Recap: Expectation

- Let X be a discrete rv with $X \in \{x_1, x_2, \cdots\}$. Then
$$E[X] = \sum_i x_i \, f_X(x_i)$$
- If X is a continuous random variable with pdf, f_X,
$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$
- Sometimes we use the following notation to denote expectation of both kinds of rv
$$E[X] = \int_{-\infty}^{\infty} x \, dF_X(x)$$
- We take the expectation to exist when the sum or integral above is absolutely convergent
- Note that expectation is defined for all random variables

Recap: Expectation of a function of a random variable

- Let X be a rv and let $Y = g(X)$. Then,
- $EY = \int y \, dF_Y(y) = \int g(x) \, dF_X(x)$
- That is, if X is discrete, then

$$EY = \sum_j y_j \ f_Y(y_j) = \sum_i g(x_i) f_X(x_i)$$

- If X and Y are continuous

$$EY = \int y \ f_Y(y) \ dy = \int g(x) \ f_X(x) \ dx$$

- This is true for all rv's.

Recap: Properties of Expectation

$$E[g(X)] = \sum_i g(x_i) f_X(x_i) \quad \text{or} \quad E[g(X)] = \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx$$

- If $X \geq 0$ then $EX \geq 0$
- $E[b] = b$ where b is a constant
- $E[ag(X)] = aE[g(X)]$ where a is a constant
- $E[aX + b] = aE[X] + b$ where a, b are constants.
- $E[ag_1(X) + bg_2(X)] = aE[g_1(X)] + bE[g_2(X)]$
- $E\left[(X - c)^2\right] \geq E\left[(X - EX)^2\right],\ \forall c$

Recap: Variance of random variable

$$\text{Var}(X) = E\left[(X - EX)^2\right] = E[X^2] - (EX)^2$$

- Properties of Variance:
  - $\text{Var}(X) \geq 0$
  - $\text{Var}(X + c) = \text{Var}(X)$
  - $\text{Var}(cX) = c^2\text{Var}(X)$

Recap: Moments of a random variable

- The k^{th} (order) moment of X is

$$m_k = E[X^k] = \int x^k \, dF_X(x)$$

- The k^{th} central moment of X is

$$s_k = E[(X - EX)^k] = \int (x - EX)^k \, dF_X(x)$$

- If moment of order k is finite then so is moment of order s for $s < k$.

Moment generating function

- The moment generating function (mgf) of rv X, $M_X : \Re \to \Re$, is defined by

$$M_X(t) = Ee^{tX} = \sum_i e^{tx_i} f_X(x_i) \text{ or } \int e^{tx} f_X(x)\, dx, \;\; t \in \Re$$

- We say the mgf exists if $E[e^{tX}] < \infty$ for t in some interval around zero
- The mgf may not exist for some random variables.

- The mgf of X is: $M_X(t) = E[e^{tX}]$.
- If $M_X(t)$ exists (for $t \in [-a, \ a]$ for some $a > 0$) then all its derivatives also exist.
- Then we can get the moments of X by successive differentiation of $M_X(t)$.

$$\left.\frac{dM_X(t)}{dt}\right|_{t=0} = \left.\frac{d}{dt} E\left[e^{tX}\right]\right|_{t=0} = E[Xe^{tX}]\big|_{t=0} = EX$$

- In general

$$\left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} = E[X^k]$$

- We can easily see this by expanding e^{tX} in Taylor series:

$$\begin{aligned} M_X(t) &= Ee^{tX} = E\left[1 + \frac{tX}{1!} + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots\right. \\ &= 1 + \frac{t}{1!}EX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \frac{t^4}{4!}EX^4 + \cdots \end{aligned}$$

- Now we can do term-wise differentiation. For example

$$\frac{d^3M_X(t)}{dt^3} = 0+0+0+\frac{3*2*1*t^0}{3!}EX^3+\frac{4*3*2*t}{4!}EX^4+\cdots$$

- Hence we get

$$\left.\frac{d^3M_X(t)}{dt^3}\right|_{t=0} = E[X^3]$$

Example – Moment generating function for Poisson

- $f_X(k) = \frac{\lambda^k}{k!} e^{-\lambda}, \ k = 0, 1, \cdots$

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \, \frac{\lambda^k}{k!} \, e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\lambda e^t\right)^k \\ &= e^{-\lambda} \, e^{\lambda e^t} = e^{\lambda(e^t - 1)} \end{aligned}$$

- Now, by differentiating it we can find EX

$$EX = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = e^{\lambda(e^t-1)} \, \lambda e^t \Big|_{t=0} = \lambda$$

(Exercise: Differentiate it twice to find EX^2 and hence show that variance is λ).

mgf of exponential rv

- $f_X(x) = \lambda e^{-\lambda x},\ x > 0$

$$\begin{aligned} M_X(t) &= E[e^{tX}] = \int_0^\infty e^{tx}\ \lambda e^{-\lambda x}\ dx \\ &= \int_0^\infty \lambda e^{-x(\lambda - t)}\ dx \end{aligned}$$

This is finite if $t < \lambda$

$$\begin{aligned} &= \left. \frac{\lambda e^{-x(\lambda-t)}}{-(\lambda - t)} \right|_0^\infty \\ &= \frac{\lambda}{\lambda - t},\ \ t < \lambda \end{aligned}$$

- We can use this to compute EX

$$EX = \left. \frac{dM_X(t)}{dt} \right|_{t=0} = \frac{d}{dt}\left(\frac{\lambda}{\lambda - t} \right)\Big|_{t=0} = \left. \frac{\lambda}{(\lambda - t)^2} \right|_{t=0} = \frac{1}{\lambda}$$

- For mgf to exist we need $E[e^{tX}] < \infty$ for $t \in [-a,\ a]$ for some $a > 0$.
- If $M_X(t)$ exists then all moments of X are finite.
- However, all moments may be finite but the mgf may not exist.
- When mgf exists, it uniquely determines the df
- We are not saying moments uniquely determine the distribution; we are saying mgf uniquely determines the distribution

Characteristic Function

- The characteristic function of X is defined by

$$\phi_X(t) = E[e^{itX}] = \int e^{itx}\, dF_X(x) \quad (i = \sqrt{-1})$$

- If X is continuous rv,

$$\phi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx}\, f_X(x)\, dx$$

- Characteristic function always exists because $|e^{itx}| = 1, \forall t, x$
- For example,

$$\left| \int_{-\infty}^{\infty} e^{itx}\, f_X(x)\, dx \right| \leq \int_{-\infty}^{\infty} \left| e^{itx} \right|\, |f_X(x)|\, dx = \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

- We would consider ϕ_X later in the course

Generating function

- Let $X \in \{0, 1, 2, \cdots\}$
- The (probability) generating function of X is defined by

$$P_X(s) = \sum_{k=0}^{\infty} f_X(k) s^k, \quad s \in \Re$$

- This infinite sum converges (absolutely) for $|s| \leq 1$.
- We have

$$P_X(s) = f_X(0) + f_X(1)s + f_X(2)s^2 + f_X(3)s^3 + \cdots$$

- The pmf can be obtained from the generating function

- $P_X(s) = f_X(0) + f_X(1)s + f_X(2)s^2 + f_X(3)s^3 + \cdots$
- Let $P'_X(s) \triangleq \frac{dP_X(s)}{ds}$ and so on
- We get

$$P'_X(s) = 0 + f_X(1) + f_X(2)\ 2s + f_X(3)\ 3s^2 + \cdots$$

$$P''_X(s) = 0 + 0 + f_X(2)\ 2 * 1 + f_X(3)\ 3 * 2s^1 + \cdots$$

Hence, we get

$$f_X(0) = P_X(0);\ f_X(1) = \frac{P'_X(0)}{1!};\ f_X(2) = \frac{P''_X(0)}{2!}$$

- The moments (when they exist) can be obtained from the generating function: $P_X(s) = \sum_{k=0}^{\infty} f_X(k) s^k$

$$P_X'(s) = \sum_{k=0}^{\infty} k \ f_X(k) \ s^{k-1} \ \Rightarrow \ P_X'(1) = EX$$

$$P_X''(s) = \sum_{k=0}^{\infty} k(k-1) \ f_X(k) \ s^{k-2} \ \Rightarrow \ P_X''(1) = E[X(X-1)]$$

- For (positive integer valued) discrete random variables, it is more convenient to deal with generating functions than mgf.

Example – Generating function for binomial rv

- $f_X(k) = \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k},\ k = 0, 1, \cdots, n$

$$\begin{aligned} P_X(s) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, p^k\, (1-p)^{n-k}\, s^k \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\, (sp)^k\, (1-p)^{n-k} \\ &= (sp + (1-p))^n = (1 + p(s-1))^n \end{aligned}$$

- From the above, we get $P'_X(s) = n(sp + (1-p))^{n-1}p$
- Thus,

$$EX = P'_X(1) = np; \quad f_X(1) = P'_X(0) = n(1-p)^{n-1}p$$

- Let $p \in (0,\ 1)$. The number $x \in \Re$ that satisfies

$$P[X \leq x] \geq p \quad \text{and} \quad P[X \geq x] \geq 1 - p$$

  is called the quantile of order p or the $100p^{th}$ percentile of rv X.

- Suppose x is a quantile of order p. Then we have
  - $p \leq P[X \leq x] = F_X(x)$
  - $1 - p \leq 1 - P[X < x] = 1 - (P[X \leq x] - P[X = x])$
    $\Rightarrow \quad 1 - p \leq 1 - F_X(x) + P[X = x]$
    $\Rightarrow F_X(x) \leq p + P[X = x]$

- Thus, x satisfies (if it is quantile of order p)

$$p \leq F_X(x) \leq p + P[X = x]$$

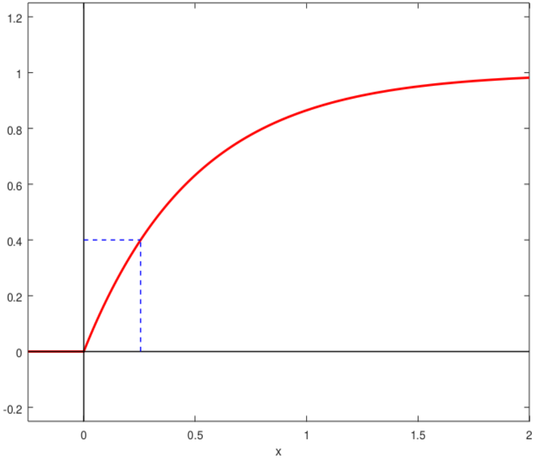
- Note that for a given p there can be multiple values for x to satisfy the above.

- If x is a quantile of order p then

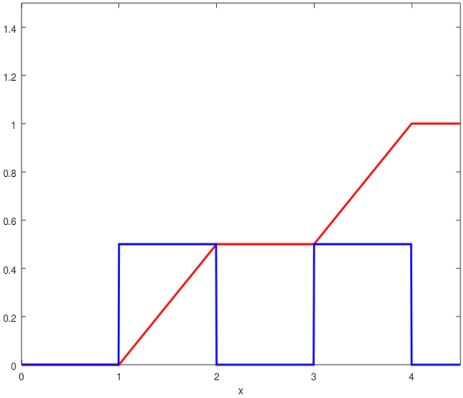
$$p \leq F_X(x) \leq p + P[X = x]$$

- If X is continuous rv, we need to satisfy $p = F_X(x)$.
- In general, for a given p, there may be multiple x that satisfy the above.
- Let us see some examples.

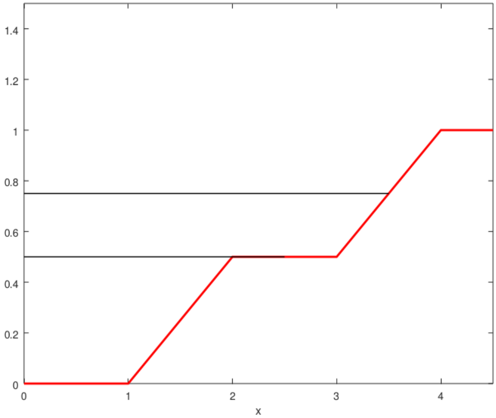
- Let X be continuous rv.
- If the df is strictly monotone then $F_X(x) = p$ would have a unique solution.



- For continuous rv, X, F_X need not be strictly monotone.
- Consider a pdf: $f_X(x) = 0.5, \ x \in [1, \ 2] \cup [3, \ 4]$
- The pdf and the corresponding df are:

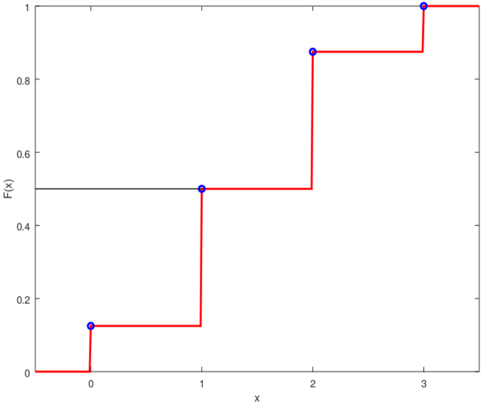


- For this df, for $p = 0.5$, the quantile of order p is not unique because there many x with $F_X(x) = 0.5$
  But for $p = 0.75$ it is unique.



- Let $X \in \{x_1, x_2, \cdots\}$
- Given a p we want to calculate quantile of order p
- Suppose there is a x_i such that $F_X(x_i) = p$.
- Then, for $x_i \leq x < x_{i+1}$, $F_X(x) = p$
- For $x_i \leq x \leq x_{i+1}$, we have $p \leq F_X(x) \leq p + P[X = x]$
- So, quantile of order p is not unique and all such x qualify.

- This situation is illustrated below

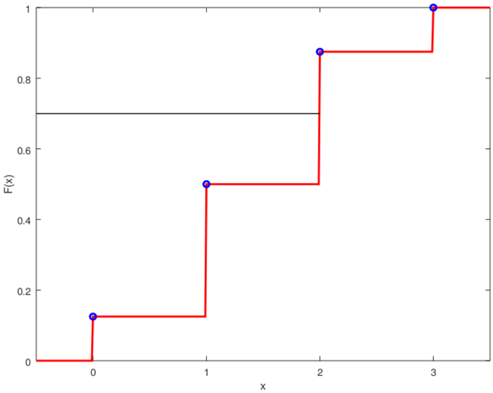


- Now suppose p is such that $F_X(x_{i-1}) < p < F_X(x_i)$.
- Let $F_X(x_{i-1}) = p - \delta_1$ and $F_X(x_i) = p + \delta_2$. (Note that $\delta_1, \delta_2 > 0$)
- Then $P[X = x_i] = F_X(x_i) - F_X(x_{i-1}) = \delta_2 + \delta_1$
- Hence we have

$$p < p + \delta_2 = F_X(x_i) < p + \delta_2 + \delta_1 = p + P[X = x_i]$$

- Hence, x_i is quantile of order p.
- For any $x < x_i$ we would have $F_X(x) \leq F_X(x_{i-1}) < p$.
- For any x, with $x_i < x < x_{i+1}$ we have $p + P[X = x] = p < F_X(x) = p + \delta_2$.
- Similarly, for $x \geq x_{i+1}$ we have $F_X(x) > p + P[X = x]$.
- Thus quantile of order p is unique here.

- This situation is illustrated below



Median of a distribution

- For $p = 0.5$ quantile of order p is called the median.
- For a continuous rv, median, x satisfies: $F_X(x) = 0.5$.
- For a discrete rv, it satisfies:
  $0.5 \leq F_X(x) \leq 0.5 + P[X = x]$.
- As we saw, median need not be unique.
- Recall that the (standard) Cauchy density is given by

$$f_X(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \; -\infty < x < \infty$$

- One can show that $\int_{-\infty}^{0} f_X(x) \, dx = 0.5$ and hence the median is at the origin.

- If we want to find c to minimize $E\left[(X-c)^2\right]$ then the solution is $c = EX$.
- We saw this earlier.
- Suppose we want to find c to minimize $E\left[|(X-c)|\right]$
- Then we would get c to be the median.
(Exercise: Show this for discrete and continuous rv)

Markov Inequality

- Let $g : \Re \rightarrow \Re$ be a non-negative function. Then
$$P[g(X) > c] \leq \frac{E[g(X)]}{c}, \quad (c > 0)$$
- **Proof**: We prove it for continuous rv. Proof is similar for discrete rv

$$\begin{aligned}
E[g(X)] &= \int_{-\infty}^{\infty} g(x)\ f_X(x)\ dx \\
&= \int_{g(x) \leq c} g(x)\ f_X(x)\ dx\ +\ \int_{g(x) > c} g(x)\ f_X(x)\ dx \\
&\geq \int_{g(x) > c} g(x)\ f_X(x)\ dx \quad \text{because } g(x) \geq 0 \\
&\geq c \int_{g(x) > c} f_X(x)\ dx\ =\ c\ P[g(X) > c]
\end{aligned}$$

Thus, $P[g(X) > c] \leq \frac{E[g(X)]}{c}$

Markov Inequality

$$P[g(X) > c] \leq \frac{E[g(X)]}{c}, \quad (c > 0)$$

- In all such results an underlying assumption is that the expectation is finite.
- Let $g(x) = |x|^k$ where k is a positive integer. We have $g(x) \geq 0, \ \forall x$. Let $c > 0$.
- We know that $|x| > c \Rightarrow |x|^k > c^k$ and vice versa.
- Now we get,

$$P[|X| > c] = P[|X|^k > c^k] \leq \frac{E\left[|X|^k\right]}{c^k}$$

- Markov inequality is often used in this form.

Chebyshev Inequality

- Markov Inequality:

$$P[|X| > c] \leq \frac{E\left[|X|^k\right]}{c^k}$$

- Take $|X|$ as $|X - EX|$ and take $k = 2$

$$P[|X - EX| > c] \leq \frac{E\left[|X - EX|^2\right]}{c^2} = \frac{\mathsf{Var}(X)}{c^2}$$

- This is known as the Chebyshev inequality.

- The Chebyshev inequality is

$$P[|X - EX| > c] \leq \frac{\mathsf{Var}(X)}{c^2}$$

- Let $EX = \mu$ and let $\mathsf{Var}(X) = \sigma^2$. Take $c = k\sigma$
- We call, σ, square root of variance, as standard deviation.
- Now, Chebyshev inequality gives us

$$P[|X - \mu| > k\sigma] \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

- This is true for all random variables and the RHS above does not depend on the distribution of X.

- **Markov inequality**: For a non-negative function, g,

$$P[g(X) > c] \leq \frac{E[g(X)]}{c}$$

- A specific instance of this is

$$P[|X| > c] \leq \frac{E\left[|X|^k\right]}{c^k}$$

- **Chebyshev inequality**

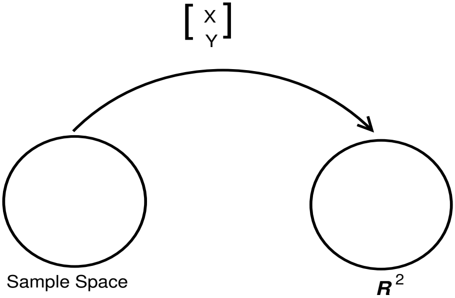
$$P[|X - EX| > c] \leq \frac{\mathsf{Var}(X)}{c^2}$$

- With $EX = \mu$ and $\mathsf{Var}(X) = \sigma^2$, we get

$$P[|X - \mu| > k\sigma] \leq \frac{1}{k^2}$$

A pair of random variables

- Let X, Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$
- Each of X, Y maps Ω to $\Re$.
- We can think of the pair of radom variables as a vector-valued function that maps Ω to $\Re^2$.

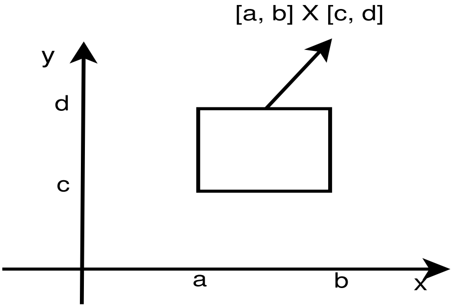


- Just as in the case of a single rv, we can think of the induced probability space for the case of a pair of rv's too.
- That is, by defining the pair of random variables, we essentially create a new probability space with sample space being $\Re^2$.
- The events now would be the Borel subsets of $\Re^2$.
- Recall that $\Re^2$ is cartesian product of $\Re$ with itself.
- So, we can create Borel subsets of $\Re^2$ by cartesian product of Borel subsets of $\Re$.

$$\mathcal{B}^2 = \sigma\left(\{B_1 \times B_2 \ : \ B_1, B_2 \in \mathcal{B}\}\right)$$

where $\mathcal{B}$ is the Borel σ-algebra we considered earlier, and $\mathcal{B}^2$ is the set of Borel sets of $\Re^2$.

- Recall that $\mathcal{B}$ is the smallest σ-algebra containing all intervals.
- Let $I_1, I_2 \subset \Re$ be intervals. Then $I_1 \times I_2 \subset \Re^2$ is known as a cylindrical set.



- $\mathcal{B}^2$ is the smallest σ-algebra containing all cylindrical sets.
- We saw that $\mathcal{B}$ is also the smallest σ-algebra containing all intervals of the form $(-\infty, x]$.
- Similarly $\mathcal{B}^2$ is the smallest σ-algebra containing cylindrical sets of the form $(-\infty, x] \times (-\infty, y]$.

- Let X, Y be random variables on the probability space $(\Omega, \mathcal{F}, P)$
- This gives rise to a new probability space $(\Re^2, \mathcal{B}^2, P_{XY})$ with P_{XY} given by

$$\begin{aligned} P_{XY}(B) &= P[(X,Y) \in B], \ \forall B \in \mathcal{B}^2 \\ &= P\left(\{\omega \ : \ (X(\omega).Y(\omega)) \in B\}\right) \end{aligned}$$

- Recall that for a single rv, the resulting probability space is $(\Re, \mathcal{B}, P_X)$ with

$$P_X(B) \ = \ P[X \in B] = P(\{\omega \ : \ X(\omega) \in B\})$$

- In the case of a single rv, we define a distribution function, F_X which essentailly assigns probability to all intervals of the form $(-\infty,\ x]$.
- This F_X uniquely determines $P_X(B)$ for all Borel sets, B.
- In a similar manner we define a joint distribution function F_{XY} for a pair of random varibles.
- $F_{XY}(x, y)$ would be $P_{XY}((-\infty, x] \times (-\infty, y])$.
- F_{XY} fixes the probability of all cylindrical sets of the form $(-\infty, x] \times (-\infty, y]$ and hence uniquely determines the probability of all Borel sets of $\Re^2$.

Joint distribution of a pair of random variables

- Let X, Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$
- The joint distribution function of X, Y is $F_{XY} : \Re^2 \to \Re$, defined by

$$\begin{aligned} F_{XY}(x, y) &= P[X \leq x, Y \leq y] \quad (= P_{XY}((-\infty, x] \times (-\infty, y])) \\ &= P\left(\{\omega \ : \ X(\omega) \leq x\} \cap \{\omega \ : \ Y(\omega) \leq y\}\right) \end{aligned}$$

- The joint distribution function is the probability of the intersection of the events $[X \leq x]$ and $[Y \leq y]$.

Properties of Joint Distribution Function

- Joint distribution function:

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

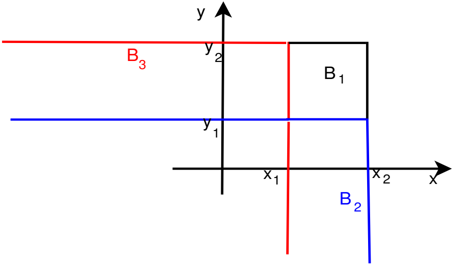
  - $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \forall x, y$;
    $F_{XY}(\infty, \infty) = 1$
    (These are actually limits: $\lim_{x \to -\infty} F_{XY}(x, y) = 0, \forall y$)
  - F_{XY} is non-decresing in each of its arguments
  - F_{XY} is right continuous and has left-hand limits in each of its arguments
- These are straight-forward extensions of single rv case
- But there is another crucial property satisfied by F_{XY}.

- Recall that, for the case of a single rv, the probability of X being in any interval is given by the difference of F_X values at the end points of the interval.
- Let $x_1 < x_2$. Then

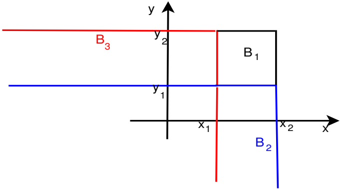
$$P[x_1 < X \leq x_2] = F_X(x_2) - F_X(x_1)$$

- The LHS above is a probability.
  The RHS is non-negative because F_X is non-decreasing.
- We will now derive a similar expression in the case of two random variables.
- Here, the probability we want is that of the pair of rv's being in a cylindrical set.

- Let $x_1 < x_2$ and $y_1 < y_2$. We want $P[x_1 < X \leq x_2, y_1 < Y \leq y_2]$.
- Consider the Borel set $B = (-\infty, x_2] \times (-\infty, y_2]$.



$$
\begin{aligned}
B &\triangleq (-\infty, x_2] \times (-\infty, y_2] = B_1 + (B_2 \cup B_3) \\
B_1 &= (x_1,\ x_2] \times (y_1,\ y_2] \\
B_2 &= (-\infty, x_2] \times (-\infty, y_1] \\
B_3 &= (-\infty, x_1] \times (-\infty, y_2] \\
B_2 \cap B_3 &= (-\infty, x_1] \times (-\infty, y_1]
\end{aligned}
$$



$$
\begin{aligned}
P[(X,Y) \in B] &= P[X \leq x_2, Y \leq y_2] = F_{XY}(x_2, y_2) \\
&= P[(X,Y) \in B_1 + (B_2 \cup B_3)] \\
&= P[(X,Y) \in B_1] + P[(X,Y) \in (B_2 \cup B_3)]
\end{aligned}
$$

$$
\begin{aligned}
P[(X,Y) \in B_2] &= P[X \leq x_2, Y \leq y_1] = F_{XY}(x_2, y_1) \\
P[(X,Y) \in B_3] &= P[X \leq x_1, Y \leq y_2] = F_{XY}(x_1, y_2) \\
P[(X,Y) \in B_2 \cap B_3] &= P[X \leq x_1, Y \leq y_1] = F_{XY}(x_1, y_1)
\end{aligned}
$$

$$
\begin{aligned}
P[(X,Y) \in B_1] &= F_{XY}(x_2, y_2) - P[(X,Y) \in (B_2 \cup B_3)] \\
&= F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)
\end{aligned}
$$

- What we showed is the following.
- For $x_1 < x_2$ and $y_1 < y_2$

$$
\begin{aligned}
P[x_1 < X \leq x_2, y_1 < Y \leq y_2] &= F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) \\
&\quad - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1)
\end{aligned}
$$

- This means F_{XY} should satisfy

  $$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) \geq 0$$

  for all $x_1 < x_2$ and $y_1 < y_2$
- This is an additional condition that a function has to satisfy to be the joint distribution function of a pair of random variables

Properties of Joint Distribution Function

- Joint distribution function: $F_{XY} : \Re^2 \rightarrow \Re$

$$F_{XY}(x, y) = P[X \leq x, Y \leq y]$$

- It satisfies
  1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \forall x, y;$
     $F_{XY}(\infty, \infty) = 1$
  2. F_{XY} is non-decreasing in each of its arguments
  3. F_{XY} is right continuous and has left-hand limits in each of its arguments
  4. For all $x_1 < x_2$ and $y_1 < y_2$

     $$F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1) \geq 0$$

- Any $F : \Re^2 \rightarrow \Re$ satisfying the above would be a joint distribution function.