E1 222 Stochastic Models and Applications
Problem Sheet 2–3

1. Let X be a random variable uniformly distributed over $\{0, 1, \cdots, N\}$. Find $E[X]$.

Hint :
$$EX = \sum_i x_i f_X(x_i) = \sum_{i=0}^{N} i \frac{1}{N+1} = \frac{1}{N+1} \frac{N(N+1)}{2} = \frac{N}{2}$$

2. A darts board consists of concentric circles with radius $\frac{k}{n}$, $k = 1, 2, \cdots, n$. Thus there are n annular regions. A dart is thrown randomly. If it hits the k^{th} annular region we get $1/(2k-1)$ rupees. What is the expected amount one gets if a dart is thrown randomly.

Hint : We can take the underlying probability space to be unit circle with uniform distribution. The area of k^{th} annular region, $k = 1, \cdots, n$, is
$$\pi \left(\frac{k}{n}\right)^2 - \pi \left(\frac{k-1}{n}\right)^2 = \pi \frac{2k-1}{n^2}$$
Area of the circle is π and hence, probability of hitting k^{th} annular region is $\frac{2k-1}{n^2}$. Hence, expected amount we get is
$$\sum_{k=1}^{n} \frac{1}{(2k-1)} \frac{2k-1}{n^2} = \frac{1}{n^2}\, n = \frac{1}{n}$$

3. Let X an exponential rv. Find EX^3.

Hint :
$$EX^3 = \int_0^\infty x^3 \, \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda} \int_0^\infty 3x^2 \, \lambda e^{-\lambda x} \, dx = \frac{6}{\lambda^3}$$

4. Let X be a rv with density function
$$f(x) = cx(1-x), \quad \text{if} \quad 0 \le x \le 1.$$
($f(x)$ is zero for all other values of x). Let $Y = 2X^3 - 3X^2 + 3X + 5$. Find the value of c and $E[Y]$.

Hint : $\int_o^1 cx(1-x)\,dx = 1 \;\;\Rightarrow\;\; c = 6$

$$EY = \int_0^1 (2X^3 - 3X^2 + 3X + 5)\; 6(x - x^2)\; dx = 6$$

5. The price of some commodity is Rs. 2 per gram this week. Next week the price would be either Rs.1 per gm or Rs. 4 per gram, each with probability 0.5. You have a capital of Rs.1000. What would be your strategy if (i) you want to maximize expected amount of money with you (next week), (ii) you want to maximize the expected quantity of the commodity with you.

Hint : We have to first decide what we mean by 'strategy'. Given what is stated in the problem, it is reasonable to consider only the following class of strategies. This week we will buy x grams of the commodity and keep the remaining money. Next week, if we want commodity we will buy whatever we get with the money we have; if we want money, we will sell the commodity we have at the prevailing price. So, a strategy is just fixing a value for x. Given that it is selling ar Rs.2 per gram this week and that we have Rs. 1000, we know $0 \leq x \leq 500$. If we buy x grams, we currently have x grams of commidity and $1000 - 2x$ rupees. Expected amount of commodity next week would be $[x + (0.5(1000 - 2x) + 0.5((1000 - 2x)/4)]$. Similarly the expected amount money would be $[1000 - 2x + (0.5(x) + 0.5(4x))]$. Now you can complete the calculation and show that if we want to maximize the commodity then we should buy all our commodity at next week's prices. On the other hand, if we want to maximize our money, we should convert all our money into commodity this week and sell it next week.

6. Children from a school went to a picnic in four buses. Different buses carried different number of students. Define two random variables, X, Y, as follows. We select one of the four drivers at random and X is the number of students in the bus driven by that driver. We select a student at random and Y is the number of students in the bus in which the selected student travelled. Can you say whether $EX > EY$ or $EY > EX$ (or the information given is not sufficient to decide which of EX, EY is greater)?

Hint : Let n_1, n_2, n_3, n_4 be the numbers of children in the four buses. Both X, Y are discrete rv's with $X, Y \in \{n_1, n_2, n_3, n_4\}$. X takes the four values with equal probability. The probability of Y taking value n_1 is $(n_1 / \sum_i n_i)$. Thus Y takes higher values with higher probability. Hence EY would be higher. You can actually write the expressions for EX and EY and show this algebraically.