E1 222 Stochastic Models and Applications
Problem Sheet 3–3

1. Let X, Y be independent discrete random variables uniformly distributed over $\{0, 1, \cdots, N\}$. Find the pmf of Z when (i). $Z = |Y - X|$, (ii). $Z = X + Y$

2. Let X, Y be *iid* geometric random variables. Find pmf of $Z = X + Y$.

3. Let (X, Y) have joint density

$$f_{XY}(x, y) = x + y, \; 0 < x < 1, \; 0 < y < 1.$$

   Find the marginal densities, the conditional densities and ρ_{XY}. Find $P[X > 2Y]$. Are X, Y independent?

4. Let X, Y have a joint distribution that is uniform over the quadrilateral with vertices at $(-1, 0), (1, 0), (0, -1)$ and $(0, 1)$. Find $P[X > Y]$. Are X, Y independent? (Hint: Can you decide on independence without calculating the marginal densities?)

5. Let X, Y be *iid* uniform over $(0, \; 1)$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$. Find the density of $Z - W$.

6. Let X, Y be iid exponential random variables with mean 1. Let $Z = X + Y$ and $W = X - Y$. Find the conditional density $f_{W|Z}$

7. Let X, Y be independent random variables each having normal density with mean zero and variance unity. Find the joint density of $aX + bY$ and $bX - aY$.

8. Let X, Y be independent Gaussian random variables with mean zero and variance unity. Define random variables D and θ by

$$D = X^2 + Y^2; \qquad \theta = \tan^{-1}(Y/X)$$

   (where, by convention, we assume θ takes values in $[0, 2\pi]$; for this we first calculate $\tan^{-1}(|Y|/|X|)$ in the range $[0, \pi/2]$ and then put that angle in the appropriate quadrant based on signs of Y and X). Find the joint density of D and θ and their marginal densities. Are D and θ independent? (Hint: Note that the (D, θ) to (X, Y) mapping is invertible. Hence you can use the formula).

9. Consider the following algorithm for generating random numbers X and Y :

    1. Generate U_1 and U_2 uniform over $[0,\ 1]$.
    2. Set $X = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$ and $Y = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$.

    What would be the joint distribution of X and Y ? (Hint: Recall that when U is uniform over $[0,\ 1]$ $-a\log(U)$ is exponential with parameter $1/a$ and $2\pi U$ is uniform over $[0, 2\pi]$).

10. Consider the following algorithm for generating random variables V_1 and V_2 :

    1. Generate X_1 and X_2 uniform over $[-1,\ 1]$.
    2. If $X_1^2 + X_2^2 > 1$ then go to step 1; else set $V_1 = X_1,\ V_2 = X_2$ and exit.

    What would be the joint distribution of V_1 and V_2 ?

11. Suppose we have access to a random number generator that can generate random numbers uniformly distributed over $(0, 1)$. Using the results of the previous problems, suggest a method for generating samples of X when X has Gaussian density with mean zero and variance unity.