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- Given a probability space $(\Omega, \mathcal{F}, P)$, a random variable is a real-valued function on Ω.
- It essentially results in an induced probability space

$$(\Omega, \mathcal{F}, P) \ \xrightarrow{X} \ (\Re, \mathcal{B}, P_X)$$

where $\mathcal{B}$ is the Borel σ-algebra and

$$P_X(B) = P[X \in B] = P\left(\{\omega \in \Omega \ : \ X(\omega) \in B\}\right)$$

Recap: σ-algebra

- An $\mathcal{F} \subset 2^\Omega$ is called a σ-algebra (also called σ-field) on Ω if it satisfies
  1. $\Omega \in \mathcal{F}$
  2. $A \in \mathcal{F} \;\; \Rightarrow \;\; A^c \in \mathcal{F}$
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- We also have $\mathcal{B} = \sigma(\{(-\infty,\ x] : x \in \Re\})$

Recap: Distribution function of a random variable

- Let X be a random variable. It distribution function, $F_X : \Re \to \Re$, is defined by

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- The distribution function, F_X, completely specifies the probability measure, P_X.

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  1. $0 \leq F_X(x) \leq 1, \ \forall x$
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  $P[a < X \leq b] = F_X(b) - F_X(a)$.

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- Note that the distribution function is defined for **all** random variables.

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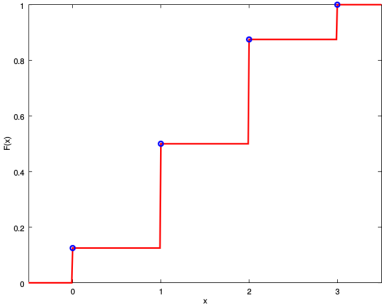

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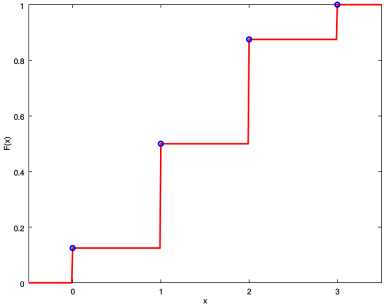

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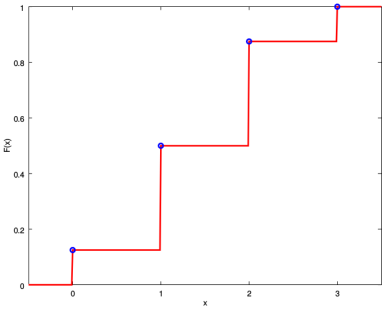

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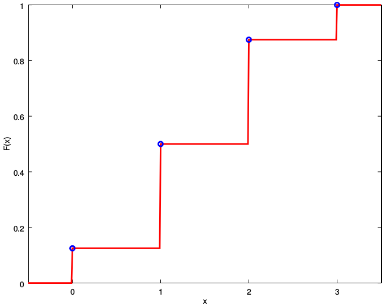

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- The jump at, e.g., $x = 2$ is $3/8$ which is the probability of X taking that value.

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- All discrete random variables would have this general form of distribution function.

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- If $a_2 \leq x < a_3$ then $[X \leq x] = [X = a_1] \cup [X = a_2] = B_1 + B_2$

- Hence we can write the distribution function as

$$F_X(x) = \begin{cases} 0 & x < a_1 \\ P(B_1) & a_1 \leq x < a_2 \\ P(B_1) + P(B_2) & a_2 \leq x < a_3 \\ & \vdots \\ \sum_{i=1}^{k} P(B_i) & a_k \leq x < a_{k+1} \\ & \vdots \\ 1 & x \geq a_n \end{cases}$$

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- Note that all this holds even when X takes countably infinitely many values.

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- We can get pmf from df and df from pmf.

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- Please remember that we have defined distribution function for any random variable. But pmf is defined only for discrete random variables

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- As we saw this is how we can specify a probability assignment on any countable sample space.

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- We next look at some standard discrete random variable models

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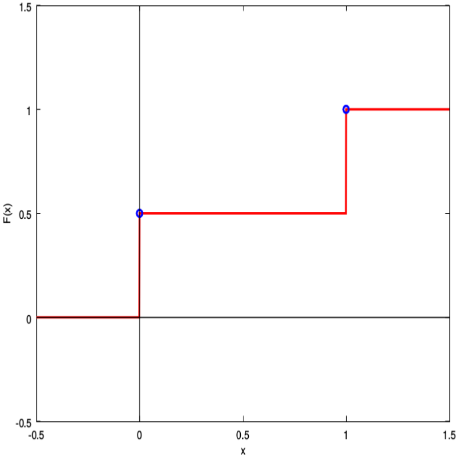
- Thus, this indicator rv has Bernoulli distribution with $p = P(B)$

One of the df examples we saw earlier is that of Bernoulli



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$$f_X(k) = \ {}^nC_k\, p^k\, (1-p)^{n-k},\ k = 0, 1, \cdots, n$$
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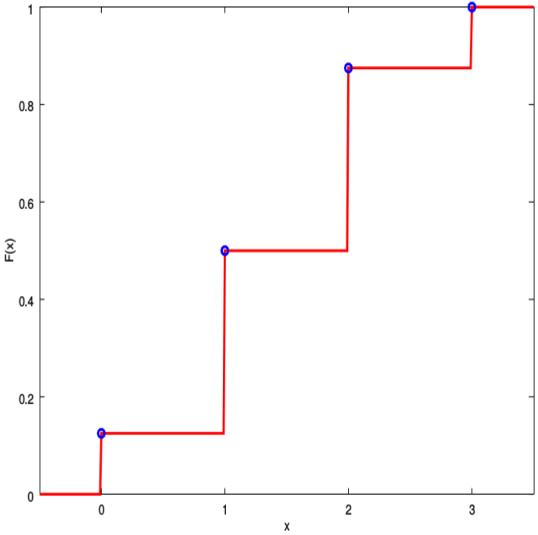
- Any one outcome (a seq of length n) with k heads would have probability $p^k(1-p)^{n-k}$. There are ${}^nC_k$ outcomes with exactly k heads.

One of the df examples we considered was that of Binomial



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- Poisson distribution is also useful in many applications

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- In general waiting for 'success' in independent Bernoulli trials.

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(Does this also tell us what is df of geometric rv?)

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- This is known as the memoryless property of geometric distribution

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&= \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n \\
\Rightarrow P[X > m+n | X > m] &= P[X > n]
\end{aligned}$$

- This is known as the memoryless property of geometric distribution
- Same as

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  Because $[X > m + n]$ is not equal to intersection of $[X > m]$ and $[X > n]$

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- This is pmf of geometric (with $q = (1-p)$)

Continuous Random Variables

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- A rv, X, is said to be continuous (or of continuous type) if its distribution function, F_X is absolutely continuous.

Absolute Continuity

- A function $g : \Re \rightarrow \Re$ is absolutely continuous on an interval, I, if given any $\epsilon > 0$ there is a $\delta > 0$ such that for any finite sequence of pair-wise disjoint subintervals, (x_k, y_k), with $x_k, y_k \in I, \ \forall k$, satisfying $\sum_k (y_k - x_k) < \delta$, we have $\sum_k |f(y_k) - f(x_k)| < \epsilon$

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- In the above, g would be differentiable almost everywhere and h would be its derivative (wherever g is differentiable).

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- As mentioned earlier, there would be many random variables that are neither discrete nor continuous.

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- This shows the the F_X is a df and hence f_X is a pdf

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- Let us find F_Y and f_Y.

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- We have $F_X = F_Y$ and thus $f_X = f_Y$. (However, note that $X(\omega) \neq Y(\omega)$ except at $\omega = 0.5$).

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- Recall that for a general rv

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- That is why f_X is called probability density function.

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- We next consider a few standard continuous random variables.

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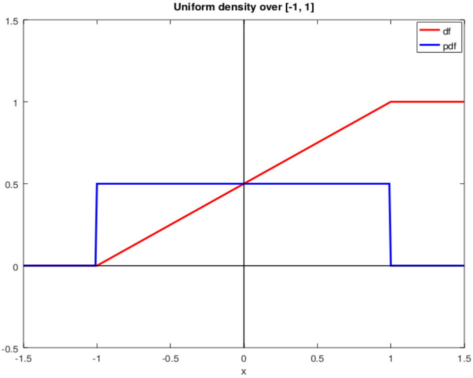

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- The earlier examples we considered are uniform over $[0,\ 1]$.

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- This also gives us: $P[X > x] = 1 - F_X(x) = e^{-\lambda x}$ for $x > 0$.

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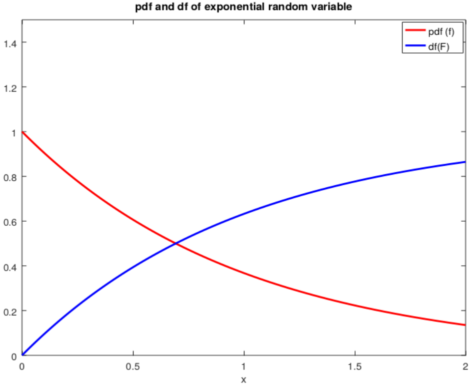

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- If the distribution of a non-negative continuous random variable is memory less then it must be exponential.

Gaussian Distribution

- The pdf of Gaussian distribution is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < \infty$$

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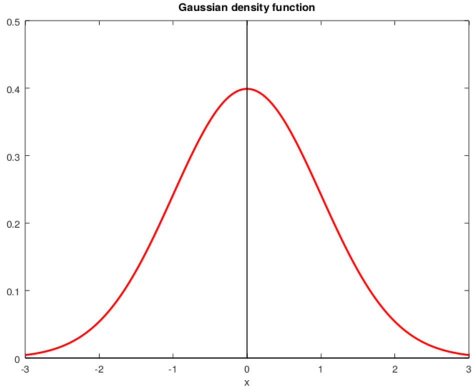

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  (Left as an exercise for you!)