Assignment 11

Note: Text in blue shows Marks. Text in green is not necessary

Q1. Let $x, y \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n\times n}$ Herm., Let $B = I - A$ and $x_0 \in \mathbb{C}^n$ be an arbitrary vector. Given A and y, we wish to solve the linear system of equations $Ax = y$ as follows

For $i = 1, 2, 3$

Do $x_i = Bx_{i-1} + y$

a) Let $\epsilon_i = x_i - x$ be the error in the i^{th} iteration. Show that $\epsilon_i = B^i(x_0 - x)$

b) Conclude that if $\rho(B) < 1$, then this algorithm works

c) Use Gersgorin thm to show that a hermitian A must be strictly diagonally dominant and $|||A|||_\infty < 2$ to ensure that this algorithm works.

Sol)
$$B = I - A$$
$$Bx = x - y$$

Let's prove (a) using Mathematical induction

$$\epsilon_i = x_i - x$$
$$\epsilon_1 = x_1 - x$$
$$= Bx_0 + y - x$$
$$= Bx_0 - Bx$$
$$\epsilon_1 = B(x_0 - x) \quad — \; 1$$

Let $\epsilon_i = B^i (x_0 - x)$ be true for $i = k$

$\therefore \epsilon_k = B^k (x_0 - x)$

$$
\begin{aligned}
\epsilon_{k+1} &= x_{k+1} - x \\
&= B x_k + y - x \\
&= B x_k - B x \\
&= B (x_k - x) \\
&= B \, \epsilon_k \\
&= B \, B^k (x_0 - x)
\end{aligned}
$$

$$\epsilon_{k+1} = B^{k+1} (x_0 - x) \qquad \text{--- (2)}$$

Thus, from the principles of mathematical Induction

$$\epsilon_i = B^i (x_0 - x) \qquad \text{--- (3)}$$

b) For the algorithm to work, $x_i \to x$ as $i \to \infty$

$\Rightarrow \epsilon_i \to 0$ as $i \to \infty$ — (4)

$\Rightarrow B^i \to 0$ as $i \to \infty$

ie B should be a convergent matrix

$\Rightarrow \rho(B) < 1$ — (5)

c) The question is not complete. It should have been

Show that if A is Herm. and positive definite, then A being strictly diagonally dominant and $|||A|||_\infty < 2$ is a sufficient condition for this algorithm to work.

$B = I - A$

$\Rightarrow \lambda_i(B) = 1 - \lambda_i(A)$ where λ_i is i^{th} Eval of A (in any order)

We have $\rho(B) < 1 \Rightarrow -1 < \lambda_i(B) < 1 \quad \forall i$

$\Rightarrow 0 < \lambda_i(A) < 2 \quad \forall i$ [$\because$ A & B are Herm. $\Rightarrow \lambda_i(A)$ & $\lambda_i(B)$ are real]

— 6

$A \succcurlyeq 0 \Rightarrow \lambda_i(A) > 0 \quad \forall i$

A being Herm. and strictly diagonally dominant

$\Rightarrow$ All Gersgorin discs are centered on $\mathbb{R}^+$ and are completely contained in the right half of complex plane

$\therefore$ Evals are in the region

$$\bigcup_{i=1}^{n} \left[a_{ii} - \sum_{j \neq i} |a_{ij}| , \; a_{ii} + \sum_{j \neq i} |a_{ij}| \right]$$

$$\subseteq \bigcup_{i=1}^{n} \left(0, \; a_{ii} + \sum_{j \neq i} |a_{ij}| \right] \qquad [\because A \text{ is strictly diag. dom.}]$$

$$= \bigcup_{i=1}^{n} \left(0, \; \sum_{j=1}^{n} |a_{ij}| \right]$$

$$= \left(0, \; \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \right]$$

$$= \left(0, \; |||A|||_\infty \right]$$

$$\subseteq (0, 2) \qquad [\because |||A|||_\infty < 2]$$

$\therefore$ The given conditions imply that $\lambda_i(A) \in (0,2) \; \forall i$

Hence, this is a sufficient condition for the algorithm to work.

— 7

2) Let y & x be unit l_2-norm left and right Evecs respectively of $C \in \mathbb{C}^{n\times n}$ corresponding to a simple Eval λ. Let $s(\lambda) \triangleq |y^H x|$. Prove that $s(\lambda) \neq 0$

Sol) Let's prove by contradiction.

Assume $s(\lambda) = 0$

i.e. $y^H x = 0 \Rightarrow x \perp y$

We have $y^H A = \lambda y^H$ & $Ax = \lambda x$

i.e. $y \in N((A-\lambda I)^H)$ & $x \in N(A-\lambda I)$

$x \perp y \Rightarrow x \in R(A-\lambda I)$

$\Rightarrow \exists\, z \in \mathbb{C}^n$ s.t $x = (A-\lambda I)z$ —— (1)

Since $x \in N(A-\lambda I)$

$(A-\lambda I)x = 0$

$\Rightarrow (A-\lambda I)^2 z = 0$

$\Rightarrow$ x & z are generalized Evecs of A corresp to λ

Since λ is a simple Eigenvalue, alg. mult $= 1$

$\Rightarrow$ There exists only 1 L.I generalized Evec

$\Rightarrow z = \alpha x$ for some $\alpha \neq 0$ —— (2)

But we have $x = (A - \lambda I) z$

$$= (A - \lambda I) \alpha x$$

$$= \alpha (A - \lambda I) x$$

$$= 0$$

which is a contradiction since Evecs cannot be zero

$\Rightarrow$ Our assumption that $s(\lambda) = 0$ is incorrect

Hence $S(\lambda) \neq 0$. —— 3