Continuous-Time Markov Chains

- Let $\{X(t),\ t \geq 0\}$ be a continuous-time discrete-state process
- Let $X(t)$ take non-negative integer values
- It is called a continuous-time markov chain if

$$Pr[X(t+s) = j \mid X(s) = i,\ X(u) \in A_u,\ 0 \leq u < s] \\ = Pr[X(t+s) = j \mid X(s) = i]$$

- Only most recent past matters
- It is called homogeneous chain if

$$Pr[X(t+s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

- Define

$$P_{ij}(t) = Pr[X(t) = j \mid X(0) = i] = Pr[X(t+s) = j \mid X(s) = i]$$

It is the probability of going from i to j in time t

- Analogous to transition probabilities in the discrete case
- Like in the discrete case, we can show that the Markov condition implies

$$Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(t) = i, X(s'),\ 0 \leq s' < t] \\ = Pr[X(s) \in B_s, s \in (t, t+\tau] \mid X(t) = i]$$

- Next we consider distribution of time spent in a state before leaving it

- By the Markov property and homogeneity we have

$$
\begin{aligned}
Pr\left[X(s) = i,\ s \in [t,\ t+\tau] \mid X(s') = i,\ 0 \leq s' \leq t\right] \\
&= Pr[X(s) = i,\ s \in [t,\ t+\tau] \mid X(t) = i] \\
&= Pr[X(s) = i,\ s \in [0,\ \tau] \mid X(0) = i]
\end{aligned}
$$

- Let $X(0) = i$ and let T_i be time spent in i before leaving it for the first time

$$
\begin{aligned}
Pr[X(s) = i,\ s \in [t,\ t+\tau] \mid X(s') = i,\ 0 \leq s' \leq t] \\
= Pr[T_i > t+\tau \mid T_i > t] \\
Pr[X(s) = i,\ s \in [0,\ \tau] \mid X(0) = i] \ &= \ Pr[T_i > \tau] \\
\Rightarrow \quad Pr[T_i > t+\tau \mid T_i > t] \ &= \ Pr[T_i > \tau]
\end{aligned}
$$

$\Rightarrow$ T_i is memoryless and hence exponential

- Once you transit into a state, the time spent in it is exponentially distributed.
- So, the chain can be viewed as follows
- Once you transit to a state, it spends time, say, $T_i \sim$ exponential(ν_i) in it.
- Then, when it leaves i, it transits to state j with probability, say, z_{ij}
- We would have $z_{ij} \geq 0$, $\sum_j z_{ij} = 1$. Also, $z_{ii} = 0$
- Note that $P_{ij}(t)$ is different from these z_{ij}

Example: Birth-Death process

- This is generalization of birth-death chains we saw earlier to continuous time
- From i the process can only go to $i + 1$ or $i - 1$
- A birth event takes it to $i + 1$ and a death event takes it to $i - 1$
- An example would be: $X(t)$ is number of people in a queuing system.
- A birth event would be a new person joining the queue.
- A death event would be a person leaving after finishing service

- Suppose, in state n, time till next arrival or birth event is exponential(λ_n).
- Let time till next departure or death event be exponential(μ_n)
  We assume that these two are independent
- Now, these λ_n and μ_n completely determine ν_n and z_{ij} and hence completely specify the chain
- $z_{i,i+1}$ is the probability that when the system changes state it goes to $i+1$
- Hence it is the probability that a birth event occurs before a death event.
- Let $W_1 \sim$ exponential(λ_i) and $W_2 \sim$ exponential(μ_i) be independent. Then

$$z_{i,i+1} = Pr[W_1 < W_2] = \frac{\lambda_i}{\lambda_i + \mu_i}; \quad \Rightarrow \quad z_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

- The time spent in state i, T_i, is exponential(ν_i)
- The chain would be in state i till either a birth event or a death event occurs
- Hence, $T_i = \min(W_1, W_2)$
- Hence, $T_i \sim$ exponential$(\lambda_i + \mu_i)$.
- Thus, $\nu_i = \lambda_i + \mu_i$
- We have taken state space to be non-negative integers.
- Hence, $\mu_0 = 0$ and $\nu_0 = \lambda_0$ and $z_{01} = 1$

- Suppose $\lambda_n = \lambda, \ \forall n$ and $\mu_n = 0, \ \forall n$
- It is called pure birth process
- The process spend time $T_i \sim$ exponential(λ) in state i and then moves to state $i + 1$
- This is the Poisson process

- Consider a queuing system
- Suppose people joining the queue is a Poisson process with rate λ
- Suppose the time to service each customer is independent and exponential with parmeter μ.
- We assume that the arrival and service processes are independent.
- Then this is a birth death process with

$$\lambda_n = \lambda, \; n \geq 0 \quad \text{and} \quad \mu_n = \mu, \; n \geq 1$$

- This is known as an $M/M/1$ queue
- A variation: $M/M/K$ queue

$$\lambda_n = \lambda, \; n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

- Consider an example of some calculations with continuous Markov chains
- Consider a Birth-Death process. Let Y_i be the time that a chain currently in i takes to reach state $i+1$ for the first time.
- We want to calculate $E[Y_i]$. (Note that $E[Y_0] = 1/\lambda_0$)
- The chain may directly go to $i+1$ or it may go to $i-1$ and then to i and then to $i+1$ or ...
- Define

$$I_i = \begin{cases} 1 & \text{if first transition out of } i \text{ is to } i+1 \\ 0 & \text{if first transition out of } i \text{ is to } i-1 \end{cases}$$

- We can find $E[Y_i]$ by conditioning on I_i.

- Time spent in i is exponential with rate $\lambda_i + \mu_i$.
- Hence, expected time till transition out of i is $1/(\lambda_i + \mu_i)$
- If this transition is to $i+1$ then that is the expected time to reach $i+1$

$$E\left[Y_i \mid I_i = 1\right] = \frac{1}{\lambda_i + \mu_i}$$

- Suppose this transition is to $i-1$.
- Then the expected time to reach $i+1$ is this time plus expected time to reach i from $i-1$ plus expected time to reach $i+1$ from i

$$E\left[Y_i \mid I_i = 0\right] = \frac{1}{\lambda_i + \mu_i} + E\left[Y_{i-1}\right] + E[Y_i]$$

- We also have

$$Pr[I_i = 1] = z_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}; \quad Pr[I_i = 0] = \frac{\mu_i}{\lambda_i + \mu_i}$$

- Now we can calculate $E[Y_i]$ as

$$\begin{aligned} E[Y_i] &= Pr[I_i = 1]\, E\left[Y_i \mid I_i = 1\right] + Pr[I_i = 0]\, E\left[Y_i \mid I_i = 0\right] \\ &= \frac{\lambda_i}{\lambda_i + \mu_i}\, \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \left(\frac{1}{\lambda_i + \mu_i} + E\left[Y_{i-1}\right] + E[Y_i] \right) \\ &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \left(E\left[Y_{i-1}\right] + E[Y_i] \right) \end{aligned}$$

$$\begin{aligned} E[Y_i] \left(1 - \frac{\mu_i}{\lambda_i + \mu_i} \right) &= \frac{1}{\lambda_i + \mu_i} + \frac{\mu_i}{\lambda_i + \mu_i} \left(E[Y_{i-1}] \right) \\ E[Y_i] &= \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[Y_{i-1}] \end{aligned}$$

- Thus we get

$$E\left[Y_i\right] = \frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i} E[Y_{i-1}], \; i \geq 1$$

- Since $E[Y_0] = 1/\lambda_0$, we have a formula for $E[Y_i]$
- For example,

$$E[Y_1] = \frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \; \lambda_0}; \quad E[Y_2] = \frac{1}{\lambda_2} + \frac{\mu_2}{\lambda_2}\left(\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \; \lambda_0}\right)$$

- Expected time to go from i to j, $i < j$ can now be computed as

$$E[Y_i] + E[Y_{i+1}] + \cdots + E[Y_{j-1}]$$

- Note that these are only for birth-death processes

- Consider the transition probabilities, $P_{ij}(t)$
- These satisfy Chapmann-Kolmogorov equation

$$P_{ij}(t+s) = Pr[X(t+s) = j \mid X(0) = i]$$

$$= \sum_k Pr[X(t+s) = j \mid X(s) = k, X(0) = i] Pr[X(s) = k \mid X(0) = i]$$

$$= \sum_k Pr[X(t+s) = j \mid X(s) = k] Pr[X(s) = k \mid X(0) = i]$$

$$= \sum_k Pr[X(t) = j \mid X(0) = k] Pr[X(s) = k \mid X(0) = i]$$

$$= \sum_k P_{kj}(t) \; P_{ik}(s)$$

- For finite chain, P is a matrix and $P(t+s) = P(t) \; P(s)$

- Chapmann-Kolmogorov equation gives

$$P_{ij}(t+s) = \sum_k P_{ik}(s)\ P_{kj}(t)$$

- Hence we get

$$\begin{aligned} P_{ij}(t+h) - P_{ij}(t) &= \sum_k P_{ik}(h)\ P_{kj}(t) - P_{ij}(t) \\ &= \sum_{k \neq i} P_{ik}(h)\ P_{kj}(t) - (1 - P_{ii}(h))P_{ij}(t) \end{aligned}$$

- Define

$$q_{ik} = \lim_{h\to 0} \frac{P_{ik}(h)}{h},\ i \neq k, \quad \text{and} \quad q_{ii} = \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h}$$

- Then, assuming limit and sum can be interchanged,

$$\lim_{h\to 0} \frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k\neq i} q_{ik}\ P_{kj}(t) - q_{ii}\ P_{ij}(t)$$

- By definition, $1 - P_{ii}(h)$ is the probability that the chain that started in i is not in i at h.
- This is equivalent to there being a transition in the time h and transitions out of i occur at the rate of ν_i. Also, two or more transitions in h is $o(h)$
- Hence

$$1 - P_{ii}(h) = \nu_i h + o(h)$$

- Thus $q_{ii} = \nu_i$. It is rate of transition out of i
- We also have

$$\nu_i = q_{ii} = \lim_{h \to 0} \frac{1 - P_{ii}(h)}{h} = \lim_{h \to 0} \frac{\sum_{j \neq i} P_{ij}(h)}{h} = \sum_{j \neq i} q_{ij}$$

- By definition, $P_{ij}(h) = q_{ij}h + o(h),\ i \neq j$
- Hence q_{ij} is the rate at which transitions out of i into j are occurring.
- Transitions out of i occur with rate ν_i and z_{ij} fraction of these are into j
- Hence, $q_{ij} = \nu_i\ z_{ij},\ i \neq j$
- Thus, we got

$$\nu_i = \sum_{j \neq i} q_{ij}, \quad z_{ij} = \frac{q_{ij}}{\sum_{j \neq i} q_{ij}}, \quad q_{ii} = \sum_{j \neq i} q_{ij}$$

- The $\{q_{ij}\}$ are called the infinitesimal generator of the process.
- A continuous time Markov Chain is specified by these q_{ij}

- Consider a Birth-Death process.
- We got earlier

$$\nu_i = \lambda_i + \mu_i, \quad z_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad z_{i,i-1} = \frac{\mu_i}{\lambda_i + \mu_i}$$

- Now we can calculate q_{ij}

$$q_{i,i+1} = (\lambda_i + \mu_i) \, \frac{\lambda_i}{\lambda_i + \mu_i} = \lambda_i, \quad q_{i,i-1} = (\lambda_i + \mu_i) \, \frac{\mu_i}{\lambda_i + \mu_i} = \mu_i$$

- This is intuitively obvious
- We specify a birth-death chain by
  birth rate (rate of transition from i to $i+1$), λ_i and
  death rate (rate of transition from i to $i-1$), μ_i.

- The Chapmann-Kolmogorov equations give us

$$P_{ij}(t+h) - P_{ij}(t) = \sum_{k \neq i} P_{ik}(h)\ P_{kj}(t) - (1 - P_{ii}(h))P_{ij}(t)$$

- Using this we derived

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik}\ P_{kj}(t) - q_{ii}\ P_{ij}(t)$$

  Called Kolmogorov Backward equation
- We can solve these ODEs to get $P_{ij}(t)$
- For a birth-death chain the equation becomes

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t)$$

Poisson process as a special case

- Consider the case: $\lambda_i = \lambda$ and $\mu_i = 0.\ \forall i$.
- This would be a Poisson process with rate λ.
- Taking $i = 0$, the differential equation becomes

$$P'_{0j}(t) = \lambda P_{1j}(t) - \lambda P_{0j}(t)$$

- $P_{0j}(t)$ is the probability of j events in an interval of length t which is same as what we had called $P_j(t)$.
- Similarly, $P_{1j}(t)$ is same as what we called $P_{j-1}(t)$ there
- Now one can see that the above ODE is what we got for Poisson process.

- Consider a two-state Birth-Death chain.
- We would have $\mu_0 = \lambda_1 = 0$. Let $\lambda_0 = \lambda$ and $\mu_1 = \mu$
- The two states can be a machine working or failed.
- λ is rate of failure. Time till next failure is exponential(λ)
- μ is rate of repair. Time for repair is exponential(μ)
- We may want to calculate $P_{00}(T)$, the probability that the machine would be working at a time T units later given it is in working condition now
- We can calculate it by solving the ODE's

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- For the two state chain, these equations are

$$\begin{aligned}
P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\
P'_{01}(t) &= \lambda_0 P_{11}(t) - \lambda_0 P_{01}(t) \\
P'_{10}(t) &= \mu_1 P_{00}(t) - \mu_1 P_{10}(t) \\
P'_{11}(t) &= \mu_1 P_{01}(t) - \mu_1 P_{11}(t)
\end{aligned}$$

- As is easy to see, we get a system of equations like this for any finite chain.
- Solving these we can show

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda + \mu}$$

- Consider a finite chain
- Then the transition probabilities can be represented as a matrix
- The Chapmann-Kolmogorov equation gives

$$P(t+s) = P(t)\ P(s)$$

- Differentiating the above with respect to t

$$P'(t+s) = P'(t)P(s)$$

- Putting $t = 0$ in the above we get

$$P'(s) = P'(0)\ P(s) = \bar{Q}\ P(s), \quad \text{where} \quad \bar{Q} = P'(0)$$

- The solution for this is

$$P(t) = e^{t\bar{Q}}, \quad \text{because} \quad P(0) = I$$

- This is the expression for calculating $P_{ij}(t)$ for any t and i, j

- Let us examine the matrix $\bar{Q} = [\bar{q}_{ij}]$

$$\bar{Q} = P'(0) = \lim_{h \downarrow 0} \frac{P(h) - P(0)}{h} = \lim_{h \downarrow 0} \frac{P(h) - I}{h}$$

- This gives us

$$\begin{aligned} \text{for } \ k \neq j, \ \ \bar{q}_{kj} &= \lim_{h \downarrow 0} \frac{P_{kj}(h) - 0}{h} = q_{kj} \\ \bar{q}_{jj} &= \lim_{h \downarrow 0} \frac{P_{jj}(h) - 1}{h} = -q_{jj} = -\nu_j \end{aligned}$$

- Thus this $\bar{Q}$ matrix has q_{ik} as off-diagonal entries and $-q_{jj}$ as diagonal entries
- So, each row here sums to zero
- We normally write it as Q and call it the infinitesimal generator of the process

- The Kolmogorov backward equation is

$$P'_{ij}(t) = \sum_{k \neq i} q_{ik} \, P_{kj}(t) - q_{ii} \, P_{ij}(t)$$

- The above can be written in a matrix form

$$P'(t) = QP(t)$$

- The off-diagonal entries of Q are q_{ik} and diagonal entries are $-q_{ii}$
- From the above equation, $P'(0) = Q$
- So, what we did is to write the backward equation in matrix form

- For the backward equation, we started with

$$P_{ij}(t+h) = \sum_k P_{ik}(h)\ P_{kj}(t)$$

- The Chapmann-Kolmogorov equation also gives us

$$P_{ij}(t+h) = \sum_k P_{ik}(t)\ P_{kj}(h)$$

- Similar algebra as earlier gives us

$$P'_{ij}(t) = \sum_{k \neq j} P_{ik}(t)\ q_{kj} - q_{jj}\ P_{ij}(t)$$

  (under some assumptions about interchanging limit and summation)
- This is known as Kolmogorov forward equation
- For finite chains, both forward and backward equations are same
- For infinite chains there are some differences

- We can define transient and recurrent states as in the discrete case.
- However, we need to be careful about defining hitting times or first passage times
- We define

$$T_i = \min\{t > 0 \ : \ X(t) \neq i\} \qquad f_i = \min\{t \ : \ t > T_i, \ X(t) = i\}$$

- For a chain started in i we take f_i as first return time to i
- A state i is said to be
  - Transient if $Pr[f_i < \infty \mid X(0) = i] < 1$
  - Recurrent if $Pr[f_i < \infty \mid X(0) = i] = 1$

- Most of the other definitions are also similar to the case of discrete chains
- The chain is said to be irreducible if for all i, j there is a positive probability of going from i to j in some finite time: $P_{ij}(t) > 0$ for some t
- A recurrent state is positive recurrent if mean time to return is finite: $E[f_i \mid X(0) = i] < \infty$
  Otherwise it is null recurrent
- An irreducible positive recurrent chain would have a unique stationary distribution
- There is no concept of periodicity in the continuous time case
- An irreducible positive recurrent chain would be called an ergodic chain

- Define

$$\pi_j(t) = Pr[X(t) = j] = \sum_i \pi_i(0) P_{ij}(t)$$

  This also analogous to the discrete case
- The above equation is true for general infinite chains.
- In the finite case, we can get a more compact expression
- For a finite chain, taking π as a row vector,

$$\pi(t) = \pi(0)\ P(t) = \pi(0)\ e^{Qt}$$

- We say π is a stationary distribution if

$$\pi(0) = \pi \quad \Rightarrow \quad \pi(t) = \pi, \ \forall t$$

- Hence, if we start the chain in the stationary distribution, $\pi'(t) = 0$
- We get from the earlier equation

$$\pi_j(t) = \sum_i \pi_i(0) P_{ij}(t) \quad \text{and hence} \quad \pi'_j(t) = \sum_i \pi_i(0) P'_{ij}(t)$$

- Using the forward equation for $P'_{ij}(t)$

$$\begin{aligned} \sum_i \pi_i(0) \left(\sum_{k \neq j} q_{kj} P_{ik}(t) - q_{jj} P_{ij}(t) \right) &= 0 \\ \Rightarrow \quad \sum_{k \neq j} q_{kj} \pi_k - \pi_j \sum_{k \neq j} q_{jk} &= 0 \end{aligned}$$

when π is a stationary distribution and $\pi(0) = \pi$

- What we showed is that any stationary distribution π has to satisfy

$$\sum_{k \neq j} q_{kj} \pi_k = \pi_j \sum_{k \neq j} q_{jk}$$

- We can interpret this (as we did in discrete case)
- q_{kj} is the rate of transition from k to j and π_k is the fraction present in k.
- Hence $\sum_{k \neq j} q_{kj} \pi_k$ is the net flow into j
- $\pi_j \sum_{k \neq j} q_{jk}$ is the net flow out of j
- At steady state the flows have to be balanced
- The above equation is known as a balance equation