E1 222 Stochastic Models and Applications
Problem Sheet 3–4

1. Recall that random variables $X_1, X_2, \cdots X_n$ are said to be *exchangeable* if any permutation of them has the same joint density. Suppose X_1, X_2, X_3 are exchangeable random variables. Show that

$$E\left[\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right] = \frac{2}{3}$$

   Can you generalize this result?
   (Hint: Is there a relation between $E\left[\frac{X_1}{X_1+X_2+X_3}\right]$ and $E\left[\frac{X_2}{X_1+X_2+X_3}\right]$?)

2. Find $E[X|Y]$ when X, Y have joint density given by

$$f_{XY}(x,y) = \frac{y}{2}e^{-xy},\ x > 0,\ 1 < y < 3$$

3. Let X and Y be iid random variables having Poisson distribution with parameter λ. Let $Z = X + Y$. Find $E[Z|Y]$.

4. Let X and Y be independent random variables each having geometric density with parameter p. Let $Z = X + Y$. Find $E[Y|Z]$.

5. Let Y be a continuous random variable with density

$$f_Y(y) = \frac{1}{\Gamma(0.5)}\sqrt{\frac{0.5}{y}}e^{-0.5y},\ y > 0.$$

   Let the conditional density of X given Y be

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}}\,\sqrt{y}\,e^{-0.5yx^2},\ -\infty < x < \infty,\ y > 0$$

   Show that $E[X|Y] = 0$. Show that marginal of X is given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

   (Note that $\Gamma(0.5) = \sqrt{\pi}$). Does EX exist?
   (Notice that $f_{X|Y}$ is Gaussian with mean zero and variance $1/\sqrt{y}$, Y is Gamma with parameters $0.5, 0.5$ and X has Cauchy distribution).

6. Define $\text{Var}[X|Y] = E\left[X^2|Y\right] - \left(E[X|Y]\right)^2$ Show that

$$\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}[X|Y]]$$

7. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed repeatedly until the same outcome occurs k consecutive times. Let N denote the number of trials needed. Show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

8. Let $X_1, X_2, \cdots$ be *iid* discrete random variables with $P[X_i = +1] = P[X_i = -1] = 0.5$. Find EX_i. Let N be a positive integer-valued random variable (which is a function of all X_i) defined as $N = \min\{k : X_k = +1\}$. Find EX_N.

9. A coin, with probability heads being p, is tossed repeatedly till we get r heads. Let N be the number of tosses needed. Calculate EN. (Hint: Try to express N as a sum of geometric random variables).

10. A fair dice is rolled repeatedly till each of the numbers $1, 2, \cdots, 6$, appears atleast once. Find the expected number of rolls needed.