E1 222 Stochastic Models and Applications
Problem Sheet 2–1

1. State whether the following sequences of sets are monotone.

   (i). $A_k = [0,\ 1 + \frac{(-1)^k}{k}],\ k = 1, 2, \cdots$

   (ii). $A_k = [1/k,\ 1],\ k = 1, 2, \cdots$

Answer: (i). not monotone, (ii). monotone

2. Let $A_k = (-1/k,\ 1],\ k = 1, 2, \cdots$. Let $B = \cap_{k=1}^{\infty} A_k$. For any $x < 0$, show that there is a K such that $x \notin A_K$. For any x, such that $0 < x < 1$, show that $x \in A_k,\ \forall k$. Now determine what B is.

Answer: $B = [0, 1]$

3. Let $A_k = [1/k,\ 1],\ k = 1, 2, \cdots$. Let $B = \cup_{k=1}^{\infty} A_k$. For any $0 < x < 1$, show that there is a K such that $x \in A_K$. Now determine what B is.

Answer: $B = (0, 1]$

4. Let $(\Omega,\ \mathcal{F},\ P)$ be a probability space and let $A_1, A_2 \in \mathcal{F}$. Consider the following random variable:

$$\begin{aligned} X(\omega) &= -1 \quad \text{if} \quad \omega \in A_1 \\ &= +1 \quad \text{if} \quad \omega \in A_1^c A_2 \\ &= \ \ 0 \quad \text{if} \quad \omega \in A_1^c A_2^c \end{aligned}$$

   What is the event $[X < 0.5]$? Find the distribution function of X.

Answer: $[X < 0.5] = A_1 \cup A_1^c A_2^c$

$$F_X(x) = \begin{cases} 0 & \text{if } \ x < -1 \\ P(A_1) & \text{if } \ -1 \le x < 0 \\ P(A_1) + P(A_1^c A_2^c) & \text{if } \ 0 \le x < 1 \\ 1 & \text{if } \ x \ge 1 \end{cases}$$

5. Consider the probability space with $\Omega = [0, 1]$ and the usual probability assignment (where probability of an interval is the length of the interval). Define X by $X(\omega) = 2\omega$ if $0 \le \omega \le 0.5$, and $X(\omega) = 2\omega - 0.5$ if $0.5 < \omega \le 1$. What is the event $[X \in (0.5,\ 0.75)]$? Find the distribution function of X.

Answer:

$$[X \in (0.5,\ 0.75)] = [0.25,\ 0.75/2] \cup [0.5,\ 1.25/2]$$

$$\begin{aligned} F_X(x) &= 0 \ \text{ if } \ x < 0 \\ &= x/2 \ \text{ if } \ 0 \le x \le 0.5 \\ &= (2x - 0.5)/2 \ \text{ if } \ 0.5 \le x \le 1 \\ &= (x + 0.5)/2 \ \text{ if } \ 1 \le x \le 1.5 \\ &= 1 \ \text{ if } \ x \ge 1.5 \end{aligned}$$

6. Let X be a random variable with $P[X = a] = 0$. Express $P[|X| \ge a]$ in terms of the distribution function of X.

Answer: $P[|X| \ge a] = 1 - (F_X(a) - F_X(-a))$