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- If X, Y are independent, $E[h(X)|Y] = E[h(X)]$

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$$P(A) = E[I_A] = E\left[\ E\left[I_A|Y\right]\ \right]$$

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- Such an N is an example of what is called a stopping time.

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- IF X, Y are *iid* then $P[X < Y] = 0.5$

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- So, we get: $P[X = k] = \frac{1}{n+1},\ k = 0, 1, \cdots, n$

Gaussian or Normal distribution

- The Gaussian or normal density is given by

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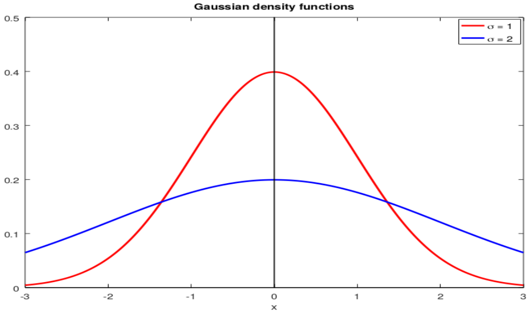
- The density is a 'bell-shaped' curve



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- We will now show that this is a joint density function.

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- Then for any $\mathbf{z} \in \Re^n$,

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- This is the multidimensional Gaussian distribution

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- Recall $|L| = 1$, $|M^{-1}| = (m_1 \cdots m_n)^{-1}$

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This shows that $Z_i \sim \mathcal{N}(0, \frac{1}{m_i})$ and Z_i are independent.

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- Thus, if $\mathbf{Y}$ has density

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Multi-dimensional Gaussian density

- $\mathbf{X} = (X_1, \cdots, X_n)^T$ are said to be jointly Gaussian if

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- This implies X_i are independent.
- If $X_1, \cdots, X_n$ are jointly Gaussian then uncorrelatedness implies independence.

- Let $\mathbf{X} = (X_1, \cdots, X_n)^T$ be jointly Gaussian:

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- Then we saw that $Z_i \sim \mathcal{N}(0, \frac{1}{m_i})$ and Z_i are independent.
- If $X_1, \cdots, X_n$ are jointly Gaussian then there is a 'linear' transform that transforms them into independent random variables.

Moment generating function

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- This is the moment generating function of multi-dimensional Normal density

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- Exercise for you – show all this starting with the joint density we have
- Note that X, Y are individually Gaussian does not mean they are jointly Gaussian (unless they are independent)

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- We will prove this using moment generating functions