# Appendix A
# Analytic derivation of the closed expressions for the fidelity and $\Delta$

As already mentioned, the fidelity between two states $\rho$ and $\rho'$ is given by

$$F(\rho, \rho') = \mathrm{Tr}\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}. \tag{1}$$

We consider unnormalized thermal states $\rho = \exp(-\beta H)$ and $\rho' = \exp(-\beta' H')$. At the end of the calculation one must, of course, normalize the expressions appropriately. We wish to find closed expressions for the fidelity and the quantity $\Delta$ with respect to these thermal states. In order to do that we will proceed by finding $e^C$, such that

$$e^A e^B e^A = e^C, \tag{2}$$

for $A = -\beta H$, $B = -\beta' H'$ and, ultimately, take the square root of the result. The previous equation is equivalent to

$$e^A e^B = e^C e^{-A}. \tag{3}$$

The Hamiltonians $H$ and $H'$ are taken to be of the form $\vec{h} \cdot \vec{\sigma}$, and thus we can write

$$e^A = a_0 + \vec{a} \cdot \vec{\sigma}, e^B = b_0 + \vec{b} \cdot \vec{\sigma}, e^C = c_0 + \vec{c} \cdot \vec{\sigma},$$

where all the coefficients are real, with the following constraints:

$$\begin{cases} 1 = \det e^A = a_0^2 - \vec{a}^2, \\ 1 = \det e^B = b_0^2 - \vec{b}^2, \\ 1 = \det e^C = c_0^2 - \vec{c}^2, \end{cases} \tag{4}$$

which are equivalent to $\mathrm{Tr}\, A = \mathrm{Tr}\, B = \mathrm{Tr}\, C = 0$, since Pauli matrices are traceless. Let us proceed by expanding the LHS and the RHS of Eq.(3),

$$\begin{aligned} &(a_0 + \vec{a} \cdot \vec{\sigma})(b_0 + \vec{b} \cdot \vec{\sigma}) = (c_0 + \vec{c} \cdot \vec{\sigma})(a_0 - \vec{a} \cdot \vec{\sigma}) \\ \Leftrightarrow\ & a_0 b_0 + a_0 \vec{b} \cdot \vec{\sigma} + \vec{a} \cdot \vec{\sigma} b_0 + (\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = c_0 a_0 - c_0 \vec{a} \cdot \vec{\sigma} + \vec{c} \cdot \vec{\sigma} a_0 - (\vec{c} \cdot \vec{\sigma})(\vec{a} \cdot \vec{\sigma}) \\ \Leftrightarrow\ & a_0 b_0 + a_0 \vec{b} \cdot \vec{\sigma} + \vec{a} \cdot \vec{\sigma} b_0 + \vec{a} \cdot \vec{b} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma} = c_0 a_0 - c_0 \vec{a} \cdot \vec{\sigma} + \vec{c} \cdot \vec{\sigma} a_0 - \vec{c} \cdot \vec{a} - i(\vec{c} \times \vec{a}) \cdot \vec{\sigma}. \end{aligned} \tag{5}$$

Now, collecting terms in 1, $\vec{\sigma}$ and $i\vec{\sigma}$, we get a system of linear equations on $c_0$ and $\vec{c}$,

$$\begin{cases} a_0 b_0 + \vec{a}\cdot\vec{b} - a_0 c_0 + \vec{a}\cdot\vec{c} = 0, \\ a_0\vec{b} + b_0\vec{a} + \vec{a}c_0 - a_0\vec{c} = 0, \\ \vec{a}\times\vec{b} - \vec{a}\times\vec{c} = 0. \end{cases} \tag{6}$$

The third equation from (6) can be written as $\vec{a}\times(\vec{b}-\vec{c}) = 0$, whose solution is given by $\vec{c} = \vec{b} + \lambda\vec{a}$, where $\lambda$ is a real number. This means that the solution depends only on two real parameters: $c_0$ and $\lambda$. Hence, we are left with a simpler system given by,

$$\begin{cases} a_0 b_0 + \vec{a}\cdot\vec{b} - a_0 c_0 + \vec{a}\cdot(\vec{b}+\lambda\vec{a}) = 0 \\ a_0\vec{b} + b_0\vec{a} + \vec{a}c_0 - a_0(\vec{b}+\lambda\vec{a}) = 0 \end{cases}. \tag{7}$$

Or,

$$\begin{cases} a_0 c_0 - \lambda\vec{a}^2 = a_0 b_0 + 2\vec{a}\cdot\vec{b} \\ (a_0\lambda - c_0)\vec{a} = b_0\vec{a} \end{cases}. \tag{8}$$

In matrix form, the above system of equations can be written as

$$\begin{bmatrix} a_0 & -\vec{a}^2 \\ -1 & a_0 \end{bmatrix}\begin{bmatrix} c_0 \\ \lambda \end{bmatrix} = \begin{bmatrix} a_0 b_0 + 2\vec{a}\cdot\vec{b} \\ b_0 \end{bmatrix}. \tag{9}$$

Inverting the matrix, we get

$$\begin{aligned} \begin{bmatrix} c_0 \\ \lambda \end{bmatrix} &= \frac{1}{a_0^2-\vec{a}^2}\begin{bmatrix} a_0 & \vec{a}^2 \\ 1 & a_0 \end{bmatrix}\begin{bmatrix} a_0 b_0 + 2\vec{a}\cdot\vec{b} \\ b_0 \end{bmatrix} \\ &= \begin{bmatrix} (2a_0^2-1)b_0 + 2a_0\vec{a}\cdot\vec{b} \\ 2(a_0 b_0 + \vec{a}\cdot\vec{b}) \end{bmatrix}, \end{aligned} \tag{10}$$

where we used the constraints (4). Because of the constraints, $c_0$ and $\lambda$ are not independent, namely, $e^C = c_0 + (\vec{b}+\lambda\vec{a})\cdot\vec{\sigma}$, and we get

$$c_0^2 - (\vec{b}+\lambda\vec{a})^2 = c_0^2 - \vec{b}^2 - 2\lambda\vec{a}\cdot\vec{b} - \vec{a}^2 = 1. \tag{11}$$

Now we want to make $A = -\beta H/2 \equiv -\xi\vec{x}\cdot\vec{\sigma}/2$ and $B = -\beta' H' \equiv -\zeta\vec{y}\cdot\vec{\sigma}$, with $\vec{x}^2 = \vec{y}^2 = 1$ and $\xi$ and $\zeta$ real parameters, meaning,

$$a_0 = \cosh(\xi/2) \text{ and } \vec{a} = -\sinh(\xi/2)\vec{x}, b_0 = \cosh(\zeta) \text{ and } \vec{b} = -\sinh(\zeta)\vec{y}. \tag{12}$$

If we write $C = \rho\vec{z}\cdot\vec{\sigma}$ (because the product of matrices with determinant 1 has to have determinant 1, it has to be of this form),

$$\begin{aligned} c_0 &= \cosh(\rho) \\ &= (2a_0^2-1)b_0 + 2a_0\vec{a}\cdot\vec{b} \\ &= (2\cosh^2(\xi/2)-1)\cosh(\zeta) + 2\cosh(\xi/2)\sinh(\xi/2)\sinh(\zeta)\vec{x}\cdot\vec{y} \\ &= \cosh(\xi)\cosh(\zeta) + \sinh(\xi)\sinh(\zeta)\vec{x}\cdot\vec{y}. \end{aligned} \tag{13}$$

For all the expressions concerning fidelity, we wish to compute $\mathrm{Tr}(e^{C/2}) = 2\cosh(\rho/2)$. If we use the formula $\cosh(\rho/2) = \sqrt{(1+\cosh(\rho))/2}$, we obtain,

$$\mathrm{Tr}(e^{C/2}) = 2\sqrt{\frac{(1+\cosh(\xi)\cosh(\zeta)+\sinh(\xi)\sinh(\zeta)\vec{x}\cdot\vec{y})}{2}}. \tag{14}$$

Hence, if we let $\xi = \beta E/2$, $\vec{x} = \vec{n}$, $\zeta = \beta' E'/2$ and $\vec{y} = \vec{n}'$, then

$$\mathrm{Tr}(\sqrt{e^{-\beta H/2}e^{-\beta' H'}e^{-\beta H/2}}) = 2\sqrt{\frac{(1+\cosh(\beta E/2)\cosh(\beta' E'/2)+\sinh(\beta E/2)\sinh(\beta' E'/2)\vec{n}\cdot\vec{n}')}{2}}. \tag{15}$$

To be able to compute the fidelities, we will just need the following expression relating the traces of quadratic many-body fermion Hamiltonians (preserving the number operator) and the single-particle sector Hamiltonian obtained by projection:

$$\mathrm{Tr}(e^{-\beta\mathcal{H}}) = \mathrm{Tr}(e^{-\beta\Psi^\dagger H\Psi}) = \det(I+e^{-\beta H}). \tag{16}$$

From the previous results, it is straightforward to derive the following formulae for the fidelities concerning the thermal states considered:

$$\begin{aligned} F(\rho,\rho') &= \prod_{k\in\mathcal{B}} \frac{\mathrm{Tr}(e^{-\mathcal{C}_k/2})}{\mathrm{Tr}(e^{-\beta\mathcal{H}_k})\,\mathrm{Tr}(e^{-\beta'\mathcal{H}'_k})} \\ &= \prod_{k\in\mathcal{B}} \frac{\det(I+e^{-C_k/2})}{\det^{1/2}(I+e^{-\beta H_k})\det^{1/2}(I+e^{-\beta' H'_k})} \\ &= \prod_{k\in\mathcal{B}} \frac{2+\sqrt{2\big(1+\cosh(E_k/2T)\cosh(E'_k/2T')+\sinh(E_k/2T)\sinh(E'_k/2T')\vec{n}_k\cdot\vec{n}'_k\big)}}{\sqrt{(2+2\cosh(E_k/2T))(2+2\cosh(E'_k/2T'))}}, \end{aligned} \tag{17}$$

where the matrix $C_k$ is such that $e^{-C_k} = e^{-\beta H_k/2}e^{-\beta' H'_k}e^{-\beta H_k/2}$ and $\mathcal{C}_k = \Psi_k^\dagger C_k\Psi_k$ is the corresponding many-body quadratic operator.

To compute $\Delta(\rho,\rho')$ one needs, in addition, $\mathrm{Tr}\,\sqrt{\rho}\sqrt{\rho'}$. This can be done along the lines of what was presented above, hence we shall omit the proof for the sake of briefness and directly provide the result:

$$\mathrm{Tr}\,\sqrt{\rho}\sqrt{\rho'} = \prod_{k\in\mathcal{B}} \frac{2+2\left(\cosh(E_k/4T)\cosh(E'_k/4T')+\sinh(E_k/4T)\sinh(E'_k/4T')\vec{n}_k\cdot\vec{n}'_k\right)}{\sqrt{(2+2\cosh(E_k/2T))(2+2\cosh(E'_k/2T'))}} \tag{18}$$