E2 212: Homework - 4

1 Topics

- Eigenvalues and Eigenvectors

2 Problems

Notation: M_n denotes an $n \times n$ matrix over a field of complex (or real) numbers, i.e. $\mathbb{C}^{n\times n}$ (or $\mathbb{R}^{n\times n}$). Note that "triple-bar" norms $|||\cdot|||$ denote vector induced matrix norms while "double-bar" norms $\|\cdot\|$ denote vector norms (possibly, on matrices).

1. Show that for any square matrix $\mathbf{A}$, the set $E_\lambda \triangleq \{\mathbf{v} \in \mathbb{R}^n : \mathbf{A}\mathbf{v} = \lambda\mathbf{v}\}$ for any $\lambda \in \mathbb{R}$, is a subspace.

2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$ matrices. Show that $\mathbf{AB}$ and $\mathbf{BA}$ have exactly the same eigenvalues.

3. Give a closed-form solution for $\mathbf{x}$ in the system of equations $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is full rank, square and symmetric, in terms of the eigendecomposition of $\mathbf{A}$.

4. The so-called power method for calculating the largest eigenvalue/vector pair of a square symmetric matrix $\mathbf{A}$ is described as follows:
   Initialize: set $\mathbf{x}_0$ to an arbitrary value. For $i = 1, 2, 3, \ldots$

   $$\mathbf{z} = \mathbf{A}\mathbf{x}_{i-1}$$
   $$\mathbf{x}_i = \frac{\mathbf{z}}{\|\mathbf{z}\|_2}$$

   As $i \to \infty$, $\mathbf{x}_i$ converges to the maximum eigenvector.

   (a) Prove that the above algorithm converges. *Hint:* express $\mathbf{x}_0$ using the eigenvectors of $\mathbf{A}$ as a basis.

   (b) Modify the algorithm to find the smallest eigenvalue/vector.

   (c) Explain what happens when the largest eigenvalue is not distinct.

5. Derive an analytical expression for the eigendecomposition of a square symmetric rank-one matrix.

6. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$. Then, prove the following:

   (a) $\operatorname{tr} \mathbf{A} = \sum_{i=1}^n \lambda_i$.

   (b) $\det(\mathbf{A}) = \prod_{i=1}^n \lambda_i$.

7. Let the eigenvectors and eigenvalues of the matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ be $\mathbf{v}_i$ and λ_i such that $\lambda_i \neq \lambda_j$ for all $j \neq i$, $i = 1, 2, \ldots, r$. Then, prove that $\mathbf{v}_1, \ldots, \mathbf{v}_r$ is a linearly independent set.

8. Let the subspace $U \subseteq \mathbb{C}^n$ be an invariant subspace under the matrix transformation $\mathbf{A} \in \mathbb{C}^{n\times n}$, i.e., $\mathbf{Au} \subseteq U$ for all $\mathbf{u} \in U$. Then, prove the following

   (a) There exists a vector $\mathbf{u} \in U$ and $\lambda \in \mathbb{C}$ such that $\mathbf{Au} = \lambda\mathbf{u}$.

(b) If $\mathbf{u}_1, \ldots, \mathbf{u}_k \in U$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\dim(U) \geqslant k$.

(c) The subspace $U^{\perp} \triangleq \{\mathbf{v} \in \mathbb{C}^n : \langle \mathbf{u}, \mathbf{v} \rangle = 0 \text{ for all } \mathbf{u} \in U\}$ is also an invariant subspace with respect to $\mathbf{A}$.

9. Let $\sigma(\mathbf{A})$ denote the spectrum of the matrix $\mathbf{A}$. Show that for a triangular matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ with entries a_{ij}, $\sigma(\mathbf{A}) = \bigcup_{i=1}^{n} \{a_{ii}\}$.

10. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues in $\mathbb{C}$. Then, prove that there exists an invertible matrix $U \in \mathbb{C}^{n \times n}$ and a diagonal matrix D such that

$$\exp\{\mathbf{A}\} = UDU^{-1}.$$

11. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, $\lambda \in \mathbb{C}$. Prove that if $(\mathbf{A} - \lambda I_n)^j \mathbf{u} = 0$ for some $j \geqslant 1$, and $\mathbf{u} \in \mathbb{C}^n$, then $(\mathbf{A} - \lambda I_n)^n \mathbf{u} = 0$.

12. Let $\lambda_1, \ldots, \lambda_n$ (not necessarily distinct) be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n \times n}$. Then,

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqslant \sum_{i,j=1}^{n} |a_{ij}|^2 .$$

13. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$, and $\beta \triangleq \max\{|a_{ij}| : i, j = 1, 2, \ldots, n\}$. Then,

$$|\det \mathbf{A}| \leqslant \beta^n n^{n/2}.$$