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- It essentially results in an induced probability space

$$(\Omega, \mathcal{F}, P) \stackrel{X}{\rightarrow} (\Re, \mathcal{B}, P_X)$$

where $\mathcal{B}$ is the Borel σ-algebra and

$$P_X(B) = P[X \in B] = P\left(\{\omega \in \Omega \ : \ X(\omega) \in B\}\right)$$

Recap: Distribution function of a random variable

- Let X be a random variable. It distribution function, $F_X : \Re \rightarrow \Re$, is defined by

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- The distribution function, F_X, completely specifies the probability measure, P_X.

Recap: Properties of distribution function

- The distribution function satisfies
  1. $0 \leq F_X(x) \leq 1, \ \forall x$
  2. $F_X(-\infty) = 0; \ F_X(\infty) = 1$
  3. F_X is non-decreasing: $x_1 \leq x_2 \ \Rightarrow \ F_X(x_1) \leq F_X(x_2)$
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- Let $X \in \{x_1, x_2, \cdots\}$
- Its distribution function, F_X is a stair-case function with jump discontinuities at each x_i and the magnitude of the jump at x_i is equal to $P[X = x_i]$

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Recap: continuous random variable

- X is said to be a continuous random variable if there exists a function $f_X : \Re \to \Re$ satisfying

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- A continuous rv takes uncountably many distinct values. However, not every rv that takes uncountably many values is a continuous rv

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- In particular,

$$P[a \leq X \leq b] = \int_a^b f_X(t)\, dt$$

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- We can determine distribution of Y given the function g and the distribution of X

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- This probability can be obtained from distribution of X.

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- Suppose $Y = g(X)$.
- Then Y is also discrete and $Y \in \{g(x_1), g(x_2), \cdots\}$.
- We can find the pmf of Y as

$$\begin{aligned} f_Y(y) &= p[Y = y] = P[g(X) = y] \\ &= P[X \in \{x_i \ : \ g(x_i) = y\}] \\ &= \sum_{\substack{i: \\ g(x_i)=y}} f_X(x_i) \end{aligned}$$

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$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \; a \leq y \leq b$$

where $a = \min(g(\infty), \; g(-\infty))$ and $b = \max(g(\infty), \; g(-\infty))$

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- This theorem is useful in some cases to find the densities of functions of continuous random variables

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- Note that expectation is defined for all random variables

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- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$
- $E\left[(X - c)^2\right] \geq E\left[(X - EX)^2\right], \ \forall c$

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  - $\text{Var}(cX) = c^2 \text{Var}(X)$

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- If moment of order k is finite then so is moment of order s for $s < k$.

Recap: Moment Generating function

- The moment generating function – $M_X : \Re \to \Re$

$$M_X(t) = Ee^{tX} = \sum_i e^{tx_i} f_X(x_i) \ \text{ or } \int e^{tx} f_X(x)\, dx, \ \ t \in \Re$$

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- We say the mgf exists if $E[e^{tX}] < \infty$ for t in some interval around zero
- If $M_X(t)$ exists (for $t \in [-a, \ a]$ for some $a > 0$) then all its derivatives also exist and

$$\left. \frac{d^k M_X(t)}{dt^k} \right|_{t=0} = E[X^k]$$

Generating function

- For $X \in \{0, 1, 2, \cdots\}$ the (probability) generating function of X is defined by

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- We can also get the moments:

$$P'_X(1) = EX, \quad P''_X(1) = E[X(X-1)]$$

quantiles of a distribution

- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

  $$P[X \leq x] \geq p \quad \text{and} \quad p[X \geq x] \geq 1 - p$$

  is called the quantile of order p or the $100p^{th}$ percentile of rv X.

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- For a given p there can be multiple values for x to satisfy the above.

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- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

$$P[X \leq x] \geq p \quad \text{and} \quad p[X \geq x] \geq 1 - p$$

  is called the quantile of order p or the $100p^{th}$ percentile of rv X.

- If x is quantile of order p, it satisfies

$$p \leq F_X(x) \leq p + P[X = x]$$

- For a given p there can be multiple values for x to satisfy the above.
- For $p = 0.5$, it is called the median.

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- With $EX = \mu$ and $\mathsf{Var}(X) = \sigma^2$, we get

$$P[|X - \mu| > k\sigma] \leq \frac{1}{k^2}$$

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- The joint distribution function is the probability of the intersection of the events $[X \leq x]$ and $[Y \leq y]$.

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- Any $F : \Re^2 \rightarrow \Re$ satisfying the above would be a joint distribution function.

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- This is a straight-forward extension of the pmf of a single discrete rv.

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- If $\omega = 0.2576$ then $X(\omega) = 2$ and $Y(\omega) = 5$
- Easy to see that $X, Y \in \{0, 1, \cdots, 9\}$.
- We want to calculate the joint pmf of X and Y

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- Hence the joint pmf of X and Y is

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  $[X = 3, Y = 6] = \{(3, \ 3)\}$,
  $[X = 4, Y = 6] = \{(4, \ 2), \ (2, \ 4)\}$
- So, $P[X = m, Y = n]$ is either $2/36$ or $1/36$ (assuming m, n satisfy other requirements)

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- Assume $1 \leq m \leq 6$ and $2 \leq n \leq 12$. Then

$$f_{XY}(m,n) = \begin{cases} \frac{2}{36} & \text{if } \ m < n < 2m \\ \frac{1}{36} & \text{if } \ n = 2m \end{cases}$$

  ($f_{XY}(m,n)$ is zero in all other cases)
- Does this satisfy requirements of joint pmf?

$$\begin{aligned} \sum_{m,n} f_{XY}(m,n) &= \sum_{m=1}^{6} \sum_{n=m+1}^{2m-1} \frac{2}{36} + \sum_{m=1}^{6} \frac{1}{36} \\ &= \frac{2}{36} \sum_{m=1}^{6} (m-1) + \frac{1}{36} 6 \\ &= \frac{2}{36}(21-6) + \frac{6}{36} = 1 \end{aligned}$$

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- Given the joint pmf, we can get the joint df as

$$F_{XY}(x,y) = \sum_{\substack{i: \\ x_i \leq x}} \sum_{\substack{j: \\ y_j \leq y}} f_{XY}(x_i, y_j)$$

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- These are very similar to the properties of the density of a single rv

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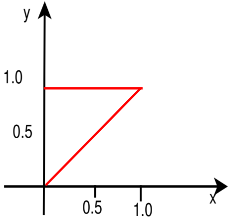
- We can say this density is uniform over the region



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- Then we can show F_{XY} is a joint distribution.

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- The only property left is the special property of F_{XY} we mentioned earlier.

$$\Delta \triangleq F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1).$$

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&\quad - \int_{-\infty}^{x_2} \int_{-\infty}^{y_1} f_{XY} \, dy \, dx + \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{XY} \, dy \, dx \\
&= \int_{-\infty}^{x_2} \left(\int_{-\infty}^{y_2} f_{XY} \, dy - \int_{-\infty}^{y_1} f_{XY} \, dy \right) dx
\end{aligned}$$

$$\Delta \triangleq F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1).$$

- We need to show $\Delta \geq 0$ if $x_1 < x_2$ and $y_1 < y_2$.
- We have

$$\begin{aligned}
\Delta &= \int_{-\infty}^{x_2} \int_{-\infty}^{y_2} f_{XY}\, dy\, dx - \int_{-\infty}^{x_1} \int_{-\infty}^{y_2} f_{XY}\, dy\, dx \\
&\quad - \int_{-\infty}^{x_2} \int_{-\infty}^{y_1} f_{XY}\, dy\, dx + \int_{-\infty}^{x_1} \int_{-\infty}^{y_1} f_{XY}\, dy\, dx \\
&= \int_{-\infty}^{x_2} \left(\int_{-\infty}^{y_2} f_{XY}\, dy - \int_{-\infty}^{y_1} f_{XY}\, dy \right) dx \\
&\quad - \int_{-\infty}^{x_1} \left(\int_{-\infty}^{y_2} f_{XY}\, dy - \int_{-\infty}^{y_1} f_{XY}\, dy \right) dx
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- Thus we have

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- In general

$$P[(X, Y) \in B] = \int_B f_{XY}(x, y) \ dx \ dy, \ \ \forall B \in \mathcal{B}^2$$

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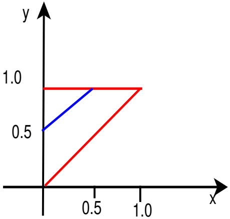
$$= 2\ \frac{y^2}{2}\Big|_{0.5}^{1} - y|_{0.5}^{1} = 1 - 0.25 - 1 + 0.5 = 0.25$$

- We can look at it geometrically



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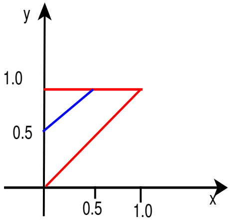

- The probability of the event we want is the area of the small triangle divided by that of the big triangle.

Marginal Distributions

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$$f_X(m) = \sum_{n=m+1}^{2m} f_{XY}(m,n) = \sum_{n=m+1}^{2m-1} \frac{2}{36} + \frac{1}{36} = \frac{2}{36}(m-1) + \frac{1}{36} = \frac{2m-1}{36}$$

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