E2 212: Homework - 6

1 Topics

- **LU factorization, Triangular factorizations and linear equations**

2 Problems

1. (Applications of LU factorization): Suggest efficient algorithms to solve the following system of equations: Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be non-singular.

   (a) $\mathbf{AX} = \mathbf{B}$, where $\mathbf{X}, \mathbf{B}$ are $n \times k$.

   (b) $\mathbf{A}^k\mathbf{x} = \mathbf{b}$. (The idea is to avoid matrix multiplications in computing $\mathbf{A}^k$ explicitly).

2. Give an algorithm for computing a non-zero $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{Ux} = 0$ where $\mathbf{U} \in \mathbb{R}^{n \times n}$ is upper triangular with $u_{nn} = 0$ and $u_{ii} \neq 0$ for all $i = 1, 2, \ldots, n-1$.

3. (Matrix forms of elementary row operations) Let $\mathbf{x}$ be such that $\mathbf{x}_k \neq 0$. Write down the matrix $\mathbf{M}$ that when multiplied with $\mathbf{x}$ produces zeros on components $[k+1 : n]$. Verify that $\mathbf{M}$ can be written as $\mathbf{M} = \mathbf{I} - \mathbf{t}\mathbf{e}_k^T$ and find the value of vector $\mathbf{t}$. In the above, $\mathbf{e}_k$ is the k^{th} standard basis vector. What would be the form of $\mathbf{M}^{-1}$?

4. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$
$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$
   Let $\mathbf{A}_{11}$ be $r \times r$ and non-singular. Show that if $\mathbf{A}_{11}$ has an LU factorization without pivoting, then after r-steps of Gaussian elimination without pivoting on $\mathbf{A}$, $\mathbf{A}(r+1 : n, r+1 : n)$ will contain the Schur's complement of $\mathbf{A}_{11}$, defined by $\mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}$.

5. Is the following statement correct: The multipliers for Gaussian elimination of matrix $\mathbf{A}^T\mathbf{A}$ are identical to the multipliers that orthogonalize the columns of $\mathbf{A}$. Why/ Why not ?

6. Show that

   (a) The inverse of an upper triangular matrix is upper triangular.

   (b) The product of two lower triangular matrices is lower triangular.

   (c) The inverse of a unit upper triangular matrix is unit upper triangular.

   (d) The product of two unit lower triangular matrices is unit lower triangular.

7. If $\mathbf{A}$ is diagonalizable and $f(\cdot)$ is a polynomial, show that $f(\mathbf{A})$ is diagonalizable.

8. Show that any 2×2 real symmetric matrix is diagonalizable.

9. If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is both normal ($\mathbf{A}\mathbf{A}^H = \mathbf{A}^H\mathbf{A}$) and nilpotent ($\mathbf{A}^k = 0$, for some k), show that $\mathbf{A} = 0$.