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$$P[(X, Y) \in B] = \sum_{\substack{i,j: \\ (x_i, y_j) \in B}} f_{XY}(x_i, y_j)$$

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$$P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] = \int_{x_1}^{x_2} \int_{y_1}^{y_2} f_{XY}\, dy\, dx$$

  and, in general,

$$P[(X,Y) \in B] = \int_{B} f_{XY}(x,y)\, dx\, dy, \;\; \forall B \in \mathcal{B}^2$$

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- Also gives us Bayes rule for discrete rv

$$f_{X|Y}(x_i|y_j) = \frac{f_{Y|X}(y_j|x_i) f_X(x_i)}{\sum_i f_{Y|X}(y_j|x_i) f_X(x_i)}$$

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- Given there is only one head, it is equally likely to occur on any toss.

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- We define conditional density of X given Y as

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- This exists if $f_Y(y) > 0$ and then it has a density:

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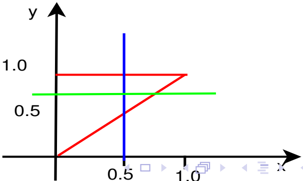
- We can see this intuitively like this



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- We can specify the marginal density of one and the conditional density of the other given the first.
- This may actually be the model of how the the rv's are generated.

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- We can verify it to be a density

$$-\int_0^1 \ln(y)\ dy = -y\ln(y)|_0^1 + \int_0^1 y\,\frac{1}{y}\ dy = 1$$

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- While often that gives the right result, one needs to be very careful

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- In both cases we have: $f_{XY}(x, y) = f_{X|Y}(x|y)f_Y(y)$
- This gives total probability rule and Bayes rule for random variables

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- This gives us

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f_X(x) = \sum_y f_{X|Y}(x|y) f_Y(y)
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- This is another version of total probability rule.
- Earlier we derived this when X, Y are discrete.
- The formula is true even when X is continuous
  Only difference is we need to take f_X as the density of X.

- When X, Y are discrete we have

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  Called a mixture density model

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- We will not be using such 'joint densities' here

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- This gives us further versions of total probability rule and Bayes rule.

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- Let us review all the total probability formulas

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  We now know it holds also when X is cont and Y is discrete. In that case f_Y is a mass function

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- Thus Bayes rule holds in all four possible scenarios
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- In general, one refers to these always as densities since the actual meaning would be clear from context.

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- We want to calculate $f_{Y|X}(1|x)$.
- We want to use the Bayes rule to calculate this

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We only need ratio of $f_{Y|X}(1|x)$ and $f_{Y|X}(0|x)$.

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- Let F_1, F_2 be one dimensional df's of continuous rv's with f_1, f_2 being the corresponding densities.
  Define a function $f : \Re^2 \to \Re$ by

$$f(x, y) = f_1(x) f_2(y) \left[1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)\right]$$

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- For this, we note the following
  $$\int_{-\infty}^{\infty} f_1(x) \ F_1(x) \ dx = \left. \frac{(F_1(x))^2}{2} \right|_{-\infty}^{\infty} = \frac{1}{2}$$

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$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy \;\; &= \;\; \int_{-\infty}^{\infty} f_1(x)\, dx \int_{-\infty}^{\infty} f_2(y)\, dy \\ &+\alpha \int_{-\infty}^{\infty} (2f_1(x)F_1(x) - f_1(x))\, dx \int_{-\infty}^{\infty} (2f_2(y)F_2(y) - f_2(y))\end{aligned}$$

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&\quad +\alpha \int_{-\infty}^{\infty} (2f_1(x)F_1(x) - f_1(x))\ dx \int_{-\infty}^{\infty} (2f_2(y)F_2(y) - f_2(y)) \\
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- So, in general, the marginals cannot determine the joint distribution.
- An important special case where this is possible is that of independent random variables

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$$F_{XY}(x, y) = P[X \leq x, Y \leq y] = P[X \leq x]P[Y \leq y] = F_X(x) \; F_Y(y)$$

- **Theorem**: X, Y are independent if and only if $F_{XY}(x, y) = F_X(x)F_Y(y)$.

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$$\begin{aligned} F_{XY}(x,y) &= \sum_{x_i \leq x, y_j \leq y} f_{XY}(x_i, y_j) = \sum_{x_i \leq x, y_j \leq y} f_X(x_i) f_Y(y_j) \\ &= \sum_{x_i \leq x} f_X(x_i) \sum_{y_j \leq y} f_Y(y_j) \end{aligned}$$

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- This is true for all the four possibilities of X, Y being continuous/discrete.