E1 222 Stochastic Models and Applications
Problem Sheet 5–2

1. Let $X(t)$ be a wide-sense stationary stochastic process with autocorrelation $R(\tau)$. Show that $\text{Prob}[|X(t+\tau)-X(t)| \geq a] < 2[R(0)-R(\tau)]/a^2$.

2. Consider a stochastic process $X(t) = e^{At}$ where A is a continuous random variable with density f_A. Express the mean $\eta(t)$ and the autocorrelation $R(t_1, t_2)$ in terms of f_A.

3. Customers arrive at a bank at a Poisson rate 4 per hour. Suppose two customers arrived during the first hour. What is the probability that (i). both arrived during the first twenty minutes, (ii). at least one arrived during the first twenty minutes.

4. Suppose vehicles pass a certain point in a highway as a Poisson process with rate 1 per minute. Suppose 5% of the vehicles are vans. What is the probability that at least one van passes by during half an hour? Given that 10 vans passed by in an hour what is the expected number of vehicles to have passed in that hour.

5. Cars pass a certain location on a street according to a Poisson process with rate λ. A woman wants to cross the street at that place and she waits till she can see that no cars would pass that place in the next T time units. Find the probability that her waiting time is zero.

6. Suppose people arrive at a bust stop in accordance with a Poisson process with rate λ. let t be some fixed time and suppose the next bus departs at t. All people who arrive till t would get on the bus that departs at t. Let X denote the total amount of waiting time of all people who got on the bus at t. (Note that a person who arrived at $s < t$ would contribute $t - s$ to the waiting time).

   - Show that $E[X|N(t)] = N(t)\frac{t}{2}$
   - Show that $\text{Var}[X|N(t)] = N(t)\frac{t^2}{12}$
   - Using these two, calculate $\text{Var}(X)$

7. Suppose $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process. Find the distribution of $B(t) + B(s)$ with $s < t$.

8. Suppose $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process. Compute $E[B(t_1)B(t_2)B(t_3)]$

9. Suppose customers arrive at a single server queuing system in accordance with a Poisson process with rate λ. However an arriving customer will join the queue with probability α_n if he sees there are n people in the system. (With the remaining probability he just departs). Represent this a birth-death process (of a continuous time markov chain) and specify the birth and death rates.