E1 222 Stochastic Models and Applications

Problem Sheet 4–1

1. Let $\{X_n,\ n = 0, 1, \cdots\}$ be a Markov chain. Show that

$$\text{Prob}[X_{n+1} = z, X_{n-1} = y | X_n = x] = \text{Prob}[X_{n+1} = z | X_n = x]\ \text{Prob}[X_{n-1} = y | X_n = x]$$

   (That is, conditioned on the 'present' the 'past' is conditinally independent of the 'future')

2. Suppose we have two boxes and $2d$ balls, of which d are black and d are red. Initially d of the balls are placed in box-1 and the remaining in box-2. At each instant n, $n = 1, 2, \cdots$, a ball is chosen at random from each box and the two balls are placed in the opposite boxes. Let X_0 denote number of black balls initially in box-1 and let X_n denote number of black balls in box-1 after the exchange at n, $n = 1, 2, \cdots$. Argue that $\{X_n\}$ is a Markov chain. Find the transition probabilities of the Markov chain $\{X_n\}$. State which are transient states and which are recurrent states. Is the chain irreducible?

3. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

   Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find the absorption probabilities from each of the transient states to each of the closed irreducible subsets of recurrent states.

4. Consider a Markov chain on nonnegative integers having transition probabilities, $P(x, x+1) = p$ and $P(x, 0) = 1 - p$ where $0 < p < 1$. Show that the chain has unique stationary distribution and find that distribution.

5. A transition probability matrix is called doubly stochastic if both the rows as well as columns sum to one. Consider a finite irreducible

Markov chain whose transition probability matrix is doubly stochastic. Show that the chain has a unique stationary distribution given by $\pi(y) = \frac{1}{n}$, $\forall y$, where n is the number of states.

6. On a road, three out of every four trucks are followed by a car while only one out of every five cars is followed by a truck. Find the ratio of trucks to cars on the road.

7. A professor keeps giving a sequence of exams to the class. The exams are of three types. Let q_i denote the probability that the class does well on exam of type i. It is known that $q_1 = 0.4$, $q_2 = 0.6$, $q_3 = 0.8$. If the class does well in the current exam, the next exam is equally likely to be any of the three types. If the class does badly on the current exam, then the next exam would always be of type 3. What proportion of exams are of type i, $i = 1, 2, 3$?

8. Let Y_n be the sum of numbers obtained on n independent rolls of a fair die, $n = 1, 2, \cdots$. Find

$$\lim_{n \to \infty} P[Y_n \text{ is divisible by } 3]$$

(Hint: Think of a 3-state Markov chain where the state at n could be the reminder obtained when Y_n is divided by 3).

9. Suppose that whether or not it rains today depends on the whether or not it rained for the previous three days. Explain how we can set up a Markov chain model for this. Suppose that if it rained on each of the previous three days then it will rain today with probability 0.6; if it did not rain on any of the previous three days then it will rain today with probability 0.2; in all other cases the weather today would be same as that of yesterday with probability 0.5. Now find the transition probability matrix for the chain.