A Brief Overview of Lyapunov Stability for Ordinary Differential Equations

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Lyapunov stability

Lyapunov stability is a way to characterize the qualitative idea of stability in models of physical systems. A stable solution to a differential equation is one which does not change much when the initial conditions are changed slightly. We shall look at two methods of checking the stability of solutions to ordinary differential equations. The first works by characterizing the solutions of linear systems to study their stability and then looking at nonlinear systems which are close to the linear case. The second uses an auxilliary function to find stability without having to characterize the solutions. Finally we look at some of the limitations of the Lyapunov notion of stability.

Introduction

We are going to look at general ordinary differential equations of the following type:

$$x'(t) = f(t, x(t)) \tag{1}$$

where $x : \Re \to \Re^n$ and $f : \Re \times \Re^n \to \Re^n$ are functions and t, which usually represents time, is the only independent variable. The derivative $x'(t)$ is simply the derivatives of each of the component functions, so if $x(t) = (x_1(t), ..., x_n(t))$ where $x_i(t)$ are real-valued functions on $\Re$ then

$$x'(t) = (x_1'(t), ..., x_n'(t)).$$

In some cases we will be considering the autonomous case

$$x'(t) = f(x(t)) \tag{2}$$

where $f : \Re^n \to \Re^n$. We are going to assume the existence of continuous solutions and uniqueness of a solution with the initial condition $x(t_0) = x_0$ where t_0 is a constant and x_0 is a constant n-vector. In the simplest case, a solution will tend to a constant as t goes to infinity. In the autonomous case, we can think of the solution as a continuous curve through n-space (called phase space), and when the solution tends to a constant, the curve will converge to a point in phase space. If a constant function c satisfies the differential equation, then the the derivative $x'(t) = f = 0$ for $x_0 = c$ and c is called an equilibrium point. Periodic solutions satisfy $x(t) = x(t + n\tau)$ for some τ and any integer n. Other, more complicated solutions also exist which may or may not be stable. Nonautonomous systems have analogous types of solutions.

We are going to be studying the stability of (1). Stability is a qualitative notion that if we perturb the initial condition slightly, say $x(t_0) = x_0 + \varepsilon$, then the new solution will be close to the original system. Stable differential equations are more useful in practice since x_0 is often a measured value and does not have infinite precision. A stable differential equation would insure that our model would come up with an answer reasonably close to the actual answer an exact initial condition.

First, we will define our notion of stability. What we shall refer to as *stable* is more correctly termed *Lyapunov stable* since there are many different mathematical notions of stability.

Definition 2.1. A solution $x(t)$ is *stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_0 - \bar{x}_0\| < \delta$ then $\|x(t) - \bar{x}(t)\| < \varepsilon$ for every $t \geq t_0$, where $\bar{x}(t)$ is the solution to

$$\bar{x}'(t) = f(t, \bar{x}(t))$$

with $x(t_0) = x_0$ and $\bar{x}(t_0) = \bar{x}_0$.

Sometimes our model works so well that if we change the initial condition slightly, the solution will still converge to the exact solution under the original initial conditions. This means that when we use a measured value, our approximate model (with an inexact initial condition) not only stays close to the actual solution, but gets closer and closer to the actual solution as t increases. This notion is called *asymptotic stability*.

Definition 2.2. A solution $x(t)$ is *asymptotically stable* if it is stable and there is a $\rho > 0$ such that if $\|x_0 - \bar{x}_0\| < \rho$ then

$$\lim_{t \to \infty} \|x(t) - \bar{x}(t)\| = 0.$$

In general, asymptotic stability is more desirable, since we know that no matter where we place our initial condition, we will converge to the actual answer if we are close enough to the actual initial condition. Unfortunately, asymptotic stability is not always possible, as we shall see in the fourth section.

We will primarily use these two notions of stability. There are two main methods used to study Lyapunov stability, known as Lyapunov's First and Second (or Direct) Methods.

Lyapunov's First Method

Lyapunov's First Method works by characterizing solutions to differential equations of the type (1) and using those characterizations to learn what we can about stability. Thus our approach will begin with simple linear systems and move on to certain kinds of nonlinear systems which exhibit similar behavior to the linear systems.

We will first look at linear differential equations

$$x'(t) = A(t)x(t) \tag{3}$$

where $A : \Re^n \to \Re^n$ is linear and $x : \Re \to \Re^n$.

When working with linear systems of the form (3) we will use the following theorem, without proof, to characterize solutions. The theorem is presented as a matrix equation where the derivative $C'(t)$ of a matrix function C is simply defined as the derivative of each of the entries of the matrix.

Theorem 3.1. If $\Phi'(t) = A(t)\Phi(t)$ with Φ an invertible $n \times n$ matrix then all solutions of (3) are linear combinations of the columns of Φ and, in particular, we can choose some $\Phi(t)$ such that $\Phi(t_0) = I$ where I is the identity matrix, so that $x(t) = \Phi(t)x_0$ is a solution to (3).

Proof. Omitted.

Definition 3.1. Φ as described in the last theorem is called the *fundamental matrix* of the differential equation.

The fundamental matrix depends only on the differential equation and thus only on the matrix function $A(t)$.

We also will need to define the operator norm for a matrix A:

Definition 3.2. If A is an $n \times n$ real matrix then

$$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$$

Note that $\|Ax\| \leq \|A\| \, \|x\|$ where $\|A\|$ is the operator norm and the other two norms are the usual Euclidean vector norm in $\Re^n$.

We are now ready to prove an important theorem. It turns out that in the linear case the property of solutions being stable is equivalent to the property of solutions being bounded.

Theorem 3.2. All solutions of (3) are stable if and only if they are bounded.

Proof. All solutions are of the form $\Phi(t)x_0$ where Φ is the fundamental matrix such that $\Phi(t_0) = I$. Let $x(t)$ and $\bar{x}(t)$ be solutions satisfying $x(t_0) = x_0$ and $\bar{x}(t_0) = \bar{x}_0$ respectively.

Assume that all solutions to (3) are bounded. Thus we can find an M such that $\|\Phi(t)\| < M$ for all $t > 0$. Given some $\varepsilon > 0$, if $\|x_0 - \bar{x}_0\| < \frac{\varepsilon}{M}$ then

$$\begin{aligned} \|x(t) - \bar{x}(t)\| &= \|\Phi(t)x_0 - \Phi(t)\bar{x}_0\| \\ &\leq \|\Phi(t)\| \, \|x_0 - \bar{x}_0\| \\ &< \varepsilon. \end{aligned}$$

Conversely, assume that all solutions are stable. Thus the particular solution $x(t) \equiv 0$ is stable, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\bar{x}_0\| < \delta$ then

$$\|\bar{x}(t)\| = \|\Phi(t)\bar{x}_0\| < \varepsilon.$$

In particular, let

$$\bar{x}_0^{(i)} = \frac{\delta}{2} e_i$$

where e_i is the ith standard basis vector of $\Re^n$. Then

$$\|\Phi(t)\bar{x}_0^{(i)}\| = \|\phi_i(t)\| \frac{\delta}{2} < \varepsilon$$

where ϕ_i is the ith column of Φ. We can sum these over all i and get:

$$\|\Phi(t)\| \leq \sum_{i=1}^{n} \|\phi_i(t)\| \leq \frac{2n\varepsilon}{\delta} = k$$

so

$$\|x(t)\| = \|\Phi(t)x_0\| \leq k\|x_0\|$$

and $x(t)$ is bounded. □

Note that boundedness is not necessarily true for stable systems in general. For instance, take the differential equation $x'(t) = 1$ whose solutions are all stable but not bounded.

Lyapunov's First Method consists mainly of showing that solutions of linear systems, especially if $A(t)$ is a constant funtion, are bounded and hence stable. It then looks at functions which are nonlinear, but simply small perturbations of linear functions. We shall state without proof several theorems. For a more detailed explanation with proofs, see [2].

Consider the constant linear equation:

$$x'(t) = A\, x(t) \tag{4}$$

where A is a constant matrix. We can easily show the following.

Theorem 3.3. If all eigenvalues of A have negative real parts then every solution to (4) is asymptotically stable.

Proof. Omitted.

We can easily extend this to equations with slight perturbations. So we characterize the following nonlinear equation

$$x'(t) = A\, x(t) + f(t, x(t)) \tag{5}$$

with the following theorem.

Theorem 3.4. If f satisfies:

1. $f(t, x)$ is continuous for $\|x\| < a,\ 0 \leq t < \infty$ and
2. $\lim_{\|x\| \to 0} \frac{\|f(x,t)\|}{\|x\|} = 0$

then the solution $x(t) \equiv 0$ is asymptotically stable.

It turns out that to characterize the general linear case where A is a function of time, we need a slightly stronger notion of stability called uniform stability. The reader is again referred to [2].

Lyapunov's Second Method

Lyapunov's Second or Direct Method is unique in that it does not require a characterization of the solutions to determine stability. This method often allows us to determine whether a differential equation is stable without knowing anything about what the solutions look like, so it is ideal for dealing with nonlinear systems.

The method uses a supplementary function called a Lyapunov function to determine properties of the asymptotic behavior of solutions to a differential equation of the general form (1). We will need the following definitions.

Definition 4.1. A function $V(t, x)$ is *positive [negative] definite* if there exists a real-valued function $\varphi(r)$ such that:

1. $\varphi(r)$ is strictly increasing on $0 \leq r \leq a$ and $\varphi(0) = 0$ and
2. $V(t, x) \geq \varphi(\|x\|)$ $[V(t, x) \leq -\varphi(\|x\|)]$ for all $(t, x) \in \{(t, x) : t_0 \leq t < \infty, \|x\| \leq b < a\}$.

Definition 4.2.

$$V' = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(t, x) + \frac{\partial V}{\partial t}$$

Note that $V'(t, x(t))$ is the time derivative of V.

Definition 4.3. A real function $V(t, x)$ is said to admit an *infinitesimal upper bound* if there exists $h > 0$ and a continuous, real-valued, strictly increasing function ψ with $\psi(0) = 0$ such that

$$|V(t, x)| \leq \psi(\|x\|) \text{ for } \|x\| < h \text{ and } t \geq t_0.$$

Using these definitions we shall construct a function V. If we can find such a V, then we can find stable solutions to the differential equation. Unfortunately, there is no general way to construct V from the differential equation (1). Note that we can define V in terms of t and x (not $x(t)$) and since our definition of V' uses the function f, the existence of V will have implications for the differential equation.

Theorem 4.1. If a continuous real-valued function $V(t,x)$ exists such that:

1. $V(t,x)$ is positive definite and
2. $V'(t,x)$ is non-positive

then $x(t) \equiv 0$ is a stable solution of (1).

Proof. We know that $V(t,x)$ is positive definite, so there exists a strictly increasing function $\varphi(r)$ such that

$$0 < \varphi(\|x\|) \leq V(t,x)$$

for $0 < \|x\| < b$ and $t > t_0$.

Given $\varepsilon > 0$ let

$$m_\varepsilon = \min_{\|x\|=\varepsilon} \varphi(\|x\|) = \varphi(\varepsilon).$$

Notice that $m_\varepsilon > 0$. Since V is continuous and $V(t,0) = 0$ for all t (by positive definiteness), we can choose a $\delta > 0$ such that

$$V(t_0, x_0) < m_\varepsilon$$

if $\|x_0\| < \delta$. Now

$$V'(t, x(t)) \leq 0$$

for $t_1 \geq t_0$ and $\|x_0\| < \delta$ implies

$$V(t_1, x(t_1)) \leq V(t_0, x(t_0)) = V(t_0, x_0) < m_\varepsilon$$

Now, suppose for some $t_1 \geq t_0$ that $\|x(t_1)\| \geq \varepsilon$ whenever $\|x_0\| < \delta$. Then

$$V(t_1, x(t_1)) \geq \varphi(\|x(t_1)\|) \geq \varphi(\varepsilon) = m_\varepsilon$$

which is a contradiction, so for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x(t)\| < \varepsilon$ if $\|x_0\| < \delta$ for $t \geq t_0$, i.e. $x(t) \equiv 0$ is stable. □

We call a function V which satisfies the hypotheses of the theorem a *Lyapunov function*. In physical systems, the Lyapunov function used is often an expression for energy, for instance stability for the differential equation for a spring:

$$\begin{aligned} x'(t) &= y \\ y'(t) &= -kx \end{aligned}$$

can be done with $V(x, y) = \frac{1}{2}y^2 + \frac{1}{2}kx^2$, which is the equation for total energy of the system.

By strengthening the requirements on V we can find conditions for asymptotic stability.

Theorem 4.2. If a continuous function $V(t, x)$ exists satisfying:

1. $V(t, x)$ is positive definite,
2. $V(t, x)$ admits an infinitesimal upper bound, and
3. $V'(t, x)$ is negative definite

then the solution $x(t) \equiv 0$ of (1) is asymptotically stable.

Proof. By the previous theorem, the solution must be stable. Suppose that $x(t)$ is not asymptotically stable, so for every $\varepsilon > 0$ there exist $\delta > 0$ and $\lambda > 0$ such that any nonzero solution $x(t)$ where $x(t_0) = x_0$ satisfies $\lambda \leq \|x(t)\| < \varepsilon$ if $t \geq t_0$ and $\|x_0\| < \delta$.

Since $V'(t, x)$ is negative definite, there is a strictly increasing function $\varphi(r)$ vanishing at the origin such that $V'(t, x) \leq -\varphi(\|x\|)$. We know that $\|x(t)\| \geq \lambda > 0$ for $t \geq t_0$ so there is a $d > 0$ such that $V'(t, x(t)) \leq -d$. This implies that

$$\begin{aligned} V(t, x(t)) &= V(t_0, x_0) + \int_{t_0}^{t} V'(s, x(s))ds \\ &\leq V(t_0, x_0) - (t - t_0)d \end{aligned}$$

which is less less than zero for large t, contradicting the assumption that $V(t, x)$ is positive definite.

Thus no such λ exists and since $V(t, x(t))$ is positive definite and decreasing with respect to t

$$\lim_{t\to\infty} V(t, x(t)) = 0$$

which implies that

$$\lim_{t\to\infty} \|x(t)\| = 0,$$

since $V(t, x(t)) \leq \psi(\|x(t)\|)$ where $\psi(r)$ is strictly increasing and vanishes at the origin (by the second hypothesis). Thus as $V(t, x(t)) \to 0$, $\psi(\|x(t)\|) \to 0$, and $\|x(t)\| \to 0$ (because ψ is strictly increasing and vanishes at 0). Thus $x(t) \equiv 0$ is asymptotically stable. $\square$

We can also use the direct method to determine instability in some cases.

Theorem 4.3. If a continuous real-valued function $V(t, x)$ exists on some set $S = \{(t, x) : t \geq t_1 \textit{ and } \|x\| < a\}$ satisfying:

1. $V(t, x)$ admits an infinitesimal upper bound,

2. $V'(t,x)$ is positive definite on S, and
3. There exists a $T > t_1$ such that if $t_0 \geq T$ and $h > 0$ then there exists $c \in \Re^n$ such that $\|c\| < h$ and $V(t_0, c) > 0$

then the solution $x(t) \equiv 0$ is not stable.

Proof. Suppose $x(t)$ is a solution not identically zero. Then, by uniqueness, $\|x(t)\| \neq 0$ for all $t \geq t_1$. If $x(t)$ is defined at t_0 then for every $t \geq t_0$ for which $x(t)$ is defined

$$V(t, x(t)) - V(t_0, x(t_0)) = \int_{t_0}^{t} V'(s, x(s))ds > 0$$

since $V'(t, x(t))$ is positive definite.

Let $t_0 \geq T$ and $\varepsilon > 0$. There exists a c such that $\|c\| < \min(a, \varepsilon)$ and $V(t_0, c) > 0$ by the third hypothesis. Since V admits an infinitesimal upper bound and is continuous, there is a $\lambda \in (0, a)$ such that $\|x\| < \lambda$ for $t \geq t_1$. Therefore,

$$|V(t,x)| < V(t_0, c).$$

Let $x(t)$ satisfy the initial condition $x(t_0) = c$. Then $V(t, x(t) > V(t_0, x(t_0)) > 0$ (from the positive definiteness argument), so $\|x(t)\| \geq \lambda$. Also by the second hypothesis we know that there is a strictly increasing function $\varphi(r)$ such that $V'(t,x) \geq \varphi(\|x\|)$. Let

$$\mu = \min_{\|x\| \in [\lambda, a]} \varphi(\|x\|) = \varphi(\lambda).$$

then

$$V'(t, x(t)) \geq \varphi(\|x(t)\|) \geq \mu.$$

Using the integral inequality, we get

$$V(t, x(t)) \geq V(t_0, x(t_0)) + (t - t_0)\mu.$$

Thus we can make $V(t, x(t))$ arbitrarily large. Since V(t,x) admits an infinitesimal upper bound, there exists a strictly increasing function $\psi(r)$ such that $|V(t, x(t))| < \psi(\|x\|)$ and $\psi(0) = 0$. So since $V(t, x(t))$ can be arbitrarily large, $\psi(\|x(t)\|)$ can also be arbitrarily large, and thus $\|x(t)\|$ gets arbitrarily large and the solution is not stable. □

Limitations of Lyapunov Stability

The Lyapunov notion of stability has some problems, some of which we shall look into here. Asymptotic stability as we have defined it does not always work well for periodic systems, so often other notions such as orbital stability are used when studying differential equations which have periodic solutions. In addition, certain classes of systems such as Hamiltonian Systems do not react well to a Lyapunov analysis. Finally, there is the question of whether Lyapunov stability actually reflects our qualitative idea of stability.

We will first look at how Lyapunov stability works with periodic solutions to the autonomous system (2). It turns out that our notion of asymptotic stability is incompatible with such solutions, since the points will not get closer together, even thought two solutions may get closer and closer to the same final state (a periodic cycle). To prove this we will first need a Lemma that solutions to autonomous systems are invariant under translation.

Lemma 5.1. If $x(t)$ is a solution to (2) then $x(t+h)$ is also a solution for any real number h.

Proof. Let $\bar{x}(t) = x(t+h)$ and $s = t+h$. Then

$$\begin{aligned} \bar{x}'(t) &= \frac{dx}{ds}\frac{ds}{dt} \\ &= x'(s) \\ &= f(x(s)) \\ &= f(x(t+h)) \\ &= f(\bar{x}(t)) \end{aligned}$$

so $\bar{x}(t) = x(t+h)$ is a solution to (2). □

Now we can use the fact that translations of solutions are also solutions to show that nontrivial periodic solutions cannot be asymptotically stable. By nontrivial periodic we mean that the solutions do not converge to a constant vector, i.e., $x(t) \neq x(t+\varepsilon)$ for some small $\varepsilon > 0$.

Theorem 5.1. If $x(t)$ is a nontrivial periodic solution of an autonomous system (2) then $x(t)$ is not asymptotically stable.

Proof. Suppose $x(t)$ is a nontrivial periodic solution. Thus there exists t_0 such that $f(x(t_0)) \neq 0$ because if not, $x'(t) = 0$ for all t and the solution is constant (trivially periodic). Suppose also that $x(t)$ is asymptotically stable.

The stability assumption says that given $\varepsilon > 0$ there exists $\hat{\delta} > 0$ such that $\|x(t) - \bar{x}(t)\| < \varepsilon$ if $\|x_0 - \bar{x}_0\| < \hat{\delta}$ where $x(t)$ satisfies $x(t_0) = x_0$ and $\bar{x}(t)$ is a solution to the same equation satisfying the initial condition $\bar{x}(t_0) = \bar{x}_0$.

By the asymptotic stability assumption we can find a $\rho > 0$ such that

$$\lim_{t\to\infty} \|x(t) - \bar{x}(t)\| = 0$$

if $\|x_0 - \bar{x}_0\| < \rho$.

Let $\delta = \min(\hat{\delta}, \rho)$. By the continuity of $x(t)$ we can fix some small $\bar{t}$ so that

$$\|x(t_0) - x(t_0 + \bar{t})\| < \delta.$$

By the Lemma, $x(t + \bar{t})$ is a solution to the differential equation, so by our asymptotic stability assumption

$$\lim_{t \to \infty} \|x(t) - x(t + \bar{t})\| = 0.$$

We know that $x'(t_0) \neq 0$ so we can find $r > 0$ so that $\|x(t_0) - x(t_0 + \bar{t})\| > r$. Also $x(t)$ is periodic, so assume it has period τ. Then

$$\|x(t_0 + n\tau) - x(t_0 + \bar{t} + n\tau)\| > r > 0$$

for all n, so the limit cannot go to zero, which contradicts our assumption of asymptotic stability. Thus $x(t)$ cannot be asymptotically stable. □

Another class of problems which for which asymptotic stability does not work is Time-Independent Hamiltonian systems. They are defined as follows.

Definition 5.1. A *time-independent Hamiltontian system (TIHS)* is the 2n dimensional system

$$\begin{aligned} x'(t) &= H_y(x(t), y(t)) \\ y'(t) &= -H_x(x(t), y(t)) \end{aligned}$$

where $x(t)$ and $y(t)$ are both functions from $\Re$ to $\Re^n$ and $H(x, y)$ is a real valued function with continuous partial derivatives.

Hamiltonian Systems are used to model a large class of physical systems and have many applications. It turns out that, like nontrivial periodic solutions to autonomous systems, no solutions to a TIHS can be asymptotically stable.

Theorem 5.2. If $(x(t), y(t))$ is a solution to a TIHS then it is not asymptotically stable.

Proof. Suppose $(x(t), y(t))$ is an asymptotically stable solution. Then we can find t_0 and $\delta > 0$ such that if $(\bar{x}(t), \bar{y}(t))$ is a solution and if $\|(x(t_0), y(t_0)) - (\bar{x}(t_0), \bar{y}(t_0))\| < \delta$ then

$$\lim_{t \to \infty} \|(x(t), y(t)) - (\bar{x}(t), \bar{y}(t))\| = 0.$$

But also,

$$\frac{d}{dt} H(x, y) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial y} \frac{dy}{dt} = H_x(H_y) + H_y(-H_x) = 0$$

so $H(x, y)$ is constant on any solution $(x(t), y(t))$. By the continuity of H, it must be constant on some δ-neighborhood N of $(x(t_0), y(t_0))$, so every point in N is an equilibrium point. Thus $(x(t), y(t))$ cannot be asymptotically stable (for initial condition $(\bar{x}(t_0), \bar{y}(t_0)) \in N$, the solution $(\bar{x}(t), \bar{y}(t))$ will not converge to $(x(t), y(t))$. $\square$

It turns out that although Lyapunov stability is extremely useful in mathematically defining our qualitative view of stability, it is neither necessary nor sufficient for something to appear stable on a physical level. Some examples can be found at the end of Chapter 4 of [1].