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- We are interested in limit of this process as $T \to 0$

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- Let us intuitively see some properties of $W(t)$

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- We will now formally define Brownian motion using these properties.

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- $\{Y(t),\ t \geq 0\}$ is called Brownian motion with a drift

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- Thus, $X(t_2) - X(t_1)$ is Gaussian with zero mean and variance $\sigma^2(t_2 - t_1)$
- Since increments are also independent, we can show that all n^{th} order distributions are Gaussian

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- Since $E[X(t)] = 0, \ \forall t$, we have

$$\text{Cov}(X(t_1), X(t_2)) = E[X(t_1)X(t_2)] = \sigma^2 \ \min(t_1, t_2)$$

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- This is how we can get n^{th} order density for any continuous-state process with independent increments

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- Since $\ln(Y_i)$ are iid, with suitable normalization, the interpolated process $\ln(X(t))$ would be Brownian motion and $X(t)$ would be geometric Brownian motion

Gaussian Processes

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- Recall that the multivariate Gaussian density is specified by the marginal means, variances and the covariances of the random variables
- Hence, a general Gaussian process is specified by the mean function and the variance and covariance functions

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Recall that, for $s < t$, conditional density of $X(s)$ conditioned on $X(t) = b$ is gaussian with mean bs/t and variance $s(t-s)/t$

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Now, for $s < t < 1$, since $E[X(s)|X(1) = 0] = 0,\ s < 1$,

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\mathsf{Cov}(X(s), X(t)|X(1) = 0) &\triangleq E[X(s)X(t) \mid X(1) = 0] \\
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Thus, for $0 < t < 1$, conditioned on $X(1) = 0$, this process has mean 0 and covariance function $s(1-t), \ s < t$

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&= s - st - st + st = s(1-t)
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White Noise

- Consider a process $\{V(t),\ t \geq 0\}$ with

$$E[V(t)] = 0; \quad \text{Var}(V(t)) = \sigma^2 \quad \text{Cov}(V(t), V(s)) = 0,\ s \neq t$$

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- The actual concept involved is rather deep

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- Any sequence of continuous random variables would be a discrete-time continuous-state process

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- When X_n is a martingale, we have

$$E[X_{n+1}] = E[X_n], \ \forall n$$

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Please note that these are 'simplified' definitions

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Please note that these are 'simplified' definitions In the above, the conditioning random variables can be another sequence Y_i if $Y_1, \cdots, Y_n$ determine $X_1, \cdots, X_n$

- Martingales are useful because of the martingale convergence theorem.

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  martingale convergence theorem: If X_n is a martingale with $\sup_n E|X_n| < \infty$ then X_n converges almost surely to a rv X which will have finite expectation. A positive supermartingale also converges almost surely

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- The algorithm is

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\begin{aligned}
p(k+1) &= p(k) + \lambda(1 - p(k)) \;\; \text{if arm 1 chosen, } b(k) = 1 \\
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&\quad = \lambda(1 - p(k)) \ Pr[b(k) = 1 \mid \text{arm 1}, p(k)] \ Pr[\text{arm 1} \mid p(k)] \\
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\end{aligned}$$

$$\Rightarrow \quad E[p(k{+}1) \mid p(k)] \geq p(k)$$

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- So, we can conclude, the algorithm converges almost surely

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- While we mentioned only discrete-time martingales, one can similarly have continuous-time martingales