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- More formally, Y is a random variable if g is a Borel measurable function.
- We can determine distribution of Y given the function g and the distribution of X

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- We have seen many specific examples of this.

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- We can find the pmf of Y as

$$\begin{aligned} f_Y(y) &= p[Y = y] = P[g(X) = y] \\ &= P[X \in \{x_i \ : \ g(x_i) = y\}] \\ &= \sum_{\substack{i: \\ g(x_i)=y}} f_X(x_i) \end{aligned}$$

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- Then Y is a continuous rv with pdf

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \; a \leq y \leq b$$

where $a = \min(g(\infty),\; g(-\infty))$ and $b = \max(g(\infty),\; g(-\infty))$

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  where $a = \min(g(\infty),\ g(-\infty))$ and $b = \max(g(\infty),\ g(-\infty))$
- This theorem is useful in some cases to find the densities of functions of continuous random variables

Expectation and Moments of a random variable

- We next consider the important notion of expectation of a random variable

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- Expectation is essentially a weighted average.
- To make the above finite and well defined, we can stipulate the following as condition for existence of expectation

$$\sum_i |x_i| \, f_X(x_i) < \infty$$

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- Now, expectation does not exist only when $EX^+ = EX^- = \infty$

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- This is what we do in this course because we deal with only discrete and continuous rv's.
- But to get a feel for the more formal definition, we look at a couple of examples.

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- But by the formal definition it exists. (Note that here $X^+ = X$ and $X^- = 0$).

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- Hence EX does not exist.

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$$EX = \int_{-\infty}^{\infty} x \, \frac{1}{\pi} \, \frac{1}{1+x^2} \, dx \;\; \stackrel{?}{=} 0$$

- Consider a continuous random variable X with pdf

$$f_X(x) = \frac{1}{\pi}\,\frac{1}{1+x^2},\;\; -\infty < x < \infty$$

- This is called (standard) Cauchy density. We can verify it integrates to 1

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{1}{1+x^2}\,dx = \frac{1}{\pi}\;\tan^{-1}(x)\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = 1$$

- What would be EX?

$$EX = \int_{-\infty}^{\infty} x\,\frac{1}{\pi}\,\frac{1}{1+x^2}\,dx \stackrel{?}{=} 0 \text{ because } \int_{-a}^{a} \frac{x}{1+x^2} = 0?$$

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&= \quad \lim_{c \to \infty} \int_{-c}^{0} g(x) \, dx + \lim_{d \to \infty} \int_{0}^{d} g(x) \, dx
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& = \quad \lim_{c\to\infty} \int_{-c}^{0} g(x)\, dx + \lim_{d\to\infty} \int_{0}^{d} g(x)\, dx
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This is not same as $\displaystyle \lim_{a\to\infty} \int_{-a}^{a} g(x)\, dx,$

which is known as Cauchy principal value

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- Here we have

$$\lim_{c\to\infty}\int_{-c}^{0}\frac{x}{1+x^2}\,dx=-\infty;\quad \lim_{d\to\infty}\int_{0}^{d}\frac{x}{1+x^2}\,dx=\infty$$

- Hence $EX=\int_{-\infty}^{\infty} x\,\frac{1}{\pi}\,\frac{1}{1+x^2}\,dx$ does not exist.
- Essentially, both halves of the integral are infinite and hence we get $\infty-\infty$ type expression which is undefined.
- However, $\lim_{a\to\infty}\int_{-a}^{a} x\,\frac{1}{\pi}\,\frac{1}{1+x^2}\,dx=0$.

Expectation of a random variable

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- We take the expectation to exist when the sum or integral above is absolutely convergent
- Note that expectation is defined for all random variables
- Let us calculate expectations of some of the standard distributions.

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- Thus, for example, $EI_A = P(A)$.

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(Left as an exercise for you!)

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&= \frac{e^{-\lambda x}}{-\lambda}\Big|_0^\infty \\
&= \frac{1}{\lambda}
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$$= \mu$$

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- This theorem is true for all rv's. But we will prove it in only some special cases.

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  That completes the proof.
- The proof goes through even when X (and Y) take countably infinitely many values (because we assume the expectation sum is absolutely convergent).

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- We can similarly show this for the case where $g'(x) < 0, \ \forall x$

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- However, this theorem is true for all random variables.
- Now, for any function, g, we can write

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\; f_X(x)\; d$$

Some Properties of Expectation

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$$2c^* = 2E[X] \;\Rightarrow\; c^* = E[X]$$

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- Thus $E[(X-c)^2] \geq E[(X-EX)^2], \ \forall c$
- So, $E[(X-c)^2]$ is minimized when $c = EX$ and the minimum value is $E[(X-EX)^2]$

Variance of a Random variable

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- This also implies: $E[X^2] \geq (EX)^2$

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- Hence the variance is now given by

$$
\mathsf{Var}(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}
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Variance of Gaussian rv

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- When $Y \sim \mathcal{N}(\mu, \sigma^2)$, $EY = \mu$ and $\text{Var}(Y) = \sigma^2$.

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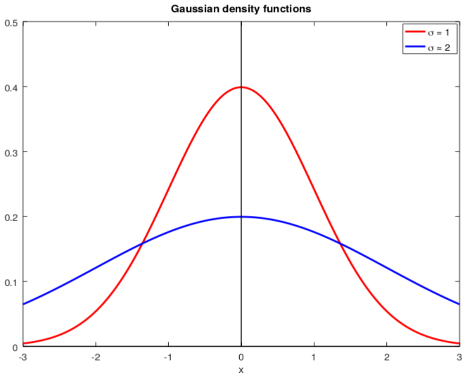

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(Left as an exercise)

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- Not all moments may exist for a given random variable. (For example, m_1 does not exist for Cauchy rv)

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- **Proof**: We prove it when X is continuous rv. Proof for discrete case is similar.

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$$