangle$ to Alice.
2. Alice prepares the operator V_a^b according to her inputs a and b. She then applies V_a^b to Bob's state: $V_a^b |e_r^x\rangle = c_{x,a,b,r} \left|e_{ax-b+r}^x\right\rangle$. She sends the resulting state back to Bob.
3. Bob measures in the basis x, subtracts r, and outputs the desired result $ax - b =: f(x)$.

Alice's output: $\perp$
Bob's output: $f(x)$

Figure 6.1: Semi-honest QOLE protocol.

It becomes clear that the protocol is secure only as long as Bob does not deviate from it; a dishonest Bob can break its security by performing the above attack. Therefore, we have to make sure that Bob sticks to the protocol. To achieve this, we apply a *commit-and-open* scheme [36] that can be briefly described as follows: Bob runs step 1. of the Semi-honest QOLE protocol (see Figure 6.1) multiple times, say m in total, for multiple values of x_i, and r_i, for $i \in [m]$ and commits to these values by means of the functionality $\mathcal{F}_{\mathbf{COM}}$ (see Figure 2.5). Then, he sends these states to Alice, who, in turn, asks him to disclose his chosen x_i's and r_i's for some of the m instances that she chooses. The functionality $\mathcal{F}_{\mathbf{COM}}$ forwards these committed values to Alice and she measures the corresponding received states in the disclosed bases. She can, thus, verify whether she got the right basis element for all the instances she chose to check. If Bob had used the Bell state $|B_{0,0}\rangle$ in one out of the m instances, then the probability of Alice getting the correct result after measuring the state in the committed basis would be $\frac{1}{d}$. In other words, Bob would get caught with high probability $1 - \frac{1}{d}$. Furthermore, if he chooses to attack all the instances, the probability of Alice getting correctly all the results is negligible, i.e. exponentially small in the number of instances, m. We explore this in detail in the next section, where we present a QOLE protocol secure against dishonest adversaries.

6.4 QOLE protocol

Our QOLE protocol is divided into two main phases: a quantum phase and a classical post-processing phase. The first phase uses quantum communication to generate several instances of OLE with random inputs. These instances may leak some information to the parties, therefore we refer to them as random weak OLE (RWOLE). The second phase is purely classical. It uses the RWOLE instances and extracts one classical OLE instance. The post-processing phase has two phases. It implements a derandomization procedure followed by an extraction phase that serves as a privacy amplification method. The full protocol is presented in Figure 6.5. Before we continue, it is worth mentioning that we consider that neither dishonest party maliciously aborts the protocol. Indeed, in our setting, such a behaviour does not provide an advantage for learning the other party's input. The only case to abort the protocol is when honest Alice catches Bob cheating during the *commit-and-open* stage.

In the next sections, we break down the protocol, show its correctness and retrieve some technical lemmas used for the security proof. In Section 6.5, we prove the protocol to be secure in the quantum-UC model against static dishonest adversaries.

Notation. During the RWOLE phase, $\mathbf{F}_0 = (F_1^0, F_2^0, \ldots, F_n^0)$ is the vector whose components are the random variables associated to Alice's functions. Each F_i^0 ranges over the set of affine functions in $\mathbb{Z}_d$ such that $P(F_i^0(x) = a_i^0 x + b_i^0)$ is uniform for all $i \in [n]$. We do not distinguish the set of affine functions in $\mathbb{Z}_d$ from $\mathbb{Z}_d^2$. The classical values $\mathbf{F}_0$ are saved in the Hilbert space $\mathcal{H}_{\mathbf{F}_0}$. The same holds for the derandomization phase, where $\mathbf{F}$ denotes the random variable for Alice's functions in the protocol $\Pi^n_{\mathbf{WOLE}}$. $\mathbf{X}_0$ and $\mathbf{Y}_0$ are the random variables for $\mathbf{x}_0, \mathbf{y}_0 \in \mathbb{Z}_d^n$ in the RWOLE phase. and $\mathbf{X}$ and $\mathbf{Y}$ the corresponding random variables for $\mathbf{x}, \mathbf{y} \in \mathbb{Z}_d^n$ in the post-processing phase. Also, we use A' and B' to denote the system that a dishonest Alice and Bob, respectively, hold at the end of the execution of the protocol.

6.4.1 RWOLE phase

We now introduce the quantum phase of the proposed QOLE protocol, which we informally call the random weak OLE (RWOLE) phase. We denote by Π^n_{RWOLE} the protocol that implements this RWOLE phase and we present it in Figure 6.2. The protocol Π^n_{RWOLE} is divided into four phases: Initialization, Test, Computation and Measurement.

If both parties are honest the protocol is correct: if Alice is honest, her functions $\mathbf{F}_0$ are chosen uniformly at random, and if Bob is honest he will obtain $\left| e^{x_i^0}_{a_i^0 x_i^0 - b_i^0 + r_i} \right\rangle_{i \in [n]}$ according to Equation (6.4).

Protocol $\Pi^n_{\mathbf{RWOLE}}$

Parameters: n, number of output qudits; t, proportion of receiver test qudits.

(Initialization Phase:)

1. Bob randomly generates $m = (1+t)n$ different pairs (x_i^0, r_i) and commits to them by sending $(\texttt{commit}, (i, x_i^0, r_i))$ to $\mathcal{F}_{\mathbf{COM}}$. He prepares the states $\left|e_{r_i}^{x_i^0}\right\rangle_{i\in[m]}$ and sends them to Alice.

(Test Phase:)

2. Alice randomly chooses a subset of indices $T \subset [m]$ of size tn and sends it to Bob.

3. Bob sends $(\texttt{open}, i)$, $i \in T$, to $\mathcal{F}_{\mathbf{COM}}$ and $\mathcal{F}_{\mathbf{COM}}$ sends to Alice $(\texttt{open}, (i, x_i^0, r_i))$, $i \in T$.

4. Alice measures the received qudits in the corresponding x_i^0 basis for $i \in T$, and checks whether the received commitments are compatible with her measurements. In case there is no error she proceeds, otherwise she aborts. After the Test Phase, we relabel and identify $[n] = [m] \setminus T$.

(Computation Phase:)

6. Alice randomly generates n pairs (a_i^0, b_i^0) and prepares $V_{a_i^0}^{b_i^0}$ for $i \in [n]$.

7. Alice applies these operators to the received states, i.e. $V_{a_i^0}^{b_i^0}\left|e_{r_i}^{x_i^0}\right\rangle = c_{x_i^0,a_i^0,b_i^0,r_i}\left|e_{a_i^0x_i^0-b_i^0+r_i}^{x_i^0}\right\rangle$, for $i \in [n]$, and sends the resulting states to Bob.

(Measurement Phase:)

8. Bob measures the received states in the basis x_i^0 for $i \in [n]$ and gets the states $\left|e_{a_i^0x_i^0-b_i^0+r_i}^{x_i^0}\right\rangle, i \in [n]$. Finally, he subtracts r_i, for $i \in [n]$ from his results.

Alice's output: (a_i^0, b_i^0), for $i \in [n]$.
Bob's output: (x_i^0, y_i^0), where $y_i^0 = g_i(x_i^0) = a_i^0x_i^0 - b_i^0$ for $i \in [n]$.

Figure 6.2: RWOLE protocol.

Security. In the case of a dishonest Alice, it is straightforward to verify that the security property of the semi-honest protocol still holds; following the same reasoning, we can conclude that she cannot learn anything about Bob's input or output values (x_i^0, y_i^0). In the case of dishonest Bob, though, these random instances of OLE might leak some information on Alice's random functions $\mathbf{F}_0$ to him. To quantify this side information of Bob, we must bound the min-entropy $H_{\min}(\mathbf{F}_0|B')_{\rho_{\mathbf{F}_0B'}}$ on the state $\rho_{\mathbf{F}_0B'}$, which is the

output state of the real execution of $\Pi^n_{\mathbf{RWOLE}}$. The following lemma shows that $\rho_{\mathbf{F}_0 B'}$ is at least $\epsilon-$close to an ideal state $\sigma_{\mathbf{F}_0 B'}$ independently of the attack that the dishonest party may perform. This ideal state $\sigma_{\mathbf{F}_0 B'}$ has the important property of having a bound on $H_{\min}(\mathbf{F}_0|B')_{\sigma_{\mathbf{F}_0 B'}}$ that is proportional to the security parameter.

Lemma 14 (Security against dishonest Bob)**.** *Let $\rho_{\mathbf{F}_0 B'}$ be the state given by the real execution of the protocol $\Pi^n_{\boldsymbol{RWOLE}}$, where $\mathbf{F}_0$ is the system saving Alice's functions, B' is Bob's (possibly quantum) system. Fix $\zeta \in]0, 1 - \frac{1}{d}]$ and let*

$$\epsilon(\zeta, n) = \exp\left(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\right).$$

Then, for any attack of a dishonest Bob, there exists an ideal classical-quantum state $\sigma_{\mathbf{F}_0 B'}$, such that

1. $\sigma_{\mathbf{F}_0 B'} \approx_\epsilon \rho_{\mathbf{F}_0 B'}$,
2. $H_{\min}(\mathbf{F}_0|B')_{\sigma_{\mathbf{F}_0 B'}} \geq \frac{n \log d}{2}(1 - h_d(\zeta))$,

where $h_d(\zeta)$ is given in Definition 1.

The proof comprises two parts corresponding to the two conditions of Lemma 14: first, we prove that the state just before the *Computation Phase* is close to the ideal state $\sigma_{\mathbf{F}_0 B'}$; and then, we prove that the operators applied by Alice to $\sigma_{\mathbf{F}_0 B'}$ increase the min-entropy by a specific amount that is proportional to the number of output qudits, n. We present the proof in B, where we follow the same reasoning as Damgård et al. in Section 4.3 of [36], and adapt it to our case. We also use certain results from [80] in order to establish the lower bound given by property 2.

6.4.2 Post-processing phase

The $\Pi^n_{\mathbf{RWOLE}}$ protocol (see Figure 6.2) generates several instances of RWOLE, which leak information to Bob about Alice's inputs. In this section, we present the post-processing phase that allows to extract one secure QOLE out of several RWOLE instances. Combining these instances is sufficient to generate a secure QOLE protocol, because Bob has only a negligible probability of attacking *all* the weak instances without being caught; indeed, if he chooses to attack one of the instances the probability of Alice not aborting is $\frac{1}{t+1} + \frac{t}{d(1+t)}$, while if he chooses to attack all instances this probability becomes $\frac{1}{d^{tn}}$, which is negligible in n, thus ensuring the asymptotic security of our protocol. The post-processing comprises two subprotocols: the first is a derandomization protocol (Figure 6.3) that integrates the randomized outputs of RWOLE into a deterministic scheme

where Alice and Bob choose their inputs; the second is an extraction protocol (Figure 6.4) that generates a secure QOLE protocol from these deterministic weak instances by means of a two-universal family of hash functions. Note that the classical post-processing phase does not give any advantage to a potentially dishonest Alice, therefore we only need to prove security against dishonest Bob.

Derandomization

Our derandomization protocol, denoted as $\Pi^n_{\mathbf{WOLE}}$ and summarized in Figure 6.3, reduces the randomized RWOLE instances into deterministic ones, which we informally call weak OLE (WOLE). The output of $\Pi^n_{\mathbf{WOLE}}$ is still a weak version of OLE because Bob is allowed to have some knowledge on Alice's inputs. The difference between RWOLE and WOLE is that the parties now choose their inputs. Our derandomization protocol is an adaptation of the derandomization protocol in [275]. We denote by $*$ the product of two matrices of the same dimensions, such that the result is also a matrix of the same dimensions whose elements are the product of the respective elements of the operand matrices.

Protocol $\Pi^n_{\mathbf{WOLE}}$

Alice's input: $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}_d^{2n}$
Bob's input: $\boldsymbol{x} \in \mathbb{Z}_d^n$

1. Alice and Bob run the $\Pi^n_{\mathbf{RWOLE}}$ protocol and receive $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ and $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, respectively.
2. Bob computes and sends to Alice $\boldsymbol{c} = \boldsymbol{x} - \boldsymbol{x}_0$.
3. Alice computes and sends to Bob $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{a}_0$ and $\boldsymbol{s} = \boldsymbol{b}_0 + \boldsymbol{a} * \boldsymbol{c} + \boldsymbol{b}$.
4. Bob computes $\boldsymbol{y} = \boldsymbol{y}_0 + \boldsymbol{x} * \boldsymbol{d} - \boldsymbol{d} * \boldsymbol{c} + \boldsymbol{s}$.

Alice's output: $\perp$
Bob's output: $\boldsymbol{y} = \boldsymbol{a} * \boldsymbol{x} + \boldsymbol{b}$

Figure 6.3: WOLE protocol.

Security. The requirements to prove security against dishonest Bob are summarized in Lemma 15, which is very similar in structure to Lemma 14. We show that the real output state $\rho_{\mathbf{F}B'}$ of the protocol $\Pi^n_{\mathbf{WOLE}}$ is $\epsilon-$close to an ideal state $\sigma_{\mathbf{F}B'}$, which has min-entropy lower-bounded by a fixed value proportional to the security parameter n. Intuitively, this means that Bob's state is indistinguishable from a state where his knowledge on Alice's

inputs is limited.

Lemma 15. *Let $\rho_{\mathbf{F}B'}$ be the state given by the real execution of the protocol $\Pi^n_{\boldsymbol{WOLE}}$, where $\mathbf{F}$ is the system saving Alice's inputs, B' is Bob's (possibly quantum) system. Fix $\zeta \in]0, 1 - \frac{1}{d}]$ and let*

$$\epsilon(\zeta, n) = \exp\left(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\right). \tag{6.5}$$

Then, for any attack of a dishonest Bob, there exists a classical-quantum state $\sigma_{\mathbf{F}B'}$ such that

1. $\sigma_{\mathbf{F}B'} \approx_\epsilon \rho_{\mathbf{F}B'}$,
2. $H_{\min}(\mathbf{F}|B')_{\sigma_{\mathbf{F}B'}} \geq \frac{n \log d}{2}(1 - h_d(\zeta))$,

where $h_d(\zeta)$ is given in Definition 1.

Proof. Alice holds the system $A = \mathbf{F}\mathbf{F}_0\mathbf{CDS}$, where $\mathbf{F} = (\mathbf{F}_{\boldsymbol{a}}, \mathbf{F}_{\boldsymbol{b}})$ refers to her inputs $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}_d^{2n}$, $\mathbf{F}_0 = (\mathbf{F}_{\boldsymbol{a}_0}, \mathbf{F}_{\boldsymbol{b}_0})$ is the subsystem obtained from the RWOLE phase, and $\mathbf{C}, \mathbf{D}$ and $\mathbf{S}$ are classical subsystems used to save the values of $\boldsymbol{c}$, $\boldsymbol{d}$, and $\boldsymbol{s}$ from the protocol, respectively. Bob holds the system $B' = \mathbf{CDS}B'_0$ where $\mathbf{C}, \mathbf{D}$ and $\mathbf{S}$ are the subsystems on Bob's side where the values of $\boldsymbol{c}$, $\boldsymbol{d}$ and $\boldsymbol{s}$ are saved, respectively, and $B'_0 = \mathbf{Y}_0 E_0$ is his (possibly quantum) system generated from the RWOLE phase.

To prove property 1., we will use Lemma 14, namely that the state $\rho_{\mathbf{F}_0 B'_0}$ resulting from the RWOLE scheme is $\epsilon-$close to the ideal state $\sigma_{\mathbf{F}_0 B'_0}$. Then, we will show that the operations applied to $\rho_{\mathbf{F}_0 B'_0}$ during the derandomization process can only decrease the distance between the real and the ideal output states of the WOLE protocol, thus keeping them at least $\epsilon-$close. We start by specifying the operators corresponding to the classical operations executed in steps 2 and 3 of $\Pi^n_{\mathbf{WOLE}}$. In step 2, a dishonest Bob can send to Alice some value $\boldsymbol{c}$ that depends on his system B'_0. So, he starts by applying a CPTP map $\mathcal{T}_{B'_0 \to \mathbf{C}B'_0} : \mathcal{P}\left(\mathcal{H}_{B'_0}\right) \to \mathcal{P}\left(\mathcal{H}_{B'_0} \otimes \mathcal{H}_{\mathbf{C}}\right)$ to his state and then projects it into the Hilbert space $\mathcal{H}_{\mathbf{C}}$. The operator for step 2 is a CPTP map

$$\mathcal{O}^{(2)} : \mathcal{P}\left(\mathcal{H}_{\mathbf{F}_0} \otimes \mathcal{H}_{B'_0}\right) \to \mathcal{P}\left(\mathcal{H}_{\mathbf{F}_0} \otimes \mathcal{H}_{\mathbf{C}} \otimes \mathcal{H}_{\mathbf{D}} \otimes \mathcal{H}_{\mathbf{S}} \otimes \mathcal{H}_{B'_0}\right)$$

described by his action on some general quantum state ρ, as

$$\mathcal{O}^{(2)}(\rho) = \mathbb{1} \otimes \sum_{\boldsymbol{d},\boldsymbol{s},\boldsymbol{c}} |\boldsymbol{c}\rangle\langle\boldsymbol{c}|_{\mathbf{C}}\, \mathcal{T}_{B'_0 \to \mathbf{C}B'_0}(\rho)\, |\boldsymbol{c}\rangle\langle\boldsymbol{c}|_{\mathbf{C}} \otimes |\boldsymbol{d}\rangle\langle\boldsymbol{d}|_{\mathbf{D}} \otimes |\boldsymbol{s}\rangle\langle\boldsymbol{s}|_{\mathbf{S}}\,.$$

In step 3, Bob takes no action. Since Alice is honest, the operator for this step simply describes her action on subsystems $\mathbf{D}$ and $\mathbf{S}$ according to her choice at subsystem $\mathbf{F}$. This

operator is a CPTP map

$$\mathcal{O}^{(3)} : \mathcal{P}\left(\mathcal{H}_{\mathbf{F}_0} \otimes \mathcal{H}_{\mathbf{C}} \otimes \mathcal{H}_{\mathbf{D}} \otimes \mathcal{H}_{\mathbf{S}} \otimes \mathcal{H}_{B_0'}\right) \to \mathcal{P}\left(\mathcal{H}_{\mathbf{F}} \otimes \mathcal{H}_{\mathbf{F}_0} \otimes \mathcal{H}_{\mathbf{C}} \otimes \mathcal{H}_{\mathbf{D}} \otimes \mathcal{H}_{\mathbf{S}} \otimes \mathcal{H}_{B_0'}\right)$$

described by his action on some general quantum state ρ, as

$$\mathcal{O}^{(3)}(\rho) = \frac{1}{d^{2n}} \sum_{\boldsymbol{a},\boldsymbol{b}} \mathcal{P}^{\boldsymbol{a},\boldsymbol{b}} \, \rho \, (\mathcal{P}^{\boldsymbol{a},\boldsymbol{b}})^{\dagger},$$

where

$$\mathcal{P}^{\boldsymbol{a},\boldsymbol{b}} = |\boldsymbol{a},\boldsymbol{b}\rangle_{\mathbf{F}} \otimes \sum_{\boldsymbol{a}_0,\boldsymbol{b}_0,\boldsymbol{c}} |\boldsymbol{a}_0,\boldsymbol{b}_0\rangle\langle\boldsymbol{a}_0,\boldsymbol{b}_0|_{\mathbf{F}_0} \otimes |\boldsymbol{c}\rangle\langle\boldsymbol{c}|_{\mathbf{C}} \otimes |\boldsymbol{a}-\boldsymbol{a}_0\rangle\langle\boldsymbol{a}-\boldsymbol{a}_0|_{\mathbf{D}} \otimes |\boldsymbol{b}_0+\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}\rangle\langle\boldsymbol{b}_0+\boldsymbol{a}\cdot\boldsymbol{c}+\boldsymbol{b}|_{\mathbf{S}}\,.$$

Note that $\mathcal{O}^{(2)}$ adds subsystems **CDS** and distributes **C** according to Bob's action. The operator $\mathcal{O}^{(3)}$ adds subsystem **F** and projects **DS** according to the information at subsystem $\mathbf{FF}_0$ and the expressions of $\boldsymbol{d}$ and $\boldsymbol{s}$. Regarding the trace distance between the real and ideal states, we have:

$$\begin{aligned}
\delta(\rho_{\mathbf{F}_0 B_0'}, \sigma_{\mathbf{F}_0 B_0'}) &\geq \delta\Big(\mathcal{O}^{(2)}(\rho_{\mathbf{F}_0 B_0'}), \mathcal{O}^{(2)}(\sigma_{\mathbf{F}_0 B_0'})\Big) \\
&\geq \delta\Big(\mathcal{O}^{(3)}\mathcal{O}^{(2)}(\rho_{\mathbf{F}_0 B_0'}), \mathcal{O}^{(3)}\mathcal{O}^{(2)}(\sigma_{\mathbf{F}_0 B_0'})\Big) \\
&= \delta(\rho_{\mathbf{F}B'}, \sigma_{\mathbf{F}B'}).
\end{aligned}$$

For the above inequalities, we took into account that $\mathcal{O}^{(2)}$ and $\mathcal{O}^{(3)}$ are CPTP maps, and as such they do not increase the trace distance (see Lemma 1). For the last equality, recall that $B' = \boldsymbol{CDS}B_0'$. Now, from Lemma 14, we have that $\sigma_{\mathbf{F}_0 B_0'} \approx_\epsilon \rho_{\mathbf{F}_0 B_0'}$. Hence, we conclude that $\delta(\rho_{\mathbf{F}B'}, \sigma_{\mathbf{F}B'}) \leq \epsilon(\zeta, n)$ for $\epsilon(\zeta, n)$ given as (6.5), i.e. $\sigma_{\mathbf{F}B'} \approx_\epsilon \rho_{\mathbf{F}B'}$.

We move on to prove property 2. Consider the bijective function $g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}} : \mathbb{Z}_d^{2n} \to \mathbb{Z}_d^{2n}$ given by

$$g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}}(\boldsymbol{x},\boldsymbol{y}) = (\boldsymbol{x}+\boldsymbol{d}, \boldsymbol{s}-\boldsymbol{y}-(\boldsymbol{x}+\boldsymbol{d})*\boldsymbol{c})$$

for fixed $\boldsymbol{c}, \boldsymbol{d}$ and $\boldsymbol{s}$. Essentially, $g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}}$ describes how the input vector $(\boldsymbol{a},\boldsymbol{b})$ is related to the RWOLE output vector $(\boldsymbol{a}_0,\boldsymbol{b}_0)$:

$$(\boldsymbol{a},\boldsymbol{b}) = g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}}(\boldsymbol{a}_0,\boldsymbol{b}_0) = (\boldsymbol{a}_0+\boldsymbol{d}, \boldsymbol{s}-\boldsymbol{b}_0-(\boldsymbol{a}_0+\boldsymbol{d})*\boldsymbol{c}). \tag{6.6}$$

Intuitively, this means that the subsystem **F** is defined by the subsystems $\mathbf{F}_0\mathbf{CDS}$. We

can rewrite the action of the operator $\mathcal{O}^{(3)}$ on some general quantum state ρ as follows:

$$\mathcal{O}^{(3)}(\rho) = \frac{1}{d^{2n}} \sum_{\boldsymbol{d},\boldsymbol{s}} \mathcal{P}^{\boldsymbol{d},\boldsymbol{s}} \, \rho \, (\mathcal{P}^{\boldsymbol{d},\boldsymbol{s}})^{\dagger},$$

where

$$\mathcal{P}^{\boldsymbol{d},\boldsymbol{s}} = \sum_{\boldsymbol{a}_0,\boldsymbol{b}_0,\boldsymbol{c}} \left| g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}}(\boldsymbol{a}_0,\boldsymbol{b}_0) \right\rangle\!\left\langle \boldsymbol{a}_0,\boldsymbol{b}_0 \right|_{\mathbf{F}} \otimes |\boldsymbol{a}_0,\boldsymbol{b}_0,\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}\rangle\langle \boldsymbol{a}_0,\boldsymbol{b}_0,\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}|_{\mathbf{F}_0 \boldsymbol{CDS}} \, .$$

Hence, for the min-entropy bound, we have that:

$$\begin{aligned}
H_{\min}(\mathbf{F} \,|\, B')_{\sigma_{\mathbf{F}B'}} &= H_{\min}(\mathbf{F}_{\boldsymbol{a}}, \mathbf{F}_{\boldsymbol{b}} \,|\, B')_{\sigma_{\mathbf{F}B'}} \\
&= H_{\min}(g^{\boldsymbol{C},\boldsymbol{D},\boldsymbol{S}}(\mathbf{F}_{\boldsymbol{a}_0}, \mathbf{F}_{\boldsymbol{b}_0}) \,|\, \mathbf{CDS} B'_0)_{\mathcal{O}^{(3)}\mathcal{O}^{(2)}\sigma_{\mathbf{F}_0 B'_0}} \\
&\geq H_{\min}(\mathbf{F}_{\boldsymbol{a}_0}, \mathbf{F}_{\boldsymbol{b}_0} \,|\, \mathbf{CDS} B'_0)_{\mathcal{O}^{(2)}\sigma_{\mathbf{F}_0 B'_0}} \qquad (6.7) \\
&\geq H_{\min}(\mathbf{F}_{\boldsymbol{a}_0}, \mathbf{F}_{\boldsymbol{b}_0} \,|\, B'_0)_{\sigma_{\mathbf{F}_0 B'_0}} \qquad (6.8) \\
&\geq \frac{n \log d}{2}(1 - h_d(\zeta)). \qquad (6.9)
\end{aligned}$$

The inequality at step (6.7) comes from Lemma 4, as $g^{\boldsymbol{c},\boldsymbol{d},\boldsymbol{s}}$ is bijective. The inequality at step (6.8) comes from Lemma 10, as the operator $\mathcal{O}^{(2)}$ takes the form of $\mathcal{O}^{(2)} = \mathbb{1} \otimes \mathcal{M}$, where $\mathcal{M}$ is a CPTP map. The last inequality comes from Lemma 14, property 2.

□

Extraction

In this section, we present our extraction protocol, $\Pi_{\mathbf{EXT}}$, that generates one OLE instance using the derandomization protocol $\Pi^n_{\mathbf{WOLE}}$ and the two-universal family of hash functions, MMH^* (see [83] and Definition 5). This family uses the inner product between two vectors in the $\mathbb{Z}_d^n$ space, and since OLE only involves linear operations, we can apply the inner product operation to all vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{y}$ without affecting the overall structure. The protocol $\Pi_{\mathbf{EXT}}$ is summarized in Figure 6.4 and uses the n instances of WOLE in such a way that Bob's knowledge on Alice's inputs decreases exponentially with respect to n[3] For this reason, n is our security parameter.

[3]This extraction step is similar to the privacy amplification step of QKD protocols.

Protocol $\Pi_{\mathbf{EXT}}$

Alice's input: $(a, b) \in \mathbb{Z}_d^2$
Bob's input: $x \in \mathbb{Z}_d$

1. Alice chooses randomly some function $g_{\boldsymbol{\kappa}} \in MMH^*$ and sends it to Bob.
2. Alice randomly generates $a_2, \ldots, a_n, b_2, \ldots, b_n \leftarrow_{\$} \mathbb{Z}_d$. She computes $a_1 = \left(a - \sum_{i=2}^{n} a_i \kappa_i\right)/\kappa_1$ and $b_1 = \left(b - \sum_{i=2}^{n} b_i \kappa_i\right)/\kappa_1$. We write $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$.
3. Alice and Bob run the derandomization protocol $\Pi^n_{\mathbf{WOLE}}((\boldsymbol{a}, \boldsymbol{b}), \boldsymbol{x})$. Bob receives $\boldsymbol{y}$ as output.
4. Bob computes $y = g_{\boldsymbol{\kappa}}(\boldsymbol{y})$.

Alice's output: $\perp$
Bob's output: y

Figure 6.4: Extraction protocol.

The correctness of $\Pi_{\mathbf{EXT}}$ is given by linearity:

$$\begin{aligned} y &= g_{\boldsymbol{\kappa}}(\boldsymbol{y}) = \boldsymbol{\kappa} \cdot (a_1 x + b_1, \ldots a_n x + b_n) \\ &= \boldsymbol{\kappa} \cdot \left(\frac{a - \sum_{i=2}^{n} a_i \kappa_i}{\kappa_1} x + \frac{b - \sum_{i=2}^{n} b_i \kappa_i}{\kappa_1},\ a_2 x + b_2, \ldots, a_n x + b_n \right) \\ &= ax + b. \end{aligned}$$

Security. By definition, the derandomization protocol leaks some information to Bob about Alice's inputs $(\boldsymbol{a}, \boldsymbol{b})$. Since $\boldsymbol{y} = \boldsymbol{a} * \boldsymbol{x} + \boldsymbol{b}$, without loss of generality, any leakage of Alice's inputs can be seen as a leakage on just $\boldsymbol{a}$. In this case, the min-entropy of $\mathbf{F} = (\mathbf{F}_{\boldsymbol{a}}, \mathbf{F}_{\boldsymbol{b}})$ should be the same as the min-entropy of $\mathbf{F}_{\boldsymbol{a}}$. Now, recall the $\mathcal{F}_{\mathbf{OLE}}$ functionality definition (see Figure 2.3), and note that Bob does not possess any knowledge about Alice's input (a, b) other than what can be deduced from his input and output (x, y). Similarly, since $y = ax + b$, Bob has some knowledge on the relation between a and b and – as b is completely determined by (a, x, y) – we only have to guarantee that a looks uniformly random to Bob. The role of the hash functions used in the above protocol $\Pi_{\mathbf{EXT}}$ is precisely to extract a uniformly random a from the leaky vector $\boldsymbol{a}$, while preserving the structure of the OLE. This result is summarized in Lemma 16 and its proof is based on Lemma 11.

Lemma 16. *Let $\rho_{FB'}$ be the state given by the real execution of the protocol $\Pi_{\boldsymbol{EXT}}$, where*

F is the system saving Alice's inputs (a, b), B' is Bob's (possibly quantum) system. Fix $\zeta \in]0, 1 - \frac{1}{d}]$ and let

$$\epsilon(\zeta, n) = \exp\left(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\right).$$

Then, for any attack of a dishonest Bob, there exists a classical-quantum state $\sigma_{FB'}$, where $F = (F_a, F_b)$, such that

1. *$\sigma_{FB'} \approx_\epsilon \rho_{FB'}$, and*

2. *$\delta(\tau_{\mathbb{Z}_d} \otimes \sigma_{B'}, \sigma_{F_a B'}) \leq K\, 2^{-n\, f_d(\zeta)}$, where $K = \frac{\sqrt{d}}{2}$, $f_d(\zeta) = \frac{\log d}{4}(1 - h_d(\zeta))$, n is the security parameter, and $h_d(\zeta)$ is given in Definition 1.*

Proof. To prove property 1, we note that the extraction operation applied to the output of $\Pi^n_{\mathbf{WOLE}}$ can be described by a projective operator on the space $F = (F_a, F_b)$. Therefore, as in the case of Lemma 15, property 1 follows from the fact that CPTP maps do not increase the trace distance (see Lemma 1).

Regarding property 2, let us first consider Bob's subsystem E to integrate Bob's inputs $\boldsymbol{x}$, i.e. $E = \mathbf{X}E'$. Then, his full system B' is identified with $\mathbf{Y}E = \mathbf{YX}E'$. We have:

$$\begin{aligned} H_{\min}(\mathbf{F} \,|\, \mathbf{Y}E)_{\sigma_{\mathbf{FY}E}} &= H_{\min}(\mathbf{F}_{\boldsymbol{a}}, \mathbf{F}_{\boldsymbol{b}} \,|\, \mathbf{YX}E')_{\sigma_{\mathbf{FY}E}} \\ &= H_{\min}(\mathbf{F}_{\boldsymbol{a}}, \mathbf{Y} - \mathbf{F}_{\boldsymbol{a}}\mathbf{X} \,|\, \mathbf{YX}E')_{\sigma_{\mathbf{F}_a\mathbf{Y}E}} \\ &= H_{\min}(\mathbf{F}_{\boldsymbol{a}} \,|\, \mathbf{YX}E')_{\sigma_{\mathbf{F}_a\mathbf{Y}E}}. \end{aligned}$$

Therefore,

$$H_{\min}(\mathbf{F}_{\boldsymbol{a}} \,|\, \mathbf{Y}E)_{\sigma_{\mathbf{F}_a\mathbf{Y}E}} \geq \frac{n \log d}{2}(1 - h_d(\zeta)).$$

Now, since MMH^* is a two-universal family of hash functions, we can directly apply Lemma 11 for $l = 1$. It follows that F_a is $\xi-$close to uniform conditioned on $\mathbf{Y}E$, i.e.

$$\delta(\tau_{\mathbb{Z}_d} \otimes \sigma_{\mathbf{Y}E}, \sigma_{F_a\mathbf{Y}E}) \leq \frac{1}{2}\sqrt{2^{\log d - \frac{n \log d}{2}(1 - h_d(\zeta))}} = K\, 2^{-n\, f_d(\zeta)} =: \xi$$

where $K = \frac{\sqrt{d}}{2}$, $f_d(\zeta) = \frac{\log d}{4}(1 - h_d(\zeta))$ and n is the security parameter.

□

Now, we are in position to combine the above subprotocols ($\Pi^n_{\mathbf{RWOLE}}$, $\Pi^n_{\mathbf{WOLE}}$ and $\Pi_{\mathbf{EXT}}$) and present the full protocol $\Pi_{\mathbf{QOLE}}$ in Figure 6.5.

Protocol $\Pi_{\mathbf{QOLE}}$

Parameters: n, security parameter; tn, number of test qudits.
Alice's input: $(a, b) \in \mathbb{Z}_d^2$
Bob's input: $x \in \mathbb{Z}_d$

(Quantum phase:)

1. Bob randomly generates $m = (1+t)n$ different pairs $(x_i^0, r_i) \in \mathbb{Z}_d^2$ and commits to them by sending $(\texttt{commit}, (i, x_i^0, r_i)_{i\in[m]})$ to $\mathcal{F}_{\mathbf{COM}}$. He also prepares the quantum states $\left|e_{r_i}^{x_i^0}\right\rangle_{i\in[m]}$ and sends them to Alice.
2. Alice randomly chooses a subset of indices $T \subset [m]$ of size tn and sends it to Bob.
3. Bob sends $(\texttt{open}, i)_{i\in T}$ to $\mathcal{F}_{\mathbf{COM}}$ and $\mathcal{F}_{\mathbf{COM}}$ sends to Alice $(\texttt{open}, (i, x_i^0, r_i))_{i\in T}$.
4. Alice measures the received quantum states in the corresponding x_i^0 basis for $i \in T$, and checks whether the received commitments are compatible with her measurements. She proceeds in case there is no error, otherwise she aborts.
5. Alice randomly generates n pairs $(a_i^0, b_i^0) \in \mathbb{Z}_d^2$ and prepares $V_{a_i^0}^{b_i^0}$ for $i \in [m] \setminus T$. We relabel $\boldsymbol{a}_0 = (a_1^0, \ldots, a_n^0)$, $\boldsymbol{b}_0 = (b_1^0, \ldots, b_n^0)$ and $\boldsymbol{x}_0 = (x_1^0, \ldots, x_n^0)$, and from now on identify $[m] \setminus T \equiv [n]$.
6. Alice $\forall i \in [n]$ applies $V_{a_i^0}^{b_i^0}$ to the received state $\left|e_{r_i}^{x_i^0}\right\rangle$, i.e. $V_{a_i^0}^{b_i^0}\left|e_{r_i}^{x_i^0}\right\rangle = c_{x_i^0,a_i^0,b_i^0,r_i}\left|e_{a_i^0 x_i^0 - b_i^0 + r_i}^{x_i^0}\right\rangle$, and sends the resulting states to Bob.
7. Bob $\forall i \in [n]$ measures the received state in the corresponding basis x_i^0, and gets the state $\left|e_{a_i^0 x_i^0 - b_i^0 + r_i}^{x_i^0}\right\rangle$. Finally, $\forall i \in [n]$ he subtracts r_i from his result and gets $y_i^0 = a_i^0 x_i^0 - b_i^0$. We write $\boldsymbol{y}_0 = (y_1^0, \ldots, y_n^0)$.

(Post-processing phase:)

8. Bob defines $\boldsymbol{x} = (x, \ldots, x)$ as the constant vector according to his input x.
9. Alice chooses randomly some function $g_{\boldsymbol{\kappa}} \in MMH^*$, and she randomly generates $a_2, \ldots, a_n, b_2, \ldots, b_n \leftarrow_{\$} \mathbb{Z}_d$. She computes $a_1 = \big(a - \sum_{i=2}^n a_i\kappa_i\big)/\kappa_1$ and $b_1 = \big(b - \sum_{i=2}^n b_i\kappa_i\big)/\kappa_1$. We write $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$.
10. Bob computes and sends to Alice $\boldsymbol{c} = \boldsymbol{x} - \boldsymbol{x}_0$.
11. Alice computes and sends to Bob $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{a}_0$ and $\boldsymbol{s} = \boldsymbol{b}_0 + \boldsymbol{a} * \boldsymbol{c} + \boldsymbol{b}$.
12. Bob computes $\boldsymbol{y} = \boldsymbol{y}_0 + \boldsymbol{x} * \boldsymbol{d} - \boldsymbol{d} * \boldsymbol{c} + \boldsymbol{s}$.
13. Finally, Alice sends $\boldsymbol{\kappa}$ to Bob and he computes $y = g_{\boldsymbol{\kappa}}(\boldsymbol{y})$.

Alice's output: $\perp$
Bob's output: y

Figure 6.5: QOLE protocol.

6.5 UC security

In this section, we will show that our protocol $\Pi_{\mathbf{QOLE}}$ (see Figure 6.5) is quantum-UC secure. More formally, we will show that $\Pi_{\mathbf{QOLE}}$ statistically quantum-UC realizes (see Definition 6) the functionality $\mathcal{F}_{\mathbf{OLE}}$ in the $\mathcal{F}_{\mathbf{COM}}-$hybrid model.

Theorem 4 (quantum-UC security of $\Pi_{\mathbf{QOLE}}$)**.** *The protocol $\Pi_{\boldsymbol{QOLE}}$ from Figure 6.5 statistically quantum-UC realizes (see Definition 6) $\mathcal{F}_{\boldsymbol{OLE}}$ in the $\mathcal{F}_{\boldsymbol{COM}}-$hybrid model.*

Theorem 4 is proved by combining Lemma 17 and Lemma 18 that we present below. In the former we prove the protocol's security for the case where Alice is dishonest and Bob is honest, while in the latter we prove security in the case where Alice is honest and Bob dishonest. In the first case, we have:

Lemma 17. *The protocol $\Pi_{\boldsymbol{QOLE}}$ (Figure 6.5) statistically quantum-UC realizes (see Definition 6) $\mathcal{F}_{\boldsymbol{OLE}}$ in the $\mathcal{F}_{\boldsymbol{COM}}-$hybrid model in the case of dishonest Alice and honest Bob.*

Proof. We start by presenting the simulator $\mathcal{S}_A$ for the case where Alice is dishonest in Figure 6.6.

To prove statistical quantum-UC security according to Definition 6, we first consider a sequence of hybrid protocols from H_0 to H_4. The first hybrid protocol, H_0, in the sequence is the real execution of the protocol $\Pi_{\mathbf{QOLE}}$, and we gradually change it until obtaining the hybrid H_4 which corresponds to the description of the simulator $\mathcal{S}_A$. By proving indistinguishaility of the hybrids throughout the sequence, we show statistical quantum-UC security for the protocol $\Pi_{\mathbf{QOLE}}$ in the case of dishonest Alice.

Hybrid H_0: This is the real execution of the protocol $\Pi_{\mathbf{QOLE}}$.

Hybrid H_1: This hybrid is identical to the previous one, H_0, except that we replace the functionality $\mathcal{F}_{\mathbf{COM}}$ with a fake commitment functionality, $\mathcal{F}_{\mathbf{FakeCOM}}$, in which Bob, i.e. the honest party, can commit no value. This fake functionality works as follows:

- Commitment phase: expects a `commit` message from Bob instead of (`commit,` x).
- Open phase: expects a message (`open`, x) (instead of open) and sends (`open`, x) to Alice.

Hybrids H_0 and H_1 are perfectly indistinguishable, as the simulator still opens the commitments in the same way.

Hybrid H_2: This hybrid is identical to the previous one, H_1, except that now $\mathcal{S}_A$ prepares entangled states $|B_{0,0}\rangle_{Q_AQ_S}$ instead of $\left|e_{r_i}^{x_i^0}\right\rangle_{i\in[m]}$, and sends the subsystem Q_A to Alice. Additionally, upon receiving the set of indices, T, from Alice, $\mathcal{S}_A$ measures the corresponding elements of subsystem Q_S using tn randomly chosen bases x_i^0 and provides $(\mathtt{open}, (i, x_i^0, r_i))$ to $\mathcal{F}_{\mathbf{FakeCOM}}$, $\forall i \in T$.

Simulator $\mathcal{S}_\mathbf{A}$

(Quantum phase:)

1. $\mathcal{S}_A$ sends `commit` to $\mathcal{F}_{\mathbf{FakeCOM}}$.
2. $\mathcal{S}_A$ generates $m = (1+t)n$ entangled states $|B_{0,0}\rangle_{Q_A Q_S}$ and sends subsystem Q_A to Alice.
3. Alice asks for a set of indices $T \subset [m]$ of size tn.
4. $\mathcal{S}_A$ measures the corresponding elements of subsystem Q_S using tn randomly chosen bases x_i^0 and provides $(\texttt{open}, (i, x_i^0, r_i))$ to $\mathcal{F}_{\mathbf{FakeCOM}},\ \forall i \in T$.
5. Upon receiving the processed system $\hat{Q}_A$ from Alice, $\mathcal{S}_A$ measures the joint system $\hat{Q}_A Q_S$ and extracts the measurement outcomes $\mathbf{F} = (\boldsymbol{a}_0, \boldsymbol{b}_0) = \big((a_1^0, \ldots, a_n^0), (b_1^0, \ldots, b_n^0)\big)$.

(Post-processing phase:)

6. $\mathcal{S}_A$ randomly generates a vector $\boldsymbol{c}'$ and sends to Alice.
7. Upon receiving $\boldsymbol{d}$ and $\boldsymbol{s}$ from Alice, $\mathcal{S}_A$ extracts $\boldsymbol{a}$ and $\boldsymbol{b}$ based on its knowledge of $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ as follows:

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{b} + \boldsymbol{a}_0 \\ \boldsymbol{b} &= \boldsymbol{s} - \boldsymbol{b}_0 - \boldsymbol{a} * \boldsymbol{c}'. \end{aligned} \tag{6.10}$$

8. Upon receiving $\boldsymbol{\kappa}$ from Alice, $\mathcal{S}_A$ extracts her inputs (a, b) as follows:

$$\begin{aligned} a &= \boldsymbol{a} \cdot \boldsymbol{\kappa} \\ b &= \boldsymbol{b} \cdot \boldsymbol{\kappa}. \end{aligned} \tag{6.11}$$

9. Finally, $\mathcal{S}_A$ sends (a, b) to the ideal functionality $\mathcal{F}_{\mathbf{OLE}}$.

Figure 6.6: Simulator $\mathcal{S}_A$ against dishonest Alice.

Claim 5. The hybrids H_1 and H_2 are indistinguishable.

Proof. From Alice's point of view, the state received is exactly the same in both hybrids. In H_1, since the elements r are chosen randomly,

$$\frac{1}{d}\sum_{r=0}^{d-1} \left|e_r^{x^0}\right\rangle\!\left\langle e_r^{x^0}\right| = \frac{\mathbb{1}_A}{d},$$

for each $x^0 = 0, \ldots, d-1$. In H_2

$$\mathrm{Tr}_{Q_S} |B_{0,0}\rangle\langle B_{0,0}| = \frac{\mathbb{1}_A}{d}.$$

Thus, the environment is not able to distinguish the two scenarios. Furthermore, upon Alice's request of the test set, T, the simulator measures in random bases, x_i^0 for $i \in T$, the corresponding qudits of subsystem Q_S. Since both entangled qudits in $Q_A Q_S$ get projected to the some random state, r_i for $i \in T$, $\mathcal{F}_{\mathbf{FakeCOM}}$ provides the correct pair $(x_i^0, r_i)_{i \in T}$ to Alice. Hence, the hybrids H_1 and H_2 are indistinguishable. □

Hybrid H_3: This hybrid is identical to the previous one, H_2, except that now $\mathcal{S}_A$ extracts Alice's elements $\mathbf{F}_0 = (\boldsymbol{a}_0, \boldsymbol{b}_0)$ by applying a joint measurement on the systems $\hat{Q}_A Q_S$ in the generalised Bell basis.

Hybrids H_2 and H_3 are perfectly indistinguishable, as the simulator only changes the measurement basis for the received state and does not communicate with Alice.

Hybrid H_4: This hybrid is identical to the previous one, H_3, except that now $\mathcal{S}_A$ generates $\boldsymbol{c}'$ uniformly at random. Additionally, upon receiving $\boldsymbol{d}$, $\boldsymbol{s}$ and $\boldsymbol{\kappa}$, the simulator extracts Alice's vectors $(\boldsymbol{a}, \boldsymbol{b})$ and inputs (a, b) by computing expressions (6.10) and (6.11). Finally, $\mathcal{S}_A$ sends (a, b) to the ideal functionality $\mathcal{F}_{\mathbf{OLE}}$. Hybrid H_4 corresponds to the description of the simulator $\mathcal{S}_A$.

Hybrids H_3 and H_4 are perfectly indistinguishable for the following reasons: first, from the proof of Claim 5, we have that the vector $\boldsymbol{x}_0$ looks uniformly random to Alice, and consequently, so does $\boldsymbol{c}$. Second, the extraction operations do not require any interaction with Alice. □

We now proceed to the case where Alice is honest and Bob is dishonest. We have:

Lemma 18. *The protocol $\Pi_{\boldsymbol{QOLE}}$ (Figure 6.5) statistically quantum-UC realizes (see Definition 6) $\mathcal{F}_{\boldsymbol{OLE}}$ in the $\mathcal{F}_{\boldsymbol{COM}}$−hybrid model in the case of honest Alice and dishonest Bob.*

Proof. We start by presenting the simulator $\mathcal{S}_B$ for the case where Bob is dishonest in Figure 6.7.

Then, we consider the following sequence of hybrid protocols, from H_0 corresponding to the execution of the real protocol to H_2 corresponding to the description of the simulator $\mathcal{S}_B$, and prove that they are indistinguishable in the case of dishonest Bob.

Simulator $\mathcal{S}_\mathbf{B}$

(Quantum phase:)

1. $\mathcal{S}_B$ receives the qudits from Bob and tests them as in the protocol $\Pi_{\mathbf{QOLE}}$.
2. $\mathcal{S}_B$ randomly chooses vectors $\boldsymbol{a}_0$ and $\boldsymbol{b}_0$ and applies $V_{a_i^0}^{b_i^0}$, $i \in [n]$ to the received qudits.
3. $\mathcal{S}_B$ extracts the input element $\boldsymbol{x}_0$ from $\mathcal{F}_{\mathbf{COM}}$.

(Post-processing phase:)

4. Upon receiving $\boldsymbol{c}$ from Bob, $\mathcal{S}_B$ extracts his input x as $\boldsymbol{x} = \boldsymbol{c} + \boldsymbol{x}_0$.
5. $\mathcal{S}_B$ sends x to $\mathcal{F}_{\mathbf{OLE}}$ and receives y.
6. $\mathcal{S}_B$ randomly generates the elements $a' \leftarrow_\$ \mathbb{Z}_d$, $\boldsymbol{\kappa} \leftarrow_\$ \mathbb{Z}_d^n$ and $a_2, \ldots, a_n, b_2, \ldots, b_n \leftarrow_\$ \mathbb{Z}_d$.
7. $\mathcal{S}_B$ computes $b' = a'x - y$, $a_1 = \big(a' - \sum_{i=2}^n a_i\kappa_i\big)/\kappa_1$ and $b_1 = \big(b' - \sum_{i=2}^n b_i\kappa_i\big)/\kappa_1$.
8. $\mathcal{S}_B$ sends $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{a}_0$, $\boldsymbol{s} = \boldsymbol{b}_0 + \boldsymbol{a} * \boldsymbol{c} + \boldsymbol{b}$ and $\boldsymbol{\kappa}$ to Bob.

Figure 6.7: Simulator $\mathcal{S}_B$ against dishonest Bob.

Hybrid H_0: This is the execution of the real protocol $\Pi_{\mathbf{QOLE}}$. In this hybrid, $\mathcal{S}_B$ behaves just like honest Alice up to step 6 of $\Pi_{\mathbf{QOLE}}$: tests the received qudits (steps 1-4), randomly generates n pairs $(a_i^0, b_i^0)_{i\in[n]}$ (step 5), and applies the respective operators $V_{a_i^0}^{b_i^0}$ for $i \in [n]$ to the received states (step 6).

Hybrid H_1: This hybrid is identical to the previous one, H_0, except that now $\mathcal{S}_B$ extracts Bob's random vector $\boldsymbol{x}_0$ from the commitment functionality $\mathcal{F}_{\mathbf{COM}}$. Additionally, upon receiving $\boldsymbol{c}$ from Bob, $\mathcal{S}_B$ extracts Bob's input x by computing $\boldsymbol{c} + \boldsymbol{x}_0$. Then, $\mathcal{S}_B$ sends the extracted element x to $\mathcal{F}_{\mathbf{OLE}}$ and receives y.

Hybrids H_0 and H_1 are perfectly indistinguishable, because $\mathcal{S}_B$ only interacts with Bob when receiving the element $\boldsymbol{c}$, and this does not change anything from Bob's point of view. The corresponding operations are either carried out locally by $\mathcal{S}_B$ or along with $\mathcal{F}_{\mathbf{COM}}$ which, by definition, is fully controlled by $\mathcal{S}_B$.

Hybrid H_2: This hybrid is identical to the previous one, H_1, except that now $\mathcal{S}_\mathcal{B}$ generates (a, b), $\boldsymbol{d}$ and $\boldsymbol{s}$ as follows: it starts by randomly generating $a' \leftarrow_\$ \mathbb{Z}_d$, $\boldsymbol{\kappa} \leftarrow_\$ \mathbb{Z}_d^n$

and $a_2, \ldots, a_n, b_2, \ldots, b_n \leftarrow_{\$} \mathbb{Z}_d$. Then, it computes b' according to the generated a', the extracted element x and the output y of $\mathcal{F}_{\mathbf{OLE}}$, as $b' = a'x - y$. It then masks a' and b' as

$$a' = \boldsymbol{a} \cdot \boldsymbol{\kappa} \qquad \text{and} \qquad b' = \boldsymbol{b} \cdot \boldsymbol{\kappa},$$

by setting a_1 and b_1 accordingly, i.e. $a_1 = \big(a' - \sum_{i=2}^{n} a_i \kappa_i\big)/\kappa_1$ and $b_1 = \big(b' - \sum_{i=2}^{n} b_i \kappa_i\big)/\kappa_1$. Finally, $\mathcal{S}_B$ sends $\boldsymbol{d} = \boldsymbol{a} - \boldsymbol{a}_0$, $\boldsymbol{s} = \boldsymbol{b}_0 + \boldsymbol{a} * \boldsymbol{c} + \boldsymbol{b}$ and $\boldsymbol{\kappa}$ to Bob. This is the last hybrid of the sequence and corresponds to the description of the simulator $\mathcal{S}_B$.

Claim 6. The hybrids H_1 and H_2 are indistinguishable.

Proof. Since, in its first two steps, $\mathcal{S}_B$ executes a RWOLE scheme, according to Lemma 14 we have that $\mathcal{S}_B$ is $\epsilon-$close to a situation where Bob's knowledge on the vectors $(\boldsymbol{a}_0, \boldsymbol{b}_0)$ is lower-bounded by the value

$$\frac{1}{n}\lambda(\zeta) = \frac{\log d}{2}(1 - h_d(\zeta))$$

for $\zeta \in]0, 1 - \frac{1}{d}]$, n the security parameter and $\epsilon(\zeta, n) = \exp\Big(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\Big)$. Also, as Bob receives $\boldsymbol{d}$ and $\boldsymbol{s}$, according to Lemma 15 his knowledge on $(\boldsymbol{a}, \boldsymbol{b})$ is also lower-bounded by the same $\lambda(\zeta)/n$. Furthermore, since $\mathcal{S}_B$ defines $\boldsymbol{a}$ such that $a' = \boldsymbol{a} \cdot \boldsymbol{\kappa}$, from Lemma 16 we can conclude that $\mathcal{S}_B$ is $(\xi + \epsilon)-$close to a scenario where a' is uniformly distributed. This comes from the properties in Lemma 16 and the triangle inequality:

$$\begin{aligned} \delta(\tau_{\mathbb{Z}_d} \otimes \sigma_{B'},\, \rho_{F_a B'}) &\leq \delta(\tau_{\mathbb{Z}_d} \otimes \sigma_{B'},\, \sigma_{F_a B'}) + \delta(\sigma_{F_a B'},\, \rho_{F_a B'}) \\ &\leq K\, 2^{-n\, f_d(\zeta)} + e^{-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}} = \xi + \epsilon \end{aligned}$$

where $K = \frac{\sqrt{d}}{2}$, $f_d(\zeta) = \frac{\log d}{4}(1 - h_d(\zeta))$. This means that the triple $(\boldsymbol{d}, \boldsymbol{s}, \boldsymbol{\kappa})$ only gives to the environment a negligible advantage in distinguishing between the real and ideal world executions. □

This finishes the proof of Lemma 18.

□

6.6 Protocol generalizations

6.6.1 QOLE in Galois fields of prime-power dimensions

So far, we have been working in Hilbert spaces of prime dimensions; this reflects the fact that, for prime d, $\mathbb{Z}_d$ is a field and, under a well-defined set of MUBs $\{|e_r^x\rangle\}_{r \in \mathbb{Z}_d}$, $\forall x \in \mathbb{Z}_d$,

we have the affine relation (6.4):

$$V_a^b \left|e_r^x\right\rangle = c_{a,b,x,r} \left|e_{ax-b+r}^x\right\rangle .$$

In this section, we generalise our protocol, $\Pi_{\mathbf{QOLE}}$, to Hilbert spaces of prime-power dimensions, $N = d^M$ (d prime and $M > 1$), taking advantage of the fact that in a Galois field of dimension d^M, $GF(d^M)$, we can build a complete set of $N + 1$ MUBs [43].

Succinctly, in $GF(d^M)$, we identify the integers $i \in \mathbb{Z}_N$ with their $d-$ary representation, i.e.

$$\mathbb{Z}_N \ni i = \sum_{n=0}^{M-1} i_n d^n \longleftrightarrow (i_0, \ldots, i_{M-1}) \in GF(d^M).$$

In these fields there are two operations, addition and multiplication, which we denote by $\oplus$ and $\odot$, respectively. Addition is straightforward, as it is given by the component-wise addition modulo d of elements, i.e. $i \oplus j = (i_0 + j_0 \mod d, \ldots, i_{M-1} + j_{M-1} \mod d)$. Considering $i = \sum_{n=0}^{M-1} i_n d^n$ as a polynomial of degree $M-1$ given by $i(p) = \sum_{n=0}^{M-1} i_n p^n$, multiplication between two elements i, j, is given by the multiplication between the corresponding polynomials $i(p)$ and $j(p)$ modulo some irreducible polynomial $m(p)$, i.e. $i \odot j = \big(i(p) \times j(p)\big) \mod m(p)$.

Analogously to prime-dimension fields, we can write the operators V_a^b in the computational basis, as

$$V_a^b = \sum_{k=0}^{N-1} \left|k \oplus a\right\rangle \omega^{(k\oplus a)\odot b} \left\langle k\right| ,$$

and the eigenstates for the corresponding $N + 1$ pairwise MUBs, as

$$\left|e_r^x\right\rangle = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \left|l\right\rangle \omega^{\ominus(r\odot l)} \alpha_{\ominus l}^{x*},$$

where $\alpha_{\ominus l}^x$ is a phase factor whose form depends on whether d is even or odd. For details, see Section 2.4.2 in [43].

Given the above, we can derive the following affine relation similar to (6.4):

$$V_a^b \left|e_r^i\right\rangle = \omega^{r\odot a} \alpha_a^{i*} \left|e_{i\odot a\ominus b\oplus r}^i\right\rangle . \tag{6.12}$$

Proof. The relation (6.12) can be easily deduced by considering the following property from [43] (Equation (2.56) in section 2.4.2)

$$\alpha_k^i \alpha_l^i = \alpha_{k\oplus l}^i \omega^{i\odot k\odot l}.$$

We have

$$\begin{aligned}
V_a^b \left|e_r^i\right\rangle &= \frac{1}{\sqrt{N}} \sum_{k,l=0}^{N-1} |k \oplus a\rangle\, \omega^{(k\oplus a)\odot b} \omega^{\ominus r \odot l} \langle k|l\rangle\, \alpha_{\ominus l}^{i*} \\
&= \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} |l\rangle\, \omega^{l\odot b} \omega^{\ominus r \odot (l\ominus a)} \alpha_{\ominus(l\ominus a)}^{i*} \\
&= \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} |l\rangle\, \omega^{l\odot b \ominus r \odot (l\ominus a)} \left(\omega^{\ominus(i\odot a \odot (\ominus l))} \alpha_a^i \alpha_{\ominus l}^i\right)^* \\
&= \omega^{r\odot a} \alpha_a^{i*} \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} |l\rangle\, \omega^{l\odot b \ominus r \odot l} \omega^{\ominus(i\odot a \odot l)} \alpha_{\ominus l}^{i*} \\
&= \omega^{r\odot a} \alpha_a^{i*} \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} |l\rangle\, \omega^{\ominus(i\odot a \ominus b \oplus r)\odot l} \alpha_{\ominus l}^{i*} \\
&= \omega^{r\odot a} \alpha_a^{i*} \left|e_{i\odot a \ominus b \oplus r}^i\right\rangle .
\end{aligned}$$

□

Notice that all the steps in the $\Pi_{\mathbf{QOLE}}$ depend on the properties of the field operations (addition and multiplication) and on the fact that expression (6.4) holds. Hence, we can use $\Pi_{\mathbf{QOLE}}$ adapted for the operations $\oplus$ and $\odot$, in order to quantum-UC-realize $\mathcal{F}_{\mathbf{OLE}}$ in fields of prime-power dimension d^M.

6.6.2 Quantum vector OLE

In the proposed protocol $\Pi_{\mathbf{QOLE}}$, we extract one instance of OLE out of n instances of WOLE. As far as efficiency is concerned, it would be desirable to generate more instances of OLE out of those n instances of WOLE. Here, we show how to use WOLE as a resource to realize the VOLE functionality, $\mathcal{F}_{\mathbf{VOLE}}$, presented in Figure 2.4. In this case, Alice fixes a k (which is specified later), defines a set of k linear functions $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k$ and Bob outputs the evaluation of all these functions on a specified element $x \in \mathbb{F}_q$ that he chooses, i.e. $\boldsymbol{f} := \boldsymbol{a}x + \boldsymbol{b}$. Since $\Pi_{\mathbf{QOLE}}$ can be extended to finite fields $\mathbb{F}_q$, where q is a prime or prime-power number (see Section 6.6.1), the $\mathcal{F}_{\mathbf{VOLE}}$ functionality can also be defined in $\mathbb{F}_q$.

In the extraction phase of $\Pi_{\mathbf{QOLE}}$, Alice randomly chooses a function $g_{\boldsymbol{\kappa}}$ and applies it to the pair $(\boldsymbol{a}, \boldsymbol{b})$. This procedure suggests that, in order to generate different input elements (a', b'), Alice can randomly choose another function $g_{\boldsymbol{\kappa'}}$ and set $a' = g_{\boldsymbol{\kappa'}}(\boldsymbol{a})$ and

$b' = g_{\boldsymbol{\kappa}'}(\boldsymbol{b})$. This is equivalent to generating a random $2 \times n$ matrix in $\mathbb{F}_q$, i.e.

$$\begin{bmatrix} — & \boldsymbol{\kappa} & — \\ — & \boldsymbol{\kappa'} & — \end{bmatrix} \begin{bmatrix} | & | \\ \boldsymbol{a} & \boldsymbol{b} \\ | & | \end{bmatrix} = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix}.$$

However, in case $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa'}$ are linearly dependent (i.e. $\boldsymbol{\kappa} = c\boldsymbol{\kappa'}$ for some $c \in \mathbb{Z}_d$), Bob would have some extra information about Alice's elements (a, b) and (a', b'), as $(a, b) = c(a', b')$. This leads to a situation beyond the $\mathcal{F}_{\mathbf{VOLE}}$ definition. To avoid this issue, let us consider the set of $k \times n$ matrices with rank k over $\mathbb{F}_q$ for $1 \leq k \leq n$, and denote it by $\mathcal{R}_{k\times n}(\mathbb{F}_q)$. For a binary finite field, $\mathcal{R}_{k\times n}(\mathbb{F}_2)$ is a two-universal hash family [276, 277]. Similarly, one can prove that the more general set $\mathcal{R}_{k\times n}(\mathbb{F}_q)$ is also a two-universal hash family from $\mathbb{F}_q^n$ to $\mathbb{F}_q^k$. During the extraction phase of the original $\Pi_{\mathbf{QOLE}}$, Alice chooses vectors $(\boldsymbol{a}, \boldsymbol{b})$ according to the random vector $\boldsymbol{\kappa}$ and the desired final elements (a, b) (see step 9 in Figure 6.5). In that case, since there is only one random vector $\boldsymbol{\kappa}$, there are $n - 1$ undefined variables for each vector $\boldsymbol{a}$ and $\boldsymbol{b}$, i.e. $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$ that can be chosen freely. For the VOLE protocol, instead of choosing just one vector $\boldsymbol{\kappa}$, Alice randomly chooses a matrix $\mathcal{K} \in \mathcal{R}_{k\times n}(\mathbb{F}_q)$ of rank k. She then defines vectors $(\boldsymbol{a}', \boldsymbol{b}') \in \mathbb{F}_q^n \times \mathbb{F}_q^n$ consistent with the final elements $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k$. That is, Alice has the following system:

$$\begin{bmatrix} —— & \boldsymbol{\kappa}_1 & —— \\ & \ldots & \\ —— & \boldsymbol{\kappa}_k & —— \end{bmatrix} \begin{bmatrix} | & | \\ \boldsymbol{a'} & \boldsymbol{b'} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a} & \boldsymbol{b} \\ | & | \end{bmatrix},$$

that can be solved by means of the Gaussian elimination method. Since $\mathcal{K} \in \mathcal{R}_{k\times n}(\mathbb{F}_q)$, there will be $n - k$ undefined variables in both vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Let U denote the set of undefined indexes in $\boldsymbol{a}'$ and $\boldsymbol{b}'$. Alice randomly chooses a'_i and b'_i for $i \in U$ and solves the above equation system. Then, they proceed similarly to the original $\Pi_{\mathbf{QOLE}}$ and execute the derandomization protocol $\Pi^n_{\mathbf{WOLE}}((\boldsymbol{a}', \boldsymbol{b}'), \boldsymbol{x})$. Finally, Bob applies Alice's chosen matrix $\mathcal{K}$ to his output vector $\boldsymbol{y}'$ to get the final element $\boldsymbol{y}$. This vectorized extraction protocol $\Pi_{\mathbf{VEXT}}$ is presented in Figure 6.8.

The correctness of the protocol is drawn immediately from linearity:

$$\boldsymbol{y} = \mathcal{K}\boldsymbol{y}' = \mathcal{K}(\boldsymbol{a}'x + \boldsymbol{b}') = \boldsymbol{a}x + \boldsymbol{b}.$$

Protocol $\Pi_{\mathbf{VEXT}}$

Alice's input: $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{F}_q^k \times \mathbb{F}_q^k$
Bob's input: $x \in \mathbb{F}_q$

1. Alice chooses randomly a matrix $\mathcal{K} \in \mathcal{R}_{k \times n}(\mathbb{F}_q)$ and sends it to Bob.
2. Using the Gaussian elimination method, Alice finds one solution of the system:

$$\mathcal{K} \begin{bmatrix} | & | \\ \boldsymbol{a'} & \boldsymbol{b'} \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ \boldsymbol{a} & \boldsymbol{b} \\ | & | \end{bmatrix}.$$

   (a) Alice finds the set U of undefined indexes in $\boldsymbol{a'}$ and $\boldsymbol{b'}$.
   (b) Alice randomly generates $a'_i, b'_i \leftarrow_{\$} \mathbb{F}_q$ for $i \in U$.
   (c) Alice solves the system for indexes $i \notin U$.

3. Alice and Bob run $\Pi^n_{\mathbf{WOLE}}((\boldsymbol{a'}, \boldsymbol{b'}), \boldsymbol{x})$, where $\boldsymbol{x} = (x, \ldots, x)$. Bob outputs $\boldsymbol{y'} \in \mathbb{F}_q^n$.
4. Bob computes $\boldsymbol{y} = \mathcal{K}\boldsymbol{y'}$.

Alice's output: $\perp$
Bob's output: $\boldsymbol{y} \in \mathbb{F}_q^k$

Figure 6.8: Extraction protocol for VOLE.

The security of the protocol is constrained by the closeness parameter,

$$\xi = \frac{1}{2}\sqrt{2^{k \log q - H_{\min}(X|E)}},$$

given by Lemma 11, where we consider l to be k and d to be q. As before, $\mathbf{F}_{\boldsymbol{a'}}$ denotes the distribution of the $\Pi^n_{\mathbf{WOLE}}$ protocol's input $\boldsymbol{a'}$ from Bob's perspective. From Lemma 11, since $\mathcal{R}_{k \times n}(\mathbb{F}_q)$ is a two-universal family of hash functions, we know that $\mathcal{K} \in \mathcal{R}_{k \times n}(\mathbb{F}_q)$ approximates $\mathcal{K}\mathbf{F}_{\boldsymbol{a'}} = \mathbf{F}_{\boldsymbol{a}}$ to uniform conditioned on Bob's side information. However, the closeness parameter has to be negligible in the security parameter n, thus setting a bound on k (the size of VOLE), i.e. for $\eta > 0$,

$$k \log q - \frac{n \log q}{2}\big(1 - h_q(\zeta)\big) < -n\eta \log q$$
$$k < n\Big(\frac{1}{2}\big(1 - h_q(\zeta)\big) - \eta\Big).$$

Since $k > 0$, we have that $0 < \eta < \frac{1}{2}\big(1 - h_q(\zeta)\big)$. This gives a bound on the proportion of elements that we can extract from n WOLEs and shows how Alice can fix k in the beginning. Note that this bound is not necessarily optimal, and one could try to improve it. We leave this as future work, as it goes beyond the scope of this paper, which is to introduce a quantum protocol for OLE that can be, in turn, adapted accordingly to also achieve VOLE.

Let us denote by $\Pi_{\mathbf{QVOLE}}$ the protocol $\Pi_{\mathbf{QOLE}}$ with the subprotocol $\Pi_{\mathbf{VEXT}}$ instead of $\Pi_{\mathbf{EXT}}$. For the security of $\Pi_{\mathbf{QVOLE}}$, we have:

Theorem 7 (quantum-UC security of $\Pi_{\mathbf{QVOLE}}$)**.** *The protocol $\Pi_{\boldsymbol{QOVLE}}$ statistically quantum-UC realizes (see Definition 6) $\mathcal{F}_{\boldsymbol{VOLE}}$ in the $\mathcal{F}_{\boldsymbol{COM}}-$hybrid model.*

The proof is much the same as the proof of Theorem 4, therefore we omit it.

6.7 Conclusion

OLE is an important primitive for secure two-party computation, and while for stronger primitives such as bit commitment, OT and coin flipping there is a plethora of both theoretical as well as concrete protocol proposals [48, 129, 278–292], up until now, there was no OLE protocol based on quantum communication. In this chapter, we present two protocols for QOLE. The first protocol is secure against semi-honest adversaries in the static corruption setting. The second proposed protocol, $\Pi_{\mathbf{QOLE}}$, builds upon the semi-honest version and extends it to the dishonest case, following a commit-and-open approach. We prove this second protocol to be secure in the quantum-UC framework when assuming ideal commitments, making it possible to be composed in any arbitrary way. We also constructed two generalizations of our protocol: the first achieves QOLE in Galois fields of prime-power dimensions and the second is a protocol for quantum vector OLE. Note that our protocol achieves everlasting security, i.e. it remains information-theoretically secure after its execution, even if the dishonest party becomes more powerful in the future.

Chapter 7

Conclusion

In this dissertation, the focus is on exploring and enhancing the utilization of quantum cryptography in secure multiparty computation (SMC) systems. Chapter 3 provides an overview of quantum oblivious transfer (QOT) protocols, which play a crucial role in many SMC protocols. Our analysis is centered around the use of oblivious keys, which facilitate the modular execution of SMC protocols by allowing the separation of quantum technology and secure computation. We also examine the threat posed by quantum hacking techniques and provide an evaluation of both practical and theoretical measures to mitigate these attacks.

In Chapter 4, we conducted a theoretical comparison of the complexity of quantum and classical OT protocols to assess their impact on the efficiency of SMC protocols. This is motivated by the close connection between the Yao garbled circuit protocol and OT. We proposed an optimized version ($\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$) of the BBCS-based QOT protocols and compared its transfer phase with that of the fastest known classical OT implementation, ALSZ13 [3]. Our conclusion was that the transfer phase of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ has the potential to be faster than that of the ALSZ13 OT extension while maintaining a much higher level of security. In contrast, the ALSZ13 protocol is only proven to be secure in the semi-honest model, whereas $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ is secure in the malicious setting. Furthermore, we compared the transfer phase of maliciously secure classical protocols, ALSZ15 [196] and KOS15 [4], with that of $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$ and found that they have a greater computation and communication complexity than $\Pi_{\mathbf{O}}^{\mathbf{BBCS}}$.

In Chapter 5, we bring theory closer to practice by presenting a secure multiparty computation (SMC) protocol that uses quantum technologies to analyse distance-based algorithms of phylogenetic trees. Our proposed system integrates the use of ready-to-use libraries, such as CBMC-GC, Libscapi, and PHYLIP, to provide a complete, quantum-resistant solution. We implement and compare the performance of both a classical-only and a quantum-assisted system using simulated symmetric and oblivious keys. The re-

sults from Chapter 4 suggest that the quantum-assisted version could have comparable efficiency to the classical-only version. However, this assumes that the overhead created by the oblivious key management system is not taken into account. Despite this difference, the quantum-assisted system provides a significantly higher level of security against quantum computer attacks.

In Chapter 6, we presented a two-phase quantum oblivious linear evaluation that, to the best of our knowledge, is the first quantum protocol proposed for this primitive.

7.1 Future work

To summarize, this dissertation presents a comprehensive investigation of the impact and potential improvement of the adoption of quantum cryptography in secure multi-party computation (SMC) systems. Our findings have implications for both theoretical and practical areas of research. In terms of practical implementation, there is room for improvement in the area of efficient oblivious key management systems. Currently, the implementation utilized a file system to store the oblivious keys, which may not be the most efficient method. However, it was chosen for its modularity. In the future, it would be beneficial to extend the implementation of Libscapi to include a function that can directly access the oblivious keys stored in memory, potentially increasing efficiency.

In the theoretical realm, there is room for further development of the protocol $\pi_{\mathbf{QOLE}}$. Currently, its security has been analysed in the absence of noise. Proving security in the presence of noise would follow a similar approach (as seen in [36]). In case of noise, during Step 4 of the quantum phase of $\pi_{\mathbf{QOLE}}$ (depicted in Figure 6.5), Alice should abort the protocol if the error measurement (err) exceeds a pre-defined value ν attributed to the noise. This results in $err = \nu + \zeta'$, where ζ' represents the potential dishonest behavior of Bob. This adjustment reduces the lower bound of the min-entropy of Alice's functions $\mathbf{F}$ given Bob's side information, i.e.

$$H_{\min}(\mathbf{F}|\mathbf{Y}E)_{\sigma_{\mathbf{FY}E}} \geq \frac{n \log d}{2} \left(1 - h_d(\nu + \zeta' + \zeta)\right).$$

While it is possible to extend the security of the protocol to include noise in the quantum states, its correctness cannot be guaranteed in such scenarios. Therefore, as future work, new protocols should be developed that account for noise and examine its impact on the security properties. As an initial suggestion, one could build upon existing work. The error-tolerant OLE combiner from [293] provides a way to integrate several possibly faulty OLE instances into one correct OLE. Although this protocol ensures the correctness of the protocol under noise, it does not include a privacy-amplification phase robust

against quantum side information. To bridge this gap, one could strive to apply a linear strong extractor to each OLE instance [294]. For example, by considering a prime-power dimension, d^M, and the natural correspondence between $\mathbb{Z}_{d^M}$ and $(\mathbb{Z}_d)^M$, one could use the inner product seeded extractor in $(\mathbb{Z}_d)^M$ [295].

The $\pi_{\mathbf{QOLE}}$ protocol can also be adapted to operate within the bounded-quantum-storage model, while preserving its security. In this adaptation, the test phase of $\pi^n_{\mathbf{RWOLE}}$ is replaced by a waiting time Δt, limiting the amount of qudits that Bob can store. Further investigation into the impact of different noisy channels on the security properties of the protocol would be valuable. Additionally, to ensure the protocol's composability, an analysis within the bounded-quantum-storage-UC model, as proposed by Unruh [152], should be performed.

Our protocol is a two-way scheme, in which Bob prepares and sends a quantum state, Alice performs an operation on it, and returns it to Bob, who finally measures the final state. There are several two-way QOT protocols in the literature [296–300], with the one proposed by Amiri et al. [297] demonstrating their experimental feasibility. This drives the motivation to further develop practical implementations of our protocol. Moreover, the security of our protocol can be enhanced by making it device-independent. One can look to the work of Kundu et al. [298] as inspiration, who built upon the work of Chailloux et al. [299]. While the above-mentioned works primarily focus on two-way QOT protocols, recent studies have also proposed one-way, non-interactive protocols for device-independent [176] and XOR QOT [301].

Finally, based on our results, one could construct quantum protocols for oblivious polynomial evaluation, which – as mentioned in the beginning of Chapter 6 – is another important primitive facilitating various applications.

Appendix A

Jukes-Cantor distance for CBMC-GC

The boolean circuit that represents the Jukes-Cantor distance receives as inputs two four-based sequences (A, C, G, T) with size 32 000. Since we are using a boolean circuit representation, the nucleotide sequences must be represented in binary. So, by convention, we use the following 2-bit encoding: $A = 00$, $C = 01$, $G = 10$ and $T = 11$. As a result, we start by defining a sequence type of size 4 000 with the `unsigned short` type elements (Figure A.1, lines $1 - 5$). In fact, the type `Array_Seq` saves $4\,000 \times 16 = 64\,000$ bits. Each element of `Array_Seq` represents a small sequence of eight elements. This is an implementation choice that renders a good compromise between accuracy level and circuit size.

As we saw in the main text, the hamming distance between two binary strings can then be easily computed by XORing them and counting the number of 1's. This last operation is commonly called `popcount`. We cannot directly apply this approach because our sequences are in fact four-based sequences. In fact, our version of the `popcount` function is only interested in computing the number of 2-bit elements that are different between both sequences.

We follow a tailored divide-and-conquer technique. The original technique is described by Henry Warren in his book "Hacker's Delight", Chapter 5 [252]. In summary, the original technique starts by counting in parallel the number of 1's inside each 2-bit block and saves it in 2-bit blocks. Then, it adds two 2-bit blocks and saves the result in a 4-bit block. It continues until we get the final sum. If we follow directly this approach we might run into wrong results as described in the main text. For our case, instead of counting the number of 1's inside every 2-bit block, we only care if there is one element 1 inside each 2-bit block. This simply indicates that the elements at that site are different. This is achieved by applying an `OR` operation (represented by `|`) to the bits inside each 2-bit

block (Figure A.1, line 14). For 4-bit blocks and above we follow the same recipe of the original divide-and-conquer technique.

The main function that computes the Hamming weight between two nucleotide sequences `INPUT_A` and `INPUT_B` is defined by the function `mpc_main`, line 21. It outputs the inverse of the hamming weight: `total/distance` line 37. Since we know the hamming weight lies between 0 and 1, it renders smaller circuits to use the native integer division operator, /, from the CBMC-GC tool and then invert the output after the Yao computation. Otherwise, we would need a fixed precision representation to output decimal numbers.

Below we describe the variables used in the `mpc_main` function:

- `INPUT_A` and `INPUT_B`: the binary input sequences of Alice and Bob, respectively. Following the CBMC-GC convention, the input elements must start with the identifier `INPUT_`.
- `OUTPUT_distance`: the inverse of the hamming weight. Following the CBMC-GC convention, the output element must start with the identifier `OUTPUT_`.
- `total`: keeps track of the number of elements that can be compared between aligned sequences.
- `distance`: keeps track of the hamming distance between both sequences.
- `count_axorb`: saves the number of elements that are different in a 16-bit block sequence (i.e. in `INPUT_A.el[i]^INPUT_B.el[i]`).

```c
#define LEN_SEQ 4000

typedef struct {
  unsigned short el[LEN_SEQ];
} Array_Seq;

const unsigned int m1  = 0x55555555; //binary: 0101...
const unsigned int m2  = 0x33333333; //binary: 00110011..
const unsigned int m4  = 0x0f0f0f0f; //binary: 4 zeros, 4 ones ...
const unsigned int m8  = 0x00ff00ff; //binary: 8 zeros, 8 ones ...

unsigned int popcount(unsigned short INPUT_B_x) {
    unsigned int x = INPUT_B_x;
    x = (x & m1 ) | ((x >>  1) & m1 ); // changed step
    x = (x & m2 ) + ((x >>  2) & m2 );
    x = (x & m4 ) + ((x >>  4) & m4 );
    x = (x & m8 ) + ((x >>  8) & m8 );
    return x;
}

void mpc_main(Array_Seq INPUT_A, Array_Seq INPUT_B){
  unsigned int distance = 0;
  int total = 0;
  for(int i=0; i<LEN_SEQ; i++){
    int count_a = popcount(INPUT_A.el[i]);
    int count_b = popcount(INPUT_B.el[i]);
    if(count_a > 0 && count_b > 0){
      int count_axorb = popcount(INPUT_A.el[i]^INPUT_B.el[i]);
      if(count_axorb == 1){
        distance = distance + 1;
      }
      total = total + 8;
    }
  }
  unsigned int OUTPUT_distance;
  if(distance > 0){
    OUTPUT_distance = total/distance;
  } else {
    OUTPUT_distance = 0;
  }
}
```

Figure A.1: Jukes-Cantor distance C code for CBMC-GC boolean circuit generation.

Appendix B

Proof of Lemma 14 (Dishonest Bob)

As we mentioned in the main text, this proof is a combination and adaptation of results from [36] and [80] to our case.

To simplify the notation, in this proof we drop the subscript 0 that refers to the RWOLE phase, e.g. we write Alice's function vector $\mathbf{F}_0$, simply as $\mathbf{F}$.

Let the values that Bob commits be fixed as $(x_i, r_i)\,\forall i \in [m]$, where $m = (1+t)n$ and tn the number of qudits $\left|e_{r_i}^{x_i}\right\rangle$ used in the *Test Phase* to check whether he is honest or not. Throughout the proof we denote by $\boldsymbol{x} = (x_1, \ldots, x_m)$ and $\boldsymbol{r} = (r_1, \ldots, r_m)$ the vectors in $\mathbb{Z}_d^m$ whose components contain Bob's commitments $\forall i \in [m]$, and by $X^m = (X_1, \ldots, X_m)$ the vector of the random variables associated to $x_i, i \in [m]$. For each pair (x_i, r_i), the corresponding qudit $\left|e_{r_i}^{x_i}\right\rangle$ belongs in the Hilbert space $\mathcal{H}_{X_i}$, and the quantum system including all the qudits is in $\mathcal{H}_{X^m} = \bigotimes_{i\in[m]} \mathcal{H}_{X_i}$. For simplicity, we refer to the quantum systems in terms of the corresponding random variables X_i, instead of the Hilbert spaces $\mathcal{H}_{X_i}$.

Recall that the set $T \subset [m]$ contains the tn indices of the test qudits, and by $\bar{T}$ we denote its complement $[m]\backslash T$. For $i \in T$, $\boldsymbol{x}_{|T}$ is the vector whose components are the bases x_i in which Alice will measure the test qudits, and $\boldsymbol{r}'_{|T}$ is the vector whose components are Alice's measurement results. The corresponding quantum system is in the Hilbert space $\bigotimes_{i\in T} \mathcal{H}_{X_i}$, which for simplicity we denote as $X_{|T}^m$ in terms of the associated random variables. Finally, $r_H(\cdot,\cdot) = d_H(\cdot,\cdot)/n$ is the relative Hamming distance between two vectors of size n, with $d_H(\cdot,\cdot)$ being their Hamming distance.

Proof. Let us start by proving property 1. of Lemma 14. After the first step of the protocol π^n_{RWOLE} (Figure 6.2), the generated state is $\rho_{X^m E}$, where E is an auxiliary quantum system that Bob holds. Without loss of generality we assume that $\rho_{X^m E} = |\phi_{X^m E}\rangle\langle\phi_{X^m E}|$, i.e, it is a pure state.[1] If Bob is honest, we have that $|\phi_{X^m E}\rangle = |e_{\boldsymbol{r}}^{\boldsymbol{x}}\rangle \otimes |\psi_E\rangle$, i.e., Bob's auxiliary quantum system E is not entangled to the states that he sends to Alice.

[1] Otherwise, we purify it and carry the purification system along with E.

The *Test Phase* of the protocol is used to guarantee that the real state is close to an ideal state that satisfies the properties 1. and 2. of Lemma 14. Let $\boldsymbol{r'}$ be the vector whose components are Alice's outcomes when measuring the state of X^m in the committed bases $\boldsymbol{x}$, and let $\mathcal{T}$ be the random variable associated to the set of indexes T of size tn. We can consider the state:

$$\rho_{\mathcal{T}X^mE} = \rho_{\mathcal{T}} \otimes |\phi_{X^mE}\rangle\langle\phi_{X^mE}| = \sum_T P_{\mathcal{T}}(T)\,|T\rangle\langle T| \otimes |\phi_{X^mE}\rangle\langle\phi_{X^mE}|\,, \tag{B.1}$$

to be the state resulting from the real execution of the protocol, and prove that it is close to some state, $\sigma_{\mathcal{T}X^mE}$, that fulfills the following property: for any choice of T and for any outcome $\boldsymbol{r'}_{|T}$ when measuring the state of ${X^m}_{|T}$ in the bases $\boldsymbol{x}_{|T}$, the relative error $r_H(\boldsymbol{r'}_{|T}, \boldsymbol{r}_{|T})$ is an upper bound on the relative error $r_H(\boldsymbol{r'}_{|\bar{T}}, \boldsymbol{r}_{|\bar{T}})$, which one would obtain by measuring the remaining subsystems ${X^m}_{|\bar{T}}$ in the bases $\boldsymbol{x}_{|\bar{T}}$. This state, $\sigma_{\mathcal{T}X^mE}$, can be written as:

$$\sigma_{\mathcal{T}X^mE} = \sum_T P_{\mathcal{T}}(T)\,|T\rangle\langle T| \otimes \left|\tilde{\phi}^T_{X^mE}\right\rangle\!\left\langle\tilde{\phi}^T_{X^mE}\right|, \tag{B.2}$$

where $\forall T$,

$$\left|\tilde{\phi}^T_{X^mE}\right\rangle = \sum_{\boldsymbol{r'}\in\mathcal{B}_T} \alpha^T_{\boldsymbol{r'}}\,|e^{\boldsymbol{x}}_{\boldsymbol{r'}}\rangle \otimes \left|\psi^{\boldsymbol{r'}}_E\right\rangle, \tag{B.3}$$

for $\mathcal{B}_T = \{\boldsymbol{r'} \in \mathbb{Z}_d^m : r_H(\boldsymbol{r'}_{|\bar{T}}, \boldsymbol{r}_{|\bar{T}}) \leq r_H(\boldsymbol{r'}_{|T}, \boldsymbol{r}_{|T}) + \zeta\}$ for $\zeta > 0$ and arbitrary coefficients $\alpha^T_{\boldsymbol{r'}}$. The state $\left|\psi^{\boldsymbol{r'}}_E\right\rangle$ is an arbitrary state on E subsystem that possibly depends on $\boldsymbol{r'}$. Then we have:

Lemma 19. *Let the quantum states $\rho_{\mathcal{T}X^mE}$ and $\sigma_{\mathcal{T}X^mE}$ be given by* (B.1) *and* (B.2)*, respectively. Then, $\forall\zeta > 0$ and fixed strings $\boldsymbol{x}, \boldsymbol{r} \in \mathbb{Z}_d^m$, we have that*

$$\rho_{\mathcal{T}X^mE} \approx_\epsilon \sigma_{\mathcal{T}X^mE},$$

where $\epsilon(\zeta, n) = \epsilon(\zeta, n) = \exp\left(-\frac{2\zeta^2t^2n^2}{(nt+1)(t+1)}\right)$. That is, the real state $\rho_{\mathcal{T}X^mE}$ is exponentially close, with respect to n, to the ideal state $\sigma_{\mathcal{T}X^mE}$.

Proof. This proof is an adaptation of the proof of Lemma 4.3 from [36] to our case.

For any T, let $\left|\tilde{\phi}^T_{X^mE}\right\rangle$ be the renormalized projection of $|\phi_{X^mE}\rangle$ into the subspace

$$\text{Span}\,\{|e^{\boldsymbol{x}}_{\boldsymbol{r'}}\rangle : \boldsymbol{r'} \in \mathcal{B}_T\} \otimes \mathcal{H}_E,$$

and let $\left|\tilde{\phi}^{T^\perp}_{X^mE}\right\rangle$ be the renormalized projection of $|\phi_{X^mE}\rangle$ into its orthogonal complement.

We can, then, write

$$|\phi_{X^mE}\rangle = \epsilon_T \left|\tilde{\phi}^T_{X^mE}\right\rangle + \epsilon^\perp_T \left|\tilde{\phi}^{T^\perp}_{X^mE}\right\rangle,$$

with $\epsilon_T = \left\langle \tilde{\phi}^T_{X^m E} \middle| \phi_{X^m E} \right\rangle$ and $\epsilon_T^\perp = \left\langle \tilde{\phi}^{T^\perp}_{X^m E} \middle| \phi_{X^m E} \right\rangle$. By construction, this state satisfies (B.3).

Furthermore, we can calculate the distance:

$$\delta\left(|\phi_{X^m E}\rangle\langle\phi_{X^m E}|, \left|\tilde{\phi}^T_{X^m E}\right\rangle\left\langle\tilde{\phi}^T_{X^m E}\right|\right) = \sqrt{1 - \left|\left\langle \tilde{\phi}^T_{X^m E} \middle| \phi_{X^m E} \right\rangle\right|^2} = \sqrt{1 - |\epsilon_T|^2} = |\epsilon_T^\perp|,$$

where, given T, $|\epsilon_T^\perp|$ is the probability amplitude for getting outcome $\boldsymbol{r'} \notin \mathcal{B}_T$ when measuring the state of X^m in bases $\boldsymbol{x}$. We continue to derive an upper bound on the distance between the real and the ideal state:

$$\begin{aligned}\delta\big(\rho_{\mathcal{T}X^m E}, \sigma_{\mathcal{T}X^m E}\big) &= \left(\sum_T P_{\mathcal{T}}(T)\delta\left(|\phi_{X^m E}\rangle\langle\phi_{X^m E}|, \left|\tilde{\phi}^T_{X^m E}\right\rangle\left\langle\tilde{\phi}^T_{X^m E}\right|\right)\right)^2 \\ &\leq \sum_T P_{\mathcal{T}}(T)|\epsilon_T^\perp|^2,\end{aligned}$$

where we used Jensen's inequality and properties of the trace norm. The last term is the probability that, when choosing T according to $P_{\mathcal{T}}$ and measuring the state of X^m in bases $\boldsymbol{x}$ we get an outcome $\boldsymbol{r'} \notin \mathcal{B}_T$. We write

$$\sum_T P_{\mathcal{T}}(T)|\epsilon_T^\perp|^2 = \Pr_{\mathcal{T}}[\boldsymbol{r'} \notin \mathcal{B}_T] = \Pr_{\mathcal{T}}[r_H(\boldsymbol{r'}_{|\bar{T}}, \boldsymbol{r}_{|\bar{T}}) - r_H(\boldsymbol{r'}_{|T}, \boldsymbol{r}_{|T}) > \zeta].$$

Then, we can use Lemma 2 which states that the above probability is negligible in n and gives us an upper bound for $\delta\big(\rho_{\mathcal{T}X^m E}, \sigma_{\mathcal{T}X^m E}\big)$. In particular, given the set $[m]$ with $(1+t)n$ elements, we apply the aforementioned corollary for a random subset $\mathcal{T}$ of size tn and its complement $\bar{\mathcal{T}}$ of size n. Denoting by $\mu_{\mathcal{T}}$ and $\mu_{\bar{\mathcal{T}}}$, respectively, the averages of these subsets, we obtain

$$\mathrm{P}[\mu_{\bar{\mathcal{T}}} - \mu_{\mathcal{T}} \geq \zeta] \leq \exp\left(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\right).$$

Hence, we have:

$$\delta\big(\rho_{\mathcal{T}X^m E}, \sigma_{\mathcal{T}X^m E}\big) \leq \sum_T P_{\mathcal{T}}(T)|\epsilon_T^\perp|^2 \leq \exp\left(-\frac{2\zeta^2 t^2 n^2}{(nt+1)(t+1)}\right) =: \epsilon, \tag{B.4}$$

concluding the proof of Lemma 19, i.e. that the real state (B.1) generated by the protocol until the *Computation Phase* is $\epsilon-$close to the ideal state (B.2). □

It is now straightforward to complete the proof of property 1. of Lemma 14.

To obtain the states $\sigma_{\mathbf{F}B'}$ and $\rho_{\mathbf{F}B'}$ from the states $\sigma_{\mathcal{T}X^m E}$ and $\rho_{\mathcal{T}X^m E}$, respectively, we need to apply the operator $V_{\boldsymbol{a}}^{\boldsymbol{b}} \operatorname{Tr}_{X^m_{|\bar{T}} E_{|T}}[\,\cdot\,] V_{\boldsymbol{a}}^{\boldsymbol{b}\dagger}$. This operator is a CPTP map, being

the composition of the two CPTP maps, $V_{\boldsymbol{a}}^{\boldsymbol{b}}$ and Tr. Since the trace distance between two density matrices does not increase under CPTP maps (see Lemma 7 in [75]), the final states indeed satisfy property 1. of Lemma 14, namely

$$\sigma_{\mathbf{F}B'} \approx_{\epsilon} \rho_{\mathbf{F}B'}.$$

For the rest of this proof dishonest Bob's system B' is identified with $\mathbf{Y}\,E$, where $\mathbf{Y}$ corresponds to the classical information leaking to him through the output of the WROLE and E is, in general, a quantum auxiliary system that he might also hold. Consequently, from now on we write $\sigma_{\mathbf{F}B'}$ as $\sigma_{\mathbf{F}\mathbf{Y}E}$. Now, we have to prove property 2. of Lemma 14, i.e., obtain the corresponding lower bound on min-entropy with respect to $\sigma_{\mathbf{F}\mathbf{Y}E}$. We start with the following Lemma 20:

Lemma 20 (Corollary 4.4 in [36])**.** *Let $err := r_H(\boldsymbol{r}'_{|T}, \boldsymbol{r}_{|T}) \leq 1 - \frac{1}{d}$ be the error measured by Alice while measuring the state of $X^m_{|T}$ according to her choice of T, and let $\sigma_{\mathbf{X}E} := |\psi\rangle\langle\psi|_{\mathbf{X}E}$ be the state to which the ideal state $\sigma_{\mathcal{T}X^mE}$ collapses after this measurement. Following* (B.2) *and* (B.3)*, we write $|\psi\rangle_{\mathbf{X}E} = \sum_{\boldsymbol{z}\in\mathcal{B}} \alpha_{\boldsymbol{z}} |e^{\boldsymbol{x}}_{\boldsymbol{z}}\rangle |\psi^{\boldsymbol{z}}_E\rangle$ for some $|\psi^{\boldsymbol{z}}_E\rangle$ and $\mathcal{B} = \{\boldsymbol{z} \in \mathbb{Z}_d^n : r_H(\boldsymbol{z}, \boldsymbol{r}_{|\bar{T}}) \leq err + \zeta\}$ with $\zeta > 0$. Then, we have:*

$$H_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \geq -h_d(err + \zeta) n \log d, \tag{B.5}$$

where $h_d(x)$ is given in Definition 1.

Proof. We start by defining the state $\tilde{\sigma}_{\mathbf{X}E} := \sum_{\boldsymbol{z}\in\mathcal{B}} |\alpha_{\boldsymbol{z}}|^2 |e^{\boldsymbol{x}}_{\boldsymbol{z}}\rangle\langle e^{\boldsymbol{x}}_{\boldsymbol{z}}| \otimes |\psi^{\boldsymbol{z}}_E\rangle\langle\psi^{\boldsymbol{z}}_E|$. Then, by applying Lemma 5, we obtain

$$H_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \geq H_{\min}(\mathbf{X}|E)_{\tilde{\sigma}_{\mathbf{X}E}} - \log|\mathcal{B}|.$$

Since $\tilde{\sigma}_{\mathbf{X}E}$ is a classical-quantum state, its min-entropy cannot be negative, i.e.

$$H_{\min}(\mathbf{X}|E)_{\tilde{\sigma}_{\mathbf{X}E}} \geq 0,$$

thus $H_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \geq -\log|\mathcal{B}|$.

Finally, to get the lower bound shown in (B.5), we apply Lemma 3 to our case, namely for the Hamming ball around $\boldsymbol{r}_{|\bar{T}}$ with radius $n(err + \zeta)$.

□

To complete the proof of property 2. of Lemma 14 and find a lower bound on $H_{\min}(\mathbf{F}|\mathbf{Y}\,E)\sigma_{\mathbf{F}\mathbf{Y}E}$ we need to relate it with $H_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}$, for which we just derived a lower bound. In what follows, we adapt the notation from [80]:

- $\Phi_{\mathbf{X}\bar{\mathbf{X}}} = |\Phi\rangle\langle\Phi|_{\mathbf{X}\bar{\mathbf{X}}}$, where $|\Phi\rangle_{\mathbf{X}\bar{\mathbf{X}}} = \sum_{\boldsymbol{s}} |e^{\boldsymbol{x}}_{\boldsymbol{s}}\rangle_{\mathbf{X}} \otimes |e^{\boldsymbol{x}}_{\boldsymbol{s}}\rangle_{\bar{\mathbf{X}}}$ for all basis choices $\boldsymbol{x} \in \mathbb{Z}_d^n$, and
- $\Phi_{(\boldsymbol{a},\boldsymbol{b})} = \big|\Phi_{(\boldsymbol{a},\boldsymbol{b})}\big\rangle\big\langle\Phi_{(\boldsymbol{a},\boldsymbol{b})}\big| = \sum_{\boldsymbol{s}\boldsymbol{s}'} V^{\boldsymbol{b}}_{\boldsymbol{a}} |e^{\boldsymbol{x}}_{\boldsymbol{s}}\rangle\langle e^{\boldsymbol{x}}_{\boldsymbol{s}'}| V^{\boldsymbol{b}\dagger}_{\boldsymbol{a}} \otimes |e^{\boldsymbol{x}}_{\boldsymbol{s}}\rangle\langle e^{\boldsymbol{x}}_{\boldsymbol{s}'}|$, with $\big|\Phi_{(\boldsymbol{a},\boldsymbol{b})}\big\rangle = (V^{\boldsymbol{b}}_{\boldsymbol{a}} \otimes \mathbb{1}) |\Phi\rangle_{\mathbf{X}\bar{\mathbf{X}}}$, for $(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Z}_d^2$

and we use the following properties of the operators V_a^b, which can be derived from (6.2):

1. $V_a^b |e_i^x\rangle\langle e_i^x| V_a^{b\dagger} = \big|e^x_{ax-b+i}\big\rangle\big\langle e^x_{ax-b+i}\big|$,
2. $V_a^b = \sum_j c_{a,b,j,x} \big|e^x_{ax-b+j}\big\rangle\big\langle e^x_j\big|$,
3. $V_a^{b\dagger} = \sum_j c^*_{a,b,j,x} \big|e^x_j\big\rangle\big\langle e^x_{ax-b+j}\big|$.

The following lemma, which is an adaptation of Theorem 12 in [80] to our case, provides a lower bound for $H_{\min}(\mathbf{F}|\mathbf{Y}\,E)_{\sigma_{\mathbf{F}\mathbf{Y}E}}$ in terms of $H_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}$:

Lemma 21. *Let $\mathbf{X}$ denote our n-qudit system and $\sigma_{\mathbf{X}E}$ be the ideal quantum state to which the system collapsed after Alice's test measurements, as introduced before. Then, we have:*

$$H_{min}(\mathbf{F}|\mathbf{Y}\,E)_{\sigma_{\mathbf{F}\mathbf{Y}E}} \geq \frac{1}{2}\big(n \log d + H_{min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}\big),$$

where

$$\sigma_{\mathbf{F}\mathbf{Y}E} = \frac{1}{d^{2n}} \sum_{(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Z}_d^{2n}} |e^{\boldsymbol{x}}_{\boldsymbol{a}}, e^{\boldsymbol{x}}_{\boldsymbol{b}}\rangle\langle e^{\boldsymbol{x}}_{\boldsymbol{a}}, e^{\boldsymbol{x}}_{\boldsymbol{b}}| \otimes V^{\boldsymbol{b}}_{\boldsymbol{a}}\, \sigma_{\mathbf{X}E}\, V^{\boldsymbol{b}\dagger}_{\boldsymbol{a}}, \tag{B.6}$$

is the state obtained when $V^{\boldsymbol{b}}_{\boldsymbol{a}}$ is applied to the system $\mathbf{X}$ according to $\mathbf{F}$.

Proof. This proof is an adaptation of the proof of Theorem 12 in [80] to our case. Let us fix $x \in \mathbb{Z}_d$ and write $|i\rangle = |e_i^x\rangle$ for short. V_a^b is a CPTP map, and it is known that the min-entropy does not decrease whenever a CPTP map is applied. However, this is not enough to prove the security of the protocol and determine a lower bound. We need a more refined expression relating $H_{\min}(\mathcal{M}(\mathbf{X})|E)$ and $H_{\min}(\mathbf{X}|E)$ for some CPTP map $\mathcal{M}$. This is given by Lemma 6, which can be applied for

$$\mathcal{M}_{\mathbf{X}\to\mathbf{F}\mathbf{Y}} = \mathcal{N}^{\otimes n},$$

where $\mathcal{N}(\sigma_X) = \frac{1}{d^2}\sum_{(a,b)\in\mathbb{Z}_d^2} \mathcal{N}_{a,b}\sigma_X\mathcal{N}_{a,b}^\dagger = \frac{1}{d^2}\sum_{(a,b)\in\mathbb{Z}_d^2} \Big(|a,b\rangle \otimes V_a^b\Big)\sigma_X\Big(\langle a,b| \otimes V_a^{b\dagger}\Big)$. The operator $\mathcal{N}$ applies the operator V_a^b to the single system X and saves its choice (a,b) to a new record in the F space.

To use Lemma 6, we have to prove that $((\mathcal{M}^\dagger \circ \mathcal{M})_{\mathbf{X}} \otimes \mathrm{id}_{\bar{\mathbf{X}}})(\Phi_{\mathbf{X}\bar{\mathbf{X}}})$ can be written as a linear combination of $\Phi_{(\boldsymbol{a},\boldsymbol{b})}$, i.e.

$$((\mathcal{M}^\dagger \circ \mathcal{M})_{\mathbf{X}} \otimes \mathrm{id}_{\bar{\mathbf{X}}})(\Phi_{\mathbf{X}\bar{\mathbf{X}}}) = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}^{2n}} \lambda_{(\boldsymbol{a},\boldsymbol{b})}\Phi_{(\boldsymbol{a},\boldsymbol{b})}.$$

First, note that

$$\mathcal{N}(|i\rangle\langle j|) = \frac{1}{d^2}\sum_{(a,b)\in\mathbb{Z}_d^2}\Big(|a,b\rangle\otimes V_a^b\Big)|i\rangle\langle j|\Big(\langle a,b|\otimes V_a^{b\dagger}\Big)$$
$$\overset{1}{=}\frac{1}{d^2}\sum_{(a,b)\in\mathbb{Z}_d^2}|a,b\rangle\langle a,b|\otimes|ax-b+i\rangle\langle ax-b+j|\,, \tag{B.7}$$

where we used the property 1. We proceed to compute $\mathcal{N}^\dagger\circ\mathcal{N}\,|i\rangle\langle j|$:

$$\begin{aligned}\mathcal{N}^\dagger\circ\mathcal{N}\,|i\rangle\langle j| &= \frac{1}{d^2}\sum_{(a',b')\in\mathbb{Z}_d^2}\mathcal{N}^\dagger_{a'b'}(\mathcal{N}\,|i\rangle\langle j|))\mathcal{N}_{a'b'}\\ &\overset{\text{eq.(B.7)}}{=}\frac{1}{d^4}\sum_{(a',b'),(a,b)\in\mathbb{Z}_d^2}\Big(\langle a',b'|\otimes V_{a'}^{b'\dagger}\Big)|a,b\rangle\langle a,b|\\ &\qquad\otimes|ax-b+i\rangle\langle ax-b+j|\Big(|a',b'\rangle\otimes V_{a'}^{b'}\Big)\\ &\overset{2,3}{=}\frac{1}{d^4}\sum_{(a,b)\in\mathbb{Z}_d^2}V_a^{b\dagger}\,|ax-b+i\rangle\langle ax-b+j|\,V_a^b\\ &=\frac{1}{d^4}\sum_{(a,b)\in\mathbb{Z}_d^2}|i\rangle\langle j|\\ &=\frac{1}{d^2}\,|i\rangle\langle j|\,,\end{aligned} \tag{B.8}$$

and

$$\begin{aligned}((\mathcal{N}^\dagger\circ\mathcal{N})_X\otimes\mathrm{id}_{\bar{X}})(\Phi_{X\bar{X}}) &= \sum_{i,j}\mathcal{N}^\dagger\circ\mathcal{N}(|i\rangle\langle j|)\otimes|i\rangle\langle j|\\ &\overset{\text{eq.(B.8)}}{=}\sum_{i,j}\Big(\frac{1}{d^2}\,|i\rangle\langle j|\Big)\otimes|i\rangle\langle j|\\ &=\frac{1}{d^2}\sum_{i,j}|i\rangle\langle j|\otimes|i\rangle\langle j|\\ &=\frac{1}{d^2}\Phi_{X\bar{X}}.\end{aligned} \tag{B.9}$$

Therefore, we have that

$$\big((\mathcal{N}^\dagger\circ\mathcal{N})_X\otimes\mathrm{id}_{\bar{X}}\big)(\Phi_{X\bar{X}}) = \frac{1}{d^2}\Phi_{(0,0)}, \tag{B.10}$$

from which we easily see that

$$\big((\mathcal{M}^\dagger\circ\mathcal{M})_{\mathbf{X}}\otimes\mathrm{id}_{\bar{\mathbf{X}}}\big)(\Phi_{\mathbf{X}\bar{\mathbf{X}}}) = \frac{1}{d^{2n}}\Phi_{(\mathbf{0},\mathbf{0})}. \tag{B.11}$$

Consequently,

$$\lambda_{(\boldsymbol{a},\boldsymbol{b})} = \begin{cases} \frac{1}{d^{2n}} & \text{if } (\boldsymbol{a},\boldsymbol{b}) = (\mathbf{0},\mathbf{0}) \\ 0 & \text{otherwise.} \end{cases}$$

Now, we want to choose the partition that gives us the best lower bound on the collision entropy, i.e. decreases the r.h.s of the following relation:

$$2^{-\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}}} \leq \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_+} \lambda_{(\boldsymbol{a},\boldsymbol{b})} 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} + (\max_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_-} \lambda_{(\boldsymbol{a},\boldsymbol{b})}) d^n.$$

Note that we dropped the conditioning on the state $\sigma_{\mathbf{X}E}$ at $\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}}$. This is because the map $\mathcal{M}$ is trace-preserving (for a detailed explanation see [80] below Theorem 1). For our case there are just two types of partitions: the case where $0 \in \mathfrak{S}_+$ and the case where $0 \in \mathfrak{S}_-$. If $0 \in \mathfrak{S}_+$:

$$\text{r.h.s} = \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_+} \lambda_s 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} = \frac{1}{d^{2n}} 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} \leq \frac{1}{d^n}.$$

The last inequality holds because $-n\log d \leq \mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \leq n\log d$. In fact,

$$\frac{1}{d^{2n}} 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} \leq \frac{1}{d^n} \iff 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} \leq d^n \iff \mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \geq -n\log d.$$

If $0 \in \mathfrak{S}_-$, r.h.s $= \frac{1}{d^n}$ which, as we have just seen by the previous inequality, does not provide a better lower bound on the collision entropy whenever $\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}} \neq -n\log d$.

So, choosing any partition such that $0 \in \mathfrak{S}_+$, we get

$$2^{-\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}}} \leq \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_+} \lambda_{(\boldsymbol{a},\boldsymbol{b})} 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}} + (\max_{(\boldsymbol{a},\boldsymbol{b})\in\mathfrak{S}_-} \lambda_{(\boldsymbol{a},\boldsymbol{b})}) d^n = \frac{1}{d^{2n}} 2^{-\mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}},$$

from which we conclude that

$$\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}} \geq 2n\log d + \mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}. \tag{B.12}$$

In order to relate $\mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}}$ with $\mathrm{H}_2(\mathbf{F}|\mathbf{Y}E)_{\sigma_{\mathbf{FY}E}}$, we use the chain rule given by Lemma 7:

$$\mathrm{H}_2(\mathbf{F}|\mathbf{Y}E)_{\sigma_{\mathbf{FY}E}} \geq \mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}} - \log\mathrm{rank}(\sigma_{\mathbf{Y}}) \geq \mathrm{H}_2(\mathbf{FY}|E)_{\sigma_{\mathbf{FY}E}} - n\log d, \tag{B.13}$$

since $\log\mathrm{rank}(\sigma_{\mathbf{Y}}) \leq n\log d$. Combining (B.12) and (B.13), we get the desired result:

$$\mathrm{H}_2(\mathbf{F}|\mathbf{Y}\,E)_{\sigma_{\mathbf{FY}E}} \geq n\log d + \mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}.$$

To express the above relation in terms of the min-entropy instead of the collision entropy, we start by noticing that the state given in (B.6) can be written as

$$\sigma_{\mathbf{FY}E} = \frac{1}{d^{2n}} \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}_d^{2n}} |\boldsymbol{a},\boldsymbol{b}\rangle\langle\boldsymbol{a},\boldsymbol{b}| \otimes V_{\boldsymbol{a}}^{\boldsymbol{b}}\, \sigma_{\mathbf{X}E}\, V_{\boldsymbol{a}}^{\boldsymbol{b}\dagger} = \frac{1}{d^{2n}} \sum_{(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}_d^{2n}} |\boldsymbol{a},\boldsymbol{b}\rangle\langle\boldsymbol{a},\boldsymbol{b}| \otimes \sigma_{\mathbf{X}E}^{\boldsymbol{a},\boldsymbol{b}}, \quad \text{(B.14)}$$

which is a classical-quantum state. Therefore, we can use Lemma 18 from [80] to obtain

$$\mathrm{H}_{\min}(\mathbf{F}|\mathbf{Y}\,E)_{\sigma_{\mathbf{FY}E}} \geq \frac{1}{2}\left(n\log d + \mathrm{H}_2(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}\right).$$

Furthermore, $\sigma_{\mathbf{X}E}$ is a general quantum state, and from Lemma 17 in [80] we have

$$\mathrm{H}_{\min}(\mathbf{F}|\mathbf{Y}\,E)_{\sigma_{\mathbf{FY}E}} \geq \frac{1}{2}\left(n\log d + \mathrm{H}_{\min}(\mathbf{X}|E)_{\sigma_{\mathbf{X}E}}\right).$$

□

To complete the proof of property 2. of Lemma 14, we combine Lemma 20 and Lemma 21 and obtain:

$$H_{\min}(\mathbf{F}|\mathbf{Y}E)_{\sigma_{\mathbf{FY}E}} \geq \frac{1}{2}(n\log d - h_d(\zeta)n\log d) = \frac{n\log d}{2}(1-h_d(\zeta)),$$

for $err = 0$.

□

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