E1 222 Stochastic Models and Applications
Problem Sheet 3.5

1. Let X, Y have joint density

$$f_{XY}(x,y) = \frac{\sqrt{3}}{2\pi} e^{-0.5(x^2+4y^2-2xy)}, \quad -\infty < x, y < \infty$$

   Find the marginal densities of X, Y and the conditional density $f_{X|Y}$.

2. Let X, Y be continuous random variables with the following joint density

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2}\left\{ \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho)^2}(x^2 - 2\rho xy + y^2) \right] + \right. \\ &\left. \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho)^2}(x^2 + 2\rho xy + y^2) \right] \right\} \end{aligned}$$

   Find f_X, f_Y and EXY. Are X, Y uncorrelated? Are X, Y jointly normal?

3. Let X, Y be jointly normal with means μ_1, μ_2, variances σ_1^2, σ_2^2 and correlation coefficient ρ. Find a necessary and sufficient condition for $X + Y$ and $X - Y$ to be independent.

4. Let X, Y be jointly normal with $EX = EY = 0$, $\mathrm{Var}(X) = \mathrm{Var}(Y) = 1$ and correlation coefficient ρ. Show that $Z = X/Y$ has Cauchy distribution. A Cauchy distribution with parameters μ and θ is given by

$$f(x) = \frac{\mu}{\pi} \frac{1}{(x-\theta)^2 + \mu^2}$$

5. Let X_1, X_2, X_3, X_4 be iid Gaussian random variables with mean zero and variance one. Show that the density function of $Y = X_1X_2 + X_3X_4$ is $f(y) = 0.5e^{-|y|}$, $-\infty < y < \infty$. (Hint: Try finding mgf of Y).