Assignment 5 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. Let $A, B \in \mathbb{C}^{n\times n}$, A be invertible and $A+B$ be singular. For any matrix norm $|||\cdot|||$, show that $|||B||| \geq 1/|||A^{-1}|||$.

A. $A+B = A + AA^{-1}B = A(I + A^{-1}B)$

Since A is invertible & $A+B$ is singular, $I + A^{-1}B$ has to be singular as well. — 1

$$\left(\because \det(A) \neq 0, \det(A+B) = 0 \Rightarrow \det(A(I+A^{-1}B)) = 0 \Rightarrow \det(I + A^{-1}B) = 0 \right)$$

$\Rightarrow \exists\, x \neq 0$ such that $(I + A^{-1}B)x = 0$

$\Rightarrow -Ix = A^{-1}Bx$ — 2

$$\Rightarrow 1 = \frac{\|A^{-1}Bx\|}{\|Ix\|} = \frac{\|A^{-1}Bx\|}{\|x\|} \leq |||A^{-1}B|||$$

$\Rightarrow 1 \leq |||A^{-1}|||\cdot|||B|||$ (Submultiplicativity)

$\Rightarrow |||B||| \geq 1/|||A^{-1}|||$ — 3

Alt: 1. $A+B = A + AA^{-1}B = A(I + A^{-1}B)$

Since A is invertible & $A+B$ is singular, $I + A^{-1}B$ has to be singular as well. — (1)

If $\exists$ a matrix norm $|||\cdot|||$ such that $|||I-C||| < 1$, then C is invertible.

Consider the contrapositive of the above: If C is singular, then there exists no matrix norm $|||\cdot|||$ such that $|||I-C||| < 1$.

Thus, for a singular C, $|||I-C||| \geq 1 \quad \forall \; |||\cdot|||$. (2)

Hence for any $|||\cdot|||$, $|||I - (I + A^{-1}B)||| \geq 1$.

$\Rightarrow \quad |||-A^{-1}B||| \geq 1 \Rightarrow |-1| \cdot |||A^{-1}B||| \geq 1$

$\Rightarrow \quad |||A^{-1}||| \cdot |||B||| \geq 1$ (Submultiplicativity)

$\Rightarrow \quad |||B||| \geq 1 / |||A|||$ (3)

2. Prove or disprove the following properties of the spectral radius $\rho(\cdot)$ on $\mathbb{C}^{n\times n}$:
ⓐ Non-negativity ⓑ Positivity ⓒ Homogeneity
ⓓ Triangle inequality ⓔ Submultiplicativity
Is the spectral radius $\rho(\cdot)$ a matrix norm on $\mathbb{C}^{n\times n}$?

A. ⓐ Non-negativity: $\rho(\cdot)$ is a non-negative quantity
<u>Proof:</u> Let $A \in \mathbb{C}^{n\times n}$.

$$\Rightarrow \rho(A) = \max\{|\lambda| : \lambda \text{ is an EVal of } A\}$$

As $|\lambda| \geq 0$, $\rho(A)$ which is a maximum of such non-negative quantities is also non-negative.

$$\Rightarrow \rho(A) \geq 0. \quad — (1) \quad ∎$$

ⓑ Positivity: $\rho(\cdot)$ does NOT satisfy positivity
Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \lambda = 0 \Rightarrow \rho(A) = 0$
$\rightarrow A \neq 0$

Thus $\rho(A) = 0 \not\Rightarrow A = 0 \quad — (2)$

ⓒ Homogeneity: $\rho(\cdot)$ satisfies the homogeneity property.
<u>Proof:</u> Let $B = cA$.

$$\Rightarrow \rho(A) = \max\{|\lambda| : \lambda \text{ is an EVal of } A\}$$

$$\& \quad \rho(B) = \max\{|\lambda| : \lambda \text{ is an EVal of } B\}$$

$$Ax = \mu x \Rightarrow Bx = cAx = (c\cdot\mu)x$$

$\Rightarrow c\mu$ is an EVal of B when μ is an EVal of A — (3)

$$\Rightarrow \rho(B) = \max\{|c\lambda| : \lambda \text{ is an EVal of } A\}$$
$$= \max\{|c|\cdot|\lambda| : \lambda \text{ is an EVal of } A\}$$
$$= |c| \cdot \max\{|\lambda| : \lambda \text{ is an EVal of } A\}$$
$$= |c|\cdot\rho(A) \qquad \text{— (4)}$$

$\Rightarrow$ Spectral radius satisfies Homogeneity. ∎

(d) Triangle Ineq.: $\rho(\cdot)$ does not satisfy triangle ineq.

Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \Rightarrow \rho(A) = \rho(B) = 0$

$$A+B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$
$$\Rightarrow \rho(A+B) = 1$$
$$\Rightarrow \rho(A+B) \nleq \rho(A) + \rho(B) \qquad \text{— (5)}$$

(e) Submul.: $\rho(\cdot)$ does not satisfy submultiplicativity

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow |AB - \lambda I| = 0$$
$$\Rightarrow (1-\lambda)(-\lambda) = 0 \Rightarrow \lambda = 0, 1$$
$$\Rightarrow \rho(AB) = 1 \Rightarrow \rho(AB) \nleq \rho(A)\cdot\rho(B) \qquad \text{— (6)}$$

Thus $\rho(\cdot)$ satisfies only the non-negativity and the homogeneity properties. It does not satisfy the other three properties. Hence $\rho(\cdot)$ is not a matrix norm. — (7)