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- $E\left[(X - c)^2\right] \geq E\left[(X - EX)^2\right],\ \forall c$

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- If moment of order k is finite then so is moment of order s for $s < k$.

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- In general

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- Hence we get

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(Exercise: Differentiate it twice to find EX^2 and hence show that variance is λ).

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- We can use this to compute EX

$$EX = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = \left. \frac{d}{dt} \left(\frac{\lambda}{\lambda - t} \right) \right|_{t=0} = \left. \frac{\lambda}{(\lambda - t)^2} \right|_{t=0} = \frac{1}{\lambda}$$

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- We would consider ϕ_X later in the course

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Hence, we get

$$f_X(0) = P_X(0);\ f_X(1) = \frac{P'_X(0)}{1!};\ f_X(2) = \frac{P''_X(0)}{2!}$$

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- For (positive integer valued) discrete random variables, it is more convenient to deal with generating functions than mgf.

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- Note that for a given p there can be multiple values for x to satisfy the above.

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- Let us see some examples.

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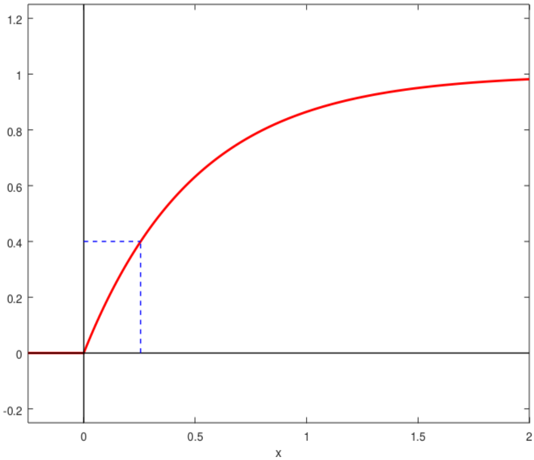

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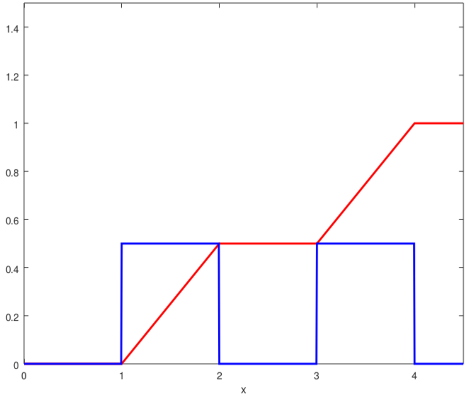

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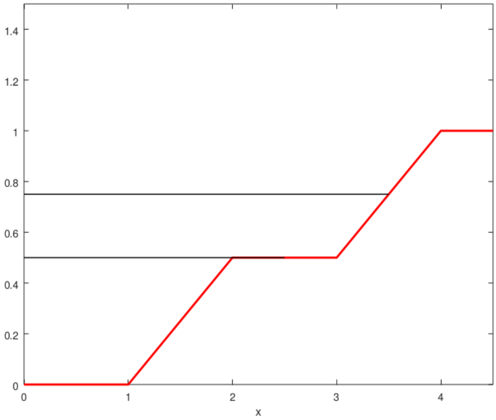

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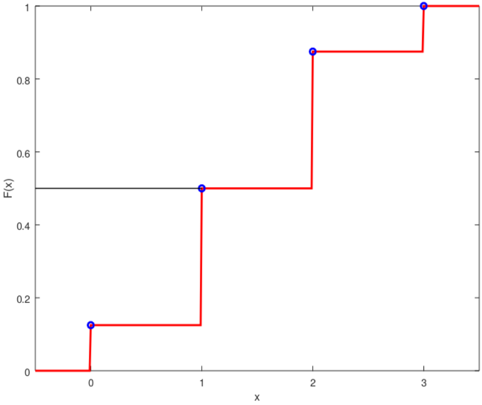
- So, quantile of order p is not unique and all such x qualify.

- This situation is illustrated below



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$$p < p + \delta_2 = F_X(x_i) < p + \delta_2 + \delta_1 = p + P[X = x_i]$$

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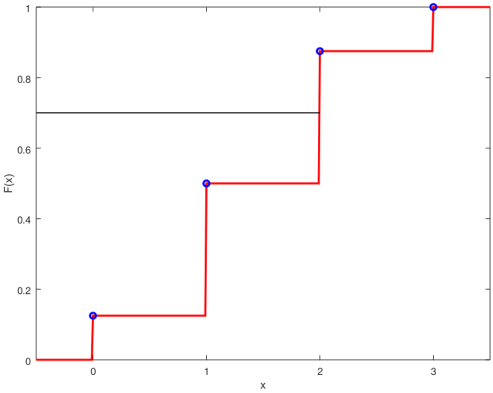
- Thus quantile of order p is unique here.

- This situation is illustrated below



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  (Exercise: Show this for discrete and continuous rv)

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- Markov inequality is often used in this form.

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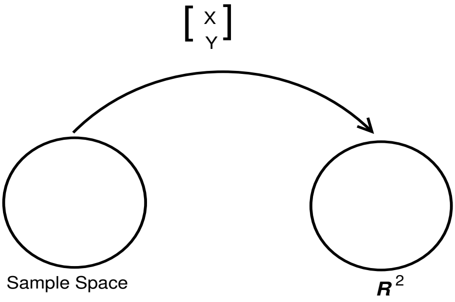

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where $\mathcal{B}$ is the Borel σ-algebra we considered earlier, and $\mathcal{B}^2$ is the set of Borel sets of $\Re^2$.

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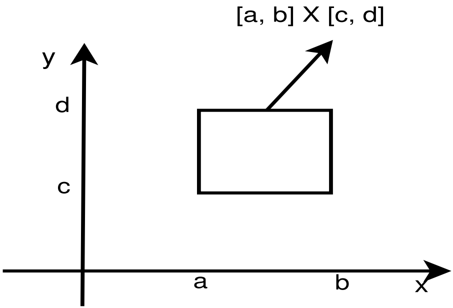

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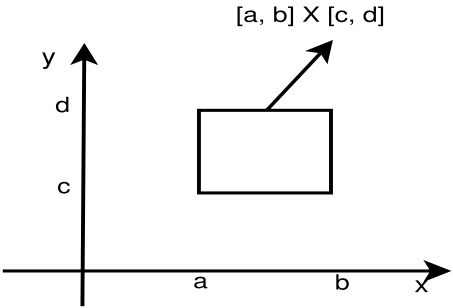

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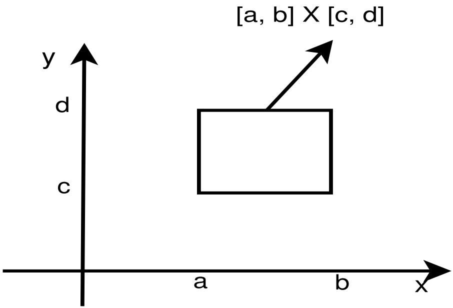

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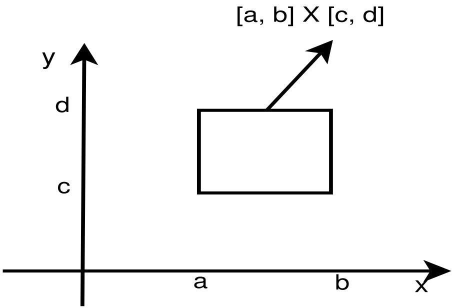

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- This gives rise to a new probability space $(\Re^2, \mathcal{B}^2, P_{XY})$ with P_{XY} given by

$$\begin{aligned} P_{XY}(B) &= P[(X,Y) \in B], \ \forall B \in \mathcal{B}^2 \\ &= P\left(\{\omega \ : \ (X(\omega).Y(\omega)) \in B\}\right) \end{aligned}$$

- Recall that for a single rv, the resulting probability space is $(\Re, \mathcal{B}, P_X)$ with

$$P_X(B) = P[X \in B] = P(\{\omega \ : \ X(\omega) \in B\})$$

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- The joint distribution function is the probability of the intersection of the events $[X \leq x]$ and $[Y \leq y]$.

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- But there is another crucial property satisfied by F_{XY}.

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- Here, the probability we want is that of the pair of rv's being in a cylindrical set.

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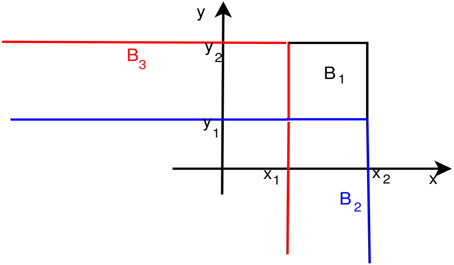

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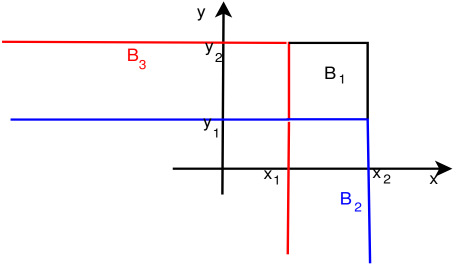

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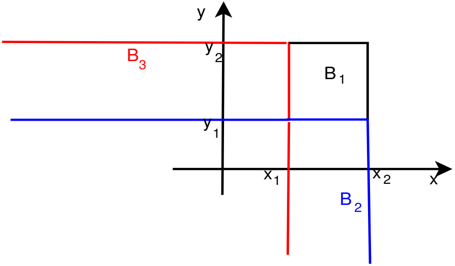

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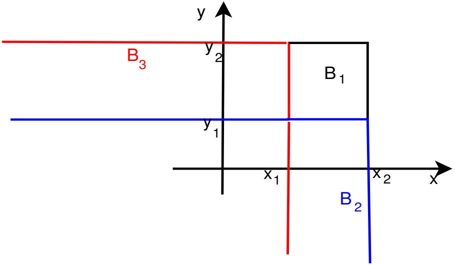

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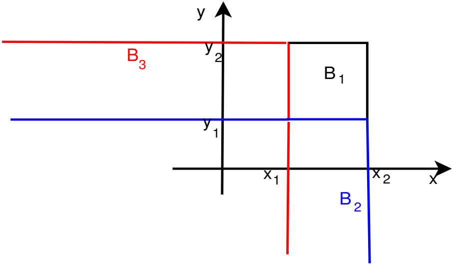

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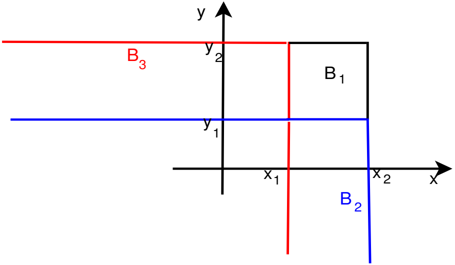

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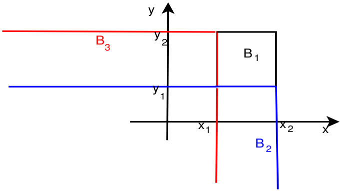
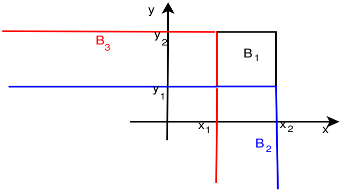

y
y_2
B_3
B_1
y_1
x_1
x_2
x
B_2



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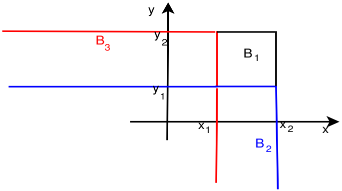

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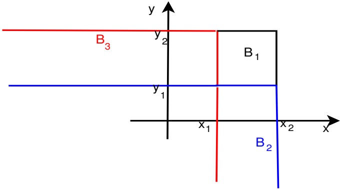

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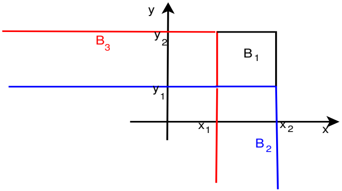

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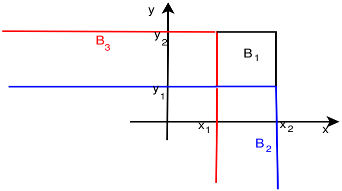

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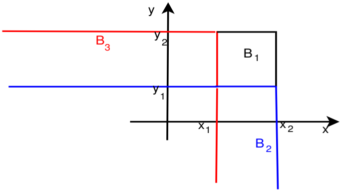

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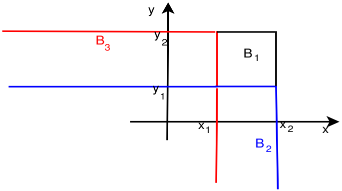

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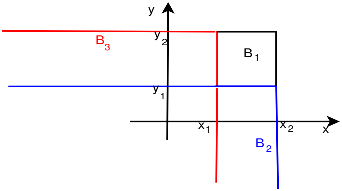

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- This is an additional condition that a function has to satisfy to be the joint distribution function of a pair of random variables

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- It satisfies
  1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \forall x, y$;
     $F_{XY}(\infty, \infty) = 1$
  2. F_{XY} is non-decreasing in each of its arguments
  3. F_{XY} is right continuous and has left-hand limits in each of its arguments
  4. For all $x_1 < x_2$ and $y_1 < y_2$

$$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) \ge 0$$

Properties of Joint Distribution Function

- Joint distribution function: $F_{XY} : \Re^2 \to \Re$

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

- It satisfies
  1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \forall x, y;$
     $F_{XY}(\infty, \infty) = 1$
  2. F_{XY} is non-decreasing in each of its arguments
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$$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) \geq 0$$

- Any $F : \Re^2 \to \Re$ satisfying the above would be a joint distribution function.