E1 222 Stochastic Models and Applications
Test II

Time: 75 minutes Max. Marks:40
Date: 13 Jan 2021

Answer **ALL** questions. All questions carry equal marks

1. a. Let $X_1, X_2, \cdots$ be a sequence of iid continuous random variables. Let their common distribution function be F and suppose it is strictly monotonically increasing. Let $M_n = \max(X_1, \cdots, X_n)$ and $Y_n = n[1 - F(M_n)],\ n = 1, 2, \cdots$. Find the limiting distribution of Y_n.

   b. Let $X_1, X_2, \cdots$ be iid continuous random variables with density $f(x) = 12x^2(1-x),\ 0 \leq x \leq 1$. Let $S_n = \sum_{i=1}^n X_i^2$. Does $\frac{1}{n}S_n$ converge almost surely? Answer Yes/No with a short justification. If your answer is yes, find the limit.

2. a. Consider a Probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{1, 2, \cdots\}$, $\mathcal{F}$ is the power set of Ω and $P(\{i\}) = q_i,\ \forall i$. Note that we would have $q_i \geq 0, \forall i$ and $\sum_i q_i = 1$. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables defined on this space given by

$$\begin{aligned} X_n(\omega) &= 1 \text{ if } n \leq \omega \\ &= 0 \text{ otherwise} \end{aligned}$$

   Does the sequence converge in (i) Probability, (ii) almost surely.

   b. A university has 300 vacancies for research students. Since not all students offered admission would accept, the university sends out offers of admission to 400 students. By past experiance the university knows that only 70% of students offered admission would accept the offer. Calculate the approximate probability that more than 300 students would accept the offer of admission.

3. a. Consider an irreducible birth-death Markov chain on the state space $\{0, 1, \cdots, N\}$. If the chain is started in state 1 what is the probability that the chain will visit state $N-1$ at sometime or the other. Can this chain have a null recurrent state? Explain your answer.

b. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.15 & 0.22 & 0.1 & 0.28 & 0.25 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0.35 \\ 0 & 0 & 0 & 0.55 & 0.45 \end{bmatrix}$$

Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find a stationary distribution of the chain

4. a. A man has n umbrellas. Everyday in the morning he goes from his house to office and takes an umbrella with him if it is raining and if he has an umbrella with him; he goes without an umbrella if it is not raining or if he has no umbrellas with him. Similarly in the evening when he goes from office to home he takes an umbrella if it is raining and he has one. The probability of rain is same in the morning and evening and it is equal to p. Construct an $n+1$ state Markov chain and using that calculate the probability that the man would be without an umbrella when it is raining. (Note that this is the generalization of the problem solved in class)

   b. Let $X_n,\ n = 1, 2, \cdots$ be discrete random variables taking values in $\{1, 2, \cdots, K\}$, $K < \infty$. Suppose $X_n \xrightarrow{P} 0$. Then show that the sequence converges in r^{th} mean to zero.