E1 222 Stochastic Models and Applications
Problem Sheet 3–2

1. Let (X, Y) have joint density

$$f_{XY}(x, y) = \frac{1}{4}[1 + xy(x^2 - y^2)], \ |x| \leq 1, \ |y| \leq 1.$$

   Find the marginal and conditional densities.

2. Let $f(x, y) = e^{-x-y}, \ x > 0, y > 0$. Show that this a density function. Find the marginals and the conditional densities.

3. Let X be uniform from 0 to 1, let Y be uniform from 0 to X and let Z be uniform from 0 to Y. What is the joint density of X, Y, Z? Find the marginal densities, f_X, f_Y, f_Z and the joint density of Y, Z.

4. Let X, Y be iid random variables which are uniform over $(0, 1)$. Find $P[X > Y]$.

5. Let A, B be two events. Let I_A and I_B denote the indicator random variables of these events. Show that I_A and I_B are independent iff A and B are independent.

6. Consider a communication system. Let Y denote the bit sent by transmiter. (Y is a binary random variable). The receiver makes a measurement, X, and based on its value decides what is sent. The decision at the receiver can be represented by a function $h : \Re \to \{0, \ 1\}$. For any specific h, let $R_0(h)$ represent the set of all $x \in \Re$ for which $h(x) = 0$ and let $R_1(h)$ represent the set of $x \in \Re$ for which $h(x) = 1$. An error occurs if a wrong decision is made. Argue that the event of error occurring is: $[h(X) = 0, Y = 1] \cup [h(X) = 1, Y = 0]$. Show that probability of error for a decision rule h is

$$\int_{R_0(h)} p_1 f_{X|Y}(x|1) dx \ + \ \int_{R_1(h)} p_0 f_{X|Y}(x|0) dx$$

   where $p_i = P[Y = i]$. Now consider a h given by

$$h(x) = 1 \ \text{ if } \ f_{Y|X}(1|x) \geq f_{Y|X}(0|x)$$

   (Otherwise $h(x) = 0$). Show that this h would achieve minimum probability of error.

7. Let X and Y be independent random variables each having an exponential distribution with the same value of parameter λ. Show that $Z = \min(X, Y)$ is exponential with parameter 2λ.

8. Let X and Y be independent Gaussian random variables with $EX = \mu_1$, $EY = \mu_2$, $\mathrm{Var}(X) = \sigma_1^2$, and $\mathrm{Var}(Y) = \sigma_2^2$. Show that $X + Y$ has gaussian density with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.