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  - $\Omega$ is the sample space
  - $\mathcal{F} \subset 2^{\Omega}$ set of events; each event is a subset of $\Omega$
  - $P : \mathcal{F} \rightarrow [0, 1]$ is a probability (measure) that satisfies the three axioms:
    - A1 $P(A) \geq 0, \; \forall A \in \mathcal{F}$
    - A2 $P(\Omega) = 1$
    - A3 If $A_i \cap A_j = \phi, \forall i \neq j$ then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$

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- When $\Omega$ is finite with $n$ elements, a special case is $q_i = \frac{1}{n}, \forall i$. (All outcomes equally likely)

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- This holds for multiple event, e.g., $P(ABC) = P(A|BC)P(B|C)P(C)$
- Given a partition, $\Omega = B_1 + B_2 + \cdots + B_m$, for any event, $A$,

$$P(A) = \sum_{i=1}^{m} P(A|B_i)P(B_i) \quad \text{(Total Probability rule)}$$

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- Bayes rule can be viewed as transforming a prior probability into a posterior probability.

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- If $A, B$ are independent then so are $A \& B^c$, $A^c \& B$ and $A^c \& B^c$.

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- But, $P(E_1 E_2 E_3) = 0.25 \neq (0.5)^3$

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- Events may be conditionally independent but not independent. (e.g., 'independent' multiple tests for confirming a disease)
- It is also possible that $A, B$ are independent but are not conditionally independent given some other event $C$.

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- Assuming conditional independence we can calculate the new posterior probability using the same information we had about true positive and false positive rate.

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$$= \frac{0.99 * 0.99 * 0.1}{0.99 * 0.99 * 0.1 + 0.05 * 0.05 * 0.9} = 0.97$$

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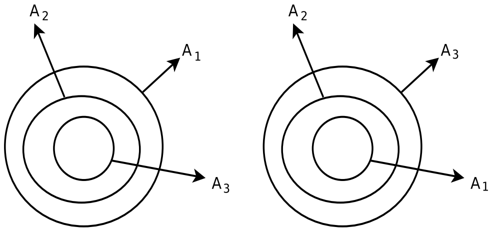


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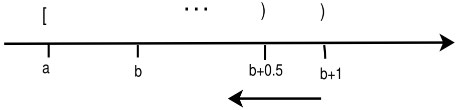

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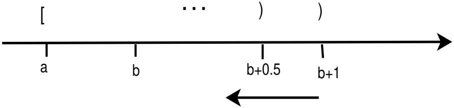


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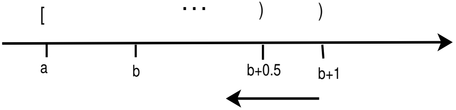


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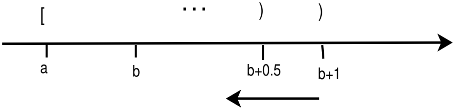


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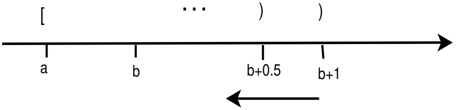


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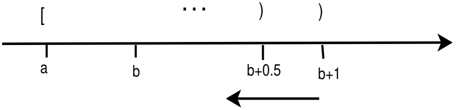


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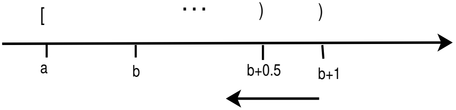


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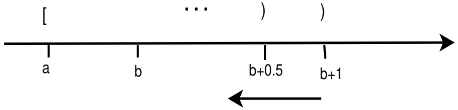


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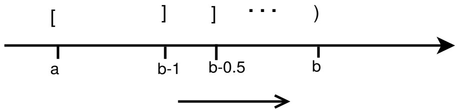

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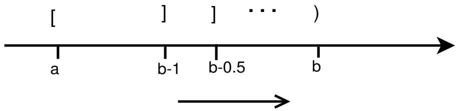


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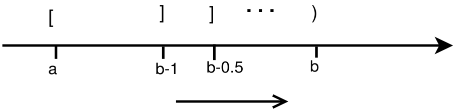


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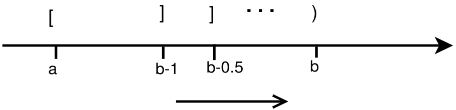


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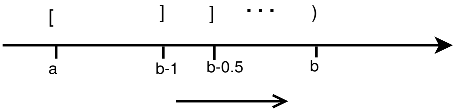


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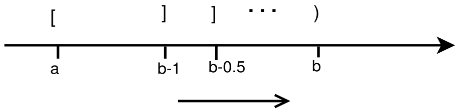


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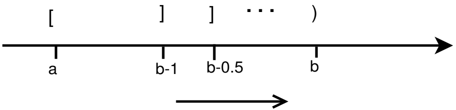


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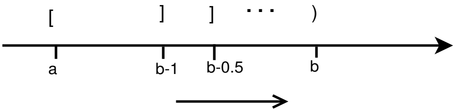


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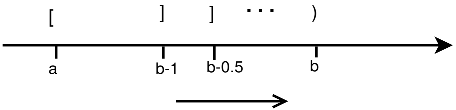


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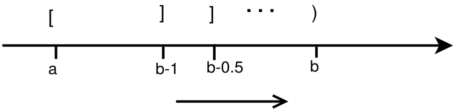


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- These examples also show how using countable unions or intersections we can convert one end of an interval from 'open' to 'closed' or vice versa.

- To summarize, limits of monotone sequences of events are defined as follows

$$A_n \downarrow \quad \lim_{n\to\infty} A_n = \cap_{k=1}^{\infty} A_k$$

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- Having defined the limits, we now ask the question

$$P\left(\lim_{n\to\infty} A_n\right) \stackrel{?}{=} \lim_{n\to\infty} P(A_n)$$

where we assume the sequence is monotone.

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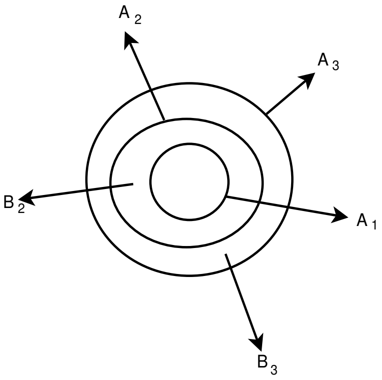

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- But for our problem, we can not put any fixed limit on the number of tosses and hence our sample space should be for infinite tosses of a coin.

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- So, uncountable $\Omega$ arise naturally if we want to consider infinite repetitions of a random experiment

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- As we already saw, the probability of this event is $(0.5)^2$ which is the length of this interval

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- This is called Borel's normal number theorem and is an interesting result about real numbers.

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  - Probability of 0 or 3 heads is $1/8$ while that of 1 or 2 heads is $3/8$

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- If we think tosses are independent then we choose $P_1$. But if we need to model some dependence among tosses, we choose a model like $P_2$.

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- It is also a useful model.

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This entire course can be considered as studying different random variables.

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- Another example: $\Omega = \{H, T\}^3$, $X(\omega)$ is numbers of $H$'s.
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- We can effectively work with $\Re$ as sample space in all probability models