E1 222 Stochastic Models and Applications
Problem Sheet #1

1. There are three chests each having two drawers. Chest 1 has a gold coin in each drawer while chest 2 has a silver coin in each drawer. Chest 3 has a gold coin in one drawer and a silver coin in the other drawer. A chest is chosen at random and one of its drawers, chosen at random, is opened. It is found to contain a gold coin. What is the probability that the other drawer has (i). a gold coin, (ii). a silver coin?

Answer: Note that each outcome of this random experiment consists of (the choice of) a chest and a drawer. So, we can take Ω as the following six element set.

$$\Omega = \{(c1, d1), (c1, d2), \cdots, (c3, d2)\}$$

We assume all these six outcomes are equally likely. Let A be the event of selected drawer having G. Let B be the event of the non-selected drawer in the selected chest having G. What we want is $P(B|A)$. We see that these events (as subsets of Ω) are:

$$A = \{(c1, d1), (c1, d2), (c3, d1)\}; \quad B = \{(c1, d1), (c1, d2), (c3, d2)\}$$

Hence we have

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(\{(c1, d1), (c1, d2)\})}{P(\{(c1, d1), (c1, d2), (c3, d1)\})} = \frac{2/6}{3/6} = \frac{2}{3}$$

Hence the probability that the other drawer (which is the non-selected drawer in the selected chest) having a S given selected one has G is 1/3.

We can get the probabilities of A, B by total probability rule too. (Let ci denote event of choosing chest ci, $i = 1, 2, 3$; this is an abuse of notation!).

$$P(A) = P(A|c1)P(c1) + P(A|c2)P(c2) + P(A|c3)P(c3) = 1\,\frac{1}{3} \;+\; 0\,\frac{1}{3} \;+\; \frac{1}{2}\,\frac{1}{3} = \frac{1}{2}$$

Similarly we get $P(B) = 0.5$. For $P(AB)$, note that $P(AB|c1) = 1$ and $P(AB|c2) = P(AB|c3) = 0$.

We can also say the following: given that A has occurred, effectively Ω is reduced to $\{(c1, d1), (c1, d2), (c3, d1)\}$ and hence the the other drawer having a G has probability 2/3. As I told you in class, while this is good for calculation, we should realize that conditional probability is defined for all events in the original sample space.

One may wonder why the probability is not 0.5. After all, the other drawer has either a G or a S. But since we are choosing a chest and a drawer, there are three outcomes for finding a G in the opened drawer and in two of them the other drawer has a G.

2. A box contains coupons labelled $1, 2, 3, \cdots, n$. Two coupons are drawn from the box with replacement. Let a, b denote the numbers on the two coupons. Find the probability that one of a, b divides the other.

Answer: Given a fixed number a which is between 1 and n, how many multiples of a are there between 1 and n? Given a, the multiples of a are $k * a$, $k = 1, 2, \cdots$. (Note a is a multiple of a). Hence, there are $\lfloor \frac{n}{a} \rfloor$ multiples of a between 1 and n. (Here $\lfloor x \rfloor$ is the floor function, that is, it is the greatest integer smaller than or equal to x). For example, there are three multiples of 6 between 1 and 20. Thus given a specific number i, $1 \leq i \leq n$, if we randomly select a number between 1 and n, the probability of it being a multiple of i is $\frac{1}{n} \lfloor \frac{n}{i} \rfloor$.

In this problem, for one of the two numbers to divide the other, we need either the first to be a multiple of the second or the other way.

Let A be the event that the second number is a multiple of the first and is not equal to the first. (Note that here the second number is strictly greater than the first). Then

$$\begin{aligned} P(A) &= \sum_{i=1}^{n} P(\text{1st is } i) P(\text{2nd is multiple of } i \text{ and greater} | \text{1st is i}) \\ &= \sum_{i=1}^{n} \frac{1}{n} \left(\frac{\lfloor \frac{n}{i} \rfloor - 1}{n} \right) \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor - \frac{1}{n} \end{aligned}$$

where we used the fact that the second number is independent of the first because we are sampling with replacement. Now let B be the event that the first number is a multiple of second and is not equal to the second. Easy to see that $P(B) = P(A)$. Let C be the event that both numbers are same. Then $P(C) = (1/n)$. Also, these three are disjoint. Their union is the event we want. hence, the required probability is

$$\frac{2}{n^2}\sum_{i=1}^{n}\left\lfloor\frac{n}{i}\right\rfloor - \frac{1}{n}$$

3. A fair dice is rolled repeatedly till we get at least one 5 and one 6. What is the probability that we need n rolls?

Answer: If it takes n rolls what can we say about the outcome of n^{th} roll and about the outcomes of the first $n-1$ rolls? We should get either a 5 or a 6 in the n^{th} roll. If we stop with a 5 (that is 5 on n^{th} roll), the first $(n-1)$ rolls should have at least one 6 and no 5. Similarly for the case of stopping with a 6. Note that the minimum n is 2.

Let A be the event that first $(n-1)$ have at least one 6 and no 5 and n^{th} is a 5. Let B be the event that first $(n-1)$ have at least one 5 and no 6 and n^{th} is a 6. These are disjoint and what we want is probability of their union.

$$\begin{aligned}
P(A) &= \sum_{k=1}^{n-1} {}^{n-1}C_k \left(\frac{1}{6}\right)^k \left(\frac{4}{6}\right)^{n-1-k} \frac{1}{6} \\
&= \frac{1}{6}\left(\sum_{k=0}^{n-1} {}^{n-1}C_k \left(\frac{1}{6}\right)^k \left(\frac{4}{6}\right)^{n-1-k} - \left(\frac{4}{6}\right)^{n-1}\right) \\
&= \frac{1}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right)
\end{aligned}$$

It is easy to see that $P(B)$ would also be the same. Hence the required probability is

$$\frac{2}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right)$$

To verify the answer we must have the above summing to 1 over all n.

$$\sum_{n=2}^{\infty}\frac{2}{6}\left(\left(\frac{5}{6}\right)^{n-1} - \left(\frac{4}{6}\right)^{n-1}\right) = \frac{2}{6}\left(\frac{5/6}{1/6} - \frac{4/6}{2/6}\right) = 1$$

4. Suppose E and F are mutually exclusive events of a random experiment. This random experiment is repeated till either E or F occurs. Show that the probability that E occurs before F is $P(E)/(P(E)+P(F))$.

Answer: Intuitively, we can say the question is asking the following. In the random experiment, given that $E \cup F$ occurred what is the probability that E occurred? That conditional probability is the answer.

To derive this formally. Let us first calculate the probability that exactly n repetitions are required and E occurred before F. That means in the first $n-1$ repetitions neither E nor F occurred and in the n^{th} one E occurred. The probability of this is

$$(1 - P(E \cup F))^{n-1} P(E)$$

For different n, these are disjoint and what we need is the union over all n. hence the required probability is

$$\sum_{n=1}^{\infty} (1 - P(E \cup F))^{n-1} P(E) = \frac{P(E)}{P(E \cup F)} = \frac{P(E)}{P(E) + P(F)}$$

This is a useful general formula.

5. Suppose n men put all their hats together in a heap and then each man selects a hat at random. Show that the probability that none of the n men selects his own hat is

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots \frac{(-1)^n}{n!}$$

Answer: Here the possible outcomes are all possible permutations of the numbers 1 to n and there are $n!$ of them. We assume all are equally likely.

Let A_i denote the event of i^{th} man getting his own hat. Then

$$P(A_i) = \frac{(n-1)!}{n!} = \frac{1}{n}, \ \forall i$$

The event $\cup_{i=1}^{n} A_i$ is the event of some man or the other getting his own hat. Hence the probability we want is $1 - P\left(\cup_{i=1}^{n} A_i\right)$.

To calculate $P\left(\cup_{i=1}^{n} A_i\right)$, we can use the general formula for probability of union of events. for this we need probabilities of intersections of subcollections of these events. We have

$$P(A_i \cap A_j) = \frac{(n-2)!}{n!} = \frac{1}{2!}\frac{(n-2)!2!}{n!} = \frac{1}{2!}\frac{1}{{}^{n}C_2}$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)!}{n!} = \frac{1}{3!}\frac{1}{{}^{n}C_3}$$

and so on.

The general formula for probability of union of events is

$$\begin{aligned} P(U_{i=1}^{n} A_i) &= \sum_i P(A_i) - \sum_i \sum_{j>i} P(A_i \cap A_j) \\ &+ \sum_i \sum_{j>i} \sum_{k>j} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(\cap_i A_i) \end{aligned}$$

Note that there are ${}^{n}C_2$ terms represented by the two summations in the second term, ${}^{n}C_3$ terms represented by the three summations in the third term and so on. Now, simple algebra will complete the problem.

6. Suppose there are three special dice, A, B, C which have the following numbers on their six faces:

   A: 1, 1, 6, 6, 8, 8

   B: 2, 2, 4, 4, 9, 9

   C: 3, 3, 5, 5, 7, 7

   The dice are fair in the sense that each of the faces have the same probability of coming up.
   (i). Suppose we roll dice A and B. What is the probability that the number that comes up on A is less than the one that comes up on B?
   (ii)Suppose your friend, with whom you go out for dinner often, offers you the following. At the end of each dinner, you choose any one of the three dice that you want. She/He would then choose one of the two dice that are remaining. Then both of you roll your respective dice. Whoever gets the smaller number would pay for the dinner. Would you take the offer?

Answer: Let us denote by $A < B$, the event that the number of A would be less than that on B. Then

$$P(A < B) = P(A \text{ shows } 1) + P(A \text{ shows } 6 \text{ or } 8 \text{ and } B \text{ shows } 9) = \frac{2}{6} + \frac{4}{6}\frac{2}{6} = \frac{5}{9} > 0.5$$

Similarly, we can show that $P(B < C) > 0.5$ and $P(C < A) > 0.5$. Thus, no matter which dice you pick up, your fried has choice to pick up a dice that gives lower number with probability greater than 0.5.

7. Consider a communication system. The transmitter sends one of two waveforms. One waveform represents the symbol 0 and the other represents the symbol 1. Due to the noise in the channel, the receiver cannot say with certainty what was sent. The receiver is designed so that, after sensing signal coming out of the noisy channel, it puts out one of the three symbols: a, b, c. The following statistical parameters of the system are determined (either through modeling or experimentation):

$$P[a|1] = 0.6, \; P[b|1] = 0.2, \; P[c|1] = 0.2$$

$$P[a|0] = 0.3, \; P[b|0] = 0.4, \; P[c|0] = 0.3$$

Here, $p[a|0]$ denotes the probability of the receiver putting out symbol a when the symbol transmitted is 0 and similarly for all others. The transmitter sends the two symbols with probabilities: $P[0] = 0.4$ and $P[1] = 0.6$. Find $P[1|a]$ and $P[0|a]$. When receiver puts out a what should we conclude about the symbol sent? We would like to build a decision device that will observe the receiver output (that is, a, b, or c) and decide whether a 0 was sent or a 1 was sent. An error occurs if the decision device says 1 when a 0 was sent or vice versa. Find a decision rule that minimizes the probability of error. What is the resulting (minimum) probability of error?

Answer: Using Bayes rule, we get

$$P[1|a] = \frac{P[a|1]P[1]}{P[a|1]P[1] + P[a|0]P[0]} = \frac{0.6 * 0.6}{0.6 * 0.6 + 0.3 * 0.4} = \frac{36}{48} = \frac{3}{4}$$

Hence, $P[0|a] = 0.25$.

Similarly, we can calculate that $P[1|b] = \frac{3}{7}$, $P[0|b] = \frac{4}{7}$ and $P[1|c] = P[0|c] = 0.5$.

Thus, intuitively, when receiver puts out a we should conclude 1, if it puts out b we should conclude 0 and when it puts out c we can choose anything. We can actually prove that this is optimal in terms of minimizing probability of error.

Any decision rule is just a function that maps $\{a, b, c\}$ to $\{0, 1\}$. Consider any function h like that. As a notation, if $h(a) = 1$ then $\bar{h}(a) = 0$. Note that $h(a), \bar{h}(a) \in \{0, 1\}$.

We want to calculate the probability of error for any such decision rule.

Let e denote the event of h making an error. Then, $P(e|a)$ is the probability that the rule h makes an error conditioned on receiver outputting a. When a is received, h would conclude that the bit sent is $h(a)$. Hence, it would make an error if $h(a)$ is not the bit sent. Hence $P(e|a) = P(\bar{h}(a)|a)$. Now, we have

$$\begin{aligned} P(e) &= P(e|a)P(a) + P(e|b)P(b) + P(e|c)P(c) \\ &= P(\bar{h}(a)|a)P(a) + P(\bar{h}(b)|b)P(b) + P(\bar{h}(c)|c)P(c) \end{aligned}$$

In the above expression, if we change h the only things that change are $P(\bar{h}(a)|a)$. For example, if $h(a) = 1$ then the first term above would contain $P(0|a)$ and if $h(a) = 0$ it would contain $P(1|a)$.

Hence, the optimal h should satisfy $P(\bar{h}(a)|a) < P(h(a)|a)$ and similarly for the others. Thus, if $P(1|a) > P(0|a)$ then optimal h should have $h(a) = 1$. Thus, the optimal h here would have $h(a) = 1$ and $h(b) = 0$. It does not matter what $h(c)$ is. So, optimal decision rule here is not unique.

8. At a telephone exchange, the probability of receiving k calls in a time interval of two minutes is given by the function $h(2, k)$. Assume that the event of receiving k_1 calls in a time interval I_1 is independent of the event of receiving k_2 calls in a time interval I_2, for all k_1 and k_2 whenever the intervals I_1 and I_2 do not overlap. Find an expression for the probability of receiving s calls in 4 minutes in terms of $h(2, k)$. Now suppose $h(2, k)$ is given by

$$h(2, k) = \frac{(2a)^k e^{-2a}}{k!}.$$

Now show that the probability of s calls in 4 minutes is given by $\frac{(4a)^s e^{-4a}}{s!}$

Answer: We have

$$\begin{aligned} P(s \text{ calls in 4 min}) &= \sum_{k=0}^{s} P(k \text{ calls in first 2 min \& } (s-k) \text{ calls in next 2 min}) \\ &= \sum_{k=0}^{s} h(2,k)\; h(2,s-k) \end{aligned}$$

where we used the independence in non-overlapping time intervals.

Now we can substitute for $h(2,k)$ and get

$$\sum_{k=0}^{s} \frac{(2a)^k e^{-2a}}{k!} \frac{(2a)^{s-k} e^{-2a}}{(s-k)!} = e^{-4a} \frac{1}{s!} \sum_{k=0}^{s} \frac{s!}{k!(s-k)!} (2a)^k (2a)^{s-k} = \frac{(4a)^s e^{-4a}}{s!}$$

9. There is a component manufacturing facility where 5% of the products may be faulty. The factory wants to pack the components into boxes so that it can guarantee that 99% of the boxes have at least 100 good components. What is the minimum number of components they should put into each box?

Answer: We are given that the probability a component is faulty is 0.05. We assume that a component being faulty is independent of another being faulty. Thus, if we have n components in a box then the probability of there being exactly k faulty components is given by

$${}^{n}C_k (0.05)^k (0.95)^{n-k}$$

So, if a box contains $100 + n$ components, then the probability that at most n components are faulty is given by

$$\sum_{k=0}^{n} {}^{100+n}C_k (0.05)^k (0.95)^{100+n-k}$$

So, given any random box with $100 + n$ components, the probability that there are at least 100 good components in the box is given by the above expression.

We want 99% of boxes to have at least 100 good components. Hence, we want to satisfy

$$\sum_{k=0}^{n} {}^{100+n}C_k (0.05)^k (0.95)^{100+n-k} \geq 0.99$$

What this says is the following. Suppose I randomly pack $100 + n$ components into boxes. Then, if I pick a box at random then the probability that that box contains at least 100 good components is at least 0.99. In this sense, the quality guarantee we are giving is a probabilistic guarantee.

So, to solve the problem, we need to find a value of n to satisfy the above. Of course, we want to find the least n to satisfy that.

Analytically solving the above is difficult. But we can solve it numerically. We can keep calculating LHS for $n = 1, 2, \cdots$ till we find an n that satisfies it.