Construction and complement circuit of a quantum stabilizer code with length 7

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Construction and Complement Circuit of a Quantum Stabilizer Code with Length 7

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***Abstract*— In this paper, a new method for the construction of a quantum stabilizer code from circulant permutation matrices is discussed. First, we choose a finite-length vector randomly, and we can construct circulant permutation matrices from the vectors. Then, the parity-check matrix can be produce from the circulant permutation matrices. Hence, the generators of stabilizer code are determined according to the parity-check matrix and quantum stabilizer group are defined from the generators. From the stabilizer group, codewords of the proposed quantum codes can also be generated. Finally, a complete efficient encoding and decoding quantum circuit of [[7,1,3]] is proposed. [[7,1,3]] is stabilizer code that construction based on our method is an seven-qubit code that protects a one-qubit state with up to one error, which is very important for quantum information processing.**

***Keywords*— Circulant permutation matrices, quantum error-correction codes, stabilizer codes, symplectic inner product.**

I. INTRODUCTION

Starting with the ideas of P. Benioff (1980) and R. Feynman (1982) [1], quantum theory of information has been developed. Until now, quantum processing devices have a great deal of potential for various tasks such as factorizing large number, searching a pattern in database, cryptography. And they showed better performances than classical computers even if the best currently known algorithms are considered in classical processing devices [2].

However, to realize that potential, methods to protect fragile quantum states from unwanted evolution, errors are needed [3]. Hence, quantum error correcting codes (QECC) have been developed to protect quantum information from these errors [4]. Stabilizer codes, first introduced by Gottesman [5], have become an important class of QECC. These codes are useful for building quantum fault tolerant circuits [6]. Stabilizer codes appends ancilla qubits to qubits to be protected, and the most important advantage of stabilizer codes is that errors can be detected and removed from operators rather than from the quantum state itself. In addition, the stabilizer formalism allows us to construct stabilizer codes from binary formalism as the classical parity-check matrix over binary or quaternary alphabet and that constraint referred to as the symplectic inner product (SIP) [5]. Therefore, a variety of stabilizer codes have been proposed and whose constructions are analogous to classical linear code such as quantum code from arbitrary binary matrix [7], quantum BCH [8], quantum Reed-Solomon codes [9], quantum convolutional codes [10], fast quantum code based on Pauli block jacket matrices [11], codeword stabilized quantum codes [12], non-binary quantum stabilizer code [13] and quantum non-additive codes [14], quantum code based on LDPC classical code [15]…

The key results of this paper are to propose new methods for construction of the parity-check matrices from circulant permutation matrices. This parity check matrices are proved to satisfy commutative constraint to obtain the generators of quantum stabilizer code. These generators should first be chosen from the parity-check matrices for the independent constraint, and stabilizer group is determined from the generators. Then, codewords of the proposed codes are determined from the stabilizer group and corresponding of coding and decoding circuit are explained.

The organization of the paper is as follow. We review the theory of quantum mechanics, the role of quantum stabilizer codes on quantum error correction as well as how general stabilizer codes are related to classical binary codes in section I. In section III, we explain the proposed construction, practical with an example and corresponding coding, decoding circuit. Finally, conclusions are presented in Section IV.

II. QUANTUM STABILIZER CODE

A. *Qubit and quantum elementary operations*

The *bit* is the fundamental concept of classical computation and classical information. Quantum computation and quantum information are built upon an analogous concept, the quantum *bit*, or *qubit* for short. Then, just as the classical bit has state either 0 or 1, the state in quantum system is instead a vector over the complex number $\boldsymbol{C}$, the state denoted as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ can be considered to have both values of $|0\rangle$ and $|1\rangle$ at the same time, where the probability of value $|0\rangle$ is $|\alpha|^2$ and probability of value $|1\rangle$ is $|\beta|^2$. This concept known as superposition is the main property of quantum computation since it allows gate operations to deal with several values in one step. Hence, the amount of information that can be represented is infinite. A qubit can be representing in vector form:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \alpha\begin{bmatrix}1\\0\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}\alpha\\\beta\end{bmatrix}.$$

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According to the probability, the condition $|\alpha|^2+|\beta|^2=1$ must be satisfied. A quantum memory register is a physical system composed of n qubits, that is multiple by tensor product of some qubits. Generally, the n-qubits state is denoted as

$$|\psi\rangle=\sum_{i_k=\{0,1\}}\alpha_{i_1 i_2 \dots i_n}|i_1\rangle\otimes|i_2\rangle\otimes\dots\otimes|i_n\rangle=\sum_i \alpha_i|i\rangle,$$

where $i=\sum_{k=1}^{n}2^{n-k}i_k$.

Hence, a state vector of n-qubit quantum system is considered as superposition of the states that make up a base in 2^n dimensional complex Hilbert space $\mathrm{H}^{\otimes n}$.

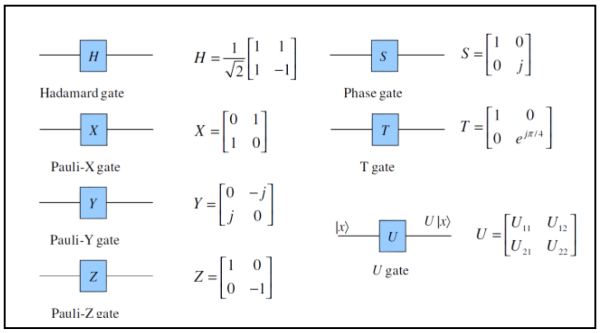

Figure 1. Summary of important quantum gates.

A particularly fruitful way to understand a quantum system is to look at the behavior of various operators acting on the states of the system. Quantum information processing requires unitary transformations operating on states, quantum gate U: $U^{-1}=U^{\dagger}$. Some important quantum gates are showing in Figure 1. In quantum depolarizing channel, (the identity matrix), **X**, **Z** and **Y** form orthogonal basis of linear space of operators acting on qubit and any error acting on qubit can be represented as the combination of ,**X**,**Z** and **Y**.

$$\mathbf{I}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\mathbf{X}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\mathbf{Z}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\text{and }\mathbf{Y}=\begin{bmatrix}0 & -j\\ j & 0\end{bmatrix}.$$

Thus, the Pauli operators **X**, **Z,** and **Y** are regarded as a bit flip, a phase flip, and a combination of bit and phase flips, respectively. Multiplication of two Pauli operators satisfies as the following equations.

$$\mathbf{X}^2=\mathbf{Y}^2=\mathbf{Z}^2=\mathbf{I};\ \mathbf{X}\times\mathbf{Y}=-\mathbf{Y}\times\mathbf{X};$$
$$\mathbf{Y}\times\mathbf{Z}=-\mathbf{Z}\times\mathbf{Y};\ \mathbf{Z}\times\mathbf{X}=-\mathbf{X}\times\mathbf{Z}.$$

The single Pauli group P_1 is a group formed by the Pauli operators, which is closed under multiplication. The n-fold tensor product of single Pauli operators forms an n-qubit Pauli group P_n . The main property of P_n is that any two elements $\mathbf{A},\mathbf{B}\in P_n$ either commute or anticommute. For n–qubit Pauli operators $\mathbf{A},\mathbf{B}\in P_n$, the operator $\circ$ for commutativity is defined as

$$\mathbf{A}\circ\mathbf{B}=\prod_{i=1}^{N}\mathbf{A}_i\bullet\mathbf{B}_i$$

,where $\mathbf{A}_i\bullet\mathbf{B}_i=\begin{cases}+1, & \text{if } \mathbf{A}_i\times\mathbf{B}_i=\mathbf{B}_i\times\mathbf{A}_i\\ -1, & \text{if } \mathbf{A}_i\times\mathbf{B}_i=-\mathbf{B}_i\times\mathbf{A}_i\end{cases}$.

Two operators **A**, **B** are commutative if and only if $\mathbf{A}\circ\mathbf{B}=+1$; otherwise, they are anti-commutative. Commutativity is an important feature of the Pauli group since this can be used to detect errors within the stabilizer formalism in the next section.

B. *Quantum error correction and stabilizer code*

A stabilizer group $\boldsymbol{S}$ is an Abelian subgroup of P_n such as a non-trivial sub-space C_S of $\mathrm{H}^{\otimes n}$ is stabilized by $\boldsymbol{S}$. The sub-space C_S defines a quantum code space such that

$$C_S=\{|\psi\rangle\in\mathrm{H}^{\otimes n}\mid\mathbf{g}|\psi\rangle=|\psi\rangle,\forall\mathbf{g}\in S\}.$$

The stabilizer group $\boldsymbol{S}$ consists of a set of Pauli operators on n qubits and is closed under multiplication. The stabilizer group also has the property that any two operators commute in the set and not include $-\mathbf{I}$. It is enough to check the commutative property on a set of generators of $\boldsymbol{S}$, i.e., on a set $\{\mathbf{g}_i\}$ that generates all elements of $\boldsymbol{S}$ under multiplication. If $\boldsymbol{S}$ generates from $\mathbf{g}=\{\mathbf{g}_1,\mathbf{g}_2,\dots,\mathbf{g}_{n-k}\}$, which are mutually independent operators, the code space C_S will encode k logical qubits into n physical qubits and it can correct $t=\lfloor(d_{\min}-1)/2\rfloor$ errors. This code C_S is called $[[n,k,d_{\min}]]$ quantum stabilizer code.

Considering a set of error operators $\{\mathbf{E}\}\subset P_n$, the collection of Pauli operators takes a state $|\psi\rangle$ to the corrupted state $\mathbf{E}|\psi\rangle$ .A given operator **E** either commutes or anticommute with each stabilizer $\mathbf{S}_i$. Then, the corrupted state $\mathbf{E}|\psi\rangle$ is diagnosed by elements $\mathbf{S}_i$ of the set $\boldsymbol{S}$. The outcome of the diagnostic procedure is a vector of $\{+1,-1\}$ indicating whether or not **E** can be detected. The indication for the error detection is expressed as

$$\mathbf{S}_i\times\mathbf{E}|\psi\rangle=\begin{cases}\mathbf{E}\times\mathbf{S}_i|\psi\rangle=\mathbf{E}|\psi\rangle, & \text{Error undetected.}\\ -\mathbf{E}\times\mathbf{S}_i|\psi\rangle=-\mathbf{E}|\psi\rangle, & \text{Error detected.}\end{cases}$$

The condition for quantum error correction is that **E** is a set of correctable error operators for C_S if

$$\mathbf{E}_i^{\dagger}\mathbf{E}_j\notin N(S)\setminus S,\ \forall\mathbf{E}_i,\mathbf{E}_j\in\mathbf{E},$$

where $\mathbf{E}_i^{\dagger}$ is conjugate transpose of $\mathbf{E}_i$ and $N(S)$ is the normalizer of S in P_n such as

$$N(S)=\{\mathrm{A}\in P_n\mid\mathrm{A}^{\dagger}\mathrm{E}\mathrm{A}\in S,\ \forall\,\mathrm{E}\in S\}.$$

Note that $N(S)$ is the collection of all operators in P_n that commutes with $\boldsymbol{S}$ and $S\subset P_n$. Then, the minimum distance $d_{\min}$ of stabilizer code is determined by

$$d_{\min}=\min(\mathrm{W}(\mathbf{E}))\ \text{s.t.}\ \mathbf{E}\in N(S)\setminus S,$$

where the weight of an operator, $\mathrm{W}(*)$, is the numbers of positions not equal to Pauli operator **I**.

Quantum stabilizer code can be expressed in the binary field since any given Pauli operator on n qubit can be composed into an **X**–containing operator and a **Z**–containing operator, for example: $\mathbf{XYYZI} = \mathbf{XXXII} \times \mathbf{IZZZI}$ .This is achieved by mapping **I**, **X**, **Y**, **Z** as follows: $\mathbf{I} \to (0,0); \mathbf{X} \to (1,0); \mathbf{Y} \to (1,1); \mathbf{Z} \to (0,1)$ .Then, the $n-k$ generators of an [[*n*,*k*]] stabilizer code can be expressed as a concatenation of a pair of $(n-k) \times n$ binary matrices $\mathbf{H_X}, \mathbf{H_Z}$. Then, the parity-check matrix **H** of the quantum stabilizer code is defined as

$$\mathbf{H} = [\mathbf{H_X} \mid \mathbf{H_Z}]. \tag{1}$$

The commutative property of the stabilizers can be transformed into the orthogonality of rows in the matric forms with respect to the symplectic product. If the *m*-th row $\mathbf{r}_m$ is expressed as $\mathbf{r}_m = [\mathbf{x}_m \mid \mathbf{z}_m]$, where $\mathbf{z}_m$ and $\mathbf{x}_m$ are binary strings for **Z** and **X,** respectively, then the symplectic product of the m_1-th row and m_2-th row in the parity-check matrix **H** in (1) is expressed as

$$\mathbf{r}_{m_1} \odot \mathbf{r}_{m_2} = [\mathbf{x}_{m_1} \mid \mathbf{z}_{m_1}] \odot [\mathbf{x}_{m_2} \mid \mathbf{z}_{m_2}] = \mathbf{x}_{m_1} * \mathbf{z}_{m_2} + \mathbf{x}_{m_2} * \mathbf{z}_{m_1}, \tag{2}$$

where $\mathbf{x}_k * \mathbf{z}_l = \sum_{i=1}^{n} \mathbf{x}_{ki} \times \mathbf{z}_{li}$.

The linear combinations among rows of parity-check matrix **H** generate the stabilizer group **S** in binary modulo-2 addition. Since the dual-space of **H** has the dimension of $2n - m = m + 2k$, the normalizer group $N(S)$ that commutes with **S** can be considered as the dual-space of **S** generated by a $(m+2k) \times 2n$ binary matrix. The last 2k rows are called the logical operators $\overline{\mathbf{X}}$ and $\overline{\mathbf{Z}}$ which satisfy the following conditions

$$\begin{cases} \overline{\mathbf{X}_i} \circ \overline{\mathbf{X}_j} = +1 \\ \overline{\mathbf{Z}_i} \circ \overline{\mathbf{Z}_j} = +1 \\ \overline{\mathbf{X}_i} \circ \overline{\mathbf{Z}_j} = +1 \text{ for } i \neq j \\ \overline{\mathbf{X}_i} \circ \overline{\mathbf{Z}_j} = -1 \text{ for } i = j \end{cases} \tag{3}$$

From a stabilizer in standard form by using Gaussian elimination, the parity-check matrix **H** can be uniquely determined as

$$\begin{array}{c} \begin{array}{cccccc} R & N-K-R & K & R & N-K-R & K \end{array} \\ \left[\begin{array}{cccccc} \mathbf{I} & \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{B} & \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{I} & \mathbf{E} \end{array} \right] \begin{array}{l} \}\ R \\ \}\ N-K-R \end{array} \end{array} . \tag{4}$$

The standard form of the logical operators is given as follows:

$$\begin{cases} \overline{\mathbf{X}} = \begin{bmatrix} \mathbf{0} & \mathbf{E}^T & \mathbf{I} & (\mathbf{E}^T\mathbf{C}_1 + \mathbf{C}_2{}^T) & \mathbf{0} & \mathbf{0} \end{bmatrix} \\ \overline{\mathbf{Z}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}_2{}^T & \mathbf{0} & \mathbf{I} \end{bmatrix} \end{cases} . \tag{5}$$

The operation of encoding a general stabilizer code can be described as

$$\left| \overline{\sigma_1 \sigma_2 \ldots \sigma_k} \right\rangle = \frac{1}{\sqrt{2^{n-k}}} \times \left(\prod_{i=1}^{n-k} (\mathbf{I} + \mathbf{g}_i) \right) \times \overline{\mathbf{X}_1}^{\sigma_1} \times \overline{\mathbf{X}_2}^{\sigma_2} \times \ldots \times \overline{\mathbf{X}_k}^{\sigma_k} \left| 00\ldots0 \right\rangle_n ,$$

where $|00\ldots0\rangle$ is the *n*-qubit state and $c_i \in \{0,1\}$. (6)

III. DESIGN QUANTUM STABILIZER CODE

A. Propose circulant permutation construction

By using two randomly chosen vectors with length 7, construction method of a quantum stabilizer code with length 7 is explained in Example 3.1.

Example 3.1: For $n = 7$, two vectors **u**= [1 1 0 0 1 0 1], **v**= [1 0 0 1 0 1 1], are considered. From the two vectors, we have the corresponding parity-check matrix as

$$\mathbf{H} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

where the left part and right part of **H** is determined by shifting **u** right and shifting **v** left, respectively.

Then, 6 generators are chosen from the first 6 rows of matrix **H** which satisfies the independent condition to generate all stabilizers set. By using Gaussian elimination, matrix **H** can transform into standard form as

$$\mathbf{H}_{st} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{7}$$

And the corresponding generators are as following:

$$\begin{aligned} \mathbf{g}_1 &= [1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1] \\ \mathbf{g}_2 &= [0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1] \\ \mathbf{g}_3 &= [0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0] \\ \mathbf{g}_4 &= [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1] \\ \mathbf{g}_5 &= [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1] \\ \mathbf{g}_6 &= [0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0] \end{aligned} . \tag{8}$$

The stabilizer code [[7,1,3]] that encode 1 information qubit into 7 physical qubit, will have logical operators $\overline{\mathbf{X}}_1$ and $\overline{\mathbf{Z}}_1$. From $\mathbf{H}_{st}$, the logical operators $\overline{\mathbf{X}}_1$ and $\overline{\mathbf{Z}}_1$ are calculated as:

$$\begin{aligned} \overline{\mathbf{X}}_1 &= [0\ \ 0\ \ 0\ \ 1\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0], \\ \overline{\mathbf{Z}}_1 &= [0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 1\ \ 0\ \ 0\ \ 0\ \ 1]. \end{aligned} \tag{9}$$

Then collection of all operators in P_7 that commutes with ***S***, $N(S)$ is generated from all generators and logical operators. Hence, the minimum distance is calculated as

$$d_{\min} = \min(\mathrm{W}(\mathbf{E})) = 3 \text{, where } \mathbf{E} \in N(S) \setminus S.$$

It means the stabilizer code [[7,1,3]] with the operators from (8) and (9) can correct one error. The codewords of [[7,1,3]] stabilizer code is calculated as equation (6) from the operators in (8) and (9).

□

B. Complete circuit of encoding and decoding

We will consider quantum circuit of [[7,1,3]] quantum stabilizer code given in example 3.1. The efficient way for encoding and decoding due to Gottesman [5](see also reference [4][16]).

According to the equation (6), we obtain the basis codewords:

$$\left|\overline{\sigma_1\sigma_2...\sigma_k}\right\rangle = \frac{1}{\sqrt{2^{n-k}}}\times\left(\prod_{i=1}^{n-k}(\mathbf{I}+\mathbf{g}_i)\right)\times\overline{\mathbf{X}}_1^{\sigma_1}\times\overline{\mathbf{X}}_2^{\sigma_2}\times...\times\overline{\mathbf{X}}_k^{\sigma_k}\left|00...0\right\rangle_n$$
$$=\left(\prod_{i=1}^{r}T_i^{\sigma_i}\mathbf{H}_i\right)\times\left(\prod_{l=1}^{k}\mathbf{U}_l\right)\left|00...0\sigma_1\sigma_2...\sigma_k\right\rangle_n.$$

where r is rank of $\mathbf{X}$ part in $\mathbf{H}_{st}$, T_i and $\mathbf{U}_l$ are calculated from logical operators $\overline{\mathbf{X}}$, $\overline{\mathbf{Z}}$ and $\mathbf{H}_{st}$.Hence, the effective encoding circuit for [[7,1,3]] are shown in figure 2.

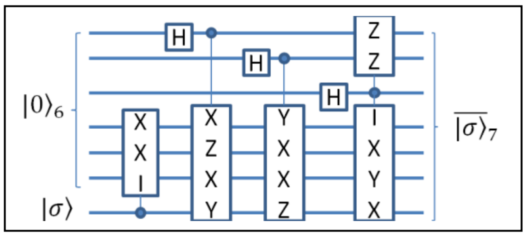

Figure 2. Encoding circuit for quantum [[7,1,3]] code.

As a stabilizer, the important for [[7,1,3]] code is to correct one error from seven-qubits. Thus, the design for decoding circuit of [[7,1,3]] code is necessary. Assume the syndrome of a receive state $\left|\psi\right\rangle$ is defined as $l=\left|l_1l_2...l_{n-k}\right\rangle$ where $l_i\in\{0,1\}$ and

$$g_i\left|\psi\right\rangle=(-1)^{l_i}\left|\psi\right\rangle \Leftrightarrow g_iE=(-1)^{l_i}Eg_i.$$

Hence, the syndrome calculation for [[7,1,3]] stabilizer code with weight one error is given in table 1. From the syndrome, the correction part of decoding is given as figure 3, the main operation are Hadamard operations, controlled gates and multi-controlled gates [17]. We will have the corrected codeword after moving to correction part; it is needed to convert into original qubit in the next section.

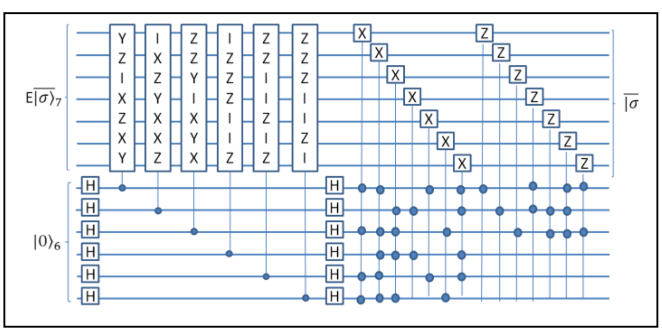

Figure 3. Error correction for decoding circuit for [[7,1,3]].

TABLE I. SYMDROME CALCULATION OF QUANTUM [[7,1,3]] STABILIZER CODE

E	Syndrome	E	Syndrome	E	Syndrome
XIIIIII	101011	**ZIIIIII**	100000	**YIIIIII**	001011
IXIIIII	101111	**IZIIIII**	010000	**IYIIIII**	111111
IIXIIII	011101	**IIZIIII**	001000	**IIYIIII**	010101
IIIXIII	010100	**IIIZIII**	110000	**IIIYIII**	100100
IIIIXII	100010	**IIIIZII**	011000	**IIIIYII**	111010
IIIIIXI	001001	**IIIIIZI**	111000	**IIIIIYI**	110001
IIIIIIX	110110	**IIIIIIZ**	101000	**IIIIIIY**	011110
XIIIIII	101011	**ZIIIIII**	100000	**YIIIIII**	001011

To recover original state, Gottesman's approach is to introduce k ancillary qubits in state$\left|0\right\rangle_k$, and decoding takes the initial state$\left|\psi_{in}\right\rangle=\left|\overline{\sigma_1\sigma_2...\sigma_k}\right\rangle_n\otimes\left|00...0\right\rangle_k$ to the final state:

$$\left|\psi_f\right\rangle=\mathbf{U}_{decode}\left|\overline{\sigma_1\sigma_2...\sigma_k}\right\rangle_n\otimes\left|00...0\right\rangle_k=\left|\overline{00...0}\right\rangle_n\otimes\left|\sigma_1\sigma_2...\sigma_k\right\rangle_k.$$

Decode operations can be performed in two stages. First, by applying properly the CNOT gates to ancillary qubits, we put the ancillary into decode state$\left|\overline{\sigma_1\sigma_2...\sigma_k}\right\rangle_n\otimes\left|\sigma_1\sigma_2...\sigma_k\right\rangle_k$. Then, by applying a controlled-$\overline{\mathbf{X}}_i$ operation to each encoded qubit i (the i-th ancillary qubit), we complete the decoding operations. Figure 4 describes the sequence of decoding circuit for [[7,1,3]] stabilizer code.

□

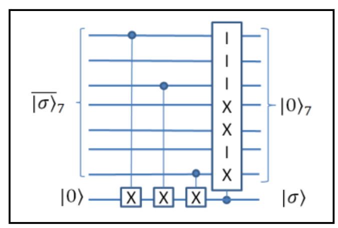

Figure 4. Decoding circuit for [[7,1,3]].

IV. CONCLUSION

In this paper, the new construction is proposed in detail based on the mathematical way circulant permutation structure. Then, the length seven [[7,1,3]] quantum stabilizer code and its quantum circuit for encoding and decoding are given based on standard form of parity-checked matrices and corresponding logical operators. Circulant permutation matrices can be created in any length, it's open up the possibility of identifying any length of quantum stabilizer codes with efficient to correct error when qubit transfers through quantum channel.

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