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- For a Markov chain, given the current state, the future evolution is independent of the history of how you reached the current state

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- Similarly, $\pi_n(x) = Pr[X_n = x]$

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- For a finite chain, the n-step transition probability matrix is n-fold product of the transition probability matrix

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(i). Let y be transient. Then

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$$P_x[N(y) = \infty] = \rho_{xy}, \ \text{ and } \ G(x,y) = \begin{cases} 0 & \text{if } \ \rho_{xy} = 0 \\ \infty & \text{if } \ \rho_{xy} > 0 \end{cases}$$

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  Theorem: If x is recurrent and x leads to y then y is recurrent and $\rho_{xy} = \rho_{yx} = 1$.

Recap: closed and irreducible sets

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- A set of states, $C \subset S$, is said to be closed if $x \in C,\ y \notin C$ implies $\rho_{xy} = 0$.
- Once the chain visits a state in a closed set, it cannot leave that set.

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  where C_i are closed and irreducible
- We can calculate absorption probabilities for C_i using

$$\rho_C(x) = \sum_{y \in C} P(x, y) + \sum_{y \in S_T} P(x, y) \; \rho_C(y)$$

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- The stationary distribution, when it exists, is related to expected fraction of time spent in different states.

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- We will first establish a limit for the above as $n \to \infty$

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- This is intuitively obvious

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- Convergence would be with probability one.

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$$= Pr[X_j \neq y, \; 1 \leq j \leq k_3 - 1, \;\; X_{k_3} = y \mid X_0 = y]$$
$$= P_y[W_y^1 = k_3]$$

- In general, we get

$$Pr[W_y^r = k_r \mid W_y^{r-1} = k_{r-1}, \cdots, W_y^1 = k_1] = P_y[W_y^1 = k_r]$$

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- Note that this is true even if $m_y = \infty$

- For all n such that $N_n(y) \geq 1$, we have

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- Hence we get

$$\lim_{n\to\infty} \frac{n}{N_n(y)} = m_y, \quad w.p.1$$

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- Let us characterize y for which $m_y = \infty$

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- Thus the limiting fraction of time spent by the chain in transient and null recurrent states is zero.

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Summing the above for $m = 1, 2, \cdots n$ and dividing by n

$$\frac{1}{n}\sum_{m=1}^{n} P^{n_1+m+n_0}(y,y) \geq P^{n_1}(y,x) \ \frac{1}{n}\sum_{m=1}^{n} P^m(x,x) \ \ P^{n_0}(x,y), \ \forall n$$

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If we now let $n \to \infty$, the RHS goes to $P^{n_1}(y,x) \ \frac{1}{m_x} \ P^{n_0}(x,y) > 0$.

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$$\Rightarrow \;\; \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n} P^{n_1+m+n_0}(y,y) = \frac{1}{m_y}$$

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which implies y is positive recurrent

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- We next show that a finite chain cannot have any null recurrent states.

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Example of null recurrent chain

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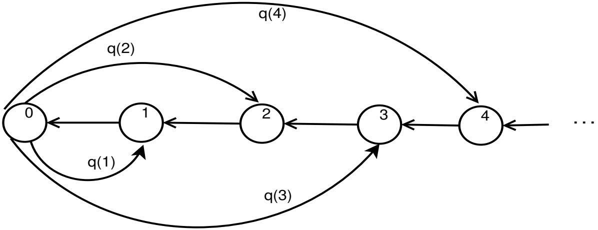

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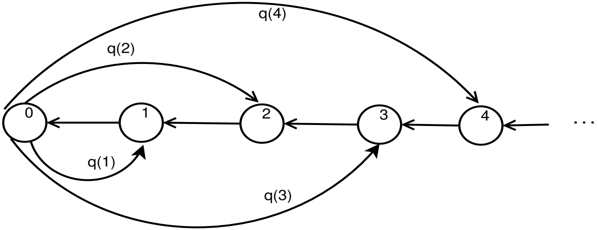

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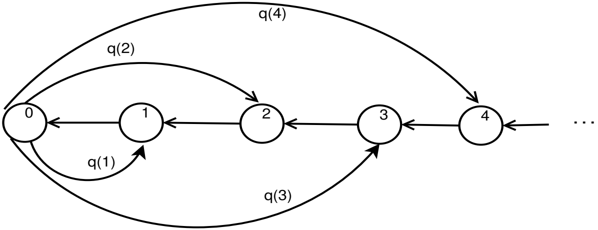

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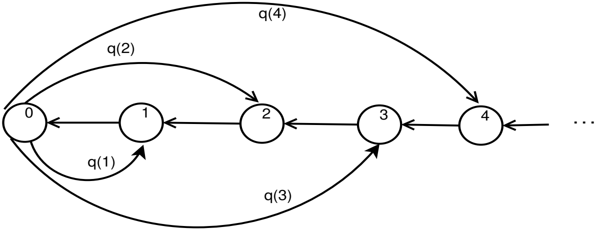

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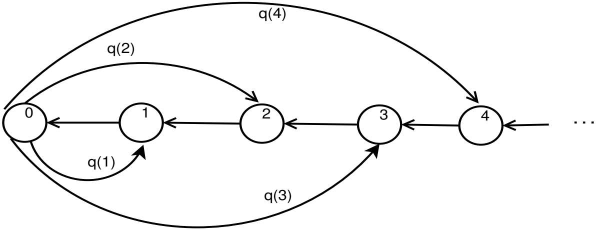

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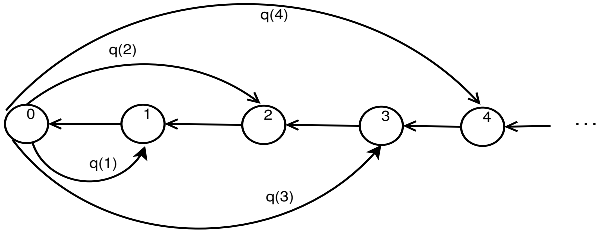

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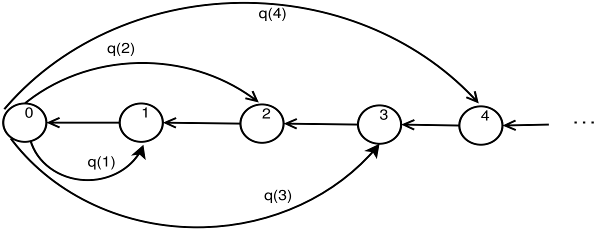

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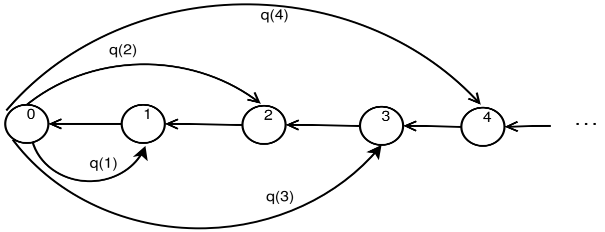

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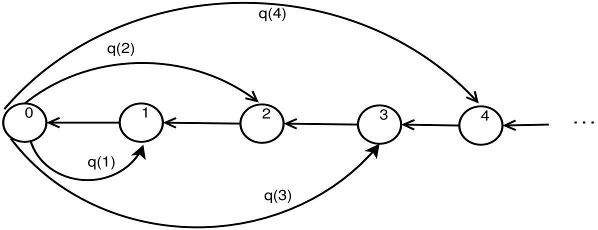

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- The proof is complete if we can take the limit inside the sum

- **Bounded Convergence Theorem**: Suppose $a(x) \geq 0, \ \forall x \in S$ and $\sum_x a(x) < \infty$. Let $b_n(x), \ x \in S$ be such that $|b_n(x)| \leq K, \ \forall x, n$ and suppose $\lim_{n\to\infty} b_n(x) = b(x), \forall x \in S$. Then

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- This answers all questions about existence and uniqueness of stationary distributions

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  For example, $a_n = (-1)^n$

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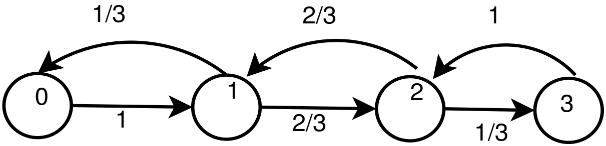
- However, P^n goes to different limits based on whether n is even or odd

- The chain is the folowing



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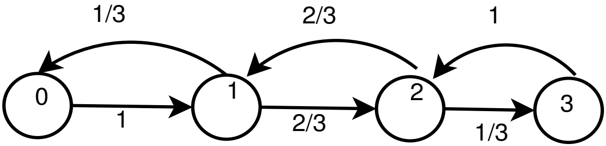

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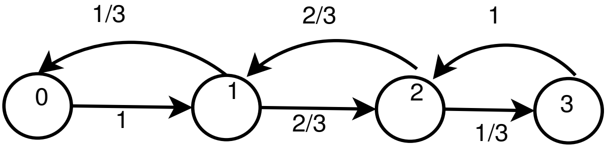

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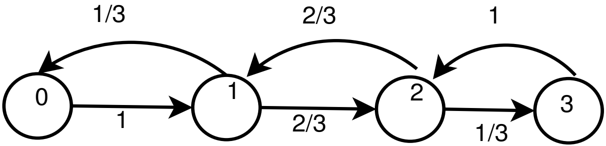

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- That is why P^n does not go to a limit
- Such a chain is called a periodic chain

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- If the period is 1 then chain is called aperiodic

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- For an aperiodic, irreducible, positive recurrent chain, there is a unique stationary distribution and π_n converges to it irrespective of what π_0 is.
- An aperiodic, irreducible, positive recurrent chain is called an ergodic chain