ASSIGNMENT V
PARTIAL DIFFERENTIAL EQUATIONS

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This homework can be presented in pairs.

1. Proof that

$$a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \leq \frac{1}{2T}\int_{-T}^{T} f(x)^2 dx.$$

known as Bessel's inequality.

2. Write in full detail the proof of Dirichlet theorem for Fourier series sketched in class.

3. Suppose that f is a 2π-periodic piecewise smooth function. For fixed x in $[-\pi, \pi]$ define

$$g(t) = \begin{cases} \frac{f(x+t)-f(x+)}{2\sin\frac{1}{2}} & \text{if } 0 < t \leq \pi \\ f'(x+) & \text{if } t = 0 \\ 0 & \text{if } -\pi \leq t < 0 \end{cases}$$

a. Show that $g(0-) = 0, g(0+) = f'(x+)$, and conclude that g is piecewise continuous on $[-\pi, \pi]$. (Hint: To prove the second part, you need to show that

$$\lim_{t\to 0+} \frac{f(x+t) - f(x+)}{t} = f'(x+)$$

For this purpose, apply the mean value theorem on $(x, x+t)$, then let $t \to 0+$.)

b. Show that

$$\frac{1}{\pi}\int_{0}^{\pi} \frac{f(x+t) - f(x+)}{2} D_N(t)dt \to 0 \quad \text{as} \quad N \to \infty$$

c. Show that

$$\frac{1}{\pi}\int_0^{\pi} \frac{f(x-t)-f(x-)}{2} D_N(t)dt \to 0 \quad \text{as} \quad N \to \infty$$

d. Combine the results of b. and c. to show that

$$s_N(x) - \frac{f(x+)+f(x-)}{2} \to 0 \quad \text{as} \quad N \to \infty$$

3. Prove Parseval's identity

$$\frac{1}{2T}\int_{-T}^{T} f(x)^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

and withi this:

a. Use Parseval's identity and the Fourier series expansion

$$\frac{x}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx, \quad -\pi < x < \pi$$

to obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

b. Show that

$$\sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$$

.

c. Show the identity

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

5. Fourier series can be defined on other intervals besides $-L \le x \le L$. Suppose that $g(y)$ is defined for $a \le y \le b$. Represent $g(y)$ using periodic trigonometric functions with period $b-a$. Determine formulas for the coefficients. Hint: Use the linear transformation

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x.$$