E1 222 Stochastic Models and Applications
Assignment 2

1. Consider the random experiment of five independent tosses of a fair coin. In any outcome (of this random experiment) we say a change-over has occurred at the i^{th} toss if the result of the i^{th} toss differs from that of the $(i-1)^{th}$ toss. Let X be a random variable whose value is the number of change-overs. For example, if the outcome of the random experiment is $HTTHH$ then the value of X would be 2. Note that the minimum value of X is 0 (e.g., when the outcome is $HHHHH$) and the maximum value of X is 4 (e.g., when the outcome is $HTHTH$). Find the probability mass function of X. Generalize this to the case of n tosses.

2. Let p be a number such that $0 < p < 1$ and let U be a random variable distributed uniformly over $(0,\ 1)$. Let X be a random variable defined by
$$X = \text{Int}\left(\frac{\log(1-U)}{\log(1-p)}\right) + 1$$
where $\text{Int}(x)$ is the largest integer smaller than or equal to x. Find the distribution of X.

3. Consider the following special case of Polya's urn scheme. We start with one white and one black ball in the urn. We choose a ball at random and then put back that ball along with one more ball of the same colour. We keep repeating this process till the number of balls in the urn are n. Let X_n denote the number of white balls in the urn when the total number of balls in the urn is n. Show that X_n is uniform over $\{1, 2, \cdots, (n-1)\}$.

4. Suppose X is a continuous random variable with $E|X| < \infty$. Suppose the density function of X is symmetric about c. (That is, $f_X(c-x) = f_X(c+x), \forall x$). Show that $EX = c$.

5. Suppose X is a discrete random variable taking positive integer values. Show that
$$E[X] = \sum_{k=0}^{\infty} P[X > k].$$