Assignment 9 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. Let $M_{n\times n}$ be a matrix which when multiplied with a vector $x_{n\times 1}$ produces zeros on components $[k+1:n]$. Further, let $x_k \neq 0$ and e_k be the kth column of I_n.

ⓐ Write down the elements of M in terms of the elements of x.

ⓑ Verify that M can be written as $I_n - te_k^T$. What are the elements of t?

ⓒ Obtain an expression for M^{-1}.

(Hint: Verify that $(I+uv^T)^{-1} = I - \dfrac{uv^T}{1+v^Tu}$, $1+v^Tu \neq 0$)

A. ⓐ Let $y = Mx$. It is given that $y_i = 0$, $k+1 \le i \le n$.

$$\Rightarrow \sum_{j=1}^{n} M_{ij} x_j = 0, \quad k+1 \le i \le n$$

$\Rightarrow$ x is orthogonal to the $k+1, k+2, \dots, n$ rows of M — ①

$$\Rightarrow M = \begin{array}{c c} & \begin{array}{ccc} \overbrace{\quad}^{k-1} & \overbrace{\quad}^{k\text{th}} & \overbrace{\quad}^{n-k} \end{array} \\ \begin{array}{c} \\ \\ n-k+1 \left\{ \right. \end{array} & \left[\begin{array}{c|c|c} I_{k-1} & \begin{matrix} * \\ \vdots \\ * \end{matrix} & 0 \\ \hline 0 & \begin{matrix} * \\ -\frac{x_{k+1}}{x_k} \\ \vdots \\ -\frac{x_n}{x_k} \end{matrix} & I_{n-k} \end{array}\right] \end{array}$$

is one such matrix which has the above mentioned property. — ②

(b) $y = M.x = Ix - te_k^T x = x - (e_k^T x)t$

$$= \begin{bmatrix} x_1 - x_k t_1 \\ x_2 - x_k t_2 \\ \vdots \\ x_n - x_k t_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ \vdots \\ * \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ * \\ \vdots \\ * \end{matrix}}\right\} k \\ \left.\vphantom{\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}}\right\} n-k \end{matrix}$$

$$\Rightarrow t_i = \begin{cases} \dfrac{x_i}{x_k}, & i \in \{k+1, \cdots, n\} \\ 0, & \text{else} \end{cases} \quad — ③$$

Don't care, so set to zero

$$\Rightarrow M = I - te_k^T = \left[\begin{array}{ccc|c} 1 & \cdots & -t_1 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & 1-t_k & \\ \hline 0 & & \begin{matrix} -t_{k+1} \\ \vdots \\ -t_n \end{matrix} & I_{n-k} \end{array}\right] \quad — ④$$

→ Can be written in this form from ①

(c) To verify: $(I+uv^T)^{-1} = I - \dfrac{uv^T}{1+v^Tu}$

$$(I+uv^T)\left(I - \frac{uv^T}{1+v^Tu}\right) = I - \frac{uv^Tuv^T}{1+v^Tu} - \frac{uv^T}{1+v^Tu} + uv^T$$

$$= I + uv^T - (v^Tu)\frac{uv^T}{1+v^Tu} - \frac{uv^T}{1+v^Tu}$$

$$= I + uv^T - \left(\frac{v^Tu+1}{1+v^Tu}\right)uv^T = I + uv^T - uv^T = I \quad — ⑤$$

Similarly, $\left(I - \frac{uv^T}{1+v^Tu}\right)(I+uv^T) = I$

$$\Rightarrow M^{-1} = (I - te_k^T)^{-1}$$
$$= I - \frac{(-t)e_k^T}{1+e_k^T(-t)} = I + \frac{te_k^T}{1-t_k}$$
$$\Rightarrow M^{-1} = I + \frac{te_k^T}{1-t_k} = I + te_k^T \quad \text{— } 6$$
$(\because e_k^T t = t_k = 0 \Rightarrow 1-t_k = 1)$

2. A is normal, B is nilpotent & $A+B=I \Rightarrow A=I$

A. $B = I - A \Rightarrow B^H = I - A^H$
$$\Rightarrow BB^H = I - A - A^H - AA^H$$
$$= I - A - A^H - A^HA = B^HB \quad \text{— } 1$$
Thus B is normal & nilpotent.

If k is the index of B, $B^k = 0$
$$Bx = \lambda x \Rightarrow B^k x = \lambda^k x = 0 \Rightarrow \lambda = 0$$
$$\Rightarrow \sigma(B) = \{0, \dots, 0\} \quad \text{— } 2$$

Since B is normal, it is unitarily diagble, $B = UDU^H$
$D = \text{diag}\{0, \cdots, 0\} \Rightarrow D = 0$ — 3

$$\Rightarrow B = U \cdot 0 \cdot U^H = 0$$
$$\left.\Rightarrow \underset{\overset{\parallel}{I}}{A+B} = A + 0 = A \right\} \Rightarrow A = I \quad \text{— } 4$$

(Alt: $B^k = UD^kU^H = 0 \Rightarrow D^k = 0 \Rightarrow D = 0$)