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- CLT is also important to get information on rate of convergence of law of large numbers.

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- For a Markov chain, given the current state, the future evolution is independent of the history of how you reached the current state

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- If S is finite then P can be represented as a matrix

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- This shows that the transition probabilities, P, and initial state probabilities, π_0, completely specify the chain.

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- This can easily be seen through a graphical notation.

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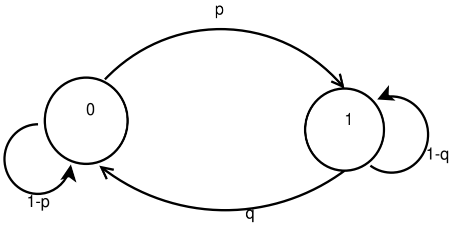

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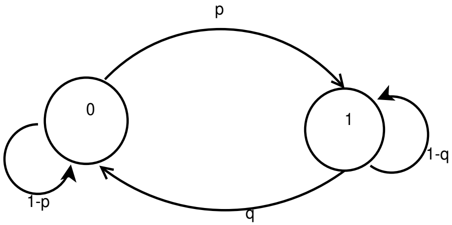

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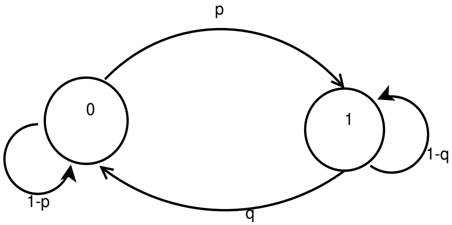

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- $S = \{0, 1, \cdots, 5\}$. The transition probabilities are

$$P = \left[\begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1-p & p \\ 2 & 0 & 0 & 1-p & p & 0 \\ 3 & 0 & 1-p & p & 0 & 0 \\ 4 & 1-p & p & 0 & 0 & 0 \end{array}\right]$$

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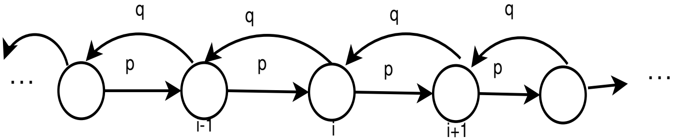

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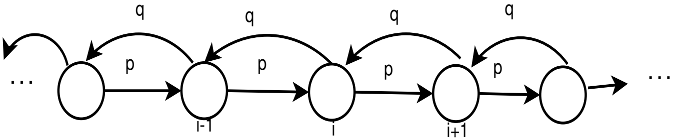

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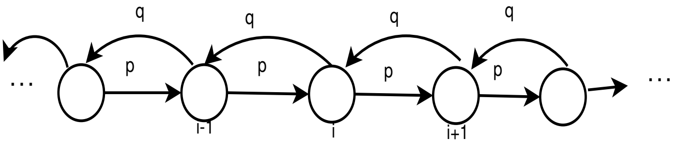

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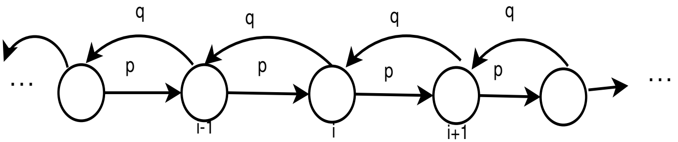

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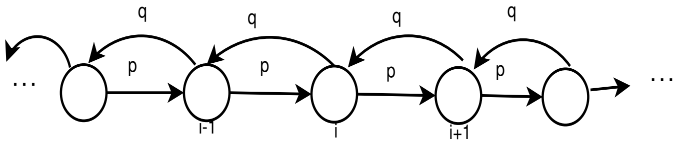

- In general, birth-death chains may have self-loops on states
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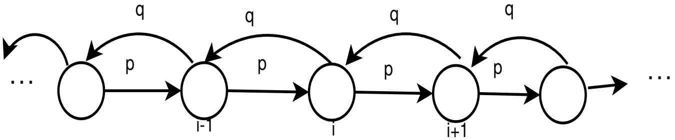

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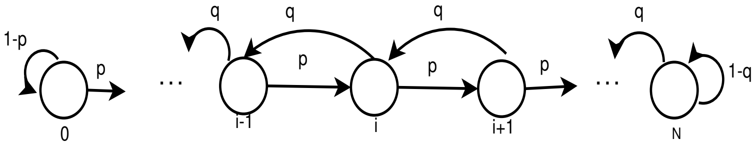
- Queuing chains can also be birth-death chains

- We can have birth-death chains with finite state space also



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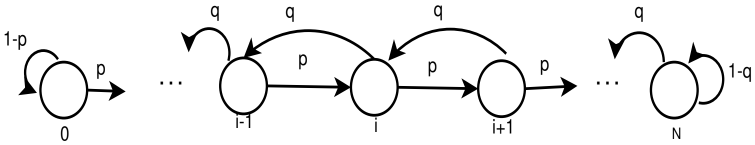

- This chain keeps visiting all the states again and again

Gambler's Ruin chain

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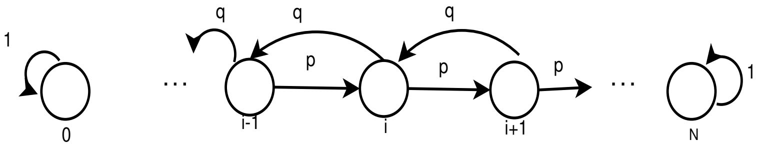

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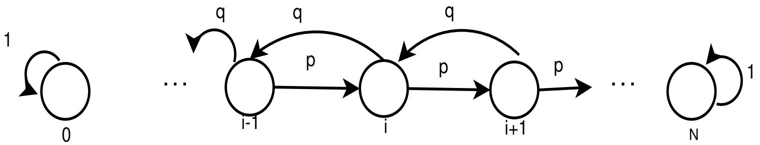

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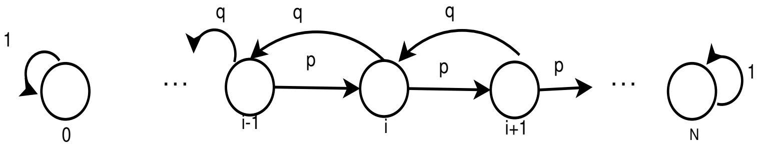

- Here, the chain is ultimately absorbed either in 0 or in N
- Here state can be the current funds that the gambler has

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- This relationship is intuitively clear

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- That is why we use P^n for n-step transition probabilities

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- For any state y define

$$I_y(X_n) = \begin{cases} 1 & \text{if } X_n = y \\ 0 & \text{otherwise} \end{cases}$$

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P_x(N_y = 0) &= 1 - P_x(N_y \geq 1) = 1 - \rho_{xy}
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- $G(x,y)$ is the expected number of visits to y for a chain that is started in x.

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$$P_x[N_y = \infty] = \rho_{xy}, \ \text{ and } \ G(x,y) = \begin{cases} 0 & \text{if } \ \rho_{xy} = 0 \\ \infty & \text{if } \ \rho_{xy} > 0 \end{cases}$$