E1 222 Stochastic Models and Applications
Problem Sheet 4–1

1. Given $P[X_n = 0] = 1 - n^{-2}$, $P[X_n = e^n] = n^{-2}$. Show that X_n converge almost surely but not in r^{th} mean.

2. Given $P[X_n = 0] = 1 - 1/n$, $P[X_n = n^{1/2r}] = 1/n$, X_n are independent. Show that $E|X_n|^r \to 0$ but the sequence does not converge to zero almost surely.

3. Let $\Omega = [0,\ 1]$ and let P be the usual length measure. Let $X_n = n^{0.25} I_{[0,\ 1/n]}$, $n = 1, 2, \cdots$, where I_A denotes indicator of event A. Does the sequence converge in (i) probability, (ii) r^{th} mean for some r?

4. Let $X_1, X_2, \cdots$, be random variables with distributions

$$
\begin{aligned}
F_{X_n}(x) \quad &= 0 \qquad \text{if} \quad x < -n \\
&= \tfrac{x+n}{2n} \qquad \text{if} \quad -n \le x \le n \\
&= 1 \qquad \text{if} \quad x \ge n
\end{aligned}
$$

   Does $\{X_n\}$ converge in distribution?

5. Let $\Omega = [0, 1]$. Consider a sequence of binary random variables: $X_{nk},\ k = 1, \cdots, n, n = 1, 2, \cdots$. That is, the sequence is $X_{11}, X_{21}, X_{22}, X_{31}, X_{32}, X_{33}, \cdots$. These random variables are defined by

$$
X_{nk}(\omega) = 1 \ \text{ iff } \ \frac{k-1}{n} \le \omega < \frac{k}{n},\ 1 \le k \le n, n = 1, 2, \cdots
$$

   Show that the sequence converges to zero in probability but it does not converge with probability one

6. Let $X_1, X_2, \cdots$ be iid Gaussian random variables with mean zero and variance unity. Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Let F_n be the distribution function of $\bar{X}_n$. Find Lim F_n. Is this a distribution function?

7. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with X_n being uniform over the set $\{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Does the sequence $\{X_n\}$ converge in distribution?

8. Let $\{X_n\}$ be a sequence of random variables converging in distribution to a continuous random variable X. Let a_n be a sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Show that X_n/a_n converges to zero in probability.

9. Find the characteristic function of X when X has (i) Poisson distribution, (ii) Geometric distribution

10. Let $X_1, X_2, \cdots$ be independent normally distributed random variables having mean zero and variance σ^2.

    (a). What is the mean and variance of X_1^2?

    (b). How should $P[X_1^2 + X_2^2 + \cdots + X_n^2 \leq x]$ be approximated in terms of standard normal distribution?

    (c). Suppose $\sigma^2 = 1$. Find (approximately) $P[80 \leq X_1^2 + \cdots + X_{100}^2 \leq 120]$.

    (d). Find c such that (approximately) $P[100 - c \leq X_1^2 + \cdots + X_{100}^2 \leq 100 + c] = 0.95$.

11. Candidates A and B are contesting an election and 55% of the electorate favour B. What is the (approximate) probability that in a sample of size 100 atleast one-half of the people sampled favour A.

12. A fair coin is tossed until 100 heads appear. Find (approximately) the probability that atleast 230 tosses will be necessary.