E1 222 Stochastic Models and Applications
Problem Sheet 2–2

1. Two fair dice are rolled and X is the maximum of the two numbers. Specify a reasonable probability space and then define this random variable as a function on the appropriate Ω. Derive its probability mass function.

Answer: Obvious choice is $\Omega = \{(a, b) : a, b \in \{1, 2, \cdots, 6\}\}$ and $X((a, b)) = \max(a, b)$. We can assume that each singleton event here would have probability 1/36. It is easy to see that $X \in \{1, \cdots, 6\}$. We can write the event $[X = k]$ as the mutually exclusive union of three sets as

$$[X = k] = \{(k, a) : a < k\} + \{(a, k) : a < k\} + \{(k, k)\}$$

This gives us the mass function as $f_X(k) = \frac{2k-1}{36},\ 1 \leq k \leq 6$.
(Verify that $\sum_k f_X(k) = 1$).
(Can we write the $[X = k]$ event as

$$[X = k] = \{(k, a) : a \leq k\} + \{(a, k) : a \leq k\}?$$

)

2. A fair dice is rolled repeatedly till the sum of all numbers obtained exceeds 6. Let X denote the number of rolls needed. Find the values of $F_X(1)$, $F_X(7)$ and $F_X(2)$.

Hint: Since we want the sum of numbers to exceed 6, the minimum number of rolls needed is 2. Hence, $F_X(1) = 0$. Similarly, $F_X(7) = 1$ because by the time you roll the dice seven times, the sum of number has to exceed 6 and hence $X \leq 7$. We have $F_X(2) = P[X \leq 2] = P[X = 2]$ because $X \geq 2$. Calculating $P[X = 2]$ is straight forward. It is very similar to the previous problem. Take the probability space of rolling two dice and define Y as the sum of the two numbers. Then $P[X = 2] = P[Y \geq 7]$. I hope now the solution is easy.

3. Let X be geometric. Calculate probabilities of the events (i). $[X \leq 10]$, (ii). $[X = 3 \ \text{ or } \ 5 \leq X \leq 7]$.

Answer:

$$P[X \leq 10] = \sum_{k=1}^{10} f_X(k) = \sum_{k=1}^{10} p(1-p)^{k-1} = 1 - (1-p)^{10}$$

$$P[X = 3 \;\text{ or }\; 5 \leq X \leq 7] = P[X \in \{3, 5, 6, 7\}]$$
$$= p\left[(1-p)^2 + (1-p)^4 + (1-p)^5 + (1-p)^6\right]$$

4. Let X be a rv with density function

$$f(x) = cx^3, \quad \text{if} \quad 0 \leq x \leq 1.$$

($f(x)$ is zero for all other values of x). Find the value of c and the distribution function of X. Find $P[X > 0.5]$.

Answer:

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 cx^3\, dx = 1 \Rightarrow c = 4$$

It is easily seen that $F_X(x) = 0$ for $x < 0$ and $F_X(x) = 1$ for $x > 1$. For $0 \leq x \leq 1$,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_0^x 4x^3\, dx = x^4$$

Now

$$P[X > 0.5] = 1 - F_X(0.5) = 1 - (1/16) = 15/16$$

5. Let X be exponential random variable. Calculate probabilities of (i). $[|X| \leq 3]$, (ii). $[X \leq 4 \;\text{ or }\; X \geq 10]$.

Answer:

$$P[|X| \leq 3] = \int_{-3}^{3} f_X(x)\, dx = \int_0^3 \lambda e^{-\lambda x}\, dx = 1 - e^{-3\lambda}$$

$$P[X \leq 4 \;\text{ or }\; X \geq 10] = P[X \leq 4] + P[X \geq 10] = (1 - e^{-4\lambda}) + e^{-10\lambda}$$

6. Let X be continuous random variable with uniform density over $(-1, 1)$. Find the density (or mass function) of the random variables: (a). $U = (1 + X)/2$, (b). $U = \frac{X}{1+X}$, (c). $U = g(X)$ where $g(x) = -1$ if $x < 0$, $g(x) = 0$ if $x = 0$, and $g(x) = 1$ if $x > 0$.

Hint: part (a) is straight forward because we have done the example of $Y = aX + b$ in the class.

For part (b), notice that if we write $U = g(X)$ here then the $g'(x) > 0$ showing that the function is monotone. So, you can use the theorem. You can also do this from first principles. Notice that the functon is monotone increasing and as x goes from -1 to 1, $\frac{x}{1+x}$ goes from $-\infty$ to 0.5. So, we need to find $F_U(u)$ for $u \leq 0.5$. For $x \in (-1, 1)$ and $y \leq 0.5$, we have $\frac{x}{1+x} \leq y$ implies $x \leq \frac{y}{1-y}$.

Find F_U and hence f_U from first principles and see that you get the same result as by applying the theorem.

For part(c) notice that here U is a discrete random variable. So, $P[U = -1] = P[X < 0]$ and so on. (What is $P[U = 0]$?)

7. Let X be a continuous random variable having uniform density over $[0, 3]$. Let $Y = (X - 1)^2$. Find the density of Y .

Hint: We have $g(x) = (x - 1)^2$. As x ranges over 0 to 3, what is the range of values for $g(x)$? The immediate and wrong answer to this question given by some students is that the range is 1 to 4 (since $g(0) = 1$ and $g(3) = 4$). A few seconds of thinking will convince you the range is 0 to 4.

Now, $F_Y(y) = P[(X-1)^2 \leq y]$. You can easily get the event $[(X-1)^2 \leq y]$ by drawing the graph of $(x - 1)^2$ Vs x only for the range $0 \leq x \leq 3$ and drawing a line through y (parallel to x-axis). You would notice that when $0 \leq y \leq 1$ the line cuts the graph in two points while when $y > 1$ the line cuts the graph at one point. Now, hopefully, you can solve the problem.

8. Let X be a random variable, g be some density function and ϕ a differentiable strictly increasing function on $(-\infty, \infty)$. Suppose that

$$P[X \leq x] = \int_{-\infty}^{\phi(x)} g(z)\, dz$$

Show that the density of $Y = \phi(X)$ is $g(y)$.

Answer: We have

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[\phi(X) \leq y] \end{aligned}$$

$$
\begin{aligned}
&= P[X \leq \phi^{-1}(y)] \\
&= \int_{-\infty}^{\phi^{-1}(\phi(y))} g(z)\, dz \\
&= \int_{-\infty}^{y} g(z)\, dz
\end{aligned}
$$

This shows that density of Y is $g(y)$.