Derivation of Casimir Effect without Zeta-Regularization

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This paper presents a new way to derive the Casimir force. Unlike traditional methods, this approach avoids using zeta-regularization to handle infinities. This work achieves this by relating the vacuum energy to changes in position and length of energy states, incorporating the uncertainty principle. The resulting formula matches the standard Casimir effect formula, demonstrating an alternative method for understanding this phenomenon.

I. INTRODUCTION

The Casimir effect, a well-established force in quantum physics, is traditionally derived using zeta-regularization, a complex mathematical technique. This paper presents a new approach that avoids zeta-regularization, focusing on the energy density between conducting plates. We relate this energy to properties of confined energy states and incorporate the uncertainty principle. This method yields the standard Casimir effect formula, offering a potentially clearer understanding of the phenomenon.

II. DERIVATION

Precondition

1. Uncertainty Principle[1]:
$$\Delta E \Delta t \geq \frac{\hbar}{2} \; ; \; \Delta x \Delta p \geq \frac{\hbar}{2} \tag{1}$$
2. The conclusion of Casimir effect assuming zeta-regularization[2]:
$$\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3} \tag{2}$$
where A is the area of the metal plates and a is the distance between the metal plates.
3. a - The distance between two uncharged conductive plates in a vacuum.
4. The plates lie parallel to the xy-plane and is orthogonal to z-axis.

Calculate the Expected Vacuum Energy

$\langle E \rangle$ represents the expected vacuum energy, which is the average $\bar{E}$ of individual energy states (E_n) for all possible states (n) in the system.

$$\langle E \rangle = \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} \bar{E} \tag{3}$$

We can express the average energy $\bar{E}$ as the energy uncertainty of a state ΔE_n divided by the ratio of its existence time Δt_n to a characteristic timescale t. This timescale t represents the time it takes energy to travel across the gap a between the plates.

$$t = \frac{a}{c} \tag{4}$$

$$\bar{E} = \Delta E_n \frac{\Delta t_n}{t} \tag{5}$$

A key concept from the Heisenberg uncertainty principle states that the uncertainty energy of a state (ΔE_n) and its corresponding uncertainty in time (Δt_n) are related by a constant factor ($\hbar/2$).

$$\langle E \rangle = \sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a} \tag{6}$$

This expresses the expected vacuum energy density by summing the product of energy uncertainty and its corresponding time uncertainty ratio for all states.

Calculate the Area

The total area A is the sum of the areas A_n for each individual state n. The concept behind this summation is that the overall area accessible to a particle is determined by considering the allowed areas for each possible quantum state.

$$A = \sum_{n=1}^{\infty} A_n \tag{7}$$

The nth area A_n relates to the change in length and the change in position.

$$A_n = L_n^2 = \left(\frac{a}{\Delta x_n^{x,y} \cdot n^z} \cdot a\right)^2 \tag{8}$$

The wavenumber k_n associated with the nth state. It relates to the momentum and wavelength of the particle in that state.

$$k_n = \frac{n\pi}{a} \; ; \; p_n = \hbar k_n = \frac{n\pi\hbar}{a} \tag{9}$$

This equation calculates the change in position along the x and y directions $\Delta x_n^{x,y}$ for the nth state. It uses the Heisenberg uncertainty principle, which states that the product of momentum uncertainty and position uncertainty has a lower bound.

$$\Delta x_n^{x,y} = \frac{\hbar/2}{p_n} = \frac{\hbar/2}{n\pi\hbar/a} = \frac{a}{2n\pi} \tag{10}$$

n^z represents the ratio of a single wave in nth state passing through the z-axis to the plates.

$$n^z = \frac{a}{2n\pi} \cdot \frac{2\pi}{a} = \frac{1}{n} \tag{11}$$

Therefore

$$A_n = (\frac{a}{\Delta x_n^{x,y} \cdot n^z} \cdot a)^2 = 4n^4\pi^2 a^2 \tag{12}$$

$$A = \sum_{n=1}^{\infty} 4n^4\pi^2 a^2 \tag{13}$$

This summation considers all possible quantum states the particle can occupy within the system defined by the distance between the plates.

Calculate Vacuum Energy on Two Plates

$$\begin{aligned} \frac{\langle E \rangle}{A} &= \sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t} \frac{1}{A_n} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a} \cdot \frac{1}{4n^4\pi^2 a^2} \\ &= \frac{\hbar c}{8\pi^2 a^3} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\hbar c}{8\pi^2 a^3} \zeta(4) = \frac{\hbar c \pi^2}{720 a^3} \end{aligned} \tag{14}$$

III. PARADOX OF THE CASIMIR EFFECT

Define

1. L_o - Distance outside the plates
2. L_i - Distance inside the plates
3. P_o - Pressure outside the plates
4. P_i - Pressure inside the plates

Derivation

Consider the scenario where L_o approaches infinity, while L_i remains very small.

$$L_o \to \infty \; ; \; L_o \gg L_i \tag{15}$$

Since L_o becomes infinitely large, the pressure outside plates(P_o) remain constant regardless of the size of L_i.

$$L_o - Li \to L_o \tag{16}$$

According to the Casimir Effect[2], the pressure difference between inside and outside the plates is:

$$P_i - P_o = -\frac{\hbar c \pi^2}{240 L_i^4} \tag{17}$$

Situation 1:

$$L_i \to 0 \; ; \; P_i \geq 0 \tag{18}$$

$$P_o = P_i + \frac{\hbar c \pi^2}{240 L_i^4} \to \mathbb{R}^+ + \infty = \infty \tag{19}$$

The pressure outside the plates (P_o) would tend towards infinity. Consequently, the pressure difference would become infinitely negative regardless of L_i.

$$P_i - P_o \to P_i - \infty = -\infty \tag{20}$$

This outcome contradicts our observations.

Situation 2:

$$L_i \to 0 \; ; \; P_i < 0 \; ; \; P_i \to -\frac{\hbar c \pi^2}{240 L_i^4} \tag{21}$$

$$P_o = P_i + \frac{\hbar c \pi^2}{240 L_i^4} \to -\frac{\hbar c \pi^2}{240 L_i^4} + \frac{\hbar c \pi^2}{240 L_i^4} = 0 \tag{22}$$

This solution implies a negative pressure inside the platese ($P_i < 0$), which can be interpreted as **negative energy density** within that region. The results are consistent with our observations.

IV. CONCLUSION

This work calculates the absolute value of the energy density. However, since the energy inside the plates is negative, as previously established, the expected vacuum energy is also negative.

$$\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3} \tag{23}$$

The current work has explored the theoretical implications of negative pressure within the Casimir effect. A crucial next step is to definitively address the question of whether negative energy truly exists.

[1] Werner Heisenberg. *The physical principles of the quantum theory.* University of Chicago Press, 1930.

[2] H. B. G. Casimir. On the attraction between two perfectly conducting plates. *Indag. Math.*, 10(4):261–263, 1948.