

Casimir Effect Calculation

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1 Proving Casimir Effect

Uncertainty Principle: $\Delta E \Delta t \approx \frac{\hbar}{2}$ and $\Delta x \Delta p \approx \frac{\hbar}{2}$

Casimir Effect: $\frac{\bar{E}}{A} = -\frac{\hbar c \pi^2}{720 a^3}$ where a is distance between two plates.

1. Compute Average Energy: \bar{E}

$$\bar{E} = \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t}$$

$$E_n = \hbar \omega_n = n \frac{\hbar c \pi}{a}$$

$$\Delta t_n = \frac{\hbar/2}{E_n} = \frac{a}{2nc\pi}$$

$$t = \frac{a}{c}$$

$$\bar{E} = \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t} = \sum_{n=1}^{\infty} n \frac{\hbar c \pi}{a} \times \frac{a}{2nc\pi} \times \frac{c}{a} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a}$$

2. Compute Effective Area: A

The ratio of one axis energy effecting on the area is $\Delta x_n \frac{\pi a}{a^2} = \frac{1}{2n}$

According to $\Delta x \Delta p \approx \frac{\hbar}{2}$, $\Delta x_n = \frac{\hbar/2}{p_n} = \frac{\hbar/2}{n\pi\hbar/a} = \frac{a}{2n\pi}$

derive from k_n (wave number) $= n \frac{\pi}{a}$, $p_n = \hbar k_n = \frac{n\pi\hbar}{a}$

The ratio of time of energy effecting on the area is $\frac{\Delta t_n}{t}$

Therefore $A = \sum_{n=1}^{\infty} \frac{\pi a^2}{4} / \left(\frac{1}{8n^3} \cdot \frac{\Delta t_n}{t} \right)$

3. Compute Energy per Effective Area: $\frac{\bar{E}}{A}$

$$\begin{aligned} \frac{\bar{E}}{A} &= \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t} \times \frac{1}{8n^3} \cdot \frac{\Delta t_n}{t} \cdot \frac{4}{\pi a^2} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a} \cdot \frac{1}{8n^3} \cdot \frac{a}{2\pi c n} \cdot \frac{c}{a} \cdot \frac{4}{\pi a^2} \\ &= \sum_{n=1}^{\infty} \frac{\hbar c}{8\pi^2 a^3 n^4} = \frac{\hbar c}{8\pi^2 a^3} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\hbar c}{8\pi^2 a^3} \zeta(4) = \frac{\hbar c \pi^2}{720 a^3} \end{aligned}$$