E2 212: Homework - 5

1 Topics

- Schur's theorem
- Unitary equivalence
- Normal matrices
- QR decomposition and Jordan form

2 Problems

1. Let $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n \times n}$ be given, and suppose $\mathbf{A}$ and $\mathbf{B}$ are simultaneously similar to upper triangular matrices: that is, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ are both upper triangular for some nonsingular $\mathbf{S}$. Show that every eigenvalue of $\mathbf{AB} - \mathbf{BA}$ must be zero.

2. If $\mathbf{A} \in \mathbb{C}^{n \times n}$, show that rank of $\mathbf{A}$ is not less than the number of nonzero eigenvalues of $\mathbf{A}$.

3. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be a nonsingular matrix. Show that any matrix that commutes with A also commutes with $\mathbf{A}^{-1}$.

4. If $\mathbf{A} \in \mathbb{C}^{n \times n}$ is normal, and if $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to distinct eigenvalues, show that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal.

5. Show that a normal matrix is unitary if and only if all its eigenvalues have absolute value 1.

6. Show that a given matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is normal if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$,
$$(\mathbf{A}\mathbf{x})^H(\mathbf{A}\mathbf{y}) = (\mathbf{A}^H\mathbf{x})^H(\mathbf{A}^H\mathbf{y}).$$

7. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be given. Define
$$K_\lambda \triangleq \{\mathbf{x} \in \mathbb{C}^n : (\mathbf{A} - \lambda I)^p\mathbf{x} = 0 \text{ for some integer } p > 0\}.$$
Prove that
   (a) If λ is an eigenvalue of $\mathbf{A}$, then K_λ is a $\mathbf{A}-$invariant subspace of $\mathbb{C}^n$, i.e., $\mathbf{A}\mathbf{y} \subseteq K_\lambda$ for all $\mathbf{y} \in K_\lambda$.
   (b) If λ is an eigenvalue of $\mathbf{A}$ with multiplicity m, then $\dim(K_\lambda) \leq m$.

8. Find an invertible matrix $\mathbf{U}$ such that $\mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ is in Jordan form when
$$\mathbf{A} = \begin{bmatrix} 1 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 1 \end{bmatrix},$$
where $i = \sqrt{-1}$.

9. Find the Jordan form of the following matrices
$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -4 & 2 \\ -i & 0 & 1 \end{bmatrix}.$$