Computational Methods of Optimization
Second Midterm(30th Dec, 2020)

Time: 60 minutes

Instructions

- Answer all questions
- See upload instructions in the form

In the following, assume that f is a $\mathcal{C}^1$ function defined from $\mathbb{R}^d \to \mathbb{R}$ unless otherwise mentioned. Let $\mathbf{I} = [e_1, \ldots, e_d]$ be a $d \times d$ matrix with e_j be the jth column. Also $\mathbf{x} = [x_1, x_2, \ldots, x_d]^\top \in \mathbb{R}^d$ and $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\top \mathbf{x}}$. Set of real symmetric $d \times d$ matrices will be denoted by $\mathcal{S}_d$. $[n]$ will denote the set $\{1, 2, \ldots, n\}$

1. (5 points) Please indicate True(T) or False(F) in the space given after each question. All questions carry equal marks
   (a) The set $\{\mathbf{x} \in \mathbb{R}^d | f(\mathbf{x}) \leq b\}$ is convex where $f : C \subset \mathbb{R}^d \to \mathbb{R}$ is a convex function and $b \in \mathbb{R}$ ____
   (b) The set $\{\mathbf{x} \in \mathbb{R}^d | \|\mathbf{x}\| = 1\}$ is not convex ____
   (c) The set $\{\mathbf{x} \in \mathbb{R}^d | 1 \leq \|\mathbf{x}\| \leq 2\}$ is not convex ____
   (d) The projection of $\mathbf{z} \in \mathbb{R}^d$ on a non-convex set C does not exist ____
   (e) Let $\mathbf{x}^*$ be the global minimum of
   $$min_{\mathbf{x} \in C} f(\mathbf{x}) \left(= \|\mathbf{x} - \mathbf{a}\|^2\right)$$
   and $\mathbf{z}^*$ be the minimum of $\sqrt{f(\mathbf{x})}$. The two minima are different ____.

2. (4 points) Pick the correct choice. All questions carry equal marks
   (a) Consider the following problem
   $$\mathbf{x}^* = argmin_{\mathbf{x} \in C} f(\mathbf{x})$$
   where $f \in \mathcal{C}^1$. Let $\mathbf{z}^*$ be the unconstrained minimum of $f(\mathbf{x})$. When is $\mathbf{z}^* = \mathbf{x}^*$?
   A. There is no relationship B. $\mathbf{x}^*$ is not an interior point of C C. $\mathbf{x}^*$ is an interior point of C
   (b) Let the columns of $d \times d$ matrix $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_d]$ be Q conjugate for a $d \times d$ matrix Q. The off-diagonal entries of matrix $B = \mathbf{U}^\top Q \mathbf{U}$ are A. $\mathbf{u}_i^\top \mathbf{u}_j$ B. 0 C. Numerical value cannot be determined

3. (a) (3 points) Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ be the columns of $\mathbf{I}_{d \times d}$ matrix and $Q \in \mathbb{R}^{d \times d}$ be a positive semi-definite matrix. Find A_{ij} such that $\mathbf{u}_i$ are Q conjugate where $\mathbf{u}_1 = \mathbf{e}_1$, $\mathbf{u}_i = \mathbf{e}_i + \sum_{j=1}^{i-1} A_{ij} \mathbf{u}_j$ for $i \geq 2$.

   (b) Let $B\mathbf{x} = b$ be a linear system of equations where $B \in \mathbb{R}^{d \times d}$, a symmetric matrix which is positive definite and $b \in \mathbb{R}^d$. Using $\mathbf{u}_i$ defined in the previous question we wish to solve the linear system of equations using Conjugate direction algorithm
      i. (3 points) State the objective function to be used and argue why it will lead to solving the linear system of equations.

      ii. (4 points) Starting at $\mathbf{x}^{(0)} = 0$ find $\mathbf{x}^{(1)}$ using the direction $\mathbf{u}_1$

4. Bunty and Babli were arguing over the following problem

$$min_{x_1 \in \mathbb{R}, x_2 \in \mathbb{R}} f(x_1, x_2) = (x_1 - 2)^2 + x_2^2 \text{ subject to } (x_1 - 2)^2 = (x_2 - 3)^5 \quad (\mathcal{P})$$

Bunty substitutes $(x_1 - 2)^2$ in the objective by $(x_2 - 3)^5$ and transforms $(\mathcal{P})$ into the following unconstrained problem

$$min_{x_2} (x_2 - 3)^5 + x_2^2 \quad (\mathcal{Q})$$

The objective function of $(\mathcal{Q})$ is not bounded from below and global minimum does not exist. Hence Bunty concludes that global minimum of $(\mathcal{P})$ does not exist. Babli disagrees with Bunty and says that $(\mathcal{Q})$ is not equivalent to $(\mathcal{P})$.

(a) (1 point) Who is correct, Bunty or Babli? Give reasons.

(b) (3 points) Babli further says that $(\mathcal{P})$ can be solved by solving a equivalent convex optimization problem. What should Bunty do to make $(\mathcal{Q})$,a convex optimization problem? State the optimization problem.

(c) i. (5 points) Find the global minimum point of the convex optimization problem. Jusitify your answer using KKT conditions.

ii. (2 points) Find the global minimum point and optimal value of $(\mathcal{P})$. Justify your answer.

5. We are interested in finding the projection of $\mathbf{z} \in \mathbb{R}^d$ on the set $C = \{\mathbf{x} \in \mathbb{R}^d | 0 \leq x_i \leq t\}$ where $t > 0$.

   (a) (3 points) State the Lagrangain of the projection problem as $\sum_{i=1}^{d} g(x_i, \lambda_{1i}, \lambda_{2i})$

   (b) (6 points) Find a KKT point for the problem.

   (c) (1 point) Find the projection.

6. Let $B \in \mathbb{R}^{d \times d}$ is a symmetric positive definite matrix. We wish to solve

$$\mathcal{P} \quad min_{\mathbf{x} \in \mathbb{R}^d} \mathbf{x}^\top B \mathbf{x} \text{ subject to } \mathbf{a}^\top \mathbf{x} = 1$$

   (a) (4 points) Show that $\mathcal{P}$ is solved if there exists $\mu \in \mathbb{R}$ so that $(\mathbf{z}^\top, \mu)^\top$ solves

$$\begin{bmatrix} 2B & \mathbf{a} \\ \mathbf{a}^\top & 0 \end{bmatrix} \begin{pmatrix} \mathbf{z} \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(b) (6 points) Solve $\mathcal{P}$. State the optimal objective function and the optimum point