Chair of Combinatorial Optimization
RWTH Aachen University

Bachelor Thesis in Computer Science
March 2026

Min-Max and Min-Max Regret Spanning Trees

Concepts, Complexity, and Results in Robust Combinatorial Optimization

Submitted by
Archit Dhama
Street + Number
Postal Code + City
Matriculation No.: 396580
archit.dhama@rwth-aachen.de

First Examiner: Prof. Dr. Christina Büsing
Chair of Combinatorial Optimization
RWTH Aachen University

Second Examiner: Prof. Dr. Arie M. C. A. Koster
Chair II of Mathematics
RWTH Aachen University

Abstract

This thesis is a self-contained primer on robust spanning trees under cost uncertainty. We formalize and contrast the *Min-Max* and *Min-Max Regret* objectives for discrete scenario sets and interval uncertainty, prove the foundational tools used later (fundamental cut/cycle, MST optimality via cut/cycle criteria), and present a curated synthesis of reported complexity and approximability results. A running micro-graph and a compact TikZ gallery illustrate how robust solutions differ from nominal MSTs.

Contents

Chapter 1

Introduction

1.1 Motivation and Problem Statement

Designing cost–efficient networks is a recurring task in transportation, telecommunications, energy, and logistics. When edge costs are known, the *minimum spanning tree* (MST) is a classical and well-understood baseline: it connects all vertices with minimum total cost. In practice, however, costs are seldom known precisely at design time. They depend on estimates, market prices, delays, outages, or policy changes, which may shift before the network is built or operated. A tree that is optimal for a single nominal estimate can therefore become unexpectedly expensive when the realized costs differ.

This thesis studies spanning trees when edge costs are *uncertain*. We consider two common ways of representing uncertainty that already cover many practical situations: (i) a *discrete set of scenarios*, each a full cost vector; and (ii) *interval ranges* for each edge, forming a Cartesian product of bounds. Given such an uncertainty description, there are (at least) two natural design goals. The first is *Min–Max*: choose a tree that keeps its *worst-case total cost* as small as possible across all scenarios. The second is *Min–Max Regret*: choose a tree whose cost is, in the worst case, close to the *scenario-optimal* MST; that is, we hedge not against absolute cost but against how far we fall short of the best we could have done if the scenario had been known in advance. Both perspectives formalize robustness, yet they prioritize different risks: Min–Max is absolute (guard the total), Regret is relative (guard the performance gap).

Even at this conceptual level, two ingredients are crucial. First, we need a clean and self-contained foundation for MST reasoning—cuts, cycles, and the fundamental exchange arguments—so that all later robust models rest on solid ground. Second, when costs lie in intervals, we must understand *where* worst cases live: Do we have to consider all points in a box, or do extremal cases suffice (e.g., bounds on selected edges)? Answering such questions allows us to reason about robustness without enumerating infinitely many possibilities.

We focus on spanning trees under discrete- and interval-uncertainty and study Min–Max and Min–Max Regret objectives; precise models and notation follow in Chapter 2.

1.2 Objectives and Scope

This thesis aims to provide a self-contained, correctness-first primer on robust spanning trees that carries the reader from classical MST reasoning to two robust objectives—*Min–Max* and *Min–Max Regret*—under two standard uncertainty styles (discrete scenarios and intervals). The emphasis is on precise modeling, short but complete foundational proofs, and a curated synthesis of established complexity and approximation results, complemented by small worked examples.

Objectives. We pursue the following concrete objectives:

- **Foundational bedrock.** Present the graph/MST basics used later (cuts, cycles, fundamental exchange arguments), with complete proofs and a tiny running example (Chapter 2).
- **Unified modeling.** Formalize *Min–Max* and *Min–Max Regret* for spanning trees under (a) *discrete* uncertainty and (b) *interval* uncertainty, using consistent notation across chapters (Chapters 3 and 4).
- **Extremal reasoning for intervals.** State and use short extremal lemmas that identify where worst cases occur within interval boxes, enabling analysis without enumerating infinitely many realizations.
- **Curated landscape.** Collect and organize established results on computational complexity and known approximation (or inapproximability) guarantees for the above models.
- **Illustrative micro-examples.** Use a single small graph throughout to compare behaviors of the models and to make definitions tangible.
- **Synthesis and outlook.** Summarize models and results in a one-page table and outline concise takeaways and avenues for future work (Chapters 5 and 6).

Scope and boundaries. We deliberately restrict attention to the following setting:

- **Problem class.** Undirected, connected graphs with real-valued edge costs; solutions are spanning trees.
- **Uncertainty descriptions.** (i) *Discrete* sets of cost scenarios, i.e., a finite set $\mathcal{U} = \{c^{(1)}, \dots, c^{(K)}\}$; (ii) *Interval* uncertainty given by per-edge bounds forming a Cartesian product $\prod_e [\,\ell_e, u_e\,]$.
- **Objectives.** *Min–Max* (minimize worst-case total cost) and *Min–Max Regret* (minimize worst-case gap to the scenario-optimal MST). Formal definitions appear in Chapter 2 and are used in Chapters 3 and 4.

A detailed per-chapter roadmap is given in Section 1.4.

1.3 Contributions

This section states what the reader *gets* from this document—concrete deliverables and value added beyond a standard overview.

- **Foundations package (Chapter 2).** A self-contained write-up of the MST toolkit (cut/cycle criteria, fundamental exchange arguments) with complete proofs, a consistent notation bank, and one reusable micro-graph figure.
- **Interval extremal toolkit.** A compact set of lemmas pinpointing where worst cases occur in interval boxes, with short proofs and guidance on how they are used later for Min–Max and Regret analyses.
- **Normalized notation and cross-referencing.** One consistent symbol set ($\mathcal{U}, \mathcal{T}, \mathrm{MST}((\cdot)\cdot)$) and a clear cross-reference policy (via `\Cref`) applied throughout, so readers can navigate definitions/results without ambiguity.
- **Organized literature map.** A tabulated map of established complexity and approximation/inapproximability results, indexed by objective (Min–Max / Regret), uncertainty (discrete / interval), and scenario parameter K where relevant.
- **Synthesis artifacts (Chapter 5).** (i) A one-page summary table (models $\times$ uncertainty $\times$ complexity/approx); (ii) a small example gallery (shared micro-graph) illustrating behaviors across objectives.
- **Optional full proof (Appendix A).** One representative hardness proof reproduced in full for pedagogical completeness.
- **Reproducible build and readable sources.** Clean Route A LaTeX build (biblatex/biber, cleveref, TikZ) and a *readable* Markdown export of chapters to facilitate review.

1.4 Thesis Roadmap

This document proceeds from foundations to robust models and, finally, to a concise synthesis.

1. **Chapter 2 (Foundations).** We fix notation and present the MST toolkit with complete proofs: the Fundamental Cut Lemma and the cycle/cut optimality criteria. A tiny running graph is introduced once and reused later.
2. **Chapter 3 (Min–Max Spanning Tree).** We formalize Min–Max for spanning trees under discrete scenarios and interval uncertainty, and we state the minimal interval extremal facts needed to reason about worst cases. A small worked example illustrates the model.
3. **Chapter 4 (Min–Max Regret Spanning Tree).** We formalize regret and its discrete/interval variants, relate regret to scenario-optimal MSTs, and use short extremal lemmas for intervals. A worked example mirrors Chapter 3 for direct comparison.

4. **Chapter 5 (Synthesis & Example Gallery).** We summarize the established landscape in a one-page table (model $\times$ uncertainty; complexity; (in)approximability; notes on K) and gather a few TikZ micro-examples that highlight behavioral contrasts.

5. **Chapter 6 (Conclusion & Outlook).** We extract the main takeaways, note limitations, and briefly indicate natural extensions (e.g., budgeted-Γ uncertainty or polyhedral sidebars) that lie beyond our core scope.

Appendices. Appendix A (optional) contains one representative hardness/inapproximability proof rendered in our notation. Appendix B lists all symbols in a compact table for quick reference. Appendix C holds auxiliary figures or expanded examples referenced from the main text.

Chapter 2

Foundations

2.1 Graphs, Trees, and Notation

2.2 Fundamental Cut Lemma

Lemma 1 (Fundamental Cut Lemma)**.** *Let T be a spanning tree and $e \in E(T)$. Then $\delta(X_e) \cap E(T) = \{e\}$.*

Proof. □

2.3 MST Optimality: Cycle and Cut Criteria

Theorem 1. *For a spanning tree T, the following are equivalent: (i) T is an MST; (ii) every non-tree edge is a most expensive edge on its fundamental cycle; (iii) every tree edge is among the cheapest edges in its fundamental cut.*

Proof sketch. □

2.4 Kruskal and Prim (Brief Recap)

2.5 Uncertainty Sets and Robust Criteria (Definitions Only)

2.6 Running Micro-Graph and Notation Table

Chapter 3

Min-Max Spanning Tree

3.1 Models

3.1.1 Discrete Scenarios

$$\min_{T \in \mathcal{T}} \max_{u \in \mathcal{U}} \sum_{e \in T} c_e^{(u)}$$

3.1.2 Interval Uncertainty

3.2 Short Properties Used Later

Lemma 2 (Interval extremal cost for a fixed tree)**.** *For interval costs, the worst-case cost of a fixed tree is attained by setting each chosen edge to its upper bound.*

Proof. □

3.3 Reported Complexity and Approximability

3.3.1 Discrete: dependence on the number of scenarios K

3.3.2 Intervals: NP-hardness

3.4 Modeling Patterns / Formulations (Pointers)

3.5 Worked Micro-Graph Example

Chapter 4

Min-Max Regret Spanning Tree

4.1 Regret Definition and Objective

4.2 Models

4.2.1 Discrete Scenarios

4.2.2 Interval Uncertainty

4.3 Short Properties Used Later

Lemma 3 (Interval extremal regret for a fixed tree)**.** *For interval costs, the worst-case regret of a fixed tree is attained at bound assignments.*

Proof. □

4.4 Reported Complexity and Approximability

4.4.1 Discrete K: constant vs. unbounded

4.4.2 Intervals: NP-hardness; 2-approximation

4.5 Modeling Pointers / Reformulations

4.6 Worked Micro-Graph Example

4.7 Pointer to Appendix A (Representative Full Proof)

Chapter 5

Synthesis and Example Gallery

5.1 One-Page Summary Table

5.2 Gallery A: Discrete Scenarios

5.3 Gallery B: Interval Uncertainty

5.4 Takeaways

Chapter 6

Conclusion and Outlook

6.1 Key Takeaways

6.2 Limitations

6.3 Outlook

Appendix A

Representative Full Proof (Optional)

Appendix B

Notation Table

Symbol	Meaning
$G = (V, E)$	Undirected connected graph
c_e	Edge cost (nominal)
$\delta(X)$	Cut induced by $X \subseteq V$
$\mathcal{T}$	Set of spanning trees
C_e	Fundamental cycle of non-tree edge e w.r.t. T
X_e	Vertex side for the fundamental cut of $e \in E(T)$
$\mathcal{U}$	Uncertainty set of scenarios
$c_e^{(u)}$	Cost of e in scenario u
$\mathrm{Regret}(T, u)$	Scenario regret

Affidavit / Eidesstattliche Versicherung

I hereby declare that I have written this thesis independently and that I have not used any sources or aids other than those indicated. All passages which are quoted or closely paraphrased from other works are marked as such. This thesis has not been submitted in the same or a substantially similar form to any other examination board.

Place, Date | Signature