E1 222 Stochastic Models and Applications
Final Examination

Time: 3 hours Max. Marks:50
Date: 21 Jan 2021

Answer **Any FIVE** questions. All questions carry equal marks

1. a. Let A, B be events with $P(A) = 0.4$, $P(B) = 0.5$, and $P(A \cap B) = 0.3$. Let I_A and I_B be the indicator random variables of events A and B respectively. Find the correlation coefficient of I_A and I_B.

   b. A rod of length 1 is broken at a random point. The piece containing the left end is once again broken at a random point. Let L be the length of the final piece containing the left end. Find the probability that L is greater than 0.25.

2. a. Consider a game where N men put all their hats in a heap and then everyone randomly chooses a hat. Let$X$ denote the number of men who get their own hat. Show that $\text{Var}(X) = 1$.

   b. Let X, Y be iid exponential random variables with parameter λ. Find the density of $Z = Y/X$

3. a. Let $X_1, \cdots, X_n$ be *iid* Poisson random variables with mean 1. Let $S_n = \sum_{k=1}^{n} X_k$. Find $\text{Prob}[S_n \le n]$. Show that

$$\lim_{n \to \infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \; = \; 0.5.$$

   (You can use the fact: if X, Y are independent Poisson random variables then $X + Y$ is also Poisson).

   b. Let $X_1, X_2, \cdots, X_n$ be iid continuous random variables each having uniform distribution over $(0, 1)$. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, $Y_3 = X_1X_2X_3$, $\cdots$, $Y_n = X_1X_2 \cdots X_n$. Find joint density of $Y_1, Y_2, \cdots, Y_n$, and conditional density of Y_k conditioned on $Y_1, Y_2, \cdots, Y_{k-1}$. Let t be a fixed number in the interval [0, 1]. Let Z denote the number of Y_i that are in the interval $[t, 1]$. Find $P[Z = 1]$.

4. a. Let X be Gaussian with mean zero and variance 1. Let Z be a discrete random variable that is independent of X and suppose $\text{Prob}[Z = 1] = \text{Prob}[Z = -1] = 0.5$. Let $Y = ZX$. Find density of Y. Are X, Y uncorrelated? Are X, Y jointly Gaussian?

   b. Let X, Y have joint density given by

$$f_{XY}(x, y) = \frac{\lambda^{a+b}}{\Gamma(a)\Gamma(b)} x^{a-1}(y-x)^{b-1}e^{-\lambda y},\ 0 < x < y < \infty$$

   where $a, b, \lambda > 0$ are parameters. Find $E[X|Y]$.

5. a. Let $X_n,\ n = 1, 2, \cdots$, be *iid* random variables uniform over $[0,\ 1]$. Let $W_n = \min(X_1, \cdots, X_n)$ and $Y_n = \max(X_1, \cdots, X_n)$. Let $Z_n = 0.5(W_n + Y_n)$. Does the sequence Z_n converge in probability?

   b. Let $X_1, X_2, \cdots$ be iid Gaussian random variables with mean zero and variance 1. Define

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \qquad Z_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

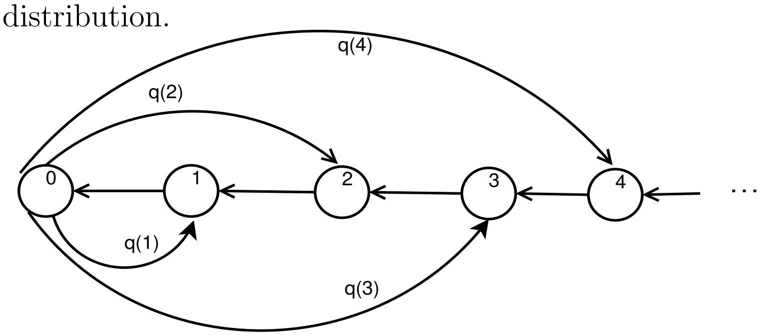
   What are the distributions of Y_n, Z_n? Do the sequences Y_n, Z_n converge to some constant in probability? If yes, state the limit. Do these sequences converge in distribution? If yes, state the limit.
   (You need not 'derive' or 'prove' any thing here. Simply state the answer along with a short justification/explanation for the answer).

6. a. Let $X(t)$ be a stochastic process defined by $X(t) = (-1)^{N(t)}\ X_0$ where $N(t)$ is a Poisson process with rate λ and X_0 is a random variable which is independent of $N(t)$ and which has the distribution $P[X_0 = +1] = P[X_0 = -1] = 0.5$. Find the mean and autocorrelation of $X(t)$.

   b. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ and assume that it is independent of a non-negative random variable, T. Suppose the mean of T is μ and its variance is σ^2. Find (i). $E[N(T)]$, (ii). $\text{Var}(N(T))$

7. a. Let $\{X_n, \ n \geq 0\}$ be a Markov Chain. Let s_0, s_1, s_2 be some specific three states. Suppose the probabilities of transition out of s_0 are given by: $P(s_0, s_0) = 0.5; P(s_0, s_1) = 0.2; P(s_0, s_2) = 0.3$. Suppose the chain is started in s_0. Let T denote the first time instant when the chain left state s_0. (That is, $T = \min\{n : n \geq 1, \ X_n \neq s_0\}$). Find the distribution of T and X_T.

   b. Consider the following Markov chain on state space $\{0, 1, \cdots\}$. Take $q(k) = (1-p)^{k-1}p, \ k = 1, 2, \cdots$ with $0 < p < 1$. Will this chain have a stationary distribution? If yes, find the stationary distribution.



8. a. Let $\{B(t), \ t \geq 0\}$ be a standard Brownian motion process. Consider a process defined by

$$V(t) = e^{-\alpha t/2} \, B\left(e^{\alpha t}\right)$$

where $\alpha > 0$ is a parameter. Find the mean and autocorrelation of $V(t)$.

   b. Suppose X is a Poisson random variable with mean λ. The λ itself is a random variable whose distribution is exponential with mean 1. Show that $P[X = n] = (0.5)^{n+1}$