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- One way to generate samples is to design an ergodic markov chain with stationary distribution π
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- We can also use the chain to generate samples from distribution π

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- For all these, we need to design a Markov chain with π as stationary distribution

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- We want to choose $\alpha(i,j)$ to make X_n an ergodic Markov chain with stationary probabilities π

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- Note that it is not necessary for a stationary distribution to satisfy detailed balance

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- Note that $\pi(i)$ above can be replaced by $b(i)$

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- We could have chosen Q to be 'uniform over neighbours'

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- Gives rise to interesting optimization technique called simulated annealing

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- This is known as Gibbs sampling

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- The index need not necessarily represent time. It can represent, for example, space coordinates.

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- We will denote the random variables as X_t or $X(t)$

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- As we saw, for a Markov chain, π_0 and P together specify all such joint distributions

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- If all X_t are continuous random variables and if all distributions have density functions, then we denote joint density of $X_{t_1}, \cdots, X_{t_n}$ by $f_X(x_1, \cdots, x_n; t_1, \cdots t_n)$

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- As we saw, in a Markov chain, the transition probabilities and initial state probabilities would determine all the distributions
- Another such useful assumption is what is called a process with independent increments

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&\quad = Pr[X(t_1) = x_1]\ Pr[X(t_2) - X(t_1) = x_2 - x_1] \cdots \\
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&Pr[X(t_1) = x_1, X(t_2) = x_2, \cdots X(t_n) = x_n] \\
&\quad = Pr[X(t_1) = x_1, X(t_2) - X(t_1) = x_2 - x_1, \cdots] \\
&\quad = Pr[X(t_1) = x_1]\ Pr[X(t_2) - X(t_1) = x_2 - x_1] \cdots \\
&\qquad\qquad \cdots Pr[X(t_n) - X(t_{n-1}) = x_n - x_{n-1}]
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- This is a rather stringent condition and is often referred to as strict-sense stationarity

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- When the process is wide-sense stationary, we write autocorrelation as

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- The question is : do 'time-averages' converge to 'ensemble-averages'
- The process is wide-sense stationary and hence all $X(n)$ have the same distribution; but they need not be independent or uncorrelated (e.g., Markov chain)

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- One sufficient condition could be that covariance between $X(t)$ and $X(t+\tau)$ decreases fast with increasing τ.

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- Note that $E[\eta_\tau] = \eta, \; \forall \tau$.
- Hence it is enough if we show
$$\sigma_\tau^2 \triangleq E\left[(\eta_\tau - \eta)^2\right] \; \rightarrow \; 0, \quad \text{as} \quad \tau \rightarrow \infty$$

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- Assuming wide-sense stationarity, $C_X(t_1, t_2) = C_X(t_1 - t_2)$
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$$\begin{aligned} \sigma_\tau^2 &= E\left[(\eta_\tau - \eta)^2\right] \\ &= E\left[\frac{1}{2\tau}\int_{-\tau}^{\tau}(X(t) - \eta)\,dt\,\frac{1}{2\tau}\int_{-\tau}^{\tau}(X(t') - \eta)\,dt'\right] \\ &= \frac{1}{4\tau^2}\int_{-\tau}^{\tau}\int_{-\tau}^{\tau} E[(X(t) - \eta)(X(t') - \eta)]\,dt\,dt' \\ &= \frac{1}{4\tau^2}\int_{-\tau}^{\tau}\int_{-\tau}^{\tau} C_X(t - t')\,dt\,dt' \end{aligned}$$

$$\text{Let} \quad I = \int_{-\tau}^{\tau} \int_{-\tau}^{\tau} C_X(t_1 - t_2) \, dt_2 \, dt_1$$

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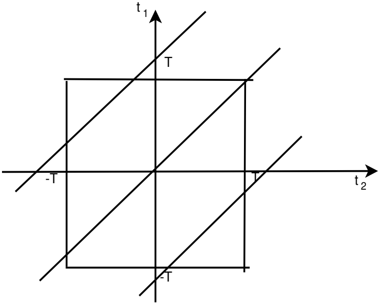

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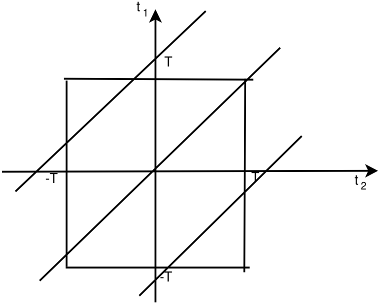

- Easy to see z goes from -2τ to 2τ

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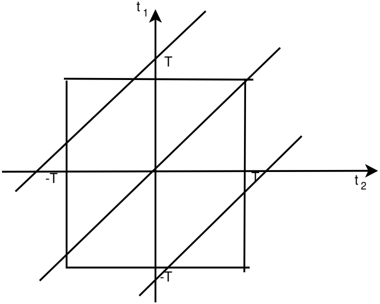

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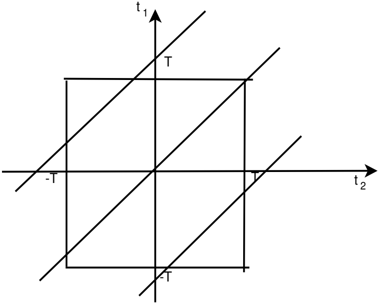

- Easy to see z goes from -2τ to 2τ
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&= \int_{-2\tau}^{0} \int_{-\tau-z}^{\tau} C_X(z)\, dt_2\, dz \;+\; \int_{0}^{2\tau} \int_{-\tau}^{\tau-z} C_X(z)\, dt_2\, dz
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&= \int_{-2\tau}^{0} C_X(z)\, (2\tau + z)\, dz \ + \ \int_{0}^{2\tau} C_X(z)\, (2\tau - z)\, dz
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- Hence, a sufficient condition for $\sigma_\tau^2 \to 0$ is

$$\int_{-\infty}^{\infty} |C_X(z)|\ dz \ < \ \infty$$

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- Hence, a sufficient condition for $\sigma_\tau^2 \to 0$ is

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- This is a sufficient condition for the process being mean-ergodic