Recap: Multi-dimensional Gaussian density

- $\mathbf{X} = (X_1, \cdots, X_n)^T$ are said to be jointly Gaussian if

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- When X, Y are jointly Gaussian, the joint densityis given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \; e^{-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} - \frac{2\rho(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right)}$$

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- We will prove this using moment generating functions

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  showing that W is Gaussian
- Shows density of X_i is Gaussian for each i. For example, if we take $\mathbf{t} = (1, 0, 0, \cdots, 0)^T$ then W above would be X_1.

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- This is a defining property of multidimensional Gaussian density

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$$\begin{aligned} \Sigma_y &= E\left[(\mathbf{Y} - \mu_y)(\mathbf{Y} - \mu_y)^T\right] \\ &= E\left[(A(\mathbf{X} - \mu_x))(A(\mathbf{X} - \mu_x))^T\right] \end{aligned}$$

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This shows $\mathbf{Y}$ is jointly Gaussian

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- For example, if you take A to be

$$A = \left[\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{array} \right]$$

then $\mathbf{Y} = (X_1, X_2)^T$

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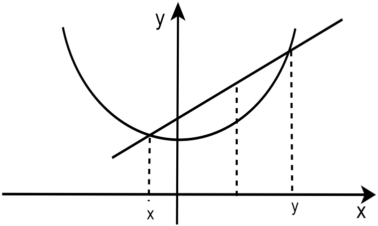

Jensen's Inequality

- Let $g : \Re \to \Re$ be a convex function. Then

$$g(EX) \leq E[g(X)]$$

- For example, $(EX)^2 \leq E\left[X^2\right]$
- Function g is convex if

$$g(\alpha x + (1-\alpha)y) \leq \alpha g(x) + (1-\alpha)g(y), \;\; \forall x, y, \;\; \forall 0 \leq \alpha \leq 1$$

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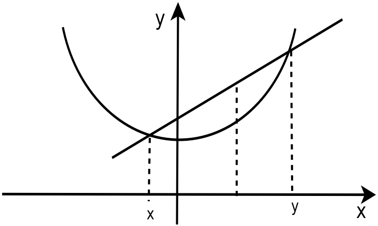

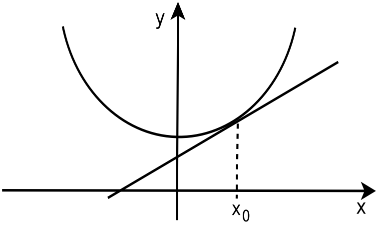

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- A generalization of Cauchy-Schwartz inequality is Holder inequality

Holder Inequality

- For all p, q with $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$E[|XY|] \leq (E|X|^p)^{\frac{1}{p}} \ (E|Y|^q)^{\frac{1}{q}}$$

(We assume all the expectations are finite)

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Proof

- First we will show, for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

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Chernoff Bounds

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$$P[X > a] \;\leq\; \frac{E[e^{sX}]}{e^{sa}} \;=\; \frac{M_X(s)}{e^{sa}},\; \forall s > 0$$

- The RHS is a function of S. We can get a tight bound by using a value of s which minimizes RHS.

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- Note we do not need knowledge of any moments of X_i to calculate the bound

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- There are other ways of defining convergence of random variables

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- Or we can look at different notions of convergence of a sequence of functions to a function.

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- Or we can ask for $\int |g_n(x) - g_0(x)|^2 \ dx \to 0$.

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- Thus there would be multiple ways to define convergence of sequence of random variables.

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- We only need marginal distributions of individual X_n to decide whether a sequence converges to a constant in probability

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- We omit the proofs