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- For X to be a random variable

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  1. $0 \leq F_X(x) \leq 1, \ \forall x$
  2. $F_X(-\infty) = 0; \ F_X(\infty) = 1$
  3. F_X is non-decreasing: $x_1 \leq x_2 \ \Rightarrow \ F_X(x_1) \leq F_X(x_2)$
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- We also have
  $F_X(x^+) - F_X(x^-) = F_X(x) - F_X(x^-) = P[X = x]$
  $P[a < X \le b] = F_X(b) - F_X(a)$.

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- Let $X \in \{x_1, x_2, \cdots\}$
- Its distribution function, F_X is a stair-case function with jump discontinuities at each x_i and the magnitude of the jump at x_i is equal to $P[X = x_i]$

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Recap: continuous random variable

- X is said to be a continuous random variable if there exists a function $f_X : \Re \to \Re$ satisfying

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- A continuous rv takes uncountably many distinct values. However, not every rv that takes uncountably many values is a continuous rv

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$$P[X \in B] = \int_B f_X(t)\,dt, \quad \forall B \in \mathcal{B}$$
- In particular,
$$P[a \leq X \leq b] = \int_a^b f_X(t)\,dt$$

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- Gaussian (Normal): Parameters: $\sigma > 0, \mu$.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty$$

Functions of a random variable

- We next look at random variables defined in terms of other random variables.

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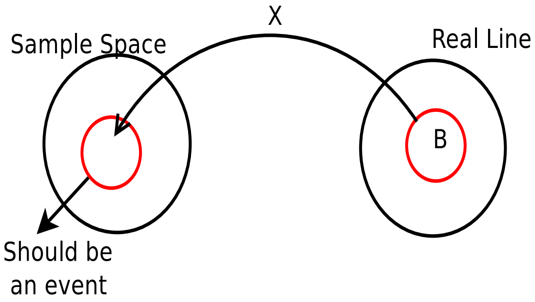

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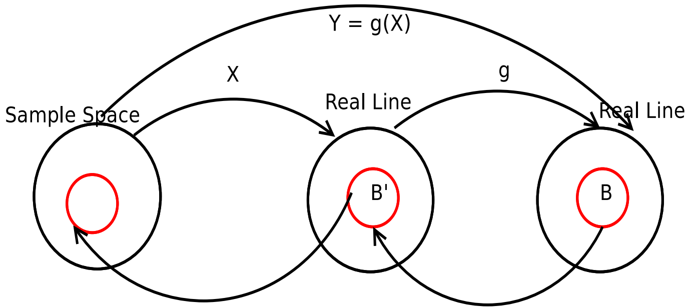

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- Thus, in principle, we can find the distribution of Y if we know that of X

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- This tells us how to find df of Y when it is an affine function of X.
- If X is continuous rv, then, $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$

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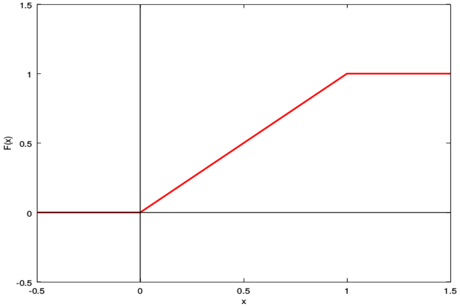

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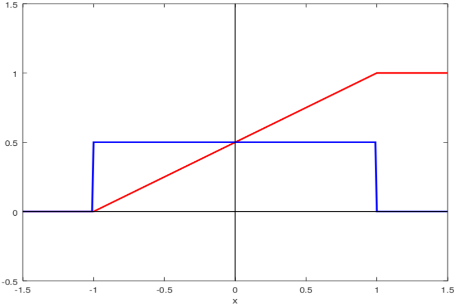

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- Hence $f_Y(y) = \frac{1}{a},\ y \in [b,\ a+b]$ and $Y \sim U[b,\ a+b]$.
- If $X \sim U[0,\ 1]$ then $Y = aX + b$, $(a > 0)$, is uniform over $[b,\ a+b]$.

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$$f_Y(b + ka) = f_X(k) = \frac{1}{N}, \ 1 \leq k \leq N$$

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- We next consider $Y = h(X)$ where

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- This is written as $Y = X^+$ to indicate the function only keeps the positive part.

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- For $y \geq 1$, $F_Y(y) = 1$.
- Thus, the df of Y is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ 0.5 & \text{if } y = 0 \\ \frac{1+y}{2} & \text{if } 0 < y < 1 \\ 1 & \text{if } y \geq 1 \end{cases}$$

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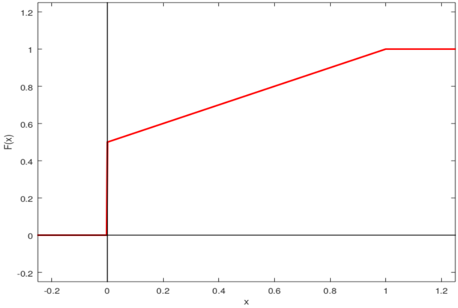
- This is plotted below



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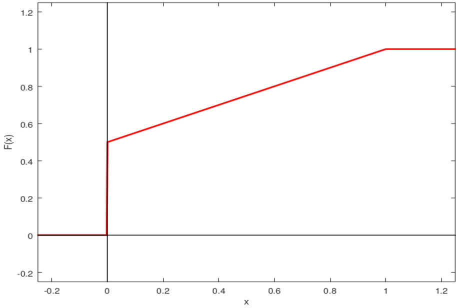

- This is neither a continuous rv nor a discrete rv.

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- This is the general formula for density of X^2 when X is continuous rv.

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&= \frac{1}{2\sqrt{y}}\,\frac{2}{\sqrt{2\pi}}\, e^{-\frac{y}{2}}
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f_Y(y) &= \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right] \\
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- This is an example of gamma density.

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- The earlier density we saw corresponds to $\alpha = \lambda = 0.5$:

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- If $\alpha = 1$, gamma density becomes exponential density.

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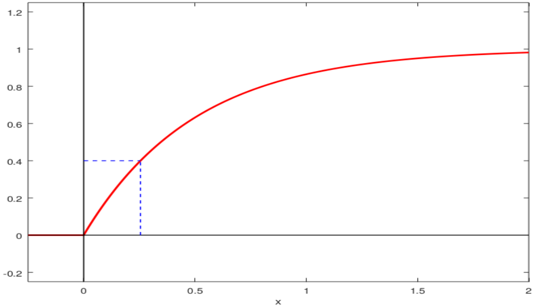
- Thus, the inverse of F is $F^{-1}(z) = \frac{-1}{\lambda}\ln(1-z)$
- So, we had $Y = F^{-1}(X)$ and the df of Y was F

- We can visualize this as shown below



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- Thus, starting with uniform rv, we can generate a rv with a desired distribution.
- Very useful in random number generation. Known as the inverse function method.
- Can be generalized to handle discrete rv also. It only involves defining an 'inverse' when F is a stair-case function. (Left as an exercise!)

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- Has interesting applications.
  E.g., histogram equalization in image processing

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- If $X \sim U[0,\ 1]$ and $Y = F^{-1}(X)$, then Y has df F.
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- Finally, we look at a theorem that gives a formula for pdf of Y in certain special cases

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- **Proof**: Since $g'(x) > 0$, g is strictly monotonically increasing and hence is invertible and g^{-1} would also be monotone and differentiable.
- So, range of Y is $[g(-\infty), \; g(\infty)]$.
- Now we have

$$F_Y(y) = P[Y \leq y] = P[g(X) \leq y] = P[X \leq g^{-1}(y)] = F_X(g^{-1}(y))$$

- Since g^{-1} is differentiable, so is F_Y and we get the pdf as

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- Let $g : \Re \rightarrow \Re$ be differentiable with $g'(x) > 0, \forall x$.
- Let X be a continuous rv with pdf f_X.
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- We can combine both cases into one result.

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$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \ a \leq y \leq b$$

where $a = \min(g(\infty),\ g(-\infty))$ and $b = \max(g(\infty),\ g(-\infty))$

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- We need to find the range of Y based on range of X.

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- Essentially, what we need is that for a any y, the equation $g(x) = y$ would have finite solutions and the derivative of g is not zero at any of these points.
- There are multiple '$g^{-1}(y)$' and we can get density of Y by summing all the terms.

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- However, the notation in the above may be confusing.

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- If $g(x) = y$ has no solution (or no solution satisfying $g'(x) \neq 0$), then at that y, $f_Y(y) = 0$.

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- This is same as what we derived from first principles earlier.