E1 222 Stochastic Models and Applications
Problem Sheet 2–4

1. We have a coin with probability p of coming up heads, $0 < p < 1$. Now consider the following procedure that determines value of a random variable, X.
   1. Flip the coin and let the result (heads or tails) be denoted by O_1 .
   2. Flip the coin again and let the result be O_2.
   3. If $O_1 = O_2$ go to step 1; else go to 4.
   4. If O_2 is heads set $X = 0$; otherwise set $X = 1$.

   Find the mass function of X.

Hint: The procedure amounts to the following. You are repeating the random experiment of tossing a coin twice, till either HT or TH occurs. The event of $[X = 0]$ is same as the event of HT occurring before TH. Now you can use problem-4 in assignment-1

As you would figure out, this is a nice way to simulate a fair coin using a biased coin.

2. For a continuous random variable, X, the real number a that satisfies $\int_{-\infty}^{a} f_X(x)\,dx = 0.5$ is called the median of X. Show that for a continuous random variable, X, the number x_0 that minimizes $E|X - x_0|$ is the median of X.

Hint: Split the integral of $E|X - x_0|$ into two parts one for $x \leq x_0$ and the other for $x > x_0$ and thus get rid of absolute value inside the integral. Now you need to find the value of x_0 for which this expression is minimized. You can differentiate it with respect to x_0. But differentiating after some algebra may be easier. You may need the Liebnitz formula for differentiating an integral:

$$\frac{d}{dx}\int_{h(x)}^{g(x)} f(x,t)\,dt = f(x,g(x))\frac{d}{dx}g(x) - f(x,h(x))\frac{d}{dx}h(x) + \int_{h(x)}^{g(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

3. Let X be a continuous random variable with $E|X|^k < \infty$ for some $k > 0$. Then show that $n^k P[|X| > n] \to 0$ as $n \to \infty$.

Hint: Write the expectation integral of $|X|^k$ as two parts – one for $|x| \leq n$ and the other for $|x| > n$. Since the integral is given to be finite, argue that the second part goes to zero as $n \to \infty$. (This is because, the limit of the first integral as $n \to \infty$ is the value of the expectation). Then try and bound the second integral in terms of $P[|X| > n]$. For this, first ask what happens to the second integral if inside the integral you replace $|x|^k$ with n^k.

4. Let X be a non-negative continuous random variable and suppose EX exists. Show that

$$EX = \int_0^\infty (1 - F(x))\, dx$$

Hint: Integrate $\int_0^n (1 - F(x))\, dx$ by parts and take limit as $n \to n$ and use the previous problem.

5. Consider the following density function (called Beta density)

$$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1},\ 0 \leq x \leq 1.$$

where $\Gamma(\cdot)$ is the gamma function and $a, b \geq 1$ are parameters. Show that this is a density as follows. By definition of gamma function, we have

$$\Gamma(a)\Gamma(b) = \int_0^\infty x^{a-1} e^{-x}\, dx \int_0^\infty y^{b-1} e^{-y}\, dy$$

First bring the integral over y inside the integral over x. Now in the inner integral change the variable from y to t using $t = y + x$. Now change the order of the x and t integrals so that the x integral becomes the inner integral. Now, in the inner integral change the variable from x to s using $x = ts$. The final expression you get can then be used to show that the above $f(x)$ is a density.

6. If X has beta density, find EX and $\mathrm{Var}(X)$.

Hint: Even if you cannot solve the previous problem you can solve this one!

All you need to know here is that beta density given above is a density for all $a, b \geq 1$. That is, use the fact that

$$\int_0^1 x^{a-1}(1-x)^{b-1}\, dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

Also, remember the identity $\Gamma(a+1) = a\Gamma(a)$.

7. A coin having probability p of coming up heads is successively tossed till the r^{th} head appears. (p and r are parameters). Let X denote the number of tosses needed. Find the mass function of X. (Hint: To calculate $P[X = n]$, think of how many heads are allowed in the first $n-1$ tosses).

8. Consider a random variable X with the mass function

$$f(x) = {}^{(\alpha+x-1)}C_x \, p^\alpha (1-p)^x, \quad x = 0, 1, \cdots$$

where $\alpha > 0$. Is this realted to the X in the previous problem? This is known as the negative binomial distribution. The motivation for the name can be seen as follows. For any positive real number α and a nonnegative integer x we have

$$\begin{aligned} {}^{-\alpha}C_x &= \frac{-\alpha(-\alpha-1)(-\alpha-x+1)}{x!} \\ &= \frac{(-1)^x(\alpha)(\alpha+1)(\alpha+x-1)}{x!} \\ &= {}^{(\alpha+x-1)}C_x \, (-1)^x \end{aligned}$$

Thus ${}^{(\alpha+x-1)}C_x \, p^\alpha(1-p)^x = {}^{-\alpha}C_x p^\alpha (-1)^x (1-p)^x$. Thus our distribution can be seen to involve binomial coefficients for negative index and hence the name. What will this distribution be for $\alpha = 1$?

9. The binomial distribution can be approximated by the Poisson distribution for large n. Consider a binomial distribution with parameters n and p. Since, the expectation is np, if we want an approximation as n tends to infinity we need to ensure that the expectation is finite. So, let us write p_n as the probability of success when we consider n trials and let us assume that as $n \to \infty$, $np_n \to \lambda$. Noting that, as $n \to \infty$, we have (i). $(1 - \frac{\lambda}{n})^n \to e^{-\lambda}$, (ii). $(1 - \frac{\lambda}{n})^{-k} \to 1$, (iii). $(n(n-1)\cdots(n-k+1))/(n^k) \to 1$, show that

$$\lim_{n\to\infty} {}^nC_k (p_n)^k (1-p_n)^{n-k} = \frac{\lambda^k}{k!} e^{-\lambda}$$