# Recap: Joint Distribution Function

- Given $X, Y$ rv's on same probability space, joint distribution function: $F_{XY} : \Re^2 \to \Re$

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

- It satisfies
  1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \forall x, y;$
     $F_{XY}(\infty, \infty) = 1$
  2. $F_{XY}$ is non-decreasing in each of its arguments
  3. $F_{XY}$ is right continuous and has left-hand limits in each of its arguments
  4. For all $x_1 < x_2$ and $y_1 < y_2$

$$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) \geq 0$$

- Any $F : \Re^2 \to \Re$ satisfying the above would be a joint distribution function.

# Recap: Joint Probability mass function

- $X \in \{x_1, x_2, \cdots\}$, $Y \in \{y_1, y_2, \cdots\}$
- The joint pmf: $f_{XY}(x, y) = P[X = x, Y = y]$.
- The joint pmf satisfies:
  A1 $f_{XY}(x, y) \geq 0, \forall x, y$ and non-zero only for $x_i, y_j$ pairs
  A2 $\sum_i \sum_j f_{XY}(x_i, y_j) = 1$
- Given the joint pmf, we can get the joint df as

$$F_{XY}(x, y) = \sum_{\substack{i: \\ x_i \leq x}} \sum_{\substack{j: \\ y_j \leq y}} f_{XY}(x_i, y_j)$$

- Any $f_{XY} : \Re^2 \to [0,\ 1]$ satisfying $A1$ and $A2$ above is a joint pmf. (The $F_{XY}$ satisfies all properties of df).
- Given the joint pmf, we can (in principle) compute the probability of any event involving the two discrete random variables.

$$P[(X, Y) \in B] = \sum_{\substack{i,j: \\ (x_i, y_j) \in B}} f_{XY}(x_i, y_j)$$

# Recap joint density

- Two cont rv $X, Y$ have a joint density $f_{XY}$ if
$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x', y')\, dy'\, dx', \ \ \forall x, y$$
- The joint density $f_{XY}$ satisfies the following
  1. $f_{XY}(x,y) \geq 0, \ \ \forall x, y$
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x', y')\, dy'\, dx' = 1$
- Any function $f_{XY} : \Re^2 \to \Re$ satisfying the above two is a joint density function. (Then the above $F_{XY}$ can be shown to be a joint df).
- We also have
$$P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\, dy\, dx$$
and, in general,
$$P[(X,Y) \in B] = \int_B f_{XY}(x,y)\, dx\, dy, \ \ \forall B \in \mathcal{B}^2$$

# Recap Marginals

- Marginal distribution functions of $X, Y$ are

$$F_X(x) = F_{XY}(x, \infty); \quad F_Y(y) = F_{XY}(\infty, y)$$

- $X, Y$ discrete with joint pmf $f_{XY}$. The marginal pmfs are

$$f_X(x) = \sum_y f_{XY}(x, y); \quad f_Y(y) = \sum_x f_{XY}(x, y)$$

- If $X, Y$ have joint pdf $f_{XY}$ then the marginal pdf are

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy; \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx$$

# Recap Conditional distributions

- Let $X, Y$ be continuous or discrete random variables

$$F_{X|Y}(x|y) = \lim_{\delta \to 0} P[X \leq x | Y \in [y, y+\delta]]$$

  $(= P[X \leq x | Y = y]$ when $Y$ is discrete)
- This is well defined for all values that $Y$ can assume.
- For each $y$, $F_{X|Y}(x|y)$ is a df in $x$.
- If $X, Y$ have a joint density or if $X$ is continuous and $Y$ is discrete, $F_{X|Y}$ would be absolutely continuous and would have a density.

# Recap Contional density (or mass) fn

- Let $X$ be a discrete random variable. Then

$$f_{X|Y}(x|y) = \lim_{\delta \to 0} P[X = x | Y \in [y,\ y + \delta]\ ]$$

  $(= P[X = x | Y = y]$ if $Y$ is discrete)
- This will be the mass function corresponding to the df $F_{X|Y}$.
- Let $X$ be a continuous rv. Then we define conditional density $f_{X|Y}$ by

$$F_{X|Y}(x|y) = \int_{-\infty}^{x} f_{X|Y}(x'|y)\ dx'$$

  This exists if $X, Y$ have a joint density or when $Y$ is discrete.

# Recap

- When $X, Y$ are both discrete or they have a joint density

$$f_{XY}(x,y) = f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

- When $X, Y$ are discrete or continuous (all four possibilities)

$$f_{X|Y}(x|y) f_Y(y) = f_{Y|X}(y|x) f_X(x)$$

  Here $f_{X|Y}, f_X$ are densities when $X$ is continuous and mass functions when $X$ is discrete. Similarly for $f_{Y|X}, f_Y$
- The above relation gives rise to the total probability rules and Bayes rule for rv's

# Recap

- If $Y$ is discrete

$$f_X(x) = \sum_y f_{X|Y}(x|y) f_Y(y)$$

- If $X$ is continuous, the $f_X, f_{X|Y}$ are densities; If $X$ is also discrete, they are mass functions
- If $Y$ is continuous

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x|y) f_Y(y) \, dy$$

- If $X$ is continuous, the $f_X, f_{X|Y}$ are densities; If $X$ is also discrete, they are mass functions (Where needed we assume the conditional density exists)

# Recap Bayes rule

- When $X, Y$ are continuous or discrete (all four possibilities)

$$f_{Y|X}(y|x) f_X(x) = f_{X|Y}(x|y)\ f_Y(y)$$

- This gives rise to Bayes rule:

$$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)\ f_Y(y)}{f_X(x)} \quad f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

- We need to interpret $f_X$ or $f_{X|Y}$ as mass functions when $X$ is discrete and as densities when $X$ is a continuous and so on

# Recap Independent Random variables

- $X$ and $Y$ are said to be independent if events $[X \in B_1]$, $[Y \in B_2]$ are independent for all $B_1, B_2 \in \mathcal{B}$.
- $X$ and $Y$ are independent if and only if
  1. $F_{XY}(x, y) = F_X(x)\ F_Y(y)$
  2. $f_{XY}(x, y) = f_X(x)\ f_Y(y)$
- This also implies $F_{X|Y}(x|y) = F_X(x)$ and $f_{X|Y}(x|y) = f_X(x)$

# More than two rv

- Everything we have done so far is easily extended to multiple random variables.
- Let $X, Y, Z$ be rv on the same probability space.
- We define joint distribution function by

$$F_{XYZ}(x, y, z) = P[X \leq x, Y \leq y, Z \leq z]$$

- If all three are discrete then the joint mass function is

$$f_{XYZ}(x, y, z) = P[X = x, Y = y, Z = z]$$

- If they are continuous , they have a joint density if

$$F_{XYZ}(x, y, z) = \int_{-\infty}^{z} \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XYZ}(x', y', z') \, dx' \, dy' \, dz'$$

- Easy to see that joint mass function satisfies
  1. $f_{XYZ}(x, y, z) \geq 0$ and is non-zero only for countably many tuples.
  2. $\sum_{x,y,z} \; f_{XYZ}(x, y, z) = 1$
- Similarly the joint density satisfies
  1. $f_{XYZ}(x, y, z) \geq 0$
  2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \; f_{XYZ}(x, y, z) \; dx \; dy \; dz = 1$
- These are straight-forward generalizations
- The properties of joint distribution function such as it being non-decreasing in each argument etc are easily seen to hold here too.
- Generalizing the special property of the df (relating to probability of cylindrical sets) is a little more complicated. (An exercise for you!)

- Now we get many different marginals:

$$F_{XY}(x,y) = F_{XYZ}(x,y,\infty); \;\; F_Z(z) = F_{XYZ}(\infty,\infty,z) \quad \text{and so on}$$

- Similarly we get

$$\begin{aligned} f_{YZ}(y,z) &= \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dx; \\ f_X(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\,dy\,dz \end{aligned}$$

- Any marginal is a joint density of a subset of these rv's and we obtain it by integrating the (full) joint density with respect to the remaining variables.
- We obtain the marginal mass functions for a subset of the rv's also similarly where we sum over the remaining variables.

- We have to be a little careful in dealing with these when some random variables are discrete and others are continuous.
- Suppose $X$ is continuous and $Y, Z$ are discrete. We do not have any joint density or mass function as such.
- However, the joint df is always well defined.
- Suppose we want marginal joint distribution of $X, Y$. We know how to get $F_{XY}$ by marginalization.
- Then we can get $f_X$ (a density), $f_Y$ (a mass fn), $f_{X|Y}$ (conditinal density) and $f_{Y|X}$ (conditional mass fn)
- With these we can generally calculate most quantities of interest.

- Like in case of marginals, there are different types of conditional distributions now.
- We can always define conditional distribution functions like

$$\begin{aligned} F_{XY|Z}(x,y|z) &= P[X \leq x, Y \leq y | Z = z] \\ F_{X|YZ}(x|y,z) &= P[X \leq x | Y = y, Z = z] \end{aligned}$$

- In all such cases, if the conditioning random variables are continuous, we define the above as a limit.
- For example when $Z$ is continuous

$$F_{XY|Z}(x,y|z) = \lim_{\delta \to 0} P[X \leq x, Y \leq y | Z \in [z, z+\delta]]$$

- If $X, Y, Z$ are all discrete then, all conditional mass functions are defined by appropriate conditional probabilities. For example,

$$f_{X|YZ}(x|y, z) = P[X = x|Y = y, Z = z]$$

- Thus the following are obvious

$$\begin{aligned} f_{XY|Z}(x, y|z) &= \frac{f_{XYZ}(x, y, z)}{f_Z(z)} \\ f_{X|YZ}(x|y, z) &= \frac{f_{XYZ}(x, y, z)}{f_{YZ}(y, z)} \\ f_{XYZ}(x, y, z) &= f_{Z|YX}(z|y, x) f_{Y|X}(y|x) f_X(x) \end{aligned}$$

- For example, the first one above follows from

$$P[X = x, Y = y|Z = z] = \frac{P[X = x, Y = y, Z = z]}{P[Z = z]}$$

- When $X, Y, Z$ have joint density, all such relations hold for the appropriate (conditional) densities. For example,

$$\begin{aligned} F_{Z|XY}(z|x,y) &= \lim_{\delta \to 0} \frac{P[Z \leq z, X \in [x, x+\delta], Y \in [y, y+\delta]]}{P[X \in [x, x+\delta, Y \in [y, y+\delta]]} \\ &= \lim_{\delta \to 0} \frac{\int_{-\infty}^{z} \int_{x}^{x+\delta} \int_{y}^{y+\delta} f_{XYZ}(x', y', z') \, dy' \, dx' \, dz'}{\int_{x}^{x+\delta} \int_{y}^{y+\delta} f_{XY}(x', y') \, dy' \, dx'} \\ &= \int_{-\infty}^{z} \frac{f_{XYZ}(x, y, z')}{f_{XY}(x, y)} \, dz' \end{aligned}$$

- Thus we get

$$f_{XYZ}(x, y, z) = f_{Z|XY}(z|x, y) f_{XY}(x, y) = f_{Z|XY}(z|x, y) f_{Y|X}(y|x) f_X(x)$$

- We can similarly talk about the joint distribution of any finite number of rv's
- Let $X_1, X_2, \cdots, X_n$ be rv's on the same probability space.
- We denote it as a vector $\mathbf{X}$ or $\underline{X}$. We can think of it as a mapping, $\quad \mathbf{X} : \Omega \to \Re^n$.
- We can write the joint distribution as

$$F_{\mathbf{X}}(\mathbf{x}) = P[\mathbf{X} \leq \mathbf{x}] = P[X_i \leq x_i, \ i = 1, \cdots, n]$$

- We represent by $f_{\mathbf{X}}(\mathbf{x})$ the joint density or mass function. Sometimes we also write it as $f_{X_1 \cdots X_n}(x_1, \cdots, x_n)$
- We use similar notation for marginal and conditional distributions

# Independence of multiple random variables

- Random variables $X_1, X_2, \cdots, X_n$ are said to be independent if the the events $[X_i \in B_i], \ i = 1, \cdots, n$ are independent.
  (Recall definition of independence of a set of events)
- Independence implies that the marginals would determine the joint distribution.

# Example

- Let a joint density be given by

$$f_{XYZ}(x,y,z) = K, \quad 0 < z < y < x < 1$$

First let us determine $K$.

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x,y,z)\; dz\; dy\; dx &= \int_0^1 \int_0^x \int_0^y K\; dz\; dy\; dx \\ &= K \int_0^1 \int_0^x y\; dy\; dx \\ &= K \int_0^1 \frac{x^2}{2}\; dx \\ &= K \frac{1}{6} \quad \Rightarrow K = 6\end{aligned}$$

# Example

- Let a joint density be given by

$$f_{XYZ}(x, y, z) = K, \quad 0 < z < y < x < 1$$

First let us determine $K$.

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\; dz\; dy\; dx &= \int_{x=0}^{1}\int_{y=0}^{x}\int_{z=0}^{y} K\; dz\; dy\; dx \\
&= K\int_{x=0}^{1}\int_{y=0}^{x} y\; dy\; dx \\
&= K\int_{0}^{1} \frac{x^2}{2}\; dx \\
&= K\frac{1}{6} \quad \Rightarrow K = 6
\end{aligned}$$

$$f_{XYZ}(x, y, z) = K, \quad 0 < z < y < x < 1$$

- Suppose we want to find the (marginal) joint distribution of $X$ and $Z$.

$$\begin{aligned} f_{XZ}(x, z) &= \int_{-\infty}^{\infty} f_{XYZ}(x, y, z) \, dy \\ &= \int_{z}^{x} K \, dy, \quad 0 < z < x < 1 \\ &= 6(x - z), \quad 0 < z < x < 1 \end{aligned}$$

- We got the joint density as

$$f_{XZ}(x,z) = 6(x-z), \quad 0 < z < x < 1$$

- We can verify this is a joint density

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XZ}(x,z)\ dz\ dx &= \int_0^1 \int_0^x 6(x-z)\ dz\ dx \\
&= \int_0^1 \left( 6x\ z|_0^x - 6\ \left.\frac{z^2}{2}\right|_0^x \right)\ dx \\
&= \int_0^1 \left( 6x^2 - 6\ \frac{x^2}{2} \right)\ dx \\
&= 3\ \left.\frac{x^3}{3}\right|_0^1 = 1
\end{aligned}$$

- The joint density of $X, Y, Z$ is

$$f_{XYZ}(x, y, z) = 6, \quad 0 < z < y < x < 1$$

- The joint density of $X, Z$ is

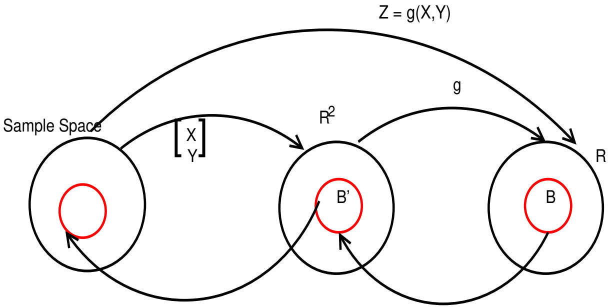
$$f_{XZ}(x, z) = 6(x - z), \quad 0 < z < x < 1$$

- Hence,

$$f_{Y|XZ}(y|x, z) = \frac{f_{XYZ}(x, y, z)}{f_{XZ}(x, z)} = \frac{1}{x - z}, \quad 0 < z < y < x < 1$$

# Functions of multiple random variables

- Let $X, Y$ be random variables on the same probability space.
- Let $g : \Re^2 \rightarrow \Re$.
- Let $Z = g(X, Y)$. Then $Z$ is a rv
- This is analogous to functions of a single rv



- let $Z = g(X, Y)$
- We can determine distribution of $Z$ from the joint distribution of $X, Y$

$$F_Z(z) = P[Z \leq z] = P[g(X, Y) \leq z]$$

- For example, if $X, Y$ are discrete, then

$$f_Z(z) = P[Z = z] = P[g(X, Y) = z] = \sum_{\substack{x_i, y_j : \\ g(x_i, y_j) = z}} f_{XY}(x_i, y_j)$$

- Let $X, Y$ be discrete rv's. Let $Z = \min(X, Y)$.

$$
\begin{aligned}
f_Z(z) &= P[\min(X,Y) = z] \\
&= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \\
&= \sum_{y>z} P[X = z, Y = y] + \sum_{x>z} P[X = x, Y = z] \\
&\qquad\qquad\qquad\qquad + P[X = z, Y = z] \\
&= \sum_{y>z} f_{XY}(z, y) + \sum_{x>z} f_{XY}(x, z) + f_{XY}(z, z)
\end{aligned}
$$

- Now suppose $X, Y$ are independent and both of them have geometric distribution with the same parameter, $p$.
- Such random variables are called **independent and identically distributed** or **iid** random variables.

- Now we can get pmf of $Z$ as (note $Z \in \{1, 2, \cdots\}$)

$$
\begin{aligned}
f_Z(z) &= P[X = z, Y > z] + P[Y = z, X > z] + P[X = Y = z] \\
&= P[X = z]P[Y > z] + P[Y = z]P[X > z] + P[X = z]P[Y = z] \\
&= p(1-p)^{z-1}(1-p)^z * 2 + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{z-1}(1-p)^z + \left(p(1-p)^{z-1}\right)^2 \\
&= 2p(1-p)^{2z-1} + p^2(1-p)^{2z-2} \\
&= p(1-p)^{2z-2}(2(1-p) + p) \\
&= (2-p)p(1-p)^{2z-2}
\end{aligned}
$$

- We can show this is a pmf

$$\begin{aligned}
\sum_{z=1}^{\infty} f_Z(z) &= \sum_{z=1}^{\infty} (2-p)p(1-p)^{2z-2} \\
&= (2-p)p \sum_{z=1}^{\infty} (1-p)^{2z-2} \\
&= (2-p)p \, \frac{1}{1-(1-p)^2} \\
&= (2-p)p \, \frac{1}{2p-p^2} = 1
\end{aligned}$$

- Let us consider the $\max$ and $\min$ functions, in general.
- Let $Z = \max(X, Y)$. Then we have

$$\begin{aligned} F_Z(z) &= P[Z \leq z] = P[\max(X,Y) \leq z] \\ &= P[X \leq z, Y \leq z] \\ &= F_{XY}(z,z) \\ &= F_X(z)F_Y(z), \quad \text{if } X, Y \text{ are independent} \\ &= \left(F_X(z)\right)^2, \quad \text{if they are iid} \end{aligned}$$

- This is true of all random variables.
- Suppose $X, Y$ are iid continuous rv. Then density of $Z$ is

$$f_Z(z) = 2F_X(z)f_X(z)$$

- Suppose $X, Y$ are iid uniform over $(0,\ 1)$
- Then we get df and pdf of $Z = \max(X, Y)$ as

$$F_Z(z) = z^2, 0 < z < 1; \quad \text{and} \quad f_Z(z) = 2z, 0 < z < 1$$

$F_Z(z) = 0$ for $z \leq 0$ and $F_Z(z) = 1$ for $z \geq 1$ and $f_Z(z) = 0$ outside $(0, 1)$

- This is easily generalized to $n$ radom variables.
- Let $Z = \max(X_1, \cdots, X_n)$

$$\begin{aligned}
F_Z(z) &= P[Z \leq z] = P[\max(X_1, X_2, \cdots, X_n) \leq z] \\
&= P[X_1 \leq z, X_2 \leq z, \cdots, X_n \leq z] \\
&= F_{X_1 \cdots X_n}(z, \cdots, z) \\
&= F_{X_1}(z) \cdots F_{X_n}(z), \quad \text{if they are independent} \\
&= (F_X(z))^n, \quad \text{if they are iid} \\
&\qquad \text{where we take } F_X \text{ as the common df}
\end{aligned}$$

- For example if all $X_i$ are uniform over $(0, 1)$ and ind, then $F_Z(z) = z^n, \ 0 < z < 1$

- Consider $Z = \min(X, Y)$ and $X, Y$ independent

$$F_Z(z) = P[Z \leq z] = P[\min(X, Y) \leq z]$$

- It is difficult to write this in terms of joint df of $X, Y$.
- So, we consider the following

$$\begin{aligned}
P[Z > z] &= P[\min(X, Y) > z] \\
&= P[X > z, Y > z] \\
&= P[X > z]P[Y > z], \quad \text{using independence} \\
&= (1 - F_X(z))(1 - F_Y(z)) \\
&= (1 - F_X(z))^2, \quad \text{if they are iid}
\end{aligned}$$

$$\text{Hence,} \quad F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z))$$

- We can once again find density of $Z$ if $X, Y$ are continuous

- Suppose $X, Y$ are iid uniform $(0,\ 1)$.
- $Z = \min(X, Y)$

$$F_Z(z) = 1 - (1 - F_X(z))^2 = 1 - (1 - z)^2, 0 < z < 1$$

- Notice that $P[X > z] = (1 - z)$.
- We get the density of $Z$ as

$$f_Z(z) = 2(1 - z), \ \ 0 < z < 1$$

- $\min$ fn is also easily generalized to $n$ random variables
- Let $Z = \min(X_1, X_2, \cdots, X_n)$

$$\begin{aligned}
P[Z > z] &= P[\min(X_1, X_2, \cdots, X_n) > z] \\
&= P[X_1 > z, \cdots, X_n > z] \\
&= P[X_1 > z] \cdots P[X_n > z], \quad \text{using independence} \\
&= (1 - F_{X_1}(z)) \cdots (1 - F_{X_n}(z)) \\
&= (1 - F_X(z))^n, \quad \text{if they are iid}
\end{aligned}$$

- Hence, when $X_i$ are iid, the df of $Z$ is

$$F_Z(z) = 1 - (1 - F_X(z))^n$$

where $F_X$ is the common df

- Let $X, Y$ be independent
- Let $Z = \max(X, Y)$ and $W = \min(X, Y)$.
- We want joint distribution function of $Z$ and $W$.

$$F_{ZW}(z, w) = P[Z \leq z, W \leq w]$$

- This is difficult to find. But we can easily find

$$P[\max(X, Y) \leq z, \min(X, Y) > w]$$

- Remaining details are left as an exercise for you!!

- Let $X, Y \in \{0, 1, \cdots\}$
- Let $Z = X + Y$. Then we have

$$\begin{aligned} f_Z(z) &= P[X+Y=z] = \sum_{\substack{x,y: \\ x+y=z}} P[X=x, Y=y] \\ &= \sum_{k=0}^{z} P[X=k, Y=z-k] \\ &= \sum_{k=0}^{z} f_{XY}(k, z-k) \end{aligned}$$

- Now suppose $X, Y$ are independent. Then

$$f_Z(z) = \sum_{k=0}^{z} f_X(k) f_Y(z-k)$$

- Now suppose $X, Y$ are independent Poisson with parameters $\lambda_1, \lambda_2$. And, $Z = X + Y$.

$$
\begin{aligned}
f_Z(z) &= \sum_{k=0}^{z} f_X(k) f_Y(z-k) \\
&= \sum_{k=0}^{z} \frac{\lambda_1^k}{k!} e^{-\lambda_1} \; \frac{\lambda_2^{z-k}}{(z-k)!} e^{-\lambda_2} \\
&= e^{-(\lambda_1+\lambda_2)} \; \frac{1}{z!} \sum_{k=0}^{z} \frac{z!}{k!(z-k)!} \; \lambda_1^k \; \lambda_2^{z-k} \\
&= e^{-(\lambda_1+\lambda_2)} \; \frac{1}{z!} (\lambda_1 + \lambda_2)^z
\end{aligned}
$$

- $Z$ is Poisson with parameter $\lambda_1 + \lambda_2$

- Let $X, Y$ have a joint density $f_{XY}$. Let $Z = X + Y$

$$\begin{aligned}
F_Z(z) &= P[Z \leq z] = P[X + Y \leq z] \\
&= \int\int_{\{(x,y):x+y\leq z\}} f_{XY}(x,y)\ dy\ dx \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\ dy\ dx \\
&\quad \text{change of variable: } t = x + y \\
&\qquad dt = dy; \quad \text{when } (y = z - x),\ t = z \\
&= \int_{x=-\infty}^{\infty} \int_{t=-\infty}^{z} f_{XY}(x, t-x)\ dt\ dx \\
&= \int_{-\infty}^{z} \left( \int_{-\infty}^{\infty} f_{XY}(x, t-x)\ dx \right)\ dt
\end{aligned}$$

- This gives us

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z-x)\ dx$$

- $X, Y$ have joint density $f_{XY}$. $Z = X + Y$. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx$$

- Now suppose $X$ and $Y$ are independent. Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z - x)\, dx$$

  Density of sum of independent random variables is the convolution of their densities.

$$f_{X+Y} = f_X * f_Y \quad \text{(Convolution)}$$

- Suppose $X$, $Y$ are iid exponential rv's.

$$f_X(x) = \lambda\, e^{-\lambda x}, \;\; x > 0$$

- Let $Z = X + Y$. Then, density of $Z$ is

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx \\ &= \int_0^z \lambda\, e^{-\lambda x}\, \lambda\, e^{-\lambda(z-x)}\, dx \\ &= \lambda^2\, e^{-\lambda z} \int_0^z dx = \lambda^2\, z\, e^{-\lambda z} \end{aligned}$$

- Thus, sum of independent exponential random variables has gamma distribution:

$$f_Z(z) = \lambda z\, \lambda e^{-\lambda z}, \;\; z > 0$$