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- In general, almost sure convergence does not imply convergence in r^{th} mean and vice versa

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- Given rv X, its characteristic function, ϕ_X, is defined by

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  - If $E|X|^r < \infty$, ϕ would be differentiable r times and

$$\phi^{(r)}(u) = E[(iX)^r e^{iuX}]$$

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  - If $F_n \to F$ then $\phi_{F_n} \to \phi_F$
  - If $\phi_{F_n} \to \psi$ and ψ is continuous at zero, then ψ would be characteristic function of some df, say, F, and $F_n \to F$

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$$\phi_{S_n}(t) = (\phi(t))^n \quad \text{and} \quad \phi_{\tilde{S}_n}(t) = \left(\phi\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$

- Recall that we can expand ϕ in a Taylor series

$$\phi(u) = \sum_{s=0}^{r-1} \mu_s \frac{(iu)^s}{s!} + \rho(u)\ \mu_r \frac{(iu)^r}{r!}, \quad \rho(u) \to 1, \text{ as } u \to 0$$

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$$\begin{aligned} \phi\left(\frac{t}{\sigma\sqrt{n}}\right) &= 1 - \frac{1}{2} \, \rho\left(\frac{t}{\sigma\sqrt{n}}\right) \; \sigma^2 \; \frac{t^2}{\sigma^2 n} \\ &= 1 - \frac{1}{2} \, \frac{t^2}{n} \; + \; \frac{1}{2} \, \frac{t^2}{n} \; \left(1 - \rho\left(\frac{t}{\sigma\sqrt{n}}\right)\right) \end{aligned}$$

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\phi\left(\frac{t}{\sigma\sqrt{n}}\right) &= 1 - \frac{1}{2}\, \rho\left(\frac{t}{\sigma\sqrt{n}}\right) \sigma^2 \, \frac{t^2}{\sigma^2 n} \\
&= 1 - \frac{1}{2}\, \frac{t^2}{n} + \frac{1}{2}\, \frac{t^2}{n} \left(1 - \rho\left(\frac{t}{\sigma\sqrt{n}}\right)\right) \\
&= 1 - \frac{1}{2}\, \frac{t^2}{n} + o\left(\frac{1}{n}\right)
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- By Continuity theorem, distribution function of $\tilde{S}_n$ converges to that of standard Normal rv

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- Thus, S_n is well approximated by a normal rv with mean $n\mu$ and variance $n\sigma^2$, if n is large

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- Then Z represents the error in the sum.

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&\approx \Phi\left(\frac{3}{\sqrt{\frac{5}{3}}}\right) - \Phi\left(\frac{-3}{\sqrt{\frac{5}{3}}}\right)
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- Hence probability that the sum differs from true sum by more than 3 is 0.02

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- The approximation is quite good in practice

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- To get a good approximation, to calculate $P[S_n \leq m]$ one uses $P[S_n \leq m + 0.5]$ in the above approximation formula

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(because $\Phi(-x) = (1 - \Phi(x))$)

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- Here, the 0.34 would be the sample mean. The other two numbers can be fixed using CLT

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- There are other ways of handling it

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- If we change ϵ to 0.05, then at variance equal to 0.5 the probability becomes about 0.02 while the Chebyshev bound would be about 0.2

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- In analyzing any experimental data the confidence intervals or the variance term is important

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- How doe we specify or characterize an infinite collection of random variables?
- We need the joint distribution of every finite subcollection of them.

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- Number of packets coming into a network switch, number people joining the queue in a bank, number of infections till date are all Markov chains.
- This is a useful model for many dynamic systems or processes

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- It can be more general and we discuss some of them

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- This shows that the transition probabilities, P, and initial state probabilities, π_0, completely specify the chain.
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- The joint distribution of $X_{i_1}, \cdots X_{i_k}$ is now calculated as a marginal distribution from the joint distribution of $X_0, \cdots, X_m$