# E1 222 Stochastic Models and Applications
## Problem Sheet 3–3

1. Let $X, Y$ be independent discrete random variables uniformly distributed over $\{0, 1, \cdots, N\}$. Find the pmf of $Z$ when (i). $Z = |Y - X|$, (ii). $Z = X + Y$

Hint: (i) $Z = |Y - X|$. So, $Z \in \{0, 1, \cdots, N\}$.
$P[Z = 0] = P[X = Y]$. For non-zero $k$, $[Z = k]$ is disjoint union of $[Y - X = k]$ and $[X - Y = k]$. The final answer will be

$$f_Z(k) = \begin{cases} \frac{2(N-k+1)}{(N+1)^2} & k = 1, \cdots N \\ \frac{1}{N+1} & k = 0 \end{cases}$$

(ii). $Z = X + Y$ implies $Z \in \{0, 1, \cdots, 2N\}$

$$P[Z = k] = P[X + Y = k] = \sum_i P[X = i, Y = k - i]$$

We need $0 \leq i \leq N$ and $0 \leq k - i \leq N$ or $\max(0, k - N) \leq i \leq \min(N, k)$. Hence, for $0 \leq k \leq N$ the summation is over 0 to $k$ and for $N + 1 \leq k \leq 2N$ the summation is over $k - N$ to $N$. The final answer is

$$f_Z(k) = \begin{cases} \frac{(k+1)}{(N+1)^2} & 0 \leq k \leq N \\ \frac{2N-k+1}{(N+1)^2} & N + 1 \leq k \leq 2N \end{cases}$$

You should always verify your answers by summing the mass function.

2. Let $X, Y$ be *iid* geometric random variables. Find pmf of $Z = X + Y$.

Hint: This is similar to the previous problem

$$P[Z = k] = P[X + Y = k] = \sum_i P[X = i, Y = k - i]$$

We need $i \geq 1$ and $k - i \geq 1$. So, the summation should be from 1 to $k - 1$

$$P[Z = k] = \sum_{i=1}^{k-1} (1 - p)^{i-1} p (1 - p)^{k-i-1} p = p^2 (1 - p)^{k-2} (k - 1)$$

3. Let $(X, Y)$ have joint density

$$f_{XY}(x, y) = x + y, \ 0 < x < 1, \ 0 < y < 1.$$

Find the marginal densities, the conditional densities and $\rho_{XY}$. Find $P[X > 2Y]$. Are $X, Y$ independent?

4. Let $X, Y$ have a joint distribution that is uniform over the quadrilateral with vertices at $(-1, 0), (1, 0), (0, -1)$ and $(0, 1)$. Find $P[X > Y]$. Are $X, Y$ independent? (Hint: Can you decide on independence without calculating the marginal densities?)

Hint: Here, drawing a figure would help. Since $X, Y$ are uniform over some region, probability of any subset would be proportional to the area of that subset. That is the easy way to find $P[X > Y]$. Of course, you can also get the answer by integrating the joint density.

Similarly, to test independence, you need not find marginal densities and ask whether the joint is a product of marginals. Looking at the figure, you can see, for example, that the range of values that $X$ can take depends on value of $Y$. Hence you intuitively know they are not independent. In such situations a easy way to show that they are not independent is to identify $A, B$ such that $P[X \in A | Y \in B] = 0$ but $P[X \in A] \neq 0$. For example, you can look at $P[X < -0.75 | Y > 0.5]$.

5. Let $X, Y$ be *iid* uniform over $(0,\ 1)$. Let $Z = \max(X, Y)$ and $W = \min(X, Y)$. Find the density of $Z - W$.

Hint: We found the expression for $f_{ZW}$ in class. You also know how to find density of $X - Y$ if you know joint density of $X, Y$. So, it is just a matter of combining the two. The final answer is

$$f_{Z-W}(u) = 2(1 - u), \ 0 < u < 1$$

6. Let $X, Y$ be iid exponential random variables with mean 1. Let $Z = X + Y$ and $W = X - Y$. Find the conditional density $f_{W|Z}$

Hint: You have a straight formula for joint density of $Z, W$ here. Since sum of two independent exponential rv would be a gamma rv, you know $f_Z$. Hence, you can write down the expression for the conditional density.

7. Let $X, Y$ be independent random variables each having normal density with mean zero and variance unity. Find the joint density of $aX + bY$ and $bX - aY$.

Hint: You can write the transformation as

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

The transformation is invertible because the $2 \times 2$ matrix above is invertible.

You can find the inverse and then use the theorem to find the joint distribution of $Z, W$. Please complete the algebra. (you can chenck your answer by reading the rest of the hint!)

This is the only way you know to solve this problem at the time this problem sheet is sent. Now you know an easier method.

Since $X, Y$ are individually Gaussian **and** independent they are jointly Gaussian. Given this and that the $2 \times 2$ matrix above is non-sngular, we conclude that $Z, W$ are jointly Gaussian. Hence, we can write down their joint density if we know their means, variances and the covariance. From the given expressions for $Z, W$ and the knowledge that $X, Y$ are mean-zero and variance 1, we know $Z, W$ have mean zero and have the same variance, namely, $a^2 + b^2$. By calculating $E[(aX + bY)(bX - aY)]$ you would see that the $Z, W$ are uncorrelated. Hence they are ind normal rv with means zero and variance $a^2 + b^2$ and hence we can directly write their joint density

8. Let $X, Y$ be independent Gaussian random variables with mean zero and variance unity. Define random variables $D$ and $\theta$ by

$$D = X^2 + Y^2; \quad \theta = \tan^{-1}(Y/X)$$

(where, by convention, we assume $\theta$ takes values in $[0, 2\pi]$; for this we first calculate $\tan^{-1}(|Y|/|X|)$ in the range $[0, \pi/2]$ and then put that angle in the appropriate quadrant based on signs of $Y$ and $X$). Find the joint density of $D$ and $\theta$ and their marginal densities. Are $D$ and $\theta$ independent? (Hint: Note that the $(D, \theta)$ to $(X, Y)$ mapping is invertible. Hence you can use the formula).

Hint: The inverse of the given transformation is $X = \sqrt{D}\ \cos(\theta)$ and $Y = \sqrt{D}\ \sin(\theta)$. The jacobian is 1/2 and by applying the formula you see that $D$ is exponential with $\lambda = \frac{1}{2}$ and $\theta$ is uniform over $[0,\ 2\pi]$ and they are independent.

Note that if you start with independent $D, \theta$ with these densities and define $X, Y$ by $X = \sqrt{D}\ \cos(\theta)$ and $Y = \sqrt{D}\ \sin(\theta)$, then the same calculation would show you that $X, Y$ are ind standard Gaussian.

9. Consider the following algorithm for generating random numbers $X$ and $Y$ :

   1. Generate $U_1$ and $U_2$ uniform over $[0,\ 1]$.
   2. Set $X = \sqrt{-2\log(U_1)}\cos(2\pi U_2)$ and $Y = \sqrt{-2\log(U_1)}\sin(2\pi U_2)$.

   What would be the joint distribution of $X$ and $Y$ ? (Hint: Recall that when $U$ is uniform over $[0,\ 1]$ $-a\log(U)$ is exponential with parameter $1/a$ and $2\pi U$ is uniform over $[0, 2\pi]$).

Hint: From the hint in the problem, you know that $-2\log(U_1)$ is exponential with $\lambda = \frac{1}{2}$. When $U_2 \sim U[0,\ 1]$, $2\pi U_2$ is uniform over $[0,\ 2\pi]$. Now, from the previous problem, you can easily see, $X, Y$ are ind Gaussian with mean zero variance 1.

This tells you a simple algorithm whereby using two uniform random numbers, you can generate Gaussian random numbers. This is a very popular method for generating Gaussian random numbers and these equations are known as Box-Muller transform.

10. Consider the following algorithm for generating random variables $V_1$ and $V_2$ :

    1. Generate $X_1$ and $X_2$ uniform over $[-1,\ 1]$.
    2. If $X_1^2 + X_2^2 > 1$ then go to step 1; else set $V_1 = X_1,\ V_2 = X_2$ and exit.

    What would be the joint distribution of $V_1$ and $V_2$ ?

Hint: This process always stops with a point on or inside the unit circle. So, $V_1, V_2$ would be uniform over the unit disc, The way to see this is as follows. Take any subset of the unit disc. Call it $A$. Now suppose you repeat the above till $(V_1, V_2)$ is either in $A$ or in $A^c$. The probability that you end in $A$ is now proportional to its area.

This is a generic method to generate random vectors uniform over some region (which is not a cylindrical set). In $\Re^2$ if $X, Y$ are uniform over a cylindrical set then they are independent and each is uniform over the corresponding interval. We can easily generate them. Thus generating $X_1, X_2$ uniform over $[-1,\ 1] \times [-1,\ 1]$ is easy. But to generate them uniform over the interior of unit circle is difficult because they are no longer independent. The above gives a method of handling this.

Try and find the expected number of rv uniform over $[-1,\ 1]$ that you need to generate to get one random vector uniform over the interior of unit circle.

11. Suppose we have access to a random number generator that can generate random numbers uniformly distributed over $(0, 1)$. Using the results of the previous problems, suggest a method for generating samples of X when X has Gaussian density with mean zero and variance unity.

Hint: We already saw the Box-Muller transform for this. One difficulty with equations you had there is that you need to compute $\cos(\theta)$ and $\sin(\theta)$ which have to be computed through Taylor series and hence are computationally expensive.

Suppose we can generate $(V_1, V_2)$ uniform over the interior of the unit circle. Let $\theta$ be the angle that the line joining origin to $(V_1, V_2)$ makes with $X$-axis. Then, that $\theta$ would be uniform over $[0,\ 2\pi]$. Hence, $\frac{V_1}{\sqrt{V_1^2+V_2^2}}$ would have the same distribution as that of $\cos(\theta)$ with $\theta$ being uniform over $[0,\ 2\pi]$. Hence we can replace $\cos(\theta)$ in Box-Muller transform by $\frac{V_1}{\sqrt{V_1^2+V_2^2}}$ with $V_1, V_2$ generated as in the previous problem. This improves computational efficiency of Box-Muller transform