E1 222 Stochastic Models and Applications
Problem Sheet 3.5

1. Let X, Y have joint density

$$f_{XY}(x, y) = \frac{\sqrt{3}}{2\pi} e^{-0.5(x^2+4y^2-2xy)}, \quad -\infty < x, y < \infty$$

Find the marginal densities of X, Y and the conditional density $f_{X|Y}$.

Hints: You can easily integrate this w.r.t. to x or y (using the trick of completing squares) and hence find f_X, f_Y and then the conditional density. But there is a much simpler way of doing this.

From the form of the density (exponential of a quadratic) it is clear that X, Y are jointly Gaussian. By writing the expression in the exponential as a quadratic form of a symmetric matrix, you know what is Σ^{-1}. By inverting this 2×2 matrix you get Σ and comparing with the standard form, can verify that this is a 2-D Gaussian density. Then you know means of X, Y are zero. Since X, Y are jointly Gaussian, you know marginals of X and Y are Gaussian. Since you know Σ matrix you know that the variances are 4/3 and 1/3. Thus, you can directly write the marginals of X and Y (and hence the conditinal density) without doing any integration.

2. Let X, Y be continuous random variables with the following joint density

$$\begin{aligned} f_{XY}(x, y) &= \frac{1}{2}\left\{\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho)^2}(x^2 - 2\rho xy + y^2)\right] + \right. \\ &\left. \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho)^2}(x^2 + 2\rho xy + y^2)\right]\right\} \end{aligned}$$

Find f_X, f_Y and EXY. Are X, Y uncorrelated? Are X, Y jointly normal?

Hint: You can find the marginal densities and EXY by integration. You would find that both X, Y are Gaussian with mean-zero and variance 1. You would get $E[XY] = 0$ and thus conclude that X, Y are uncorrelated. The next question is whether X, Y are jointly Gaussian. The form of joint density is not exactly the 2-D Gaussian though it very close to that form. How do we decide whether or not X, Y are jointly Gaussian. Here, we know X, Y are individually Gaussian and they are uncorrelated. However, they are not independent because the product of the marginals is not equal to the joint density. Hence, they are not jointly Gaussian.

This is an example where X, Y are individually Gaussian and uncorrelated but are not jointly Gaussian.

If X, Y are jointly Gaussian it is called a 2D Gaussian vector. As you know a Gaussian vector has a special property. The defining property of a Gaussian vector is $\mathbf{X}^T\mathbf{a}$ is Gaussian for every non-zero $\mathbf{a}$. This example shows that simply taking a vector where individual rv are Gaussian would not give you a Gaussian vector.

There is interesting structure in the example. Each term in the f_{XY} here is a 2-D Gaussian density. That is why we know that if we integrate it, say, w.r.t. x, we will get a Gaussian density in y from each term and if we add them and divide by 2 we will get a Gaussian density. The first term is a joint Gaussian density with covariance ρ and the second term is a joint Gaussian with covariance $-\rho$. Hence EXY integral from first term would give us ρ and the second term would give us $-\rho$ and that is why we would get $EXY = 0$. So, we could have guessed the marginals and EXY without doing any integration.

3. Let X, Y be jointly normal with means μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation coefficient ρ. Find a necessary and sufficient condition for $X + Y$ and $X - Y$ to be independent.

Hint: Since X, Y are jointly Gaussian, from the result we proved we know that $X + Y$ and $X - Y$ are jointly Gaussian. Hence they would be independent iff they are uncorrelated. So, the condition needed is $E[(X+Y)(X-Y)] = E[X+Y]E[X-Y]$ which is same as the condition that X and Y have same variance.

4. Let X, Y be jointly normal with $EX = EY = 0$, $\text{Var}(X) = \text{Var}(Y) = 1$ and correlation coefficient ρ. Show that $Z = X/Y$ has Cauchy distribution. A Cauchy distribution with parameters μ and θ is given by

$$f(x) = \frac{\mu}{\pi} \frac{1}{(x-\theta)^2 + \mu^2}$$

Hint: This can be solved by simply applying the formula for density of X/Y.

5. Let X_1, X_2, X_3, X_4 be iid Gaussian random variables with mean zero and variance one. Show that the density function of $Y = X_1X_2 + X_3X_4$ is $f(y) = 0.5e^{-|y|}$, $-\infty < y < \infty$. (Hint: Try finding mgf of Y).

Hint: Finding the density of Y from first principles or using the theorem is involved because we have 4 rv here. But we are asked to show that Y has the given density. Hence we can do it using mgf. Finding mgf corresponding to density f is a simple integration. We can also find mgf of Y easily.

$E[e^{tY}] = E[e^{tX_1X_2}\ e^{tX_3X_4}] = E[e^{tX_1X_2}]\ E[e^{tX_3X_4}]$ by independence. Now

$$E[e^{tX_1X_2}] = \int\int \frac{1}{2\pi}\ e^{-\frac{x^2}{2}}\ e^{-\frac{y^2}{2}}\ e^{txy}\ dx\ dy$$

which can be calculated easily using the trick of completing squares.