String-Plate Coupling

Mangesh Chandrakant Sonawane

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Supervisor 1: Dr. Michele Ducceschi

Abstract

The dissertation work in this project constitutes and discusses techniques for modelling a coupled system of string and plate. This project demonstrated the purpose of physics based sound synthesis. The principal method that follows this project is based on Finite Difference Time Domain(FDTD) method. This method is chosen mainly due to it's flexibility, computational efficiency and programming simplicity, with a possible extension to non-linearity.

The way in which sub structures are connected to string is one of the important area of my interest so as to create a realistic sound synthesis. The important aspect for the sound production for this model can be considered is the achieved vibratory motion at the points where the sub-structures are connected. When string vibrates, this resonating body's force acts on the sub-structure of plate and then indeed it's resultant force acts backs on the string.

By applying this method, it can be used to model a simplified piano, as well as it is also similar to the soundboard of piano where string and plates are coupled, but with few changes in the connection types and parameters.

Declaration

I do hereby declare that this dissertation was composed by myself and that the work described within is my own, except where explicitly stated otherwise.

Mangesh Chandrakant Sonawane
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Chapter 1

Introduction

Musical instruments has a vast history which can be traced back to to the civilisation times. Ever since then, there has been tremendous evolution of instruments. String instruments can be found in the history as back as 6 BC. Pythagoras was assumed to have used lyre which is an ancestor of harp to discover the relationship between mathematics and music[2]. It was also assumed that Pythagoras loved the harmonics of string instruments so much that he prohibited his students from listening any other instruments. Since then there has been tremendous increase in the research for physics of string instruments. Various scholars have then been performing research for discovering the mathematical rules that governs the physics for the specified instruments particularly related to strings and thus a discrete interdisciplinary branch of musical acoustic was being made[3].

Use of mathematical equations describes the driving and dynamic performance of string instrument. It describes the state of time and space according to how the system moves. A string instrument consists of various sub structures connected to each other. It mainly comprises of string connected to a body or which can also be considered as soundboard. This body or soundboard can be stimulated with the help of beams and plates. If we consider guitar as an example, it mainly consists of three sub-structures.The first structure is the string itself. Second structure consists of bridge onto which string is placed. And the third structure is the resonating body. This resonating body consists of top and back plates along with cavity and ribs[4]. The vibration and it's motion for each object is defined using equation of motions which consists of partial differential equations. These equations can be solved in digital domain using computer based methods and thus they are called as physical modelling of sound. It generates a waveform which is the output sound of the modelled virtual instrument. The equations governing the total system that is being stimulated for instrument, consists of parameters based on the material as well as the geometrical properties of the vibrating object. Generally, these parameters are under the control

of the user and can be alter as per requirements. These change in parameters makes the user to an advantage of changing the properties of the instrument similar to that of real world instrument maker.
Connection of objects with the string instruments as well as its motion of equation analysis to the energy analysis is one of the main area of interest in this dissertation in order to create realistic sound model of the piano soundboard. The motion gained at the connection of structures is one of the important aspect for the effective sound production. According to Newton's third law of motion, there is always equal as well as opposite force for any action[5]. Hence, the same law applies to a system with the connection to another structure. When a force applied to string starts vibrating and acts on the resonating body, then the resultant force acts back on the string too. In this project, linearly coupled model for string and plate shall be considered. Most of the string instruments demonstrates transverse as well as longitudinal motions. But, in the string instrument like piano, the strings are set in parallel to the plates and different resonating body present in the instrument. Therefore, longitudinal motion will not propel the resonating body of soundboard as much as the transverse motion will, due to complex geometry present between the bridge and soundboard.

1.1 Numerical Modelling

Physical modelling of string instruments can be achieved using three main methods namely *'Finite Difference, Finite Element,* and *Digital Wave Guides'* .Hiller and Ruiz[6, 7] developed first synthesis model of 1D string using finite difference method in about 70's. Even though it was big step in string synthesis, it required lot of computational power to carry out the synthesis at that time. 20 years later when there was a tremendous development in computers, Chaigne developed strings models using Finite Difference methods. He presented his comparison between guitar, violin as well as piano[8]. He determined each model using three basic requirements: motion of string, boundary conditions and domain of mechanism. The piano model developed by him initially involved hammer string mechanism using Finite difference method. Later in extension to his work, he developed coupling for soundboard using mechanical oscillators. Dr. Stefan Bilbao represents a detailed study of finite difference for various systems like string, plates, beams with applications to musical acoustics in his book along with coupling systems.

Each method for modelling sound synthesis depends on it's use according to it's strength and weakness. Modelling of string and plate coupled system using finite difference method, the equation of the motion for each sub structure is being discretised. Any changes to the boundary conditions get updated locally, without any need of reformulating the whole system. Reducing the complexity for computational purposes

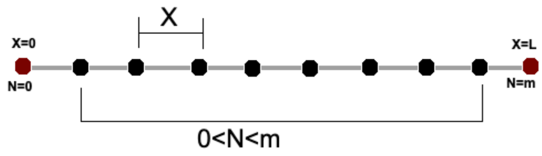

Figure 1.1: This illustrates a discrete length string 'L' with m segments and m+1 grid points. Red coloured points indicated the boundary points[1]

needs lower sample rate. However, reducing sample rate would adversely effect resultant sound as resolution of system is dependent on the sample rate of system.

1.2 Finite Difference Time Domain

Finite Difference method is considered as one of the most popular techniques in musical acoustics. It is one of the direct method used for calculation of partial differential equations. It has been used extensively in physical modelling of musical instruments in order to stimulate the instrument. It enables to create an algorithm that can be executed and implemented in digital domain. Material used and its parameters are in great control of user with the help of finite difference method. However control over these parameters are limited as it needs to follow the stability bounds of the total system. Finite difference has a grid system which updates points on either side. Hence the boundary points can be calculated without actually updating or recalculating the equation at all other points. Finite difference method can also be extended in modelling of non-linearities. The derivatives of partial differential equations of the system are approximated by calculating variations between adjacent points and thus equation is updated which is a linear combination of values at each grid.

1.3 Difference Operators

Now let us consider a system with displacement *w(x,t)* which depends on space *'x'* and time *'t'*. The continuous form of any equation can be written in form of discrete. In this discrete form, time is sampled into discrete points. Time is sampled in form of $m\Delta t$ and space is sampled as $n\Delta x$. Here, m and n are the grid points. Hence the discrete form $w(m\Delta t, n\Delta x)$ can be written as w_n^m. Finite difference time domain method relies on adjacent grid points. It generally follows principle of future=present-past, where future points are obtained with the help of present grid points and past grid points respectively. The previous time step would be w_n^{m-1} and w_{n-1}^m in time and space respectively. Similarly the future points would be w_n^{m+1} and w_{n+1}^m in time ans space respectively.

There are three main operators used in finite difference method. First operator is the forward difference operator in which difference between next time step and current time step is being used. Second is the backward difference operator in which difference between current and previous time step is calculated. Third is the centre difference operator in which difference between previous and next time step is being calculated[9]. Taylor series expansion leads to finite difference time domain method. Hence the accuracy of the method depends on it's order of expansion. Higher the order of expansion, there is a high chance of more accuracy. However this can lead to high computational power needs due to increase in the size of update equation.
Let us consider the following equation for time and space operators :

$$\text{here, } 1 = w_n^m \; \delta_{t+}w \triangleq \frac{1}{k}(w_n^{m+1} - 1) = \frac{\partial}{\partial t} \qquad \delta_{x+}w \triangleq \frac{1}{h}(w_{n+1}^m - 1) = \frac{\partial w}{\partial x}$$
$$\delta_{t-}w \triangleq \frac{1}{k}(1 - w_n^{m-1}) = \frac{\partial}{\partial t} \qquad \delta_{x-}w \triangleq \frac{1}{h}(1 - w_{n-1}^m) = \frac{\partial w}{\partial x}$$
$$\delta_{t\cdot}w \triangleq \frac{1}{k}(w_n^{m+1} - w_n^{m-1}) = \frac{\partial}{\partial t} \qquad \delta_{x\cdot}w \triangleq \frac{1}{h}(w_{n+1}^m - w_{n-1}^m) = \frac{\partial w}{\partial x}$$

With the help of these basic operator, high order derivatives can be approximated using the combinations:

$$\delta_{xx} = \delta_{x+}\delta_{x-}$$
$$\delta_{xxxx} = (\delta_{xx})(\delta_{xx})$$

In case of 2D system $w(x, y, t)$, the finite difference scheme consists of laplacian (Δ) and biharmonic ($\Delta\Delta$) operators, which can be expanded as:

$$\Delta w = \left[\frac{\partial^2 w}{\partial^2 x} + \frac{\partial^2 w}{\partial^2 y}\right]$$
$$\Delta\Delta w = \left[\frac{\partial^2 w}{\partial^2 x} + \frac{\partial^2 w}{\partial^2 y}\right]^2$$
$$\Delta\Delta w = \left[\frac{\partial^2 w}{\partial^2 x} + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial^2 y}\right]$$

Biharmonic operator is just the square of laplacian operator. These operators will be widely used in plate system. In the upcoming sections, plate and string displacement will be represented using u and w respectively. Superscript n would represent time steps and subscript would represent spatial co-ordinate or index. In order to construct laplacian and biharmonic operator function, following division of equation can be utilised in order to achieve simplicity and efficiency. The matrix obtained will be the sparse matrix. Due to linear indesing of grid, there arises issues regarding the D_{xx}

or D_{yy} matrix generation. Use of function like kron in MATLAB helps to overcome this issue.

$$\begin{aligned}
\Delta\Delta w &= \left[\frac{\partial^2 w}{\partial^2 x} + \frac{\partial^2 w}{\partial^2 y}\right]^2 \\
\Delta\Delta w &= (w_{xx} + w_{yy})^2 \\
\delta_{\Delta\Delta} w &= (\delta_{xx} w + \delta_{yy} w)^2 \\
\frac{1}{h^4} D_{\Delta\Delta} &= (\frac{1}{h^2} D_{xx} + \frac{1}{h^2} D_{yy})^2
\end{aligned}$$

Biharmonic operator can further be expanded using operators. But matrix form makes it look better and makes it easy to solve. Creating a sparse matrix for biharmonic helps to make the equation look simpler and approximation can be done using a single call function for biharmonic.

$$\begin{aligned}
\delta_{\Delta\Delta} &= \frac{1}{h^4}(e_{x+1} - 2 + e_{x-1} e_{y+1} - 2 + e_{y-1})^2 \\
\delta_{\Delta\Delta} &= \frac{1}{h^4}\Big(20 - 8(e_{x+,y} + e_{x-,y} + e_{x,y+} + e_{x,y-}) + \\
&\quad (e_{x+,y+} + e_{x+,y-} + e_{x-,y+} + e_{x-,y-}) + \\
&\quad (e_{x+2,y} + e_{x-2,y} + e_{x,y+2} + e_{x,y-2})\Big)
\end{aligned}$$

Chapter 2

Kirchhoff Plate Theory

In this project, the main system that would be considered is plate and string alone. Coupling of these two system would be implemented using finite difference time domain methodology. Here, coupling refers to connection of two system of an instrument. This system stimulates the sound produced by soundboard of piano.

2.1 Plate

Plate vibration is treated as 2D wave equations for sound synthesis.But apart from its membrane, the material used often leads to the stiffness in plate. These has a great interest in the music acoustic for generating real world sound synthesis. It is also an amazing thing that stiffness tends to decrease the computational work required for the implementation. The derivations of equation for plates shall be derived and discussed in this section.

2.1.1 Plate Equation

Kirchhoff model for thin plate is describes as:

$$\rho H u_{tt} = -D\Delta\Delta u \tag{2.1}$$

Here u(m) is defined as the displacement of plate. ρ (kgm^{-3}) is the density of material, H (m) is the thickness of the plate, D is defined by following constant:

$$D = \frac{EH^3}{12(1-\nu)}$$

This is also termed as plate's flexural rigidity[9]. In this, E is the Young's modulus, ν is the Poisson's ratio which is less than 0.5. Equation (2.1) can be rearranged to a simpler form. The term ρ and H can taken to the right hand side of equation for simplicity.Not to forget that when force excitation term is added, ρ and H term must be included in denominator. Thus it can be rewritten as:

$$u_{tt} = -\kappa^2 \Delta\Delta u \tag{2.2}$$

Here,

$$\kappa = \sqrt{\tfrac{EH^2}{12\rho(1-\nu)}}$$

This plate model is to be considered as an isotropic plate and not anisotropic plate, which implies the velocity of wave is equal in same directions[9].

2.1.2 Energy Analysis

Energy analysis is used to check the stability of the system. It helps determine the behaviour of the system. The enerygy analysis for plate over domain D can be calculated using few manupulation. First we take the inner product of the equation w.r.t. u_t

$$\rho H < u_{tt}, u_t >= -D < \Delta\Delta u, u_t > \tag{2.3}$$

The reuslt can be extracted as

$$\frac{dH}{dt} = B_0 \tag{2.4}$$

$$H = \frac{\rho H}{2}\|u_t\|^2 \qquad\qquad B_0 = \frac{-D}{2}\|\Delta u\|^2 \tag{2.5}$$

Thin plate may consists of three valid boundary condition. These are free condition, clamped and simply supported condition. Free condition defines the edges of plates are confined and constrained. Clamped condition defines the boundary of the plate to have zero displacement. Table 2.1 shows the conditions.

In simply supported and clamped condition, when we expand at 0th point, we get a ghost point of -1 which is not in the domain. Thus these conditions help to resolve them.

Clamped	$u = u_x = 0$
Simply Supported	$u = u_{xx} = 0$
Free	$u_0 = u_m = 0$

Table 2.1: Boundary conditions for plate

Simply supported condition:

$$
\begin{aligned}
\delta_{xx}u_0 &= 0 \\
\delta_{xx}u_0 &= \frac{1}{h^2}(u_1 - 2u_0 + u_{-1}) \\
\frac{1}{h^2}(u_1 - 2u_0 + u_{-1}) &= 0 \\
u_{-1} &= 2u_0 - u_1 \\
but, \qquad u_0 &= 0 \\
u_{-1} &= -u_1
\end{aligned}
$$

Clamped condition:

$$
\begin{aligned}
\delta_x u_0 &= 0 \\
\delta_x u_0 &= \frac{1}{2h}(u_1 - u_{-1}) \\
\frac{1}{2h}(u_1 - u_{-1}) &= 0 \\
u_{-1} &= u_1
\end{aligned}
$$

2.1.3 Plate Finite Difference

Equation for finite difference time domain of plate can be discretised as follows using basic operator principle:

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u \\
\rho H \delta_{tt} u &= -D\delta_{\Delta\Delta} u \\
\delta_{tt} u &= -\kappa^2 \delta_{\Delta\Delta} u \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &- \kappa^2 \delta_{\Delta\Delta} u \\
u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u + 2u^n - u^{n-1} \\
u^{n+1} &= -\frac{k^2\kappa^2}{h^4} D_{\Delta\Delta} u + 2u^n - u^{n-1} \\
u^{n+1} &= -\mu^2 D_{\Delta\Delta} u + 2u^n - u^{n-1} \qquad (2.6)
\end{aligned}
$$

here, $\mu = \frac{k\kappa}{h^2}$.

Plate equation of motion can be further derived with inclusion of loss terms in it. Inclusion of damping terms σ_1 and σ_2 governs the motion of equation for plate loss. κ is the stiffness term as used in previous section. Here σ_0 is the generic frequency independent loss whereas σ_1 is frequency dependent loss.

Finite Difference scheme for plate with frequency independent loss:

$$
\begin{aligned}
\rho H \delta_{tt} u^n &= -D\delta_{\Delta\Delta} u - 2\rho H \sigma_0 \delta_{t\cdot} u \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0}{2k} u^{n+1} + \frac{2\sigma_0}{2k} u^{n-1} \\
u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u^n + k^2 \sigma_0 u^{n-1} + 2u^n - u^{n-1} \\
u^{n+1} + k^2\sigma_0 u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u^n + k^2 \sigma_0 u^{n-1} + 2u^n - u^{n-1} \\
(1 + k^2\sigma_0) u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2I) u^n - (1 - k^2\sigma_0) u^{n-1}
\end{aligned} \tag{2.7}
$$

Finite Difference scheme for plate with frequency dependent loss:

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u - 2\rho H \sigma_0 u_t + 2\rho H \sigma_1 \Delta u_t \\
u_{tt} &= -\kappa^2 \Delta\Delta u - 2\sigma_0 u_t + 2\sigma_1 \Delta u_t \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n \\
u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0 k^2}{2k}(u^{n+1} - u^{n+1}) + 2\sigma_1 k^2 \delta_\Delta \delta_{t-} u^n \\
u^{n+1} + \sigma_0 k u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n + 2u^n + 2\sigma_1 k \delta_\Delta u^n + ... \\
&\sigma_0 k u^{n-1} - u^{n-1} - 2\sigma_1 k \delta_\Delta u^{n-1} \\
(I + k\sigma_0 I) u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2\sigma_1 D_\Delta + 2I) u^n - ((1 - k\sigma_0) I + 2\sigma_1 D_\Delta) u^{n-1}
\end{aligned} \tag{2.8}
$$

Now, in order to use the plate system in the coupling system, there is a need for a force term. This force acts as the vibratory motion achieved through string vibrations. It has dirac delta along with it. Dirac delta spreads the force. It actsa s a function of time at appropriate point on plate. The condition is as follow [9]:

$$
\delta(x, x_f) = \begin{cases} \infty, & dS = 0 \\ 0, & dS \neq 0 \end{cases},
$$

$$
\int_D \delta(x - x_f) = 1
$$

But since in discrete space , J which is a spreading function, cannot be considered as infinite point now. It's magnitude needs to be changed to the grid spacing value of $\frac{1}{h^2}$.

$$J(x_i, y_i) = \frac{1}{h_x h_y} \begin{cases} 1, & m = m_i, n = n_i \\ 0, & else \end{cases},$$

Let us take a look at the equations for the plate equation with force term included in it which would be used for coupling system. There isn't much change in the finite difference scheme for this system.

Finite difference scheme without loss, inclusion of force term:

$$\begin{aligned}
u_{tt} &= -\kappa^2 \Delta\Delta u + \frac{1}{\rho_p H}\delta(x - x_f)f \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n + \frac{1}{\rho_p H h_p^2} J f \\
\frac{1}{k^2}(u^{n+1} - 2u + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n + \frac{1}{\rho_p H h_p^2} J f \\
u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u + 2u^n - u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f \\
u^{n+1} &= -\mu^2 D_{\Delta\Delta} u^n + 2u^n - u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f \qquad (2.9)
\end{aligned}$$

Finite difference scheme for frequency dependent loss with force

$$\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u - 2\rho H \sigma_0 u_t + 2\rho H \sigma_1 \Delta u_t + \delta(x - x_f) f(t) \\
u_{tt} &= -\kappa^2 \Delta\Delta u - 2\sigma_0 u_t + 2\sigma_1 \Delta u_t + \frac{1}{\rho_p H}\delta(x - x_f) f(t) \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n + -2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} J f \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} J f \\
u^{n+1} &= k^2(-\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} J f) + 2u^n - u^{n-1} \\
u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0 k^2}{2k}(u^{n+1} - u^{n+1}) + ... \\
&\quad 2\sigma_1 k^2 \delta_\Delta \delta_{t-} u^n + \frac{k^2}{\rho_p H h_p^2} J f + 2u^n - u^{n-1} \\
u^{n+1} + \sigma_0 k u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n + 2u^n + 2\sigma_1 k \delta_\Delta u^n + ... \\
&\quad \sigma_0 k u^{n-1} - u^{n-1} - 2\sigma_1 k \delta_\Delta u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f \\
(I + k\sigma_0 I) u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2\sigma_1 D_\Delta + 2I) u^n - ... \\
&\quad ((1 - k\sigma_0) I + 2\sigma_1 D_\Delta) u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f \qquad (2.10)
\end{aligned}$$

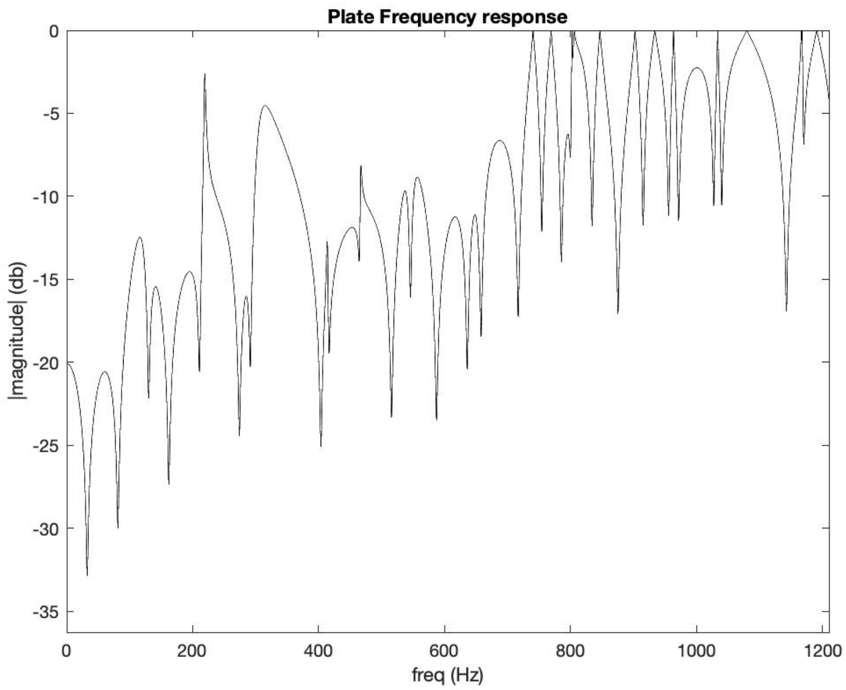

Figure 2.1: Plate frequency response

Figure 2.1 shows the frequency response obtained through scheme 2.10. Frequency response in lower octave seems to be close match to that of NSS[9]. However, increase in the frequency seems to drift the value in frequency spectrum. It also shows the behaviour of plate within the bounds of required behaviour that can be obtained using modal analysis [9].

Figure 2.2 show the displacement of plate at each time sample. It shows a gradual decay in it's displacement which exhibits the real time plate behaviour. It can also imply of no errors in the finite difference scheme implemented for the use of plate. According to NSS[9], the value of two decay terms can be set as

$$\sigma_0 = \frac{6\ln(10)}{\xi(\omega_2)-\xi(\omega_1)}\left(\frac{\xi(\omega_2)}{T_{60}(\omega_1)} - \frac{\xi(\omega_2)}{T_{60}(\omega 2)}\right), \quad \sigma_1 = \frac{6\ln(10)}{\xi(\omega_2)-\xi(\omega_1)}\left(-\frac{1}{T_{60}(\omega_1)} + \frac{1}{T_{60}(\omega 2)}\right)$$

For this we first derive the characteristic equation:

$$w = e^{st}e^{j\beta\eta}, \quad s = \sigma + j\omega$$

$$\sigma(\omega) = -\sigma_0 - \sigma_1\xi(\omega), \quad \xi(\omega) = \frac{-c^2+\sqrt{c^4+\kappa^2\omega^2}}{2\kappa^2}$$

This is the same approach used in Numerical Sound Synthesis [9] book. Here, for plate,

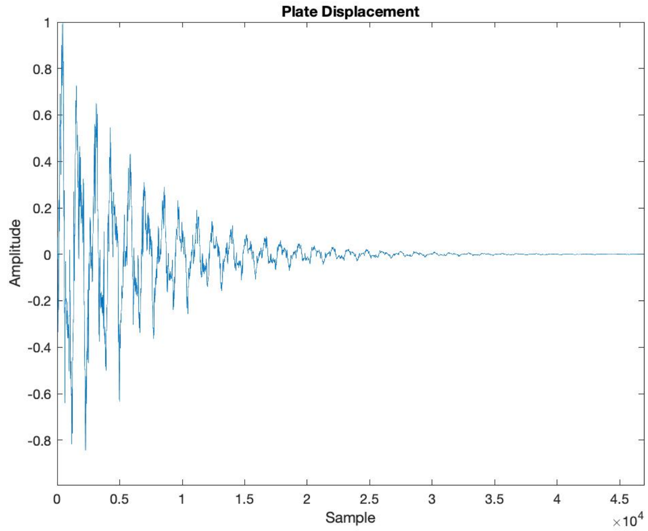

Figure 2.2: Plate Displacement

it can be implied as $\xi(\omega) = \frac{\omega}{\kappa}$

In the below 2.11

$$\begin{aligned}(I + k\sigma_0 I)u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2\sigma_1 D_{\Delta} + 2I)u^n - ... \\ &((1 - k\sigma_0)I + 2\sigma_1 D_{\Delta})u^{n-1} + \frac{k^2}{\rho_p H h_p^2} Jf \end{aligned} \tag{2.11}$$

In this FDTD scheme of plate, if we set σ_1 equals to zero, then the scheme deduces to the generic loss FDTD scheme which is as follows:

$$\begin{aligned} \rho H \delta_{tt} u^n &= -D\delta_{\Delta\Delta} u^n - 2\rho H \sigma_0 \delta_{t\cdot} u^n \\ \delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n \\ \frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0}{2k} u^{n+1} + \frac{2\sigma_0}{2k} u^{n-1} \\ u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u^n + k^2\sigma_0 u^{n-1} + 2u^n - u^{n-1} \\ u^{n+1} + k^2\sigma_0 u^{n+1} &= -k^2\kappa^2 \delta_{\Delta\Delta} u^n + k^2\sigma_0 u^{n-1} + 2u^n - u^{n-1} \\ (1 + k^2\sigma_0)u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2I)u^n - (1 - k^2\sigma_0)u^{n-1} \end{aligned} \tag{2.12}$$

Thus, the stability condition for this scheme would follow as follow[9]:

$$h_p \geq \sqrt{4\sigma_{1p}k + \sqrt{(4\sigma_{1p}k)^2 + 12\kappa_p^2 k^2}}$$

Chapter 3

String-Plate Coupling Theory

This chapter deals with the coupling system of plate and string. In regards to physical modelling of musical instruments, coupling refers to the connection between sub-structures of a system. For example, let us consider a guitar. It has a set of six string which is connected to a bridge. This bridge is then connected to a resonating body like plates,ribs, cavity,etc. Every other substructure vibrating exerts a force on other substructure causing the vibratory motion in that object. Coupling conditions can be calculated using operator or using the equation expansion methods. We will solve using both of them where appropriate.

In this project, connection of string is carried out with plate. All the term related to string will be represented using subscript s and all the terms related to plate will be represented using p. The connection between plate and string is considered as rigid connection. This chapter represents the FDTD scheme working for the coupling system and various problems and solutions related to it. The equation of plate that we will consider is

$$u_{tt} = -\kappa^2 \Delta\Delta u + \frac{1}{\rho_p H}\delta(x - x_f)f \tag{3.1}$$

3.1 String Equations

In real world, the richness of sound obtained through string instrument is not just because of inclusion of string but it is obtained due to the the unique shape of the instrument overall. There are multiple types of string instruments designed with different structures and shapes. These overall properties affects the resulting sound produced through that instrument. Foe example, string and plates are parallel in a piano mechanism whereas strings are suspended through the frame of plate in harp. String consists of restoring force. In case of ideal string, the restoring force is just the tension of the string. There also can be two restoring force in string, one is the tension of the string and other is the stiffness of string. Thus, these strings are called as stiff strings.

3.1.1 1D String

Let us consider an ideal string where there is only a single restoring force.

$$\rho_s A_s w_{tt} = T_s w_{\eta\eta} \tag{3.2}$$

$$w_{tt} = c^2 w_{\eta\eta}, \quad c = \sqrt{\frac{T_s}{\rho_s A}} \tag{3.3}$$

In this equation, w represents the displacement of string, A is the cross sectional area of string, ρ is the density of string, T is the tension of the string and the constant c is the wave speed. The FDTD of the ideal string will be as follows:

$$\begin{aligned} w_{tt} &= c^2 w_{\eta\eta} \\ \delta_{tt} w &= c^2 \delta_{\eta\eta} w \\ \frac{1}{k^2}(w^{n+1} - 2w + w^{n-1}) &= c^2 \delta_{\eta\eta} w \\ w^{n+1} &= \lambda^2 D_{\eta\eta} w + 2w - w^{n-1} \end{aligned} \tag{3.4}$$

Coupling

The total energy in the coupling system must be equal to zero in order to conserve the total energy. Coupling conditions can be derived using the energy analysis of plate and string combined. Energy analysis in continuous form would be as follows:

$$String \quad Equation:$$

$$\rho_s A_s w_{tt} = T_s w_{\eta\eta} \tag{3.5}$$

$$Plate \quad Equation:$$

$$\rho_p H_p u_{tt} = -D\Delta\Delta u + \delta(x - x_f) f(t)$$

$$Taking \quad inner \quad product \quad of \quad both \quad equation:$$

$$\rho_s A_s < w_{tt}, w_t > = T_s < w_{\eta\eta}, w_t > \tag{3.6}$$

$$\rho_p H_p < u_{tt}, u_t > = -D < \Delta\Delta u, u_t > ... \tag{3.7}$$

$$+ < \delta(x - x_f), u_t > f \tag{3.8}$$

$$Adding \quad 3.6 \quad and \quad 3.8, \quad we \quad get$$

$$\frac{dH}{dt} = B$$

$$\frac{d}{dt}\left[\frac{\rho_s A_s}{2}\|w_t\|^2 + \frac{\rho_p H_p}{2}\|u_t\|^2 + \frac{T}{2}\|w_\eta\|^2 + \frac{D}{2}\|\Delta u\|^2\right] = T_s[w_t w_\eta]_0^L + < \delta(x - x_f), u_t > \tag{3.9}$$

Since the energy must be conserved , B=0, and the string is connected to only one end, the other end i.e.$L = 0$ hence:

$$T_s[\dot{w}w']_0^L + < \delta(x - x_f), u_t >= 0 \tag{3.10}$$

$$T\dot{w}_L w'_L - T\dot{w}_0 w'_0 + < \delta(x - x_f), u_t >= 0 \tag{3.11}$$

but

$$T\dot{w}_L w'_L = 0 \tag{3.12}$$

$$-T\dot{w}_0 w'_0 + < \delta(x - x_f), u_t >= 0 \tag{3.13}$$

$$T\dot{w}_0 w'_0 =< \delta(x - x_f), u_t > \tag{3.14}$$

The boundary condition obtained through equation 3.14 are :

$$f = T_s w_{\eta(0)} \tag{3.15}$$

$$w_{t(0)} = J u_t \tag{3.16}$$

The same energy analysis can be carried out in discrete time using summation by parts and following boundary condition is being obtained for it:

$$f = T_s w_0 \tag{3.17}$$

$$\delta_{t\cdot} w_0 = < \delta(x - x_f), \delta_t u > \tag{3.18}$$

Now taking inner product of plate equation w.r.t. J, we get:

$$\rho_p H_p < u_{tt}, J >= - D < \Delta\Delta u, J > + < \delta(x - x_f), J > f \tag{3.19}$$

But since according to 3.18, velocity is same, hence the acceleration will also be constant Therefore we can say that,

$$< \delta_{tt} u, J >= \delta_{tt} w_0 \tag{3.20}$$

$$but \quad \delta_{tt} w_0 = c^2 \delta_{\eta\eta} w_0 \tag{3.21}$$

Hence substituting 3.21 in 3.20 and thus substituting that in 3.19, we get;

$$\rho_p H_p c^2 \delta_{\eta\eta} w_0 = -D\delta_{\Delta\Delta} u + \|J\|^2 f \tag{3.22}$$

$$rearranging \quad terms \quad we \quad get$$

$$f = \left[\frac{\rho_p H_p T_s}{\rho_s A} \delta_{\eta\eta} w_0 + \kappa^2 \delta_{\Delta\Delta} u \right] J \tag{3.23}$$

Thus this is the required force term which is the coupling force. It is used in string as well as plate equation for the coupling force update to generate couple system for ideal string and plate system.

Force

The operator J in the equation of motion just helps to locate the grid point foe string displacement vector w. It can be either a full raised cosine or half raised cosine that will help in simulation for plucks. The important part is that while applying the force, there will be a term $\frac{k^2}{\rho_s A_s h}$ in front of force. The force applied for the string can be assumed as a series of vectors for each string, containing full or half raised cosine at required points of excitation. One possible problem can be the use of two excitation overlapping the same string. It is also important that the excitation is not over written. Overwritten or overlapping might create a sudden pluck.

3.1.2 Stiff String

Until now, we only considered coupling of string which has only single restoring force and that was tension. Now we consider two restoring force i.e. tension as wella s stiffness in the string. These restoring forces are derived from ideal string equation and beam equation. Thus the equation of motion for stiff string without loss is:

$$\rho_s A_s w_{tt} = T_s w_{\eta\eta} - E_s I_s w_{\eta\eta\eta} \tag{3.24}$$

$$w_{tt} = \frac{T}{\rho_s A_s} w_{\eta\eta} - \frac{E_s I_s}{\rho_s A_s} w_{\eta\eta\eta\eta} \tag{3.25}$$

$$w_{tt} = c^2 w_{\eta\eta} - \kappa^2 w_{\eta\eta\eta\eta} \tag{3.26}$$

here,

$$c = \sqrt{\frac{T_s}{\rho_s A_s}}$$

and

$$\kappa = \sqrt{\frac{E_s I_S}{\rho_s A_s}}$$

Finite difference scheme for the stiff string without loss is as follow:

$$
\begin{aligned}
w_{tt} =& c^2 w_{\eta\eta} - \kappa^2 w_{\eta\eta\eta\eta} \\
\delta_{tt} w =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w \\
\frac{1}{k^2}(w^{n+1} - 2w^n + w^{n-1}) =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w \\
w^{n+1} - 2w^n + w^{n-1} =& c^2 k^2 \delta_{\eta\eta} w - \kappa^2 k^2 \delta_{\eta\eta\eta\eta} w \\
w^{n+1} =& c^2 k^2 \delta_{\eta\eta} w - \kappa^2 k^2 \delta_{\eta\eta\eta\eta} w + 2w^n - w^{n-1} \\
w^{n+1} =& \frac{c^2 k^2}{h^2} D_{\eta\eta} w - \frac{\kappa^2 k^2}{h^4} D_{\eta\eta\eta\eta} w + 2w^n - w^{n-1} \\
w^{n+1} =& (\lambda^2 D_{\eta\eta} - \mu^2 D_{\eta\eta\eta\eta} + 2) w^n - w^{n-1} \qquad (3.27)
\end{aligned}
$$

Finite difference scheme for stiff string with loss

$$
w_{tt} = c^2 w_{\eta\eta} - \kappa^2 w_{\eta\eta\eta\eta} - 2\sigma_0 w_t + 2\sigma_1 w_{t\eta\eta}
$$

here,

$$
c = \sqrt{\frac{T_s}{\rho_s A_s}}
$$

,

$$
\kappa = \sqrt{\frac{E_s I_S}{\rho_s A_s}}
$$

and

$$
I_s = \frac{\pi r_s^4}{4}
$$

$$
\begin{aligned}
\delta_{tt} w =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w - 2\sigma_0 \delta_t w + 2\sigma_1 \delta_{t\eta\eta} w \\
\frac{1}{k^2}(w^{n+1} - 2w^n + w^{n-1}) =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w - 2\sigma_0 \delta_{t\cdot} w + 2\sigma_1 \delta_{t-} \delta_{\eta\eta} w \\
(1 + \sigma_0 k) w^n + 1 =& w^n \Big(\lambda^2 D_{\eta\eta} - \mu^2 D_{\eta\eta\eta\eta} + \frac{2k\sigma_1}{h^2} D_{\eta\eta} + 2 \Big) ... \\
& w^{n-1} \Big((1 - k\sigma_0) + \frac{2k\sigma_1}{h^2} D_{\eta\eta} \Big) \qquad (3.28)
\end{aligned}
$$

Coupling

As the total energy must be conserved, we first find the energy analysis of total system of plate and string and then with the help of energy analysis, we solve the boundary condition which leads to the force term and this force term is to be replaced in plate/string FDTD. Let us consider the two equation of motion :

$$\begin{aligned} &String \quad Equation: \\ &\rho_s A_s w_{tt} = T_s w_{\eta\eta} - E_s I_s w_{\eta\eta\eta\eta} \qquad (3.29) \\ &Plate \quad Equation: \\ &\rho_p H_p u_{tt} = -D\Delta\Delta u + \delta(x - x_f) f(t) \end{aligned}$$

$$\begin{aligned} &Taking \quad inner \quad product \quad of \quad both \quad equation: \\ &\rho_s A_s < w_{tt}, w_t > = T_s < w_{\eta\eta}, w_t > ... \\ &\qquad - E_s I_s < w_{\eta\eta\eta\eta}, w_t > \qquad (3.30) \\ &\rho_p H_p < u_{tt}, u_t > = -D < \Delta\Delta u, u_t > ... \\ &\qquad + < \delta(x - x_f), u_t > f \end{aligned}$$

Adding 3.30 and 3.31, we get

$$\frac{dH}{dt} = T_s \left[w_t w_\eta \right]_0^L - E_s I_s \left[w_t w_{\eta\eta\eta} \right]_0^L + E_s I_s \left[w_{t\eta} w_{\eta\eta} \right]_0^L + < \delta(x - x_f), u_t > f \qquad (3.31)$$

Extracting boundary conditions from energy analysis, we get the following terms in continuous form:

$$\partial_\eta^2 w_0 = 0 \qquad (3.32)$$

$$< \delta(x - x_f), u_t > = w_t \qquad (3.33)$$

$$(T_s \partial_\eta - E_s I_s \partial_\eta^3) w_0 = f \qquad (3.34)$$

Now, we take the FDTD scheme for discrete form:
Stiff String

$$\delta_{tt} w = c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w \qquad (3.35)$$

Plate:

$$\rho H \delta_{tt} u^n = -D \delta_{\Delta\Delta} u^n + \delta(x - x_f) f \qquad (3.36)$$

And the boundary conditions obtained through energy analysis using summation by parts are as follows:

$$\delta_{\eta\eta} w_0 = 0 \qquad (3.37)$$

$$< \delta(x - x_f), \delta_t u > = \delta_t w_0 \qquad (3.38)$$

$$(T_s \delta_\eta - E_s I_s \delta_{\eta\eta\eta}) w_0 = f \qquad (3.39)$$

Now using three conditions obtained, we can calculate two ghost points and force term. Equation3.37 will be used to solve the value for ghost point of w_{-1}. Equation 3.39 will be used to get the value of ghost pointw_{-2}, however there can be an alternative way to get rid of 4th order term using operator method i.e. change in stiff string equation by splitting the 4th order term. And equation 3.38 will be used to obtain force term. First let us obtain the force term. For this we take inner product of either equation w.r.t. spreading vector J. Here we take plate equation:

$$\rho_p H_p < \delta_{tt} u, J >= - D < \delta_{\Delta\Delta} u, J > + < \delta(x - x_f), J > f \tag{3.40}$$

Therefore, using 3.38

$$< \delta_{tt} u, J >= \delta_{tt} w_0 \tag{3.41}$$

But we know that,

$$\delta_{tt} w_0 = c^2 \delta_{\eta\eta} w_0 - \kappa^2 \delta_{\eta\eta\eta\eta} w_0 \tag{3.42}$$

Now, substituting equation 3.42 in 3.41 and then into 3.40, we get,

$$\rho_p H_p \Big[c^2 \delta_{\eta\eta} w_0 - \kappa^2 \delta_{\eta\eta\eta\eta} w_0 \Big] = - D \delta_{\Delta\Delta} u + \|J\|^2 f \tag{3.43}$$

$$\rho_p H_p \left[\frac{T_s}{\rho_s A_s} \delta_{\eta\eta} w_0 - \frac{E_s I_s}{\rho_s A_s} \delta_{\eta\eta\eta\eta} w_0 \right] = - D \delta_{\Delta\Delta} u + \|J\|^2 f \tag{3.44}$$

$$\tag{3.45}$$

Rearranging the terms and setting the term$\frac{T_s}{\rho_s A_s} \delta_{\eta\eta} w_0 = 0$ according to 3.37, we get

$$\rho_p H_p \Big[\frac{E_s I_s}{\rho_s A_s} \delta_{\eta\eta\eta\eta} w_0 \Big] = -D \delta_{\Delta\Delta} u + \frac{f}{h^2} \tag{3.46}$$

$$\Big[\frac{E_s I_s}{\rho_s A_s} \delta_{\eta\eta\eta\eta} w_0 \Big] = \frac{-D}{\rho_p H_p} \delta_{\Delta\Delta} u + \frac{f}{\rho_p H_p h^2} \tag{3.47}$$

$$-\frac{E_s I_s}{\rho_s A_s} \Big(\frac{w_{-2} - 4w_{-1} + 6w_0 - 4w_1 + w_2}{h^4} \Big) = \frac{-D}{\rho_p H_p} \delta_{\Delta\Delta} u + \frac{f}{\rho_p H_p h^2} \tag{3.48}$$

Now, here in this equation we have two ghost point on the left hand side of equation $w_{-1} and w_{-2}$, so now let us find the value of these two ghost point in order to get a full working value for force term. Using condition 3.37

$$\delta_{\eta\eta} w_0 = 0$$

$$\frac{w_{-1} - 2w_0 + w_2}{h^2} = 0$$

$$\boxed{w_{-1} = 2w_0 - w_1} \tag{3.49}$$

This is the value for our first ghost point.
Now using 3.39, we can find ghost point value of w_{-2}

$$(T_s\delta_\eta - E_sI_s\delta_{\eta\eta\eta})w_0 = f$$

$$T_s\left(\frac{w_0 - w_{-1}}{h}\right) - E_sI_s\left(\frac{-w_{-2} + 3w_{-1} - 3w_0 + w_1}{h^3}\right) = f$$

$$f - \frac{T_s}{h}\big(w_0 - (2w_0 - w_1)\big) + (3w_{-1} - 3w_0 + w_1)\frac{E_sI_s}{h^3} = \frac{E_sI_s}{h^3}(w_{-2})$$

$$f - \frac{T_s}{h}(w_0 - 2w_0 + w_1) + \frac{E_sI_s}{h^3}\Big(3(2w_0 - w_1) - 3w_0 + w_1\Big) = \frac{E_sI_s}{h^3}(w_{-2})$$

$$\boxed{w_{-2} = \frac{h^3 f}{E_sI_s} - \frac{h^2T_s}{E_sI_s}(-w_0 + w_1) + (3w_0 - 2w_1)} \tag{3.50}$$

Substituting 3.50 and 3.49 in 3.48, we obtain the force equation as follows:

$$-\frac{E_sI_s}{\rho_sA_s}\left(\frac{w_{-2} - 4w_{-1} + 6w_0 - 4w_1 + w_2}{h^4}\right) = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2}$$

$$-\frac{E_sI_s}{\rho_sA_sh^4}\left[\frac{h^3f}{E_sI_s} - \frac{h^2T_s}{E_sI_s}(-w_0+w_1) + 3w_0 - 2w_1 - 4(2w_0 - w_1) + 6w_0 - 4w_1 + w_2\right] = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2}$$

$$-\frac{E_sI_s}{\rho_sA_sh^4}\left[\frac{h^3f}{E_sI_s} - \frac{h^2T_s}{E_sI_s}(-w_0+w_1) + w_0 - 2w_1 + w_2\right] = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2}$$

$$-\frac{E_sI_s}{\rho_sA_sh^4}\left[\frac{h^3f}{E_sI_s} + \frac{h^2T_s}{E_sI_s}(w_0) - \frac{h^2T_s}{E_sI_s}(w_1) + w_0 - 2w_1 + w_2\right] = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2}$$

$$-\frac{E_sI_s}{\rho_sA_sh^4}\left[\frac{h^3f}{E_sI_s} + \left(\frac{h^2T_s}{E_sI_s} + 1\right)(w_0) - \left(\frac{h^2T_s}{E_sI_s} + 2\right)(w_1) + w_2\right] = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2}$$

$$-\frac{\cancel{E_sI_sh^3}}{\rho_sA_s\cancel{h^4}\cancel{E_sI_s}}f = \frac{-D}{\rho_pH_p}\delta_{\Delta\Delta}u + \frac{f}{\rho_pH_ph^2} + \frac{E_sI_s}{\rho_sA_sh^4}\left[\left(\frac{h^2T_s}{E_sI_s} + 1\right)(w_0) - \left(\frac{h^2T_s}{E_sI_s} + 2\right)(w_1) + w_2\right]$$

$$\frac{f}{\rho_sA_sh_s} + \frac{f}{h_p^2\rho_pH_p} = \frac{D}{\rho_pH_p}\delta_{\Delta\Delta} - \frac{E_sI_s}{\rho_sA_sh^4}\left[\left(\frac{h^2T_s}{E_sI_s} + 1\right)(w_0) - \left(\frac{h^2T_s}{E_sI_s} + 2\right)(w_1) + w_2\right]$$

$$f\left(\frac{1}{\rho_sA_sh_s} + \frac{1}{h_p^2\rho_pH_p}\right) = \frac{D}{\rho_pH_p}\delta_{\Delta\Delta} - \frac{E_sI_s}{\rho_sA_sh^4}\left[\left(\frac{h^2T_s}{E_sI_s} + 1\right)(w_0) - \left(\frac{h^2T_s}{E_sI_s} + 2\right)(w_1) + w_2\right]$$

Let

$$\mathcal{M} = \frac{\rho_sA_sh_s + h_p^2\rho_pH_p}{(\rho_sA_sh_s)(h_p^2\rho_pH_p)}$$

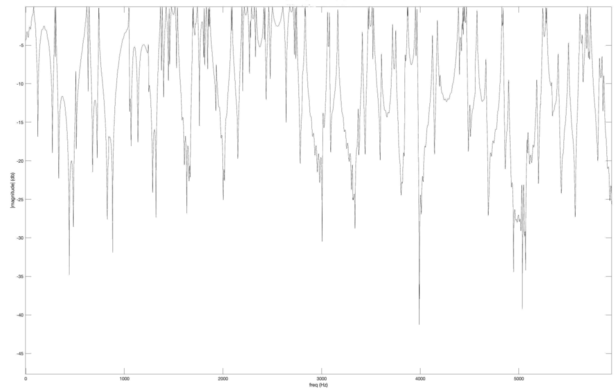

Figure 3.1: Total energy of coupled system

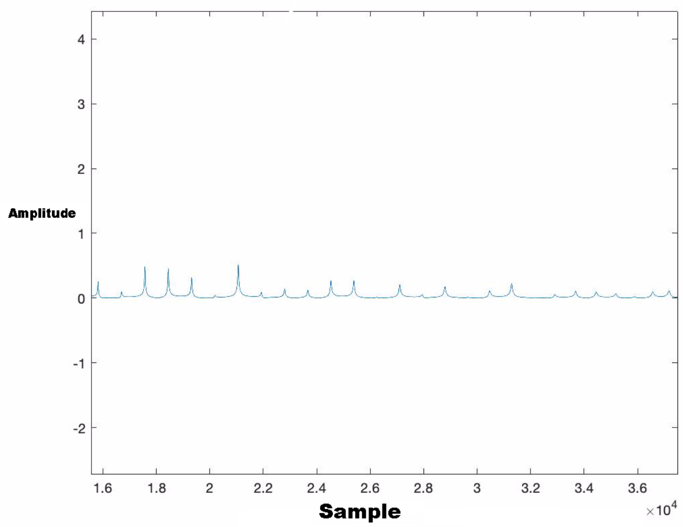

Figure 3.2: Difference in energy of total coupled system conserved and is approximately equal to zero.

$$\boxed{f = \mathcal{M}\left[\frac{D}{\rho_p H_p}\delta_{\Delta\Delta} - \frac{E_s I_s}{\rho_s A_s h^4}\left(\left(\frac{h^2 T_s}{E_s I_s} + 1\right)(w_0) - \left(\frac{h^2 T_s}{E_s I_s} + 2\right)(w_1) + w_2\right)\right]} \tag{3.51}$$

Thus, the force resolved now can be used to the updated FDTD equation. This force will be the coupling force acting in the system. However, in order to skip the step for finding w_{-2}, The original FDTD can be manipulated as follows using operator method:

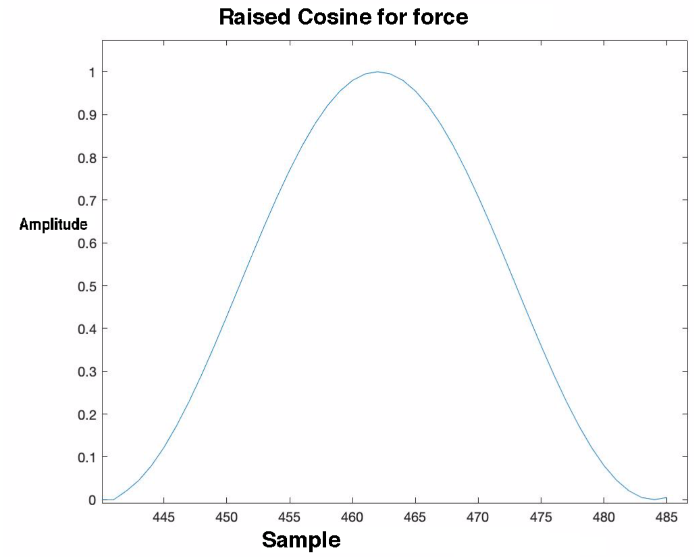

Figure 3.3: Difference in energy of total coupled system conserved and is approximately equal to zero

$$
\begin{array}{rcl}
\rho_s A_s \delta_{tt} w_0 & = & T_s \delta_{\eta\eta} w_0 - E_s I_s \delta_{\eta\eta\eta\eta} w_0 \\
\rho_s A_s \delta_{tt} w_0 & = & \frac{2T_s}{h}(\delta_{\eta\cdot} - \delta_{\eta-}) w_0 - E_s I_s \Big(\frac{2}{h}(\delta_{\eta\cdot} - \delta_{\eta-}) \delta_{\eta\eta}\Big) w_0 \\
\rho_s A_s \delta_{tt} w_0 & = & \frac{2T_s}{h}\delta_{\eta\cdot} w_0 - \frac{2T_s}{h}\delta_{\eta-} w_0 - \frac{2E_s I_s}{h}(\delta_{\eta\cdot}\delta_{\eta\eta}) w_0 - \frac{2E_s I_s}{h}\delta_{\eta-}\delta_{\eta\eta} w_0 \\
\rho_s A_s \delta_{tt} w_0 & = & \frac{2}{h}(T_s \delta_{\eta\cdot} w_0 - E_s I_s(\delta_{\eta\cdot}\delta_{\eta\eta}) w_0 - (T_s \delta_{\eta-} - E_s I_s \delta_{\eta-}\delta_{\eta\eta}) w_0) \\
But, \quad f & = & T_s \delta_{\eta-} w_0 - E_s I_s \delta_{\eta\eta\eta} w_0 \\
\rho_s A_s \delta_{tt} w_0 & = & \frac{2}{h}(T_s \delta_{\eta\cdot} w_0 - E_s I_s(\delta_{\eta\cdot}\delta_{\eta\eta} w_0) - (T_s \delta_{\eta-} - E_s I_s \delta_{\eta-}\delta_{\eta\eta}) w_0) \\
\rho_s A_s \delta_{tt} w_0 & = & \frac{2}{h}(T_s \delta_{\eta\cdot} w_0 - E_s I_s(\delta_{\eta\cdot}\delta_{\eta\eta}) w_0 - f)
\end{array}
\tag{3.52}
$$

The energy change i.e. $H(n) - H^0$ of the total system shows that the energy level is approximately equal to zero. Figure3.2 represents it statistical. The value tends to contain relative error which causes variation in change of energy. In order to simulate the string, external force will be used for either struck or pluck mechanism. A full hann window of raised cosine3.3 will be used for struck mechanism and half hann window will be used for pluck mechanism.

Chapter 4

Conclusion

The work in this project has presented methods and steps for string-plate connection which is the result of force from both string and the plate with applications to sound synthesis. Model of string-plate has been formulated using FDTD method. Initially, the methodology was experimented with ideal string and then used on stiff string too. One of the key feature to note is that inversion does not exists, hence stable explicit update equation exists for the total system. Numerical simulations for string-plate system verifies the stable model and that the methodology followed models correctly. Resonating body such as thin plate was modelled in Chapter 2. It has its applications in virtual piano instrument.

In real string instrument, the richness of sound and unique timber is due to the shape and size of instrument. Various parameters of the instrument can be altered digitally to obtain the required sound. Change in the material and its property is also achievable using numeric approach to sound synthesis. This project approximates only for the soundboard of piano. However inclusion of multiple substructure can increase the quality of its sound. Substructures like bridge, additional beam,etc can make it sound more natural.

4.1 Extension

A natural extension can be carried out for this project is connection of multiple strings with plate. Multiple string can be coupled with each other in order to obtain a more realistic system like piano that can be coupled with plate. These type of system tends to have adverse effect on bridge or body that is connected to it depending on the configuration of the instrument.
Another extension yo this project would be study of geometric non-linearities in string. This could be applicable for in depth work for non-linear coupling of transverse and longitudinal motion throughout the string. This would introduce a phenomena of

changeover and intonation of tension in the string.

Until now the system has been modelled using ideal conditions like simple shape such as thin plate with rectangular shape and beam. However, synthesis for such shapes can be derived and implemented analytically, but these shapes which are considered as ideal are not realistic. Use of finite element method for such system can give realistic result for complex geometric shapes.

String instruments are generally mounted on bridge, specifically guitar instrument.This is then indeed connected to a resonating body like plate. In order to synthesis and model such system, strings need to be coupled with bridge and then with resonating body at all points. This makes the coupling conditions to be modified and reformulated in such cases. Another possible extension of this project would be to consider application to specific instrument like Harp. This could be further advanced in more realistic mechanism of excitation such as finger string interaction.

Appendix A

Finite Difference Scheme

A.1 Plate FDTD

A.1.1 Plate without loss

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u \\
\rho H \delta_{tt} u &= -D\delta_{\Delta\Delta} u \\
\delta_{tt} u &= -\kappa^2 \delta_{\Delta\Delta} u \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &- \kappa^2 \delta_{\Delta\Delta} u \\
u^{n+1} &= -k^2\kappa^2\delta_{\Delta\Delta} u + 2u^n - u^{n-1} \\
u^{n+1} &= -\frac{k^2\kappa^2}{h^4} D_{\Delta\Delta} u + 2u^n - u^{n-1} \\
u^{n+1} &= -\mu^2 D_{\Delta\Delta} u + 2u^n - u^{n-1}
\end{aligned} \tag{A.1}
$$

A.1.2 Plate without loss(force)

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u + \delta(x - x_f) f \\
u_{tt} &= -\kappa^2 \Delta\Delta u + \frac{1}{\rho_p H}\delta(x - x_f) f \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n + \frac{1}{\rho_p H h_p^2} J f \\
\frac{1}{k^2}(u^{n+1} - 2u + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n + \frac{1}{\rho_p H h_p^2} J f \\
u^{n+1} &= -k^2\kappa^2\delta_{\Delta\Delta} u + 2u^n - u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f \\
u^{n+1} &= -\mu^2 D_{\Delta\Delta} u^n + 2u^n - u^{n-1} + \frac{k^2}{\rho_p H h_p^2} J f
\end{aligned} \tag{A.2}
$$

A.1.3 Plate with frequency dependent loss

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u - 2\rho H\sigma_0 u_t + 2\rho H\sigma_1 \Delta u_t \\
u_{tt} &= -\kappa^2 \Delta\Delta u - 2\sigma_0 u_t + 2\sigma_1 \Delta u_t \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n \\
u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0 k^2}{2k}(u^{n+1} - u^{n+1}) + 2\sigma_1 k^2 \delta_\Delta \delta_{t-} u^n \\
u^{n+1} + \sigma_0 k u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n + 2u^n + 2\sigma_1 k \delta_\Delta u^n + ... \\
&\quad \sigma_0 k u^{n-1} - u^{n-1} - 2\sigma_1 k \delta_\Delta u^{n-1} \\
(I + k\sigma_0 I)u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2\sigma_1 D_\Delta + 2I)u^n - ((1 - k\sigma_0)I + 2\sigma_1 D_\Delta)u^{n-1}
\end{aligned}
\tag{A.3}
$$

A.1.4 Plate with frequency dependent loss (Force)

$$
\begin{aligned}
\rho H u_{tt} &= -D\Delta\Delta u - 2\rho H\sigma_0 u_t + 2\rho H\sigma_1 \Delta u_t + \delta(x - x_f)f(t) \\
u_{tt} &= -\kappa^2 \Delta\Delta u - 2\sigma_0 u_t + 2\sigma_1 \Delta u_t + \frac{1}{\rho_p H}\delta(x - x_f)f(t) \\
\delta_{tt} u^n &= -\kappa^2 \delta_{\Delta\Delta} u^n + -2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} Jf \\
\frac{1}{k^2}(u^{n+1} - 2u^n + u^{n-1}) &= -\kappa^2 \delta_{\Delta\Delta} u - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} Jf \\
u^{n+1} &= k^2(-\kappa^2 \delta_{\Delta\Delta} u^n - 2\sigma_0 \delta_{t\cdot} u^n + 2\sigma_1 \delta_\Delta \delta_{t-} u^n + \frac{1}{\rho_p H h_p^2} Jf) + 2u^n - u^{n-1} \\
u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n - \frac{2\sigma_0 k^2}{2k}(u^{n+1} - u^{n+1}) + ... \\
&\quad 2\sigma_1 k^2 \delta_\Delta \delta_{t-} u^n + \frac{k^2}{\rho_p H h_p^2} Jf + 2u^n - u^{n-1} \\
u^{n+1} + \sigma_0 k u^{n+1} &= -\kappa^2 k^2 \delta_{\Delta\Delta} u^n + 2u^n + 2\sigma_1 k \delta_\Delta u^n + ... \\
&\quad \sigma_0 k u^{n-1} - u^{n-1} - 2\sigma_1 k \delta_\Delta u^{n-1} + \frac{k^2}{\rho_p H h_p^2} Jf \\
(I + k\sigma_0 I)u^{n+1} &= (-\mu^2 D_{\Delta\Delta} + 2\sigma_1 D_\Delta + 2I)u^n - ... \\
&\quad ((1 - k\sigma_0)I + 2\sigma_1 D_\Delta)u^{n-1} + \frac{k^2}{\rho_p H h_p^2} Jf
\end{aligned}
\tag{A.4}
$$

A.2 String FDTD

A.2.1 Ideal String

$$\begin{aligned}
\rho_s A_s w_{tt} &= T_s w_{\eta\eta} \\
w_{tt} &= c^2 w_{\eta\eta} \\
\delta_{tt} w &= c^2 \delta_{\eta\eta} w \\
\frac{1}{k^2}(w^{n+1} - 2w + w^{n-1}) &= c^2 \delta_{\eta\eta} w \\
w^{n+1} &= \lambda^2 D_{\eta\eta} w + 2w - w^{n-1}
\end{aligned} \tag{A.5}$$

here,

$$\lambda = \frac{kc}{h^2}$$

A.2.2 Stiff String

$$\begin{aligned}
\rho_s A_s w_{tt} =& T_s w_{\eta\eta} - E_s I_s w_{\eta\eta\eta} \\
w_{tt} =& \frac{T}{\rho_s A_s} w_{\eta\eta} - \frac{E_s I_s}{\rho_s A_s} w_{\eta\eta\eta\eta} \\
w_{tt} =& c^2 w_{\eta\eta} - \kappa^2 w_{\eta\eta\eta\eta} \\
\delta_{tt} w =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w \\
\frac{1}{k^2}(w^{n+1} - 2w^n + w^{n-1}) =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w \\
w^{n+1} - 2w^n + w^{n-1} =& c^2 k^2 \delta_{\eta\eta} w - \kappa^2 k^2 \delta_{\eta\eta\eta\eta} w \\
w^{n+1} =& c^2 k^2 \delta_{\eta\eta} w - \kappa^2 k^2 \delta_{\eta\eta\eta\eta} w + 2w^n - w^{n-1} \\
w^{n+1} =& \frac{c^2 k^2}{h^2} D_{\eta\eta} w - \frac{\kappa^2 k^2}{h^4} D_{\eta\eta\eta\eta} w + 2w^n - w^{n-1} \\
w^{n+1} =& (\lambda^2 D_{\eta\eta} - \mu^2 D_{\eta\eta\eta\eta} + 2) w^n - w^{n-1}
\end{aligned} \tag{A.6}$$

A.2.3 Stiff string with loss

$$\begin{aligned}
w_{tt} =& c^2 w_{\eta\eta} - \kappa^2 w_{\eta\eta\eta\eta} - 2\sigma_0 w_t + 2\sigma_1 w_{t\eta\eta} \\
\delta_{tt} w =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w - 2\sigma_0 \delta_t w + 2\sigma_1 \delta_{t\eta\eta} w \\
\frac{1}{k^2}(w^{n+1} - 2w^n + w^{n-1}) =& c^2 \delta_{\eta\eta} w - \kappa^2 \delta_{\eta\eta\eta\eta} w - 2\sigma_0 \delta_{t\cdot} w + 2\sigma_1 \delta_{t-} \delta_{\eta\eta} w \\
(1 + \sigma_0 k) w^n + 1 =& w^n \Big(\lambda^2 D_{\eta\eta} - \mu^2 D_{\eta\eta\eta\eta} + \frac{2k\sigma_1}{h^2} D_{\eta\eta} + 2 \Big) ... \\
& w^{n-1} \Big((1 - k\sigma_0) + \frac{2k\sigma_1}{h^2} D_{\eta\eta} \Big)
\end{aligned} \tag{A.7}$$

Appendix B

MATLAB Code

B.1 Kirchhoff Plate Model using Finite Differenc Time Domain

```
% --------------------------------------------------------------------------%
% A Kirchhoff Thin Plate FDTD Model with and without loss
%Author:Mangesh Chandrakant Sonawane
%MSc Acoustic and Music Technology dissertation project
% --------------------------------------------------------------------------%
clear all
close all

% boundary condition type: 1: simply supported, 2: clamped
bctype = 1;
% output type: 1: displacement, 2: velocity
outtype = 1;
% loss type: 0: independent, 1: independent 2: Freq dependent
losstype = 1;

% --------------------------------------------------------------------------%
                    % Parameters for simulation
% --------------------------------------------------------------------------%
% sample rate (Hz)
SR = 44100;
% duration
Tf = 10;
```

```
% number of time steps
Nf = floor(SR*Tf);
% Poisson Ratios (< .5)
poissnratio = 0.5;
%excitation (normalised)
exc = [0.45, 0.45];
wi = 0.2;
%initial displacement
u0 = 0;
%initial velocity
v0 = 1;
% readout position as percentage
rp = [.45, .65];
%
    -------------------------------------------------------------------------%

                        % physical parameters
%
    -------------------------------------------------------------------------%

% Young's modulus
E = 2e11;
% density (kg/m^3)
rho = 7850;
% thickness (m)
H = .005;
% plate length (m)
L = .9;
% plate length X axis (m)
Lx = .7;
% plate length Y axis (m)
Ly = .7;
 % loss [freq.(Hz), T60;...]
loss = [500, 2; 1500, 1];
 %  Motion Coefficients
D = (E*(H)^3)/(12*(1-(poissnratio^2)));
kappa = sqrt(D / (rho*  H) );
%Time step
k = 1/SR;
%Spacing
hmin = 2*sqrt(k*kappa);
N = floor(L./hmin);
h = L./(N);
%Nx = floor(Lx/hmin)-1
%Ny =floor(Ly/hmin)-1
%h=sqrt(Lx*Ly)/sqrt(Nx*Ny);
mu = (kappa * k)/(h^2);
%Updated value
N = N+1;
```

```
% total grid size.
ss = N*N;
% ------------------------------------------------------------------------%
                        % Loss coefficients
% ------------------------------------------------------------------------%
if losstype ==0
  % no loss
  sigma0 = 0;
  sigma1 = 0;
end

if losstype ==1
  % frequency independant loss
  sigma0 = 6*log(10)/loss(1,2);
  sigma1 = 0;
end

if losstype == 2
  %frequency  dependent
  z1 = 2*kappa*(2*pi*loss(1,1))/(2*kappa.^2);
  z2 = 2*kappa*(2*pi*loss(2,1))/(2*kappa.^2);

  sigma0 = 6*log(10)*(-z2/loss(1,2) + z1/loss(2,2))./(z1-z2);
  sigma1 = 6*log(10)*(1/loss(1,2) - 1/loss(2,2))./(z1-z2);
end
% ------------------------------------------------------------------------%
                          % Read In/Out
% ------------------------------------------------------------------------%
lo = rp*N;
lo = floor(sub2ind([N N],lo(1), lo(2)));
li = exc*N;
li = floor(sub2ind([N N],li(1), li(2)));
% ------------------------------------------------------------------------%
                        % Create Force Signal
% ------------------------------------------------------------------------%
[X,Y] = meshgrid([1:N]*h, [1:N]*h);
% distance of points from excitation
```

```
dist = sqrt((X-(exc(1)*L)).^2 + (Y-(exc(2)*L)).^2); %NSS
ind = sign(max(-dist+(wi*0.5),0));
rc = 0.5*ind.*(1+cos(2*pi*dist/wi));        %NSS
rc = rc(:);
%--------------------------------------------------------------------------%
                              %operators
%--------------------------------------------------------------------------%
biharmonic_ope = biharmonic(N,N,bctype); % biharmonic matrix
laplacian_ope = laplacian(N,N,bctype);  % Laplacian matrix
%--------------------------------------------------------------------------%
                        % Coefficient Matrices
%--------------------------------------------------------------------------%
A = (1/(1+k*sigma0))*speye(ss);
B = (-(mu^2)*biharmonic_ope + (2*sigma1*k/(h^2))*laplacian_ope + 2*speye(ss)) * A;
C = (-(2*sigma1*k/(h^2))*laplacian_ope - (1-sigma0*k)*speye(ss))  * A;
%--------------------------------------------------------------------------%
                            % initialise output
%--------------------------------------------------------------------------%
u2 = u0*rc;
u1 = (u0+(k*v0))*rc;
u  = u2;
y = zeros(Nf,1);
%--------------------------------------------------------------------------%
                                 % Main Calculation
%--------------------------------------------------------------------------%

  for n = 1:Nf

    % main operation
    u = B*u1 + C*u2;
    % read output
    if (outtype==1)
```

```
      y(n,:) = u(lo);
    elseif (outtype==2)
      y(n,:) = (SR*(u(lo)-u1(lo)));
    end
     u2 = u1; u1 = u;
  end

y= y/max(abs(y));

figure
%       yfft = 10*log10(abs(fft(y)));
%       plot([0:Nf-1]'/Nf*SR,-abs(yfft), 'k')
%       xlabel('freq (Hz)')
%       ylabel('|magnitude| (db)');
%       title('Plate Frequency response')
plot(y)
      xlabel('Sample');
      ylabel('Amplitude');
      title('Plate Displacement');
soundsc(y,SR)
audiowrite('Plate_Loss.wav', y,SR)
%
    --------------------------------------------------------------------------%
```

B.2 String-Plate Connection

```
% -------------------------------------------------------------------------%
% FDTD Model for string plate connection. The connection here is assumed
% to be rigid with and without loss.
%Author:Mangesh Chandrakant Sonawane
%MSc Acoustic and Music Technology dissertation project
% -------------------------------------------------------------------------%

% -------------------------------------------------------------------------%
                                % Flags
% -------------------------------------------------------------------------%

 bctype_p = 1; %1: simply supported, 2: clamped
 bctype_S = 1; %1: simply supported, 2: clamped
 outtype=1;    % output type: 1: displacement, 2: velocity
 losstype = 2; % loss type: 0:no loss 1: independant, 2: dependant
 intype = 1;    % type of input: 1: struck, 2: plucked
 energyAn=0;

% -------------------------------------------------------------------------%
                        % Simulation Parameters
% -------------------------------------------------------------------------%

 % simulation
 Tf = 2; % duration in seconds
 % readout position as percentage
 readout_position = [.85 .85; .95 .95];
% sample rate (Hz)
 SR = 44.1e3;
 k = 1/SR;
% -------------------------------------------------------------------------%
                              % Physical Parameters
% -------------------------------------------------------------------------%
```

```
% Plate

% Young's modulus
 E_Plate = 2e11;
% density (kg/m^3)
 rho_Plate =7850;
% Poisson Ratio (< .5)
 nu_Plate=.3;
% thickness (m)
 H_Plate = .005;
% plate length X axis (m)
 Lx_Plate = .5;
% plate length X axis (m)
 Ly_Plate = .4;
% loss [freq.(Hz), T60;...]
 loss_Plate = [100, 2; 1000, 1.5];
% coordinates of string coupling [Xs1,Ys1; Xs2,Ys2];
 ctr_Plate = [.35 .35; .71 .78];

% String

%Frequency for note
 f0_String = 440; %A6
% string radius (m)
 r_String = 3e-4;
% string lengths
 L_String = 1;
% Young's modulus
 E_String = 2e11;%6e11;
% density (kg/m^3)
 rho_String = 7850;%1350;
% loss [freq1(Hz), T60_1; freq_2, T60_2]
 loss_String = [100, 6; 900 5];
% I/O string
 xi_String = 0.75;  % coordinate of excitation (normalised, 0-1)
% peak amplitude of excitation (N)
 famp_String = 1;
% duration of excitation (s)
 dur_String = 0.001;
% start time of excitation (s)
 st_exc_st = 0.04;
 % Tension in Newtons using frequency of particular note
 st_T = (((2*f0_String.*r_String).*L_String).^2)*pi*rho_String;
 %st_T=99.279e1; %Alternate for frequency

% -----------------------------------------------------------------------%
```

```
                        % Plate Derived Parameters
%
    -------------------------------------------------------------------------%

 % Coefficients
 D_Plate = (E_Plate*(H_Plate)^3)/(12*(1-(nu_Plate^2)));
 kappa_Plate = sqrt(D_Plate / (rho_Plate*H_Plate) );
 hmin_Plate = 2*sqrt(k*kappa_Plate);
 Nx_Plate = floor((Lx_Plate)/hmin_Plate);
 Ny_Plate = floor((Ly_Plate)/hmin_Plate);
 %Grid Spacing
 h_Plate = sqrt(Lx_Plate*Ly_Plate)/sqrt(Nx_Plate*Ny_Plate);
 mu_Plate = (kappa_Plate * k)/(h_Plate^2);
 % % number of time steps
 Nf = floor(SR*Tf);
 Nx_Plate = Nx_Plate +1;
 Ny_Plate = Ny_Plate +1;
% total grid
 tg = Nx_Plate*Ny_Plate;

%
    -------------------------------------------------------------------------%
                        % String Derived Parameters
%
    -------------------------------------------------------------------------%

% Cross-sectional area
 A_String = pi*r_String^2;
% Moment of intertia
 I_String = 0.25*pi*r_String^4;
 % wave speed
 c_String = sqrt(st_T/(rho_String*A_String));
 % stiffness constant
 K_String = sqrt(E_String*I_String/(rho_String*A_String));
 % minimal grid spacing for stability
 hmin_String = sqrt(0.5* (c_String.^2*k^2+sqrt(c_String.^4*...
     k^4+16*K_String.^2.*k.^2)) );
% number of grid points to update
 N_String = floor(L_String/hmin_String);
% actual grid spacing used
 h_String = L_String/N_String;
% Courant number (?)
 lambda_String = c_String*k/h_String;
% numerical stiffness constant (?)
 mu_String = K_String*k/h_String^2;
%Grid points update
```

```
N_String = N_String +1;


%
    ---------------------------------------------------------------------------%
                                  % Loss
%
    ---------------------------------------------------------------------------%

if losstype==0
sigma0_String=0;
sigma1_string=0;
sigma0_Plate=0;
sigma1_Plate=0;
end

if losstype==1
% frequency independant loss
sigma0_String = 6*log(10)/loss_String(1,2);
sigma1_string = 0;
sigma0_Plate = 6*log(10)/loss_Plate(1,2);
sigma1_Plate = 0;
end

if losstype == 2


    % String
coeff1_String = (-c_String.^2 + sqrt(c_String.^4 + 4*K_String.^2.*...
    (2*pi*loss_String(1,1))^2))./(2*K_String.^2);
coeff2_String = (-c_String.^2 + sqrt(c_String.^4 +4*K_String.^2.*...
    (2*pi*loss_String(2,1))^2))./(2*K_String.^2);

sigma0_String = 6*log(10)*(-coeff2_String/loss_String(1,2)...
    + coeff1_String/loss_String(2,2))./(coeff1_String -coeff2_String);
sigma1_string = 6*log(10)*(1/loss_String(1,2)...
    - 1/loss_String(2,2))./(coeff1_String-coeff2_String);

% Plate
coeff1_Plate = 2*kappa_Plate*(2*pi*loss_Plate(1,1))/(2*kappa_Plate.^2);
coeff2_Plate = 2*kappa_Plate*(2*pi*loss_Plate(2,1))/(2*kappa_Plate.^2);
sigma0_Plate = 6*log(10)*(-coeff2_Plate/loss_Plate(1,2) + coeff1_Plate
   /...
    loss_Plate(2,2))./(coeff1_Plate -coeff2_Plate);
sigma1_Plate = 6*log(10)*(1/loss_Plate(1,2) - 1/loss_Plate(2,2))...
    ./(coeff1_Plate-coeff2_Plate);

```

```
 end
%
    ------------------------------------------------------------------------%
                            % Read In/Out
%
    ------------------------------------------------------------------------%
 lo = readout_position .*[ Nx_Plate Ny_Plate ];
 lo = [floor(sub2ind([Nx_Plate Ny_Plate],lo(1), lo(3))),...
     floor(sub2ind([Nx_Plate Ny_Plate],lo(2), lo(4)))];
%
    ------------------------------------------------------------------------%
                         % Create Force Signal
%
    ------------------------------------------------------------------------%

 f_String = zeros(Nf,1);
 durint = floor(dur_String*SR);
 exc_st_int = (floor(st_exc_st*SR))+1;
 durf = exc_st_int:exc_st_int+durint -1;
 fcoeff_string = (k^2/(h_String*rho_String*A_String));
 f_String(durf) = famp_String*0.5*(1-cos((2/intype)*pi.*(durf...
     /durint)))*fcoeff_string;

%
    ------------------------------------------------------------------------%
                          % Plate Coefficient
%
    ------------------------------------------------------------------------%
%Laplacian
laplacian = triangle(Ny_Plate ,Nx_Plate ,1,bctype_p);
%Biharmonic
biharmonic = triangle(Ny_Plate ,Nx_Plate ,2,bctype_p);
I = speye(tg);
I([1 end],:) = 0;
out1 = I;
pl_mI = out1;
A_coeff_Plate =(1/(1+k*sigma0_Plate))*speye(tg);
B_coeff_Plate =(-(mu_Plate^2)*biharmonic + 2*pl_mI + (2*k*...
    sigma1_Plate/h_Plate^2)*laplacian) * A_coeff_Plate;
C_coeff_Plate = (-(1-sigma0_Plate*k)*pl_mI - (2*k*sigma1_Plate/...
```

```
    h_Plate^2)*laplacian) * A_coeff_Plate;
%----------------------------------------------------------------------------%
                        % String Coefficients
%----------------------------------------------------------------------------%

% scheme coefficients
ls=N_String;
outMx = spdiags([-ones(ls,1),ones(ls,1)],0:1,speye(ls));
Dn=outMx;
%Second order operator
Dnn = Delta(N_String,'xx');
%Fourth order operator
Dnnnn = Delta(N_String,'xxxx',bctype_S);
Dnnnn(2,1) = -2;
st_mI = speye(N_String);
% String matrix
s_A = (1/(1+k*sigma0_String));
s_B = ((lambda_String^2)*Dnn - (mu_String^2)*Dnnnn + (2*st_mI) ...
     + ((2*sigma1_string*k/h_String^2)*Dnn)) * s_A ;
s_C = -((1-sigma0_String*k)*st_mI + ((2*sigma1_string*k/h_String^2)...
     *Dnn) ) * s_A;
%coupling string boundary
 s_B([1 end],:) = 0;
 s_C([1 end],:) = 0;
 s_B(1,:) = ((lambda_String^2)*Dn(1,:) - (mu_String^2)*Dnn(2,:)...
     + (2*st_mI(1,:))) * s_A;
 s_C(1,:) = -((1-sigma0_String*k)*st_mI(1,:));

%----------------------------------------------------------------------------%
                  % Matrices update and Couple Matrices
%----------------------------------------------------------------------------%

 %Total Number of Grid points
 Nt = tg+ N_String;
% coupling points
 w0p = sub2ind([Nx_Plate, Ny_Plate], floor(ctr_Plate(1,1)...
     *Nx_Plate),floor(ctr_Plate(1,2)*Ny_Plate));
% Mass Ratio Coefficients
 M_Plate = 1/((rho_Plate*H_Plate*h_Plate^2)*(1+k*sigma0_Plate));
 M_String = 1/((rho_String*A_String*h_String)*(1+k*sigma0_String));
% Spreading operators
```

```
 J = sparse(zeros(Nt,1));
 J([w0p tg+1]) = [M_Plate -M_String];
 pJ = sparse(tg,1);
 pJ(w0p) = 1;
 sJ = sparse(N_String ,1);
 sJ(1) = 1;
% Mass ratio
 M_Coef1 = 1/( 1/(rho_String*A_String*h_String*(1+k*sigma0_String)) +...
 (1/(rho_Plate*H_Plate*(h_Plate^2)*(1+k*sigma0_Plate)))*(pJ'*pJ));
% Coupling Vector
 F = M_Coef1*[-pJ'*B_coeff_Plate,sJ'*s_B];
 F1 = M_Coef1*[pJ'*(2*pl_mI + (2*sigma1_Plate*k/h_Plate^2)*laplacian)...
     /(1+k*sigma0_Plate),...
-2*sJ'*st_mI/(1+k*sigma0_String)];
%Sparsed Matrix as coefficient for main calculation
 B=blkdiag(B_coeff_Plate ,s_B) + J*F;
 C=blkdiag(C_coeff_Plate ,s_C) + J*F1;

% -------------------------------------------------------------------------%
                             % Initialise I/O
% -------------------------------------------------------------------------%
% initialise output
u =zeros(tg,1);
w=zeros(N_String ,1);
% Joined vectors
 uw = [u;w];
 uw1 = uw;
 uw2 = uw;
% input
 fvect = [u;w];
% Index of excitation
 li = floor(xi_String*N_String) + tg;
% output
 y = zeros(Nf,2);
%Energy analysis initialise
if energyAn==1
  % total energy
  plate_Energy = zeros(Nf,1);
  % kineteic energy
  plate_KE =zeros(Nf,1);
  % potential enery
  plate_PE = zeros(Nf,1);
  % total energy
  string_Energy = zeros(Nf,1);
  % kineteic energy
```

```
    string_KE = zeros(Nf,1);
    % potential energy
    string_PE = zeros(Nf,1);
    fEnergy = zeros(Nf,1);
end



%
    -----------------------------------------------------------------------%
                            % Main Calculation
%
    -----------------------------------------------------------------------%

 tic

 for n = 1:Nf
 % update input forces
 fvect(li) = f_String(n);

 % main operation
 uw = B*uw1 + C*uw2 + fvect;

 % read output
 if (outtype==1)
 y(n,:) = uw(lo);

 elseif (outtype==2)
 y(n,:) = (SR*(uw(lo)-uw1(lo)));

 end


 if energyAn==1
      u = uw(1:tg);
      u1 = uw1(1:tg);
      w = uw(tg+1:end);
      w1 = uw1(tg+1:end);

      plate_KE(n) = 0.5*(h_Plate^2)/k^2*((u-u1)'*(u-u1));
      plate_PE(n) = 0.5*(kappa_Plate^2)/(h_Plate^2)*((laplacian*u)'...
          * (laplacian*u1));
      plate_Energy(n)=plate_KE(n)+plate_PE(n);
     % pEnergy(n) = pl_coE*[sum(((u-u1).^2)); (laplacian*u)' * (
    laplacian*u1)];

      string_KE(n) = 0.5*(h_String)/k^2 *((w-w1)'*(w-w1));
```

```
        string_PE(n) = 0.5*(c_String^2)/h_String*((Dn*w)' * (Dn*w1));
        string_Energy(n) = string_KE(n)+string_PE(n);
        %fEnergy(n)=pEnergy(n)+sEnergy(n);
        fEnergy(n) = (k*c_String^2/(2*h_String^2))*(w(2)-w(1)) ...
            + SR*(w(1)-w1(1)) + (kappa_Plate^2*k*.5)*B_coeff_Plate(w0p,:) ...
            *u - SR*(u(w0p) - u1(w0p));
 end


 % Update
 uw2=uw1;
 uw1=uw;

 end
 y= y/max(abs(y));
soundsc(y,SR);
 toc
%  n=1:Nf;
%  plot(n,y(n));
%  xlabel('Samples');
%  ylabel('magnitude');

% figure
%       yfft = 10*log10(abs(fft(y)));
%       plot([0:Nf-1]'/Nf*SR, abs(yfft), 'k')
%       xlabel('freq (Hz)')
%       ylabel('|magnitude| (db)');
%       title('total system output stectra')
%
% plot(y);
% xlabel('Sample');
% ylabel('Amplitude');
% title('String-Plate connection displacement');
```

Appendix C

Project Archive

This appendix contains list of all the materials and files included in this submission. It also explains the organisation and explanation for it's structure. All the codes included has been compiled and tested using MATLAB 2018 on Mac operating system. All the care has been taken into account in order to run the scripts without any alteration or addition of any extra materials.

C.1 Structure

In the file, there are three components. First is the source code written in MATLAB, second is the audio examples extracted from the code generated and third is the document.

C.2 Document

This file contains all the documentation related to the dissertation. All the methodology, steps, diagrams have been included in this file.

C.3 Source_Code

In this folder, there are two sub folder: 'Plate_FDTD' and 'String-Plate_Coupling'.

Plate_FDTD

In this folder, FDTD model of plate has been coded.It consists of a function named 'triangle.m' which consists of laplacian and biharmonic matrix. Another MATLAB file named 'thin_plate_loss.m' is the model of Kirchhoff plate equation.

String-Plate_Coupling

In this file there are three MATLAB files. First is the 'couple.m' which is the main file to be executed for coupled system of string and plate. Other two files which are the functions, named 'Delta.m' is used for operator matrix creation and 'triangle.m ' is used for laplacian and biharmonic operator matrix.

C.4 Audio Example

This file consists of four audio examples. First is the output of sound obtained from plate model named as 'Plate_Loss.wav'. Second is the output from the string plate coupled system which are without loss, with frequency independent loss and frequency dependent loss named respectively: 'Couple_no_Loss_A4.wav' 'Coupled_frequency_independent_Nylon_A4.wav' 'Coupled_frequency_dependent_Nylon_A4.wav'

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