Assignment 3

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1. apply separation of variables to find the solution of the heat distribuiton through a solid block.

$$\begin{aligned} u_t &= c^2\nabla^2 u \\ u(0,y,z,t) &= k_1 \\ u(a,y,z,t) &= k_2 \\ u(x,0,z,t) &= k_3 \\ u(x,b,z,t) &= k_4 \\ u(x,y,0,t) &= k_5 \\ u(x,y,c,t) &= k_6 \\ u(x,y,c,0) &= f(x) \end{aligned}$$

where $0 < x < a$, $0 < y < b$ y $0 < z < c$.

2. Apply separation of variables to solve the one dimensional wave equation, for finite vibrating string with fixed ends, i.e.,

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(0,t) &= 0 \\ u(l,t) &= 0 \\ u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}$$

where $0 < x < l$, $t > 0$ and $\phi(x)$, $\psi(x)$ well defined for $x \in (0,l)$. Explain in full detail every step in your process.

3. Prove that:

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = L\delta_{nm},$$

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$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = 0$$

We will start one by one.

Proof of the first integral. We have to prove two cases: when $n \neq m$ and when $n = m$, also notice that from now on we will assume that $n, m \in \mathbb{Z}$. We will start with the first case, by using the trigonometric identity $\sin(u)\sin(v) = \frac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$ on the initial integral we obtain:

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{\pi x}{L}(m-n)\right) - \cos\left(\frac{\pi x}{L}(m+n)\right)\right],$$

Which is easier to solve,

$$\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{\pi x}{L}(m-n)\right) - \cos\left(\frac{\pi x}{L}(m+n)\right)\right],$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\frac{\pi x}{L}(m-n)\right) - \frac{L}{\pi(m+n)}\sin\left(\frac{\pi x}{L}(m+n)\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) - \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right) - \frac{L}{\pi(m-n)}\sin\left(-\pi(m-n)\right) + \frac{L}{-\pi(m+n)}\sin\left(-\pi(m+n)\right)\right] \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) - \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right) + \frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) - \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right)\right] \\
&= \frac{1}{2}\left[\frac{2L}{\pi(m-n)}\sin(\pi(m-n)) - \frac{2L}{\pi(m+n)}\sin(\pi(m+n))\right] \\
&= \frac{L}{\pi(m-n)}\sin(\pi(m-n)) - \frac{L}{\pi(m+n)}\sin(\pi(m+n)),
\end{aligned}$$

notice than $m+n$ or $m-n$ gives us some z integer and such that $\sin(z\pi) = 0$, therefore when $n \neq m$ the integral is always zero. Now we solve for the case when $n = m$, if that is the case notice that the initial integral can be rewritten as:

$$\int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right) dx,$$

by using the trigonometric indentity $\sin^2(u) = \frac{1-\cos(2u)}{2}$ we obtain

$$\int_{-L}^{L} \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{L}x\right)\right],$$

Which again is an easier integral to solve

$$\begin{aligned}
&= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[L - \frac{L}{2n\pi}\sin(2n\pi) + L + \frac{L}{2n\pi}\sin(-2n\pi)\right] \\
&= \frac{1}{2}\left[2L - \frac{L}{2n\pi}\sin(2n\pi) - \frac{L}{2n\pi}\sin(2n\pi)\right] \\
&= \frac{1}{2}\left[2L - \frac{L}{n\pi}\sin(2n\pi)\right] \\
&= L - \frac{L}{2n\pi}\sin(2n\pi),
\end{aligned}$$

but same as before for any integer z $\sin(z\pi) = 0$, and therefore when $n = m$, then

$$\int_{-L}^{L} \frac{1}{2}\left[1 - \cos\left(\frac{2n\pi}{L}x\right)\right] = L.$$

and by the two previous results we have verified that

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right) dx = L\delta_{nm}$$

Proof of the second integral.

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) dx = L\delta_{nm}$$

we will repeat the same procedure as on the first proof, with this in mind for the first case $m \neq n$ we will rewrite the integral as

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right) dx = \int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{\pi x}{L}(m-n)\right) + \cos\left(\frac{\pi x}{L}(m+n)\right)\right],$$

by using the trigonometric identity $\cos(u)\cos(v) = \frac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$ on the initial integral.

Which can be solved almost the same means we use on the previous one, but here we will repeat the solving procedure,

$$\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{\pi x}{L}(m-n)\right) + \cos\left(\frac{\pi x}{L}(m+n)\right)\right],$$

$$\begin{aligned}
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\frac{\pi x}{L}(m-n)\right) + \frac{L}{\pi(m+n)}\sin\left(\frac{\pi x}{L}(m+n)\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) + \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right) - \frac{L}{\pi(m-n)}\sin\left(-\pi(m-n)\right) - \frac{L}{-\pi(m+n)}\sin\left(-\pi(m+n)\right)\right] \\
&= \frac{1}{2}\left[\frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) + \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right) + \frac{L}{\pi(m-n)}\sin\left(\pi(m-n)\right) + \frac{L}{\pi(m+n)}\sin\left(\pi(m+n)\right)\right] \\
&= \frac{1}{2}\left[\frac{2L}{\pi(m-n)}\sin(\pi(m-n)) + \frac{2L}{\pi(m+n)}\sin(\pi(m+n))\right] \\
&= \frac{L}{\pi(m-n)}\sin(\pi(m-n)) + \frac{L}{\pi(m+n)}\sin(\pi(m+n)),
\end{aligned}$$

notice than $m+n$ or $m-n$ gives us some z integer and such that $\sin(z\pi) = 0$, therefore when $n \neq m$ the integral is always zero. Now we solve for the case when $n = m$, if that is the case notice that the initial integral can be rewritten as:

$$\int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right)dx,$$

by using the trigonometric indentity $\cos^2(u) = \frac{1+\cos(2u)}{2}$ we obtain

$$\int_{-L}^{L} \frac{1}{2}\left[1 + \cos\left(\frac{2n\pi}{L}x\right)\right],$$

Which again is an easier integral to solve

$$\begin{aligned}
&= \frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\left(\frac{2n\pi}{L}x\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[L + \frac{L}{2n\pi}\sin(2n\pi) + L - \frac{L}{2n\pi}\sin(-2n\pi)\right] \\
&= \frac{1}{2}\left[2L - \frac{L}{2n\pi}\sin(2n\pi) + \frac{L}{2n\pi}\sin(2n\pi)\right] \\
&= \frac{1}{2}\left[2L - \frac{2L}{n\pi}\sin(2n\pi)\right] \\
&= L - \frac{L}{n\pi}\sin(2n\pi),
\end{aligned}$$

but same as before for any integer z $\sin(z\pi) = 0$, and therefore when $n = m$, then

$$\int_{-L}^{L} \frac{1}{2}\left[1 + \cos\left(\frac{2n\pi}{L}x\right)\right] = L.$$

and by the two previous results we have verified that

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = L\delta_{nm}$$

Proof of the third integral.

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = 0$$

We will start by assuming that $n \neq m$, and therefore by using the trigonometric identity $\sin(u)\cos(v) = \frac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$ we rewrite the integral as

$$\int_{-L}^{L} \frac{1}{2}\left[\sin\left(\frac{\pi x}{L}(m+n)\right) + \sin\left(\frac{\pi x}{L}(m-n)\right)\right],$$

Now we solve the integral,

$$\begin{aligned}
&= \frac{1}{2}\left[-\frac{L}{\pi(m+n)}\cos\left(\frac{\pi x}{L}(m+n)\right) - \frac{L}{\pi(m-n)}\cos\left(\frac{\pi x}{L}(m-n)\right)\right]_{-L}^{L} \\
&= \frac{1}{2}\left[-\frac{L}{\pi(m+n)}\cos\left(\pi(m+n)\right) - \frac{L}{\pi(m-n)}\cos\left(\pi(m-n)\right) + \frac{L}{\pi(m+n)}\cos\left(-\pi(m+n)\right) + \frac{L}{\pi(m-n)}\cos\left(-\pi(m-n)\right)\right] \\
&= \frac{1}{2}\left[-\frac{L}{\pi(m+n)}\cos\left(\pi(m+n)\right) - \frac{L}{\pi(m-n)}\cos\left(\pi(m-n)\right) + \frac{L}{\pi(m+n)}\cos\left(\pi(m+n)\right) + \frac{L}{\pi(m-n)}\cos\left(\pi(m-n)\right)\right] \\
&= \frac{1}{2}\left[0\right] \\
&= 0
\end{aligned}$$

as we saw assuming that $n \neq m$ we obtain zero, now we evaluate as $n = m$, then the initial integral becomes

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx,$$

now we solve the integral, notice that we can perform the substitution $u = \sin\left(\frac{n\pi}{L}x\right)$, then $\frac{du}{dx} = \frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)$, we obtain

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = \int_{?}^{?} \frac{L}{n\pi}udu = \left[\frac{L}{2n\pi}u^2\right]_{-?}^{?} = \left[\frac{L}{2n\pi}\sin^2\left(\frac{n\pi}{L}x\right)\right]_{-L}^{L} = \frac{L}{n\pi}\sin\left(n\pi\right) = 0,$$

$\sin(n\pi) = 0$, therefore we prove that

$$\int_{-L}^{L} \sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = 0,$$

no matter if $n = m$ or $m \neq n$, and that concludes our proof.