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- Called multidimensional change of variable formula

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$$

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f_{(X/Y)}(z) = \int_{-\infty}^{\infty} |t|\ f_{XY}(zt, t)\ dt = \int_{-\infty}^{\infty} \left|\frac{t}{z^2}\right|\ f_{XY}\left(t, \frac{t}{z}\right)\ dt
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- If the random variables are exchangeable then the joint distribution function remains the same on permutation of arguments.
- Exchangeable random variables are identically distributed but they may not be independent.

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- In general, $E\left[g_1(\mathbf{X}) + g_2(\mathbf{X})\right] = E\left[g_1(\mathbf{X}\right] + E\left[g_2(\mathbf{X})\right]$

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- Hence, Cov$(X, Y) = E[XY] - EX \; EY = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$

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- X, Y are uncorrealted does not imply they are independent.

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- Hence $-1 \leq \rho_{XY} \leq 1,\ \forall X, Y$

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\frac{(E[XY])^2}{E[X^2]} + \beta^2 E[Y^2] - 2\beta \frac{(E[XY])^2}{E[X^2]} &\geq 0, \quad \forall \beta \in \Re
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- Correlation coefficient of X, Y is ± 1 only when Y is a linear function of X

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&= \mathrm{Var}(Y)\,\left(1 - \frac{(\mathrm{Cov}(X,Y))^2}{\mathrm{Var}(Y)\,\mathrm{Var}(X)}\right) \\
&= \mathrm{Var}(Y)\left(1 - \rho_{XY}^2\right)
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- The best mean-square approximation of Y as a 'linear' function of X is

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- Informally, covariance captures the 'linear dependence' between the two random variables.

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- This gives $\mathbf{a}^T\Sigma_X\mathbf{a} \geq 0, \ \forall \mathbf{a}$

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- It is positive semidefinite.
- Let $\mathbf{a} \in \Re^n$ and let $Y = \mathbf{a}^T \mathbf{X}$.
- Then, $EY = \mathbf{a}^T E\mathbf{X}$. We get variance of Y as

$$
\begin{aligned}
\text{Var}(Y) &= E[(Y - EY)^2] = E\left[\left(\mathbf{a}^T\mathbf{X} - \mathbf{a}^T E\mathbf{X}\right)^2\right] \\
&= E\left[\left(\mathbf{a}^T(\mathbf{X} - E\mathbf{X})\right)^2\right] \\
&= E\left[\mathbf{a}^T(\mathbf{X} - E\mathbf{X})\,(\mathbf{X} - E\mathbf{X})^T\mathbf{a}\right] \\
&= \mathbf{a}^T E\left[(\mathbf{X} - E\mathbf{X})\,(\mathbf{X} - E\mathbf{X})^T\right]\mathbf{a} \\
&= \mathbf{a}^T\Sigma_X\mathbf{a}
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- If X_i are independent, variance of sum is sum of variances.

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- Hence, Σ_X fails to be positive definite only if there is a non-zero linear combination of X_i's that is a constant.

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- Hence the direction is the eigen vector corresponding to the highest eigen value.

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- We can similarly define joint moments of multiple random variables

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- More generally

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- We will now define it formally

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- The last property above follows directly from the definition.

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- Similarly

$$E[\ E[Y|X]\ ] = E\left[\frac{1+X}{2}\right] = \frac{2}{3} = E[Y]$$

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- We can choose a Y that is useful.