Taller 5

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1. Proof that.

$$a_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \leq \frac{1}{2T}\int_{-T}^{T} f(x)^2 dx.$$

Known as the Bessel's inequality.

La serie de Fourier asociada a la función $f(x)$ es

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

y sea la función $g(x)$ con su serie de Fourier asociada

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right),$$ y N tiende a infinito.

Vamos a calcular el error cuadrático medio integrado entre esas dos funciones, entonces:

$$[f(x) - g(x)]^2 = f^2(x) - 2f(x)g(x) + g^2(x)$$

$$\mathcal{E}_2[f,g] = \frac{1}{L}\int_{-L}^{L}[f(x) - g(x)]^2 dx \quad (1)$$

$$= \frac{1}{L}\left(\int_{-L}^{L} f^2(x)dx - \underbrace{2\int_{-L}^{L} f(x)g(x)dx}_{(2)} + \underbrace{\int_{-L}^{L} g^2(x)dx}_{(3)}\right)$$

(3) $$\int_{-L}^{L}\left(\frac{a_0}{2} + \sum_{n=1}^{N} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)^2 dx$$

$$= \frac{a_0^2}{2}L + \sum_{n=1}^{N} a_n^2 \int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)dx + b_n^2 \int_{-L}^{L}\sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$= L\left(\frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 + b_n^2\right)$$

2. Write in full detail the proof, of Dirichlet theorem for Fourier series sketched in class.

If f and f' are piecewise continuous functions on the interval $-T \leq x \leq T$, and outside the interval the function is well defined but it is periodic with period $2T$, then f has a representation in a Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi}{T} x\right) + b_n \sin\left(\frac{n\pi}{T} x\right) \right]$$

where the coefficients are given by:

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi}{T} x\right) dx,$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi}{T} x\right) dx,$$

$$a_0 = \frac{1}{T} \int_{-T}^{T} f(x)\, dx.$$

The Fourier series converge to $f(x)$, at any point where the function is continuous and converge to

$$\frac{1}{2}\left[f(x^+) + f(x^-) \right]$$

when the function is discontinuous.

Proof: For the last part, let f be some piecewise continuous function on the interval well defined, the actual Fourier series are given to assure the initial condition

$$u(x,0) = f(x)$$

Then

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)\right]$$

From the second period $F(x)$ the nth partial sum of the series is

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}\left[a_k \cos(kx) + b_k \sin(kx)\right]$$

where,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos(kt)\,dt\ ,\quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin(kt)\,dt \quad (2)$$

by substitution (2) in (1), and factor $F(t)$ we obtain:

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\left[\cos(kx)\cos(kt) + \sin(kx)\sin(kt)\right]\right]dt$$

Notice that

$$\sin(kx)\sin(kt) = \frac{1}{2}\left[\cos(k(x-t)) - \cos(k(x+t))\right]$$

$$\cos(kx)\cos(kt) = \frac{1}{2}\left[\cos(k(x-t)) + \cos(k(x+t))\right]$$

$$= \cos(k(x-t))$$

Therefore

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right]dt$$

Now we define the Dirichlet Kernel by

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos(ku),$$

with this result we rewrite $S_n(x)$ as

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, D_n(t-x)\, dt,$$

Now we making the substitution $u = t - x$,

$$S_n(x) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(x+u)\, D_n(u)\, du,$$

as $D_n(u)$ has a period 2π, like as $f(x+u)$, then

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)\, D_n(u)\, du,$$

as $D_n(-u) = D_n(u)$, replacing u by $-u$ on the previous equation,

$$S_n(x) = -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x-u)\, D_n(u)\, du,$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x-u)\, D_n(u)\, du$$

Now we add the two previous result we obtain,

$$2 S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left[f(x+u) + f(x-u) \right] D_n(u)\, du$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \left[f(x+u) + f(x-u) \right] D_n(u)\, du$$

Notice that $\frac{1}{\pi}\int_0^{\pi} D_n(u)\, du = \frac{1}{2}$, let's check this result,

$$\frac{1}{\pi}\int_0^{\pi} D_n(u)\, du = \frac{1}{\pi}\int_0^{\pi}\left[\frac{1}{2} + \sum_{k=1}^{n} \cos(k u)\right] du$$

$$= \frac{1}{\pi}\left[\frac{1}{2}u + \sum_{k=1}^{n} \frac{\sin(k u)}{k}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} + 0\right]$$

$$= \frac{1}{2}$$

Now we substract $S_n(x) - f(x)$, we obtain

$$S_n(x) - f(x) = \frac{1}{\pi}\int_0^{\pi}\left[f(x+u) + f(x-u) - 2f(x)\right] D_n(u)\, du,$$

By some identities we have

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos(k u) = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\left(\frac{u}{2}\right)}$$

If $\sin\left(\frac{u}{2}\right) \neq 0$ then the difference

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{\pi} \sin\left(n + \frac{1}{2}\right) u \, g(u) \, du$$

where,

$$g(x,u) = \frac{f(x+u) + f(x-u) - 2f(x)}{2\sin\left(\frac{u}{2}\right)},$$

Now to prove that the series converge, therefore we need to proof $\lim_{n\to\infty} S_n(x) = f(x)$,

$$\lim_{n\to\infty} \int_0^{\pi} g(u) \sin\left(n + \frac{1}{2}\right) u \, du = 0, \text{ from this}$$

$$\int_0^{\pi} g(u) \cdot \cos\left(\frac{u}{2}\right) \cdot \sin(nu) \, du + \int_0^{\pi} g(u) \sin\left(\frac{u}{2}\right) \cos(nu) \, du = 0$$

If we set

$$A_n = \frac{2}{\pi} \int_0^{\pi} g(u) \cdot \sin\left(\frac{u}{2}\right) \cdot \cos(nu) \, du,$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} g(u) \cdot \cos\left(\frac{u}{2}\right) \cdot \sin(nu) \, du,$$

Therefore it holds, that $\frac{\pi}{2}(A_n + B_n) = 0$

By Bessel's inequality $A_n \to 0$, $B_n \to 0$

Using limit in the report we have

$$\lim_{u \to 0^+} \frac{f(x^+ + u) - f(x^-)}{u}, \ldots$$

$$\lim_{u \to 0^+} \frac{f(x+u) - f(x)}{u}$$

and then

$$f(x) = \frac{f(x^-) + f(x^+)}{2}$$

3 .

3. Suppose that f is a 2π-periodic piecewise smooth function. For fixed x in $[-\pi, \pi]$ define

$$g(t) = \begin{cases} \dfrac{f(x+t) - f(x^+)}{2\sin\left(\frac{t}{2}\right)} & \text{if } 0 < t \leq \pi \\ f'(x^+) & \text{if } t = 0 \\ 0 & \text{if } -\pi \leq t < 0 \end{cases}$$

a) Show that $g(0^-) = 0$.

If we approach from 0^- we see that,

$$g(t) = 0 \text{ if } -\pi \leq t < 0.$$

Show that $g(0+) = f'(x^+)$. If we approach from 0^+, we see that

$$g(t) = \frac{f(x+t) - f(x^+)}{2\sin\left(\frac{t}{2}\right)}$$

If we approach from 0^+,

4. Prove the Parseval identity

$$\frac{1}{2T}\int_{T}^{T} f(x)^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

$$\int_{-T}^{T} f(x)^2\, dx = \int_{-T}^{T}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)\right)^2 dx$$

$$= \frac{a_0^2}{2} L + \sum_{n=1}^{\infty} a_n^2 \int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right) dx + b_n^2 \int_{-L}^{L}\sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= L\left(\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right)$$

$$\frac{1}{L}\int_{-T}^{T} f^2(x)\, dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

3.a $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Expandiendo la funcion $f(x) = x$ entre $[-\pi, \pi]$ con series de Fourier: (Guía):

$$x = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin(nx)$$

Si $f(x) = x$ es par, esto implica que $a_0 = a_n = 0$

y $b_n = \dfrac{2}{n}(-1)^{n+1}$

Ahora, usando la identidad de Parseval se tiene:

$$\sum_{n=1}^{\infty}\left[\frac{2}{n}(-1)^{n+1}\right]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx$$

$$4\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi}\left[\frac{x^3}{3}\Big|_{-\pi}^{\pi}\right]$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4\pi}\left(\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right)$$

$$= \frac{1}{4\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{6}$$

De 3.a se tiene que

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \underbrace{\sum_{m=1}^{\infty} \frac{1}{(2m)^2}}_{1} + \underbrace{\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}}_{2}$$

3.b 1 $\displaystyle \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{1}{4} \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{24}$

3.c 2 $\displaystyle \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m=1}^{\infty} \frac{1}{(2m)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \frac{\pi^2}{8}.$

5.

Fourier series can be defined on other intervals besides $-L \le x \le L$. Suppose that $g(y)$ is defined for $a \le y \le b$. Represent $g(y)$ using periodic trigonometric functions with period $b-a$. Determine formulas for the coefficients

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x \quad (1)$$

Supongamos que existe $f(x)$ y su expansión en series de Fourier:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x/L) + \sum_{n=1}^{\infty} B_n \sin(n\pi x/L) \quad (2)$$

y despejando a x de (1), tenemos lo siguiente

$$x = \frac{L(2y-b-a)}{b-a} \quad (3)$$

y lo sustituimos en (2), como tendremos una función en términos de y...

$$g(y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L} \cdot \frac{L(2y-b-a)}{b-a}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} \cdot \frac{L(2y-b-a)}{b-a}\right)$$

$$= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi(2y-b-a)}{b-a}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(2y-b-a)}{b-a}\right)$$

Fijémonos que los periodos de $\sin(n\pi x/L)$ y $\cos(n\pi x/L)$ de $g(y)$ es $L = b-a$, entonces, por las fórmulas de los coeficientes de la serie de Fourier, se tiene lo siguiente:

$$A_0 = \frac{1}{2L}\int_a^b g(y) \cdot \frac{2L}{b-a}\, dy$$ con sustitución de (3) $b-a\, dx = 2L\, dy$, $dx = \frac{2L}{b-a} dy$

$$A_0 = \frac{1}{b-a}\int_a^b g(y)\, dy$$

$$A_n = \frac{1}{L} \int_a^b g(y) \cos\left(\frac{n\pi(2y-b-a)}{b-a}\right) \frac{2L}{b-a}\, dy$$

$$A_n = \frac{2}{b-a} \int_a^b g(y) \cos\left(\frac{n\pi(2y-b-a)}{b-a}\right) dy.$$

$$B_n = \frac{1}{L} \int_a^b g(y) \sin\left(\frac{n\pi(2y-b-a)}{b-a}\right) \frac{2L}{b-a}\, dy$$

$$B_n = \frac{2}{b-a} \int_a^b g(y) \sin\left(\frac{n\pi(2y-b-a)}{b-a}\right) dy$$