Assignment 1 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

Given: $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

1ⓐ PT $\operatorname{rank}(AB) \leq \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}$

A.

$\rightarrow \forall x \in \mathcal{N}(B), \quad Bx = 0$ (Definition of $\mathcal{N}(B)$)

$\Rightarrow ABx = 0$ (Premultiply with A)

$\Rightarrow x \in \mathcal{N}(AB)$ (Definition of $\mathcal{N}(AB)$)

$\Rightarrow \mathcal{N}(B) \subseteq \mathcal{N}(AB)$ — ①

$(\because \forall x \in U \Rightarrow x \in V \Rightarrow U \subseteq V)$

$\Rightarrow \dim(\mathcal{N}(B)) \leq \dim(\mathcal{N}(AB))$ $(U \subseteq V \Rightarrow \dim(U) \leq \dim(V))$

$\Rightarrow p - \dim(\mathcal{N}(B)) \geq p - \dim \mathcal{N}(AB)$ (Rank-Nullity Thm.)

$\Rightarrow \operatorname{rank}(AB) \leq \operatorname{rank}(B)$ — ②

$\rightarrow \forall x \in R(AB), \exists y \text{ st } ABy = x$

$\Rightarrow x = A(By) = Az$ (Definition of $R(A)$)

$\Rightarrow x \in R(A)$ — ③ $\left(\because \forall x \in U \Rightarrow x \in V \Rightarrow U \subseteq V\right)$

$\Rightarrow R(AB) \subseteq R(A)$

Thus, $\dim(R(AB)) \leq \dim(R(A))$, — ④ $\left(U \subseteq V \Rightarrow \dim(U) \leq \dim(V)\right)$

and $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

$\rightarrow$ ②, ④ $\Rightarrow$ $\text{rank}(AB) \leq \text{rank}(A)$
and $\text{rank}(AB) \leq \text{rank}(B)$
$\Rightarrow \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ — ⑤

Alt.: 1. After either $\text{rank}(AB) \leq \text{rank}(A)$ or $\text{rank}(AB) \leq \text{rank}(B)$ is proven, you can also do:
$\text{rank}(A'B') \leq \text{rank}(A')$ for some compatible A', B'

Let $B' = A^T$, $A' = B^T$.

$\Rightarrow \text{rank}(B^T A^T) \leq \text{rank}(B^T)$
$= \text{rank}(B)$ (Row rank = Col. Rank)

$\Rightarrow \text{rank}(B^T A^T) \leq \text{rank}(B)$
$\text{rank}(B^T A^T) = \text{rank}((B^T A^T)^T)$ (Row rank = Col. Rank)
$= \text{rank}(AB) \leq \text{rank}(B)$

2. You can also prove and use
$\text{rank}(AB) = \text{rank}(B) - \dim(\mathcal{N}(A) \cap \mathcal{R}(B))$.
This proof is also correct.

1(b) PT $\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$

A. From 1(a), $\text{rank}(A^T A) \leq \text{rank}(A)$ — ①

$\forall x \in \mathcal{N}(A^T A)$, $A^T A x = 0$ (Definition)

$\Rightarrow x^T A^T A x = 0 \Rightarrow \|Ax\|_2^2 = 0$ — ②

$\Leftrightarrow Ax = 0 \Rightarrow x \in \mathcal{N}(A)$

$\Rightarrow \mathcal{N}(A^T A) \subseteq \mathcal{N}(A)$ $\left(\because \forall x \in U \Rightarrow x \in V \Rightarrow U \subseteq V \right)$

$\Rightarrow \dim(\mathcal{N}(A^T A)) \leq \dim(\mathcal{N}(A))$ $(U \subseteq V \Rightarrow \dim(U) \leq \dim(V))$

$\Rightarrow n - \dim(\mathcal{N}(A)) \leq n - \dim(\mathcal{N}(A^T A))$ (Rank-Nullity Thm)

$\Rightarrow \text{rank}(A) \leq \text{rank}(A^T A)$ — ③

①, ② $\Rightarrow \text{rank}(A) = \text{rank}(A^T A)$

Consider $B = A^T \Rightarrow \text{rank}(B) = \text{rank}(B^T B)$

$$\begin{array}{ccc} \text{rank}(B) & = & \text{rank}(B^T B) \\ \| & & \| \\ \text{rank}(A^T) & = & \text{rank}(A A^T) \\ \| & & \\ \text{rank}(A) & & \end{array}$$

(Row Rank = Col. Rank)

$\Rightarrow \text{rank}(A) = \text{rank}(AA^T)$ — ④

$\Rightarrow \text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T)$ — ⑤

<u>Alt:</u> 1. Using $\text{rank}(AB) = \text{rank}(B) - \dim(\mathcal{N}(A) \cap \mathcal{R}(B))$ with $B = A^T$ results in $\mathcal{N}(A) \cap \mathcal{R}(A^T) = \phi$

$\Rightarrow \text{rank}(AA^T) = \text{rank}(A)$

2. To prove $\text{rank}(A) = \text{rank}(AA^T)$, similar arguments used in ①, ② can be used.

<u>Wrong:</u> $A^T A$ is a symmetric matrix.

$(A^T A)^T = A^T (A^T)^T = A^T A \neq AA^T$ and thus using this, you cannot arrive at the correct answer.