E1 222 Stochastic Models and Applications
Assignment 3

1. Let X, Y be iid geometric random variables with parameter p. Let $Z = X - Y$ and $W = \min(X, Y)$. Find the joint mass function of Z, W. Show that Z, W are independent.

Answer: From their definitions, it is easy to see that W takes positive integer values while possible values for Z are both positive and negative integers (including zero). Also, when $Z > 0$ Y is smaller than X and when $Z < 0$, X is the smaller one.

Let $z > 0, w \geq 1$ be integers. Then

$$P[Z = z, W = w] = P[Y = w, X = w + z] = P[X = z + w]P[Y = w]$$

This gives us

$$f_{ZW}(z, w) = (1 - p)^{z+w-1}p(1 - p)^{w-1}p = (1 - p)^{2w+z-2}p^2$$

Now consider the case where z, w are integers with $z < 0, w \geq 1$. Then

$$P[Z = z, W = w] = P[X = w, Y = w - z] = P[x = w]P[Y = w - z]$$

and hence

$$f_{ZW}(z, w) = (1 - p)^{w-1}p(1 - p)^{w-z-1}p = (1 - p)^{2w-z-2}p^2$$

Finally when $z = 0$, we have

$$P[Z = z, W = w] = P[X = w, Y = w] = (1-p)^{w-1}p(1-p)^{w-1}p = (1-p)^{2w-2}p^2$$

Combining all these we now have

$$f_{ZW}(z, w) = (1-p)^{2w+|z|-2}\, p^2, w = 1, 2, \cdots,\ z \in \{\cdots, -2, -1, 0, 1, 2, \cdots\}$$

The marginal for Z is given by

$$f_Z(z) = \sum_{w=1}^{\infty} p^2(1-p)^{|z|}(1-p)^{2w-2} = p^2(1-p)^{|z|}\frac{1}{1 - (1 - p)^2} = \frac{p(1 - p)^{|z|}}{2 - p}$$

The marginal for W is given by

$$f_W(w) = \sum_{z=-\infty}^{\infty} p^2(1-p)^{2w-2}(1-p)^{|z|} = p^2(1-p)^{2w-2}\left(2\sum_{z=1}^{\infty}(1-p)^z + 1\right)$$

$$= p^2(1-p)^{2w-2}\left(\frac{2(1-p)}{p} + 1\right) = p(1-p)^{2w-2}(2-p)$$

From this, we can see that $f_{ZW}(z,w) = f_Z(z)f_W(w)$ and hence Z, W are independent.

2. Let X be a random variable having Gaussian density with mean zero and variance 1. Show that $Y = X^2$ has gamma density with parameters $\frac{1}{2}$ and $\frac{1}{2}$.
   Now, let $X_1, \cdots, X_n$ be iid random variables having Gaussian density with mean zero and variance σ^2. Show that $Y = \frac{X_1^2+\cdots+X_n^2}{\sigma^2}$ has Gamma density with parameters $\frac{n}{2}$ and $\frac{1}{2}$. (This rv, Y, is said to have chi-squared distribution with n degrees of freedom).

Hint: The first part is solved in class. (See lecture 6). We also showed in class that if X, Y are independent Gamma rv with parameters α_1, λ and α_2, λ then $X + Y$ is gamma with parameters $(\alpha_1 + \alpha_2, \lambda)$.

3. Let X be uniform over $(0, 1)$ and let Y be a discrete random variable taking non-negative integer values. Suppose X, Y are independent. let $Z = X + Y$. Show that Z is a continuous random variable.

Answer: To show that Z is a continuous rv, we need to show that F_Z is continuous everywhere and that it is differentiable at all but countably many points. (In an exam in this course, it is enough if you show continuity of F_Z).

Let $[z]$ denote the integer part of z (which is the largest integer less than or equal to z). We know that Z cannot take negative values and hence $F_Z(z) = 0$ for $z < 0$. Take $z \geq 0$.

$$\begin{aligned} F_Z(z) &= P[X + Y \leq z] \\ &= P[Y \leq [z] - 1] + P[Y = [z], X \leq z - [z]], \quad \text{because } X \in (0,1) \\ &= \sum_{k \leq [z]-1} f_Y(k) + f_Y([z])\,(z - [z]) \\ &\quad \text{because } X, Y \text{ are independent and } X \sim U(0,1) \end{aligned}$$

If z is not an integer, then, for sufficiently small δ, we would have $[z] = [z+\delta]$. Hence $F_Z(z+\delta) - F_Z(z)$ would be proportional to δ and hence F_Z would be continuous and differentiable at these points.

So, we only need to establish continuity at integer points. Also, df is always right continuous and hence we only need to show left continuity. Let $z = m - \delta$ where m is a positive integer and $\delta > 0$. Hence, $[z] = m - 1$. We need to show that $\lim_{\delta \downarrow 0} F_Z(z) = F_Z(m)$. Now, for this z,

$$F_Z(z) = \sum_{k=0}^{m-2} f_Y(k) + f_Y(m-1)\,(1-\delta)$$

and

$$F_Z(m) = \sum_{k=0}^{m-1} f_Y(k) + f_Y(m)(m-m)$$

which shows $\lim_{\delta \downarrow 0} F_Z(z) = F_Z(m)$.

4. Let X, Y, Z be iid continuous random variables. Show that $P[X < Y] = 0.5$ irrespective of what is the common density function of these random variables. Now calculate $P[X < Y < Z]$ and show that its value is same irrespective of what is the common density function of these random variables. Based on all this, can you guess what is the value of $P[X < Y, Z < Y]$. Explain.

Answer: We have already shown in class $P[X < Y] = 0.5$. Let f be the common density of the three random variables and let F be the corresponding distribution function. Then

$$\begin{aligned}
P[X < Y < Z] &= \int_{-\infty}^{\infty} \int_{-\infty}^{z} \int_{-\infty}^{y} f_{XYZ}(x,y,z)\; dx\; dy\; dz \\
&= \int_{-\infty}^{\infty} \left(f(z) \int_{-\infty}^{z} \left(f(y) \int_{-\infty}^{y} f(x)\; dx \right)\; dy \right)\; dz \\
&= \int_{-\infty}^{\infty} \left(f(z) \int_{-\infty}^{z} f(y)F(y)\; dy \right)\; dz \\
&= \int_{-\infty}^{\infty} f(z) \frac{F(z)^2}{2}\; dz \\
&= \frac{1}{2}\,\frac{1}{3}(F(\infty)^3 - F(-\infty)^3) = \frac{1}{6}
\end{aligned}$$

Given the above calculation, it is easy to see that $P[Z < Y < X]$ would also be the same. Thus all orderings of them would have equal probability. Hence, $P[X < Y, Z < Y] = P[X < Z < Y] + P[Z < X < Y] = \frac{2}{6}$

Comment If $X_1, X_2, \cdots, X_n$ are iid then any one specific ordering of them would have probability $\frac{1}{n!}$. This is because there are $n!$ orderings and all of them are equally likely.

5. Let $X_1, X_2, \cdots, X_n$ be random variables with mean zero and variance unity. Suppose the correlation coefficient of any pair of random variables, X_i and X_j, $i \neq j$, is ρ. Show that $\rho \geq \frac{-1}{n-1}$. Will this result remain true if $EX_i = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$; but correlation coefficient between any pair of them is still ρ.

Answer: We have

$$\text{Var}(X_1 + \cdots + X_n) = \sum_i \text{Var}(X_i) + \sum_i \sum_{j \neq i} \text{Cov}(X_i, X_j)$$

Hence, here we get

$$n + n(n-1)\rho \geq 0 \quad \Rightarrow \quad \rho \geq \frac{-1}{n-1}$$

If $X' = aX + b$ and $Y' = cY + d$ $(a, c > 0)$, then straight-forward algebra shows that $\rho_{X'Y'} = \rho_{XY}$. This shows that the answer to the second part of the question is yes because you can consider $X_i' = \frac{X_i - \mu}{\sigma}$ and so on.

6. Let X and Y be two discrete random variables with

$$P[X = x_1] = p_1, \quad P[X = x_2] = 1 - p_1;$$

$$P[Y = y_1] = p_2, \quad P[Y = y_2] = 1 - p_2.$$

Show that X and Y are independent if and only if they are uncorrelated. (Hint: Consider the special case where $x_1 = y_1 = 0$ and $x_2 = y_2 = 1$).

Answer: First consider the special case of X, Y taking values in $\{0, 1\}$ as suggested by the hint. We need to show (for this special case) that if the two random variables are uncorrelated then they are independent.

When X, Y are uncorrelated, we have $E[XY] = EX\ EY$. Given this we need to show that the joint mass function is product of the marginals.

When X, Y are binary, $E[XY] = P[X = 1, Y = 1]$, $EX = P[X = 1]$ and $EY = P[Y = 1]$. So, uncorrelatedness implies

$$P[X = 1, Y = 1] = P[X = 1]P[Y = 1] \quad \Rightarrow f_{XY}(1, 1) = f_X(1)f_Y(1)$$

Now we have:

$$P[X = 1, Y = 0] = P[X = 1] - P[X = 1, Y = 1] = P[X = 1] - P[X = 1]P[Y = 1]$$

$$= P[X = 1](1 - P[Y = 1]) = P[X = 1]P[Y = 0]$$

which shows $f_{XY}(1, 0) = f_X(1)f_Y(0)$. Similarly we can show the remaining two possibilities to complete the proof that X, Y are independent.

(The above is essentially the same as the fact that if two events A, B are independent then so are A, B^c, and A^c, B and so on).

Now consider the general case given in the problem. Define

$$X' = \frac{X - x_1}{x_2 - x_1}, \quad Y' = \frac{Y - y_1}{y_2 - y_1}$$

Since this is a linear (affine) transform, it is easily verified that X, Y being uncorrelated implies X', Y' are uncorrelated. This in turn implies that X', Y' are independent (because they take values in $\{0, 1\}$ which is the case proved above). By inverting the above transformations we can easily see that X is a function of X' and Y is a function of Y'. Hence X', Y' independent implies X, Y independent. This completes the solution of the problem.

7. An interval of length 1 is broken at a point uniformly distributed over $(0, 1)$. Let c be a fixed point in $(0, 1)$. Find the expected length of the subinterval that contains the point c. Show that this probability is maximized when $c = 0.5$.

Answer: Let X be uniform over $(0, 1)$. Let Y denote the length of the piece containing c. Y is a function of X. If $X > c$ (so that the rod is broken at a point to the right of c) then $Y = X$; otherwise (that is, if the rod is broken at a point to the left of c), $Y = (1 - X)$. Hence, we have $Y = XI_{[X \geq c]} + (1 - X)I_{[X<c]}$ where I_A denotes indicator of event A.

Let $g(X) = XI_{[X \geq c]}$. Then, $g(x) = x$ if $x \geq c$ and is zero otherwise. Hence $E[XI_{[X \geq c]}] = E[g(X)] = \int_0^1 g(x) f_X(x)\, dx = \int_c^1 x f_X(x)\, dx$. Also note that $f_X(x) = 1$ for $x \in (0, 1)$. Hence we get

$$EY = \int_c^1 x\, dx + \int_0^c (1 - x)\, dx = \frac{1}{2} + c - c^2$$

Now differentiating it with respect to c and equating to zero, you get $c = 0.5$. You can conclude this is a maximum by differentiating once more.