E1 222 Stochastic Models and Applications
Test I

`Time: 75 minutes` `Max. Marks:40`
`Date: 23 Nov 2020`

Answer **ALL** questions. All questions carry equal marks

1. a. Let A, B, C be events in a probability space. If B and C are independent show that

$$P(A|B) = P(A|BC)P(C) + P(A|BC^c)P(C^c)$$

   b. Two numbers are drawn at random with replacement from $\{1, 2, \cdots, N\}$. Calculate the probability that one of the numbers is less than or equal to half of the other number. Assume N is even.

2. a. Let X be a continuous random variable with density function

$$f_X(x) = K(3x - x^2), \ \ 0 < x < 2$$

   Find the value of K, F_X, EX, and $P[X < 1]$.

   b. Suppose X is a continuous random variable with density $f_X(x) = -\ln(x), \ 0 < x < 1$. Find the distribution function of X. Let $Y = X - X\ln(X)$. Find the density of Y and EY.

3. a. Let X be a continuous random variable having exponential density with parameter λ. For any given $\epsilon > 0$, let X_ϵ be defined by

$$X_\epsilon = \epsilon k \ \text{ if } \ \epsilon k \leq X < \epsilon(k+1), \ k \text{ integer}.$$

   Find EX_ϵ and its limit as $\epsilon \to 0$.

   b. Let X be a discrete random variable having geometric distribution with parameter p. Let $M > 0$ be an integer. Define $Y = \max(X, M)$. Find distribution of Y.

4. a. Let X be a non-negative integer valued random variable. Let $\Phi_X(t) = Et^X$ be its probability generating function and assume that $\Phi_X(t)$ is finite for all t. Show that for any positive integer, y,

$$P[X \leq y] \leq \frac{\Phi_X(t)}{t^y}, \ 0 \leq t \leq 1$$

b. Let X, Y be random variables with joint density given by

$$f_{XY}(x,y) = e^{-y}, \;\; 0 < x < y < \infty$$

Find the marginal densities of X, Y and $P[X \leq \frac{y}{2} \mid Y = y]$