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- For a Markov chain, given the current state, the future evolution is independent of the history of how you reached the current state

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- The P and π_0 together determine all joint distributions

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- These are known as Chapman-Kolmogorov equations
- For a finite chain, the n-step transition probability matrix is n-fold product of the transition probability matrix
- We also have

$$\pi_n(x) \triangleq Pr[X_n = x] = \sum_x \pi_0(x) P^n(x,y)$$

Recap: Hitting times

- We define hitting time for y as the random variable

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$$P^n(x, y) = \sum_{m=1}^{n} P_x(T_y = m) P^{n-m}(y, y)$$

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  Definition: A state y is called transient if $\rho_{yy} < 1$; it is called recurrent if $\rho_{yy} = 1$.
- Intuitively, all transient states would be visited only finitely many times while recurrent states are visited infinitely often.

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- We can get distribution of $N(y)$ as

$$P_x(N(y) = m) = \rho_{xy}\, \rho_{yy}^{m-1}(1 - \rho_{yy}),\ m \geq 1$$

and $P_x(N(y) = 0) = 1 - \rho_{xy}$

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- $G(x,y)$ is the expected number of visits to y for a chain that is started in x.

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- This completes proof of the theorem

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- Easy to check it is an equivalence relation.

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- In our example, each equivalence class corresponds to a rational number.
- Here, C_i contains all fractions that are equal to that rational number

- The state space of any Markov chain can be partitioned into the transient and recurrent states: $S = S_T + S_R$:

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- All states in any C_i lead to each other or communicate with each other
- If $i \neq j$ and $x \in C_i$ and $y \in C_j$, then, $\rho_{xy} = \rho_{yx} = 0$. x and y do not communicate with each other.

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where each C_i is a closed and irreducible set of states.

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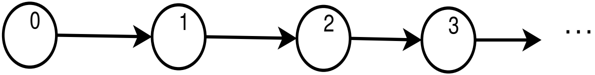
- Here is a trivial example of non-irreducible transient chain:



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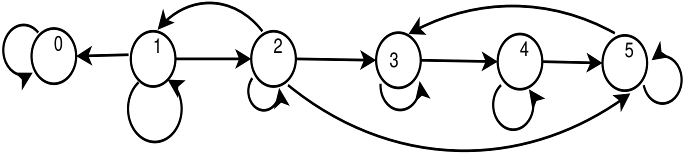

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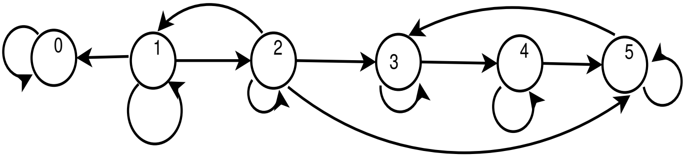

$$\begin{array}{c|cccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 \\
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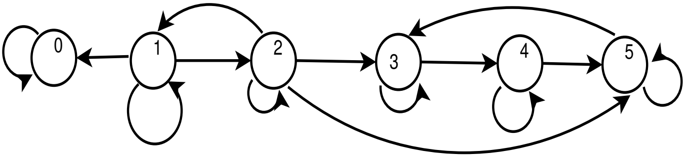

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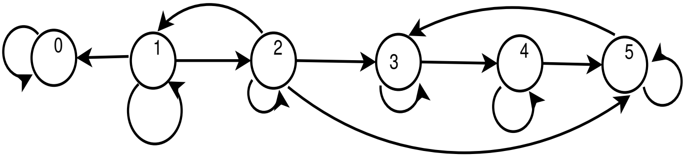

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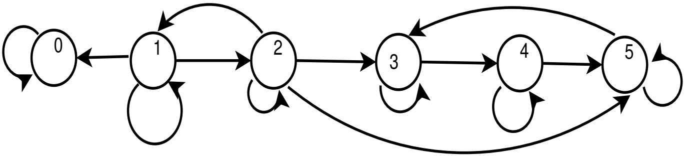

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- By solving this set of linear equations we can get $\rho_c(x), \ x \in S_T$

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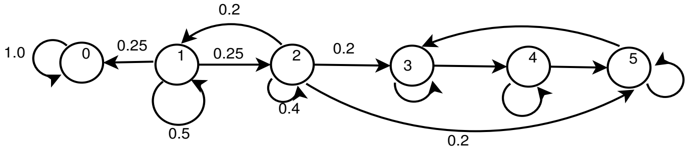

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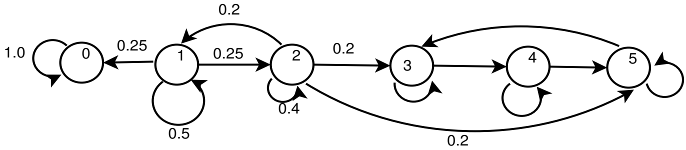

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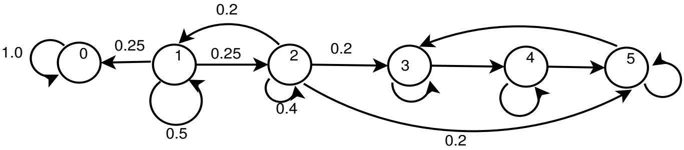

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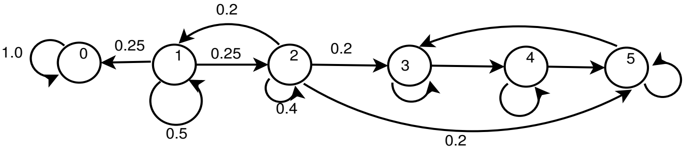

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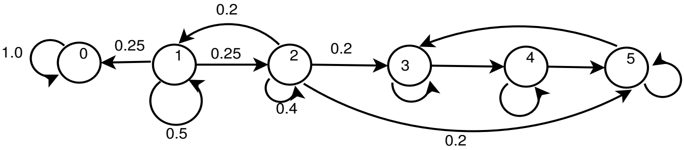

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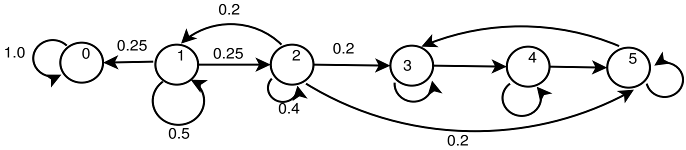

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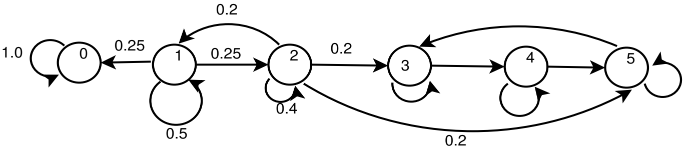

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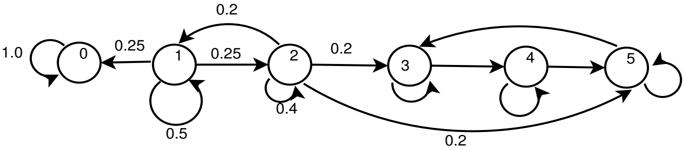

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$$\pi_{n+1}(y) = \sum_{x \in S} Pr[X_{n+1} = y | X_n = x] Pr[X_n = x] = \sum_{x \in S} \pi_n(x) P(x,y)$$

- Hence, if $\pi_0 = \pi$ then $\pi_1 = \pi$
  and hence $\pi_n = \pi, \ \forall n$
- Hence the name, stationary distribution.
- It is also called the invariant distribution or the invariant measure

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which shows π is a stationary distribution

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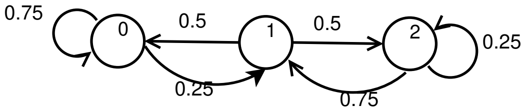
- What about infinite chains?

Example



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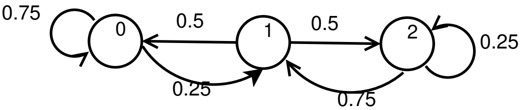

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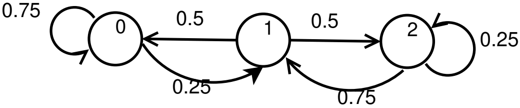

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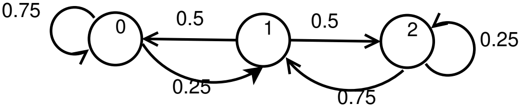

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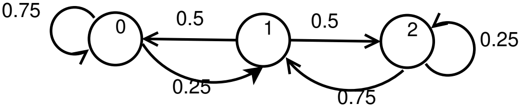

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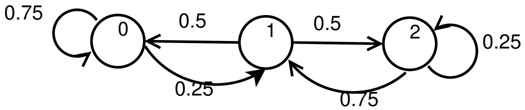

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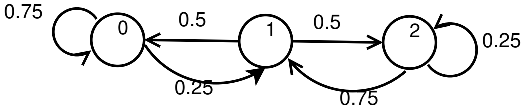
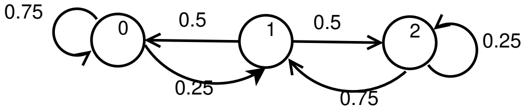
in addition, $\quad \pi(0) + \pi(1) + \pi(2) = 1$



- We can also write the equations for π as

$$\left[\begin{array}{ccc} \pi(0) & \pi(1) & \pi(2) \end{array}\right] \left[\begin{array}{ccc} 0.75 & 0.25 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.75 & 0.25 \end{array}\right] = \left[\begin{array}{ccc} \pi(0) & \pi(1) & \pi(2) \end{array}\right]$$



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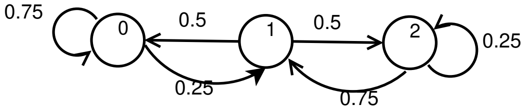

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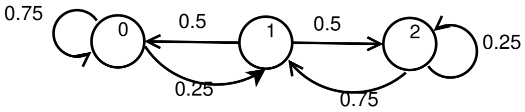
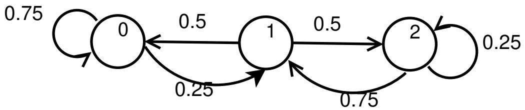
- We have to solve these along with $\pi(0) + \pi(1) + \pi(2) = 1$


0.75
0
0.5
1
0.5
2
0.25
0.25
0.75



$$\begin{aligned}
0.75\pi(0) + 0.5\pi(1) &= \pi(0) \\
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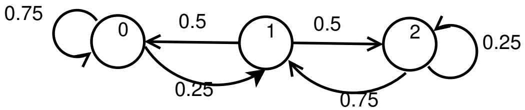

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0.75\pi(0) + 0.5\pi(1) &= \pi(0) \quad \Rightarrow \quad \pi(1) = \frac{1}{2}\,\pi(0) \\
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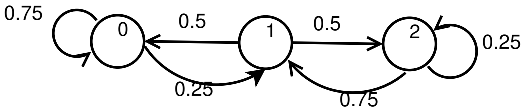

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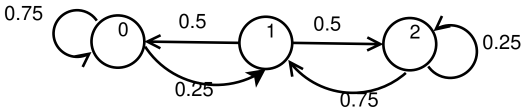

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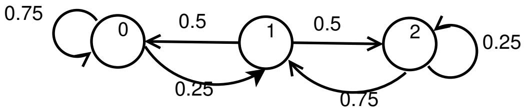

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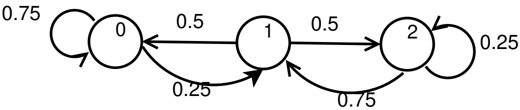
- Now, $\pi(0)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1$ gives $\pi(0) = \frac{6}{11}$



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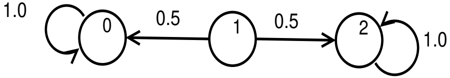
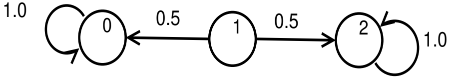
- Now, $\pi(0)\left(1 + \frac{1}{2} + \frac{1}{3}\right) = 1$ gives $\pi(0) = \frac{6}{11}$
- We get a unique solution: $\left[\frac{6}{11} \;\; \frac{3}{11} \;\; \frac{2}{11}\right]$

Example2



Example2



- The stationary distribution has to satisfy

$$\begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix} \begin{bmatrix} 1.0 & 0 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1.0 \end{bmatrix} = \begin{bmatrix} \pi(0) & \pi(1) & \pi(2) \end{bmatrix}$$

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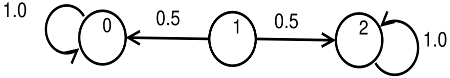

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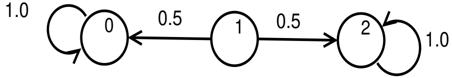

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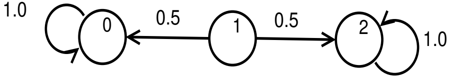
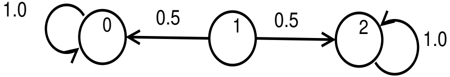
- We also have to add the equation $\pi(0) + \pi(1) + \pi(2) = 1$
- We now do not have a unique stationary distribution

Example2



Example2



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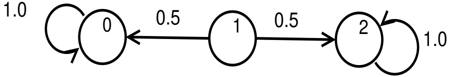

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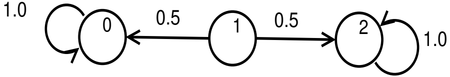

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$$\pi(0) + 0.5\pi(1) = \pi(0)$$

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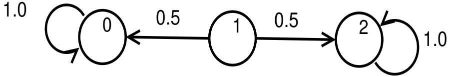

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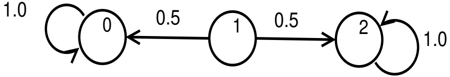

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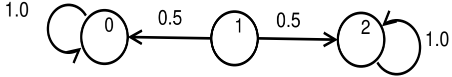

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- We get the following linear equations

$$\begin{aligned}
\pi(0) + 0.5\pi(1) &= \pi(0) \quad \Rightarrow \quad \pi(1) = 0 \\
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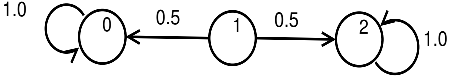
Example2



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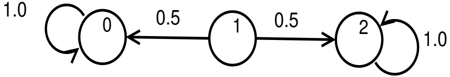

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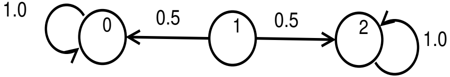

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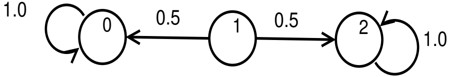

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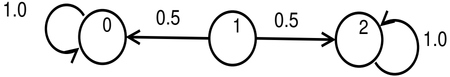

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- Any distribution $[a \;\; 0 \;\; 1-a]$ with $0 \le a \le 1$ is a stationary distribution
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- The stationary distribution, when it exists, is related to expected fraction of time spent in different states.

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- We will first establish a limit for the above as $n \to \infty$