Midterm exam

You are allowed your notes, but no phones, laptops or other devices. All answers must be justified.

Exercise 1: For a positive integer n the *hypercube* $Q_n = (V, E)$ is the graph whose vertices correspond to the $(0,1)$-vectors of length n, i.e., $V = \{0,1\}^n$, where two vertices are adjacent if and only if the two corresponding vectors differ in exactly one entry.

1. Draw Q_n for $1 \leq n \leq 4$.
2. Determine those n, such that Q_n is Eulerian.
3. Show that Q_n is bipartite for all $n \geq 1$.
4. Compute the girth of Q_n for all $n \geq 1$.

Exercise 2: Prove or disprove the following statements:

1. Every Eulerian bipartite graph has an even number of edges.
2. Every Eulerian graph with an even number of vertices has an even number of edges.

Exercise 3: Let T be a *binary tree*, i.e., one vertex (the *root*) has degree 2, and all other vertices have degree 1 or 3. Show that if T has ℓ leafs, then it has $2\ell - 1$ vertices in total.
Apply this to answer the following:
We want to break a $n \times m$ chocolate bar into single pieces of size 1×1. At each step we are allowed to break a piece along a horizontal or vertical line. How may steps are needed?

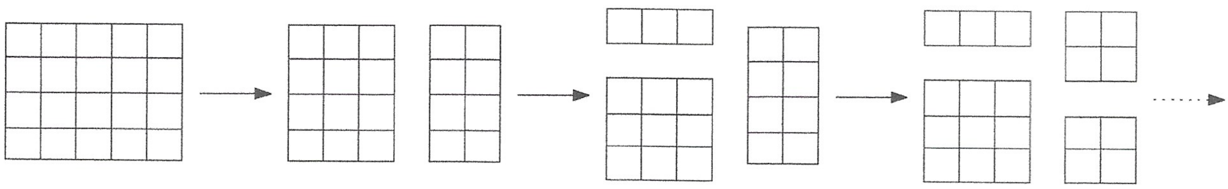

Figure 1: Some steps in breaking a 4×5 chocolate bar.

Exercise 4: Let $k \leq n$ be positive integers and $d = (d_1 \geq \ldots \geq d_n)$ a sequence of non-negative integers. Prove that the following two statements are equivalent:

(i) d is graphic and $\sum_{i=0}^{n} d_i \geq 2(n-k)$,

(ii) d is the degree sequence of a simple graph with at most k connected components.