E1 222 Stochastic Models and Applications
Problem Sheet 3–1

1. Let

$$\begin{aligned} F(x,y) &= 0, \quad \text{if } x<0, \text{ or } \ y<0, \text{ or } \ x+y<1 \\ &= 1, \quad \text{otherwise} \end{aligned}$$

Show that F satisfies the following: $F(-\infty,-\infty)=0$; $F(\infty,\infty)=1$; F is non-decreasing in each variable. Is $F(x,y)$ a distribution function? (Hint: If it were the joint distri- bution of two random variables, X, Y, what would be $P[1/3 < X \leq 1, \ 1/3 < Y \leq 1]$).

Answer: I hope it is straight-forward to see that $F(x,-\infty) = F(-\infty, y) = 0$ and $F(\infty,\infty) = 1$ from the definition of F. Next we want to show $F(x_1, y) \leq F(x_2, y)$, $\forall x_1, x_2$ with $x_1 < x_2$ and $\forall y$. If $y < 0$ then both the quantities are zero and hence the inequality is satisfied. Similarly, if $x_1 < 0$ then $F(x_1, y) = 0$ while $F(x_2, y) \geq 0$ and hence the inequality is again satisfied. So, let us consider the case $y > 0, x_2 > x_1 > 0$. If $x_1 + y \geq 1$ then $x_2 + y \geq 1$ and hence the inequality is satisfied because both terms are 1. If $x_1 + y < 1$ then $F(x_1, y) = 0$ and hence once again the inequality is satisfied. Similarly you can show $F(x, y_1) \leq F(x, y_2)$, $\forall y_1, y_2$ with $y_1 < y_2$ and $\forall x$. We can also show that the function is right continuous in each variable because all inequalities are strict in the first line of function definition. However, this function is not a distribution function:

$$F(1,1) - F(1/3,1) - F(1,1/3) + F(1/3,1/3) = 1 - 1 - 1 + 0 < 0$$

2. Let F_1 and F_2 be two one dimensional continuous distribution functions with f_1 and f_2 being the corresponding densities. Define a function $f : \Re^2 \to \Re$ by

$$f(x,y) = f_1(x) f_2(y) \left[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)\right]$$

where α is a real number. Show that $f(x,y)$ is a two dimensional density function for all $\alpha \in (-1,1)$. Show that the two marginals of $f(x,y)$ are f_1 and f_2. What does this imply about determining the joint density from the marginals? (Note that $\int_{-\infty}^{\infty} F_1(x) f_1(x) \, dx = \frac{1}{2}$).

Answer: Since $0 \leq F_1(x) \leq 1$, we have $-1 \leq (2F_1(x) - 1) \leq 1$. Same is true of F_2. If $\alpha \in (-1, 1)$ then $-1 \leq \alpha(2F_1(x) - 1)(2F_2(y) - 1) \leq 1$ and hence $f(x, y) \geq 0,\ \forall x, y$. Now to show it is a density we need to show

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy = 1$$

The first term in the integral above is

$$\int_{-\infty}^{\infty} f_1(x)\ dx \int_{-\infty}^{\infty} f_2(y)\ dy = 1$$

because both f_1 and f_2 are densities. The second term in the integral of $f(x, y)$ is zero because

$$\int_{-\infty}^{\infty} f_1(x)(2F_1(x)-1)\ dx = 2\int_{-\infty}^{\infty} f_1(x)F_1(x)\ dx - \int_{-\infty}^{\infty} f_1(x)\ dx = 2\frac{1}{2} - 1 = 0$$

So, $f(x, y)$ is a density. Let us say it is the joint density of X, Y. Now suppose we want to find marginal of Y. Then

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\ dx = f_2(y)$$

because $\int_{-\infty}^{\infty} f_1(x)\ dx = 1$ and $\int_{-\infty}^{\infty} f_1(x)(2F_1(x) - 1)\ dx = 0$. Similarly we can show the other marginal is $f_1(x)$.

So, no matter what the value of α is, the marginals are same. But changing α changes $f(x, y)$. Thus, there are infinitely many joint densities all having the same marginals.

Note that we can easily generalize this example to create such joint densities for any number of random variables.

3. Let F_{XY} be a joint distribution function with F_X and F_Y being the corresponding marginal distribution functions. Show that

$$1 - (1 - F_X(x) + 1 - F_Y(y)) \leq F_{XY}(x, y) \leq \min(F_X(x), F_Y(y)),\ \forall x, y$$

Answer: For any events A, B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Hence $P(AB) \leq P(A)$ and $P(AB) \leq P(B)$. This gives us $F_{XY}(x, y) \leq \min(F_X(x), F_Y(y)),\ \forall x, y$. We also have $P(A \cap B) = 1 - P(A^c \cup B^c) \geq 1 - (P(A^c) + P(B^c))$ because

$P(A^c \cup B^c) \leq (P(A^c) + P(B^c))$. This gives us $1 - (1 - F_X(x) + 1 - F_Y(y)) \leq F_{XY}(x, y)$.

We can show something more. Suppose as earlier F_{XY} is a 2D distribution function. Suppose F_1 and F_2 are distribution functions satisfying

$$1 - (1 - F_1(x) + 1 - F_2(y)) \leq F_{XY}(x, y) \leq \min(F_1(x), F_2(y)), \ \forall x, y$$

Then F_1 and F_2 are the marginals from F_{XY}. By putting $y = \infty$ in the above we get $F_1(x) \leq F_{XY}(x, \infty) \leq F_1(x)$, $\forall x$, showing F_1 is marginal from F_{XY}. Similarly you can show for F_2.

4. Consider n Bernoulli trials. Let X denote the number of successes and let Y denote the trial number of first success. Find $f_{Y|X}(y|1)$.

Answer: Done in class

5. Let X, Y be independent discrete random variables each being uniform over $\{0, 1, \cdots, N\}$. Find $P[X > Y]$ and $P[X < Y]$.

Answer: The event $[X > Y]$ is the mutually exclusive union of events of the type $[X = k, Y > k]$. Hence

$$P[X > Y] = \sum_{k=0}^{N} \frac{1}{N+1} \frac{N-k}{N+1} = \frac{1}{(N+1)^2} \left(N(N+1) - \frac{N(N+1)}{2} \right) = \frac{N}{2(N+1)}$$

Now, what would be $P[X < Y]$? The intuitive idea is that since X, Y are iid, essentially it is arbitrary which is called X and which is called Y. Hence, the two probabilities should be same. The calculation gives you the same answer. What would be $P[X = Y]$?