# E0 230 : Computational Methods of Optimization
# Assignment 4

**Instructions:**

- Attempt all questions
- For Numerical Answer questions, insert only the digits **with no leading or trailing spaces** in the MS Teams Form provided
- You can submit the MS Teams Form **only once**. No further changes can be made, so click on *Submit* wisely.
- For the file `a4.csv`, each column is a 3d vector of the form $[x_i, y_i]$; $x_i \in \mathbb{R}^2$, and $y_i \in \{\pm 1\}$.
- Late Submissions will be penalised.

1. Consider the problem

$$x^* = \arg\min \frac{1}{2}\|Cx + d\|^2 \text{ such that } Ax = b$$

   (a) Suppose we want to project a point $u$ onto the set $\{x : Ax = b\}$, where $A$ is full rank and $b$ is nonzero. What is the closed form solution to the projection $x$?
      A. $x = u - A^T(AA^T)^{-1}(b - Au)$
      B. $x = u - A^T(A^TA)^{-1}(b - Au)$
      C. $x = u - A^T(AA^T)^{-1}(Au - b)$
      D. $x = u - A^T(A^TA)^{-1}(Au - b)$

   (b) Suppose $A$ is of full rank, $C^TC$ is invertible and $b$ is nonzero. What is the closed form solution to this problem?
      A. $x^* = (C^TC)^{-1}A^T(A(C^TC)^{-1}A^T)^{-1}(b + A(C^TC)^{-1}C^Td) - (C^TC)^{-1}C^Td$
      B. $x^* = (C^TC)^{-1}A(A^T(C^TC)^{-1}A)^{-1}(b - A^T(C^TC)^{-1}Cd) - (C^TC)^{-1}C^Td$
      C. $x^* = (C^TC)^{-1}A^T(A^TA)^{-1}b - (C^TC)^{-1}C^Td$
      D. $x^* = (C^TC)^{-1}A(AA^T)^{-1}b - (C^TC)^{-1}C^Td$

   (c) Suppose

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \; C = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; d = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

   Starting at $(0, 0, 0)$, how many iterations of projeted gradient descent with stepsize $\eta = 0.5$ are required to reach a point $z$ such that $\|z - x^*\| \leq 0.001$?

   (d) Starting at $(0, 0, 0)$, how many iterations of projeted gradient descent with stepsize $\eta = 0.25$ are required to reach a point $z$ such that $\|z - x^*\| \leq 0.001$?

   (e) Starting at $(0, 0, 0)$, how many iterations of projeted gradient descent with stepsize $\eta = 2.5$ are required to reach a point $z$ such that $\|z - x^*\| \leq 0.001$?

Consider the sets $S(0, 1) := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $S(a, r) := \{x \in \mathbb{R}^n : \|x - a\|^2 \leq r^2\}$.

2. (a) What is the projection $x$ of a point $y$ onto $S(0, 1)$?
      A. $x = \frac{1}{\|y\|}y$ for all $y$.
      B. $x = y\frac{1}{\max\{1, \|y\|\}}$ for all $y$
      C. $x = y\frac{1}{\min\{1, \|y\|\}}$ for all $y$

   (b) What is the projection $x$ of a point $y$ onto $S(a, r)$?
      A. $x = \frac{r}{\|y-a\|}(y - a)$ for all $y \in \mathbb{R}^n \backslash S(a, r)$
      B. $x = \frac{r}{\|y-a\|}(y - a) + a$ for all $y \in \mathbb{R}^n \backslash S(a, r)$
      C. $x = \frac{r}{\|y-a\|}(y - a) - a$ for all $y \in \mathbb{R}^n \backslash S(a, r)$

   (c) Suppose $S \subset \mathbb{R}^2$, $S := S(0, 1) \cap S((-1, 1), 1/\sqrt{2})$. Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix},$$

   and consider the problem

$$x^* = \arg\min_{x \in S} \|Ax + b\|.$$

   We'll solve this problem with projected gradient descent. How many iterations are needed to reach a point $z$ such that $\|z - x^*\| \leq 0.0001$ with a stepsize $\alpha = 0.01$, starting at $x_0 = (3, 0)$ and what is the final answer?
   Use the following algorithm to project any point $z \in \mathbb{R}^2$ onto $S$:
   Algorithm:

1. Initialize: $C = S(0,1)$, $D = S((-1,1), 1/\sqrt{2})$, $z \in \mathbb{R}^2$ $w_0 = P_C(z)$
2. while $\|w_k - y_k\| \geq 0.0001$, Compute:
   (a) $y_k = P_D(x_k)$
   (b) $w_{k+1} = P_C(y_k)$
3. Output $w = w_{k+1} = P_S(z)$

3. Consider the problem

$$\begin{aligned} &\arg\min_{w,b} \ \frac{1}{2}\|w\|^2 \\ &\text{such that } \ y_i(w^T x_i + b) \geq 1 \end{aligned}$$

where $y_i \in \{\pm 1\}$, $x_i \in \mathbb{R}^n$ are given scalars. We have provided data $\{(x_i, y_i)\}_{i=1}^{10}$ in `a4.csv`.

(a) Consider the problem stated above. We will use active set methods to solve this problem. Initialize the problem with $w0 = [132.1473, -53.9619, 0]$,. What is the initial working set?

(b) Find the feasible direction for this set.

(c) What is the next step you need to take?

A. Find the blocking constraints

B. Check the feasibility of $x + u$

C. Compute the Lagrange multipliers/

(d) If you chose (a), what is the optimal stepsize? If you chose (b), is the point feasible? If you chose (c), what is the smallest Lagrange multiplier?

(e) What is the working set after you complete this iteration?

4. Consider the problems

$$m_1 = \arg\min_m \sum_i \|x_i - m\|_1$$

and

$$m_\infty = \arg\min_m \sum_i \|x_i - m\|_\infty.$$

We will use the data given in `a4.csv` to answer the following questions. You may use any LP solver you please, such as `linprog`, `sedumi`, or `cvx/cvxpy`.

(a) Reformulate the problem of finding $m_\infty$ into a linear program, and solve it.

A. $m_\infty = (-.116, .167)$

B. $m_\infty = (0.821, 0.733)$

C. $m_\infty = (.137, .924)$

D. $m_\infty = (-.137, .924)$

(b) Now consider the problem

$$x^* = \arg\min \|x\|_1 \text{ such that } Ax = b.$$

We aim to solve this problem with linear programming. Consider the following change of variables: $x = u - v$, where $u_i = \max\{x_i, 0\}$ and $v_i = \max\{-x_i, 0\}$. Using this change of variables, convert the problem into a linear program and solve it. Use the following problem data:

$$A = \begin{bmatrix} 0.8147 & 0.6324 & 0.9575 & 0.9572 \\ 0.9058 & 0.0975 & 0.9649 & 0.4854 \\ 0.1270 & 0.2785 & 0.1576 & 0.8003 \\ 0.9134 & 0.5469 & 0.9706 & 0.1419 \end{bmatrix}, \quad b = \begin{bmatrix} 0.4218 \\ 0.9157 \\ 0.7922 \\ 0.9595 \end{bmatrix}$$

A. $x = (0.1060, 0.5418, 0.0233, 0.5866)$
B. $x = (3.1400, -12.9328, 4.2818, -1.8395)$.
C. $x = (17.268, 0.838, -15.893, 1.088)$
D. $x = (2.8825, -0.5676, 0.9225, 10.0653)$

(c) We will now try and convert the problem of finding $m_1$ into a linear program as well. Suppose we use the substitution of variables we defined in the previous part. Reformulate the problem of finding $m_1$ into a linear program, and find the answer.

A. $m_1 = (-0.8407, 0.1580)$
B. $m_1 = (0.0032, 0.0264)$
C. $m_1 = (-0.1660, -0.0313)$
D. $m_1 = (0.0264, 0.0108)$