E1 222 Stochastic Models and Applications
Problem Sheet 3–2

1. Let (X, Y) have joint density

$$f_{XY}(x,y) = \frac{1}{4}[1 + xy(x^2 - y^2)],\ |x| \leq 1,\ |y| \leq 1.$$

Find the marginal and conditional densities.

Answer: The marginal density for X is given by

$$f_X(x) = \frac{1}{4}\int_{-1}^{1}(1 + x^3y - xy^3)dy = \frac{1}{2},\ -1 \leq x \leq 1$$

Similarly, marginal of Y is also uniform over $[-1,\ 1]$. Now the conditionals are given by joint density divided by marginal density.

2. Let $f(x,y) = e^{-x-y},\ x > 0, y > 0$. Show that this a density function. Find the marginals and the conditional densities.

Answers: Since $\int_0^\infty e^{-x}\ dx = \int_0^\infty e^{-y}\ dy = 1$, it is easy to see $f(x,y)$ is a density. Also, by the same reason, the marginals are $e^{-x}, x > 0$ and $e^{-y}, y > 0$. Here the joint density is product of the marginals and hence the two random variables are independent.

3. Let X be uniform from 0 to 1, let Y be uniform from 0 to X and let Z be uniform from 0 to Y. What is the joint density of X, Y, Z? Find the marginal densities, f_X, f_Y, f_Z and the joint density of Y, Z.

Answer: From the given verbal description we conclude

$$\begin{aligned} f_{XYZ}(x,y,z) &= f_X(x) f_{Y|X}(y,x) f_{Z|Y,X}(z|y,x) \\ &= 1\frac{1}{x}\frac{1}{y} = \frac{1}{xy},\quad 0 < z < y < x < 1 \end{aligned}$$

The rest of the problem is an exercise in integration. This is a good problem to test your skills in multiple integrals. The final answers are

$$f_Y(y) = -\ln(y),\ 0 < y < 1; \quad f_Z(z) = \frac{1}{2}(\ln(z))^2,\ 0 < z < 1$$

$$f_{YZ}(y,z) = -\ln(y)\frac{1}{y},\ 0 < z < y < 1$$

4. Let X, Y be iid random variables which are uniform over $(0,1)$. Find $P[X > Y]$.

Answer:

$$P[X > Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{XY}(x, y)\, dy\, dx = \int_{0}^{1} \int_{0}^{x} dy\, dx = 0.5$$

Since X, Y are iid, by intuition, we expect $P[X > Y] = P[Y > X]$. Since X, Y are continuous random variables and independent, intuitively, $P[X = Y] = 0$. Hence we expect $P[X > Y] = 0.5$.

5. Let A, B be two events. Let I_A and I_B denote the indicator random variables of these events. Show that I_A and I_B are independent iff A and B are independent.

Answer: Note that I_A, I_B take values in $\{0, 1\}$. We have

$$P[I_A = 1, I_B = 1] = P[AB]; \quad \text{and} \quad P[I_A = 1] = P[A],\ P[I_B = 1] = P[B]$$

Hence $f_{I_A I_B}(1, 1) = f_{I_A}(1) f_{I_B}(1)$ if and only if A, B are independent. Since independence of A, B implies independence of A, B^c etc, we similarly show this factorization for other combinations and thus show that I_A, I_B are independent if and only if A, B are independent.

6. Consider a communication system. Let Y denote the bit sent by transmiter. (Y is a binary random variable). The receiver makes a measurement, X, and based on its value decides what is sent. The decision at the receiver can be represented by a function $h : \Re \to \{0,\ 1\}$. For any specific h, let $R_0(h)$ represent the set of all $x \in \Re$ for which $h(x) = 0$ and let $R_1(h)$ represent the set of $x \in \Re$ for which $h(x) = 1$. An error occurs if a wrong decision is made. Argue that the event of error occurring is: $[h(X) = 0, Y = 1] \cup [h(X) = 1, Y = 0]$. Show that probability of error for a decision rule h is

$$\int_{R_0(h)} p_1 f_{X|Y}(x|1) dx \ + \ \int_{R_1(h)} p_0 f_{X|Y}(x|0) dx$$

where $p_i = P[Y = i]$. Now consider a h given by

$$h(x) = 1 \ \text{ if } \ f_{Y|X}(1|x) \geq f_{Y|X}(0|x)$$

(Otherwise $h(x) = 0$). Show that this h would achieve minimum probability of error.

Answer: An error occurs when $h(X)$, which is the decision of the receiver, differs from Y, which is what was sent. Since both are binary, the event representing error is given by $[h(X) = 0, Y = 1] \cup [h(X) = 1, Y = 0]$. Hence we have

$$\begin{aligned} P[\text{error}] &= P[h(X) = 0, Y = 1] + P[h(X) = 1, Y = 0] \\ &= P[h(X) = 0 | Y = 1] P[Y = 1] + P[h(X) = 1 | Y = 0] P[Y = 0] \\ &= \int_{R_0(h)} p_1 \ f_{X|Y}(x|1) \ dx \ + \ \int_{R_1(h)} p_0 \ f_{X|Y}(x|0) \ dx \end{aligned}$$

because, by definition of $R_0(h)$, $[h(X) = 0]$ is same as $[X \in R_0(h)]$ and similarly for the other term. Also, note that $R_0(h) \cap R_1(h) = \phi$ and $R_0(h) \cup R_1(h) = \Re$. So, every $x \in \Re$ is accounted for in one of the two integrals above.

For the second part of the problem. By Bayes rule, $f_{Y|X}(1|x) \geq f_{Y|X}(0|x)$ is same as $p_1 \ f_{X|Y}(x|1) \geq p_0 \ f_{X|Y}(x|0)$. Hence for the given h here, in $R_0(h)$ we have $p_1 \ f_{X|Y}(x|1) \leq p_0 \ f_{X|Y}(x|0)$ and the other way in $R_1(h)$. Hence, the probability of error for this h can be written as

$$P[\text{error}] = \int_{\Re} \min(p_1 \ f_{X|Y}(x|1), p_0 \ f_{X|Y}(x|0)) \ dx$$

Comparing this with the expression for probability of error for any other function, we can see that the given h is optimal.

7. Let X and Y be independent random variables each having an exponential distribution with the same value of parameter λ. Show that $Z = \min(X, Y)$ is exponential with parameter 2λ.

Answer: We have

$$P[Z > z] = P[X > z, Y > z]] = e^{-\lambda z} \ e^{-\lambda z} = e^{-2\lambda z}$$

because X, Y are independent exponential random variables with the same parameter. Hence, $f_Z(z) = P[Z \leq z] = 1 - e^{-2\lambda z}$ and Z is exponential with parameter 2λ.

8. Let X and Y be independent Gaussian random variables with $EX = \mu_1$, $EY = \mu_2$, $\text{Var}(X) = \sigma_1^2$, and $\text{Var}(Y) = \sigma_2^2$. Show that $X + Y$ has gaussian density with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.