Recap: Monotone Sequences of Sets

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Recap: Monotone Sequential Continuity

- We showed that

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

when $A_n \downarrow$ or $A_n \uparrow$

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- It essentially means we can treat all outcomes as real numbers.
- We can effectively work with $\Re$ as sample space in all probability models

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- P_X is a new probability measure (which depends on P and X) that assigns probability to different subsets of $\Re$.

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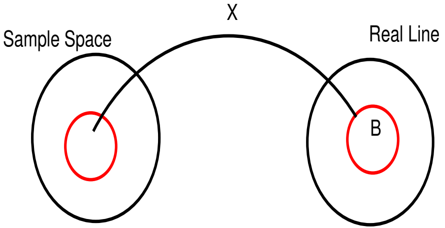

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- We can easily verify P_X is a probability measure. It satisfies the axioms.

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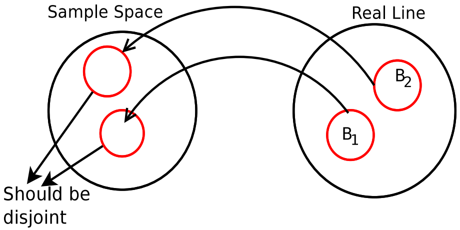

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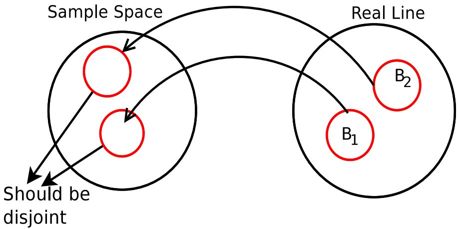

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- Hence

$$P_X((0, 1]) = \frac{3}{8};\ P_X((-1.2, 2.78)) = \frac{7}{8}$$

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- We briefly consider this and then move on to studying random variables.

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- If we could take $\mathcal{B} = 2^{\Re}$ then everything would be simple. But that is not feasible.
- What this means is that if we want every subset of real line to be an event, we cannot construct a probability measure (to satisfy the axioms).

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- We also saw that this sample space comes up when we consider infinite tosses of a coin.
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  (Recall that for countable Ω we can take $\mathcal{F} = 2^{\Omega}$).
- Now the question is what is the best $\mathcal{B}$ we can have?

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- It is defined as the intersection of all σ-algebras containing G (and hence is well defined).

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- It contains all intervals, all complements, countable unions and intersections of intervals and all sets that can be obtained through complements, countable unions and/or intersections of such sets and so on.

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- Thus, $\sigma(G)$ is also the smallest σ-algebra containing all intervals.

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- Are there any subsets of real line that are not Borel?

Borel σ-algebra

- We have defined $\mathcal{B}$ as

$$\mathcal{B} = \sigma\left(\{(-\infty,\ x]\ :\ x \in \Re\}\right)$$

- It is also the smallest σ-algebra containing all intervals.
- Elements of $\mathcal{B}$ are called Borel sets
- Intervals (including singleton sets), complements of intervals, countable unions and intersections of intervals, countable unions and intersections of such sets on so on are all Borel sets.
- Borel σ-algebra contains enough sets for our purposes.
- Are there any subsets of real line that are not Borel?
- YES!! Infinitely many non-Borel sets would be there!

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- We always assume this.

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- How does one represent this probability measure

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- The distribution function, F_X completely specifies the probability measure, P_X.

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- Let us look at a simple example.

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Please note that x is a 'dummy variable'

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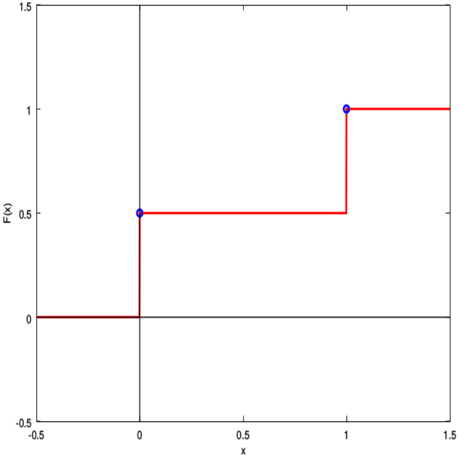
- A plot of this distribution function:



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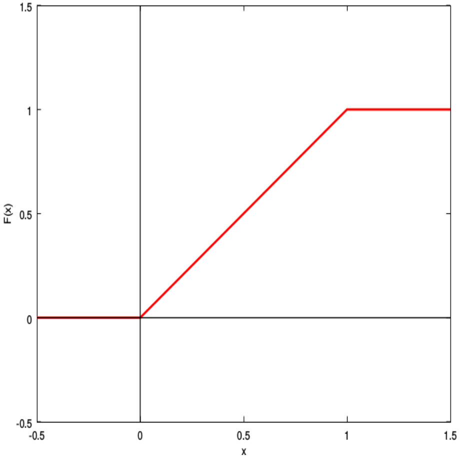

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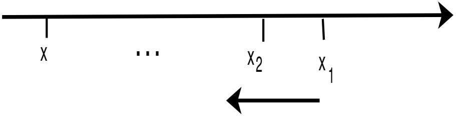

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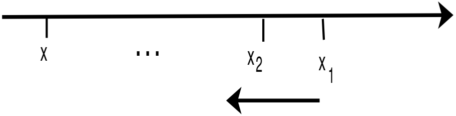

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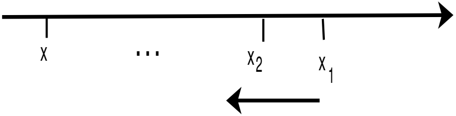

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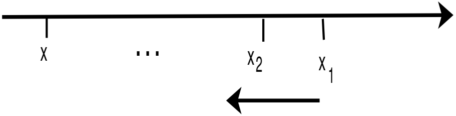

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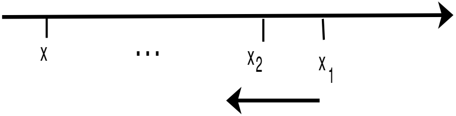

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- Any real-valued function of a real variable satisfying the above four properties would be a distribution function of some random variable.

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- Note that the distribution function is defined for **all** random variables.

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- Thus the family of discrete random variables includes all probability models on finite or countably infinite sample spaces.

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Discrete Random Variable Example

- Consider three independent tosses of a fair coin.
- $\Omega = \{H, T\}^3$ and $X(\omega)$ is the number of H's in ω.
- This rv takes four distinct values, namely, $0, 1, 2, 3$.
- We denote this as $X \in \{0, 1, 2, 3\}$
- Let us find the distribution function of this rv
- Let us take some examples of $[X \leq x]$

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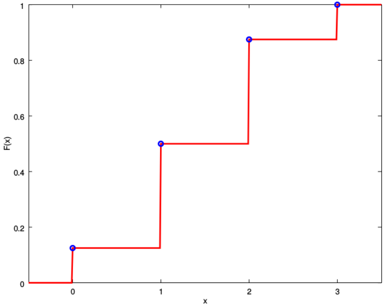

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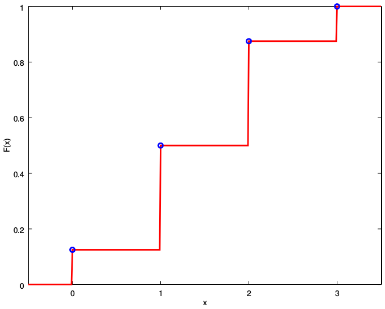

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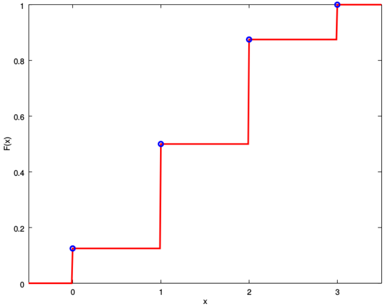

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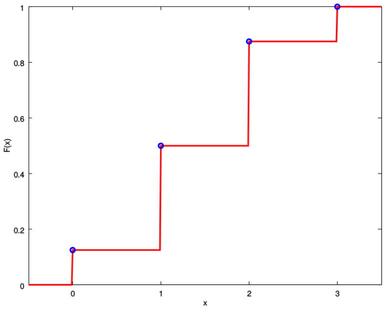

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- It has jumps at $x = 0, 1, 2, 3$, which are the values that X takes. In between these it is constant.
- The jump at, e.g., $x = 2$ is $3/8$ which is the probability of X taking that value.