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$$Pr[X(t{+}s) = j \mid X(s) = i] = Pr[X(t) = j \mid X(0) = i],\ \forall s$$

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$\Rightarrow$ T_i is memoryless and hence exponential

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- Note that $P_{ij}(t)$ is different from these z_{ij}

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- Hence, $\mu_0 = 0$ and $\nu_0 = \lambda_0$ and $z_{01} = 1$

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- This is the Poisson process

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$$\lambda_n = \lambda, \ n \geq 0 \quad \text{and} \quad \mu_n = \begin{cases} n\mu & 1 \leq n \leq K \\ K\mu & n > K \end{cases}$$

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$$E[Y_i \mid I_i = 0] = \frac{1}{\lambda_i + \mu_i} + E[Y_{i-1}] + E[Y_i]$$

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- We specify a birth-death chain by birth rate (rate of transition from i to $i+1$), λ_i and death rate (rate of transition from i to $i-1$), μ_i.

- The Chapmann-Kolmogorov equations give us

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$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i)P_{ij}(t)$$

Poisson process as a special case

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- Now one can see that the above ODE is what we got for Poisson process.

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- We can calculate it by solving the ODE's

$$P'_{ij}(t) = \lambda_i P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\lambda_i + \mu_i) P_{ij}(t)$$

- For the two state chain, these equations are

$$\begin{aligned}
P'_{00}(t) &= \lambda_0 P_{10}(t) - \lambda_0 P_{00}(t) \\
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- Solving these we can show

$$P_{00}(t) = \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda + \mu}$$

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- We normally write it as Q and call it the infinitesimal generator of the process

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- So, what we did is to write the backward equation in matrix form

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- An irreducible positive recurrent chain would be called an ergodic chain

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when π is a stationary distribution and $\pi(0) = \pi$

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- The above equation is known as a balance equation