E1 222 Stochastic Models and Applications
Problem Sheet #1

1. There are three chests each having two drawers. Chest 1 has a gold coin in each drawer while chest 2 has a silver coin in each drawer. Chest 3 has a gold coin in one drawer and a silver coin in the other drawer. A chest is chosen at random and one of its drawers, chosen at random, is opened. It is found to contain a gold coin. What is the probability that the other drawer has (i). a gold coin, (ii). a silver coin?

Hint: Note that each outcome of this random experiment consists of (the choice of) a chest and a drawer.

2. A box contains coupons labelled $1, 2, 3, \cdots, n$. Two coupons are drawn from the box with replacement. Let a, b denote the numbers on the two coupons. Find the probability that one of a, b divides the other.

Hint: Given a fixed number a which is between 1 and n, think of how many multiples of a are there between 1 and n.

3. A fair dice is rolled repeatedly till we get at least one 5 and one 6. What is the probability that we need n rolls?

Hint: If it takes n rolls what can we say about the outcome of n^{th} roll and about the outcomes of the first $n-1$ rolls?

4. Suppose E and F are mutually exclusive events of a random experiment. This random experiment is repeated till either E or F occurs. Show that the probability that E occurs before F is $P(E)/(P(E) + P(F))$.

Hint: First try to calculate the probability of the event: n repetitions were needed and E occured before F.

5. Suppose n men put all their hats together in a heap and then each man selects a hat at random. Show that the probability that none of the n men selects his own hat is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \frac{(-1)^n}{n!}$$

Hint: If A_i is the event that i^{th} man gets his hat, think about the event $\cup_i A_i$ and its relation with the event you want.

6. Suppose there are three special dice, A, B, C which have the following numbers on their six faces:

   A: 1, 1, 6, 6, 8, 8

   B: 2, 2, 4, 4, 9, 9

   C: 3, 3, 5, 5, 7, 7

   The dice are fair in the sense that each of the faces have the same probability of coming up.
   (i). Suppose we roll dice A and B. What is the probability that the number that comes up on A is less than the one that comes up on B?
   (ii)Suppose your friend, with whom you go out for dinner often, offers you the following. At the end of each dinner, you choose any one of the three dice that you want. She/He would then choose one of the two dice that are remaining. Then both of you roll your respective dice. Whoever gets the smaller number would pay for the dinner. Would you take the offer?

7. Consider a communication system. The transmitter sends one of two waveforms. One waveform represents the symbol 0 and the other represents the symbol 1. Due to the noise in the channel, the receiver cannot say with certainty what was sent. The receiver is designed so that, after sensing signal coming out of the noisy channel, it puts out one of the three symbols: a, b, c. The following statistical parameters of the system are determined (either through modeling or experimentation):

$$P[a|1] = 0.6, \; P[b|1] = 0.2, \; P[c|1] = 0.2$$

$$P[a|0] = 0.3, \; P[b|0] = 0.4, \; P[c|0] = 0.3$$

   Here, $p[a|0]$ denotes the probability of the receiver putting out symbol a when the symbol transmitted is 0 and similarly for all others. The transmitter sends the two symbols with probabilities: $P[0] = 0.4$ and $P[1] = 0.6$. Find $P[1|a]$ and $P[0|a]$. When receiver puts out a what should we conclude about the symbol sent? We would like to build a decision device that will observe the receiver output (that is, a, b, or c) and decide whether a 0 was sent or a 1 was sent. An error occurs if the decision device says 1 when a 0 was sent or vice versa. Find a decision rule that minimizes the probability of error. What is the resulting (minimum) probability of error?

8. At a telephone exchange, the probability of receiving k calls in a time interval of two minutes is given by the function $h(2, k)$. Assume that the event of receiving k_1 calls in a time interval I_1 is independent of the event of receiving k_2 calls in a time interval I_2, for all k_1 and k_2 whenever the intervals I_1 and I_2 do not overlap. Find an expression for the probability of receiving s calls in 4 minutes in terms of $h(2, k)$. Now suppose $h(2, k)$ is given by

$$h(2, k) = \frac{(2a)^k e^{-2a}}{k!}.$$

Now show that the probability of s calls in 4 minutes is given by $\frac{(4a)^s e^{-4a}}{s!}$

9. There is a component manufacturing facility where 5% of the products may be faulty. The factory wants to pack the components into boxes so that it can guarantee that 99% of the boxes have at least 100 good components. What is the minimum number of components they should put into each box?