Casimir Effect Calculation

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1 Proving Casimir Effect

Uncertainty Principle: $\Delta E \Delta t \approx \frac{\hbar}{2}$ and $\Delta x \Delta p \approx \frac{\hbar}{2}$

Casimir Effect: $\frac{\bar{E}}{A} = -\frac{\hbar c \pi^2}{720 a^4}$ where a is distance between two plates.

1. Compute Average Energy: $\bar{E}$

$$\bar{E} = \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t}$$

$$E_n = \hbar \omega_n = n \frac{\hbar c \pi}{a}$$

$$\Delta t_n = \frac{\hbar/2}{E_n} = \frac{a}{2nc\pi}$$

$$t = \frac{a}{c}$$

$$\bar{E} = \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t} = \sum_{n=1}^{\infty} n \frac{\hbar c \pi}{a} \times \frac{a}{2nc\pi} \times \frac{c}{a} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a}$$

2. Compute Effective Area: A

The ratio of one axis energy effecting on the area is $\sqrt[3]{\frac{\Delta x_n^3}{4a^2/2\pi a}} = \frac{1}{\sqrt[3]{2}n}$

According to $\Delta x \Delta p \approx \frac{\hbar}{2}$, $\Delta x_n = \frac{\hbar/2}{p_n} = \frac{\hbar/2}{n\pi\hbar/a} = \frac{a}{2n\pi}$

derive from k_n (wave number) $= n\frac{\pi}{a}$, $p_n = \hbar k_n = \frac{n\pi\hbar}{a}$

The ratio of time of energy effecting on the area is $\frac{\Delta t_n}{t}$

Therefore, $A = \sum_{n=1}^{\infty} \pi a^2 / (\frac{1}{(\sqrt[3]{2}n)^3} \cdot \frac{\Delta t_n}{t})$

3. Compute Energy per Effective Area: $\frac{\bar{E}}{A}$

$$\frac{\bar{E}}{A} = \sum_{n=1}^{\infty} \frac{E_n \Delta t_n}{t} \times \frac{1}{(\sqrt[3]{2}n)^3} \cdot \frac{\Delta t_n}{t} \cdot \frac{1}{2\pi a^2} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a} \cdot \frac{1}{2n^3} \cdot \frac{a}{2\pi c n} \frac{c}{a} \cdot \frac{1}{\pi a^2}$$

$$= \sum_{n=1}^{\infty} \frac{\hbar c}{8\pi^2 a^4 n^4} = \frac{\hbar c}{8\pi^2 a^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\hbar c}{8\pi^2 a^4} \zeta(4) = \frac{\hbar c \pi^2}{720 a^4}$$