Derivation of Casimir Effect without zeta-regularization

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1 Abstract

The Casimir effect describes the attractive force arising due to quantum fluctuations of the vacuum electromagnetic field between closely spaced conducting plates.

Traditionally, zeta-regularization is employed in calculations to address infinities that emerge during the derivation. This paper presents a novel derivation of the Casimir effect that circumvents the need for zeta-regularization.

We derive a formula for the average vacuum energy density between two perfectly conducting plates separated by distance a. Our approach relates the expected vacuum energy to the change in length and position associated with each energy state. The uncertainty principle is incorporated to calculate the area linked to each state.

The final result aligns with the standard Casimir effect formula obtained with zeta-regularization. This work demonstrates that the Casimir effect can be derived without relying on zeta-regularization, offering an alternative perspective on this well-established phenomenon.

2 Introduction

The Casimir effect, a cornerstone of quantum field theory, predicts an attractive force arising between closely spaced conducting plates due to quantum fluctuations of the vacuum electromagnetic field. This seemingly counterintuitive phenomenon has been experimentally verified and holds significant implications for miniaturized devices and nanotechnology.

Traditionally, the derivation of the Casimir effect relies on zeta-regularization, a mathematical technique used to handle infinities that can arise in quantum field theory calculations. While zeta-regularization is a powerful tool, it can introduce complexities and may obscure the underlying physical principles.

This paper presents an alternative approach to deriving the Casimir effect that avoids the use of zeta-regularization. Our derivation focuses on the average vacuum energy density between two perfectly conducting plates separated by a distance a.

We achieve this by: Relating the expected vacuum energy to the change in length and position associated with each energy state confined between the plates. Incorporating the uncertainty principle to calculate the area linked to each energy state. Through this approach, we arrive at a formula for the average vacuum energy density that aligns with the standard Casimir effect formula obtained with zeta-regularization. This work demonstrates that the key physical principles behind the Casimir effect can be understood without resorting to zeta-regularization, offering a potentially more transparent perspective on this fascinating phenomenon.

3 Condition

The following theories and conclusions are used to derive Casimir Effect.

1. Uncertainty Principle:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

2. Zero-point energy (ground state energy):

$$E_{\mathbb{Z}} = \frac{1}{2}\hbar\omega$$

3. The expectation value of the vacuum energy:

$$\langle E \rangle = \frac{1}{2}\sum_{n=1}^{\infty} E_n$$

4. The conclusion of Casimir effect assuming zeta-regularization:

$$\frac{\langle E \rangle}{A} = -\frac{\hbar c \pi^2}{720 a^3}$$

where A is the area of the metal plates. (Wikipedia contributors, 2024)

4 Derivation

Let

1. The distance between two uncharged conductive plates in a vacuum

$$a$$

2. The wavenumber orthogonal to the plates

$$k_n = \frac{n\pi}{a}$$

3. The plates lie parallel to the xy-plane and is orthogonal to z-axis

Calculate the Expected Vacuum Energy

$$\langle E \rangle = \frac{1}{2}\sum_{n=1}^{\infty} E_n = \frac{1}{2}\sum_{n=1}^{\infty} \bar{E} = \frac{1}{2}\sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t}$$

$$\omega_n = c \cdot k_n = n \cdot \frac{c\pi}{a}$$

$$\Delta E_n = \frac{1}{2}\hbar\omega_n = n \cdot \frac{\hbar c\pi}{2a}$$

$$\frac{\Delta t_n}{t} = \frac{\hbar/2}{\Delta E_n} \cdot \frac{1}{t} = \frac{a}{nc\pi} \cdot \frac{c}{a} = \frac{1}{n\pi}$$

$$\sum_{n=1}^{\infty} \Delta E_n \frac{\Delta t_n}{t} = \sum_{n=1}^{\infty} \frac{n\hbar c\pi}{2a} \cdot \frac{1}{n\pi} = \sum_{n=1}^{\infty} \frac{\hbar c}{2a}$$

- **Summation notation:** Define the expected vacuum energy (denoted by $\langle E \rangle$) as the average of individual energy states (represented by E_n). Since we're summing over infinitely many states, summation notation is used.

- **Relating energy and time:** The key idea is that the energy uncertainty (ΔE_n) is related to the time uncertainty (Δt_n) through a constant factor ($\hbar/2$).

- **Introducing wavenumber:** Define the wavenumber (k_n) for the nth state, which relates to the wave's frequency (ω_n) and the distance between the plates (a).

- **Calculating energy uncertainty:** Calculate the energy uncertainty for each state (ΔE_n) using the wavenumber and the proportionality constant from the uncertainty principle.
- **Deriving time uncertainty ratio:** Calculate the ratio of the time uncertainty for each state (Δt_n) to the total time period (t). This ratio is inversely proportional to the energy uncertainty.
- **Summing the expected energy:** This expresses the expected vacuum energy density by summing the product of energy uncertainty and its corresponding time uncertainty ratio for all states, divided by the total area (which will be calculated later). This summation represents the average energy density across all possible quantum states.

Calculate the Area

$$A = \sum_{n=1}^{\infty} A_n$$

The nth area A_n relates to the change in length and the change in position.

$$\begin{aligned} A_n &= a_n^2 = (\frac{L_n}{\Delta x_n^{x,y}} \cdot a)^2 \\ L_n &= na \\ p_n &= \hbar k_n = \frac{n\pi\hbar}{a} \end{aligned}$$

The length of x, y axis waves locating on the nth area is

$$\Delta x_n^{x,y} = \frac{\hbar/2}{p_n} = \frac{\hbar/2}{n\pi\hbar/a} = \frac{1}{2n\pi a}$$

Therefore

$$A_n = (\frac{L_n}{\Delta x_n^{x,y}} \cdot a)^2 = 4n^4\pi^2 a^2$$

- **Relating area and uncertainty:** The concept is that the area for a particular state is related to the change in length (L_n) and the change in position ($\Delta x_n^{x,y}$) along the x and y axes.
- **Defining state variables:** Define the change in length for the nth state (L_n) and the momentum (p_n) associated with that state.

- **Calculating x,y position uncertainty:** Due to the confinement in the z-axis, this calculates the change in position along the x and y directions for the nth state ($\Delta x_n^{x,y}$). This is derived using the uncertainty principle and the momentum along those directions.
- **Expressing Area:** The formula for the area (A_n) of the nth state relates the area to the change in length, change in position along x and y, and the distance between the plates.

Calculate Vacuum Energy on Two Plates

$$\begin{aligned}
\frac{\langle E\rangle}{A} &= 2\cdot\frac{1}{2}\sum_{n=1}^{\infty}\Delta E_n\frac{\Delta t_n}{t}\frac{1}{A_n}\\
&= \sum_{n=1}^{\infty}\frac{\hbar c}{2a}\cdot\frac{1}{4n^4\pi^2a^2} = \sum_{n=1}^{\infty}\frac{\hbar c}{8n^4\pi^2a^3} = \frac{\hbar c}{8\pi^2a^3}\sum_{n=1}^{\infty}\frac{1}{n^4}\\
&= \frac{\hbar c}{8\pi^2a^3}\zeta(4) = \frac{\hbar c}{8\pi^2a^3}\cdot\frac{\pi^4}{90} = \frac{\hbar c\pi^2}{720a^3}
\end{aligned}$$

5 Conclusion

This derivation for the Casimir effect without zeta-regularization is that the average vacuum energy density (energy per unit area, $\langle E\rangle/A$) between two perfectly conducting plates separated by distance a is:

$$\frac{\hbar c\pi^2}{720a^3}$$

This paper's derivation of the Casimir effect, lacking the negative sign, differs from the standard approach which attributes the negative sign to the existence of negative energy between the plates.

While this paper calculates the absolute value of the energy density, the standard derivation focuses on the energy difference, where the negative sign reflects the lower energy state within the cavity due to the presence of the plates.

References

Wikipedia contributors. (2024). *Casimir effect — Wikipedia, the free encyclopedia.* `https://en.wikipedia.org/w/index.php?title=Casimir_effect&oldid=1212199656`. ([Online; accessed 24-March-2024])