

E2 212: Homework - 7

1 Topics

- Geršgorin disc theorem
- Perturbation of eigenvalues

2 Problems

1. If $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mathbf{B} \in \mathbb{C}^{n \times n}$ are Hermitian matrices with all the eigenvalues of $\mathbf{A} - \mathbf{B}$ being nonnegative, then prove that $\lambda_i(\mathbf{A}) \geq \lambda_i(\mathbf{B})$ for all $i = 1, 2, \dots, n$, where the eigenvalues of both matrices are arranged in increasing order, as usual.
2. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ have entries $a_{ij} = 1$ for all $i \neq j$, and $a_{ii} = n$ for each i . Show that \mathbf{A} is invertible.
3. If $\mathbf{A} \in \mathbb{R}^{n \times n}$ and the Geršgorin discs for the rows of \mathbf{A} are mutually disjoint, then, show that all of the eigenvalues of \mathbf{A} must be real.
4. Consider the following iterative algorithm for solving the n -by- n system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{y}$, where \mathbf{A} and \mathbf{y} are given:

- Define $\mathbf{B} = \mathbf{I} - \mathbf{A}$ and rewrite the system as $\mathbf{x} = \mathbf{B}\mathbf{x} + \mathbf{y}$.
- Choose an initial approximation $\mathbf{x}^{(0)}$ to the solution in any way you wish.
- For $m = 0, 1, 2, \dots$ calculate $\mathbf{x}^{(m+1)} = \mathbf{B}\mathbf{x}^{(m)} + \mathbf{y}$.

- (a) Denote by $\epsilon^{(m)} = \mathbf{x}^{(m)} - \mathbf{x}$ the error in the m th approximation to the solution, and show that

$$\epsilon^{(m)} = \mathbf{B}^m(\mathbf{x}^{(0)} - \mathbf{x}).$$

- (b) Conclude that if $\rho(\mathbf{I} - \mathbf{A}) < 1$, then this algorithm works in the sense that $\mathbf{x}^{(m)} \rightarrow \mathbf{x}$ as $m \rightarrow \infty$ regardless of the choice of the initial approximation $\mathbf{x}^{(0)}$.
- (c) Use Geršgorin theorem to give a simple explicit condition on \mathbf{A} that is sufficient for this algorithm to work.
5. Without computing the eigenvalues, decide how many are positive, negative and zero for the following matrix ? Why ?

$$\mathbf{A} = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & n-1 \\ 1 & \dots & n-1 & n \end{pmatrix}.$$

6. Suppose that $\mathbf{A} \in \mathbb{R}^{n \times n}$ is a real matrix whose n Geršgorin discs are all mutually disjoint. Show that all the eigenvalues of \mathbf{A} are real.
7. Suppose that \mathbf{A} is idempotent ($\mathbf{A}^2 = \mathbf{A}$), but $\mathbf{A} \neq \mathbf{I}$. Show that \mathbf{A} cannot be strictly diagonally dominant.

8. Let $\mathbf{A} = [a_{ij}]$ be a $n \times n$ matrix. Show that

$$\rho(\mathbf{A}) \leq \min_{p_1, p_2, \dots, p_n > 0} \max_{1 \leq i \leq n} \frac{1}{p_i} \sum_{j=1}^n p_j |a_{ij}|.$$

Hint: Since the eigenvalues do not change under similarity transformations, we can tighten the Geršgorin regions by considering similarity transformations.

9. If $\mathbf{A} \in \mathbb{C}^{n \times n}$, and $\rho(\mathbf{A})$ represents the spectral radius of \mathbf{A} . Using Geršgorin theorem, prove that

$$(a) \quad \rho(\mathbf{A}) \leq \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

$$(b) \quad \rho(\mathbf{A}) \leq \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$$

10. Let λ be a simple eigenvalue (eigenvalue with algebraic multiplicity 1) of an $n \times n$ matrix \mathbf{A} . Let $s(\lambda) = |\mathbf{y}^H \mathbf{x}|$ where $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ and $\mathbf{y}^H \mathbf{A} = \lambda\mathbf{y}^H$ and \mathbf{x} and \mathbf{y} have unit Euclidean norm. Prove the following:

$$(a) \quad S(\lambda) \leq 1.$$

$$(b) \quad S(\lambda) \neq 0.$$