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- All stationary distributions of the chain are convex combinations of these

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- In an irreducible chain, all states have the same period
- An irreducible chain is called aperiodic if the period is 1
- For an irreducible aperiodic positive recurrent chain, π_n converges to π, the unique stationary distribution, irrespective of what π_0 is.
- Also, for an irreducible, aperiodic, positive recurrent chain, $P^n(x,y)$ converges to $\frac{1}{m_y}$

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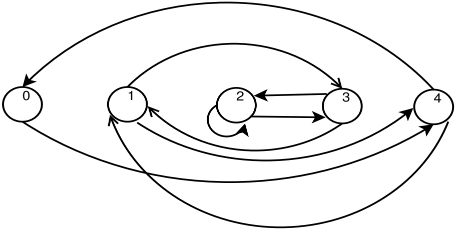

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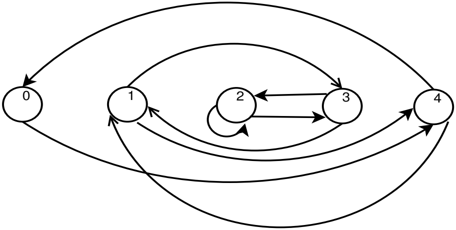

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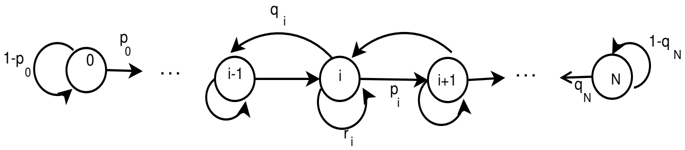
$$\pi(i) = \frac{1}{5-p} \; i = 1,2,3,4 \quad \text{and} \quad \pi(0) = \frac{1-p}{5-p}$$

Birth-Death chains

- The following is a finite birth-death chain



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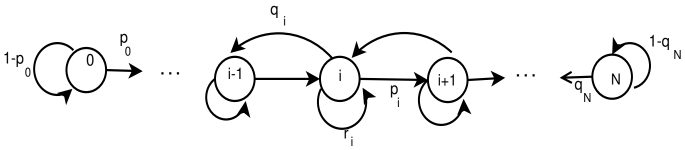

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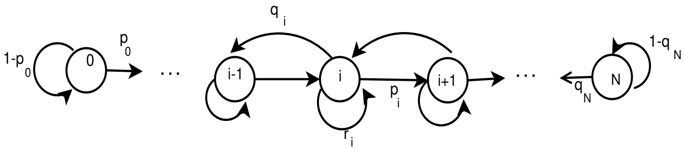

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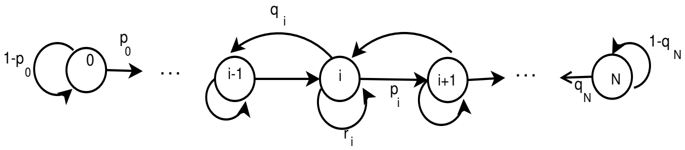

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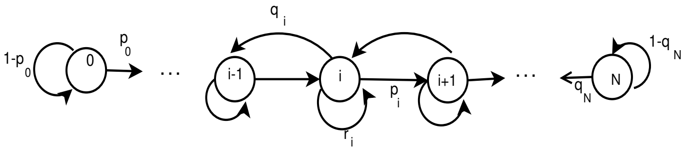

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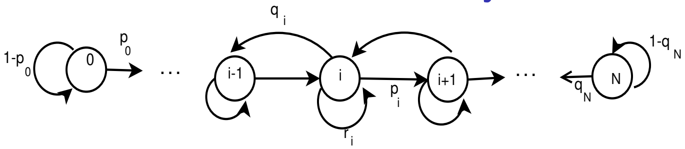
- We can derive a general form for its stationary probabilities

birth-death chains – stationary distribution



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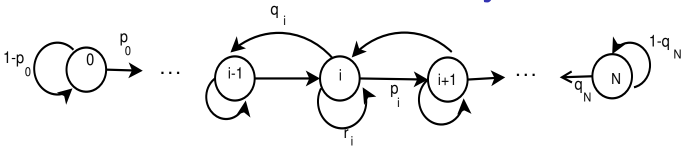

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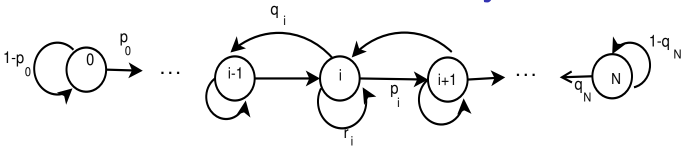

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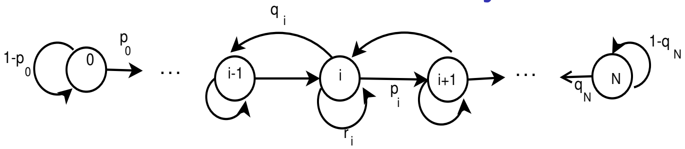

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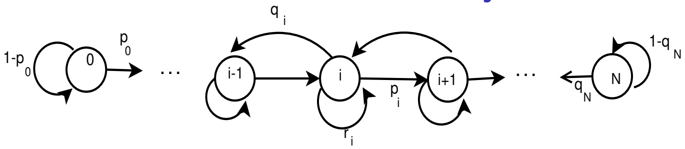

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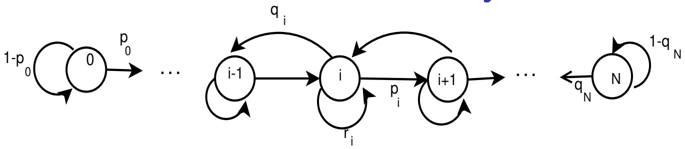

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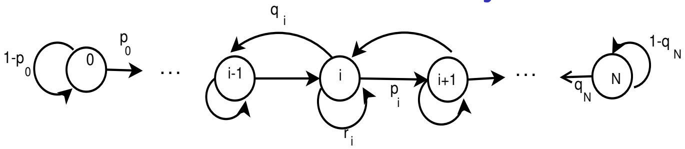

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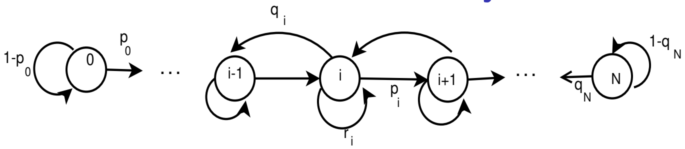

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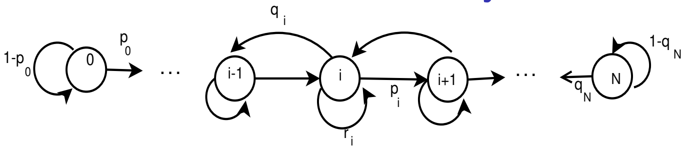

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&\Rightarrow \pi(2)q_2 - \pi(1)p_1 = 0 \\
\pi(2) &= \pi(1)p_1 + \pi(2)(1-p_2-q_2) + \pi(3)q_3 \\
&\Rightarrow \pi(2)q_2 - \pi(1)p_1 = \pi(3)q_3 - \pi(2)p_2 = 0
\end{aligned}$$

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- Iterating like this, we get

$$\pi(n) = \eta_n\,\pi(0), \quad \text{where} \quad \eta_n = \frac{p_0 p_1 \cdots p_{n-1}}{q_1 q_2 \cdots q_n}, \; n = 1, 2, \cdots, N$$

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$$\begin{aligned}
\pi(1)q_1 - \pi(0)p_0 &= 0 \Rightarrow \pi(1) = \frac{p_0}{q_1}\,\pi(0) \\
\pi(2)q_2 - \pi(1)p_1 &= 0 \Rightarrow \pi(2) = \frac{p_1}{q_2}\,\pi(1) = \frac{p_0 p_1}{q_1 q_2}\,\pi(0)
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- With $\eta_0 = 1$, we get $\pi(0) \sum_{j=0}^{N} \eta_j = 1$ and hence

$$\pi(0) = \frac{1}{\sum_{j=0}^{N} \eta_j} \quad \text{and} \quad \pi(n) = \eta_n\,\pi(0), \; n = 1, \cdots, N$$

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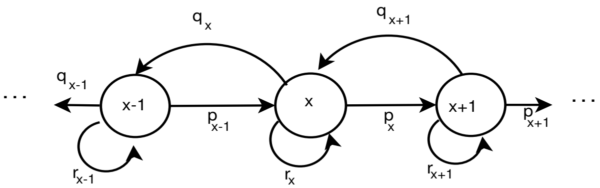
- Note that this process is applicable even for infinite chains with state space $\{0, 1, 2, \cdots\}$ (but there may not be a solution)

- Consider a birth-death chain



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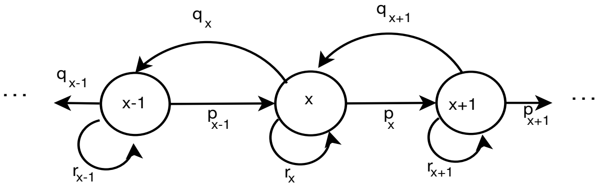

- The chain may be infinite or finite

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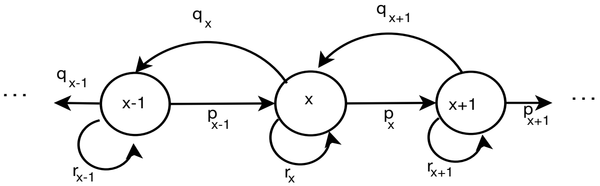
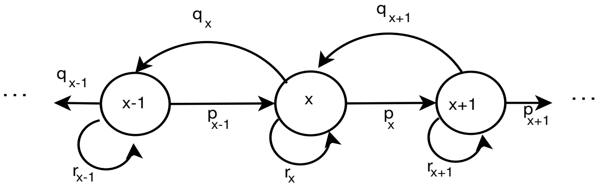


- The chain may be infinite or finite
- Let $a, b \in S$ with $a < b$. Assume $p_x, q_x > 0$, $a < x < b$.

- Consider a birth-death chain



- The chain may be infinite or finite
- Let $a, b \in S$ with $a < b$. Assume $p_x, q_x > 0$, $a < x < b$.
- Define

$$U(x) = P_x[T_a < T_b], \ a < x < b, \quad U(a) = 1, \ U(b) = 0$$

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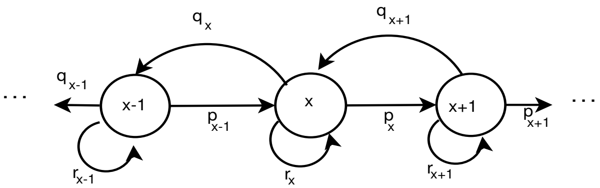

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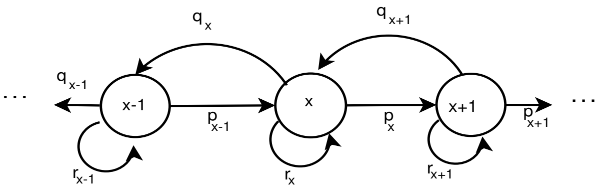
- We want to derive a formula for $U(x)$

- Consider a birth-death chain



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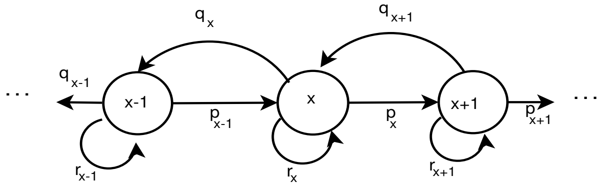
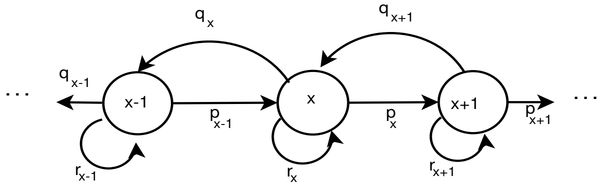
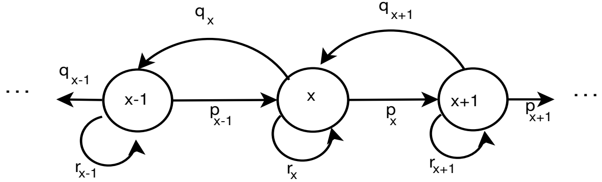
$$U(x) = P_x[T_a < T_b], \; a < x < b, \quad U(a) = 1, \; U(b) = 0$$

- We want to derive a formula for $U(x)$
- This can be useful, e.g., in the gambler's ruin chain

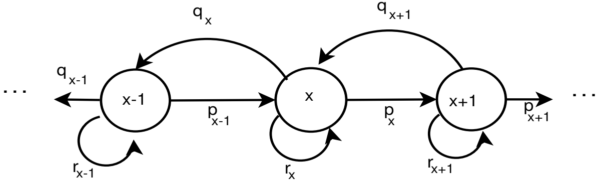

q x
q x+1
q x-1
x-1
x
x+1
p x-1
p x
p x+1
r x-1
r x
r x+1



$$U(x) \;=\; P_x[T_a < T_b] = Pr[T_a < T_b | X_0 = x]$$



$$
\begin{aligned}
U(x) &= P_x[T_a < T_b] = Pr[T_a < T_b | X_0 = x] \\
&= \sum_{y=x-1}^{x+1} Pr[T_a < T_b | X_1 = y] \; Pr[X_1 = y | X_0 = x]
\end{aligned}
$$



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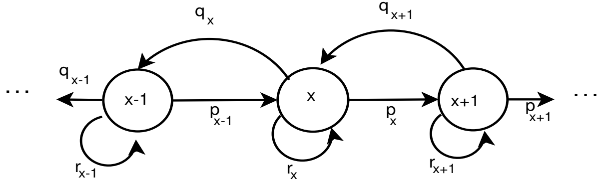
&= U(x-1)q_x + U(x)r_x + U(x+1)p_x
\end{aligned}
$$



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&= U(x-1)q_x + U(x)(1 - p_x - q_x) + U(x+1)p_x
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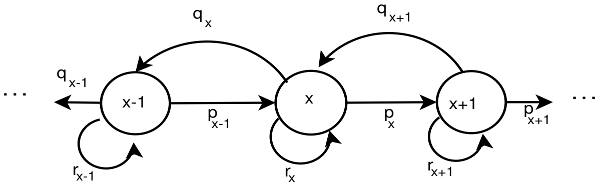
$$



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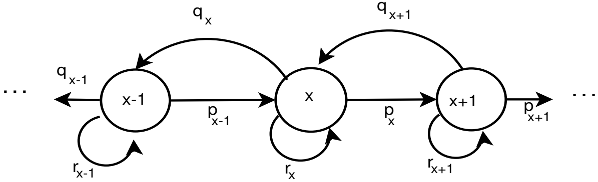
\end{aligned}
$$

$$
\Rightarrow \qquad q_x[U(x) - U(x-1)] = p_x[U(x+1) - U(x)]
$$



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\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & q_x[U(x) - U(x-1)] = p_x[U(x+1) - U(x)] \\
\Rightarrow \qquad & U(x+1) - U(x) = \frac{q_x}{p_x} \; [U(x) - U(x-1)]
\end{aligned}
$$

$$U(x+1) - U(x) = \frac{q_x}{p_x} \left[U(x) - U(x-1)\right]$$

$$
\begin{aligned}
U(x+1) - U(x) &= \frac{q_x}{p_x} \left[U(x) - U(x-1) \right] \\
&= \frac{q_x}{p_x} \frac{q_{x-1}}{p_{x-1}} \left[U(x-1) - U(x-2) \right]
\end{aligned}
$$

$$
\begin{aligned}
U(x+1) - U(x) &= \frac{q_x}{p_x}\left[U(x) - U(x-1)\right] \\
&= \frac{q_x}{p_x}\,\frac{q_{x-1}}{p_{x-1}}\left[U(x-1) - U(x-2)\right] \\
&= \frac{q_x q_{x-1} \cdots q_{a+1}}{p_x p_{x-1} \cdots p_{a+1}}\left[U(a+1) - U(a)\right]
\end{aligned}
$$

$$\begin{aligned}
U(x+1) - U(x) &= \frac{q_x}{p_x}\,[U(x) - U(x-1)] \\
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\end{aligned}$$

Let $\gamma_y = \dfrac{q_y q_{y-1}\cdots q_{a+1}}{p_y p_{y-1}\cdots p_{a+1}},\ a < y < b,\ \ \gamma_a = 1$

$$
\begin{aligned}
U(x+1) - U(x) &= \frac{q_x}{p_x}\,[U(x) - U(x-1)] \\
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&= \frac{q_x q_{x-1} \cdots q_{a+1}}{p_x p_{x-1} \cdots p_{a+1}}\,[U(a+1) - U(a)]
\end{aligned}
$$

$$
\text{Let} \quad \gamma_y = \frac{q_y q_{y-1} \cdots q_{a+1}}{p_y p_{y-1} \cdots p_{a+1}},\ a < y < b, \quad \gamma_a = 1
$$

Now we get

$$
U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}[U(a+1) - U(a)]
$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a} \left[U(a+1) - U(a)\right]$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\left[U(a+1) - U(a)\right]$$

- By taking $x = b-1,\ b-2, \cdots, a+1,\ a,$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\left[U(a+1) - U(a)\right]$$

- By taking $x = b-1,\ b-2, \cdots, a+1,\ a$,

$$U(b) - U(b-1) \;=\; \frac{\gamma_{b-1}}{\gamma_a}\left[U(a+1) - U(a)\right]$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\,[U(a+1) - U(a)]$$

- By taking $x = b-1,\ b-2, \cdots, a+1,\ a,$

$$\begin{aligned} U(b) - U(b-1) &= \frac{\gamma_{b-1}}{\gamma_a}\,[U(a+1) - U(a)] \\ U(b-1) - U(b-2) &= \frac{\gamma_{b-2}}{\gamma_a}\,[U(a+1) - U(a)] \end{aligned}$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\,[U(a+1) - U(a)]$$

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U(b-1) - U(b-2) &= \frac{\gamma_{b-2}}{\gamma_a}\,[U(a+1) - U(a)] \\
&\vdots \\
U(a+1) - U(a) &= \frac{\gamma_a}{\gamma_a}\,[U(a+1) - U(a)]
\end{aligned}$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a} \left[U(a+1) - U(a) \right]$$

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&\vdots \\
U(a+1) - U(a) &= \frac{\gamma_a}{\gamma_a} \left[U(a+1) - U(a) \right]
\end{aligned}$$

- Adding all these we get

$$\frac{1}{\gamma_a} \left[U(a+1) - U(a) \right] \sum_{x=a}^{b-1} \gamma_x = U(b) - U(a)$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a}\left[U(a+1) - U(a)\right]$$

- By taking $x = b-1,\ b-2, \cdots, a+1,\ a$,

$$\begin{aligned} U(b) - U(b-1) &= \frac{\gamma_{b-1}}{\gamma_a}\left[U(a+1) - U(a)\right] \\ U(b-1) - U(b-2) &= \frac{\gamma_{b-2}}{\gamma_a}\left[U(a+1) - U(a)\right] \\ &\vdots \\ U(a+1) - U(a) &= \frac{\gamma_a}{\gamma_a}\left[U(a+1) - U(a)\right] \end{aligned}$$

- Adding all these we get

$$\frac{1}{\gamma_a}\left[U(a+1) - U(a)\right] \sum_{x=a}^{b-1} \gamma_x = U(b) - U(a) = 0 - 1$$

$$U(x+1) - U(x) = \frac{\gamma_x}{\gamma_a} \left[U(a+1) - U(a)\right]$$

- By taking $x = b-1,\ b-2, \cdots, a+1,\ a$,

$$\begin{aligned}
U(b) - U(b-1) &= \frac{\gamma_{b-1}}{\gamma_a} \left[U(a+1) - U(a)\right] \\
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- Adding all these we get

$$\frac{1}{\gamma_a} \left[U(a+1) - U(a)\right] \sum_{x=a}^{b-1} \gamma_x = U(b) - U(a) = 0 - 1$$

$$\Rightarrow \quad U(a) - U(a+1) = \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x}$$

- Using these, we get

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$$U(x) - U(x+1) \quad = \quad \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)]$$

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$$
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U(x) - U(x+1) &= \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)] \\
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\end{aligned}
$$

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$$
\begin{aligned}
U(x) - U(x+1) &= \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)] \\
&= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x}
\end{aligned}
$$

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$$\begin{aligned} U(x) - U(x+1) &= \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)] \\ &= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x} \end{aligned}$$

- Putting $x = b-1,\ b-2, \cdots, y$ in the above

- Using these, we get

$$
\begin{aligned}
U(x) - U(x+1) &= \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)] \\
&= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x}
\end{aligned}
$$

- Putting $x = b-1,\ b-2, \cdots, y$ in the above

$$
U(b-1) - U(b) = \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x}
$$

- Using these, we get

$$U(x) - U(x+1) = \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)]$$
$$= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x}$$

- Putting $x = b-1,\ b-2, \cdots, y$ in the above

$$U(b-1) - U(b) = \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x}$$
$$U(b-2) - U(b-1) = \frac{\gamma_{b-2}}{\sum_{x=a}^{b-1} \gamma_x}$$

- Using these, we get

$$\begin{aligned} U(x) - U(x+1) &= \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)] \\ &= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x} \end{aligned}$$

- Putting $x = b-1,\ b-2, \cdots, y$ in the above

$$\begin{aligned} U(b-1) - U(b) &= \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x} \\ U(b-2) - U(b-1) &= \frac{\gamma_{b-2}}{\sum_{x=a}^{b-1} \gamma_x} \\ &\vdots \\ U(y) - U(y+1) &= \frac{\gamma_y}{\sum_{x=a}^{b-1} \gamma_x} \end{aligned}$$

- Using these, we get

$$U(x) - U(x+1) = \frac{\gamma_x}{\gamma_a}[U(a) - U(a+1)]$$
$$= \frac{\gamma_x}{\gamma_a} \frac{\gamma_a}{\sum_{x=a}^{b-1} \gamma_x} = \frac{\gamma_x}{\sum_{x=a}^{b-1} \gamma_x}$$

- Putting $x = b-1,\ b-2, \cdots, y$ in the above

$$U(b-1) - U(b) = \frac{\gamma_{b-1}}{\sum_{x=a}^{b-1} \gamma_x}$$
$$U(b-2) - U(b-1) = \frac{\gamma_{b-2}}{\sum_{x=a}^{b-1} \gamma_x}$$
$$\vdots$$
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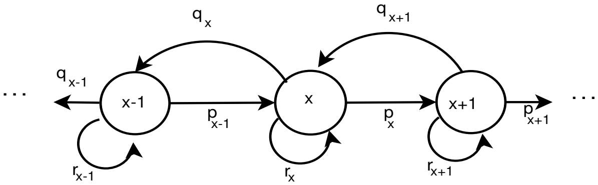
- Adding these we get

$$U(y) - U(b) = U(y) = \frac{\sum_{x=y}^{b-1} \gamma_x}{\sum_{x=a}^{b-1} \gamma_x},\ a < y < b$$

- We are considering birth-death chains



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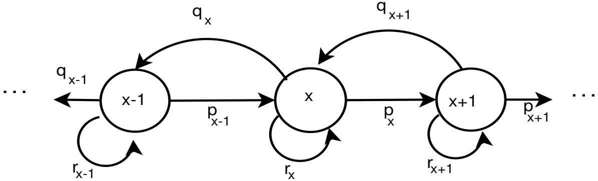

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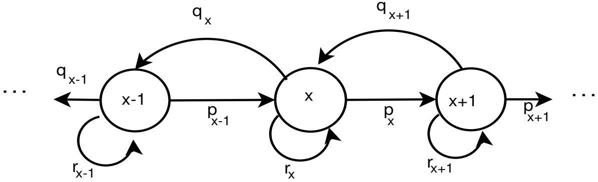

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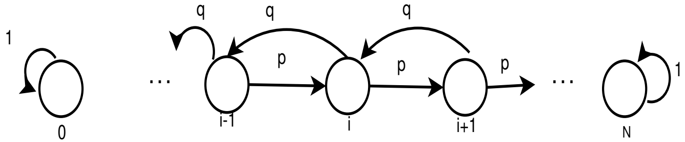

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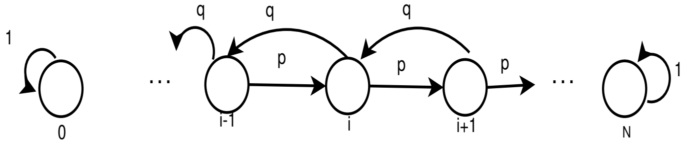

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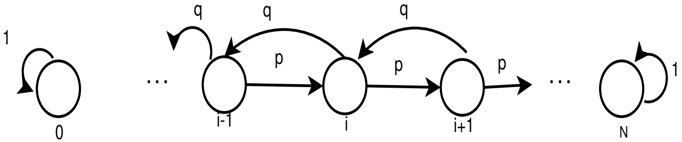

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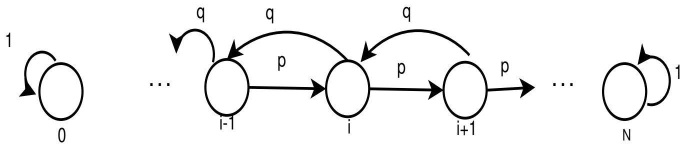

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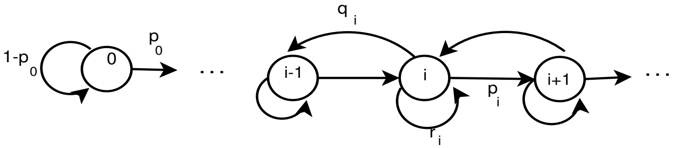

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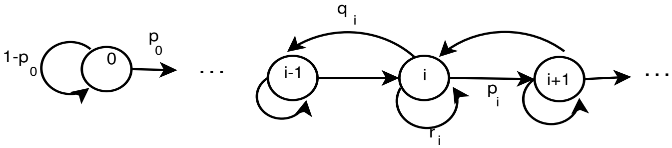

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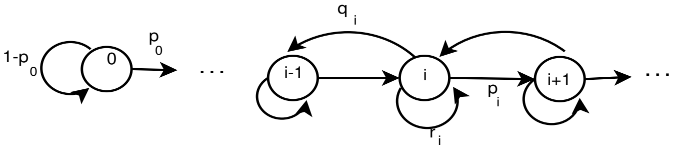

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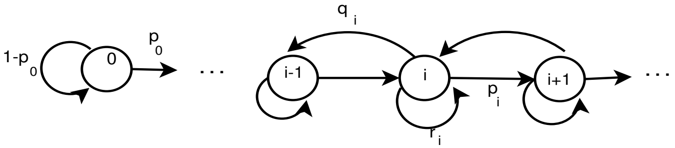

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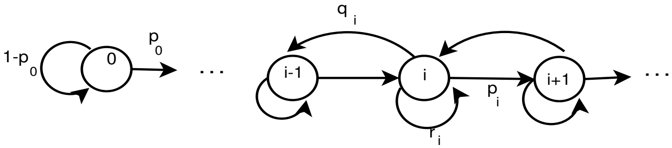

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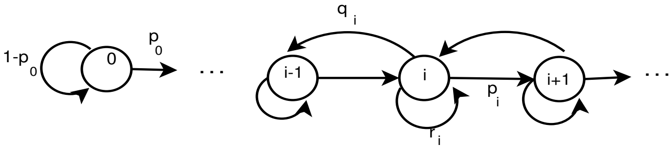

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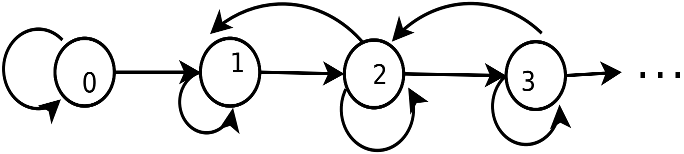

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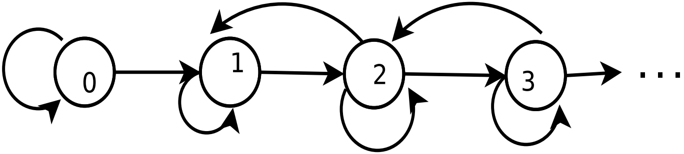

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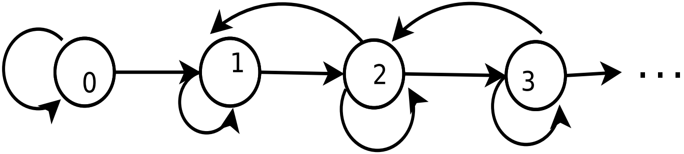

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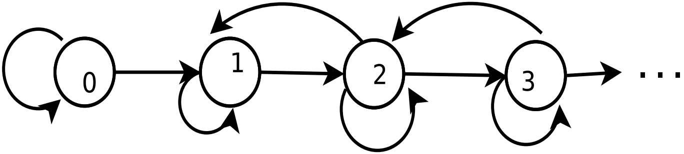

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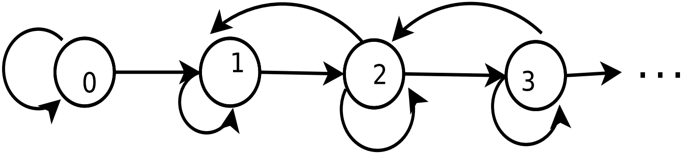

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- This chain with $q < p$ is an example of an irreducible chain that is wholly transient

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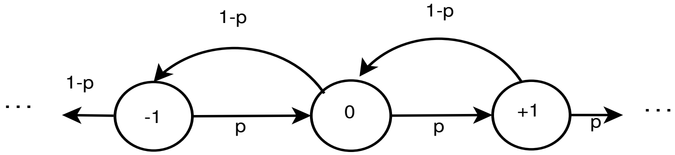

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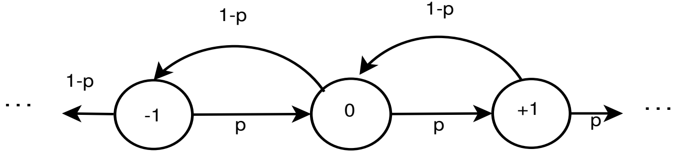

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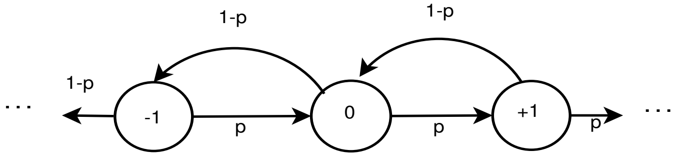

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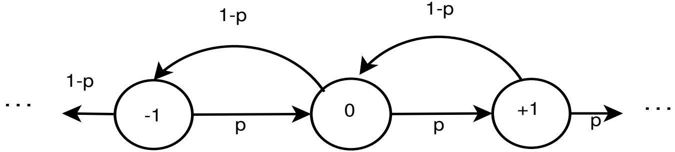

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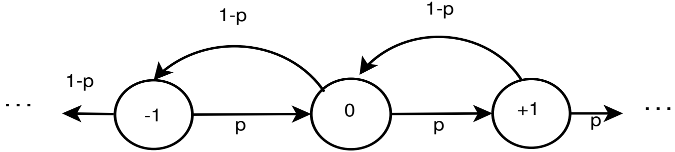

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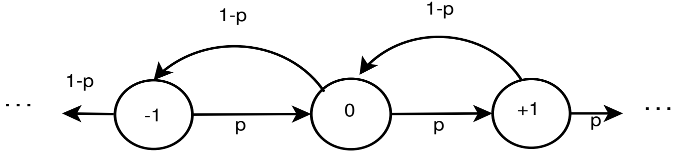

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Let P be the transition probabilities of a homogeneous irreducible Markov chain with state space S. Let $h : S \to \Re$ with $h(x) \geq 0$ and

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- For this to be true for infinite S, we need some extra conditions

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- One way to generate samples is to design an ergodic markov chain with stationary distribution π
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- We can also use the chain to generate samples from distribution π

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- For all these, we need to design a Markov chain with π as stationary distribution

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- We want to choose $\alpha(i,j)$ to make X_n an ergodic markov chain with stationary probabilities π

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- Note that it is not necessary for a stationary distribution to satisfy detailed balance

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