RWTH Aachen
Faculty of Computer Science
Chair of Combinatorial Optimization

Bachelor Thesis in Computer Science

# Min-Max and Regret Spanning Trees

## Complexity and Approximation under Discrete and Interval Uncertainty

Submitted by

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# Abstract

This thesis is a self-contained primer on robust spanning trees under cost uncertainty. We formalize and contrast the *Min-Max* and *Min-Max Regret* objectives for discrete scenario sets and interval uncertainty, prove the foundational tools used later (fundamental cut/cycle, MST optimality via cut/cycle criteria), and present a curated synthesis of reported complexity and approximability results. A running micro-graph and a compact TikZ gallery illustrate how robust solutions differ from nominal MSTs.

## Contents

# Chapter 1

# Introduction

## 1.1 Motivation and Problem Statement

Designing cost–efficient networks is a recurring task in transportation, telecommunications, energy, and logistics. When edge costs are known, the *minimum spanning tree* (MST) is a classical and well-understood baseline: it connects all vertices with minimum total cost. In practice, however, costs are seldom known precisely at design time. They depend on estimates, market prices, delays, outages, or policy changes, which may shift before the network is built or operated. A tree that is optimal for a single nominal estimate can therefore become unexpectedly expensive when the realized costs differ.

This thesis studies spanning trees when edge costs are *uncertain*. We consider two common ways of representing uncertainty that already cover many practical situations: (i) a *discrete set of scenarios*, each a full cost vector; and (ii) *interval ranges* for each edge, forming a Cartesian product of bounds. Given such an uncertainty description, there are (at least) two natural design goals. The first is *Min–Max*: choose a tree that keeps its *worst-case total cost* as small as possible across all scenarios. The second is *Min–Max Regret*: choose a tree whose cost is, in the worst case, close to the *scenario-optimal* MST; that is, we hedge not against absolute cost but against how far we fall short of the best we could have done if the scenario had been known in advance. Both perspectives formalize robustness, yet they prioritize different risks: Min–Max is absolute (guard the total), Regret is relative (guard the performance gap).

Even at this conceptual level, two ingredients are crucial. First, we need a clean and self-contained foundation for MST reasoning—cuts, cycles, and the fundamental exchange arguments—so that all later robust models rest on solid ground. Second, when costs lie in intervals, we must understand *where* worst cases live: Do we have to consider all points in a box, or do extremal cases suffice (e.g., bounds on selected edges)? Answering such questions allows us to reason about robustness without enumerating infinitely many possibilities.

We focus on spanning trees under discrete- and interval-uncertainty and study Min–Max and Min–Max Regret objectives; precise models and notation follow in Chapter 2. We also fix basic complexity and approximation vocabulary early on, enabling concise statements in later chapters.

## 1.2 Objectives and Scope

This thesis aims to provide a self-contained, correctness-first primer on robust spanning trees that carries the reader from classical MST reasoning to two robust objectives—*Min–Max* and *Min–Max Regret*—under two standard uncertainty styles (discrete scenarios and intervals). The emphasis is on precise modeling, short but complete foundational proofs, and a curated synthesis of established complexity and approximation results, complemented by small worked examples.

**Objectives.** We pursue the following concrete objectives:

- **Foundational bedrock.** Present the graph/MST basics used later (cuts, cycles, fundamental exchange arguments), with complete proofs and a tiny running example (Chapter 2).
- **Algorithmic preliminaries (complexity & approximation).** A concise primer (decision vs. optimization; P/NP; reductions; weak/strong NP-hardness; approximation ratios; PTAS/FPTAS; APX) so that results in later chapters can be stated tersely; this primer will live in Chapter 2.
- **Unified modeling.** Formalize *Min–Max* and *Min–Max Regret* for spanning trees under (a) *discrete* uncertainty and (b) *interval* uncertainty, using consistent notation across chapters (Chapters 3 and 4).
- **Extremal reasoning for intervals.** State and use short extremal lemmas that identify where worst cases occur within interval boxes, enabling analysis without enumerating infinitely many realizations.
- **Curated landscape.** Collect and organize established results on computational complexity and known approximation (or inapproximability) guarantees for the above models.
- **Illustrative micro-examples.** Use a single small graph throughout to compare behaviors of the models and to make definitions tangible.
- **Synthesis and outlook.** Summarize models and results in a one-page table and outline concise takeaways and avenues for future work (Chapters 5 and 6).

**Scope and boundaries.** We deliberately restrict attention to the following setting:

- **Problem class.** Undirected, connected graphs with real-valued edge costs; solutions are spanning trees.
- **Uncertainty descriptions.** (i) *Discrete* sets of cost scenarios, i.e., a finite set $\mathcal{U} = \{c^{(1)}, \ldots, c^{(K)}\}$; (ii) *Interval* uncertainty given by per-edge bounds forming a Cartesian product $\prod_e [\,\ell_e, u_e\,]$.
- **Objectives.** *Min–Max* (minimize worst-case total cost) and *Min–Max Regret* (minimize worst-case gap to the scenario-optimal MST). Formal definitions appear in Chapter 2 and are used in Chapters 3 and 4.

A detailed per-chapter roadmap is given in Section 1.4.

## 1.3 Contributions

This section states what the reader *gets* from this document—concrete deliverables and value added beyond a standard overview.

- **Foundations package (Chapter 2).** A self-contained write-up of the MST toolkit (cut/cycle criteria, fundamental exchange arguments) with complete proofs, a consistent notation bank, and one reusable micro-graph figure.
- **Compact complexity/approximation primer.** A one-page glossary of the terms we use for algorithmic results, included in Chapter 2 to keep Chapters 3–4 focused on model specifics.
- **Interval extremal toolkit.** A compact set of lemmas pinpointing where worst cases occur in interval boxes, with short proofs and guidance on how they are used later for Min–Max and Regret analyses.
- **Normalized notation and cross-referencing.** One consistent symbol set ($\mathcal{U}, \mathcal{T}, \mathrm{MST}(()\cdot)$) and a clear cross-reference policy (via `\Cref`) applied throughout, so readers can navigate definitions/results without ambiguity.
- **Organized literature map.** A tabulated map of established complexity and approximation/inapproximability results, indexed by objective (Min–Max / Regret), uncertainty (discrete / interval), and scenario parameter $K$ where relevant.
- **Synthesis artifacts (Chapter 5).** (i) A one-page summary table (models $\times$ uncertainty $\times$ complexity/approx); (ii) a small example gallery (shared micro-graph) illustrating behaviors across objectives.
- **Optional full proof (Appendix A).** One representative hardness proof reproduced in full for pedagogical completeness.
- **Reproducible build and readable sources.** Clean Route A LaTeX build (biblatex/biber, cleveref, TikZ) and a *readable* Markdown export of chapters to facilitate review.

## 1.4 Thesis Roadmap

This document proceeds from foundations to robust models and, finally, to a concise synthesis.

1. **Chapter 2 (Foundations).** We fix notation, prove the MST toolkit (Fundamental Cut Lemma; cycle/cut criteria), introduce a tiny running graph, and include a short primer on complexity and approximation to support the statements used in later chapters.

2. **Chapter 3 (Min–Max Spanning Tree).** We formalize Min–Max for spanning trees under discrete scenarios and interval uncertainty, and we state the minimal interval extremal facts needed to reason about worst cases. A small worked example illustrates the model.

3. **Chapter 4 (Min–Max Regret Spanning Tree).** We formalize regret and its discrete/interval variants, relate regret to scenario-optimal MSTs, and use short extremal lemmas for intervals. A worked example mirrors Chapter 3 for direct comparison.

4. **Chapter 5 (Synthesis & Example Gallery).** We summarize the established landscape in a one-page table (model $\times$ uncertainty; complexity; (in)approximability; notes on $K$) and gather a few TikZ micro-examples that highlight behavioral contrasts.

5. **Chapter 6 (Conclusion & Outlook).** We extract the main takeaways, note limitations, and briefly indicate natural extensions (e.g., budgeted-$\Gamma$ uncertainty or polyhedral sidebars) that lie beyond our core scope.

**Appendices.** Appendix A (optional) contains one representative hardness/inapproximability proof rendered in our notation. Appendix B lists all symbols in a compact table for quick reference. Appendix C holds auxiliary figures or expanded examples referenced from the main text.

# Chapter 2

# Foundations

## 2.1 Graphs, Trees, and Notation

## 2.2 Minimum Spanning Tree Optimality

**Lemma 1** (Fundamental Cycle).

*Proof.* □

**Lemma 2** (Fundamental Cut).

*Proof.* □

**Theorem 1** (MST Cycle Criterion).

*Proof.* □

**Theorem 2** (MST Cut Criterion).

*Proof.* □

**Theorem 3** (MST Characterisation).

*Proof.* □

## 2.3 Kruskal and Prim: Algorithmic Remarks

## 2.4 Uncertainty Models and Robust Optimisation

## 2.5 Algorithmic Preliminaries

## Summary

# Chapter 3

# Min-Max Spanning Tree

## 3.1 Problem Formulation

## 3.2 Interval Worst-Case Characterisation

**Lemma 3** (Interval Extremal Cost)**.**

*Proof.* □

## 3.3 Complexity and Approximation

### 3.3.1 Discrete Scenarios: Constant K

**Theorem 4** (Partition Reduction)**.**

*Proof.* □

**Theorem 5** (Pseudo-Polynomial Algorithm)**.**

### 3.3.2 Discrete Scenarios: Unbounded K

**Theorem 6** (Strong NP-Hardness)**.**

## 3.4 Discussion

## Summary

# Chapter 4

# Min-Max Regret Spanning Tree

## 4.1 Regret Formulation

## 4.2 Interval Regret Extremal Behaviour

**Lemma 4** (Interval Extremal Regret).

*Proof.* □

## 4.3 Complexity for Discrete Scenarios

### 4.3.1 Constant K

**Theorem 7** (K=2 Weak NP-Hardness).

*Proof.* □

**Theorem 8** (Pseudo-Polynomial for Constant K).

### 4.3.2 Unbounded K

**Theorem 9** (Strong NP-Hardness).

## 4.4 Interval Regret Approximation

**Theorem 10** (Interval NP-Hardness).

**Theorem 11** (2-Approximation via Midpoint).

**Lemma 5** (Lower Bound on Regret).

*Proof.* □

**Lemma 6** (Upper Bound Relating Solutions).

*Proof.* □

*Proof of Theorem 11.* □

## 4.5 Discussion

## Summary

# Chapter 5

# Synthesis and Example Gallery

## 5.1 Comprehensive Results Classification

### 5.1.1 Complexity and Approximation Landscape

### 5.1.2 Key Patterns and Insights

## 5.2 Example Gallery

### 5.2.1 Micro-Graph Solutions Across Objectives

### 5.2.2 Extremal Behaviour Visualisation

### 5.2.3 Worked Example: Partition Reduction

## 5.3 Key Takeaways

## Summary

# Chapter 6

# Conclusion and Outlook

## 6.1 Summary

### 6.1.1 Thesis Achievements

### 6.1.2 Core Findings

## 6.2 Limitations

### 6.2.1 Scope Restrictions

### 6.2.2 Proofs and Coverage

### 6.2.3 Micro-Graph and Empirical Work

## 6.3 Outlook

### 6.3.1 Extensions in Uncertainty Modelling

### 6.3.2 Extensions in Solution Concepts

### 6.3.3 Open Problems

# Appendix A

# Notation

# Affidavit / Eidesstattliche Versicherung

I hereby declare that I have written this thesis independently and that I have not used any sources or aids other than those indicated. All passages which are quoted or closely paraphrased from other works are marked as such. This thesis has not been submitted in the same or a substantially similar form to any other examination board.

Place, Date Signature