

E2 212: Homework - 9

1 Topics

- SVD
- Least squares

Note: Most of the problems below are from Golub and Van Loan, Horn and Johnson, or David Lewis' books.

2 Problems

1. Show that $\sigma_1, \dots, \sigma_r$ are the non-zero singular values of the matrix \mathbf{A} iff $\{\sigma_1, \dots, \sigma_r, -\sigma_1, \dots, -\sigma_r\}$ are the non-zero eigenvalues of the matrix

$$\tilde{\mathbf{A}} = \begin{pmatrix} 0 & \mathbf{A} \\ \mathbf{A}^T & 0 \end{pmatrix}.$$

2. Let \mathbf{A} be an $m \times n$ real matrix with the “economy” singular value decomposition $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^T$. Show that the unitary matrix $\mathbf{B} = \mathbf{U}\mathbf{V}^T$ is the closest unitary matrix to \mathbf{A} , i.e., for any unitary matrix \mathbf{P} , $\|\mathbf{B} - \mathbf{A}\| \leq \|\mathbf{P} - \mathbf{A}\|$.
3. Let \mathbf{A} be an $m \times n$ matrix with rank r and whose non-zero singular values are $\sigma_1, \sigma_2, \dots, \sigma_r$. Show that the Euclidean (Frobenius) norm of \mathbf{A} is given by

$$\|\mathbf{A}\| = \sqrt{\sum_{i=1}^r \sigma_i^2}.$$

4. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, and $\mathbf{x} \in \mathbb{R}^n$. Show that

$$\|\mathbf{Ax} - \mathbf{b}\|^2 = \sum_{i=1}^n \left(\sum_{j=1}^n a_{ij}x_j - b_i \right)^2.$$

Show that the equations $\partial f / \partial x_i = 0$, $i = 1, \dots, n$, where $f(\mathbf{x}) \triangleq \|\mathbf{Ax} - \mathbf{b}\|^2$, are equivalent to the normal equations $\mathbf{A}^T \mathbf{Ax} = \mathbf{A}^T \mathbf{b}$.

5. Let $\hat{\mathbf{w}}$ be the solution to

$$\min_{\mathbf{w}} \|\mathbf{y} - H\mathbf{w}\|^2, \tag{1}$$

and let $\hat{\mathbf{y}} \triangleq H\hat{\mathbf{w}}$. Show that

- (a) The fundamental orthogonality principle holds, i.e., $\hat{\mathbf{w}}$ is a solution if, and only if, the residual $\tilde{\mathbf{y}} \triangleq \mathbf{y} - H\hat{\mathbf{w}}$ is orthogonal to $\mathcal{R}(H)$.

(b) The norms of $\{\mathbf{y}, \hat{\mathbf{y}}, \tilde{\mathbf{y}}\}$ satisfy the relation $\|\mathbf{y}\|^2 = \|\hat{\mathbf{y}}\|^2 + \|\tilde{\mathbf{y}}\|^2$.

6. Let

$$\mathbf{A} = \begin{pmatrix} \mathbf{R} & \mathbf{w} \\ 0 & \mathbf{v} \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

where \mathbf{R} is a $k \times k$ block, $\mathbf{c}, \mathbf{w} \in \mathbb{R}^k$, $\mathbf{v}, \mathbf{d} \in \mathbb{R}^{m-k}$, and the zero is a block of appropriate dimension. If $\mathbf{A} \in \mathbb{R}^{m \times (k+1)}$ has full column rank, then show that

$$\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{b}\|_2^2 = \|\mathbf{d}\|_2^2 - (\mathbf{v}^T \mathbf{d} / \|\mathbf{v}\|_2)^2.$$

7. Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ and a set of vectors $\mathbf{x}_i \in \mathbb{R}^m$, $\mathbf{y}_i \in \mathbb{R}^n$, $i = 1, 2, \dots, k$,

(a) Find a set of coefficients a_i 's such that $\|\mathbf{A} - \sum_{i=1}^k a_i \mathbf{x}_i \mathbf{y}_i^T\|_F^2$ is minimized.

(b) What are the set of $\mathbf{x}_i, \mathbf{y}_i$ that minimize the minimum in (a) for $k < \min(m, n)$?

8. Given the system of equations $\mathbf{Ax} = \mathbf{b}$, \mathbf{A} being a tall matrix, what is \mathbf{P} such that $[\min_{\mathbf{x}} \|\mathbf{Ax} - \mathbf{Pb}\|]$ is minimized?