Recap: Convergence in Probability

- A sequence of random variables, X_n, is said to **converge in probability** to a random variable X_0 is

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- We only need marginal distributions of individual X_n to decide whether a sequence converges to a constant in probability

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- We can also write it as

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- Almost sure convergence is a stronger mode of convergence

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- Hence, $X_n \stackrel{a.s.}{\to} X$ iff

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- We need a bound: $P[|\frac{S_n}{n} - \mu|] \leq c_n$ such that $\sum_n c_n < \infty$.

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- Hence we get

$$E\left[\left(\sum_{i=1}^{n}(X_i - \mu)\right)^4\right] = nE[(X_i - \mu)^4] + 3n(n-1)\sigma^4 \leq C'n^2$$

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- Since $\sum_n \frac{C}{n^2} < \infty$, we get $\frac{S_n}{n} \stackrel{a.s.}{\rightarrow} \mu$

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- Strong law of large numbers says that sample mean converges to the expectation with probability one.

Convergence in r^{th} mean

- We say that a sequence X_n converges in r^{th} mean to X if $E[|X_n|^r] < \infty,\ \forall n,\ E[|X|^r] < \infty$ and

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- However, if all X_n take values in a bounded interval, then almost sure convergence implies r^{th} mean convergence

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- The converse is not true. (e.g., sequence of iid random variables)

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- In general, almost sure convergence does not imply convergence in r^{th} mean and vice versa

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- CLT is about (normalized) sums of of independent random variables converging to the Gaussian distribution

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- We use characteristic functions for proving CLT

Characteristic Function

- Given rv X, its characteristic function, ϕ_X, is defined by

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- Since $|e^{iux}| \leq 1$, ϕ_X exists for all random variables

Properties of characteristic function

$$\phi_X(u) = E\left[e^{iuX}\right] = \int e^{iux}\, dF_X(x) \quad (i = \sqrt{-1})$$

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- If $Y = aX + b$, $\phi_Y(u) = e^{iub}\phi_X(ua)$
- If $E|X|^r < \infty$, ϕ would be differentiable r times and

$$\phi^{(r)}(u) = E[(iX)^r e^{iuX}]$$

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    - If $F_n \rightarrow F$ then $\phi_{F_n} \rightarrow \phi_F$
    - If $\phi_{F_n} \rightarrow \psi$ and ψ is continuous at zero, then ψ would be characteristic function of some df, say, F, and $F_n \rightarrow F$

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