Poisson Process

- This is the next process we study
- This is a discrete-state continuous-time process
- The index set is the interval $[0, \infty)$ and all random variables are discrete and take non-negative integer values.

- A random process $\{N(t),\ t \geq 0\}$ is called a counting process if
    1. $N(t) \geq 0$ and is integer-valued
    2. If $s < t$ then, $N(s) \leq N(t)$

  $N(t)$ represents number of 'events' till t
- The counting process has independent increments if for all $t_1 < t_2 \leq t_3 < t_4$, $N(t_2) - N(t_1)$ is independent of $N(t_4) - N(t_3)$
- In particular, for all $s > t$, $N(s) - N(t)$ is independent of $N(t) - N(0)$
- The process is said to have stationary increments if $N(t_2) - N(t_1)$ has the same distribution as $N(t_2 + \tau) - N(t_1 + \tau)$, $\forall \tau, \forall t_2 > t_1$

- We start with two definitions of Poisson process
- **Definition 1** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
    1. $N(0) = 0$
    2. The process has stationary and independent increments
    3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \; n = 0, 1, \cdots$
- $N(t)$ is Poisson with parameter λt
- $E[N(t)] = \lambda t$ and hence λ is called rate
- Since the process has stationary increments and $N(0) = 0$, $(N(t+s) - N(s))$ would be Poisson with parameter λt for all $s, t > 0$.

- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
  2. The process has stationary and independent increments
  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
     $Pr[N(h) \geq 2] = o(h)$
- We say $g(h)$ is $o(h)$ if

$$\lim_{h \to 0} \frac{g(h)}{h} = 0$$

- This definition tells us when Poisson process may be a good model
- We will show that both definitions are equivalent

- We first show Definition 2 $\Rightarrow$ Definition 1
- For this we need to calculate distribution of $N(t)$
- Let $P_n(t) = Pr[N(t) = n]$

$$
\begin{aligned}
P_0(t+h) &= Pr[N(t+h) = 0] \\
&= Pr[N(t) = 0,\ N(t+h) - N(t) = 0] \\
&= Pr[N(t) = 0]\ Pr[N(t+h) - N(t) = 0] \\
&\qquad \text{(because of independent increments)} \\
&= Pr[N(t) = 0]\ Pr[N(h) = 0] \quad \text{(stationary increments)} \\
&= P_0(t)(1 - \lambda h + o(h))
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{P_0(t+h) - P_0(t)}{h} &= -\lambda P_0(t) + \frac{o(h)}{h} \\
\Rightarrow \frac{d}{dt} P_0(t) &= -\lambda P_0(t)
\end{aligned}
$$

- Now we can solve this differential equation to get $P_0(t)$

$$\begin{aligned} \frac{d}{dt}P_0(t) &= -\lambda P_0(t) \\ \Rightarrow \quad \frac{1}{P_0(t)}\frac{d}{dt}P_0(t) &= -\lambda \\ \Rightarrow \quad \ln(P_0(t)) &= -\lambda t + c \\ \Rightarrow \quad P_0(t) &= Ke^{-\lambda t} \end{aligned}$$

- Since $P_0(0) = Pr[N(0) = 0] = 1$, we get $K = 1$ and hence
$$P_0(t) = Pr[N(t) = 0] = e^{-\lambda t}$$
- Next we consider $P_n(t)$ for $n > 0$

$$
\begin{aligned}
P_n(t+h) &= Pr[N(t+h)=n] \\
&= Pr[N(t)=n,\ N(t+h)-N(t)=0] + \\
&\qquad Pr[N(t)=n-1,\ N(t+h)-N(t)=1] + \\
&\qquad \sum_{k=2}^{n} Pr[N(t)=n-k,\ N(t+h)-N(t)=k] \\
&= P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + o(h) \\
&= P_n(t)(1-\lambda h + o(h)) + P_{n-1}(t)(\lambda h + o(h)) + o(h)
\end{aligned}
$$

$$
\Rightarrow \quad \frac{P_n(t+h)-P_n(t)}{h} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(h)}{h}
$$

$$
\Rightarrow \quad \frac{d}{dt}P_n(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)
$$

$$\frac{d}{dt}P_n(t) + \lambda P_n(t) = \lambda P_{n-1}(t)$$

- We need to solve this linear ODE to obtain P_n
- The integrating factor is $e^{\lambda t}$. Let $P_n'(t) = \frac{d}{dt}P_n(t)$

$$\begin{aligned} e^{\lambda t}\left(P_n'(t) + \lambda P_n(t)\right) &= e^{\lambda t}\,\lambda P_{n-1}(t) \\ \Rightarrow \;\; \frac{d}{dt}\left(P_n(t)\,e^{\lambda t}\right) &= \lambda\,e^{\lambda t}\,P_{n-1}(t) \end{aligned}$$

- We need P_{n-1} to solve for P_n. Take $n = 1$

$$\frac{d}{dt}\left(P_1(t)\,e^{\lambda t}\right) = \lambda\,e^{\lambda t}\,P_0(t) = \lambda e^{\lambda t} e^{-\lambda t} = \lambda$$

$$\Rightarrow \;\; e^{\lambda t}P_1(t) = \lambda t + c \;\; \Rightarrow \;\; P_1(t) = e^{-\lambda t}(\lambda t + c)$$

- Since $P_1(0) = Pr[N(0) = 1] = 0$, $c = 0$
  Hence $P_1(t) = \lambda t\, e^{-\lambda t}$

- We showed: $P_0(t) = e^{-\lambda t}$ and $P_1(t) = \lambda t e^{-\lambda t}$
- We need to show: $P_k(t) = e^{-\lambda t} \ \frac{(\lambda t)^k}{k!}$
- Assume it is true till $k = n - 1$

$$\frac{d}{dt}\left(P_n(t)\, e^{\lambda t}\right) = \lambda\, e^{\lambda t}\, P_{n-1}(t) = \lambda e^{\lambda t} e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} = \lambda^n \frac{t^{n-1}}{(n-1}$$

$$\Rightarrow \quad e^{\lambda t} P_n(t) = \lambda^n \frac{t^n}{n} \frac{1}{(n-1)!} + c \quad \Rightarrow \quad P_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

where $c = 0$ because $P_n(0) = 0$.

- This completes the proof that Definition 2 implies Definition 1

- **Definition 1** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
    1. $N(0) = 0$
    2. The process has stationary and independent increments
    3. $Pr[N(t) = n] = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \ n = 0, 1, \cdots$
- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
    1. $N(0) = 0$
    2. The process has stationary and independent increments
    3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
       $Pr[N(h) \geq 2] = o(h)$

- Now we prove Definition 1 implies Definition 2
- We need to only show point(3) of Definition 2 using point (3) of Definition 1

$$\text{Let} \quad Pr[N(t) = k] = e^{-\lambda t} \, \frac{(\lambda t)^k}{k!}$$

$$Pr[N(h) = 1] = \lambda \, h \, e^{-\lambda h} = \lambda \, h + \lambda \, h \left(e^{-\lambda h} - 1 \right) \, = \lambda \, h + o(h)$$

because

$$\lim_{h \to 0} \frac{\lambda \, h \left(e^{-\lambda h} - 1 \right)}{h} = \lim_{h \to 0} \lambda \left(e^{-\lambda h} - 1 \right) \, = 0$$

- We showed $Pr[N(h) = 1] = \lambda h + o(h)$

- Now we need to show $Pr[N(h) \geq 2] = o(h)$

$$\begin{aligned} Pr[N(h) \geq 2] &= 1 - Pr[N(h) = 0] - Pr[N(h) = 1] \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} \end{aligned}$$

- This goes to zero as $h \to 0$
- We can use L'Hospital rule

$$\lim_{h \to 0} \frac{1 - e^{-\lambda h} - \lambda h e^{-\lambda h}}{h} = \lim_{h \to 0} \frac{\lambda e^{-\lambda h} - \lambda e^{-\lambda h} + \lambda^2 h e^{-\lambda h}}{1} = 0$$

- This completes the proof that Definition 2 implies Definition 1

These two definitions are equivalent

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  3. $Pr[N(t) = n] = e^{-\lambda t} \, \frac{(\lambda t)^n}{n!}, \; n = 0, 1, \cdots$
- **Definition 2** A counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if
  1. $N(0) = 0$
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  3. $Pr[N(h) = 1] = \lambda h + o(h)$ and
     $Pr[N(h) \geq 2] = o(h)$

- Since the process has stationary increments, for $t_2 > t_1$,

$$\begin{aligned} Pr[N(t_2) - N(t_1) = k]] &= Pr[N(t_2 - t_1) - N(0) = k] \\ &= e^{-\lambda(t_2 - t_1)} \frac{(\lambda(t_2 - t_1))^k}{k!} \end{aligned}$$

- The first order distribution of the process is: $N(t) \sim \text{Poisson}(\lambda t)$
- This, along with stationary and independent increments property determines all distributions

$$\begin{aligned} &Pr[N(t_1) = n_1, N(t_2) = n_2, N(t_3) = n_3] \\ &\quad = Pr[N(t_1) = n_1] \; Pr[N(t_2) - N(t_1) = n_2 - n_1] \\ &\quad\quad Pr[N(t_3) - N(t_2) = n_3 - n_2] \\ &= Pr[N(t_1) = n_1] \; Pr[N(t_2 - t_1) = n_2 - n_1] \; Pr[N(t_3 - t_2) = n_3 - n_2 \end{aligned}$$

where we assumed $t_1 < t_2 < t_3$

- We can easily calculate mean and autocorrelation of the process

$$\eta_N(t) = E[N(t)] = \lambda\, t \quad \Rightarrow \quad \text{not stationary}$$

With $t_2 > t_1$, we have

$$\begin{aligned}
R_N(t_1, t_2) &= E[N(t_2)N(t_1)] \\
&= E[N(t_1)(N(t_2) - N(t_1) + N(t_1))] \\
&= E[N(t_1)(N(t_2) - N(t_1))] + E[N(t_1)^2] \\
&= E[N(t_1)]\ E[N(t_2) - N(t_1)] + E[N(t_1)^2] \\
&= E[N(t_1)]\ E[N(t_2 - t_1)] + E[N(t_1)^2] \\
&= \lambda t_1(\lambda(t_2 - t_1)) + (\lambda t_1 + \lambda^2 t_1^2) \\
&= \lambda t_1 + \lambda^2 t_1 t_2
\end{aligned}$$

$$\Rightarrow \quad R_N(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda\ \min(t_1, t_2)$$

Inter-arrival or waiting times

- Let T_1 denote the time of first event and let T_n denote the time between n^{th} and $(n-1)st$ events.
- Let $S_n = \sum_{i=1}^{n} T_i$ – time of n^{th} event

$$Pr[T_1 > t] = Pr[N(t) = 0] = e^{-\lambda t}$$

$$\Rightarrow \quad T_1 \sim \text{exponential}(\lambda)$$

$$\begin{aligned} Pr[T_2 > t | T_1 = s] &= Pr[0 \ \text{ events in } \ (s, \ s+t] \mid T_1 = s] \\ &= Pr[0 \ \text{ events in } \ (s, \ s+t] \ ] = e^{-\lambda t} \\ \Rightarrow \ Pr[T_2 > t] &= \int Pr[T_2 > t | T_1 = s] \ f_{T_1}(s) \ ds = e^{-\lambda t} \end{aligned}$$

- T_n are iid exponential with parameter λ

- The time of n^{th} event is

$$S_n = \sum_{i=1}^{n} T_i$$

Since T_i are iid, exponential, S_n is Gamma with parameters n, λ

- Let $s < t$.

$$\begin{aligned} Pr[T_1 < s | N(t) = 1] &= \frac{Pr[T_1 < s,\ N(t) = 1]}{Pr[N(t) = 1]} \\ &= \frac{Pr[1 \text{ event in } (0,\ s),\ 0 \text{ in } [s,\ t]]}{Pr[N(t) = 1]} \\ &= \frac{\lambda s e^{-\lambda s}\ e^{-\lambda(t-s)}}{\lambda t e^{-\lambda t}} \\ &= \frac{s}{t} \end{aligned}$$

- Conditioned on $N(t) = 1$, T_1 is uniform over $[0, t]$

- This can be used, e.g., in simulating Poisson process
- We can cut time axis into small intervals of length h.
- In each interval we can decide whether or not there is an event, with prob λh.
- If there is an event, we choose its time uniformly in the interval.
- Called Bernoulli approximation of Poisson process
- We could also generate Poisson process by generating independent exponential random variables

Examples

- We look at a few simple example problems using Poisson process.

$$
\begin{aligned}
E[N(4) - N(2) | N(1) = 3] &= E[N(4) - N(2)] \\
&= E[N(2) - 0] = 2\lambda
\end{aligned}
$$

- Another example;

$$
E[S_4] = E\left[\sum_{i=1}^{4} T_i\right] = \frac{4}{\lambda}
$$

- The memoryless property of exponential rv can be useful

$$
Pr[S_3 > t | N(1) = 2] = \begin{cases} 1 & \text{if } \ t < 1 \\ e^{-\lambda(t-1)} & \text{if } \ t \geq 1 \end{cases}
$$

- We can explicitly derive this (taking $t > 1$)

$$
\begin{aligned}
Pr[S_3 > t | N(1) = 2] &= \frac{Pr[S_3 > t,\ N(1) = 2]}{Pr[N(1) = 2]} \\
&= \frac{Pr[2 \text{ event in } (0,\ 1],\ 0 \text{ in } (1,\ t)]}{Pr[N(1) = 2]} \\
&= \frac{Pr[2 \text{ event in } (0,\ 1]]\ Pr[0 \text{ in } (1,\ t)]}{Pr[2 \text{ event in } (0,\ 1]]} \\
&= e^{-\lambda(t-1)}
\end{aligned}
$$

- Here is another example

$$E[S_4 | N(1) = 2] = 1 + E[S_2] = 1 + \frac{2}{\lambda}$$

Exercise for you: calculate $Pr[S_4 > t | N(1) = 2]$ and use it to find the above expectation

Example

- Given a specific T_0 we want to guess which is the last event before T_0.
- Consider a strategy: we will wait till $T_0 - \tau$ and pick the next event as the last one before T_0.
- The probability of winning for this is

$$Pr[\text{exactly 1 event in } (T_0 - \tau,\ T_0)] = \lambda \tau e^{-\lambda \tau}$$

- We pick τ to maximize this

$$\lambda e^{-\lambda \tau} - \lambda^2 \tau e^{-\lambda \tau} = 0 \ \ \Rightarrow \ \ \tau = \frac{1}{\lambda}$$

- Intuitively reasonable because expected inter-arrival time is $\frac{1}{\lambda}$

- Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ
- Suppose each event can be one of two types – Typ-I or Typ-II
  - $N_1(t) =$ number of Typ-I events till t
  - $N_2(t) =$ number of Typ-II events till t
  - Note that $N(t) = N_1(t) + N_2(t),\ \forall t$
- Suppose that, independently of everything else, an event is of Typ-I with probability p and Typ-II with probability $(1-p)$

  Theorem: $\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are Poisson processes with rate λp and $\lambda(1-p)$ respectively, and they are independent

$$
\begin{aligned}
&Pr[N_1(t) = n, N_2(t) = m] \\
&= \sum_k Pr[N_1(t) = n, N_2(t) = m \mid N(t) = k] \; Pr[N(t) = k] \\
&= Pr[N_1(t) = n, N_2(t) = m \mid N(t) = m+n] \; Pr[N(t) = m+n] \\
&= \; {}^{m+n}C_n \; p^n \; (1-p)^m \; e^{-\lambda t} \; \frac{(\lambda t)^{m+n}}{(m+n)!} \\
&= \frac{(m+n)!}{m! \; n!} \; p^n \; (1-p)^m \; e^{-\lambda(p+1-p)t} \; \frac{(\lambda t)^m (\lambda t)^n}{(m+n)!} \\
&= \frac{(\lambda p t)^n}{n!} \; e^{-\lambda p t} \; \frac{(\lambda(1-p)t)^m}{m!} \; e^{-\lambda(1-p)t}
\end{aligned}
$$

- This shows that $N_1(t)$ and $N_2(t)$ are independent Poisson

- The interesting issue here is that $N_1(t)$ and $N_2(t)$ are independent.
- Suppose customers arrive at a bank as a Poisson process with rate 12 per hour.
- Independently of everything, an arriving customer is male or female with equal probability.
- Q: Given that on some day 6 male customers came in the first half hour, what is the expected number of female customers in that half hour?
- The answer is 3 because the two processes are independent

- The theorem is easily generalized to multiple types for events
- Consider Poisson process with rate λ
- Suppose, independently of everything, an event is Typ-i with probability p_i, $i = 1, \cdots K$.
- Note we have $\sum_{i=1}^{K} p_i = 1$
- Let $N_i(t)$ be the number of Typ-i customers till t
- Then, these are independent Poisson processes with rates λp_i, $i = 1, \cdots , K$

- Superposition of independent Poisson processes also gives Poisson process.
- If N_1 and N_2 are independent Poisson processes with rates λ_1 and λ_2 then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$
- We know that sum of independent Possion rv's is Poisson

- Suppose number of radioactive particles emitted is Poisson with rate λ.
- We are counting particles using a sensor
- Suppose (independent of everything) an emitted particle is detected by our sensor with probability p
- Given that we detected K particles till t what is the expected number of particles emitted?
- Let these processes be $N(t), N_1(t), N_2(t)$

$$\begin{aligned} E[N(t) | N_1(t) = K] &= E[N_1(t) + N_2(t) | N_1(t) = K] \\ &= K + E[N_2(t)] = K + \lambda(1-p)t \end{aligned}$$

where we have used independence of N_1 and N_2

- There is an interesting generalization of this.
- Events are of different types
- The type of an event can depend on the time of occurrence but it is independent of everything else.
- Suppose an event occurring at time t is Typ-i with probability $p_i(t)$.
- $p_i(t) \geq 0, \ \forall i, t$ and $\sum_{i=1}^{K} p_i(t) = 1, \ \forall t$
- $N_i(t)$ is the number of Typ-i events till t

  Theorem; Then, at any t, $N_i(t), i = 1, \cdots K$ are independent Poisson random variables with

$$E[N_i(t)] = \lambda \int_0^t p_i(s) \, ds$$

Example: Tracking infections

- We use a simple model
- Individuals get infected as a Poisson process with rate λ
- Time between getting infected and showing symptoms is a random variable with known distribution function G
  An individual infected at s would show symptoms by t with probability $G(t - s)$
- The incubation times of different infected individuals are iid
- Define
  - $N(t)$ – total number infected till t
  - $N_1(t)$ – number showing symptoms by t
  - $N_2(t)$ – infected by t but not showing symptoms

- Define two types of events. We take t as current time and treat it as fixed
  - An event occurring at s is Typ-1 with probability $G(t-s)$
  - It is Typ-2 with probability $1-G(t-s)$
- Then, Typ-1 individuals are those showing symptoms by t
- From our theorem,

$$E[N_1(t)] = \lambda \int_0^t G(t-s)\, ds = \lambda \int_0^t G(y)\, dy$$

$$E[N_2(t)] = \lambda \int_0^t (1-G(t-s))\, ds = \lambda \int_0^t (1-G(y))\, dy$$

- Suppose we have n_1 people showing symptoms at t
- We can approximate

$$n_1 \approx E[N_1(t)] = \lambda \int_0^t G(y)\ dy$$

- Hence we can estimate

$$\hat{\lambda} = \frac{n_1}{\int_0^t G(y)\ dy}$$

- Using this we can approximate

$$E[N_2(t)] \approx \hat{\lambda} \int_0^t (1 - G(y))\ dy$$

- The Poisson process we considered is called homogeneous because the rate is constant.
- For a non-homogeneous Poisson process the rate can be changing with time.
- But we can still use a definition similar to definition 2

$$Pr[N(t+h) - N(t) = 1] = \lambda(t)h + o(h)$$

- We still stipulate independent increments though we cannot have stationary increments now
- One can show that $N(t+s) - N(t)$ is Poisson with parameter $m(t+s) - m(t)$ where $m(\tau) = \int_0^\tau \lambda(s)\, ds$
- Suppose Y_i are iid and ind of $N(t)$. Then

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

is called a compound Poisson process