E2 212: Homework - 7

1 Topics

- **Variational characterization of eigenvalues**

2 Problems

1. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix, let $\mathbf{x} \in \mathbb{C}^n$ be a given nonzero vector, and let $\alpha \triangleq \mathbf{x}^H\mathbf{A}\mathbf{x}/\mathbf{x}^H\mathbf{x}$. Show that there exists at least one eigenvalue of $\mathbf{A}$ in the interval $(-\infty, \alpha]$ and at least one in $[\alpha, \infty)$.

2. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Show that the following three optimization problems all have the same solution:

   (a) $\max_{\mathbf{x}^H\mathbf{x}=1} \mathbf{x}^H\mathbf{A}\mathbf{x}$

   (b) $\max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}$

   (c) $\min_{\mathbf{x}^H\mathbf{A}\mathbf{x}=1} \mathbf{x}^H\mathbf{x}$

3. Show that, if λ_i is any eigenvalue of $\mathbf{A} \in \mathbb{C}^{n\times n}$ (not necessarily Hermitian), then, one has the bounds

$$\min_{\mathbf{x}\neq 0}\left|\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right| \leq |\lambda_i| \leq \max_{\mathbf{x}\neq 0}\left|\frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}\right|, i = 1, 2, \ldots, n. \tag{1}$$

4. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian and if $\mathbf{x}^H\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x}$ in a k-dimensional subspace, then prove that $\mathbf{A}$ has at least k nonnegative eigenvalues.

5. Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ be two Hermitian matrices. Prove that

$$|\lambda_k(\mathbf{A}+\mathbf{B}) - \lambda_k(\mathbf{A})| \leq \rho(\mathbf{B})$$

   for all $k = 1, 2, \ldots, n$, where $\rho(\mathbf{B})$ is the spectral radius of $\mathbf{B}$, and where the eigenvalues of $\mathbf{A}$ and $\mathbf{A}+\mathbf{B}$ are arranged, as usual, in increasing order.

6. If $\mathbf{A} \in \mathbb{C}^{n\times n}$ and $\mathbf{B} \in \mathbb{C}^{n\times n}$ are Hermitian matrices with eigenvalues arranged in increasing order, and if $1 \leq k \leq n$, show that

$$\lambda_k(\mathbf{A}+\mathbf{B}) \leq \min\{\lambda_i(\mathbf{A}) + \lambda_j(\mathbf{B}) : i + j = k + n\}.$$

7. Explain why the Courant-Fischer theorem discussed in class is equivalent to

$$\lambda_k = \min_{\{\mathcal{S}:\, \dim \mathcal{S}=k\}} \max_{\{\mathbf{x}:0\neq\mathbf{x}\in\mathcal{S}\}} \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}$$

   and

$$\lambda_k = \max_{\{\mathcal{S}:\, \dim \mathcal{S}=n-k+1\}} \min_{\{\mathbf{x}:0\neq\mathbf{x}\in\mathcal{S}\}} \frac{\mathbf{x}^H\mathbf{A}\mathbf{x}}{\mathbf{x}^H\mathbf{x}}$$

   where $\mathbf{A} \in \mathbb{C}^{n\times n}$ is a Hermitian matrix with eigenvalues $\lambda_1 \leq \ldots \leq \lambda_n$, $k \in \{1, \ldots, n\}$ and $\mathcal{S}$ denotes a subspace of $\mathbb{C}^n$.

8. Let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. What are the eigenvalues of $\mathbf{A}$? What is $\max\{\mathbf{x}^T\mathbf{A}\mathbf{x}/\mathbf{x}^T\mathbf{x} : 0 \neq \mathbf{x} \in \mathbb{R}^2\}$? Does this contradict the Rayleigh-Ritz theorem?

9. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ be Hermitian. Use Weyl's theorem to show that $\lambda_1(\mathbf{B}) \leq \lambda_i(\mathbf{A}+\mathbf{B}) - \lambda_i(\mathbf{A}) \leq \lambda_n(\mathbf{B})$. Hence, conclude that $|\lambda_i(\mathbf{A}+\mathbf{B}) - \lambda_i(\mathbf{A})| \leq \rho(\mathbf{B})$ for all $i = 1, \ldots, n$. This is an example of a *perturbation theorem* for the eigenvalues of a Hermitian matrix.

10. Let $\lambda, a \in \mathbb{R}$, $\mathbf{y} \in \mathbb{C}^n$, and $\mathbf{A} = \begin{bmatrix} \lambda\mathbf{I}_n & \mathbf{y} \\ \mathbf{y}^H & a \end{bmatrix} \in \mathbb{C}^{(n+1)\times(n+1)}$. Use the Cauchy interlacing theorem to show that λ is an eigenvalue of $\mathbf{A}$ with multiplicity at least $n-1$. What are the other two eigenvalues?