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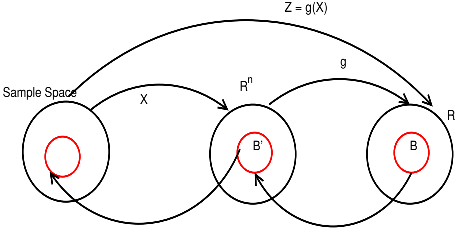

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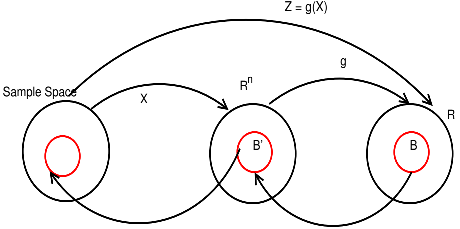

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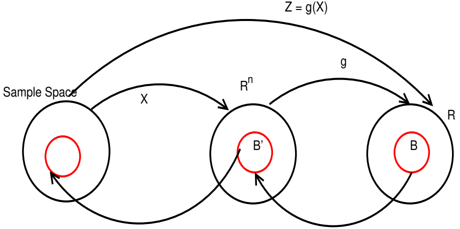

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- Remaining details are left as an exercise for you!!

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- Is this correct for all values of z, w?

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- We calculated the order statistics for the case $n = 2$.
- It can be shown that

$$f_{X_{(1)} \cdots X_{(n)}}(x_1, \cdots x_n) = n! \prod_{i=1}^{n} f(x_i), \quad x_1 < x_2 < \cdots < x_n$$

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- Hence we get

$$F_{X_{(k)}}(y) = \sum_{j=k}^{n} {}^nC_j(F(y))^j(1 - F(y))^{n-j}$$

We can get the density by differentiating this.

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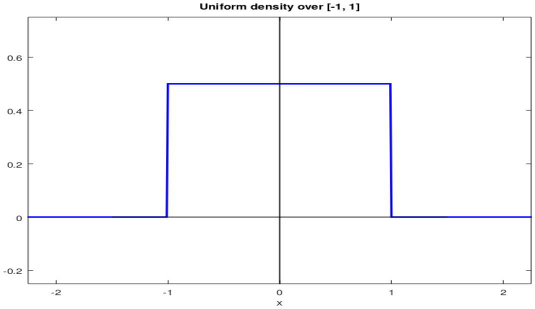

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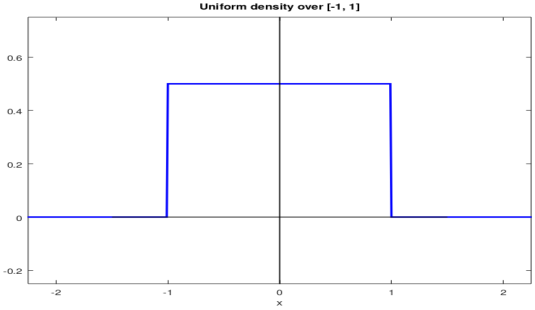

- f_Z is convolution of this density with itself.

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- $f_X(x) = 0.5, \quad -1 < x < 1.$ f_Y is also same
- Note that Z takes values in $[-2,\ 2]$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(t)\ f_Y(z-t)\ dt$$

- For the integrand to be non-zero we need
  - $-1 < t < 1 \quad \Rightarrow \quad t < 1, \quad t > -1$
  - $-1 < z - t < 1 \ \Rightarrow \ t < z+1, \quad t > z-1$
  - Hence we need:
    $t < \min(1, z+1), \quad t > \max(-1, z-1)$
  - Hence, for $z < 0$, we need $-1 < t < z+1$
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- Thus we get

$$f_Z(z) = \begin{cases} \int_{-1}^{z+1} \frac{1}{4} dt = \frac{z+2}{4} & \text{if} \quad -2 \leq z < 0 \\ \int_{z-1}^{1} \frac{1}{4} dt = \frac{2-z}{4} & \text{if} \ \ 2 \geq z \geq 0 \end{cases}$$

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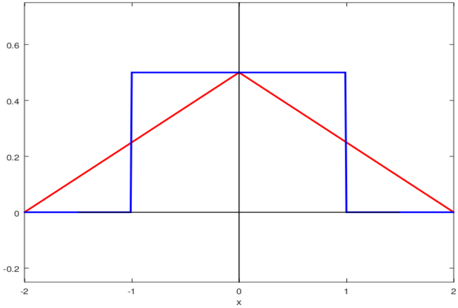

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- Exercise for you: Find density of $X_1 + X_2 + X_3$ where X_1, X_2, X_3 are iid uniform over $(0,\ 1)$.

Sum of independent gamma rv

- Gamma density with parameters $\alpha > 0$ and $\lambda > 0$ is given by
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- There is a calculation trick that is often useful with Gaussian density

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because

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\frac{1}{\sqrt{2\pi K}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-b)^2}{2K}\right)\, dx = 1
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  We denote the partial derivatives of these functions by $\frac{\partial x_i}{\partial y_j}$ etc.

- The jacobian of the inverse transformation is

$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

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- We assume that J is non-zero in the range of the transformation
- **Theorem**: Under the above conditions, we have

$$f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = |J| f_{X_1 \cdots X_n}\left(h_1(y_1, \cdots, y_n), \cdots, h_n(y_1, \cdots, y_n)\right)$$

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$$J = \frac{\partial(x_1, \cdots, x_n)}{\partial(y_1, \cdots, y_n)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix}$$

- We assume that J is non-zero in the range of the transformation
- **Theorem**: Under the above conditions, we have

$$f_{Y_1 \cdots Y_n}(y_1, \cdots, y_n) = |J| f_{X_1 \cdots X_n}\left(h_1(y_1, \cdots, y_n), \cdots, h_n(y_1, \cdots, y_n)\right)$$

Or, more compactly, $\quad f_{\mathbf{Y}}(\mathbf{y}) = |J| f_{\mathbf{X}}(h(\mathbf{y}))$

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- Called multidimensional change of variable formula

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f_W(w) &= \int_{-\infty}^{\infty} \frac{1}{2} f_{XY}\left(\frac{z+w}{2}, \frac{z-w}{2}\right) dz \\
& \qquad \text{change the variable: } t = \frac{z+w}{2} \quad \Rightarrow dt = \frac{1}{2}\, dz \\
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\end{aligned}$$

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- This we have when $X, Y \sim U(0, 1)$ iid

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- We cannot always interchange density and mass functions!!

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- The f_Z should be same in both cases.

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- We can show that the density of quotient is same in both these approches.

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- The df or density should be "symmetric" in its variables if the random variables are exchangeable.

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- So, the joint density is not the product of marginals

Expectation of functions of multiple rv

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- Similarly, if all X_i are discrete

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- Expectation is a linear operator.
- This is true for all random variables.