Varying delays and sampling intervals

- Finite number of values
- $h_k \in \{h^1, h^2, ..., h^M\}$
- $\tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$

Sampled-data system

Time-Varying delays and sampling intervals

- Previously: Constant delays and sampling intervals $\xi_{k+1} = \left(x_k^T u_{k-1}^T\right)^T$:

$$\xi_{k+1} = H(h,\tau)\xi_k \text{ with}$$

$$H(h,\tau) = \begin{pmatrix} e^{Ah} - \int_0^{h-\tau} e^{Ah} ds B\hat{K} & \int_0^{h-\tau} e^{As} ds B - \int_0^{h-\tau} e^{As} ds B K_u \\ -\hat{K} & -K_u \end{pmatrix}$$

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- Now: $h_k \in \{h^1, h^2, ..., h^M\}$ and $\tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$
- Discrete-time sampled-data dynamics in extended-state formulation

$$\xi_{k+1} = H(h_k, \tau_k)\xi_k$$

- Swithced linear systemns:
  - Depending on h_k and τ_k a different linear model is active!

Sampled-data system

Switched linear system: $\xi_{k+1} = H(h_k \tau_k)\xi_k$

with $h_k \in \{h^1, h^2, ..., h^M\}$ and $\tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$

- SLS exponentially stable if Lyapunov funciton $V(\xi) = \xi^T P \xi$ is found, i.e.
  - V is positive definite, i.e. $\xi^T P \xi > 0$ when $\xi \neq 0$ and
  - Decrease of V along trajectories (irrespective of h_k, τ_k)

$$V(\xi_{k+1}) < V(\xi_k) \quad \Leftrightarrow \quad \xi_{k+1}^T P \xi_{k+1} < \xi_k^T P \xi_k, \xi_k \neq 0$$

or

$$\xi_k^T \left[H(h_k, \tau_k)^T P H(h_k, \tau_k) - P \right] \xi_k < 0 \text{ for } \xi_k \neq 0$$

Sampled-data system

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- Gives rise to linear matrix inequalities (LMIs)

$$P \succ 0 \qquad \text{(positive definite)}$$

$$H(h, \tau)^T P H(h, \tau) - P \succ 0 \qquad h_k \in \{h^1, h^2, ..., h^M\}, \quad \tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$$

- "Appendix" LMIs

Sampled-data system

Switched linear system: $\xi_{k+1} = H(h_k \tau_k)\xi_k$

with $h_k \in \{h^1, h^2, ..., h^M\}$ and $\tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$

- SLS exponentially stable if

$$P \succ 0 \qquad \text{(positive definite)}$$

$$H(h,\tau)^T P H(h,\tau) - P \succ 0 \qquad h_k \in \{h^1, h^2, ..., h^M\},\ \tau_k \in \{\tau^1, \tau^2, ..., \tau^L\}$$

- LMIs can be solved using efficient LMI solvers (e.g. in Matlab)
- LMIs based on a common quadratic Lyapunov function.
- Results using switched/multiple Lyapunov functions exist $V(x,h,\tau) = x^T P_{ij} x$, when $h = h^i, i = 1, ..., M$ and $\tau = \tau^j, j = 1, ..., L$
- See, e.g., [Daafouz et al, TAC, 2002]

Matrices and inequalities

- A symmetric matrix $P \in \mathbb{R}^{nxn}$ is positive definite, if $x^T P x > 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$ .We write $P \succ 0$
- A symmetric matrix $P \in \mathbb{R}^{nxn}$ is positive semi-definite, if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$ We write $P \succeq 0$
- A symmetric matrix $P \in \mathbb{R}^{nxn}$ is negative definite, if $x^T P x < 0$ for all $x \in \mathbb{R}^n$ with $x \neq 0$ .We write $P \prec 0$
- A symmetric matrix $P \in \mathbb{R}^{nxn}$ is negative semi-definite, if $x^T P x \leq 0$ for all $x \in \mathbb{R}^n$ We write $P \preceq 0$

Matrices and inequalities

Characterisations in terms of eigenvalues and determinats of leading principal submatrices!

Equivalent (using symmetry of P):

- P positive definite ($x^T Px > 0$ for all $x \neq 0$)
- all eigenvalues are positive
- all leading principal minors $det P_{JJ} > 0$ for all $J = \{1, ..., j\}$ for $j = 1, ..., n.$

Equivalent (using symmetry of P):

- P positive semi-definite ($x^T Px \geq 0$ for all x)
- all eigenvalues are positive or 0
- all leading principal minors $det P_{JJ} \geq 0$ for all $J = \{1, ..., j\}$ for $j = 1, ..., n.$

Question: Why symmetry without loss of generality?

Matrices and inequalities

If P postive definite, then

- P is invertible (non-singular) Question: Why?
- P^{-1} is positive definite. Question: Why?

$$\lambda_{min}(P)||x||^2 \leq x^T P x \leq \lambda_{max}(P)||x||^2$$

where $||x||^2 = x^T x$ and $\lambda_{min}(P), \lambda_{max}(P)$ denote the smallest and largest eigenvalue of P

Partial ordering on matrices:

- $P \succ Q$ means that $P - Q \succ 0$ $(P - Q$ is a positive definite matrix).

"Appendix: LMIs"

Linear matrix inequalities (LMIs)

- Given the ordering induced by "positive definiteness" we can formulate inequalities in terms of matrices
- Example $V(x) = x^{\text{`}} P x$ Lyap. function for $x_{k+1} = A x_k$ yields

$$P \succ Q \quad \text{and} \quad A' P A - P \prec 0$$

- linear MI as the matrices we solve for appear linearly (no $P^2, P_1 A P_2$)
- Important property of LMIs:
  - There are efficient numerical algorithms to solve LMIs
  - They can be used for many analysis and synthesis problems for linear, switched linear and piecewise linear systems
- Example linear systems: linear H_∞ -control can be solved via LMIs