Derivation of Casimir Effect without zeta-regularization

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1 Condition

The following theories and conclusions are used to derive Casimir Effect.

1. Uncertainty Principle:
$$\Delta E \Delta t \geq \frac{\hbar}{2}$$
$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

2. Zero-point energy (ground state energy):
$$E_{\mathbb{Z}} = \frac{1}{2}\hbar\omega$$

3. The expectation value of the vacuum energy:
$$\langle E \rangle = \frac{1}{2}\sum_{n=1}^{\infty} E_n$$

4. The conclusion of Casimir effect assuming zeta-regularization:
$$\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3}$$
where A is the area of the metal plates. (Wikipedia contributors, 2024)

2 Derivation

Let

1. The distance between two uncharged conductive plates in a vacuum

$$a$$

2. The wavenumber orthogonal to the plates

$$k_n = \frac{n\pi}{a}$$

3. The plates lie parallel to the xy-plane and is orthogonal to z-axis

Calculate the Expected Vacuum Energy

$$\begin{aligned}
&\langle E \rangle = \frac{1}{2}\sum_{n=1}^{\infty} E_n = \frac{1}{2}\sum_{n=1}^{\infty} \bar{E} = \frac{1}{2}\sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t} \\
&\omega_n = c \cdot k_n = n \cdot \frac{c\pi}{a} \\
&\Delta E_n = \frac{1}{2}\hbar\omega_n = n \cdot \frac{\hbar c\pi}{2a} \\
&\frac{\Delta t_n}{t} = \frac{\hbar/2}{\Delta E_n} \cdot \frac{1}{t} = \frac{a}{nc\pi} \cdot \frac{c}{a} = \frac{1}{n\pi} \\
&\sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t} = \sum_{n=1}^{\infty} \frac{n\hbar c\pi}{2a} \cdot \frac{1}{n\pi} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a}
\end{aligned}$$

- **Summation notation:** Define the expected vacuum energy (denoted by $\langle E \rangle$) as the average of individual energy states (represented by E_n). Since we're summing over infinitely many states, summation notation is used.

- **Relating energy and time:** The key idea is that the energy uncertainty (ΔE_n) is related to the time uncertainty (Δt_n) through a constant factor ($\hbar/2$).

- **Introducing wavenumber:** Define the wavenumber (k_n) for the nth state, which relates to the wave's frequency (ω_n) and the distance between the plates (a).

- **Calculating energy uncertainty:** Calculate the energy uncertainty for each state (ΔE_n) using the wavenumber and the proportionality constant from the uncertainty principle.

- **Deriving time uncertainty ratio:** Calculate the ratio of the time uncertainty for each state (Δt_n) to the total time period (t). This ratio is inversely proportional to the energy uncertainty.

- **Summing the expected energy:** This expresses the expected vacuum energy density by summing the product of energy uncertainty and its corresponding time uncertainty ratio for all states, divided by the total area (which will be calculated later). This summation represents the average energy density across all possible quantum states.

Calculate the Area

$$A = \sum_{n=1}^{\infty} A_n$$

The nth area A_n relates to the change in length and the change in position.

$$\begin{aligned}
&A_n = a_n^2 = (\frac{L_n}{\Delta x_n^{x,y}} \cdot a)^2 \\
&L_n = na \\
&p_n = \hbar k_n = \frac{n\pi\hbar}{a}
\end{aligned}$$

The length of x, y axis waves locating on the nth area is

$$\Delta x_n^{x,y} = \frac{\hbar/2}{p_n} = \frac{\hbar/2}{n\pi\hbar/a} = \frac{1}{2n\pi a}$$

Therefore

$$A_n = (\frac{L_n}{\Delta x_n^{x,y}} \cdot a)^2 = 4n^4\pi^2 a^2$$

- **Relating area and uncertainty:** The concept is that the area for a particular state is related to the change in length (L_n) and the change in position ($\Delta x_n^{x,y}$) along the x and y axes.

- **Defining state variables:** Define the change in length for the nth state (L_n) and the momentum (p_n) associated with that state.

- **Calculating x,y position uncertainty:** Due to the confinement in the z-axis, this calculates the change in position along the x and y directions for the nth state ($\Delta x_n^{x,y}$). This is derived using the uncertainty principle and the momentum along those directions.
- **Expressing Area:** The formula for the area (A_n) of the nth state relates the area to the change in length, change in position along x and y, and the distance between the plates.

Calculate Vacuum Energy on Two Plates

$$\begin{aligned}
\frac{\langle E\rangle}{A} &= 2\cdot\frac{1}{2}\sum_{n=1}^{\infty}\Delta E_n\frac{\Delta t_n}{t}\frac{1}{A_n} \\
&= \sum_{n=1}^{\infty}\frac{\hbar c}{2a}\cdot\frac{1}{4n^4\pi^2a^2} = \sum_{n=1}^{\infty}\frac{\hbar c}{8n^4\pi^2a^3} = \frac{\hbar c}{8\pi^2a^3}\sum_{n=1}^{\infty}\frac{1}{n^4} \\
&= \frac{\hbar c}{8\pi^2a^3}\zeta(4) = \frac{\hbar c}{8\pi^2a^3}\cdot\frac{\pi^4}{90} = \frac{\hbar c\pi^2}{720a^3}
\end{aligned}$$

3 Conclusion

This derivation for the Casimir effect without zeta-regularization is that the average vacuum energy density (energy per unit area, $\langle E\rangle/A$) between two perfectly conducting plates separated by distance a is:

$$\frac{\hbar c\pi^2}{720a^3}$$

This paper's derivation of the Casimir effect, lacking the negative sign, differs from the standard approach which attributes the negative sign to the existence of negative energy between the plates.

While this paper calculates the absolute value of the energy density, the standard derivation focuses on the energy difference, where the negative sign reflects the lower energy state within the cavity due to the presence of the plates.

References

Wikipedia contributors. (2024). *Casimir effect — Wikipedia, the free encyclopedia.* `https://en.wikipedia.org/w/index.php?title=Casimir_effect&oldid=1212199656`. ([Online; accessed 24-March-2024])