Assignment 3 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. Let $A \in \mathbb{R}^{m \times n}$ and $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$. Let $\|x\|_A \triangleq \|Ax\|$. Prove or disprove: (6)
   (a) $\|\cdot\|_A$ is a norm on $\mathbb{R}^n$ when $\text{rank}(A) = n$.
   (b) $\|\cdot\|_A$ is a norm on $\mathbb{R}^n$ when $\text{rank}(A) = k < n$.

A. (a) $\|\cdot\|_A$ is a norm when $\text{rank}(A) = n$

Given: $\|\cdot\|$ is a vector norm on $\mathbb{R}^m$.

Proof: (1) $\|x\|_A = \|Ax\| \geq 0$ ($\because \|y\| \geq 0$, since $\|\cdot\|$ is a norm)

$\Rightarrow \|x\|_A \geq 0$

$\Rightarrow$ Non-negativity satisfied. — (1)

(1a) $\text{rank}(A) = n$ $\rightarrow$ (Rank-nullity thm)

$\Rightarrow \dim(\mathcal{N}(A)) = n - \text{rank}(A) = 0$

$\Rightarrow \mathcal{N}(A) = \{\underline{0}\}$

If $x = 0$, $\|x\|_A = \|A.0\| = 0$.

If $\|x\|_A = \|Ax\| = 0$ ($\because \|y\| = 0 \Leftrightarrow y = 0$, since $\|\cdot\|$ is a norm)

$\Leftrightarrow Ax = 0$

Since $\mathcal{N}(A) = \{\underline{0}\}$, $Ax = 0 \Rightarrow x = 0$

Thus $\|x\|_A = 0 \Rightarrow x = 0$

$\Rightarrow$ Positivity satisfied — (2)

(2) $\|cx\|_A = \|cAx\|$

$= |c| \cdot \|Ax\|$ ($\because \|cy\| = |c| \cdot \|y\|$, since $\|\cdot\|$ is a norm)

$= |c| \cdot \|x\|_A$

$\Rightarrow$ Homogeneity satisfied — (3)

(3) $\|x+y\|_A = \|A(x+y)\|$

$= \|Ax + Ay\|$ ($\because \|y+z\| \leq \|y\| + \|z\|$, since $\|\cdot\|$ is a norm)

$\leq \|Ax\| + \|Ay\|$

$= \|x\|_A + \|y\|_A$

$\Rightarrow$ Triangle inequality satisfied — (4)

Thus, $\|\cdot\|_A$ is a vector norm.

(b) When $\text{rank}(A) = k < n$,

$\dim(\mathcal{N}(A)) = n - \text{rank}(A)$ (Rank-nullity thm)

$= n - k$

Thus $\mathcal{N}(A) \neq \{\underline{0}\}$. — (1)

Thus $\exists\, x \in \mathbb{R}^m$ such that $Ax = 0$ and $x \neq 0$.

If $x = 0$, $\|x\|_A = \|A \cdot 0\| = 0$

If $\|x\|_A = \|Ax\| = 0$ $\left(\because \|y\| = 0 \Leftrightarrow y = 0, \right.$ since $\|\cdot\|$ is a norm$\left. \right)$

$\Leftrightarrow Ax = 0$

$\not\Rightarrow x = 0$ $(\because$ Above$)$

Thus, Positivity is not satisfied. — 2

Hence $\|\cdot\|_A$ is not a norm when rank$(A) = k < n$.

For example, if $A = [1 \ 0]$, $x = [0 \ 1]^T$,

$Ax = 0$ for $x \neq \underline{0}$.

Note: 1. $\|\cdot\|$ is a given vector norm. It need not necessarily be the l_p norm. Do not assume that it is the l_p norm.

2. There are NO alternate solutions to such proofs.

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2. Let $C \in \mathbb{C}^{m \times n}$, $D \in \mathbb{C}^{n \times m}$. Prove or disprove: (4)

(a) $|CD| = |DC|$, when $m = n$.

(b) $|CD| = |DC|$, when $m > n$.

A. (a) When $m = n$, both C and D are square matrices and determinants can be defined for both C and D. — 1

$$\Rightarrow |CD| = |C| \cdot |D| \qquad \rightarrow \left(\because \text{ Done in class : Proof by } d(A) = \det(AB)/\det(B) \right)$$
$$= |D| \cdot |C|$$
$$= |DC| \quad — ②$$

Thus $\det(CD) = \det(DC)$

(b) When $m > n$, C and D are not square and hence determinant cannot be defined for both.

Let $C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \end{bmatrix}$

$CD = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow |CD| = 0$

$DC = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1] \Rightarrow |DC| = 1$

— ①

Thus $|CD| \neq |DC|$ in general, when C and D are rectangular matrices. — ②

<u>Note:</u> Take simple counter-examples. Complicated counter-examples may lead to errors.