E1 222 Stochastic Models and Applications
Test III

`Time: 75 minutes` `Max. Marks:40`
`Date: 13 Jan 2021`

Answer **ALL** questions. All questions carry equal marks

1. a. Let $X_1, X_2, \cdots$ be a sequence of iid continuous random variables. Let their common distribution function be F and suppose it is strictly monotonically increasing. Let $M_n = \max(X_1, \cdots, X_n)$ and $Y_n = n[1 - F(M_n)],\ n = 1, 2, \cdots$. Find the limiting distribution of Y_n.

Answer: Given $M_n = \max(X_1, \cdots, X_n)$. Hence distribution of M_n is

$$F_{M_n}(z) = P[M_n \leq z] = P[X_i \leq z,\ 1 = 1, \cdots, n] = (F(z))^n$$

Since we are given that F is strictly increasing and X_i are continuous rv, we know F is invertible. Now the distribution of Y_n is

$$\begin{aligned}
F_{Y_n}(y) &= P[n[1 - F(M_n)] \leq y] \\
&= P\left[F(M_n) \geq 1 - \frac{y}{n}\right] \\
&= P\left[M_n \geq F^{-1}\left(1 - \frac{y}{n}\right)\right] \\
&= 1 - P\left[M_n \leq F^{-1}\left(1 - \frac{y}{n}\right)\right] \\
&= 1 - \left(F\left(F^{-1}\left(1 - \frac{y}{n}\right)\right)\right)^n \\
&= 1 - \left(1 - \frac{y}{n}\right)^n
\end{aligned}$$

Hence

$$\lim_{n\to\infty} F_{Y_n}(y) = 1 - \lim_{n\to\infty}\left(1 - \frac{y}{n}\right)^n = 1 - e^{-y}$$

Hence limiting distribution of Y_n is exponential.

b. Let $X_1, X_2, \cdots$ be iid continuous random variables with density $f(x) = 12x^2(1 - x),\ 0 \leq x \leq 1$. Let $S_n = \sum_{i=1}^{n} X_i^2$. Does $\frac{1}{n}S_n$ converge almost surely? Answer Yes/No with a short justification. If your answer is yes, find the limit.

Answer: Yes. Since X_i are iid, so are X_i^2 and, hence, by strong law of large numbers, $\frac{1}{n}S_n$ converges to $E[X_i^2]$.

$$E\left[X_i^2\right] = \int_0^1 x^2 \; 12x^2(1-x)\; dx = 12\left[\left.\frac{x^5}{5}\right|_0^1 - \left.\frac{x^6}{6}\right|_0^1\right] = \frac{12}{30} = \frac{2}{5}$$

So, $\frac{1}{n}S_n$ converges to $\frac{2}{5}$

2. a. Consider a Probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{1, 2, \cdots\}$, $\mathcal{F}$ is the power set of Ω and $P(\{i\}) = q_i, \; \forall i$. Note that we would have $q_i \geq 0, \forall i$ and $\sum_i q_i = 1$. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables defined on this space given by

$$\begin{aligned} X_n(\omega) &= 1 \;\text{ if }\; n \leq \omega \\ &= 0 \;\text{ otherwise} \end{aligned}$$

Does the sequence converge in (i) Probability, (ii) almost surely.

Answer: By the definition of X_n, we have

$$P[X_n = 1] = P\left(\{\omega \;:\; \omega \geq n\}\right] = P\left(\{n, n+1, \cdots\}\right) = 1 - \sum_{k=1}^{n-1} q_k$$

This goes to zero as n goes to infinity. We can verify that $X_n \xrightarrow{P} 0$ as:

$$\lim_{n\to\infty} P[|X_n - 0| > \epsilon] = \lim_{n\to\infty} P[X_n = 1] = 1 - \lim_{n\to\infty}\sum_{k=1}^{n-1} q_k = 0$$

We cannot use Borel-Cantelli lemma here for checking for almost sure convergence. Whether or not $\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} q_k < \infty$ depends on q_k.

But since we are given the random variables as functions on Ω, we can ask when does $X_n(\omega) \to 0$. Fix any ω. Then $X_n(\omega)$ is 1 till $n \leq \omega$; after that it stays zero. So, $X_n(\omega) \to 0$ as $n \to \infty$, $\forall \omega$. Hence, X_n converges almost surely to zero

b. A university has 300 vacancies for research students. Since not all students offered admission would accept, the university sends out offers of admission to 400 students. By past experiance the university knows that only 70% of students offered admission would accept the offer. Calculate the approximate probability that more than 300 students would accept the offer of admission.

Answer: Let X_i be indicator random variables representing whether or not the i^{th} student offered admission would accept. We assume that these are independent. Then X_i are iid binary random variables with $[X_i = 1] = 0.7$. Let $S_{400} = \sum_{i=1}^{400} X_i$. S_{400} would be the number of students who have accepted the offer. Hence,

$$
\begin{aligned}
P[S_{400} > 300] &= P\left[\frac{S_{400} - ES_{400}}{\sqrt{\mathrm{Var}(S_{400})}} > \frac{300 - ES_{400}}{\sqrt{\mathrm{Var}(S_{400})}}\right] \\
&= P\left[\frac{S_{400} - ES_{400}}{\sqrt{\mathrm{Var}(S_{400})}} > \frac{300 - 400 * 0.7}{\sqrt{400 * 0.7 * 0.3}}\right] \\
&= P\left[\frac{S_{400} - ES_{400}}{\sqrt{\mathrm{Var}(S_{400})}} > \frac{1}{\sqrt{0.7 * 0.3}}\right] \\
&= P\left[\frac{S_{400} - ES_{400}}{\sqrt{\mathrm{Var}(S_{400})}} > 2.18\right] \\
&\approx 1 - \Phi(2.18) = 1 - 0.985 = 0.015
\end{aligned}
$$

Comment: I did not use the so called continuity correction in the above. We could have, for example, calculated $P[S_n > 300.5] \approx 0.013$ as the required probability.

3. a. Consider an irreducible birth-death Markov chain on the state space $\{0, 1, \cdots, N\}$. If the chain is started in state 1 what is the probability that the chain will visit state $N-1$ at sometime or the other. Can this chain have a null recurrent state? Explain your answer.

Answer: Since this is a finite irreducible chain, all states are recurrent. Hence, starting from any state the probability of visiting any other state at some finite time is 1.

Since this is a finite chain it cannot have any null recurrent states. This can be established as follows. We can prove that (i). any finite closed set has to have at least one positive recurrent state and (ii). if x is positive recurrent and x leads to y then y is positive recurrent.

In a finite chain all sets of closed irreducible sets of recurrent states would be finite. Since it is a finite closed set, it should have at least one positive recurrent state and since it is irreducible, now all states in the set have to be positive recurrent. Thus each of the finite sets closed irreducibles sets of recurrent states have to be wholly positive recurrent and hence we cannot have a null recurrent state.

Comment: I am only looking for some logical explanation such as the one above. I am not looking for a formal prrof of the two results listed above. Since this is an open-notes exam, I assumed it should be clear to you that I would not be asking you to copy proofs from your notes.

b. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.15 & 0.22 & 0.1 & 0.28 & 0.25 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0.35 \\ 0 & 0 & 0 & 0.55 & 0.45 \end{bmatrix}$$

Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find a stationary distribution of the chain

Answer: Let us label the state space as $\{0, 1, \cdots, 4\}$. From the transition probability matrix, it is easy to see that the set $\{1, 2\}$ is closed and irreducible. Since this is a finite closed set, there has to be at least one recurrent state and since it is irreducible, both states in the set are recurrent. Since state 0 leads to this closed set, we can conclude 0 is transient. From the matrix we can also similarly see that the set $\{3, 4\}$ is the other closed and irreducible set of recurrent states. Thus we get

$$S_T = \{0\}, \quad S_R = \{1, 2\} + \{3, 4\}$$

For a chain like this there is always a stationary distribution concentrated on any one set of closed and irreducible set. If we pick the set $\{1, 2\}$, the transition probability matrix for this subset is doubly stochastic and hence we know what is the unique stationray distribution if we consider only this subset. Thus a stationary distribution for the chain is $\pi = [0 \;\; 0.5 \;\; 0.5 \;\; 0 \;\; 0]$

4. a. A man has n umbrellas. Everyday in the morning he goes from his house to office and takes an umbrella with him if it is raining and if he has an umbrella with him; he goes without an umbrella if it is not raining or if he has no umbrellas with him. Similarly in the evening when he goes from office to home he takes an umbrella if it is raining and he has one. The probability of rain is same in the morning and evening and it is equal to p. Construct an $n+1$ state Markov chain and using that calculate the probability that the man would be without an umbrella when it is raining. (Note that this is the generalization of the problem solved in class)

Answer: As discussed in the class a useful way to formulate this is to take the number of umbrellas with the man as the state. Thus we have the state space $\{0, 1, \cdots, n\}$.

From state 0, you can only go to n. Consider a state i, $i \geq 1$. That means the man has i umbrellas with him. So, if it is not raining the state would change to $n - i$ because he would not carry an umbrell; if it is raining the state would change to $n-i+1$ because he would carry an umbrella. Thus the transition probabilities are given by

$$P(0, n) = 1; \quad P(i, n-i) = 1-p, \;\; P(i, n-i+1) = p, \;\; i = 1, \cdots, n$$

It is easily seen that the chain is irreducible. This can be seen as follows. Suppose you want to go from state i to j and assume $i < j$. You make a transition out of i with probability $1-p$ and from the next state make a transition with probability p. Then you would be in state $i+1$. That is, currently the man has i umbrellas, if he goes to the other place without carrying an umbrella and comes back carrying an umbrella, then he would have $i+1$ umbrellas with him. Like this you can go from i to j. Similar arguments apply when $j < i$

Since this is a finite irreducible chain it has a unique stationary distribution. The stationary distribution has to satisfy

$$\pi(j) = \sum_i P(i, j)\pi(i)$$

For a given $j \neq 0$, $P(i, j) \neq 0$ only if $i = n-j$ or $i = n-j+1$. You can come to state 0 only from n Thus the stationary distribution has to satisfy

$$\begin{aligned}
\pi(0) &= (1-p)\pi(n), \quad \text{and} \\
\pi(j) &= P(n-j,j)\pi(n-j) + P(n-j+1,j)\pi(n-j+1) \\
&= (1-p)\pi(n-j) + p\pi(n-j+1), \ j = 1, \cdots, n-1 \\
\pi(n) &= \pi(0) + p\pi(1)
\end{aligned}$$

It is easy to see that all these equations would be satisfied if choose $\pi(i) = \alpha, \ i = 1, \cdots, n$ and $\pi(0) = (1-p)\alpha$. Then we need $n\alpha + (1-p)\alpha = 1$ which gives $\alpha = \frac{1}{n+1-p}$. Thus the stationary distribution is

$$\pi(j) = \frac{1}{n+1-p}, \ j = 1, \cdots, n, \quad \pi(0) = \frac{1-p}{n-1-p}$$

The probability that the man would be without an umbrella when it is raining is $p\pi(0)$.

b. Let $X_n, \ n = 1, 2, \cdots$ be discrete random variables taking values in $\{0, 1, 2, \cdots, K\}$, $K < \infty$. Suppose $X_n \xrightarrow{P} 0$. Then show that the sequence converges in r^{th} mean to zero.

Answer: Since we are given $X_n \xrightarrow{P} 0$, we have

$$\lim_{n\to\infty} P[X_n \neq 0] = 0$$

Now we have

$$0 \leq E\left[|X_n - 0|^r\right] = \sum_{j=1}^{K} j^r P[X_n = j] \leq K^r \sum_{j=1}^{K} P[X_n = j] = K^r P[X_n \neq 0]$$

Since $\lim_{n\to\infty} P[X_n \neq 0] = 0$, $\lim_{n\to\infty} E\left[|X_n - 0|^r\right] = 0$ which proves that X_n converge in r^{th} mean.

Comment: I hope it is easy to see that the restriction that all X_i take values in the same set is not needed. All we need is that there is a $K < \infty$ such that $|X_n| < K$ for all n.

In each question, each part would be graded 5 marks