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- We also have
$$P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\, dy\, dx$$
and, in general,
$$P[(X,Y) \in B] = \int_B f_{XY}(x,y)\, dx\, dy, \;\; \forall B \in \mathcal{B}^2$$

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- If X, Y have a joint density or if X is continuous and Y is discrete, $F_{X|Y}$ would be absolutely continuous and would have a density.

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This exists if X, Y have a joint density or when Y is discrete.

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- When X, Y are both discrete or they have a joint density

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- The above relation gives rise to the total probability rules and Bayes rule for rv's

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- We need to interpret f_X or $f_{X|Y}$ as mass functions when X is discrete and as densities when X is a continuous and so on

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- This also implies $F_{X|Y}(x|y) = F_X(x)$ and $f_{X|Y}(x|y) = f_X(x)$

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- These are straight-forward generalizations
- The properties of joint distribution function such as it being non-decreasing in each argument etc are easily seen to hold here too.

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  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\, dx\, dy\, dz = 1$
- These are straight-forward generalizations
- The properties of joint distribution function such as it being non-decreasing in each argument etc are easily seen to hold here too.
- Generalizing the special property of the df (relating to probability of cylindrical sets) is a little more complicated. (An exercise for you!)

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- Any marginal is a joint density of a subset of these rv's and we obtain it by integrating the (full) joint density with respect to the remaining variables.
- We obtain the marginal mass functions for a subset of the rv's also similarly where we sum over the remaining variables.

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- With these we can generally calculate most quantities of interest.

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- For example when Z is continuous

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F_{XY|Z}(x,y|z) = \lim_{\delta \to 0} P[X \leq x, Y \leq y | Z \in [z, z+\delta]]
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- For example, the first one above follows from

$$P[X = x, Y = y|Z = z] = \frac{P[X = x, Y = y, Z = z]}{P[Z = z]}$$

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- Thus we get

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f_{XYZ}(x,y,z) = f_{Z|XY}(z|x,y) f_{XY}(x,y) = f_{Z|XY}(z|x,y) f_{Y|X}(y|x) f_X(x)
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- We use similar notation for marginal and conditional distributions

Independence of multiple random variables

- Random variables $X_1, X_2, \cdots, X_n$ are said to be independent if the the events $[X_i \in B_i],\ i = 1, \cdots, n$ are independent.

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  (Recall definition of independence of a set of events)
- Independence implies that the marginals would determine the joint distribution.

Example

- Let a joint density be given by

$$f_{XYZ}(x, y, z) = K, \quad 0 < z < y < x < 1$$

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&= K\int_{x=0}^{1}\int_{y=0}^{x} y\, dy\, dx \\
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$$\begin{aligned} f_{XZ}(x,z) &= \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\, dy \\ &= \int_{z}^{x} K\, dy, \quad 0 < z < x < 1 \\ &= 6(x-z), \quad 0 < z < x < 1 \end{aligned}$$

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&= \int_0^1 \left(6x^2 - 6\ \frac{x^2}{2} \right)\ dx \\
&= 3\ \frac{x^3}{3}\bigg|_0^1 = 1
\end{aligned}$$

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- Hence,

$$f_{Y|XZ}(y|x, z) = \frac{f_{XYZ}(x, y, z)}{f_{XZ}(x, z)} = \frac{1}{x - z}, \quad 0 < z < y < x < 1$$

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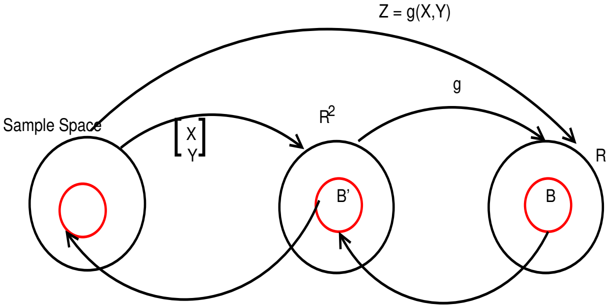
- This is analogous to functions of a single rv



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- For example, if X, Y are discrete, then

$$f_Z(z) = P[Z = z] = P[g(X, Y) = z] = \sum_{\substack{x_i, y_j: \\ g(x_i, y_j) = z}} f_{XY}(x_i, y_j)$$

- Let X, Y be discrete rv's. Let $Z = \min(X, Y)$.

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$$\begin{aligned} f_Z(z) &= P[\min(X, Y) = z] \\ &= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \end{aligned}$$

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f_Z(z) &= P[\min(X,Y) = z] \\
&= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \\
&= \sum_{y>z} P[X = z, Y = y] + \sum_{x>z} P[X = x, Y = z] \\
&\qquad + P[X = z, Y = z]
\end{aligned}
$$

- Let X, Y be discrete rv's. Let $Z = \min(X, Y)$.

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- Now suppose X, Y are independent and both of them have geometric distribution with the same parameter, p.
- Such random variables are called **independent and identically distributed** or **iid** random variables.

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$$
\begin{aligned}
f_Z(z) &= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \\
&= P[X = z]P[Y > z] + P[Y = z]P[X > z] + P[X = z]P[Y = z] \\
&= p(1-p)^{z-1}
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&= 2p(1-p)^{z-1}(1-p)^z + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{2z-1} + p^2(1-p)^{2z-2}
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&= 2p(1-p)^{z-1}(1-p)^z + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{2z-1} + p^2(1-p)^{2z-2} \\
&= p(1-p)^{2z-2}(2(1-p) + p)
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&= 2p(1-p)^{z-1}(1-p)^z + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{2z-1} + p^2(1-p)^{2z-2} \\
&= p(1-p)^{2z-2}(2(1-p) + p) \\
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\end{aligned}
$$

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&= (2-p)p \, \frac{1}{1-(1-p)^2} \\
&= (2-p)p \, \frac{1}{2p-p^2} = 1
\end{aligned}$$

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&= F_{X_1}(z) \cdots F_{X_n}(z), \quad \text{if they are independent} \\
&= \left(F_X(z)\right)^n, \quad \text{if they are iid} \\
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\end{aligned}
$$

- For example if all X_i are uniform over $(0, 1)$ and ind, then $F_Z(z) = z^n,\ 0 < z < 1$

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P[Z > z] &= P[\min(X, Y) > z] \\
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&= (1 - F_X(z))(1 - F_Y(z))
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Hence, $\quad F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z))$

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$$\text{Hence,} \quad F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z))$$

- We can once again find density of Z if X, Y are continuous

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- Notice that $P[X > z] = (1 - z)$.

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$$F_Z(z) = 1 - (1 - F_X(z))^2 = 1 - (1 - z)^2, 0 < z < 1$$

- Notice that $P[X > z] = (1 - z)$.
- We get the density of Z as

$$f_Z(z) = 2(1 - z), \ \ 0 < z < 1$$

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- $\min$ fn is also easily generalized to n random variables
- Let $Z = \min(X_1, X_2, \cdots, X_n)$

$$\begin{aligned} P[Z > z] &= P[\min(X_1, X_2, \cdots, X_n) > z] \\ &= P[X_1 > z, \cdots, X_n > z] \\ &= P[X_1 > z] \cdots P[X_n > z], \quad \text{using independence} \\ &= (1 - F_{X_1}(z)) \cdots (1 - F_{X_n}(z)) \\ &= (1 - F_X(z))^n, \quad \text{if they are iid} \end{aligned}$$

- Hence, when X_i are iid, the df of Z is

$$F_Z(z) = 1 - (1 - F_X(z))^n$$

where F_X is the common df

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- Remaining details are left as an exercise for you!!

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- Now suppose X, Y are independent. Then

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&= \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{z} f_{XY}(x, t - x)\, dt\, dx \\
&= \int_{-\infty}^{z} \left(\int_{-\infty}^{\infty} f_{XY}(x, t - x)\, dx \right) dt
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- This gives us

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Density of sum of independent random variables is the convolution of their densities.

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$$f_{X+Y} = f_X * f_Y \quad \text{(Convolution)}$$

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- Thus, sum of independent exponential random variables has gamma distribution:

$$f_Z(z) = \lambda z\, \lambda e^{-\lambda z}, \quad z > 0$$