E1 222 Stochastic Models and Applications
Final Examination

Time: 3 hours Max. Marks:50
Date: 21 Jan 2021

Answer **Any FIVE** questions. All questions carry equal marks

1. a. Let A, B be events with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$. Let I_A and I_B be the indicator random variables of events A and B respectively. Find the correlation coefficient of I_A and I_B.

Answer: We have

$$E[I_A] = P[I_A = 1] = P(A) = 0.4, \quad E[I_B] = P(B) = 0.5$$

and

$$E[I_A\ I_B] = P[I_A = 1, I_B = 1] = P(A \cap B) = 0.3$$

Hence,

$$\text{Cov}(I_A, I_B) = E[I_A I_B] - E[I_A]\ E[I_B] = 0.3 - 0.4 * 0.5 = 0.1$$

Also, $\text{Var}(I_A) = 0.4 * 0.6 = 0.24$ and $\text{Var}(I_B) = 0.5 * 0.5 = 0.25$. Hence

$$\rho_{I_A I_B} = \frac{0.1}{\sqrt{0.24 * 0.25}} \approx 0.41$$

b. A rod of length 1 is broken at a random point. The piece containing the left end is once again broken at a random point. Let L be the length of the final piece containing the left end. Find the probability that L is greater than 0.25.

Answer: Let us take the rod as the interval [0, 1]. Let X_1 denote the first point where the rod is broken and let X_2 denote the second point. Note that $L = X_2$. We are given that X_1 is uniform from 0 to 1 and X_2 is uniform from 0 to X_1. Thus

$$f_{X_1}(x_1) = 1,\ 0 \le x_1 \le 1 \quad \text{and} \quad f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1},\ 0 \le x_2 \le x_1 \le 1$$

This gives us

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_2|X_1}(x_2|x_1)\ f_{X_1}(x_1)\ dx_1 = \int_{x_2}^{1} \frac{1}{x_1}\ dx_1 = -\ln(x_2),\ 0 \le x_2 \le 1$$

Hence

$$P[L > 0.25] = \int_{0.25}^{1} (-\ln(x_2))\ dx_2 = -\ x_2\ \ln(x_2)|_{0.25}^{1} + \int_{0.25}^{1} dx_2 = 0.25*\ln(0.25) + 0.75 \approx 0.4$$

2. a. Consider a game where N men put all their hats in a heap and then everyone randomly chooses a hat. Let$X$ denote the number of men who get their own hat. Show that $\mathrm{Var}(X) = 1$.

Answer: Let $Y_i,\ i = 1, 2, \cdots, N$ be defined by

$$Y_i = \begin{cases} 1 & \text{if } i^{th} \text{ man gets his own hat} \\ 0 & \text{Otherwise} \end{cases}$$

Now we have $X = \sum_{i=1}^{N} Y_i$. Hence

$$\mathrm{Var}(X) = \sum_{i=1}^{N} \mathrm{Var}(Y_i) + \sum_{i=1}^{N} \sum_{j \neq i} \mathrm{Cov}(Y_i, Y_j)$$

We have

$$P[Y_i = 1] = \frac{(N-1)!}{N!} = \frac{1}{N} \quad \Rightarrow \quad E[Y_i] = \frac{1}{N},\ \ \mathrm{Var}(Y_i) = \frac{1}{N}\left(1 - \frac{1}{N}\right)$$

For $i \neq j$,

$$P[Y_i = 1, Y_j = 1] = \frac{(N-2)!}{N!} = \frac{1}{N(N-1)} \quad \Rightarrow \quad E\left[Y_i\ Y_j\right] = \frac{1}{N(N-1)} \quad \Rightarrow \quad \mathrm{Cov}(Y_i, Y_j) = \frac{1}{N(N-1)} - \frac{1}{N^2}$$

Hence, we get variance of X as

$$\begin{aligned} \mathrm{Var}(X) &= \sum_{i=1}^{N} \frac{1}{N}\left(1 - \frac{1}{N}\right) + \sum_{i=1}^{N} \sum_{j \neq i} \left(\frac{1}{N(N-1)} - \frac{1}{N^2}\right) \\ &= \left(1 - \frac{1}{N}\right) + 1 - \frac{N-1}{N} \\ &= 1 \end{aligned}$$

b. Let X, Y be iid exponential random variables with parameter λ. Find the density of $Z = Y/X$

Answer: Let $Z = Y/X$ and $W = X$. This is an invertible transformation: $X = W$ and $Y = ZW$. The Jacobian of the inverse transformation is $\begin{vmatrix} 0 & 1 \\ w & z \end{vmatrix} = -w$. Hence we have

$$f_{ZW}(z, w) = |w| f_{XY}(w, zw)$$

using this we get

$$f_Z(z) = \int_{-\infty}^{\infty} |w| \; f_{XY}(w, zw) \; dw$$

We could have also used this formula without explicitly deriving it here.

Since X, Y are exponential, the integrand in the above integral is zero unless we have $w \geq 0$ and $zw \geq 0$. Hence we must have $z \geq 0$ and w ranges from 0 to ∞.

$$\begin{aligned} f_Z(z) &= \int_0^{\infty} w \; \lambda e^{-\lambda w} \; \lambda e^{-\lambda z w} \; dw \\ &= \int_0^{\infty} w \; \lambda^2 \; e^{-\lambda(1+z)w} \; dw \\ &= \frac{\lambda}{1+z} \int_0^{\infty} w \; \lambda(1+z) \; e^{-\lambda(1+z)w} \; dw \\ &= \frac{\lambda}{1+z} \; \frac{1}{\lambda(1+z)} \\ & \quad \text{using formula for expectation of exponential rv} \\ &= \frac{1}{(1+z)^2} \end{aligned}$$

Hence the density of Z is given by

$$f_Z(z) = \frac{1}{(1+z)^2}, \; 0 \leq z < \infty$$

3. a. Let $X_1, \cdots, X_n$ be *iid* Poisson random variables with mean 1. Let $S_n = \sum_{k=1}^{n} X_k$. Find Prob$[S_n \leq n]$. Show that

$$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \; = \; 0.5.$$

(You can use the fact: if X, Y are independent Poisson random variables then $X + Y$ is also Poisson).

Answer: If X and Y are independent Poisson with means λ_1 and λ_2 then $X+Y$ is Poisson with mean $\lambda_1+\lambda_2$.

Since each of the X_i are Poisson with mean 1, S_n is Poisson with mean n. So, we get

$$P[S_n \le n] = \sum_{k=0}^{n} \frac{n^k}{k!}\, e^{-n}$$

Now consider a sequence of iid random variable, $X_1, \cdots$, which are all poisson with mean 1. Let $S_n = \sum_{k=1}^{n} X_k,\ n = 1, 2, \cdots$ Since S_n is Poisson with parameter n, we know $ES_n = n$ and $\text{Var}(S_n) = n$. Now, by central limit theorem we get

$$\lim_{n\to\infty} P[S_n \le n] = \lim_{n\to\infty} P\left[\frac{S_n - ES_n}{\sqrt{\text{Var}(S_n)}} \le \frac{n-n}{\sqrt{n}}\right] = \lim_{n\to\infty} P\left[\frac{S_n - ES_n}{\sqrt{\text{Var}(S_n)}} \le 0\right] = \Phi(0) = 0.5$$

This shows that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \;=\; 0.5.$$

Comment: Please note that central limit theorem does not allow you to conclude that $P[S_n \le n] = 0.5$. It only allows you to conclude $\lim_{n\to\infty} P[S_n \le n] = 0.5$

b. Let $X_1, X_2, \cdots, X_n$ be iid continuous random variables each having uniform distribution over $(0,1)$. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, $Y_3 = X_1X_2X_3$, $\cdots$, $Y_n = X_1X_2\cdots X_n$. Find joint density of $Y_1, Y_2, \cdots, Y_n$, and conditional density of Y_k conditioned on $Y_1, Y_2, \cdots, Y_{k-1}$. Let t be a fixed number in the interval $[0,\ 1]$. Let Z denote the number of Y_i that are in the interval $[t, 1]$. Find $P[Z = 1]$.

Answer: The given transformation is

$$\begin{aligned}
Y_1 &= X_1 \\
Y_2 &= X_1\, X_2 \\
Y_3 &= X_1X_2X_3 \\
&\;\vdots \\
Y_n &= X_1\, X_2\, \cdots\, X_n
\end{aligned}$$

The inverse transformation is

$$\begin{aligned} X_1 &= Y_1 \\ X_2 &= Y_2/Y_1 \\ X_3 &= Y_3/Y_2 \\ &\vdots \\ X_n &= Y_n/Y_{n-1} \end{aligned}$$

The Jacobian of the inverse transformation is

$$\left| \begin{array}{cccccc} 1 & 0 & 0 & 0 & \cdots & 0 \\ -y_2/y_1^2 & 1/y_1 & 0 & 0 & \cdots & 0 \\ 0 & -y_3/y_2^2 & 1/y_2 & 0 & \cdots & 0 \\ & & & \vdots & & \\ 0 & 0 & \cdots & 0 & -y_n/y_{n-1}^2 & 1/y_{n-1} \end{array} \right| = \frac{1}{y_1 y_2 \cdots y_{n-1}}$$

Thus, we get

$$f_{Y_1 Y_2 \cdots Y_n}(y_1, y_2, \cdots, y_n) = \left| \frac{1}{y_1 y_2 \cdots y_{n-1}} \right| f_{X_1 X_2 \cdots X_n}(y_1,\ y_2/y_1,\ y_3/y_2,\ \cdots,\ y_n/y_{n-1})$$

Since $X_1, \cdots, X_n$ are iid uniform over $(0,1)$, their joint density function is 1 if all arguments are between 0 and 1 and is zero otherwise. So, in the above expression, for the joint density of $Y_1, \cdots, Y_n$ to be non-zero, we need

$$0 < y_1 < 1, \quad 0 < y_2/y_1 < 1 \ \Rightarrow\ 0 < y_2 < y_1, \quad 0 < y_3/y_2 < 1 \ \Rightarrow\ 0 < y_3 < y_2, \cdots$$

Thus we get the joint density as

$$f_{Y_1 Y_2 \cdots Y_n}(y_1, y_2, \cdots, y_n) = \frac{1}{y_1 y_2 \cdots y_{n-1}}, \ \ 0 < y_n < y_{n-1} < \cdots < y_2 < y_1 < 1$$

From the above derivation, it is easy to see that for any $k > 1$,

$$f_{Y_1 \cdots Y_k}(y_1, \cdots, y_k) = \frac{1}{y_1 \cdots y_{k-1}}, \ \ 0 < y_k < \cdots < y_1 < 1$$

So, we can calculate the required conditional density as

$$\begin{aligned} f_{Y_k | Y_1 \cdots Y_{k-1}}(y_k \mid y_1, \cdots y_{k-1}) &= \frac{f_{Y_1 \cdots Y_k}(y_1, \cdots, y_k)}{f_{Y_1 \cdots Y_{k-1}}(y_1, \cdots, y_{k-1})} \\ &= \frac{1}{y_{k-1}}, \ \ 0 < y_k < y_{k-1} < \cdots < y_1 < 1 \end{aligned}$$

For the last part of the question, it is given that Z is the number of Y_i that are above t. We know that $Y_1 > Y_2 > \cdots > Y_n$. Hence the event $[Z = 1]$ is same as the event $[Y_1 \geq t \text{ and } Y_2 < t]$.

$$P[Z = 1] = \int_t^1 \int_0^t f_{Y_1Y_2}(y_1, y_2)\, dy_2\, dy_1 = \int_t^1 \int_0^t \frac{1}{y_1}\, dy_2\, dy_1 = -t\ln(t)$$

4. a. Let X be Gaussian with mean zero and variance 1. Let Z be a discrete random variable that is independent of X and suppose $\text{Prob}[Z = 1] = \text{Prob}[Z = -1] = 0.5$. Let $Y = ZX$. Find density of Y. Are X, Y uncorrelated? Are X, Y jointly Gaussian?

Answer: The distribution function of Y is given by

$$\begin{aligned}
P[Y \leq y] &= P[ZX \leq y] \\
&= P[ZX \leq y \mid Z = 1]P[Z = 1] + P[ZX \leq y \mid Z = -1]P[Z = -1] \\
&= P[X \leq y \mid Z = 1]P[Z = 1] + P[-X \leq y \mid Z = -1]P[Z = -1] \\
&= P[X \leq y]P[Z = 1] + P[X \geq -y]P[Z = -1] \\
&\qquad \text{since } X, Z \text{ are independent} \\
&= 0.5F_X(y) + 0.5(1 - F_X(-y)) \\
&= 0.5F_X(y) + 0.5F_X(y) \\
&\qquad \text{since } X \sim \mathcal{N}(0, 1), \text{ we have } 1 - F_X(-y) = F_X(y) \\
&= F_X(y)
\end{aligned}$$

Thus Y is also Gaussian with mean zero and variance 1.

We know $EX = EY = 0$. We can calculate $E[XY] = E[ZX^2] = EZ\ E[X^2] = 0$, because Z, X are independent and $EZ = 0$. Hence, $\text{Cov}(X, Y) = EXY - EX\ EY = 0$ showing that X, Y are uncorrelated.

If X, Y are jointly Gaussian, since they are uncorrelated they must be independent. But since Y can be either X or $-X$ only, they are not independent. For example $P[Y > 2 \mid 0 \leq X \leq 1] = 0$ but $P[Y > 2] \neq 0$. Hence, X, Y are not jointly Gaussian.

Just to reiterate, the argument is as follows. We have established that X, Y **are uncorrelated and are not independent**. Hence, they are not jointly Gaussian.

b. Let X, Y have joint density given by

$$f_{XY}(x,y) = \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} x^{a-1}(y-x)^{b-1}e^{-\lambda y},\ 0 < x < y < \infty$$

where $a, b, \lambda > 0$ are parameters. Find $E[X|Y]$.

Answer: We need the conditional density $f_{X|Y}$ for which we need the marginal density f_Y.

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx \\
&= \int_0^y \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} x^{a-1}(y-x)^{b-1}e^{-\lambda y}\, dx,\ y > 0 \\
& \quad \text{change the variable: } \ z = x/y \quad \text{we get } dx = y dz, \quad \text{limits become 0 to 1} \\
&= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)}\ e^{-\lambda y} \int_0^1 (yz)^{a-1}\ (y - yz)^{b-1}\ y\ dz \\
&= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)}\ e^{-\lambda y}\ y^{a+b-1} \int_0^1 z^{a-1}\ (1-z)^{b-1}\ dz \\
&= \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)}\ e^{-\lambda y}\ y^{a+b-1}\ \frac{\Gamma(a)\ \Gamma(b)}{\Gamma(a+b)} \\
& \quad \text{by using the beta density: } \ \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)} \int_0^1 z^{a-1}\ (1-z)^{b-1}\ dz = 1 \\
&= \lambda^{a+b}\ y^{a+b-1}\ e^{-\lambda y}\ \frac{1}{\Gamma(a+b)},\ y > 0
\end{aligned}$$

Thus Y has gamma density with parameters $a+b$ and λ

Now, the conditional density is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)}\ \frac{x^{a-1}(y-x)^{b-1}}{y^{a+b-1}},\ 0 < x < y < \infty$$

The conditional expectation is given by

$$\begin{aligned}
E[X \mid Y = y] &= \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)} \int_0^y x\ \frac{x^{a-1}(y-x)^{b-1}}{y^{a+b-1}}\ dx \\
&= \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)} \int_0^y \frac{x}{y}\ \left(\frac{x}{y}\right)^{a-1}\ \left(1 - \frac{x}{y}\right)^{b-1}\ dx
\end{aligned}$$

$$\text{change variable: } z = \frac{x}{y}$$

$$= y \frac{\Gamma(a+b)}{\Gamma(a)\,\Gamma(b)} \int_0^1 z\, z^{a-1}\ (1-z)^{b-1}\ dz$$

$$= \frac{a}{a+b}\, y$$

where we have used the formula for mean of beta density.

Thus, $E[X \mid Y] = \frac{a}{a+b} Y$

5. a. Let $X_n,\ n = 1, 2, \cdots$, be *iid* random variables uniform over $[0,\ 1]$. Let $W_n = \min(X_1, \cdots, X_n)$ and $Y_n = \max(X_1, \cdots, X_n)$. Let $Z_n = 0.5(W_n + Y_n)$. Does the sequence Z_n converge in probability?

Answer: X_n are iid uniform over $[0, 1]$. Given $Y_n = \max(X_1, \cdots, X_n)$. Then $Y_n \xrightarrow{P} 1$. This can be seen as

$$P[|1-Y_n| > \epsilon] = P[Y_n < 1-\epsilon] = P[X_i < 1-\epsilon,\ i = 1, \cdots, n] = (1-\epsilon)^n$$

which goes to zero as $n \to \infty$ for all $\epsilon > 0$.

Similarly, $W_n = \min(X_1, \cdots, X_n)$ converges to zero in probability:

$$P[|W_n - 0| > \epsilon] = P[W_n > \epsilon] = P[X_i > \epsilon,\ i = 1, \cdots, n] = (1-\epsilon)^n$$

which goes to zero as $n \to \infty$ for all $\epsilon > 0$.

Since $W_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{P} 1$, $(W_n + Y_n) \xrightarrow{P} 0 + 1 = 1$. Hence $Z_n = 0.5(W_n + Y_n)$ converges in probability to 0.5.

Comment: While correcting the paper I will accept the answer if you simply state that $(W_n + Y_n) \xrightarrow{P} 0 + 1 = 1$.

As we mentioned in class, $X_n \xrightarrow{P} c_1$ and $Y_n \xrightarrow{P} c_2$ implies $(X_n + Y_n) \xrightarrow{P} c_1 + c_2$. This result can be easily proved as follows.

By definition, $X_n \xrightarrow{P} c$ is same as $(X_n - c) \xrightarrow{P} 0$. Hence, without loss of generality, we can show the result assuming $c_1 = c_2 = 0$.

The result follows because we have

$$P[\ |X_n + Y_n| > \epsilon\ ] \leq P[\ |X_n| > \epsilon/2\ ] + P[\ |Y_n| > \epsilon/2\ ]$$

This follows because for $X_n + Y_n$ to be greater than ϵ at least one of X_n or Y_n have to be greater than $\epsilon/2$; if both are less that $\epsilon/2$ their sum cannot exceed ϵ. Since some of you may be new to such arguments (which are common in Analysis), here is a more detailed derivation of this.

Since $|x+y| \leq |x|+|y|$, we have the following, for any two random variables X, Y:

$$([\,|X| \leq \epsilon/2\,] \;\cap\; [\,|Y| \leq \epsilon/2\,]) \;\subset [\,|X+Y| \leq \epsilon\,]$$

Please note that the above is a relation among sets. The above follows because if $|X(\omega)| \leq \epsilon/2$ and $|Y(\omega)| \leq \epsilon/2$, then we have $|X(\omega) + Y(\omega)| \leq |X(\omega)| + |Y(\omega)| \leq \epsilon/2 + \epsilon/2 = \epsilon$.

If $A \subset B$ then, $B^c \subset A^c$. Hence, we get

$$[\,|X+Y| > \epsilon\,] \subset ([\,|X| > \epsilon/2\,] \;\cup\; [\,|Y| > \epsilon/2\,])$$

We know that if $A \subset B$ then $P(A) \leq P(B)$ and also that $P(A \cup B) \leq P(A) + P(B)$.

Hence, now considering the sequneces X_n, Y_n,

$$P[\,|X_n + Y_n| > \epsilon\,] \leq P[\,|X_n| > \epsilon/2\,] + P[\,|Y_n| > \epsilon/2\,]$$

If $X_n \xrightarrow{P} 0$ and $Y_n \xrightarrow{P} 0$, then the RHS above goes to zero as $n \to \infty$ which implies $X_n + Y_n \xrightarrow{P} 0$,

b. Let $X_1, X_2, \cdots$ be iid Gaussian random variables with mean zero and variance 1. Define

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \qquad Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

What are the distributions of Y_n, Z_n? Do the sequences Y_n, Z_n converge to some constant in probability? If yes, state the limit. Do these sequences converge in distribution? If yes, state the limit.

(You need not 'derive' or 'prove' any thing here. Simply state the answer along with a short justification/explanation for the answer).

Answer: We have $EX_i = 0$ and hence $EY_n = EZ_n = 0$. Also

$$\text{Var}(Y_n) = \left(\frac{1}{\sqrt{n}}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i) = 1, \ \forall n, \quad \text{and} \ \ \text{Var}(Z_n) = \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i) = \frac{1}{n}, \ \forall n$$

Since X_i are iid Gaussian, any linear combination of them would be Gaussian. Hence Y_n would be Gaussian with zero mean and variance 1 for all n. Similarly, Z_n would be Gaussian with mean zero and variance $1/n$.

Please note that the fact that Z_n and Y_n are Gaussian for every n has nothing to do with CLT. This comes about because linear combinations of independent Gaussians are Gaussian. The CLT tells you only about the limit distribution. It cannot tell you what the actual distribution is of a sum of finite number iid random variables. But, CLT can be used to **approximate** this distribution if the number in the sum is large.

Since all Y_n have the same density, the sequence cannot converge to any constant in probability. This is because for a given ϵ and any fixed finite c, $P[|Y_n - c| > \epsilon]$ is a fixed non-zero number independent of n and hence does not go to zero as n tends to infinity. Another way of looking at this is that the variance of Y_n is the same non-zero value for all n.

Since all Y_n have the same distribution, namely, standard Normal, the sequence converges in distribution to that.

Z_n converges to zero in probability by weak law of large numbers.

Since convergence in probability implies convergence in distribution, the sequence Z_n converges in distribution to the degenerate distribution representing the constant 0. (Note that the limit distribution here is: $F(x) = 0$ if $x < 0$ and $F(x) = 1$ if $x \geq 0$. This is the distribution function of a discrete random variable that takes only one value, namely, zero. The variance of this distribution is zero. That is why it is termed the degenerate distribution. See problem 6 in problem sheet 4.1)

6. a. Let $X(t)$ be a stochastic process defined by $X(t) = (-1)^{N(t)}\, X_0$ where $N(t)$ is a Poisson process with rate λ and X_0 is a random variable which is independent of $N(t)$ and which has the distribution $P[X_0 = +1] = P[X_0 = -1] = 0.5$. Find the mean and autocorrelation of $X(t)$.

Answer: The mean of $X(t)$ is, using independence of X_0 and the process $N(t)$ and the fact that $EX_0 = 0$,

$$E[X(t)] = E\left[(-1)^{N(t)}\ X_0\right] = E[X_0]\ E\left[(-1)^{N(t)}\right] = 0$$

The autocorrelation function is given by, for $t, s > 0$,

$$\begin{aligned}
R_X(t, t+s) &= E\left[X_0^2\ (-1)^{N(t)+N(t+s)}\right] \\
&= E\left[(-1)^{N(t)+N(t+s)}\right], \quad \text{since } X_0^2 = 1 \\
&= E\left[(-1)^{N(t)+N(t+s)-N(t)+N(t)}\right] \\
&= E\left[(-1)^{2N(t)}(-1)^{N(t+s)-N(t)}\right] \\
&= E\left[(-1)^{N(t+s)-N(t)}\right] \\
&= E\left[(-1)^{N(s)}\right], \quad \text{since the process has stationary increments} \\
&= \sum_{k=0}^{\infty}(-1)^k\ \frac{(\lambda s)^k}{k!} e^{-\lambda s} \\
&= e^{-\lambda s}\sum_{k=0}^{\infty}\frac{(-\lambda s)^k}{k!} \\
&= e^{-2\lambda s}
\end{aligned}$$

Thus $E[X(t)] = 0$ and $R_X(t, t+s) = e^{-2\lambda s}$.

b. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ and assume that it is independent of a non-negative random variable, T. Suppose the mean of T is μ and its variance is σ^2. Find (i). $E[N(T)]$, (ii). $\mathrm{Var}(N(T))$

Answer: For the first part, we can use $E[N(T)] = E[\ E[N(T) \mid T]\ ]$

$$E[N(T) \mid T = t] = E[N(t) \mid T = t] = E[N(t)] = \lambda t$$

In the above we have used the independence of T and $N(t)$. Thus, we have $E[N(T) \mid T] = \lambda T$. Hence, $E[N(T)] = E[\lambda T] = \lambda\ \mu$. Similarly,

$$E[N^2(T) \mid T = t] = E[N^2(t) \mid T = t] = E[N^2(t)] = \lambda t + (\lambda t)^2$$

Thus,

$$E[N^2(T) \mid T] = \lambda T + \lambda^2 T^2$$

which gives us

$$E[N^2(T)] = \lambda\, \mu + \lambda^2(\sigma^2 + \mu^2)$$

Hence

$$\text{Var}(N(T)) = E[N^2(T)] - (E[N(T)])^2 = \lambda\mu + \lambda^2(\sigma^2+\mu^2) - \lambda^2\mu^2 = \lambda\mu + \lambda^2\sigma^2$$

We could have also got the variance of $N(T)$ using the formula

$$\text{Var}(N(T)) = \text{Var}(E[N(T)|T]) + E[\text{Var}(N(T)|T)]$$

7. a. Let $\{X_n,\ n \geq 0\}$ be a Markov Chain. Let s_0, s_1, s_2 be some specific three states. Suppose the probabilities of transition out of s_0 are given by: $P(s_0, s_0) = 0.5; P(s_0, s_1) = 0.2; P(s_0, s_2) = 0.3$. Suppose the chain is started in s_0. Let T denote the first time instant when the chain left state s_0. (That is, $T = \min\{n : n \geq 1,\ X_n \neq s_0\}$). Find the distribution of T and X_T.

Answer: The event $[T = k]$ is same as $[X_1 = s_0, \cdots, X_{k-1} = s_0, X_k \neq s_0]$. Hence

$$P[T = k | X_0 = s_0] = (P(s_0, s_0))^{k-1}(1 - P(s_0, s_0)) = 0.5^{k-1}0.5,\ k = 1, 2, \cdots$$

Thus, T is geometric with parameter 0.5.

X_T is the state of the chain at the instant when the chain has left s_0. Hence $X_T \in \{s_1, s_2\}$.

To get the distribution of X_T we can argue intuitively as follows. Consider the chain started in s_0. We can think of the chain making a transition as a random experiment whose out come is the next state. If we consider the chain only till the first instant when it is not in s_0, for this random experiment $\Omega = \{s_0, s_1, s_2\}$. We can think of the situation as independent repetitions of this experiment till one of the events $\{s_1\}$ or $\{s_2\}$ occurs. The event of $[X_T = s_1]$ is same as $\{s_1\}$ occurring before $\{s_2\}$ in this repetition of our random experiment. Hence we can conclude

$$P[X_T = s_1] = \frac{P(s_0, s_1)}{P(s_0, s_1) + P(s_0, s_2)} = \frac{0.2}{0.2 + 0.3} = 0.4$$

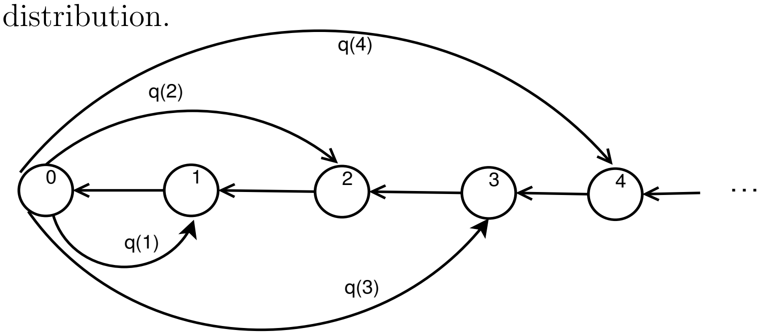
Hence $P[X_T = s_2] = 0.6$.

We can get it more formallylike as follows. Note all probabilities are conditioned on chain starting in s_0 though we do not explicitly show it.

$$\begin{aligned}
P[X_T = s_1] &= \sum_{k=1}^{\infty} P[X_T = s_1, T = k] \\
&= \sum_{k=1}^{\infty} P[X_1 = s_0, \cdots, X_{k-1} = s_0, X_k = s_1] \\
&= \sum_{k=1}^{\infty} (P(s_0, s_0))^{k-1} P(s_0, s_1) \\
&= \frac{P(s_0, s_1)}{1 - P(s_0, s_0)} \\
&= \frac{0.2}{0.2 + 0.3} = 0.4
\end{aligned}$$

And, hence, $P[X_T = s_2] = 0.6$.

b. Consider the following Markov chain on state space $\{0, 1, \cdots\}$. Take $q(k) = (1-p)^{k-1}p,\ k = 1, 2, \cdots$ with $0 < p < 1$. Will this chain have a stationary distribution? If yes, find the stationary distribution.



Answer: This is an irreducible Markov chain. It would have a unique stationary distribution if and only if it is positive recurrent. We can show the chain to be positive recurrent by calculating mean return time to state 0. Alternately, we can directly show that there is a distribution that satisfies the conditions for stationary distribution.

Suppose π is a stationarydistribution. Then it has to satisfy

$$\pi(j) = \sum_i \pi(i) P(i,j)$$

Taking $j = 0, 1, \cdots$, we get

$$\begin{aligned}
\pi(0) &= \pi(1) \\
\pi(1) &= \pi(0)q(1) + \pi(2) \quad \Rightarrow \quad \pi(2) = \pi(1) - q(1)\pi(0) = (1 - q(1))\pi(0) \\
&\quad \text{where we have used } \ \pi(1) = \pi(0) \\
\pi(2) &= \pi(0)q(2) + \pi(3) \quad \Rightarrow \quad \pi(3) = (1 - q(1) - q(2))\,\pi(0) \\
&\quad \text{where we used } \ \pi(2) = (1 - q(1))\pi(0) \\
&\ \vdots \\
\pi(k-1) &= \pi(0)q(k-1) + \pi(k) \quad \Rightarrow \quad \pi(k) = \left(1 - \sum_{j=1}^{k-1} q(j)\right)\pi(0) \\
&\ \vdots
\end{aligned}$$

Since $\left(1 - \sum_{j=1}^{k-1} q(j)\right) = \sum_{j=k}^{\infty} q(j)$, we get

$$\pi(k) = \pi(0)\sum_{j=k}^{\infty} q(j) = \pi(0)\sum_{j=k}^{\infty} p(1-p)^{j-1} = \pi(0)(1-p)^{k-1}, \ k = 1, 2, \cdots$$

This would be a stationary distribution if we can find $\pi(0)$ (with $0 < \pi(0) \leq 1$) such that $\sum_{j=0}^{\infty} \pi(j) = 1$. Thus we need

$$\pi(0) + \pi(0)\sum_{k=1}^{\infty}(1-p)^{k-1} = \pi(0) + \pi(0)\frac{1}{p} = 1$$

This implies $\pi(0) = \frac{p}{1+p}$. Hence the stationary distribution is

$$\pi(0) = \frac{p}{1+p}, \quad \pi(k) = \frac{p(1-p)^{k-1}}{1+p}, \ k = 1, 2, \cdots$$

Comment: As I said above, we could have actaully calculated mean recurrence time for state 0. Consider the chain started in state 0. Let T be time to return to 0. Then it is easy to see that

$$P[T = 1] = 0, \quad \text{and} \quad P[T = k] = q(k-1), \ k = 2, \cdots$$

Hence

$$m_0 = E_0[T] = \sum_{k=1}^{\infty}(k+1)q(k) = \sum_{k=1}^{\infty}(k+1)(1-p)^{k-1}p = \frac{1}{p}+1$$

Since $m_0 < \infty$, we can conclude that this irreducible chain is positive recurrent. Also, we know $\pi(0) = 1/m_0 = p/(1+p)$ which is same as what we calculated above. However, to get $\pi(1), \pi(2)$, etc. we need to write the equations for a stationary distribution and solve them.

8. a. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion process. Consider a process defined by

$$V(t) = e^{-\alpha t/2}\ B\left(e^{\alpha t}\right)$$

where $\alpha > 0$ is a parameter. Find the mean and autocorrelation of $V(t)$.

Answer: The mean of the process is given by

$$E[V(t)] = e^{-\alpha t/2}\ E\left[B\left(e^{\alpha t}\right)\right] = 0$$

because $E\left[B(t)\right] = 0,\ \forall t$.

The autocorrelation function is, for $t, s > 0$

$$\begin{aligned}
R(t,t+s) &= e^{-\alpha t/2}\ e^{-\alpha(t+s)/2} E\left[B\left(e^{\alpha t}\right)\ B\left(e^{\alpha(t+s)}\right)\right] \\
&= e^{-\alpha t/2}\ e^{-\alpha(t+s)/2}\ e^{\alpha t} \\
&\qquad \text{because} \quad E[B(t_1)B(t_2)] = \min(t_1,t_2) \\
&= e^{-\alpha s/2}
\end{aligned}$$

b. Suppose X is a Poisson random variable with mean λ. The λ itself is a random variable whose distribution is exponential with mean 1. Show that $P[X=n] = (0.5)^{n+1}$

Answer: Using the conditional expectation arguement, we have

$$\begin{aligned}
P[X=n] &= \int_{-\infty}^{\infty} P[X=n \mid \lambda]\ f(\lambda)\ d\lambda \\
&= \int_{-\infty}^{\infty} \frac{\lambda^n}{n!}\ e^{-\lambda}\ f(\lambda)\ d\lambda
\end{aligned}$$

$$
\begin{aligned}
& \quad \text{because, conditioned on } \lambda,\ X \text{ is Poisson} \\
&= \int_0^\infty \frac{\lambda^n}{n!}\, e^{-\lambda}\, e^{-\lambda}\, d\lambda \\
& \quad \text{because } \lambda \text{ is exponential with parameter 1} \\
&= \int_0^\infty \frac{(2\lambda)^n}{2^n\ n!} e^{-2\lambda}\ \frac{1}{2} d(2\lambda) \\
&= \frac{1}{2^{n+1}\ n!} \int_0^\infty (2\lambda)^n e^{-2\lambda}\ d(2\lambda) \\
&= \frac{1}{2^{n+1}\ n!}\ \Gamma(n+1) \\
&= \frac{1}{2^{n+1}\ n!}\ n! \\
&= (0.5)^{n+1}
\end{aligned}
$$