EI 222 Test 2

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1) (a) $f_{XY}(x,y) = \frac{1}{1-x}$, $0 < x < y < 1$

$$f_X(x) = \int_x^1 f_{XY}(x,y)\, dy$$

$$= \int_x^1 \frac{1}{1-x}\, dy$$

$$= \frac{1}{1-x} (y)_x^1$$

$$= \frac{1-x}{1-x}$$

$$f_X(x) = 1 \qquad 0 \leq x \leq 1$$

$$E[X] = \int_0^1 x f_X(x)\, dx$$

$$= \int_0^1 x\, dx$$

$$= \left(\frac{x^2}{2}\right)_0^1$$

$$\boxed{E[X] = \frac{1}{2}}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} \qquad x < y < 1$$

$$f_{Y|X}(y|x) = \frac{1}{1-x}$$

$$E[Y|X=x] = \int_x^1 y\, f_{Y|X}(y|x)\, dy$$

$$= \int_x^1 y \frac{1}{1-x}\, dy$$

$$= \frac{1}{1-x} \left(\frac{y^2}{2}\right)_x^1$$

$$= \frac{1}{1-x} \frac{(1-x^2)}{2}$$

$$E[Y|X=x] = \frac{1+x}{2}$$

$$\boxed{E[Y|X] = \frac{1+X}{2}}$$

$$E[Y] = E[E[Y|X]]$$

$$= E\left[\frac{1+X}{2}\right]$$

$$= \frac{1}{2} + E\left[\frac{X}{2}\right]$$

$$= \frac{1}{2} + \frac{1}{2} E[X]$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)$$

$$\boxed{E[Y] = \frac{3}{4}}$$

(b) $X, Y \in \{0, 1, 2, \ldots\}$

$$P[X=i, Y=j] = e^{-(a+bi)} \frac{(bi)^j a^i}{j!\, i!}$$

$$\text{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[X] = \sum_j \sum_i i \, P[X=i, Y=j]$$

$$= \sum_j \sum_i i \, e^{-(a+bi)} \frac{(bi)^j a^i}{j! \, i!} =$$

2) (a) X, Y - iid exponential with $\lambda = 1$

$f_X(x) = 1e^{-x}$ $\quad f_Y(y) = 1e^{-y}$ $\quad f_{XY}(x,y) = f_X(x) f_Y(y)$

$Z = X + Y$ $\qquad W = \frac{X}{X+Y}$ $\qquad 0 < W < 1$

$Z > 0$

$$WZ = X$$

$$Y = Z - X$$

$$Y = Z - W \cdot Z$$

$$Y = Z(1-W) \qquad X = WZ$$

$$J = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial Y}{\partial Z} \\ \frac{\partial X}{\partial W} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} W & 1-W \\ Z & -Z \end{vmatrix}$$

$$= -ZW - Z(1-W)$$

$$= -Z$$

$|J| = Z$ as $Z > 0$

$$f_{ZW}(z,w) = z \, f_{XY}(wz, z - zw)$$

$$= z \, f_X(wz) \, f_Y(z - zw)$$

$$= z e^{-wz} e^{-z(1-w)}$$

$$f_{ZW}(z,w) = z e^{-z} \qquad z > 0$$

$$f_Z(z) = \int_0^1 f_{ZW}(z,w) \, dw \qquad 0 < w < 1$$

$$= \int_0^1 z e^{-z} \, dw$$

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$$\boxed{f_Z(z) = z e^{-z} \qquad z > 0}$$

$$f_W(w) = \int_0^\infty f_{ZW}(z, w) \, dz \qquad \begin{matrix} z > 0 \\ 0 < w < 1 \end{matrix}$$

$$= \int_0^\infty z e^{-z} \, dz$$

$$= \left(-z e^{-z}\right)_0^\infty - \int_0^\infty \left(-e^{-z}\right) dz$$

$$= 0 + \int_0^\infty e^{-z} \, dz$$

$$= \left(-e^{-z}\right)_0^\infty$$

$$\boxed{f_W(w) = 1 \qquad 0 < w < 1}$$

$$f_{ZW}(z, w) = z e^{-z} \qquad \begin{matrix} z > 0 \\ 0 < w < 1 \end{matrix}$$

$$f_Z(z) f_W(w) = z e^{-z}$$

$$\Rightarrow f_{ZW}(z, w) = f_Z(z) f_W(w)$$

$\Rightarrow$ Z and W are independent.

(b) $X_1, X_2, \ldots$ iid CRV

Record occurred at m if $X_m > \max(X_{m-1}, \ldots X_1)$

$N = \min\{n : n > 1$ and record occurred at time $n\}$

Let $I_k = 1$ if X_k is largest of

Let $I_k = 1$ if X_k is largest of $X_1, X_2, \ldots X_k$

$\quad\quad\; 0$ otherwise

Since X_i are iid, all ordering are equally likely

$$P[I_k = 1] = P[X_k \text{ is largest}] = \frac{(k-1)!}{k!}$$

$$P[I_k = 1] = \frac{1}{k}$$

$N = \min\{n : n \geq 1 \text{ \& a record occurs at } n\}$

N - where the first record occurs

$N = 1$

$N = k$
record doesn't occur in 1st $k-1$ & occurs at k

$$P[N = 2] = \frac{1}{2}$$

$$P[N = k] = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{k-1}\right)\frac{1}{k}$$

$$EN = \sum_{k=2}^{\infty} k\, P[N = k]$$

$$= \sum_{k=2}^{\infty} \cancel{k}\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{k-1}\right)\frac{1}{\cancel{k}}$$

$$= \sum_{k=2}^{\infty} \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) \cdots \left(1 - \frac{1}{k-1}\right)$$

$$= \sum_{k=2}^{\infty} \left(\frac{1}{\cancel{2}}\right)\left(\frac{\cancel{2}}{\cancel{3}}\right) \cdots \left(\frac{\cancel{k-1}}{\cancel{k}}\right)\left(\frac{\cancel{k}}{k-1}\right)$$

$$= \sum_{k=2}^{\infty} \frac{1}{k-1} = \sum_{k'=1}^{\infty} \frac{1}{k'} = \infty$$

$$\boxed{EN = \infty}$$

3) (a) Coin $P(H) = p \quad 0 < p < 1$

X - no. of tosses needed to get atleast one head and one tail

Y - no. of tosses needed to get a head immediately followed by a tail

Y - no. of tosses needed to get a head immediately followed by a tail

$Y = n$ $\quad$ _ _ _ _ T H

$X = \cancel{2}$	H T	or	T H
3	H H T		T T H
4	H H H T		T T T H
⋮	⋮		⋮
5	H H H _ H T		T T T _ T H

$$P[X = n] = p^{n-1}(1-p) + (1-p)^{n-1}p$$

Z_n - n consecutive heads $\quad P[Z_n] = p$

$$E[Y \mid Z_n] = (N+1)(1-p) + (N + EY)p$$

Z_n - get n consecutive tails from start

$Z_n = n$ if $\underbrace{T\,T\,\_\,\_\,T}_{n \text{ times}}$

$$P[Z_n] = (1-p)^n$$

$$E[Y \mid Z_n = n] = (n+1)p + (n + EY)(1-p)$$

$$E[Y] = E[E[Y \mid Z_n]] = E[(Z_n+1)p + (Z_n + EY)(1-p)]$$

$$E[Y] = E[\cancel{Z_n p} + p + Z_n - \cancel{Z_n p} + EY(1-p)]$$

$$E[Y] = p + E[Z_n] + E[Y](1-p)$$

$$p\,E[Y] = p + E[Z_n]$$

$$E[Z_n] = \sum_{k=1}^{\infty} k\,P[Z_n = k]$$

$$E[Z_n] = \sum_{k=1}^{\infty} k\,(1-p)^k$$

$$E[Z_n] = \sum_{k=1}^{\infty} k\,(1-p')^k$$

$$\sum_{k=1}^{\infty} (1-p)^k = \frac{1-p}{p}$$

$$\sum_{k=1}^{\infty} \cancel{-}k\,(1-p)^{k-1} = \frac{p\,(-1) - (1-p)\,1}{p^2}$$

$$= \frac{\cancel{-p} - 1 + \cancel{p}}{p^2}$$

$$\sum_{k=1}^{\infty} k\,(1-p)^{k-1} = \frac{1}{p^2}$$

$$\sum_{k=1}^{\infty} k\,(1-p)^k = \frac{1-p}{p^2}$$

$$E[Z_n] = \frac{1-p}{p^2}$$

$$p\,E[Y] = p + E[Z_n]$$

$$E[Y] = 1 + \frac{1}{p} E[Z_n] = 1 + \frac{1}{p}\left(\frac{1-p}{p^2}\right)$$

$$\boxed{E[Y] = 1 + \frac{1}{p^3} - \frac{1}{p^2}}$$

Y - sequence of ~~tails~~ & one head

Y' - sequence of heads & one tail

$p' = 1-p$

$$E[Y'] = 1 + \frac{1}{(1-p)^3} - \frac{1}{(1-p)^2}$$

$$E[X] = p\,E[Y'] + (1-p)\,E[Y]$$

$$E[X] = p\,E[Y'] + (1-p)\,E[Y]$$

$\downarrow$ Start with head HH -- HT

$\downarrow$ Start with tail TT -- TH

$$E[X] = p\left(1 + \frac{1}{(1-p)^3} - \frac{1}{(1-p)^2}\right) + (1-p)\left(1 + \frac{1}{p^3} - \frac{1}{p^2}\right)$$

$$E[X] = 1 + \frac{p}{(1-p)^3} + \frac{1-p}{p^3} - \frac{p}{(1-p)^2} - \frac{(1-p)}{p^2}$$

(b) X, Y - 2 RV

ST: $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, E[Y|X])$

$$\begin{aligned}
RHS &= \mathrm{Cov}(X, E[Y|X]) \\
&= E[(X - EX)(E[Y|X] - E[E[Y|X]])] \\
&= E[(X - EX)(E[Y|X] - E[Y])] \quad \text{as } E[Y] = E[E[Y|X]] \\
&= E[(X-EX)E[Y|X] - (X-EX)EY] \\
&= E[(X-EX)E[Y|X]] - E[(X-EX)EY] \\
&= E[E[(X-EX)Y|X]] - E[(X-EX)EY]
\end{aligned}$$

as $E[h(X)Y|X] = h(X)E[Y|X]$

$$= E[(X-EX)Y] - E[(X-EX)EY]$$

$= E[(X - EX)Y - (X - EX)EY]$

$= E[(X - EX)(Y - EY)]$

$= Cov(X, Y)$
$= LHS$

$\therefore Cov(X, Y) = Cov(X, E[Y|X])$

4) (b) X, Y - jointly gaussian with means 0, variance 1 and ρ

$\rho_{XY} = \rho$ $\quad \sigma_X^2 = \sigma_Y^2 = 1$ $\quad \mu_X = \mu_Y = 0$

$Z = aX + bY$
$W = bX + aY$ $\quad \Sigma_{XY} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$

$\det A = a^2 - b^2 \neq 0$ as $a \neq b$

$$\Rightarrow \begin{bmatrix} Z \\ W \end{bmatrix} = A \begin{bmatrix} X \\ Y \end{bmatrix}$$

As X and Y are jointly Gaussian and A is invertible, rank(A) = 2, Z, W are jointly Gaussian

Let Σ' be Cov matrix of $[Z, W]$ joint distribution

$$\Sigma_{ZW} = A \Sigma_{XY} A^T$$
$$= \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a+\rho b & \rho a+b \\ \rho a+b & a+\rho b \end{bmatrix}$$

$$\Sigma_{ZW} = \begin{bmatrix} a(a+\rho b)+b(\rho a+b) & \\ ab+\rho b^2+\rho a^2+ab & \end{bmatrix}$$

$$\mathrm{Cov}(Z,W) = 2ab+\rho(a^2+b^2)$$

For uncorrelatedness, $\mathrm{Cov}(Z,W) = 0$

As Z, W are jointly Gaussian, Z, W are uncorrelated $\Rightarrow$ Z, W are independent.

$\therefore$ $\mathrm{Cov}(Z,W) = 0$

$$\boxed{2ab+\rho(a^2+b^2) = 0}$$

Sufficient condition on a, b so that Z, W are independent.

1(a) $X \in \{0, 1, \ldots\}$

$p(i) = P[X=i]$, $i = 0, 1, \ldots$

$Y_1, Y_2, \ldots Y_n$ iid

$P[Y_k = i] = q(i)$ $\quad i = 0, 1, \ldots$ $\quad \forall k$

$h: \mathbb{R} \to \mathbb{R}$

$$S = \frac{1}{n} \sum_{k=1}^{n} \frac{p(Y_k)\, h(Y_k)}{q(Y_k)}$$

$$ES = \frac{1}{n} \sum_{k=1}^{n} E\left[\frac{p(Y_k)\, h(Y_k)}{q(Y_k)}\right]$$

as $Y_1, \ldots Y_n$ are iid

$$E\left[\frac{p(Y_k)\,h(Y_k)}{q(Y_k)}\right] \quad \text{is same } \forall\, k$$

$$ES = \frac{1}{n} \cdot n\, E\left[\frac{p(Y_1)\,h(Y_1)}{q(Y_1)}\right]$$

$$ES = E\left[\frac{p(Y_1)}{q(Y_1)}\,h(Y_1)\right]$$

$$= \sum \left(\frac{p(Y_1 = j)}{q(Y_1 = j)}\,h(Y_1 = j)\right) q(j)$$

$$= \sum_{j=0}^{\infty} \frac{p(j)}{\cancel{q(j)}}\,h(Y_1 = j)\,\cancel{q(j)}$$

$$ES = \sum_{j=0}^{\infty} p(j)\,h(Y_1 = j)$$

$$ES = \sum_{j=0}^{\infty} p(j)\,h(X = j)$$

as X also takes values from $0, 1, \ldots \infty$

$$ES = E[h(X)]$$