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  For now, we take $\mathcal{F} = 2^\Omega$ (power set of Ω)

Probability axioms

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- A1 $P(A) \geq 0, \forall A \in \mathcal{F}$
- A2 $P(\Omega) = 1$
- A3 If $A_i \cap A_j = \phi, \forall i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

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- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

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- Thus, 'equally likely' is one way of specifying the probability function (in case of finite Ω).
- An obvious point worth remembering: specifying P for singleton events fixes it for all other events.

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- This can be done for finite Ω too.

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- Easy to see we have $q_i \geq 0$ and $\sum_{i=0}^{\infty} q_i = 1$.

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  (There are many issues that need more attention here).

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- For the pieces to make a triangle, sum of lengths of any two should be more than the third.

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- So the event of interest is:
  $A = \{(x, y) \;:\; y > 0.5;\; x < 0.5;\; y < x + 0.5\}$

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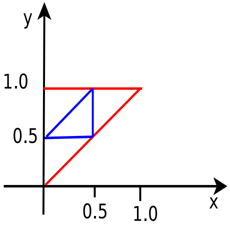

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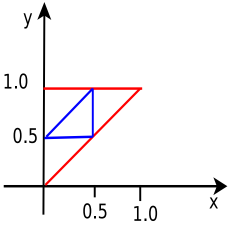

- The required probability is area of A divided by area of Ω which gives the answer as 0.25

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- If A_1, A_2 are mutually exclusive then A_1B and A_2B are also mutually exclusive and hence

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- Once we understand condional probability is a new probability assignment, we go back to the 'standard notation'

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- It is not as if we talk about conditional probability only for subsets of B. Conditional probability is also with respect to the original probability space. Every element of $\mathcal{F}$ has conditional probability defined.

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- There are probabilistic methods to capture causation (but far beyond the scope of this course!)

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- This is a very useful in many situations. ("arguing by cases")

Example: Polya's Urn

An urn contains r red balls and b black balls. We draw a ball at random, note its color, and put back that ball along with c balls of the same color. We keep repeating this process. Let R_n (B_n) denote the event of drawing a red (black) ball at the n^{th} draw. We want to calculate the probabilities of all these events.

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- These different cases are important in understanding the role of false positives rate.

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  - The knowledge we need is $P(T_+|D)$, $P(T_+|D^c)$ which can be determined through experiment or modelling of channel.

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- Independence is an important (often confusing!) concept.

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Similarly we can show for others.

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- We always have an underlying probability space $(\Omega, \mathcal{F}, P)$
- Once that is given, the probabilities of all events are fixed.
- Hence whether or not two events are independent is a matter of 'calculation'

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- For example, in the previous problem, once we saw that F and C are independent, we can conclude M and C are also independent (because in this example we are taking $F^c = M$).

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- Hence, in multiple tosses, assuming all outcomes are equally likely implies outcome of one toss is independent of another.

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  (We will look at it more formally when we consider multiple random variables).

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- This is often used, at an intuitive level, to justify assumption of independence.

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- But, $P(E_1E_2E_3) = 0.25 \neq (0.5)^3$