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This shows: $f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{x}\right| f_{XY}\left(x, \frac{z}{x}\right)\ dx$

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- The best mean-square approximation of Y as a 'linear' function of X is

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In the above, LHS is expectation of a function of Y.

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$$EX = 1; \quad \text{Var}(X) = 1; \quad EY = 2; \quad \text{Var}(Y) = 2$$

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- Let X_n be the number of people who got their own hat in the first round with n people.

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If we can guess value of $E[R_n]$ then we can prove it using mathematical induction

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- Hence a good guess is $E[R_n] = n$.
- We verify it using mathematical induction. We know $E[R_1] = 1$

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\begin{aligned}
E[R_n] &= \sum_{i=0}^{n} P[X_n = i] + \sum_{i=0}^{n} E[R_{n-i}] \; P[X_n = i] \\
&= 1 + E[R_n] \; P[X_n = 0] + \sum_{i=1}^{n} E[R_{n-i}] \; P[X_n = i]
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- So, final number of comparisons depends on the 'number of rounds'

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- M_n – number of comparisons.
- Define: $X = j$ if pivot is j^{th} smallest
- Given $X = j$ we know $M_n = (n-1) + M_{j-1} + M_{n-j}$.

$$\begin{aligned} E[M_n] &= E[\ E[M_n|X]\ ] = \sum_{j=1}^{n} E[M_n|X=j]\ P[X=j] \\ &= \sum_{j=1}^{n} E[(n-1) + M_{j-1} + M_{n-j}]\ \frac{1}{n} \end{aligned}$$

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- This is a recurrence relation. (A little complicated to solve)

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- Let us prove this.

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&\qquad\qquad +2\big(g(X)-E[Y \mid X]\big)\big(E[Y \mid X]-Y\big)
\end{aligned}$$

- Now we can take expectation on both sides.
- We first show that expectation of last term on RHS above is zero.

First consider the last term

$$E\Big[(g(X) - E[Y \mid X])(E[Y \mid X] - Y)\Big]$$

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= \; & E\left[\; E\left\{(g(X) - E[Y \mid X])(E[Y \mid X] - Y) \mid X\right\} \;\right]
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$$\text{because} \quad E[Z] = E[\,E[Z|X]\,]$$

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because $\quad E[h_1(X)h_2(Z)|X] = h_1(X)\; E[h_2(Z)|X]$

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$$

- Since the above is true for all functions g, we get

$$
g^*(X) \; = \; E\left[Y \mid X\right]
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Sum of random number of random variables

- Let $X_1, X_2, \cdots$ be iid rv on the same probability space. Suppose $EX_i = \mu, \ \forall i$.

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since $E[h(X,Y)|Y=y] = E[h(X,y)|Y=y]$

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- Actually, we did not use independence of X_i.