E1 222 Stochastic Models and Applications
Test II

Time: 75 minutes   Max. Marks:40
Date: 23 Dec 2020

Answer **ALL** questions. All questions carry equal marks

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = \frac{1}{1-x}, \ \ 0 < x < y < 1$$

Find $E[X], E[Y]$ and $E[Y|X]$

Answer: We can get the marginal densities as follows

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy = \int_x^1 \frac{1}{1-x}\ dy = \frac{1}{1-x}\ \ y|_x^1 = 1,\ 0 < x < 1 \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx = \int_0^y \frac{1}{1-x}\ dx = -\int_1^{1-y} \frac{1}{t}\ dt = -\log(1-y),\ 0 < y < 1 \end{aligned}$$

Since X is uniform over $(0,\ 1)$, we know $EX = \frac{1}{2}$. We can get the conditional density $f_{Y|X}$ as

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{1-x},\ \ 0 < x < y < 1$$

Conditioned on X, Y is uniform over $(X,\ 1)$. Hence we get $E[Y|X] = \frac{1+X}{2}$. We can also get it directly as follows

$$E[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)\ dy = \int_x^1 y\ \frac{1}{1-x}\ dy = \frac{1}{1-x}\left(\frac{1}{2} - \frac{x^2}{2}\right) = \frac{1+x}{2}$$

Since we know EX and $E[Y|X]$, we can get EY as

$$E[Y] = E[\ E[Y|X]\ ] = E\left[\frac{1+X}{2}\right] = \frac{1+EX}{2} = \frac{3}{4}$$

As you can see, we do not need to even calculate density of Y. We can also get EY directly from the density of Y

$$\begin{aligned} E[Y] &= -\int_0^1 y\ \log(1-y)\ dy = \int_1^0 (1-t)\ \log(t)\ dt = -\int_0^1 \log(t) dt + \int_0^1 t\log(t)\ dt \\ &= 1 + \left(\frac{t^2}{2}\log(t)\Big|_0^1 - \int_0^1 \frac{t}{2}\ dt\right) = 1 - \frac{1}{4} = \frac{3}{4} \end{aligned}$$

b. Let X, Y be discrete random variables, taking non-negative integer values, with joint mass function

$$P[X = i, Y = j] = e^{-(a+bi)} \frac{(bi)^j \; a^i}{j! \; i!}$$

Find $\text{Cov}(X, Y)$.

Answer: We can rewrite the joint mass function as

$$P[X = i, Y = j] = e^{-a} \frac{a^i}{i!} \; e^{-bi} \frac{(bi)^j}{j!}$$

Hence, X is Poisson with parameter a and Y, conditioned on X, is Poisson with parameter bX.

We can explicitly calculate them also as follows

$$P[X = i] = \sum_{j=0}^{\infty} e^{-a} \frac{a^i}{i!} \; e^{-bi} \frac{(bi)^j}{j!} = e^{-a} \frac{a^i}{i!} \sum_{j=0}^{\infty} e^{-bi} \frac{(bi)^j}{j!} = e^{-a} \frac{a^i}{i!}$$

and hence

$$f_{Y|X}(j|i) = P[Y = j | X = i] = \frac{P[X = i, Y = j]}{P[X = i]} = e^{-bi} \frac{(bi)^j}{j!}$$

I hope all of you are able to see this without doing the above calculation.

Hence, we have $EX = a$ and $E[Y|X] = bX$. Thus, $EY = ab$.

$$E[XY] = E[\; E[XY|X] \;] = E[X \; E[Y|X] \;] = E[bX^2] = b(a + a^2) = ab + a^2 b$$

You can calculate $E[XY] = \sum_{i,j} i \; j \; P[X = i, Y = j]$ but that would need more involved calculation. Now we get

$$\text{Cov}(X, Y) = E[XY] - E[X] \; E[Y] = ab + a^2 b - a^2 b = ab$$

Marks for part(a) – 5
Marks for part(b) – 5

2. a. Let X, Y be iid continuous random variables having exponential distribution with $\lambda = 1$. Let $Z = X + Y$ and $W = \frac{X}{X+Y}$. Show that Z and W are independent.

Answer: The given transformation is invertible and the inverse transform is $X = ZW,\ Y = Z - ZW$. The jacobian of the inverse transform is

$$\begin{vmatrix} w & z \\ 1-w & -z \end{vmatrix} = -wz - z + zw = -z$$

Hence the joint density of Z, W is

$$f_{ZW}(z, w) = |z|\ f_{XY}(zw, z - zw)$$

Since both X, Y are exponential, for the joint density above to be non-zero we need $zw > 0$ and $z > zw$. This gives us $z > 0$ and $0 < w < 1$. Hence, the joint density is

$$f_{ZW}(z, w) = z\ e^{-zw}\ e^{-(z-zw)} = z\ e^{-z}, \quad z > 0,\ 0 < w < 1$$

From this joint density, we can immediately see the marginals as: Z is Gamma and W is uniform over $(0,\ 1)$ and the joint is product of the marginals. We can get this also by explicit calculation as

$$f_Z(z) = \int_0^1 z\ e^{-z}\ dw = z\ e^{-z},\ z > 0; \quad \text{and} \quad f_W(w) = \int_0^\infty z\ e^{-z} = 1,\ 0 < w < 1$$

This shows that Z, W are independent because the joint density is the product of the marginals.

Comment: This shows that $\frac{X}{X+Y}$ is uniform over $(0,\ 1)$ when X, Y are iid exponential, which is a useful general result.

b. Let $X_1, X_2, \cdots$ be *iid* continuous random variables. We say a record has occurred at m if $X_m > \max(X_{m-1}, \cdots, X_1)$. Let $N = \min\{n\ :\ n > 1$ and a record occurs at time $n\}$. Show that $EN = \infty$.

Answer: We essentially need the mass function of N to get the expectation. Given the definition of N, it is easier to calculate $P[N > m]$. The event $[N > 2]$ implies that there is no record at time 2 which implies $X_2 < X_1$. Similarly $[N > 3]$ implies $X_3 < \max(X_1, X_2) = X_1$ because there is no record at 3 or 2. Thus, it is easy to see that the event of there being no record up to and including m, is same as $X_1 > X_i,\ i \leq m$. Since all possible orderings of iid continuous random variables are equally likely (see answer to problem 4 in Assignment-3), we get

$$P[N > m] = \frac{(m-1)!}{m!} = \frac{1}{m}$$

Since N is a positive integer valued random variable, we get

$$E[N] = \sum_{k=1}^{\infty} P[N > k] = \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

Comment: You have proved the above formula in one of your assignments.
Since you have $P[N > k]$ we can also calculate its mass function as $P[N = k] = \frac{1}{k-1} - \frac{1}{k} = \frac{1}{k(k-1)}$. Now, $EN = \sum_{k=2}^{\infty} k \, \frac{1}{k(k-1)} = \sum_{k=2}^{\infty} \frac{1}{(k-1)} = \infty$
You can also use the idea of the indicator random variables that we mentioned for Q8 in problem sheet 3.6. The event of $[N = k]$ is that there is no record at2, no record at 3, and so on till $k-1$ and there is a record at k. Using those indicator random variables and their independence this probability is $(1 - 1/2)(1 - 1/3)\cdots(1 - 1/(k-1))(1/k) = \frac{1}{k(k-1)}$ which is same as above.

Marks for part(a) – 5
Marks for part(b) – 5

3. a. Consider repeated independent tosses of a coin whose probability of heads is p, $0 < p < 1$. Let X denote the number of tosses needed to get at least one head and one tail. Let Y denote the number of tosses needed to get a head immediately followed by a tail. Find EX and EY.

Answer: Let Z be the indicator random variable of whether or not the first toss was head. For EX, we can condition on Z. If first toss is a head then we need to wait for a tail and the expected number of additional tosses for it would be $\frac{1}{1-p}$ and if the first toss is a tail we need to wait for a head. Thus

$$\begin{aligned} EX &= E[\, E[X|Z] \,] \\ &= E[X|Z=1]\, p + E[X|Z=0]\, (1-p) \end{aligned}$$

$$
\begin{aligned}
&= \left(1+\frac{1}{1-p}\right)p+\left(1+\frac{1}{p}\right)(1-p) \\
&= 1+\frac{p}{1-p}+\frac{1-p}{p} \\
&= \frac{p(1-p)+p^2+(1-p)^2}{p(1-p)} = \frac{(p+1-p)^2-p(1-p)}{p(1-p)} = \frac{1}{p(1-p)}-1
\end{aligned}
$$

To calculate EY. We have to first wait for a head. Then we have to wait for a tail. (After we get a head, till we get a tail everything would be heads and hence the first tail would be the first time we get a head followed by a tail). Hence we get

$$E[Y] = \frac{1}{p}+\frac{1}{1-p} = \frac{1}{p(1-p)}$$

Comment: You had solved a problem in one of the problem sheets about expected number of rolls of a fair dice to get all numbers at least once. There we said that you first wait for any of the 6 numbers then any of the five numbers and so on and thus the expected number of rolls is $\frac{6}{6}+\frac{6}{5}+\frac{6}{4}+\cdots$. Can we use a similar argument here for EX? Initially we wait for any of the two outcomes and then for the remaining one. But we do not know the expected number of tosses for the 'remaining' one because we do not know if the remaining one is head or tail. That is why we needed to condition on the first toss. If the coin is fair, then we can get the answer as $\frac{1}{1}+\frac{2}{1} = 3$ which is same as what the above formula gives for $p = 0.5$

In this problem, it is not difficult to find the distributions of the random variables. For $[X = k]$, by k^{th} toss we have got one head and one tail for the first time. So, either k^{th} toss is a head and everything before it is a tail or it is a tail and everything before it is a head. Hence

$$P[X=k] = (1-p)^{k-1}p + p^{k-1}(1-p),\ k = 2, 3, \cdots$$

Now we can calculate EX as

$$EX = \sum_{k=2}^{\infty} k\,(1-p)^{k-1}\,p \;+\; \sum_{k=2}^{\infty} k\,p^{k-1}\,(1-p)$$

$$
\begin{aligned}
&= (1-p)\sum_{k=2}^{\infty} k\,(1-p)^{k-2}\,p \;+\; p\sum_{k=2}^{\infty} k\,p^{k-2}\,(1-p) \\
&= (1-p)\sum_{k'=1}^{\infty} (k'+1)\,(1-p)^{k'-1}\,p \;+\; p\sum_{k'=1}^{\infty} (k'+1)\,p^{k'-1}\,(1-p) \\
&= (1-p)\,\left(\frac{1}{p}+1\right) \;+\; p\left(\frac{1}{1-p}+1\right) = 1+\frac{1-p}{p}+\frac{p}{1-p}
\end{aligned}
$$

Similarly we can find mass function of Y also. What is the event of $Y=k$? That should mean we got a Tail on k^{th} toss and a head on toss number $k-1$. What about tosses 1 to $k-2$? If there is any tail in these then all the tails have to be in the beginning because we cannot have a tail after a head in these tosses. Thus we can have s tails in the beining followed by heads till toss $k-2$ and s can be between 0 and $k-2$. Hence we have, for $k=2,3,\cdots$,

$$
\begin{aligned}
P[Y=k] &= \sum_{s=0}^{k-2}(1-p)^s p^{k-2-s} p(1-p) \\
&= p^{k-1}\,(1-p)\,\sum_{s=0}^{k-2}\left(\frac{1-p}{p}\right)^s \\
&= p^{k-1}\,(1-p)\,\frac{1-\left(\frac{1-p}{p}\right)^{k-1}}{1-\frac{1-p}{p}} \\
&= \frac{p^k(1-p)-(1-p)^k p}{2p-1}
\end{aligned}
$$

To calculate EY, we first note

$$
\sum_{k=2}^{\infty} k\,p^k(1-p) = p^2\sum_{k=2}^{\infty} k\,p^{k-2}(1-p) = p^2\sum_{k'=1}^{\infty}(k'+1)\,p^{k'-1}(1-p) = p^2\left(\frac{1}{1-p}+1\right)
$$

Similarly

$$
\sum_{k=2}^{\infty} k\,(1-p)^k\,p = (1-p)^2\,\left(\frac{1}{p}+1\right)
$$

Thus we get

$$
EY = \frac{1}{2p-1}\,\left(p^2\left(\frac{1}{1-p}+1\right)-(1-p)^2\,\left(\frac{1}{p}+1\right)\right)
$$

$$
\begin{aligned}
&= \frac{1}{2p-1}\left(\frac{p^3-(1-p)^3}{p(1-p)} + \left(p^2-(1-p)^2\right)\right)\\
&= \frac{1}{2p-1}\left(\frac{(p-(1-p))(p^2+p(1-p)+(1-p)^2)}{p(1-p)} + (2p-1)\right)\\
&= \frac{(p+(1-p))^2-p(1-p)}{p(1-p)} + 1\\
&= \frac{1}{p(1-p)}
\end{aligned}
$$

b. For any two random variables, X, Y, show that $\text{Cov}(X,Y) = \text{Cov}(X, E[Y|X])$

Answer:

$$
\begin{aligned}
\text{Cov}(X, E[Y|X]) &= E\left[X E[Y|X]\right] - EX\ E[E[Y|X]]\\
&= E\left[\ E[XY|X]\ \right] - EX\ EY = E[XY] - EX\ EY = \text{Cov}(X,Y)
\end{aligned}
$$

Marks for part (a) – 7
Marks for part (b) – 3

4. a. Let X be a discrete random variable taking non-negative integer values with mass function, $p(i),\ i = 0, 1, \cdots$. Let $Y_1, Y_2, \cdots, Y_n$ be *iid* discrete random variables taking non-negative integer values and with mass function $q(i), i = 0, 1, \cdots$. Assume $p(i), q(i) > 0, \forall i$. Let $h : \Re \to \Re$ be some function. Define

$$
S = \frac{1}{n}\sum_{k=1}^{n} \frac{p(Y_k)h(Y_k)}{q(Y_k)}.
$$

Find ES.

Answer: For any function g, $E[g(Y_k)] = \sum_m g(m)\ q(m)$. Hence

$$
E\left[\frac{p(Y_k)h(Y_k)}{q(Y_k)}\right] = \sum_{m=0}^{\infty} \frac{p(m)h(m)}{q(m)}\ q(m) = \sum_{m=0}^{\infty} p(m)h(m) = E[h(X)]
$$

Hence we get

$$
ES = \frac{1}{n}\sum_{k=1}^{n} E\left[\frac{p(Y_k)h(Y_k)}{q(Y_k)}\right] = \frac{1}{n}\sum_{k=1}^{n} E[h(X)] = E[h(X)]
$$

Comment: Suppose $X_1, X_2, \cdots, X_n$ are iid with mass function $p(i)$. That is, they are iid realizations or iid samples of X. Then we know that $S = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$ is an unbiased estimator of $E[h(X)]$. (That is, $ES = E[h(X)]$). But suppose generating sample of X is difficult. Though we know the expression for $p(i)$, it may be computationally costly to generate samples from there. Suppose $q(i)$ is another distribution from which we can easily generate samples and suppose $Y_1, \cdots, Y_n$ are the samples. We want to use the 'average' of $h(Y_i)$ to estimate $E[h(X)]$. What the above says is that we can do so but then the average is not taken with same weight to all samples. The weight we give to $h(Y_k)$ in this average is equal to $\frac{p(Y_k)}{q(Y_k)}$. If we weight different samples like this then we get an estimate of $E[h(X)]$. This problems illustrates a special case of a general technique known as importance sampling.

b. Let X, Y be jointly Gaussian with means zero, variances 1 and correlation coefficient ρ. Assume $\rho \neq 0$. Let $Z = aX + bY$ and $W = bX + aY$, where $a, b \in \Re$, $a \neq 0, b \neq 0$. Find a sufficient condition on a, b for Z and W to be independent.

Answer: We are given

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Hence, if $a^2 - b^2 \neq 0$ then Z, W are jointly Gaussian. Then they would be independent if they are uncorrelated.

Since $EX = EY = 0$, we have $EZ = EW = 0$. hence

$$\text{Cov}(Z, W) = E[ZW] = E[abX^2 + abY^2 + XY(a^2+b^2)] = 2ab + \rho(a^2+b^2)$$

Hence, Z, W are uncorrelated and hence independent if

$$a^2 - b^2 \neq 0, \quad \text{and} \quad \frac{-2ab}{a^2+b^2} = \rho$$

Comment: Since X, Y are jointly Gaussian, you know there is a linear transform to make them independent. For example, the transformation that would diagonalize the covariance matrix. But that is not what is required here. For the particular form of linear transform given, you are asked to find conditions on a, b to make Z, W independent.

Marks for part (a) – 5
Marks for part (b) – 5