CMO

Sheet 3 Solutions — E0230

For Tutors Only — Not For Distribution

Assignment (Due: 2 January 2021)

Instructions

- Answer all questions
- For numerical questions, answer without any leading or trailing spaces
- You can submit MS Teams Form **only once**
- If the value of answer is an **integer**, give answer in box provided as an integer alone **without any decimal point**. For example 0, 10
- Otherwise, if the value of answer is a **decimal number**, give answer in x.yy format **rounded to 2 decimal places**. For example 0.50, 1.50
- Follow this instruction unless otherwise specified in the question
- Late submissions will be penalised

1. Consider the problem of minimizing the function $f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$.
   (a) Determine the point $\hat{x} = (\hat{x_1}, \hat{x_2})$ corresponding to $\nabla f(\hat{x}) = 0$.
      (i) (1 point) $\hat{x_1} =$ ____________
      (ii) (1 point) $\hat{x_2} =$ ____________
   (b) (3 points) Is $\hat{x}$ a local minimum or a maximum for f ?
   (c) Apply Newton's method for two iterations starting from $x^{(0)} = (0, 0)$ and give the resultant value of $x^{(2)}$.
      (i) (2.5 points) $x_1^{(2)} =$ ____________
      (ii) (2.5 points) $x_2^{(2)} =$ ____________
   (d) (3 points) Does Newton's method converge if the initial point is $(100, 100)$?
   (e) (2 points) Does the steepest descent method with fixed step size $\alpha = 0.05$ and initial point $(100, 100)$ converge?

   Solution:

   (a) $x^* = (1, 1)$.

   (b) Local minimum, since Hessian is positive definite.

   (c) $x^{(2)} = (1, 1)$.

   (d) Yes.

   (e) No.

2. (3 points) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x-5)^4$. Let $x^{(k)}$ be the sequence obtained by applying Newton's method to this function, with $x^{(0)} = 10$, and let x^* be the global minimum of f. Determine the value of

$$\lim_{k\to\infty} \frac{|x^{(k+1)} - x^*|}{|x^{(k)} - x^*|}$$

   Solution: $x^* = 5$, $f'(x) = 4(x - 5)^3$ and $f''(x) = 12(x - 5)^2$, so the Newton's step becomes

$$x^{(k+1)} = x^{(k)} - \frac{4(x-5)^3}{12(x-5)^2}$$

$$\implies |x^{(k+1)} - x^*| = |x^{(k)} - 5 - \frac{x^{(k)} - 5}{3}| = \frac{2}{3}|x^{(k)} - 5|.$$

   So, $\lim_{k\to\infty} \frac{|x^{(k+1)}-x^*|}{|x^{(k)}-x^*|} = \frac{2}{3} = 0.67$.

3. Consider the problem of minimizing $f(x) = (\sqrt[3]{x})^4$.
   (a) (1 point) Determine $x^* = \text{argmin}_x f(x)$.
   (b) (2 points) Determine the maximum value of a such that Newton's method converges when started from the interval $[-a, a]$.

Solution:

(a) $x^* = 0$.

(b) The Newton's step becomes $x^{(k+1)} = -2x^{(k)}$, so $a = 0$.

4. Apply the DFP Quasi-Newton updates for minimizing the function

$$f(x) = x^T \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T x,$$

starting with $x^{(0)} = (0 \quad 0)^T$ and $B^{(0)} = I_{2\times 2}$.

(a) What is the global minimum x^* of f?
   (i) (1 point) $x_1^* =$ ____________
   (ii) (1 point) $x_2^* =$ ____________

(b) (1 point) How many iterations were required to obtain the result?

(c) Give the value of $B^{(1)} = \begin{pmatrix} B_{1,1}^{(1)} & B_{1,2}^{(1)} \\ B_{2,1}^{(1)} & B_{2,2}^{(1)} \end{pmatrix}$:
   (i) (0.5 points) $B_{1,1}^{(1)} =$ ____________
   (ii) (0.5 points) $B_{1,2}^{(1)} =$ ____________
   (iii) (0.5 points) $B_{2,1}^{(1)} =$ ____________
   (iv) (0.5 points) $B_{2,2}^{(1)} =$ ____________

Solution:

(a) $x^* = (-1, 3/2)$.

(b) 2 iterations.

(c) $B^{(1)} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{pmatrix}$.

5. Consider the python functions for a function $f(x)$, its gradient $\nabla f(x)$ and Hessian inverse $(\nabla^2 f(x))^{-1}$ given in the files f.pkl, grad_f.pkl and hessian_inv.pkl, which take as input numpy arrays of length 2 and give their output as numpy arrays. Apply Newton's

method starting from initial value $x^{(0)} = (0, 0)$, where the step size during each iteration is determined using backtracking, with parameters $\alpha = 0.1$ and $\beta = 0.7$. This means that at each step, an initial step size of $t = 1$ is chosen and updated as $t \leftarrow \beta t$, until it satisfies

$$f(x + tu) \leq f(x) + \alpha t \nabla f(x)^T u,$$

where u is the update direction. Iterate for k iterations until the update distance $||x^{(k)} - x^{(k-1)}|| < \epsilon = 0.001$.

(a) What is the final value $x^{(k)}$ obtained?

  (i) (2.5 points) $x_1^{(k)} =$ ____________

  (ii) (2.5 points) $x_2^{(k)} =$ ____________

(b) (5 points) How many iterations were required to obtain the result?
$k =$ ____________

Solution:

(a) Final solution : $(2, 4)$.

(b) Number of iterations: 18.

6. Apply the conjugate gradient algorithm to minimize the function $f(x) = \frac{1}{2}x^TQx - b^Tx$, with the following values of Q and b:

$$Q = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, b = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix},$$

with initial value $x_0 = (0 \quad 0 \quad 0)^T$.

(a) Give the final solution $x^* = (x_1^* \quad x_2^* \quad x_3^*)$.

  (i) (1 point) $x_1^* =$ ____________

  (ii) (1 point) $x_2^* =$ ____________

  (iii) (1 point) $x_3^* =$ ____________

(b) (4 points) How many iterations were required for converging to the solution?

Solution:

(a) $x^* = (3, -1, -1)$.

(b) Number of iterations: 2.

7. Apply the conjugate gradient algorithm to minimize the function $f(x) = \frac{1}{2}x^T Q x - b^T x$, with the following values of Q and b:

$$Q = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 5 & 0 & 2 \\ -1 & 0 & 6 & 0 \\ 1 & 2 & 0 & 3 \end{pmatrix}, b = \begin{pmatrix} 0 \\ 2 \\ -1 \\ 1 \end{pmatrix},$$

with initial value $x^{(0)} = (0 \quad 0 \quad 0 \quad 0)^T$.

(a) Give the final solution $x^* = (x_1^* \quad x_2^* \quad x_3^* \quad x_4^*)$.

(i) (0.5 points) $x_1^* =$ ____________

(ii) (0.5 points) $x_2^* =$ ____________

(iii) (0.5 points) $x_3^* =$ ____________

(iv) (0.5 points) $x_4^* =$ ____________

(b) (5 points) How many iterations were required for converging to the solution?

Solution:

(a) $x^* = (-65, 24, -11, 6)$.

(b) Number of iterations: 4.