E1 222 Stochastic Models and Applications
Problem Sheet 3–4

1. Recall that random variables $X_1, X_2, \cdots X_n$ are said to be *exchangeable* if any permutation of them has the same joint density. Suppose X_1, X_2, X_3 are exchangeable random variables. Show that

$$E\left[\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right] = \frac{2}{3}$$

Can you generalize this result?
(Hint: Is there a relation between $E\left[\frac{X_1}{X_1+X_2+X_3}\right]$ and $E\left[\frac{X_2}{X_1+X_2+X_3}\right]$?)

Hint: I hope the hint in the problem was sufficient. Here is an extended hint.

Suppose X, Y are exchangeable. We have

$$E\left[\frac{X}{X+Y}\right] = \int\int \frac{a}{a+b}\, f_{XY}(a,b)\; da\; db \quad \text{and} \quad E\left[\frac{Y}{X+Y}\right] = \int\int \frac{b}{a+b}\, f_{XY}(a,b)\; da\; db$$

By changing variables in the second integral as a to b and b to a, and noting that $f_{XY}(a,b) = f_{XY}(b,a)$ because X, Y are exchangeable, we conclude that both the above expectations are equal. Also, we easily see that the sum of the two expectations above is 1.

Now I hope you can easily solve the given problem.

2. Find $E[X|Y]$ when X, Y have joint density given by

$$f_{XY}(x,y) = \frac{y}{2} e^{-xy}, \; x > 0, \; 1 < y < 3$$

Hint: We can write the joint density as $\left(\frac{1}{2}\right)\; y e^{-xy}$. From this I hope it is easy to see that $f_Y(y) = \frac{1}{2}$ and $f_{X|Y}(x|y) = y e^{-xy}$ given the ranges for the variables in the problem.

So, conditional density of X given $Y = y$ is exponential with parameter y. Hence, $E[X|Y] = \frac{1}{Y}$. You can get this by actually finding f_Y by integrating joint density and hence finding conditioal density and so on.

3. Let X and Y be iid random variables having Poisson distribution with parameter λ. Let $Z = X + Y$. Find $E[Z|Y]$.

Hint:

$$E[Z|Y] = E[X+Y|Y] = E[X|Y] + E[Y|Y] = E[X] + Y = \lambda + Y$$

4. Let X and Y be independent random variables each having geometric density with parameter p. Let $Z = X + Y$. Find $E[Y|Z]$.

$$Z = E[Z|Z] = E[X+Y|Z] = E[X|Z] + E[Y|Z]$$

Since X, Y are iid we should have $E[X|Z] = E[Y|Z]$. So, each of them should be equal to $Z/2$.

You can verify this by direct calculation. $f_{Y|Z}(y|z) = P[Y = y|Z = z] = P[Y = y, Z = z]/P[Z = z]$. You know $P[Z = z]$ from previous problem sheet. $P[Y = y, Z = z] = P[Y = y, X = z - y]$. Now the rest is simple algebra

5. Let Y be a continuous random variable with density

$$f_Y(y) = \frac{1}{\Gamma(0.5)}\sqrt{\frac{0.5}{y}}e^{-0.5y}, \ y > 0.$$

Let the conditional density of X given Y be

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}}\sqrt{y}\, e^{-0.5yx^2}, \ -\infty < x < \infty, \ y > 0$$

Show that $E[X|Y] = 0$. Show that marginal of X is given by

$$f_X(x) = \frac{1}{\pi(1+x^2)}$$

(Note that $\Gamma(0.5) = \sqrt{\pi}$). Does EX exist?
(Notice that $f_{X|Y}$ is Gaussian with mean zero and variance $1/\sqrt{y}$, Y is Gamma with parameters $0.5, 0.5$ and X has Cauchy distribution).

Comment No hint needed here. If you multiply $f_{X|Y}$ and f_Y you get a simple expression which is easily integrated over y to get f_X.

This is an example where $E[E[X|Y]] = E[X]$ does not hold because one of the expectations does not exist.

6. Define $\text{Var}[X|Y] = E\left[X^2|Y\right] - (E[X|Y])^2$ Show that

$$\text{Var}(X) = \text{Var}(E[X|Y]) + E[\text{Var}[X|Y]]$$

7. Suppose that independent trials, each of which is equally likely to have any of m possible outcomes, are performed repeatedly until the same outcome occurs k consecutive times. Let N denote the number of trials needed. Show that

$$E[N] = \frac{m^k - 1}{m - 1}$$

Hint: This is a slight variation of the problem done in class where N_k is number of tosses to get m consecutive heads and we want EN_k. We conditioned on N_{k-1}. Suppose $N_{k-1} = n$. If next toss is head you are done; otherwise you wasted n plus one tosses and have to start all over again.

Now suppose we wanted N_k which is number of tosses needed to get either k consecutive heads or k consecutive tails and assume coin is fair. Once again condition on N_{k-1}. Suppose you know $N_{k-1} = n$. If the next toss results in the same outcome as all these, we are done. Otherwise we wasted n tosses. Note that you have not wasted the $n + 1$ toss. because you can start counting another run from that toss onwards.

Hope the hint is enough to make you to now get an idea on how to solve this problem.

8. Let $X_1, X_2, \cdots$ be *iid* discrete random variables with $P[X_i = +1] = P[X_i = -1] = 0.5$. Find EX_i. Let N be a positive integer-valued random variable (which is a function of all X_i) defined as $N = \min\{k : X_k = +1\}$. Find EX_N.

Hint: Suppose N takes value 5. What do we know about X_5 from the definition of N? Hope it is clear that $E[X_N] = 1$.

9. A coin, with probability heads being p, is tossed repeatedly till we get r heads. Let N be the number of tosses needed. Calculate EN.
(Hint: Try to express N as a sum of geometric random variables).

Hint: Number of tosses till first head is a geometric rv. After first head, you need to wait for second head and it is once again a geometric rv. Thus the N here is a sum of r geometric random variables each with mean $1/p$. hence $EN = r/p$.

10. A fair dice is rolled repeatedly till each of the numbers $1, 2, \cdots, 6$, appears atleast once. Find the expected number of rolls needed.

Hint: The expected number of rolls is $\frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \cdots + \frac{6}{1}$