DFT & DTMF Frequencies Identification (December 2023)

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***Abstract*— This work presents a study of the DFT (Discrete Fourier Transform) and its application in the identification of the frequencies that make up the DTMF (Dual-Tone Multi-Frequency) system. To carry out this analysis, MATLAB will be used as a support tool. We hope you enjoy our study. For any questions or clarifications, we are at your disposal through our emails.**

Keywords— DTMF, DFT, FFT, Zero-Padding, FTDS (Fourier Transform of a Discrete System)

I. Introduction

IN the digital age, DTMF is a key technology that uses two tones to represent digits on telephone keypads. This signalling is essential in various applications, from telephone dialling to digital security. This paper seeks to identify the pitch frequencies and amplitudes in a signal stored in a .MAT file. The Discrete Fourier Transform (DFT) will be performed using the FFT (Fast Fourier Transform) algorithm of MATLAB to achieve this goal precisely and efficiently, using mathematical relationships.

II. DFT (Discrete Fourier Transform)

To calculate the DFT of a Discrete System, it is necessary to take the signal to be analysed, of length L (windowed signal) and determine the length of the DFT, N, where $N \geq L$. If $N > L$, requires extending the initial signal by adding trailing zeros, known as **Zero-Padding**, in order to match the signal and DFT lengths. This allows the calculation of the DFT improving its resolution without altering the intrinsic information of the studied signal. However, this procedure increases the processing time by calculating more points of the DFT.

To compute the DFT of an Infinite-length digital sinusoidal signal $x[n]$ with a fundamental frequency f_0, a window will must be applied. In this study, a rectangular window $w[n]$ of length L is utilized to calculate the $DFT_N\{x'[n]\}$.

The equations that define the signal $x[n]$ and the rectangular window $w[n]$ are:

For the signal $x[n]$:

$$x[n] = A\cos(\Omega_0\, n) \qquad -\infty < n < \infty$$

Equation I. Infinite Cosine Expression.

With an amplitude $A = 1$ in the case of DTMF tones, nevertheless expressions will be generalized.

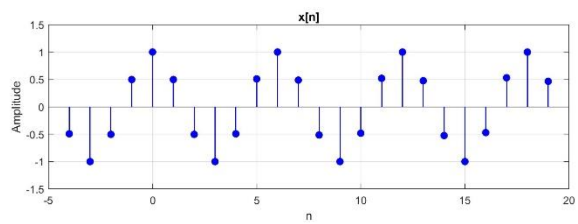

Figure I. Representation of an Infinite Discrete Cosine.

For the rectangular window $w[n]$:

$$w[n] = \begin{cases} 1 & 0 \leq n \leq L-1 \\ 0 & others \end{cases}$$

Equation II. Analytical Window Expression.

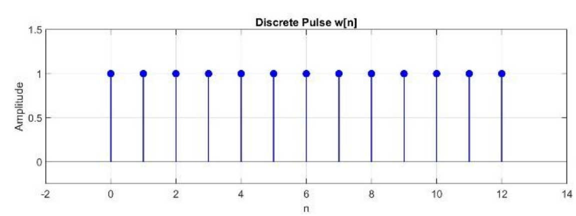

Figure III. Representation of a Finite Discrete Pulse.

When applying the window $w[n]$ to the signal $x[n]$.

$$x'[n] = x[n] \cdot w[n] = \begin{cases} A\cos(\Omega_0\, n) & 0 \leq n \leq L-1 \\ 0 & others \end{cases}$$

Equation I. Finite -Length Discrete Cosine Expression.

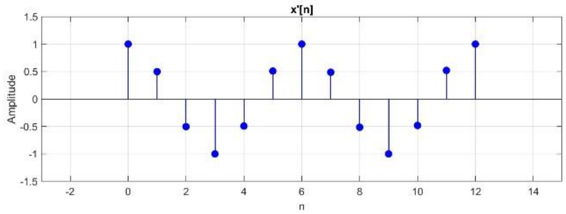

Figure II. Finite - Length Discrete Cosine.

If the entire process is executed and the transition to the transformed domain is made the $FTDS$ of a product becomes a convolution in frequency.

$$\text{FTDS}\left\{x'[n]\right\} = FTDS\left\{x[n] \cdot w[n]\right\} = X(e^{j\Omega}) \circledast W(e^{j\Omega})$$

Equation IV. Analytical Windowed Signal Expression.

Where $FTDS$ the signal $x[n]$is:

$$FTDS\left\{x[n]\right\} =$$

$$X(e^{j\Omega}) = A\pi \sum_{k=-\infty}^{\infty} \delta(\Omega - \Omega_0 - 2\pi k) + \delta(\Omega + \Omega_0 - 2\pi k)$$

Equation V. Fourier Transform of a Discrete Cosine.

What graphically looks like two deltas of amplitude Aπ between interval $-\pi \leq \Omega \leq \pi$.
And for the case of the amplitude DTMF tones π.

In this case we have focused on one of the frequencies used by DTFM, the specific frequency is $f_0 = 1336\,Hz$, where $\Omega_0 = 2\pi\, f_0 T_s$, and how the DTFM tones are sampled at one of $f_s = 8000\,Hz$ the digital frequencies where the two δ are centered: $\Omega_0 = -\frac{167}{500}\pi = -0'334\pi$ and $\Omega_0 = \frac{167}{500}\pi = 0'334\pi$.

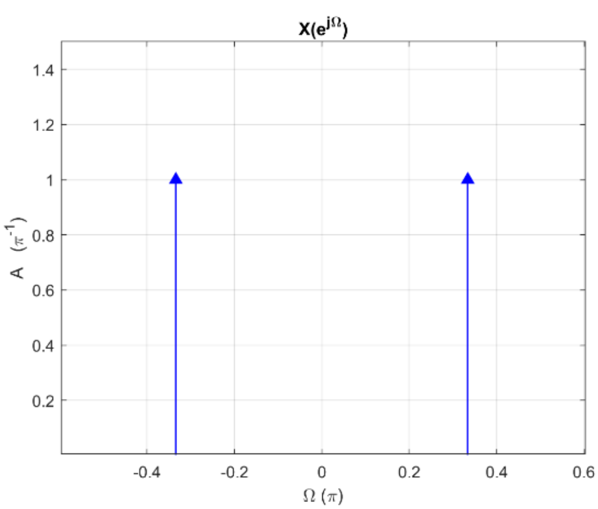

Figure V. Discrete Cosine Spectrum $-\pi \leq \Omega \leq \pi$.

And the one $FTDS$ of a discrete pulse $w[n]$ is a Digital Sinc centred at the origin so:

$$FTDS\left\{w[n]\right\} = W(e^{j\Omega}) = e^{-j\frac{\Omega(L-1)}{2}} \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}$$

Equation VI. Fourier Transform of a Discrete Pulse Expression.

With a maximum value for $\Omega = 0$ of L.

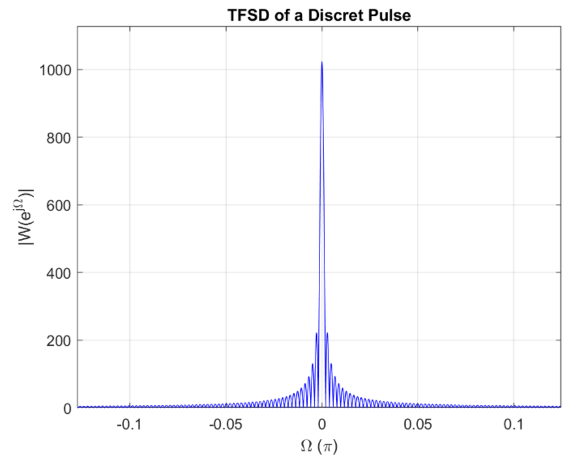

Figure VI. Discrete Pulse Spectrum of L length.

If these two functions are convoluted, we obtain the $FTDS\left\{x'[n]\right\}$.

$$\text{FTDS}\left\{x'[n]\right\} = \text{FTDS}\left\{x[n] \cdot w[n]\right\} = X(e^{j\Omega}) \circledast W(e^{j\Omega})$$

Equation VII. Fourier Transform of The Discrete Windowed Signal.

Graphically it would see two displaced Digital Sincs Ω_0 and $-\Omega_0$.

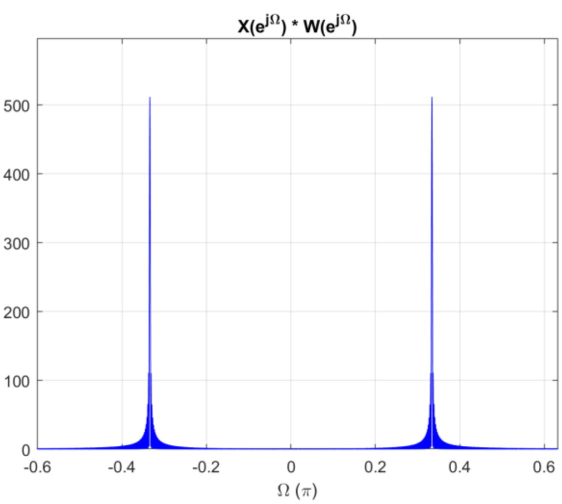

Figure VII. Graphical representation of $X(e^{j\Omega})$ and $W(e^{j\Omega})$ Convolution.

If particularized this result for each value of k, with a DFT of length N we will get the $X'[k]$, $DFT_N\{x'[n]\}$.

$$X['k] = DFT_N\{x'[n]\} =$$

$$\sum_{n=0}^{N-1} x'[n] \cdot e^{-j\frac{2\pi}{N}kn} = X(e^{j\Omega}) \circledast W(e^{j\Omega}) \Big|_{\Omega = \frac{2\pi}{N}k}$$

Equation VIII. $DFT_N\{x'[n]\}$ Expression.

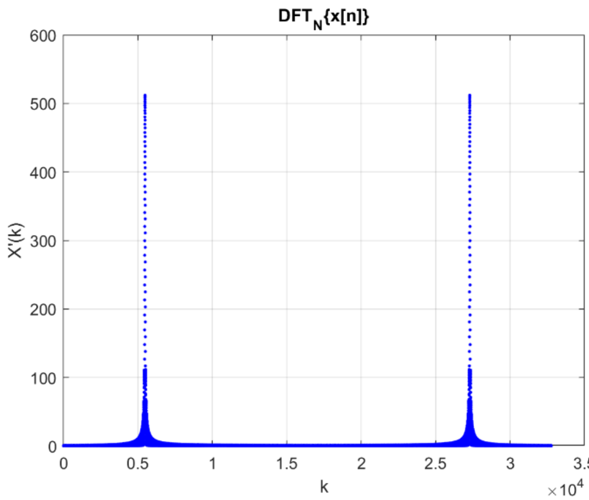

Figure VIII. Graphic Representation of $DFT_N\{x'[n]\}$.

From the figure above The Digital Sinc on the left refers to the positive δ frequency of the tone and while the Sinc on the right refers to the δ negative frequency of the tone. However, since in the discrete domain the samples are in the range $0 \leq k \leq N-1$, the graph is represented in this way.

Well, what the FFT algorithm does is: it calculates numerically the DFT, in a very fast and efficient way, given a signal $x'[n]$ and with a determined length of the DFT, N. So, the $X'[k]$, $DFT_N\{x'[n]\}$ it would look like:

$$X'[k] = DFT_N\{x'[n]\} = fft(x'[n], N) =$$

$$FTDS\ \{x'[n]\}|_{\Omega = \frac{2\pi}{N}k} = FTDS\ \{x[n] \cdot w[n]\}|_{\Omega = \frac{2\pi}{N}k}$$

Equation IX. Relations between DFT, FFT and FTDS.

The length of the signal can be determined through the programming software that we are working on in our case, it tells us that the length of the digital signal $x'[n]$ stored in a .MAT file is, $L = 1024$.

The recording time of the signal will be easy to determine, given the known length and sampling frequency.

$$t = L\, T_s$$

Equation X. Recording Time for a Digital Signal of Length L Sampled at f_s.

III. Comparison Zero-Padding vs Non-Zero-Padding

The following image shows a comparison of the discrete spectrum, between the application and non-application of this technique.

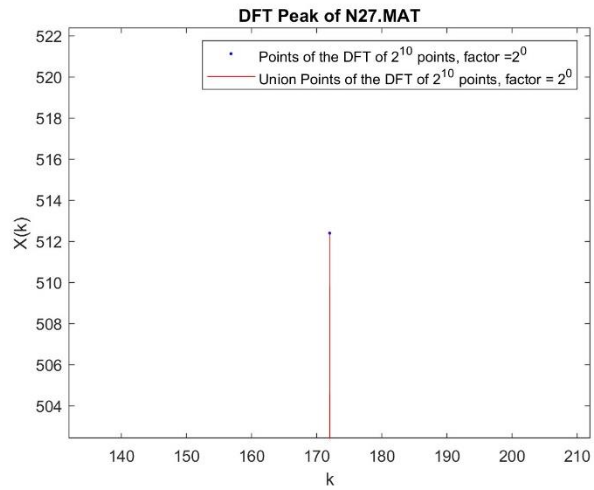

Figure X. Non-Zero-Padding DFT peak.

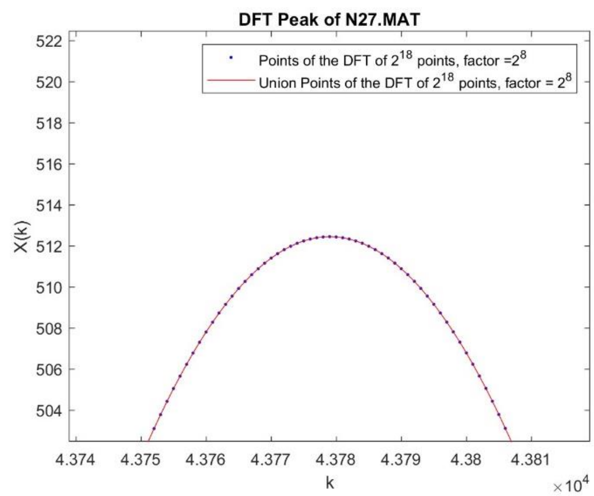

Figure XI. Zero-Padding DFT peak.

It should be noted that the DFT is only isolated points of the spectrum, only the points (in red) have been joined to give a clearer graphical idea.

The objective of this exercise is to find the frequencies where the amplitude of the signal is maximum. When **Zero-Padding** is applied, more points are obtained, allowing a more detailed representation in the frequency domain. This is due to the decrease in the step of discrete frequencies $\Delta\Omega = \frac{2\pi}{N}$ with the length of the window, increasing the resolution. This improvement can be seen in the Figure XI facilitating a more precise identification of the frequency for this maximum amplitude. In summary, increasing the window length improves the approximation of the frequencies of the transmitted tones.

IV. Frequency Identification

Once the DFT of the signal to be studied is obtained, it will be possible to identify its digital and analogue frequencies.
By making use of MATLAB's max () function, it can be determined which point of all the points of the DFT has a maximum value. This function can provide us both the maximum value of that point and its position k.
Using the equation that relates the DFT and the FTDS, the digital frequencies $\Omega = \frac{2\pi}{N}k$ can be determined and with the formula that links the digital frequency and the analogue frequency, $\Omega = 2\pi\, f\, T_s$ it is possible to know f
By manipulating these two expressions you can get:

$$f = \frac{f_s}{N}k\,, \qquad \frac{f}{f_s} = \frac{k}{N}$$

Equation XI. Expression to obtain Analogue f.

From the above formulas it can be deduced that the frequency f found by this method will be a very approximate but quite accurate frequency of the real one, since the sampling frequency is not a multiple of any of the DTMF frequencies. And especially the accuracy of this result will increase with the window length N.

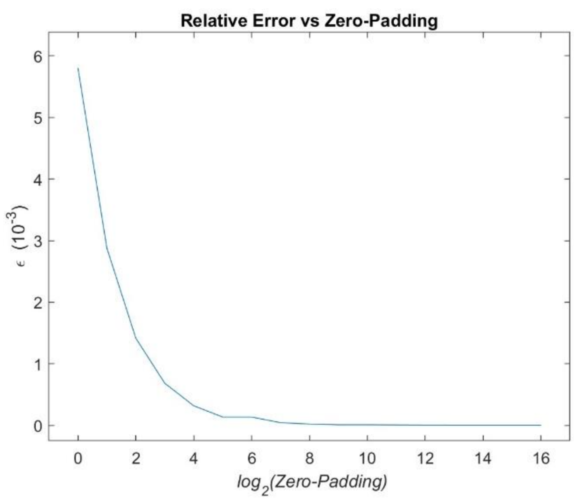

Figure. Relative Error (ε_r) vs Zero-Padding.

V. Tons Amplitudes

As mentioned in the DFT section, the amplitudes of the TFDS of a tone are $A\pi$ and after convolving the functions $X(e^{j\Omega})$ and $W(e^{j\Omega})$ the amplitudes are reduced by a factor 2π. So, the amplitude of the DFT is:

$$A_{DFT} = \frac{A\pi}{2\pi}L = \frac{A}{2}L, \qquad A = \frac{A_{DFT}\cdot 2}{L}$$

Equation XII. Tone Amplitude Expression.

If you find the DFT maximum value you can know the amplitude of the tone.

VI. Files .MAT - Key Mapping

Key	File	Key	File
1	N16. MAT	B	.MAT NOT FOUND
	N17. MAT		
	N24. MAT	7	N8. MAT
	N32. MAT		N18. MAT
2	N6. MAT		N22. MAT
	N28. MAT		N34. MAT
3	N5. MAT	8	N12. MAT
	N14. MAT		N21. MAT
	N23. MAT		N35. MAT
	N29. MAT	9	N11. MAT
A	N4. MAT		N25. MAT
4	N2. MAT		N30. MAT
	N3. MAT	C	.MAT NOT FOUND
	N15. MAT	*	.MAT NOT FOUND
5	N10. MAT		
	N19.MAT	0	N1. MAT
	N27. MAT		N9. MAT
	N33. MAT		N13. MAT
6	N7. MAT		N20. MAT
	N18. MAT	#	.MAT NOT FOUND
	N22. MAT	D	.MAT NOT FOUND

Table I. Files .MAT - Key Mapping

VII. Conclusions

The experimental study carried out in the laboratory shows the effectiveness of the Zero-Padding technique in improving the accuracy while detecting the frequencies and amplitudes of the overlapping tones in the audio files.
It has proven to obtain a more precise resolution, providing more points in the spectrum, and improving the identification of the frequency.
As a conclusion, the importance of this technique to obtain more accurate results in the detection of DTMF tones is highlighted.

VIII. References

[1] E. Bertran, “Señales y sistemas de tiempo discreto”, Edicions UPC, Edition 1, 2003, pages 28-30.