E1 222 Stochastic Models and Applications
Problem Sheet 5–2

1. Let $X(t)$ be a wide-sense stationary stochastic process with autocorrelation $R(\tau)$. Show that $\text{Prob}[|X(t+\tau)-X(t)| \geq a] \leq 2[R(0)-R(\tau)]/a^2$.

Hint: Since process is wide-sense stationary

$$E[X^2(t)] = R(0), \forall t \quad \text{and} \quad E[X(t+\tau)X(t)] = R(\tau)$$

Markov inequality gives

$$\text{Prob}[|X(t+\tau) - X(t)| \geq a] \leq \frac{E\left[|X(t+\tau) - X(t)|^2\right]}{a^2}$$

The expectation is given by

$$E\left[|X(t+\tau) - X(t)|^2\right] = E\left[X^2(t+\tau)\right] + E\left[X^2(t)\right] - 2E\left[X(t+\tau)X(t)\right]$$

2. Consider a stochastic process $X(t) = e^{At}$ where A is a continuous random variable with density f_A. Express the mean $\eta(t)$ and the autocorrelation $R(t_1, t_2)$ in terms of f_A.

Hint:

$$\eta(t) = E\left[e^{At}\right] = \int e^{at} \, f_A(a) \, da = M_A(t)$$

$$R(t_1, t_2) = E\left[e^{At_1} \, e^{At_2}\right] = M_A(t_1 + t_2)$$

3. Customers arrive at a bank at a Poisson rate 4 per hour. Suppose two customers arrived during the first hour. What is the probability that (i). both arrived during the first twenty minutes, (ii). at least one arrived during the first twenty minutes.

Hint: Take unit of time as hour. Then $\lambda = 4$

For the first part you need

$$\begin{aligned} Pr[N(1/3) = 2 | N(1) = 2] &= \frac{Pr[2 \text{ in } [0,\ 1/3] \text{ and } 0 \text{ in } (1/3,\ 1]]}{Pr[N(1) = 2]} \\ &= \frac{e^{-(1/3)\lambda} \, ((1/3)\lambda)^2/2! \; e^{-(2/3)\lambda}}{e^{-\lambda}\lambda^2/2!} \\ &= \frac{1}{9} \end{aligned}$$

You can get the second part as $1 - Pr[\text{both arrived in } (1/3,\ 1]] = \frac{5}{9}$

4. Suppose vehicles pass a certain point in a highway as a Poisson process with rate 1 per minute. Suppose 5% of the vehicles are vans. What is the probability that at least one van passes by during half an hour? Given that 10 vans passed by in an hour what is the expected number of vehicles to have passed in that hour.

Hint: Let $N(t)$ be the total vehicles, $N_1(t)$ be vans and $N_2(t)$ be the other vehicles. We know $N(t) = N_1(t) + N_2(t)$ and $N_1(t)$ and $N_2(t)$ are independent and Poisson with rates 0.05 and 0.95 per minute.

For the first part, you need to calculate $1 - Pr[N_1(30) = 0]$ and $N_1(t)$ is Poisson with rate 0.05

For the second part, you need $E[N(60) \mid N_1(60) = 10] = E[N_1(60) + N_2(60) \mid N_1(60) = 10]$ which will be 10 plus $E[N_2(60)]$ and thus the answer is $10 + 57$

5. Cars pass a certain location on a street according to a Poisson process with rate λ. A woman wants to cross the street at that place and she waits till she can see that no cars would pass that place in the next T time units. Find the probability that her waiting time is zero.

Hint: The waiting time being zero is same as there being no event in the interval $[0,\ T]$

6. Suppose people arrive at a bust stop in accordance with a Poisson process with rate λ. let t be some fixed time and suppose the next bus departs at t. All people who arrive till t would get on the bus that departs at t. Let X denote the total amount of waiting time of all people who got on the bus at t. (Note that a person who arrived at $s < t$ would contribute $t - s$ to the waiting time).

- Show that $E[X|N(t)] = N(t)\frac{t}{2}$
- Show that $\text{Var}[X|N(t)] = N(t)\frac{t^2}{12}$
- Using these two, calculate $\text{Var}(X)$

Hint: May be I should not have given this problem. It needs a result that I mentioned only in passing in the class. Given $N(t) = K$, the K arrival times, considered as unordered, would be independent and uniform over $[0,\ t]$. More precisely: suppose Z_i is the time of the i^{th} event. Then, conditioned on $N(t) = K$, the joint distribution of $Z_1, \cdots Z_K$ is same as the order statistics of $U_1, \cdots, U_K$ where U_i are iid uniform over $[0,\ t]$. In the class we only proved that conditioned on $N(t) = 1$, the time of that event is uniform over $[0,\ t]$.

Let $Z_1, \cdots, Z_n$ be the times of arrival of the n customers till t. Then the total waiting time is $\sum_{i=1}^{n}(t - Z_i)$. This expression is invariant to any permutation of Z_i.

Hence, conditioned on $N(t) = n$, the expected total waiting time would be $E\left[\sum_{i=1}^{n}(t - U_i)\right]$ where U_i are iid uniform over $[0,\ t]$. This gives us $E[X|N(t)] = N(t)\frac{t}{2}$

Similarly the conditional variance of the total waiting time would be $\mathrm{Var}\left(\sum_{i=1}^{n}(t - U_i)\right) = \sum_{i=1}^{n}\mathrm{Var}(t - U_i) = \sum_{i=1}^{n}\mathrm{Var}(U_i)$ This gives us $\mathrm{Var}[X|N(t)] = N(t)\frac{t^2}{12}$.

Now we can use the formula $\mathrm{Var}(X) = \mathrm{Var}(E[X|Z]) + E[\mathrm{Var}(X|Z)]$. (You have proved this formula in Problem Sheet 3–4).

7. Suppose $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process. Find the distribution of $B(t) + B(s)$ with $s < t$.

Hint: $B(t) + B(s) = 2B(s) + (B(t) - B(s))$. We know $B(t)$ and $B(t) - B(s)$ are independent and also that they are Gaussian. We also know that sum of independent Gaussians is Gaussian. We know $2B(s) \sim \mathcal{N}(0, 4s)$ and $B(t) - B(s) \sim \mathcal{N}(0, t - s)$. Hence $B(t) + B(s)$ is Gaussian with mean zero and variance $t + 3s$

8. Suppose $\{B(t),\ t \geq 0\}$ is a standard Brownian motion process. Compute $E[B(t_1)B(t_2)B(t_3)]$

Hint: This problem shows some more ways in which the independent increments property can be used.

Assume $t_1 < t_2 < t_3$ (without loss of generality). We first derive a couple of relations that we need.

$$
\begin{aligned}
E[B(t_3) \mid B(t_2)] &= E[(B(t_3) - B(t_2)) \ + \ B(t_2) \mid B(t_2)] \\
&= E[B(t_3) - B(t_2) \mid B(t_2)] + E[B(t_2) \mid B(t_2)] \\
&= E[B(t_3) - B(t_2)] + B(t_2), \quad \text{by independent increments} \\
&= B(t_2)
\end{aligned}
$$

(Recall that $B(t_2) - B(t_1)$ is Gaussian with zero mean and variance $t_2 - t_1$).

It is easy to see that by the above, we would also get $E[B(t_3) \mid B(t_2), B(t_1)] = B(t_2)$

$$
\begin{aligned}
E\left[B^2(t_2) \mid B(t_1)\right] &= E\left[(B(t_2) - B(t_1))^2 + 2B(t_2)B(t_1) - B^2(t_1) \mid B(t_1)\right] \\
&= E\left[(B(t_2) - B(t_1))^2\right] + 2B(t_1)E[B(t_2) \mid B(t_1)] - B^2(t_1) \\
&= (t_2 - t_1) + 2B(t_1)B(t_1) - B^2(t_1) = (t_2 - t_1) + B^2(t_1)
\end{aligned}
$$

where we have used the previous result (and also used the fact that $B(t_2) - B(t_1)$ has zero mean and variance $t_2 - t_1$).

We are now ready for the main problem.

$$
\begin{aligned}
E[B(t_1)B(t_2)B(t_3)] &= E\,[\ E[B(t_1)B(t_2)B(t_3) \mid B(t_2), B(t_1)]\ ] \\
&= E\,[\ B(t_1)B(t_2)E[B(t_3) \mid B(t_2), B(t_1)]\ ] \\
&= E\left[B(t_1)B^2(t_2)\right] \\
&= E\left[\ E[B(t_1)B^2(t_2) \mid B(t_1)]\ \right] \\
&= E\left[\ B(t_1)E[B^2(t_2) \mid B(t_1)]\ \right] \\
&= E\left[B(t_1)\left(t_2 - t_1 + B^2(t_1)\right)\right] \\
&= (t_2 - t_1)E[B(t_1)] + E\left[B^3(t_1)\right] \\
&= 0
\end{aligned}
$$

Note that we have used both the earlier results in the derivation above and the last step follows because $B_1(t)$ is Gaussian with zero mean and hence its first and third moments are zero.

9. Suppose customers arrive at a single server queuing system in accordance with a Poisson process with rate λ. However an arriving customer will join the queue with probability α_n if he sees there are n people in the system. (With the remaining probability he just departs). Represent this a birth-death process (of a continuous time markov chain) and specify the birth and death rates.

Hint: Consider the arrival process when the chain is in state n. The customer events occur as a Possion process with rate λ. But there are two types of customers – those that join the queue and those that do not. Hence, when the chain is in state n, the birth events occur as a Poisson process with rate $\lambda\alpha_n$. Hence, for the chain, the birth and death rates are: $\lambda_n = \lambda\alpha_n$ and $\mu_n = \mu$.