Assignment 10 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. Let $\lambda, a \in \mathbb{R}$, and $A = \begin{bmatrix} \lambda I_n & y \\ y^H & a \end{bmatrix} \in \mathbb{C}^{(n+1)\times(n+1)}$ where $y \in \mathbb{C}^n$. (6 points)

(a) Use Cauchy interlacing theorem to show λ is an EVal(A) with multiplicity at least $n-1$.

(b) Find the other 2 EVals(A).

A. (a) From Cauchy interlacing theorem, for $A = \begin{bmatrix} B & y \\ y^H & a \end{bmatrix}$, with $\lambda_i(B)$ and $\hat{\lambda}_i(A)$ arranged in $\uparrow$ value,

$$\hat{\lambda}_1 \le \lambda_1 \le \hat{\lambda}_2 \le \lambda_2 \le \cdots \le \hat{\lambda}_n \le \lambda_n \le \hat{\lambda}_{n+1}$$

Since $B = \lambda I_n$, $\lambda_i(B) = \lambda \;\; \forall\, 1 \le i \le n$

$$\Rightarrow \hat{\lambda}_1 \le \lambda \le \hat{\lambda}_2 \le \lambda \le \cdots \le \hat{\lambda}_n \le \lambda \le \hat{\lambda}_{n+1} \quad - (1)$$

Thus, $\hat{\lambda}_1 \le \lambda \le \hat{\lambda}_{n+1}$

& $\hat{\lambda}_2 = \hat{\lambda}_3 = \cdots = \hat{\lambda}_n = \lambda$

Thus λ is an EVal(A) with multip. $n-1$ $\Big\}$ $-(2)$

(b) $\operatorname{Tr}(A) = \sum_{i=1}^{n+1} \hat{\lambda}_i = (n-1)\lambda + \hat{\lambda}_1 + \hat{\lambda}_{n+1}$

$\quad\;\; \|$

$n\lambda + a \quad \Rightarrow \lambda + a = \hat{\lambda}_1 + \hat{\lambda}_{n+1} \quad - (3)$

$$\hat{\lambda}_1 \cdot \hat{\lambda}_{n+1} \cdot \lambda^{n-1}$$

$$\det(A) = \begin{vmatrix} \lambda I_n & y \\ y^H & a \end{vmatrix} = |\lambda I_n| \cdot |a - y^H (\lambda I)^{-1} y|$$

↳ From hint

$$= \lambda^n \cdot \left| a - \frac{\|y\|^2}{\lambda} \right|$$

$$\Rightarrow \quad \hat{\lambda}_1 \cdot \hat{\lambda}_{n+1} = a\lambda - \|y\|^2 \qquad -(4)$$

$$\hat{\lambda}_1 = \lambda + a - \hat{\lambda}_{n+1}$$

$$\Rightarrow -\hat{\lambda}_{n+1}^2 + \lambda \hat{\lambda}_{n+1} + a \hat{\lambda}_{n+1} = a\lambda - \|y\|^2$$

$$\Rightarrow \quad \hat{\lambda}_{n+1}^2 - (\lambda + a)\hat{\lambda}_{n+1} - (\|y\|^2 - a\lambda) = 0$$

$$\Rightarrow \quad \hat{\lambda}_{n+1} = \frac{(\lambda + a) + \sqrt{a^2 + \lambda^2 + 2a\lambda - 4a\lambda + 4\|y\|^2}}{2} \qquad -(5)$$

$$\Rightarrow \hat{\lambda}_1 = \frac{(\lambda + a) - \sqrt{(\lambda - a)^2 + 4\|y\|^2}}{2} \qquad -(6)$$

$$(\because \hat{\lambda}_1 \le \lambda \le \hat{\lambda}_{n+1})$$

2. Let $A \in \mathbb{C}^{n \times n}$ be Hermitian and a_{ij} be (i,j)th element of A.

ⓐ Prove that $\lambda_{min}(A) \leq a_{ii} \leq \lambda_{max}(A)$

ⓑ If A is PD, prove that $a_{ii} > 0$

$\underbrace{\text{PD}}_{\text{Positive Definite}}$ $\forall\, i = 1, 2, \dots, n$ (4 points)

A. ⓐ From the Rayleigh-Ritz theorem,

$$\lambda_{min}(A) \leq \frac{x^H A x}{x^H x} \leq \lambda_{max}(A) \quad \text{—①}$$

Take $x = e_i \rightarrow$ ith col. of I_n

$$\Rightarrow \lambda_{min}(A) \leq a_{ii} \leq \lambda_{max}(A) \quad \text{—②}$$

ⓑ A is PD $\Rightarrow$ $x^H A x > 0 \;\; \forall\; 0 \neq x \in \mathbb{C}^n$

Take $x = e_i \rightarrow$ ith col. of I_n —③

$\Rightarrow e_i^H A e_i = a_{ii} > 0$

Doing this repeatedly for all $i \in \{1, 2, \dots, n\}$ yields the final result $a_{ii} > 0 \;\forall\, i$ —④

(Alternately, for a PD matrix, all EVals > 0
$\Rightarrow$ From Q2a, $a_{ii} \geq \lambda_{min} > 0 \Rightarrow a_{ii} > 0$)