Assignment 8

Note: Text in blue shows Marks. Text in green is not necessary

1) Let $A \in \mathbb{C}^{n\times n}$ be given. Define

$K_\lambda \triangleq \{x \in \mathbb{C}^n \mid (A-\lambda I_n)^p x = 0$ for some $p \in \mathbb{Z}^+\}$.

If λ is an Eval of A with alg mult a_λ, then prove

(a) K_λ is an A-invariant subspace of $\mathbb{C}^n$ (3M)

(b) $\dim(K_\lambda) \leq a_\lambda$ (2M)

Sol (a) To prove: $\forall\ y \in K_\lambda,\ Ay \in K_\lambda$

$\therefore \forall\ y \in K_\lambda, \quad (A-\lambda I_n)^p y = 0$ ——— (1)

$\therefore (A-\lambda I_n)^{p+1} y = 0$ (1M)

i.e. $(A-\lambda I_n)^p (A-\lambda I_n) y = 0$

$(A-\lambda I_n)^p (Ay-\lambda y) = 0$

$(A-\lambda I_n)^p Ay - \cancel{\lambda (A-\lambda I_n)^p y}^{\to 0} = 0$ (from (1)) (2M)

$\therefore (A-\lambda I_n)^p Ay = 0$

$\Rightarrow Ay \in K_\lambda$ (3M)

(b) Let $r_p = \text{rank}[(A-\lambda I)^p]$

From Jordan form equations, we have (4M)

$r_p \geq n - a_\lambda$ with equality when $p =$ size of largest block corresp to λ

From rank-nullity thm

$n - \dim(K_\lambda) \geq n - a_\lambda$

$\therefore \dim(K_\lambda) \leq a_\lambda$ — (5M)

2) Let $B \in \mathbb{C}^{n \times n}$. Prove or disprove:

a) If B is normal, then $R(B) \perp N(B)$ 3M

b) If $R(B) \perp N(B)$, then B is normal 2M

Sol a) Given B is normal ie $BB^H = B^H B$

To prove: $R(B) \perp N(B)$

W.K.T $R(B) \perp N(B^H)$

$\therefore$ To Prove: $N(B) = N(B^H)$ – 1M

$\forall u \in N(B)$, $Bu = 0$

$\therefore \|Bu\| = 0$ [Positivity]

$\Rightarrow \|Bu\|^2 = 0$

ie $\langle Bu, Bu \rangle = 0$

$\Rightarrow \langle B^H Bu, u \rangle = 0$

$\Rightarrow \langle BB^H u, u \rangle = 0$ [Normality]

$\Rightarrow \langle B^H u, B^H u \rangle = 0$

$\Rightarrow \|B^H u\|^2 = 0$

$\Rightarrow \|B^H u\| = 0$

$\Rightarrow B^H u = 0$ [Positivity]

$\Rightarrow u \in N(B^H)$ – 2M

$\therefore \forall u \in N(B)$, $u \in N(B^H)$

$\therefore N(B) \subseteq N(B^H)$

$|||^{ly}$ Starting with $u \in N(B^H)$, we can show $u \in N(B)$

$\therefore N(B^H) \subseteq N(B)$

$\Rightarrow N(B) = N(B^H)$ – 3M

b) Let's disprove using a counter example

i.e We need to find a B which is not normal, but $R(B) \perp N(B)$

Since $R(B^H) \perp N(B)$, we need to find a B s.t $R(B) = R(B^H)$. Symmetric / Hermitian matrix won't work.

For invertible matrices, $R(B) = R(B^H)$

$\therefore$ We need to find an invertible matrix, which is not normal

W.K.T normal matrices are diagonalizable

$\therefore$ We need to find an invertible matrix, which is not diagonalizable i.e a Jordan Block.

Consider $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$N(B) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ $\therefore R(B) \perp N(B)$ — 4M

Check Normality of B

$$BB^H = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$B^H B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$\therefore BB^H \neq B^H B \Rightarrow B$ is not normal — 5M

Another Example:

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$