E1 222 Stochastic Models and Applications

P.S. Sastry
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Reference Material

- V.K. Rohatgi and A.K.Md.E. Saleh, An Introduction to probability and Statistics, Wiley, 2nd edition, 2018
- S.Ross, 'Introduction to Probability Models', Elsevier, 12th edition, 2019.

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- P G Hoel, S Port and C Stone, Introduction to Probability Theory, 1971.
- P G Hoel, S Port and C Stone, Introduction to Stochastic Processes, 1971.

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  - continuity, differentiability, derivatives, functions of several variables, partial derivatives, integration, multiple integrals or integration over $\Re^n$, convergence of sequences and series, Taylor series

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  But we would review the basic probability in the first two classes.

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- Probability theory is also needed for Statistics that deals with making inferences from data.

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This is only a 'sample' of possible application scenarios!

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$(\Omega, \mathcal{F}, P)$ is called the **Probability Space**

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- For these axioms to make sense, we are assuming

$$\text{(i). } \Omega \in \mathcal{F} \quad \text{and} \quad \text{(ii). } A_1, A_2, \cdots \in \mathcal{F} \Rightarrow (\cup_i A_i) \in \mathcal{F}$$

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- As defined, the co-domain of the function P is $\Re$. However, the axioms imply that it takes values in $[0, 1]$.

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- We can show $P(A^c) = 1 - P(A)$ as
  $$1 = P(\Omega) = P(A + A^c) = P(A) + P(A^c)$$

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$$\begin{aligned} P(U_{i=1}^{n} A_i) &= \sum_i P(A_i) - \sum_i \sum_{j>i} P(A_i \cap A_j) \\ &+ \sum_i \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_i A_i) \end{aligned}$$

Known as inclusion-exclusion formula

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- Thus, 'equally likely' is one way of specifying the probability function (in case of finite Ω).
- An obvious point worth remembering: specifying P for singleton events fixes it for all other events.