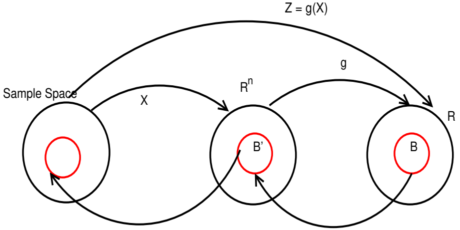
Recap

- Given $X_1, \cdots, X_n$, random variables on the same probability space, $Z = g(X_1, \cdots, X_n)$ is a rv (if $g : \Re^n \to \Re$ is borel measurable).



- We can determine distribution of Z from the joint distribution of all X_i

$$F_Z(z) = P[Z \leq z] = P[g(X_1, \cdots, X_n) \leq z]$$

Recap

- $X_1, \cdots, X_n$ are said to be independent if events $[X_1 \in B_1], \cdots, [X_n \in B_n]$ are independent.
- If $X_1, \cdots, X_n$ are indepedent and all of them have the same distribution function then they are said to be iid – independent and identically distributed

Recap

- Let $X_1, \cdots, X_n$ be independent and $Z = \max(X_1, \cdots, X_n)$

$$\begin{aligned} F_Z(z) &= \prod_{i=1}^{n} F_{X_i}(z) \\ &= (F(z))^n, \quad \text{if they are iid} \end{aligned}$$

Recap

- Let $X_1, \cdots, X_n$ be independent and $Z = \min(X_1, \cdots, X_n)$

$$\begin{aligned} F_Z(z) &= 1 - \prod_{i=1}^{n} (1 - F_{X_i}(z)) \\ &= 1 - (1 - F(z))^n, \quad \text{if they are iid} \end{aligned}$$

Recap

- Let X, Y be random variables with joint density f_{XY}
- $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(t, z-t)\, dt$$

- If X, Y are independent

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\, f_Y(z-t)\, dt$$

  Density of sum of independent random variables is the convolution of their densities.
- Sum of independent exponential random variables has gamma density.

Recall problem from last class

- Let X, Y be independent
- Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.
- We want joint distribution function of Z and W.

$$F_{ZW}(z, w) = P[Z \leq z, W \leq w]$$

- This is difficult to find. But we can easily find

$$P[\max(X, Y) \leq z, \min(X, Y) > w]$$

- Remaining details are left as an exercise for you!!

- X, Y iid with df F and density f
  $Z = \max(X, Y)$ and $W = \min(X, Y)$.
- We want joint distribution function of Z and W.
- We can use the following

$$P[Z \leq z] = P[Z \leq z, W \leq w] + P[Z \leq z, W > w]$$

$$\begin{aligned} P[Z \leq z, W > w] &= P[w < X, Y \leq z] = (F(z) - F(w))^2 \\ P[Z \leq z] &= P[X \leq z, Y \leq z] = (F(z))^2 \end{aligned}$$

- So, we get F_{ZW} as

$$\begin{aligned} F_{ZW}(z, w) &= P[Z \leq z, W \leq w] \\ &= P[Z \leq z] - P[Z \leq z, W > w] \\ &= (F(z))^2 - (F(z) - F(w))^2 \end{aligned}$$

- Is this correct for all values of z, w?

- We have $P[w < X, Y \leq z] = (F(z) - F(w))^2$ only when $w \leq z$.
- Otherwise it is zero.
- Hence we get F_{ZW} as

$$F_{ZW}(z,w) = \begin{cases} (F(z))^2 & \text{if} \quad w > z \\ (F(z))^2 - (F(z) - F(w))^2 & \text{if} \quad w \leq z \end{cases}$$

- We can get joint density of Z, W as

$$\begin{aligned} f_{ZW}(z,w) &= \frac{\partial^2}{\partial z \, \partial w} F_{ZW}(z,w) \\ &= 2 f(z) f(w), \quad w \leq z \end{aligned}$$

Order Statistics

- Let $X_1, \cdots, X_n$ be iid with density f.
- Let $X_{(k)}$ denote the k^{th} smallest of these.
- That is, $X_{(k)} = g_k(X_1, \cdots, X_n)$ where $g_k : \Re^n \rightarrow \Re$ and the value of $g_k(x_1, \cdots, x_n)$ is the k^{th} smallest of the numbers $x_1, \cdots, x_n$.
- $X_{(1)} = \min(X_1, \cdots, X_n), \quad X_{(n)} = \max(X_1, \cdots, X_n)$
- The joint distribution of $X_{(1)}, \cdots X_{(n)}$ is called the order statistics.
- We calculated the order statistics for the case $n = 2$.
- It can be shown that

$$f_{X_{(1)} \cdots X_{(n)}}(x_1, \cdots x_n) = n! \prod_{i=1}^{n} f(x_i), \;\; x_1 < x_2 < \cdots < x_n$$

- Let $X_1, \cdots, X_n$ be iid with df F and density f.
- $P[X_i \le y] = F(y)$ for any i and y.
- Since they are independent, we have, e.g.,

$$P[X_1 \le y, X_2 > y, X_3 \le y] = (F(y))^2(1 - F(y))$$

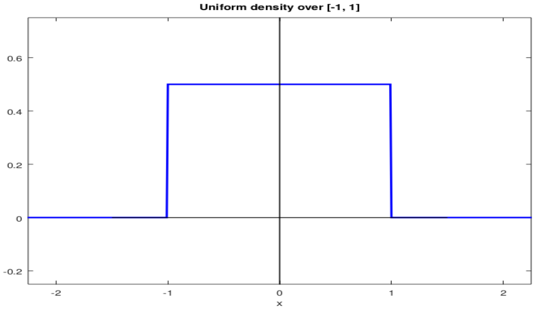
- Hence, probability that exactly k of these n random variables are less than or equal to y is ${}^nC_k (F(y))^k (1 - F(y))^{n-k}$
- Now the event $[X_{(k)} \le y]$ is same as the event "at least k of these are less than or equal to y"
- Hence we get

$$F_{X_{(k)}}(y) = \sum_{j=k}^{n} {}^nC_j (F(y))^j (1 - F(y))^{n-j}$$

We can get the density by differentiating this.

Distribution of sums of independent rv

- Suppose X, Y are iid uniform over $(-1,\ 1)$.
- let $Z = X + Y$. We want f_Z.
- The density of X, Y is



- f_Z is convolution of this density with itself.

- $f_X(x) = 0.5, \quad -1 < x < 1$. f_Y is also same
- Note that Z takes values in $[-2,\ 2]$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\ f_Y(z-t)\ dt$$

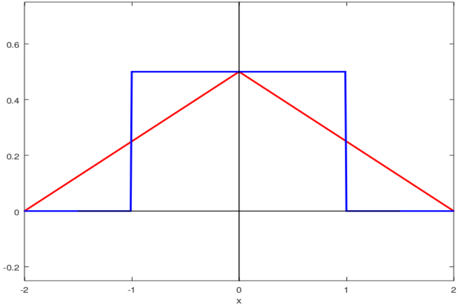
- For the integrand to be non-zero we need
  - $-1 < t < 1 \quad \Rightarrow \quad t < 1, \quad t > -1$
  - $-1 < z - t < 1 \ \Rightarrow \quad t < z + 1, \quad t > z - 1$
  - Hence we need:
    $t < \min(1, z+1), \quad t > \max(-1, z-1)$
  - Hence, for $z < 0$, we need $-1 < t < z + 1$
    and, for $z \geq 0$ we need $z - 1 < t < 1$
- Thus we get

$$f_Z(z) = \begin{cases} \int_{-1}^{z+1} \frac{1}{4} dt = \frac{z+2}{4} & \text{if } \ -2 \leq z < 0 \\ \int_{z-1}^{1} \frac{1}{4} dt = \frac{2-z}{4} & \text{if } \ 2 \geq z \geq 0 \end{cases}$$

- Thus, the density of sum of two ind rv's that are uniform over $(-1,\ 1)$ is

$$f_Z(z) = \begin{cases} \frac{z+2}{4} & \text{if } \ -2 < z < 0 \\ \frac{2-z}{4} & \text{if } \ 0 < z < 2 \end{cases}$$

- This is a triangle with vertices $(-2, 0), (0, 0.5), (2, 0)$



Independence of functions of random variable

- Suppose X and Y are independent.
- Then $g(X)$ and $h(Y)$ are independent
- This is because $[g(X) \in B_1] = [X \in \tilde{B}_1]$ for some Borel set, $\tilde{B}_1$ and similarly $[h(Y) \in B_2] = [Y \in \tilde{B}_2]$
- Hence, $[g(X) \in B_1]$ and $[h(Y) \in B_2]$ are independent.

Independence of functions of random variable

- This is easily generalized to functions of multiple random variables.
- If $\mathbf{X}, \mathbf{Y}$ are vector random variables (or random vectors), independence implies $[\mathbf{X} \in B_1]$ is independent of $[\mathbf{Y} \in B_2]$ for all borel sets B_1, B_2 (in appropriate spaces).
- Then $g(\mathbf{X})$ would be independent of $h(\mathbf{Y})$.
- That is, suppose $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ are independent.
- Then, $g(X_1, \cdots, X_m)$ is independent of $h(Y_1, \cdots, Y_n)$.

- Let X_1, X_2, X_3 be independent continuous rv
- $Z = X_1 + X_2 + X_3$.
- Can we find density of Z?
- Let $W = X_1 + X_2$.
- Then $Z = W + X_3$ and W and X_3 are independent.
- Exercise for you: Find density of $X_1 + X_2 + X_3$ where X_1, X_2, X_3 are iid uniform over $(0,\ 1)$.

Sum of independent gamma rv

- Gamma density with parameters $\alpha > 0$ and $\lambda > 0$ is given by
$$f(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x > 0$$
We will call this $Gamma(\alpha, \lambda)$.
- The α is called the shape parameter and λ is called the rate parameter.
- For $\alpha = 1$ this is the exponential density.
- Let $X \sim Gamma(\alpha_1, \lambda)$, $Y \sim Gamma(\alpha_2, \lambda)$. Suppose X, Y are independent.
- Let $Z = X + Y$. Then $Z \sim Gamma(\alpha_1 + \alpha_2, \lambda)$.

$$f_Z(z) \;=\; \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^z \frac{1}{\Gamma(\alpha_1)}\, \lambda^{\alpha_1}\, x^{\alpha_1-1}\, e^{-\lambda x} \frac{1}{\Gamma(\alpha_2)}\, \lambda^{\alpha_2}\, (z-x)^{\alpha_2-1}\, e^{-\lambda(z-x)}\, dx$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}\, e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^z z^{\alpha_1-1} \left(\frac{x}{z}\right)^{\alpha_1-1} z^{\alpha_2-1} \left(1-\frac{x}{z}\right)^{\alpha_2-1}\, dx$$

change the variable: $t = \frac{x}{z}\ \ (\Rightarrow\ \ z^{-1}dx = dt)$

$$= \frac{\lambda^{\alpha_1+\alpha_2}\, e^{-\lambda z}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\, z^{\alpha_+\alpha_2-1} \int_0^1 t^{\alpha_1-1}\, (1-t)^{\alpha_2-1}\, dt$$

$$= \frac{1}{\Gamma(\alpha_1+\alpha_2)}\, \lambda^{\alpha_1+\alpha_2}\, z^{\alpha_1+\alpha_2-1}\, e^{-\lambda z}$$

Because

$$\int_0^1 t^{\alpha_1-1}\, (1-t)^{\alpha_2-1}\, dt = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

- If X, Y are independent gamma random variables then $X + Y$ also has gamma distribution.
- If $X \sim Gamma(\alpha_1, \lambda)$, and $Y \sim Gamma(\alpha_2, \lambda)$, then $X + Y \sim Gamma(\alpha_1 + \alpha_2, \lambda)$.
- Exercise for you: Show that sum of independent Gaussian random variables has gaussian density.
- The algebra is a little involved.
- First take the two gaussians to be zero-mean.
- There is a calculation trick that is often useful with Gaussian density

A Calculation Trick

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2K}\left[x^2 - 2bx + c\right]\right)\, dx \\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2K}\left[(x-b)^2 + c - b^2\right]\right)\, dx \\
&= \int_{-\infty}^{\infty} \exp\left(-\frac{(x-b)^2}{2K}\right)\, \exp\left(-\frac{(c-b^2)}{2K}\right)\, dx \\
&= \exp\left(-\frac{(c-b^2)}{2K}\right)\sqrt{2\pi K}
\end{aligned}$$

because

$$\frac{1}{\sqrt{2\pi K}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-b)^2}{2K}\right)\, dx = 1$$

- We next look at a general theorem that is quite useful in dealing with functions of multiple random variables.
- This result is only for continuous random variables.

- Let $X_1, \cdots, X_n$ be continuous random variables with joint density $f_{X_1 \cdots X_n}$. We define $Y_1, \cdots Y_n$ by

$$Y_1 = g_1(X_1, \cdots, X_n) \quad \cdots \quad Y_n = g_n(X_1, \cdots, X_n)$$

  We think of g_i as components of $g : \Re^n \to \Re^n$.
- We assume g is continuous with continuous first partials and is invertible.
- Let h be the inverse of g. That is

$$X_1 = h_1(Y_1, \cdots, Y_n) \quad \cdots \quad X_n = h_n(Y_1, \cdots, Y_n)$$

- Each of g_i, h_i are $\Re^n \to \Re$ functions and we can write them as

$$y_i = g_i(x_1, \cdots, x_n); \quad \cdots \quad x_i = h_i(y_1, \cdots, y_n)$$

  We denote the partial derivatives of these functions by $\frac{\partial x_i}{\partial y_j}$ etc.

- The jacobian of the inverse transformation is

$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

- We assume that J is non-zero in the range of the transformation
- **Theorem**: Under the above conditions, we have

$$f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = |J| f_{X_1 \cdots X_n}\left(h_1(y_1, \cdots, y_n), \cdots, h_n(y_1, \cdots, y_n)\right)$$

Or, more compactly, $f_{\mathbf{Y}}(\mathbf{y}) = |J| f_{\mathbf{X}}(h(\mathbf{y}))$

Proof of Theorem

- Let $B = (-\infty,\ y_1] \times \cdots \times (-\infty,\ y_n] \subset \Re^n$. Then

$$\begin{aligned} F_{\mathbf{Y}}(\mathbf{y}) &= F_{Y_1 \cdots Y_n}(y_1, \cdots y_n) = P[Y_i \le y_i,\ i = 1, \cdots, n] \\ &= \int_B f_{Y_1 \cdots Y_n}(y_1', \cdots, y_n')\, dy_1' \ \cdots\ dy_n' \end{aligned}$$

- Define

$$\begin{aligned} g^{-1}(B) &= \{(x_1, \cdots, x_n) \in \Re^n \ :\ g(x_1, \cdots, x_n) \in B\} \\ &= \{(x_1, \cdots, x_n) \in \Re^n \ :\ g_i(x_1, \cdots, x_n) \le y_i,\ i = 1 \cdots n\} \end{aligned}$$

- Then we have

$$\begin{aligned} F_{Y_1 \cdots Y_n}(y_1, \cdots y_n) &= P[g_i(X_1, \cdots, X_n) \le y_i,\ i = 1, \cdots n] \\ &= \int_{g^{-1}(B)} f_{X_1 \cdots X_n}(x_1', \cdots, x_n')\, dx_1' \ \cdots\ dx_n' \end{aligned}$$

Proof of Theorem

- $B = (-\infty,\ y_1] \times \cdots \times (-\infty,\ y_n]$.
- $g^{-1}(B) = \{(x_1, \cdots, x_n) \in \Re^n\ :\ g(x_1, \cdots, x_n) \in B\}$

$$\begin{aligned} F_{\mathbf{Y}}(y_1, \cdots, y_n) &= P[g_i(X_1, \cdots, X_n) \le y_i,\ i = 1, \cdots, n] \\ &= \int_{g^{-1}(B)} f_{X_1 \cdots X_n}(x_1', \cdots, x_n')\, dx_1' \ \cdots\ dx_n' \end{aligned}$$

change variables: $y_i' = g_i(x_1', \cdots, x_n'), i = 1, \cdots n$

$$(x_1', \cdots x_n') \in g^{-1}(B) \ \Rightarrow\ (y_1', \cdots, y_n') \in B$$

$$x_i' = h_i(y_1', \cdots, y_n'), \quad dx_1' \cdots dx_n' = |J| dy_1' \cdots dy_n'$$

$$F_{\mathbf{Y}}(y_1, \cdots, y_n) = \int_B f_{X_1 \cdots X_n}(h_1(\mathbf{y}'), \cdots, h_n(\mathbf{y}'))\ |J| dy_1' \cdots dy_n'$$

$$\Rightarrow\ f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = f_{X_1 \cdots X_n}(h_1(\mathbf{y}), \cdots, h_n(\mathbf{y}))\ |J|$$

- $X_1, \cdots X_n$ are continuous rv with joint density

$$Y_1 = g_1(X_1, \cdots, X_n) \quad \cdots \quad Y_n = g_n(X_1, \cdots, X_n)$$

- The transformation is continuous with continuous first partials and is invertible and

$$X_1 = h_1(Y_1, \cdots, Y_n) \quad \cdots \quad X_n = h_n(Y_1, \cdots, Y_n)$$

- We assume the Jacobian of the inverse transform, J, is non-zero
- Then the density of $\mathbf{Y}$ is

$$f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = |J| f_{X_1 \cdots X_n}\left(h_1(y_1, \cdots, y_n), \cdots, h_n(y_1, \cdots, y_n)\right)$$

- Called multidimensional change of variable formula

- Let X, Y have joint density f_{XY}. Let $Z = X + Y$.
- We want f_Z. For the theorem we need two functions.
- To use the theorem, we need an invertible transformation of $\Re^2$ onto $\Re^2$ of which one component is $x + y$.
- Take $Z = X + Y$ and $W = X - Y$. This is an invertible.
- $X = (Z + W)/2$ and $Y = (Z - W)/2$. The Jacobian is

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

- Hence we get

$$f_{ZW}(z, w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

- Now we get density of Z as

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) \, dw$$

- let $Z = X + Y$ and $W = X - Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) \, dw$$

change the variable: $t = \dfrac{z+w}{2} \quad \Rightarrow dt = \dfrac{1}{2}\, dw$

$$\Rightarrow \quad w = 2t - z \quad \Rightarrow z - w = 2z - 2t$$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_{XY}(t, z-t) \, dt \\ &= \int_{-\infty}^{\infty} f_{XY}(z-t, t) \, dt, \quad \text{by using } t = \frac{z-w}{2} \text{ above} \end{aligned}$$

- We get same result as earlier. If, X, Y are independent

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t) \, f_Y(z-t) \, dt$$

- let $Z = X + Y$ and $W = X - Y$. We got

$$f_{ZW}(z, w) = \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right)$$

- Now we can calculate f_W also.

$$f_W(w) = \int_{-\infty}^{\infty} \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) \, dz$$

$$\text{change the variable: } \; t = \frac{z+w}{2} \quad \Rightarrow dt = \frac{1}{2} \, dz$$

$$\Rightarrow \quad z = 2t - w \quad \Rightarrow z - w = 2t - 2w$$

$$f_W(w) = \int_{-\infty}^{\infty} f_{XY}(t, t-w) \, dt$$

$$= \int_{-\infty}^{\infty} f_{XY}(t+w, t) dt, \quad \text{using } \; t = \frac{z-w}{2} \; \text{ above}$$

Example

- Let X, Y be iid $U(0,\ 1)$. Let $Z = X - Y$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\ f_Y(t-z)\ dt$$

- For the integrand to be non-zero (note $Z \in (-1, 1)$)
  - $0 < t < 1 \ \Rightarrow \ t > 0,\ \ t < 1$
  - $0 < t - z < 1 \ \Rightarrow \ t > z,\ \ t < 1 + z$
  - $\Rightarrow \quad \max(0,\ z) < t < \min(1,\ 1+z)$
- Thus, we get density as

$$f_Z(z) = \begin{cases} \int_0^{1+z} 1\ dt = 1 + z, & \text{if } -1 < z < 0 \\ \int_z^1 1\ dt = 1 - z, & 0 < z < 1 \end{cases}$$

- This we have when $X, Y \sim U(0, 1)$ iid

$$f_{X-Y}(z) = 1 - |z|, \quad -1 < z < 1$$

- We showed that

$$
\begin{aligned}
f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_{XY}(t, z-t)\, dt = \int_{-\infty}^{\infty} f_{XY}(z-t, t)\, dt \\
f_{X-Y}(w) &= \int_{-\infty}^{\infty} f_{XY}(t, t-w)\, dt = \int_{-\infty}^{\infty} f_{XY}(t+w, t) dt
\end{aligned}
$$

- Suppose X, Y are discrete. Then we have

$$
\begin{aligned}
f_{X+Y}(z) &= P[X+Y=z] = \sum_k P[X=k, Y=z-k] \\
&= \sum_k f_{XY}(k, z-k) \\
f_{X-Y}(w) &= P[X-Y=w] = \sum_k P[X=k, Y=k-w] \\
&= \sum_k f_{XY}(k, k-w)
\end{aligned}
$$

Distribution of product of random variables

- We want density of $Z = XY$.
- We need one more function to make an invertible transformation
- A possible choice: $Z = XY \quad W = Y$
- This is invertible: $X = Z/W \quad Y = W$

$$J = \begin{vmatrix} \frac{1}{w} & \frac{-z}{w^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{w}$$

- Hence we get

$$f_{ZW}(z, w) = \left|\frac{1}{w}\right| \, f_{XY}\left(\frac{z}{w}, w\right)$$

- Thus we get the density of product as

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| \, f_{XY}\left(\frac{z}{w}, w\right) \, dw$$

example

- Let X, Y be iid $U(0,\ 1)$. Let $Z = XY$.

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| \, f_X\left(\frac{z}{w}\right) \, f_Y(w) \, dw$$

- We need: $0 < w < 1$ and $0 < \frac{z}{w} < 1$. Hence

$$f_Z(z) = \int_z^1 \left|\frac{1}{w}\right| \, dw = \int_z^1 \frac{1}{w} \, dw = -\ln(z), \;\; 0 < z < 1$$

- X, Y have joint density and $Z = XY$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} \left| \frac{1}{w} \right| \; f_{XY}\left(\frac{z}{w}.w\right) \; dw$$

Suppose X, Y are discrete and $Z = XY$

$$f_Z(0) \;\; = \;\; P[X = 0 \text{ or } Y = 0] = \sum_x f_{XY}(x, 0) + \sum_y f_{XY}(0, y$$

$$f_Z(k) \;\; = \;\; \sum_{y \neq 0} P\left[X = \frac{k}{y}, Y = y\right] = \sum_{y \neq 0} f_{XY}\left(\frac{k}{y}, y\right), \; k \neq 0$$

- We cannot always interchange density and mass functions!!

- We wanted density of $Z = XY$.
- We used: $Z = XY$ and $W = Y$.
- We could have used: $Z = XY$ and $W = X$.
- This is invertible: $X = W$ and $Y = Z/W$.

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{w} & \frac{-z}{w^2} \end{vmatrix} = -\frac{1}{w}$$

- This gives

$$\begin{aligned} f_{ZW}(z, w) &= \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right) \\ f_Z(z) &= \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right) \, dw \end{aligned}$$

- The f_Z should be same in both cases.

Distributions of quotients

- X, Y have joint density and $Z = X/Y$.
- We can take: $Z = X/Y \quad W = Y$
- This is invertible: $X = ZW \quad Y = W$

$$J = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$$

- Hence we get

$$f_{ZW}(z, w) = |w| \ f_{XY}(zw, w)$$

- Thus we get the density of quotient as

$$f_Z(z) = \int_{-\infty}^{\infty} |w| \ f_{XY}(zw, w) \ dw$$

example

- Let X, Y be iid $U(0,\ 1)$. Let $Z = X/Y$.
  Note $Z \in (0, \infty)$

$$f_Z(z) = \int_{-\infty}^{\infty} |w|\ f_X(zw)\ f_Y(w)\ dw$$

- We need $0 < w < 1$ and $0 < zw < 1\ \ \Rightarrow w < 1/z$.
- So, when $z \leq 1$, w goes from 0 to 1; when $z > 1$, w goes from 0 to $1/z$.
- Hence we get density as

$$f_Z(z) = \begin{cases} \int_0^1 w\ dw = \frac{1}{2}, & \text{if } 0 < z \leq 1 \\ \int_0^{1/z} w\ dw = \frac{1}{2z^2}, & 1 < z < \infty \end{cases}$$

- X, Y have joint density and $Z = X/Y$

$$f_Z(z) = \int_{-\infty}^{\infty} |w| \ f_{XY}(zw, w) \, dw$$

- SupposeX, Y are discrete and $Z = X/Y$

$$\begin{aligned} f_Z(z) &= P[Z = z] = P[X/Y = z] \\ &= \sum_y P[X = yz, Y = y] \\ &= \sum_y f_{XY}(yz, y) \end{aligned}$$

- We chose: $Z = X/Y$ and $W = Y$.
- We could have taken: $Z = X/Y$ and $W = X$
- The inverse is: $X = W$ and $Y = W/Z$

$$J = \begin{vmatrix} 0 & 1 \\ -\frac{w}{z^2} & \frac{1}{z} \end{vmatrix} = -\frac{w}{z^2}$$

- Thus we get the density of quotient as

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \left|\frac{w}{z^2}\right| f_{XY}\left(w, \frac{w}{z}\right) \, dw \\ & \quad \text{put } \; t = \frac{w}{z} \;\; \Rightarrow dt = \frac{dw}{z}, \;\; w = tz \\ &= \int_{-\infty}^{\infty} |t| f_{XY}(tz, t) \, dt \end{aligned}$$

- We can show that the density of quotient is same in both these approches.

Exchangeable Random Variables

- $X_1, X_2, \cdots, X_n$ are said to be exchangeable if their joint distribution is same as that of any permutation of them.
- let $(i_1, \cdots, i_n)$ be a permutation of $(1, 2, \cdots, n)$. Then joint df of $(X_{i_1}, \cdots, X_{i_n})$ should be same as that $(X_1, \cdots, X_n)$
- Take $n = 3$. Suppose $F_{X_1 X_2 X_3}(a, b, c) = g(a, b, c)$. If they are exchangeable, then

$$\begin{aligned} F_{X_2 X_3 X_1}(a, b, c) &= P[X_2 \leq a, X_3 \leq b, X_1 \leq c] \\ &= P[X_1 \leq c, X_2 \leq a, X_3 \leq b] \\ &= g(c, a, b) = g(a, b, c) \end{aligned}$$

- The df or density should be "symmetric" in its variables if the random variables are exchangeable.

- Consider the density of three random variables

$$f(x, y, z) = \frac{2}{3}(x + y + z), \quad 0 < x, y, z < 1$$

- They are exchangeable (because $f(x, y, z) = f(y, x, z)$)
- If random variables are exchangeable then they are identically distributed.
  $F_{XYZ}(a, \infty, \infty) = F_{XYZ}(\infty, \infty, a) \;\; \Rightarrow \;\; F_X(a) = F_Z(a)$
- The above example shows that exchangeable random variables need not be independent. The joint density is not factorizable.

$$\int_0^1 \int_0^1 \frac{2}{3}(x + y + z)\, dy\, dz = \frac{2(x + 1)}{3}$$

- So, the joint density is not the product of marginals

Expectation of functions of multiple rv

- **Theorem**: Let $Z = g(X_1, \cdots X_n) = g(\mathbf{X})$. Then

$$E[Z] = \int_{\Re^n} g(\mathbf{x}) \; dF_{\mathbf{X}}(\mathbf{x})$$

- That is, if they have a joint density, then

$$E[Z] = \int_{\Re^n} g(\mathbf{x}) \; f_{\mathbf{X}}(\mathbf{x}) \; d\mathbf{x}$$

- Similarly, if all X_i are discrete

$$E[Z] = \sum_{\mathbf{x}} g(\mathbf{x}) \; f_{\mathbf{X}}(\mathbf{x})$$

- Let $Z = X + Y$. Let X, Y have joint density f_{XY}

$$
\begin{aligned}
E[X+Y] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+y)\; f_{XY}(x,y)\; dx\; dy \\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{XY}(x,y)\; dy\; dx \\
&\qquad + \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{XY}(x,y)\; dx\; dy \\
&= \int_{-\infty}^{\infty} x\; f_X(x)\; dx + \int_{-\infty}^{\infty} y\; f_Y(y)\; dy \\
&= E[X] + E[Y]
\end{aligned}
$$

- Expectation is a linear operator.
- This is true for all random variables.