Assignment 2 Solutions:

Equations / statements marked in blue carry 1 point each. Alternate solutions are accepted (as long as they are well reasoned). Message any of the TAs if you have a problem.

1. $A, B \in \mathbb{C}^{m\times n}$. If $\text{rank}(A) = r$, $\text{rank}(B) = k \le r$, PT

$$r - k \le \text{rank}(A+B) \le r + k.$$

A.

$\longrightarrow \forall x \in R(A+B),\ \exists y$ st $x = (A+B)y$ (Definition)

$$\left.\begin{array}{l} \Rightarrow x = Ay + By = y_1 + y_2 \\ \text{Here } y_1 \in R(A),\ y_2 \in R(B). \\ \\ \Rightarrow x \in S = \underbrace{R(A) + R(B)}_{\text{Sum of two subspaces}} \end{array}\right\} - (1)$$

$(\because S = S_1 + S_2 = \{v_1 + v_2 : v_1 \in S_1, v_2 \in S_2\})$

$(\because \forall x \in U \Rightarrow x \in V \Rightarrow U \subseteq V)$

$$\left.\begin{array}{l} \Rightarrow R(A+B) \subseteq R(A) + R(B) \\ \Rightarrow \text{rank}(A+B) = \dim(R(A+B)) \\ \qquad \le \dim(R(A) + R(B)) \end{array}\right\} - (2)$$

$(U \subseteq V \Rightarrow \dim(U) \le \dim(V))$

Since $\dim(R(A)+R(B)) = \dim(R(A)) + \dim(R(B)) - \dim(R(A) \cap R(B))$

$$\left.\begin{array}{l} \Rightarrow \dim(R(A)+R(B)) \le \dim(R(A)) + \dim(R(B)) \\ \qquad = \text{rank}(A) + \text{rank}(B) \end{array}\right\} - (3)$$

(2), (3) $\Rightarrow$ $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$
$= r + k$ — (4)

Similarly, $\text{rank}(\underset{\text{rank}(A)}{\underset{\parallel}{(A+B) - B}}) \leq \text{rank}(A+B) + \text{rank}(B)$ — (5)

$\Rightarrow$ $\text{rank}(A) - \text{rank}(-B) \leq \text{rank}(A+B)$

$\left(\because \text{rank}(B) = \text{rank}(-B)\right)$

$\Rightarrow$ $\text{rank}(A) - \text{rank}(B) \leq \text{rank}(A+B)$
$\Rightarrow$ $r - k \leq \text{rank}(A+B)$ — (6)

(4), (6) $\Rightarrow$ $r - k \leq \text{rank}(A+B) \leq r + k$

2(a) If CD is invertible, are C and D invertible?

A. From Sylvester's Inequality,

$n \underset{\text{(Given)}}{=} \text{rank}(CD) \leq \min\{\text{rank}(C), \text{rank}(D)\}$
$\leq n$ — (1)

$\left(\because \text{rank}(C), \text{rank}(D) \leq n\right)$

$\Rightarrow$ $\text{rank}(C) = \text{rank}(D) = n$
$\Rightarrow$ C, D are invertible — (2)

Alt: 1. Use of determinants:

$|CD| \neq 0 \Rightarrow |C| \cdot |D| \neq 0$

$\Rightarrow |C| \neq 0 \neq |D|$.

2. Use $N(D) \subseteq N(CD) = \{\underline{0}\}$ to show that $N(D) = \{\underline{0}\}$ (ie D is invertible).

$$\left(\begin{array}{l} x \in N(D) \Rightarrow x \in N(CD) \\ \Rightarrow N(D) \subseteq N(CD) \\ \text{Further, since } CD \text{ is invertible,} \\ CDx = 0 \text{ for only } x = 0 \\ \Rightarrow N(D) \subseteq N(CD) = \{\underline{0}\} \end{array}\right)$$

Further, if $P = (CD)^{-1}$, $PCD = I = CDP$

$\Rightarrow C(DP) = I$.

Since D is shown to be invertible,

$PCD = I \Rightarrow PC = D^{-1}$ (Right multiply by D^{-1})

$\Rightarrow (DP)C = DD^{-1} = I$ (Left multiply by D)

$\Rightarrow C(DP) = I$ & $(DP)C = I$

$\Rightarrow$ Inverse of C exists and $C^{-1} = DP$. (Definition)

Wrong!: You cannot define C^{-1} or D^{-1} if they are non-invertible / singular matrices. If you want to prove that C^{-1} or D^{-1} exists and can be defined, you cannot use it in the proof to prove the same!

$$
\left.
\begin{aligned}
& CD\,(CD)^{-1} = I \\
\Rightarrow\ & CD\,D^{-1}C^{-1} = I \\
\Rightarrow\ & C(I)C^{-1} = I \\
\Rightarrow\ & CC^{-1} = I
\end{aligned}
\right\} \text{This is wrong!}
$$

2(b) If $C+D$ is invertible, are C and D invertible?

A. Let $C = O_n$, $D = I_n$ (C is singular, D is invertible)

$\Rightarrow C+D = I_n$ is invertible! — (1) $\left(I_n = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n \times n},\ O_n = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{n \times n} \right)$

$\Rightarrow$ C and D need not be invertible for $C+D$ to be invertible. — (2)

Alt: Any counter-example is enough. Always use a counter-example to disprove. (Note: Disproving is not the same as proof by contradiction).

Note: From (1), we have the inequality

$$
\underbrace{|\operatorname{rank}(C) - \operatorname{rank}(D)|}_{\geq 0} \leq \underset{n}{\underset{\parallel}{\operatorname{rank}(C+D)}} \leq \underbrace{\operatorname{rank}(C) + \operatorname{rank}(D)}_{\leq 2n}
$$

$0 \leq n \leq 2n \rightarrow$ Meaningless bound!