E1 222 Stochastic Models and Applications
Test II

`Time: 75 minutes` `Max. Marks:40`
`Date: 23 Dec 2020`

Answer **ALL** questions. All questions carry equal marks

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x,y) = \frac{1}{1-x}, \quad 0 < x < y < 1$$

Find $E[X], E[Y]$ and $E[Y|X]$

b. Let X, Y be discrete random variables, taking non-negative integer values, with joint mass function

$$P[X=i, Y=j] = e^{-(a+bi)} \frac{(bi)^j \; a^i}{j! \; i!}$$

Find $\mathrm{Cov}(X,Y)$.

2. a. Let X, Y be iid continuous random variables having exponential distribution with $\lambda = 1$. Let $Z = X + Y$ and $W = \frac{X}{X+Y}$. Show that Z and W are independent.

b. Let $X_1, X_2, \cdots$ be *iid* continuous random variables. We say a record has occurred at m if $X_m > \max(X_{m-1}, \cdots, X_1)$. Let $N = \min\{n \ : \ n > 1 \text{ and a record occurs at time } n\}$. Show that $EN = \infty$.

3. a. Consider repeated independent tosses of a coin whose probability of heads is p, $0 < p < 1$. Let X denote the number of tosses needed to get at least one head and one tail. Let Y denote the number of tosses needed to get a head immediately followed by a tail. Find EX and EY.

b. For any two random variables, X, Y, show that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, E[Y|X])$

4. a. Let X be a discrete random variable taking non-negative integer values with mass function, $p(i),\ i = 0, 1, \cdots$. Let $Y_1, Y_2, \cdots, Y_n$ be *iid* discrete random variables taking non-negative integer values and with mass function $q(i), i = 0, 1, \cdots$. Assume $p(i), q(i) > 0, \forall i$. Let $h : \Re \to \Re$ be some function. Define

$$S = \frac{1}{n} \sum_{k=1}^{n} \frac{p(Y_k)h(Y_k)}{q(Y_k)}.$$

Find ES.

b. Let X, Y be jointly Gaussian with means zero, variances 1 and correlation coefficient ρ. Assume $\rho \neq 0$. Let $Z = aX + bY$ and $W = bX + aY$, where $a, b \in \Re$, $a \neq 0, b \neq 0$. Find a sufficient condition on a, b for Z and W to be independent.