Assignment 6 Solutions:

Equations/statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. Let $A \in \mathbb{C}^{n\times n}$. Let $U \subseteq \mathbb{C}^n$ be an A-invariant subspace, i.e., $A\cdot u \in U \ \forall\ u \in U$.

(a) Prove that $\exists$ a $u \in U$ and $\lambda \in \mathbb{C}$ such that $Au = \lambda u$. — (4)

(b) Let $0 \neq x \in \mathbb{C}^n$ & k be the smallest integer such that $\{x, Ax, \dots, A^k x\}$ is a dependent set. Prove that $V = \text{span}\{x, Ax, A^2x, \dots A^{k-1}x\}$ is A-invariant. — (2)

A. (a) Let $\{x_1, \dots, x_r\}$ be a basis for U.

$X = [x_1, \dots, x_r]$ has LI cols.

$x_i \in U \Rightarrow Ax_i \in U$ ($\because$ Def. of invariant subspace)

$\Rightarrow Ax_i = \sum_{j=1}^{r} c_{ij} x_j$ (For some c_{ij}) — (1)

$$\Rightarrow \underbrace{[Ax_1, Ax_2, \dots, Ax_r]}_{= A[x_1,\dots,x_r]} = \left[\sum_j c_{1j}x_j, \sum_j c_{2j}x_j, \dots, \sum_j c_{nj}x_j\right]$$

$$\underset{n\times n}{A}\ \underset{n\times r}{X} = \underset{n\times r}{[x_1, \dots, x_r]} \ \underset{r\times r}{\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{r1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1r} & c_{2r} & \cdots & c_{rr} \end{bmatrix}}$$

$\Rightarrow AX = XC$ — (2)

C is square $\Rightarrow$ It has atleast one EVec. y.

Let this be $Cy = \lambda y$. — (3)

Also, $Xy \neq 0$ since cols. of X are LI.

$\Rightarrow AXy = XCy \Rightarrow A(Xy) = \lambda(Xy)$

$\Rightarrow Au = \lambda u.$ — (4)

(b) Since $\{x, Ax, \ldots, A^{k-1}x, A^k x\}$ is a LD set,

$$\sum_{i=0}^{k} \alpha_i A^i x = 0 \text{ for some } \alpha_i \neq 0$$

$$\Rightarrow A^k x = \frac{\sum_{i=0}^{k-1} \alpha_i A^i x}{\alpha_k}, \alpha_k \neq 0 \quad \text{— (1)}$$

Let $y \in V$. Then $y = c_0 x + c_1 Ax + \cdots + c_{k-1} A^{k-1} x$.

$Ay = c_0 Ax + c_1 A^2 x + \cdots + c_k A^k x$

$\Rightarrow Ay = c_0' x + c_1' Ax + \cdots + c_{k-1}' A^{k-1} x$, $(\because (1))$

$\Rightarrow Ay \in V \Rightarrow V$ is an A-invariant subspace — (2)

2. PT $\det(\exp(C)) = \exp(\mathrm{tr}(C))$ for some $C \in \mathbb{C}^{n\times n}$ 4

A. $\det(\exp(C)) = \det\left(I + C + \frac{C^2}{2!} + \cdots\right)$ Schur $C = UTU^H$

$$= \det\left(UU^H + UTU^H + \frac{UT^2U^H}{2!} + \cdots\right) \quad 1$$

$$= \det\left(U\left(I + T + \frac{T^2}{2!} + \cdots\right)U^H\right)$$

$$= (\det U)(\det(\exp(T)))(\det U^H)$$

$$= \det\left[I + T + \frac{T^2}{2!} + \cdots\right] \cdot \underbrace{\det(UU^H)}_{\to 1} \quad 2$$

$$= \prod_{i=1}^{n}\left(1 + T_{ii} + \frac{T_{ii}^2}{2!} + \cdots\right) \quad \left(\because T \text{ is } \nabla \Rightarrow \exp(T) \text{ is } \nabla \Rightarrow \det(T) = \prod_i T_{ii}\right)$$

$$= \prod_{i=1}^{n} \exp(T_{ii}) \quad 3$$

$$= \exp\left(\sum_{i=1}^{n} T_{ii}\right) = \exp\left(\sum_{i=1}^{n} \lambda_i(C)\right)$$

$$= \exp(\mathrm{Tr}(C)) \quad 4$$