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A novel construction for quantum stabilizer codes based on binary formalism

Duc Manh Nguyen* and Sunghwan Kim†

Coding and Information Theory Laboratory, University of Ulsan, Ulsan, Korea
**nguyenmanhduc18@gmail.com*
†sungkim@ulsan.ac.kr

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In this research, we propose a novel construction of quantum stabilizer code based on a binary formalism. First, from any binary vector of even length, we generate the parity-check matrix of the quantum code from a set composed of elements from this vector and its relations by shifts via subtraction and addition. We prove that the proposed matrices satisfy the condition constraint for the construction of quantum codes. Finally, we consider some constraint vectors which give us quantum stabilizer codes with various dimensions and a large minimum distance with code length from six to twelve digits.

Keywords: Quantum stabilizer codes; binary formalism; symplectic inner-products.

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1. Introduction

Quantum computation is problem solving and data computation using a system based on quantum mechanics, which is a result of the effort to generalize classical computation. Quantum information systems can transmit data fundamentally, securely and solve complex problems better than classical information systems.[1] In 1994, Peter Shor invented a quantum algorithm for factoring integers into prime factors which runs on polynomial time.[2] Hence, quantum computers could break public-key cryptography schemes such as RSA. In addition, in 1996, Lov Grover devised a quantum search algorithm for searching unstructured databases,[3] yielding quadratic speedup over classical search algorithms. Hence, quantum algorithms promised big improvements on performance in comparison to classical computation; consequently, many problems have been considered in quantum computation.[4–6]

†Corresponding author.

The well-verified phenomena of quantum mechanics inspired scientists to propose different models for quantum computation, namely the circuit model, Zidan's model, adiabatic model and topological model. Zidan's model solves the quantum computing problems by measuring the degree of entanglement among two auxiliary qubits via two techniques.[7–9] The first technique solves the quantum computation problem based on the detection of entanglement,[8] which was used to solve some problems in quantum computing and quantum machine learning. The second technique of Zidan's model solves the quantum problem using Zidan's roots, which was used to solve an extended version of the Deutsch–Jozsa algorithm.[9] This extension was intractable for more than 27 years using the quantum circuit model. However, the effects from imperfectly applied quantum gates, decoherence and other quantum channel noise would affect the practical design of quantum computation. To overcome such problems, quantum error correction code (QECC), which is based on the theory of error correction code, was developed to protect quantum states. Since the first QECC was discovered by Shor for 9-qubit code[10] and Stean for 7-qubit code,[11] the method of containing redundancy to circumvent the no-cloning theorem has been popularly used. Therefore, the importance of QECC in paving the way to build a practical quantum computer is no longer in doubt.

In 1998, Gottesman introduced the idea of using stabilizer formalism to append ancilla qubits to qubits we want to protect. As a consequence, the quantum stabilizer codes restore a noisy, decohered quantum state to a pure quantum state using quantum stabilizer operators, rather than from the quantum state itself.[12] In addition, the stabilizer formalism allows one to import some classical binary or quaternary codes for construction of quantum codes. However, the classical codes must satisfy the self-orthogonal constraint or the symplectic inner product (SIP). Therefore, there are many studies on the construction of quantum stabilizer codes. First, the construction based on novel classical codes such as: quantum LDPC codes,[13] quantum Turbo codes,[14] quantum BCH codes,[15] quantum Reed Solomon codes,[16] and quantum MDS codes.[17] In addition, the use of combinatorial design to build up parity-check matrices for the construction of quantum stabilizer codes has also been discussed, such as quantum residue codes,[18] quantum codes from difference sets,[19,20] quantum codes from group association schemes,[21] fast quantum codes based on Pauli block jacket matrices,[22] quantum matrices product codes,[23] or quantum codes based on graphs.[24] Moreover, the quantum stabilizer codes have been proven to correspond to additive codes over the Galois field of order 4 ($GF(4)$).[25] As a consequence, many researches have focused on the construction of additive ($GF(4)$) codes with respect to the Hermitian product[26] and the trace-inner product.[27] However, all the proposed constructions only give the results with limitations on the code length or the dimension of quantum codes. Our novel framework for an adaptive construction allows for any code length of varying dimensions.

This research proposes a novel construction of quantum stabilizer codes based on a binary formalism. The advantages of the proposed construction is that it allows quantum stabilizer codes for any code length of varying dimensions. The paper is organized as follows: In Sec. 2, we present the preliminary quantum information and quantum code based on binary formalism. In Sec. 3, the proposed construction is discussed with 10 examples for code lengths from six to twelve digits. Finally, the paper is concluded in Sec. 4.

2. Preliminary

2.1. *Quantum information and quantum computation*

In quantum computation, a quantum bit or qubit is the basic unit of quantum information, which is the quantum version of the classical binary bit realized with a two-state device. Since a qubit uses the superposition principle of the two basic states, it can be expressed by a two-dimensional Hilbert space (H_2) complex number, where a Hilbert space is spanned by two basic states as

$$H_2 = \text{span}\left\{ |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

Therefore, a superposition state of a quantum system is denoted as $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are complex numbers that satisfy the equation: $|\alpha|^2 + |\beta|^2 = 1$. In a quantum system, if quantum states are used n times in a single qubit (n-qubits) physical system, the system consists of the n times tensor product of a two-dimensional Hilbert space (${H_2}^{\otimes n}$).

In classical computation, Boolean functions are performed over a single bit. In the case of quantum computation, reversible operations represented by unitary matrices are performed over a qubit. Representative quantum operations are Pauli operators which include four operators, $\mathbf{I}$, $\sigma_{\mathbf{X}}$, $\sigma_{\mathbf{Y}}$ and $\sigma_{\mathbf{Z}}$:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_{\mathbf{X}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_{\mathbf{Y}} = j\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_{\mathbf{Z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $j = \sqrt{-1}$. Operators $\sigma_{\mathbf{X}}$, $\sigma_{\mathbf{Z}}$ and $\sigma_{\mathbf{Y}}$ are regarded as a bit flip, a phase flip and a combination of bit and phase flips, respectively. Multiplication between single Pauli operators has a commutative property. Hence, Pauli group P_1 on a qubit is a group composed of Pauli operators and their multiplications with the factor ± 1, $\pm j$ and $P_1 = \pm\{\mathbf{I},\ \sigma_{\mathbf{X}},\ j\sigma_{\mathbf{X}},\ \sigma_{\mathbf{Y}},\ j\sigma_{\mathbf{Y}},\ \sigma_{\mathbf{Z}},\ j\sigma_{\mathbf{Z}}\}$. The Pauli group on n qubits P_n is defined as the n times tensor product of the Pauli operators. Then, the elements of P_n are either commutative or anti-commutative. The commutative operator "$\circ$" for two operators $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A} \circ \mathbf{B} = \sum_{i=1}^{n} \mathbf{A}_i \bullet \mathbf{B}_i,$$

where

$$\mathbf{A}_i \bullet \mathbf{B}_i = \begin{cases} 0, & \text{if} \quad \mathbf{A}_i \times \mathbf{B}_i = \mathbf{B}_i \times \mathbf{A}_i, \\ 1, & \text{if} \quad \mathbf{A}_i \times \mathbf{B}_i = -\mathbf{B}_i \times \mathbf{A}_i \end{cases}.$$

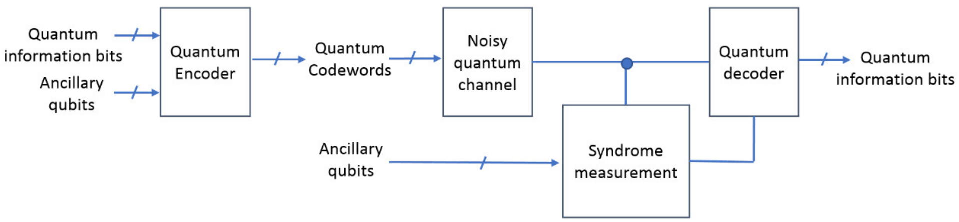

Fig. 1. Quantum error correction operating process.

Then, two operators $\mathbf{A}$ and $\mathbf{B}$ are said to be commutative if $\mathbf{A} \circ \mathbf{B} = \mathbf{0}$ and they are noncommutative if $\mathbf{A} \circ \mathbf{B} \neq 0$. Quotient group P_n/C, where $\mathrm{C} = \{\pm\mathbf{I}, \pm j\mathbf{I}\}$ is defined as the center of P_n. Therefore, the notation $\mathbf{X} \leftrightarrow \sigma_{\mathbf{X}}$, $\mathbf{Y} \leftrightarrow -j\sigma_{\mathbf{Y}}$, $\mathbf{Z} \leftrightarrow \sigma_{\mathbf{Z}}$ is used in the rest of the paper.

2.2. *Quantum error correction codes and binary formalism*

In classical error correcting code, it is easy to make a copy of the information. In contrast, it is impossible to make a copy of quantum information due to the non-cloning theorem.[29] Therefore, quantum information can be extended to highly entangled quantum states with the help of ancillary qubits and unitary transforms. Classical error correcting codes use syndrome measurement to diagnose errors which corrupt an encoded state. QECC also employs syndrome detection with the help of quantum stabilizer operators. A block diagram of the QECC process is shown in Fig. 1. The quantum information can be protected from noisy quantum channels with the help of ancillary qubits, the quantum stabilizer operators and syndrome measurement.

The quantum code for the binary case with parameter $[[n, k, d_{\min}]]$ encodes k information qubits into the system of n qubits, and it can correct the $\lfloor \frac{d_{\min}-1}{2} \rfloor$ error. The first quantum code is the Shor code with parameters [[9,1,3]], which is based on repetition codes. The second quantum code is [[7,1,3]], which is based on the classical hamming code [7,4,3] with a CSS structure. Next, the stabilizer formalism is used to express the quantum codes. With the stabilizer formalism, quantum codes are viewed via group theory of the quantum stabilizer operator; thus, we are working with quantum operators rather than with quantum states. Let $H^{\otimes n}$ be the state space of n-qubits. The quantum stabilizer group S is an Abelian subgroup of P_n and is closed under multiplication. Further, there is no trivial subspace $C_S \subset H^{\otimes n}$ that is fixed (or stabilized) by S. The stabilized C_S defines a codeword such that $C_S = \{|\psi\rangle \in H^{\otimes n} : \mathbf{g}|\psi\rangle = |\psi\rangle, \forall \mathbf{g} \in S\}$. The quantum code with parameter $[[n, k, d_{\min}]]$ corresponds to the group S with its generators, $g = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_{n-k}\}$. The constraints for the generators in g are such that any two elements must be commutative with each other. Since the number of generators $(n-k)$ is less than or equal to n, $(n-k)$ is a well-defined quantity; it is called the rank of the stabilizer. If the stabilizer has the rank n $(k = 0)$, the stabilizer will be referred to as a full rank stabilizer and the corresponding quantum code is denoted as $[[n, 0, d_{\min}]]$. The codewords, or the stabilizer state of the full rank stabilizers group, will be called the graph state; this has many applications in one-way quantum computers, secure state distribution, secret sharing, etc.

Considering a set of error operators $\{\mathbf{E}\} \subset P_n$, the collection of Pauli operators takes a state $|\psi\rangle$ to the corrupted state $\mathbf{E}|\psi\rangle$. A given operator $\mathbf{E}$ either is commutative or anti-commutative with each stabilizer operator $\mathbf{S}_i$. Then the corrupted state $\mathbf{E}|\psi\rangle$ is diagnosed by elements $\mathbf{S}_i$ of the set S. The outcome of the diagnostic procedure is a vector of $\{+1, -1\}$ indicating whether or not $\mathbf{E}$ can be detected. The indication for the error detection is expressed as follows:

$$\mathbf{S}_i \times \mathbf{E}|\psi\rangle = \begin{cases} \mathbf{E} \times \mathbf{S}_i|\psi\rangle = \mathbf{E}|\psi\rangle, & \text{Error undetected.} \\ -\mathbf{E} \times \mathbf{S}_i|\psi\rangle = -\mathbf{E}|\psi\rangle, & \text{Error detected.} \end{cases}$$

The condition for quantum error correction is that $\mathbf{E}$ is a set of correctable error operators for C_S if

$$\mathbf{E}_i{}^\dagger \mathbf{E}_j \notin N(S) \backslash S, \quad \forall \mathbf{E}_i, \quad \mathbf{E}_j \in \mathbf{E},$$

where $\mathbf{E}_i{}^\dagger$ is conjugate transpose of $\mathbf{E}_i$ and $N(S)$ is the normalize of S in P_n such as

$$N(S) = \{\mathbf{A} \in P_n \,|\, \mathbf{A}^\dagger \mathbf{E} \mathbf{A} \in S, \ \forall \mathbf{E} \in S\}.$$

Note that $N(S)$ is the collection of all operators in P_n that commutes with S. Then the minimum distance of stabilizer code is determined by $d_{\min} = \min(\mathrm{W}(\mathbf{E}))$ s.t. $\mathbf{E} \in N(S) \backslash S$, where the weight of an operator, $\mathrm{W}(*)$, is the numbers of positions not equal to Pauli operator $\mathbf{I}$.

Since we can express the generator of the quantum stabilizer code as a binary field, due to the fact that any n-qubit Pauli operator can be expressed as a multiplication of an $\mathbf{X}$-containing operator and an $\mathbf{Z}$-containing operator, we define the mapping between Pauli operators and binary vectors as $\mathbf{I} \leftrightarrow (0, 0)$, $\mathbf{X} \leftrightarrow (1, 0)$, $\mathbf{Z} \leftrightarrow (0, 1)$ and $\mathbf{Y} \leftrightarrow (1, 1)$. Let $\mathbf{A}$ and $\mathbf{B}$ be the Pauli operators of length n. As the binary mapping, the binary forms of $\mathbf{A}$ and $\mathbf{B}$ are as follows: $\mathbf{A} \leftrightarrow [x_{\mathbf{A}} \,|\, z_{\mathbf{A}}]$ and $\mathbf{B} \leftrightarrow [x_{\mathbf{B}} \,|\, z_{\mathbf{B}}]$. Then, the commutative product between $\mathbf{A}$ and $\mathbf{B}$ is given as follows:

$$\mathbf{A} \circ \mathbf{B} = [x_{\mathbf{A}} \,|\, z_{\mathbf{A}}] \circ [x_{\mathbf{B}} \,|\, z_{\mathbf{B}}] = x_{\mathbf{A}} \times z_{\mathbf{B}}{}^T + x_{\mathbf{B}} \times z_{\mathbf{A}}{}^T, \tag{1}$$

where $\mathbf{T}$ is a transpose of a matrix.

As a consequence, $(n-k)$ generators of a $[[n, k]]$ code are formed in a binary field as $\mathbf{H} = [\mathbf{H}_{\mathbf{X}} | \mathbf{H}_{\mathbf{Z}}]$, where $\mathbf{H}_{\mathbf{X}}, \mathbf{H}_{\mathbf{Z}}$ are $(n-k) \times n$ binary matrices and "|" denotes the row concatenation. Hence, $\mathbf{H}$ represents binary matrices with the size $(n-k) \times 2n$. The commutative constraint between generators must change to the symplectic product constraint as

$$\mathbf{H}_{\mathbf{Z}} \times \mathbf{H}_{\mathbf{X}}{}^T + \mathbf{H}_{\mathbf{X}} \times \mathbf{H}_{\mathbf{Z}}{}^T = \mathbf{0}_{n-k}, \tag{2}$$

where $\mathbf{0}_m$ is the matrix of all zero elements with size $m \times m$. From the binary form of a stabilizer code, by using Gaussian elimination, the parity-check matrix $\mathbf{H}$ can be uniquely determined in standard form as follows:

$$\begin{bmatrix} \overbrace{\mathbf{I}}^{r} & \overbrace{\mathbf{A}_1}^{n-k-r} & \overbrace{\mathbf{A}_2}^{k} & \overbrace{\mathbf{B}}^{r} & \overbrace{\mathbf{C}_1}^{n-k-r} & \overbrace{\mathbf{C}_2}^{k} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{D} & \mathbf{I} & \mathbf{E} \end{bmatrix} \begin{matrix} \} & r \\ \} & n-k-r. \end{matrix} \tag{3}$$

The linear combinations among rows of parity-check matrix $\mathbf{H}$ generate the stabilizer group S in binary modulo-2 addition. Since the dual-space of $\mathbf{H}$ has the dimension of $n-k+2k$, the normalize group *N(S)* that commutes with S can be considered as the dual-space of S generated by a $(n-k+2k) \times 2n$ binary matrix. The last $2k$ rows are called the logical operators $\overline{\mathbf{X}}_i$ and $\overline{\mathbf{Z}}_i$ $(i = \overline{1,k})$ which satisfy the constraint commutative conditions, and which are used to build up the quantum encoding circuits as discussed.[29]

3. Quantum Stabilizer Code Construction

3.1. *Proposed construction for quantum codes*

In this section, we propose a new construction of quantum stabilizer codes based on a binary formalism. The proposed construction is given as follows:

Definition 1. Let $\mathbf{f} = [a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n]$ be a binary vector of length $2n$. We define a new vector of length $2n$ which is based on the elements of vector $\mathbf{f}$ as follows:

$$\mathbf{f}^{(k)} = [a_{1\oplus k} a_{2\oplus k} \ldots a_{n\oplus k} b_{1\ominus k} b_{2\ominus k} \ldots b_{n\ominus k}],$$

where "$\oplus$" and "$\ominus$" are the additional modulo n and subtraction modulo n, respectively.

Theorem 1. *Let* $\mathbf{f} = [a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n]$ *be a binary vector of length* $2n$*, we will prove that for any integer* k *and* l*, the commutative product of* $\mathbf{f}^{(k)}$ *and* $\mathbf{f}^{(l)}$ *will be zero. Hence, the Pauli operators which correspond to* $\mathbf{f}^{(k)}$ *and* $\mathbf{f}^{(l)}$ *are commutative.*

Proof. As Definition 1, we have

$$\mathbf{f}^{(k)} = [a_{1\oplus k} a_{2\oplus k} \ldots a_{n\oplus k} b_{1\ominus k} b_{2\ominus k} \ldots b_{n\ominus k}]$$

and

$$\mathbf{f}^{(l)} = [a_{1\oplus l} a_{2\oplus l} \ldots a_{n\oplus l} b_{1\ominus l} b_{2\ominus l} \ldots b_{n\ominus l}].$$

Hence, we have the commutative product between two vectors are given as follows:

$$\begin{aligned} \mathbf{f}^{(k)} \circ \mathbf{f}^{(l)} = {} & [a_{1\oplus k} a_{2\oplus k} \ldots a_{n\oplus k} b_{1\ominus k} b_{2\ominus k} \ldots b_{n\ominus k}] \\ & \circ [a_{1\oplus l} a_{2\oplus l} \ldots a_{n\oplus l} b_{1\ominus l} b_{2\ominus l} \ldots b_{n\ominus l}] \end{aligned}$$

$$
\begin{aligned}
&= [a_{1\oplus k}a_{2\oplus k}\dots a_{n\oplus k}] \times [b_{1\ominus l}b_{2\ominus l}\dots b_{n\ominus l}]^T \\
&\quad + [a_{1\oplus l}a_{2\oplus l}\dots a_{n\oplus l}] \times [b_{1\ominus k}b_{2\ominus k}\dots b_{n\ominus k}]^T \\
&= (a_{1\oplus k}b_{1\ominus l} + a_{2\oplus k}b_{2\ominus l} + \cdots + a_{n\oplus k}b_{n\ominus l}) \\
&\quad + (a_{1\oplus l}b_{1\ominus k} + a_{2\oplus l}b_{2\ominus k} + \cdots + a_{n\oplus l}b_{n\ominus k}) \\
&= \sum_{i=1}^{n} a_i b_{i\ominus l\ominus k} + \sum_{j=1}^{n} a_j b_{j\ominus k\ominus l} = \sum_{i=1}^{n} (2 \times a_i b_{i\ominus k\ominus l}) = 0.
\end{aligned}
$$

□

As proved in Theorem 1, the commutative product of any two elements belonging to the set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$ are zero. Hence, any corresponding Pauli operators from $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$ are commutative to the others. We will use this property for a new construction of quantum stabilizer codes as follows:

Construction 1. Let $\mathbf{f} = [a_1a_2\dots a_nb_1b_2\dots b_n]$ be a binary vector of length $2n$, we choose from set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(n-1)}\}$ to get k vectors which are independent to each other: $\{\mathbf{f}^{(i_1)}, \mathbf{f}^{(i_2)}, \dots, \mathbf{f}^{(i_k)}\}$. We assume that the corresponding Pauli operator of $\{\mathbf{f}^{(i_1)}, \mathbf{f}^{(i_2)}, \dots, \mathbf{f}^{(i_k)}\}$ are $S_k = \{\mathbf{g}^{(i_1)}, \mathbf{g}^{(i_2)}, \dots, \mathbf{g}^{(i_k)}\}$. Then, S_k satisfies the condition to construct QECCs. The minimum distance $d_{\min}$ is calculated from $\{\mathbf{f}^{(i_1)}, \mathbf{f}^{(i_2)}, \dots, \mathbf{f}^{(i_k)}\}$ as the discussion in Sec. 2.2. Hence, a quantum stabilizer code with parameters $[[n, k, d_{\min}]]$ has been determined.

3.2. *Investigate quantum stabilizer codes with binary generators*

In this section, we will investigate the quantum stabilizer codes from the proposed construction. Via the proof of Construction 1, for any vector with a certain code length, we will have corresponding quantum stabilizer codes. Therefore, we have many candidates for the generator matrices of quantum stabilizer code. Hence, in this section, we will show the details of quantum stabilizer codes with various dimensions and a large minimum distance. In the following examples, we will consider code lengths from six to twelve digits.

Example 1. We consider the binary vector of length 12 $\mathbf{f}^{(0)} = [1\,0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1]$. Then, we can set up the full set of six elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(5)}\}$, where

$$
\begin{aligned}
\mathbf{f}^{(0)} &= [0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1], \\
\mathbf{f}^{(1)} &= [1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1], \\
\mathbf{f}^{(2)} &= [0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0], \\
\mathbf{f}^{(3)} &= [0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0], \\
\mathbf{f}^{(4)} &= [1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0], \\
\mathbf{f}^{(5)} &= [1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1].
\end{aligned}
\tag{4}
$$

We use *Quantum Code* and *Minimum Weight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As a result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(5)}\}$ corresponds to quantum stabilizer code $[[6, 0, 3]]$. The binary standard form for a generator matrix of the proposed $[[6, 0, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&1&0 \\ 0&1&0&0&0&0&0&1&1&1&0&0 \\ 0&0&1&0&0&0&1&1&1&0&0&0 \\ 0&0&0&1&0&0&1&1&0&0&0&1 \\ 0&0&0&0&1&0&1&0&0&0&1&1 \\ 0&0&0&0&0&1&0&0&0&1&1&1 \end{bmatrix}. \tag{5}$$

Example 2. We consider the binary vector of length 14 $\mathbf{f}^{(0)} = [0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0]$. Then, we can set up the full set of seven elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(6)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0], \\
\mathbf{f}^{(1)} &= [1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0], \\
\mathbf{f}^{(2)} &= [0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1], \\
\mathbf{f}^{(3)} &= [0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0], \\
\mathbf{f}^{(4)} &= [1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1], \\
\mathbf{f}^{(5)} &= [0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1], \\
\mathbf{f}^{(6)} &= [1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(6)}\}$ corresponds to quantum stabilizer code $[[7, 0, 3]]$. The binary standard form for a generator matrix of the proposed $[[7, 0, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&1&1&0&0 \\ 0&1&0&0&0&0&0&1&0&1&1&0&0&0 \\ 0&0&1&0&0&0&0&0&1&1&0&0&0&1 \\ 0&0&0&1&0&0&0&1&1&0&0&0&1&0 \\ 0&0&0&0&1&0&0&1&0&0&0&1&0&1 \\ 0&0&0&0&0&1&0&0&0&0&1&0&1&1 \\ 0&0&0&0&0&0&1&0&0&1&0&1&1&0 \end{bmatrix}.$$

Example 3. We consider the binary vector of length 14 $\mathbf{f}^{(0)} = [1\,0\,1\,1\,1\,0\,0\,0\,0\,1\,0\,1\,1\,1]$. Then, we can set up the set of six elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(5)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1],\\
\mathbf{f}^{(1)} &= [0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0],\\
\mathbf{f}^{(2)} &= [0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0],\\
\mathbf{f}^{(3)} &= [1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1],\\
\mathbf{f}^{(4)} &= [1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0],\\
\mathbf{f}^{(5)} &= [1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(5)}\}$ corresponds to quantum stabilizer code $[[7, 1, 3]]$. The binary standard form for a generator matrix of the proposed $[[7, 1, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&1&0&1&1&0&0&0&0&0&0&0\\
0&1&0&1&1&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&1&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&1&0&1&1\\
0&0&0&0&0&0&0&0&1&0&1&1&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&1&1
\end{bmatrix}.$$

Example 4. We consider the binary vector of length 16

$$\mathbf{f}^{(0)} = [0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1].$$

Then, we can set up full set of eight elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(7)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1],\\
\mathbf{f}^{(1)} &= [0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0],\\
\mathbf{f}^{(2)} &= [1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1],\\
\mathbf{f}^{(3)} &= [0\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1],\\
\mathbf{f}^{(4)} &= [1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0],\\
\mathbf{f}^{(5)} &= [0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0],\\
\mathbf{f}^{(6)} &= [1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1],\\
\mathbf{f}^{(7)} &= [0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(7)}\}$ corresponds to quantum stabilizer code $[[8, 0, 3]]$. The binary standard form for a generator matrix of the proposed $[[8, 0, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0 \\ 0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&1 \\ 0&0&1&0&0&0&0&0&0&1&1&0&1&0&1&0 \\ 0&0&0&1&0&0&0&0&1&1&0&1&0&1&0&0 \\ 0&0&0&0&1&0&0&0&1&0&1&0&1&0&0&1 \\ 0&0&0&0&0&1&0&0&0&1&0&1&0&0&1&1 \\ 0&0&0&0&0&0&1&0&1&0&1&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&0&1&0&0&1&1&0&1 \end{bmatrix}.$$

Example 5. We consider the binary vector of length 16

$$\mathbf{f}^{(0)} = [0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0].$$

Then, we can set up a set of seven elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(6)}\}$ where

$$\begin{aligned} \mathbf{f}^{(0)} &= [0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0], \\ \mathbf{f}^{(1)} &= [1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1], \\ \mathbf{f}^{(2)} &= [0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1], \\ \mathbf{f}^{(3)} &= [1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1], \\ \mathbf{f}^{(4)} &= [1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0], \\ \mathbf{f}^{(5)} &= [1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1], \\ \mathbf{f}^{(6)} &= [0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0]. \end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(6)}\}$ corresponds to quantum stabilizer code $[[8, 1, 3]]$. The binary standard form for a generator matrix of the proposed $[[8, 1, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&1&1&1&0&0&1&1&1&1 \\ 0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&1&0&1&1&0&1&1&1&1 \\ 0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&0&0&1&1&1&1&0&1&1&0&1 \\ 0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&1&1&1&1&1&0&0&1&1&1 \end{bmatrix}.$$

Example 6. We consider the binary vector of length 18

$$\mathbf{f}^{(0)} = [1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1].$$

Then, we can set up a set of nine elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(8)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1],\\
\mathbf{f}^{(1)} &= [0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0],\\
\mathbf{f}^{(2)} &= [0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1],\\
\mathbf{f}^{(3)} &= [1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0],\\
\mathbf{f}^{(4)} &= [0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0],\\
\mathbf{f}^{(5)} &= [0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0],\\
\mathbf{f}^{(6)} &= [1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1],\\
\mathbf{f}^{(7)} &= [1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0],\\
\mathbf{f}^{(8)} &= [0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(8)}\}$ corresponds to quantum stabilizer code $[[9, 0, 3]]$. The binary standard form for a generator matrix of the proposed $[[9, 0, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0\\
0&1&0&0&0&0&0&0&1&0&0&1&1&1&1&0&0&0\\
0&0&1&0&0&0&0&0&1&0&1&1&0&1&1&0&0&0\\
0&0&0&1&0&0&0&0&1&0&0&1&1&0&0&1&1&1\\
0&0&0&0&1&0&0&0&1&0&1&1&1&0&0&1&1&0\\
0&0&0&0&0&1&0&0&1&0&0&0&0&1&1&0&1&1\\
0&0&0&0&0&0&1&0&1&0&0&0&0&1&1&1&1&0\\
0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1
\end{bmatrix}.$$

In addition, a set of eight elements $\{\mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(8)}\}$ corresponds to quantum stabilizer code $[[9, 1, 3]]$. The binary standard form for a generator matrix of the proposed $[[9,1,3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&1&1&1&0&1&0&1&1&1 \\ 0&1&0&0&0&0&0&0&1&0&0&1&1&1&1&0&0&0 \\ 0&0&1&0&0&0&0&0&1&1&0&0&1&0&0&1&1&1 \\ 0&0&0&1&0&0&0&0&1&0&0&1&1&0&0&1&1&1 \\ 0&0&0&0&1&0&0&0&1&1&0&0&0&1&1&0&0&1 \\ 0&0&0&0&0&1&0&0&1&0&0&0&0&1&1&0&1&1 \\ 0&0&0&0&0&0&1&0&1&1&1&1&1&0&0&0&0&1 \\ 0&0&0&0&0&0&0&1&1&0&0&0&0&1&0&1&0&0 \end{bmatrix}.$$

Example 7. We consider the binary vector of length 20

$$\mathbf{f}^{(0)} = [1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0].$$

Then, we can set up a set of 10 elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(9)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0], \\
\mathbf{f}^{(1)} &= [1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0], \\
\mathbf{f}^{(2)} &= [0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1], \\
\mathbf{f}^{(3)} &= [0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0], \\
\mathbf{f}^{(4)} &= [1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0], \\
\mathbf{f}^{(5)} &= [0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0], \\
\mathbf{f}^{(6)} &= [0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1], \\
\mathbf{f}^{(7)} &= [1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0], \\
\mathbf{f}^{(8)} &= [1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0], \\
\mathbf{f}^{(9)} &= [0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(9)}\}$ corresponds to quantum stabilizer code $[[10,0,3]]$. The binary standard form for a generator matrix of the proposed $[[10,0,3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&0&0&0 \\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1 \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1 \end{bmatrix}.$$

In addition, a set of nine elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(6)}, \mathbf{f}^{(8)}, \mathbf{f}^{(9)}\}$ corresponds to quantum stabilizer code $[[10, 1, 3]]$. The binary standard form for a generator matrix of the proposed $[[10, 1, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&0&0&0 \\ 0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&1&1&0&1&1 \\ 0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&1&0&0&0&0&1&0&0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&1&0&0&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0&0&1&1&0 \end{bmatrix}.$$

Example 8. We consider the binary vector of length 22

$$\mathbf{f}^{(0)} = [1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 1, 0, 1].$$

Then, we can set up a set of 11 elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(10)}\}$ where

$$\mathbf{f}^{(0)} = [1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1],$$

$$\mathbf{f}^{(1)} = [0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1],$$

$$\mathbf{f}^{(2)} = [1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1],$$

$$\mathbf{f}^{(3)} = [0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0],$$

$$\mathbf{f}^{(4)} = [1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1],$$

$$\mathbf{f}^{(5)} = [1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0],$$
$$\mathbf{f}^{(6)} = [0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1],$$
$$\mathbf{f}^{(7)} = [1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0],$$
$$\mathbf{f}^{(8)} = [1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0],$$
$$\mathbf{f}^{(9)} = [1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1],$$
$$\mathbf{f}^{(10)} = [0\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0].$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(10)}\}$ corresponds to quantum stabilizer code $[[11, 0, 4]]$. The binary standard form for a generator matrix of the proposed $[[11, 0, 4]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1\\
0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1&1&1&0&1&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&1&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&1&0&1&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&0&0&1&1
\end{bmatrix}.$$

Example 9. We consider the binary vector of length 22

$$\mathbf{f}^{(0)} = [0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1.$$

Then, we can set up a set of 10 elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(9)}\}$ where

$$\mathbf{f}^{(0)} = [0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1],$$
$$\mathbf{f}^{(1)} = [0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1],$$
$$\mathbf{f}^{(2)} = [0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1],$$
$$\mathbf{f}^{(3)} = [1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0],$$
$$\mathbf{f}^{(4)} = [1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0],$$
$$\mathbf{f}^{(5)} = [0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1],$$

$$\mathbf{f}^{(6)} = [0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1],$$
$$\mathbf{f}^{(7)} = [1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0],$$
$$\mathbf{f}^{(8)} = [1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0],$$
$$\mathbf{f}^{(9)} = [1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0].$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As the result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \ldots, \mathbf{f}^{(9)}\}$ corresponds to quantum stabilizer code $[[11, 1, 3]]$. The binary standard form for a generator matrix of the proposed $[[11, 1, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&1&1&1&0&1&1&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&1&0&1&1&0&1&0\\
0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&1&1&1&1&1&0&1\\
0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&0&0&1&1\\
0&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&1&1&0\\
0&0&0&0&0&1&0&0&0&0&1&0&1&1&1&0&0&1&0&1&0&1\\
0&0&0&0&0&0&1&0&0&0&1&0&0&1&1&1&1&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&1&0&1&1&0&0&1&1&0&1&0&1&1&0\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&1&0&0&1&0&0
\end{bmatrix}.$$

In addition, a set of nine elements $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(9)}\}$ corresponds to quantum stabilizer code $[[11, 2, 3]]$. The binary standard form for a generator matrix of the proposed $[[11, 2, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&1&1&1&0&1&1&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&1&0&1&1&0&1&0&1&1&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&1&0&0&1&0&1&0&1&0&1&1\\
0&0&0&1&0&0&0&0&0&0&1&1&1&0&0&0&1&1&0&0&1&1\\
0&0&0&0&1&0&0&0&1&0&0&1&1&0&0&1&1&1&1&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&1&1&0&1&0&0&0&0&1&1\\
0&0&0&0&0&0&1&0&0&0&1&0&0&1&1&1&1&0&0&0&1&1\\
0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&0&0&1&1&1&1\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&0&1&1&0&0&1&0&0
\end{bmatrix}.$$

Example 10. We consider the binary vector of length 24

$$\mathbf{f}^{(0)} = [0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0].$$

Then, we can set up a set of 12 elements $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(11)}\}$ where

$$\begin{aligned}
\mathbf{f}^{(0)} &= [0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0],\\
\mathbf{f}^{(1)} &= [1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1],\\
\mathbf{f}^{(2)} &= [0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1],\\
\mathbf{f}^{(3)} &= [1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1],\\
\mathbf{f}^{(4)} &= [0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1],\\
\mathbf{f}^{(5)} &= [1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0],\\
\mathbf{f}^{(6)} &= [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0],\\
\mathbf{f}^{(7)} &= [0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1],\\
\mathbf{f}^{(8)} &= [1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0],\\
\mathbf{f}^{(9)} &= [1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1],\\
\mathbf{f}^{(10)} &= [1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0],\\
\mathbf{f}^{(11)} &= [1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 0\ 1].
\end{aligned}$$

We use *Quantum Code* and *MinimumWeight* functions of the Magma calculation tool[30] to check the independence and calculate the minimum distance parameter. As a result, set $\{\mathbf{f}^{(0)}, \mathbf{f}^{(1)}, \dots, \mathbf{f}^{(11)}\}$ corresponds to quantum stabilizer code $[[12, 0, 4]]$. The binary standard form for a generator matrix of the proposed $[[12, 0, 4]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&1\\
0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&1&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&1&1&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&1&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&0&1&0&0&0&0&1&0&0
\end{bmatrix}.$$

In addition, a set of 11 elements $\{\mathbf{f}^{(1)}, \mathbf{f}^{(2)}, \ldots, \mathbf{f}^{(11)}\}$ corresponds to quantum stabilizer code $[[12, 1, 4]]$. The binary standard form for a generator matrix of the proposed $[[12, 1, 4]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&0&0&0&0&1&0&0&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&1 \\ 0&0&1&0&0&0&0&0&0&0&0&1&1&0&1&0&1&0&1&0&0&0&1&1 \\ 0&0&0&1&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&1&1&1&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1 \\ 0&0&0&0&0&1&0&0&0&0&0&1&1&1&1&1&1&0&1&1&1&1&1&0 \\ 0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&1&1&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&1&0&1&1&1&0&1&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0&0&1&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0 \end{bmatrix}.$$

In addition, a set of 10 elements $\{\mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \ldots, \mathbf{f}^{(11)}\}$ corresponds to quantum stabilizer code $[[12, 2, 3]]$. The binary standard form for a generator matrix of the proposed $[[12, 2, 3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix} 1&0&0&0&0&0&0&0&1&0&0&1&1&0&1&0&0&1&0&1&1&1&0&1 \\ 0&1&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&1 \\ 0&0&1&0&0&0&0&0&1&0&0&0&1&1&0&1&1&1&1&1&0&1&1&1 \\ 0&0&0&1&0&0&0&0&1&0&0&1&1&1&1&1&0&0&0&1&1&0&1&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&1&1&0&1 \\ 0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&1&1&0&1&0&1&0 \\ 0&0&0&0&0&0&1&0&0&0&0&1&1&1&0&0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&1&0&0&1&0&0&1&1&1&0&1&1&1&1&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&1&1&1&0&1&0&0&1&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&1&1&0&0&1&1&1&0&0&0&1&1&0 \end{bmatrix}.$$

Then, a set of nine elements $\{\mathbf{f}^{(2)}, \mathbf{f}^{(3)}, \ldots, \mathbf{f}^{(10)}\}$ corresponds to quantum stabilizer code $[[12, 3, 3]]$. The binary standard form for a generator matrix of the

Table 1. List of quantum stabilizer codes from proposed construction.

Considered vector	Quantum codes
[1 0 1 1 0 0 0 0 0 1 1 1]	[[6,0,3]]
[0 1 0 1 0 0 1 0 1 0 1 1 0 0]	$[[7,0,3]]^*$
[1 0 1 1 1 0 0 0 0 1 0 1 1 1]	$[[7,1,3]]^*$
[0 0 1 0 1 0 1 0 0 1 1 0 0 1 0 1]	[[8,0,3]]
[0 0 0 1 1 1 0 1 1 1 1 0 1 0 0 0]	$[[8,1,3]]^*$
[1 0 1 1 0 0 1 0 0 0 1 0 0 0 1 0 0 1]	[[9,0,3]], $[[9,1,3]]^*$
[1 0 1 1 0 0 1 0 0 1 0 1 0 0 0 1 0 0 1 0]	[[10,0,3]], [[10,1,3]]
[1 0 1 1 1 0 1 1 0 1 0 1 1 0 1 0 1 0 0 1 0 1]	[[11,0,4]]
[0 1 1 1 1 0 0 1 1 0 0 1 1 0 0 1 1 0 0 0 1 1]	[[11,1,3]], [[11,2,3]]
[0 1 1 1 1 0 0 1 0 1 0 1 1 1 1 1 1 0 0 1 0 1 0 1 0]	[[12,0,4]], [[12,1,4]], [[12,2,3]], [[12,3,3]], [[12,4,3]]

proposed $[[12,3,3]]$ is as follows:

$$\mathbf{G} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}.$$

In Table 1, we summarize the quantum stabilizer codes with lengths ranging from six to twelve, which were shown in Examples 1–10. While the first column of Table 1 indicates the vector we considered for generator matrices, the second column of Table 1 indicates the quantum codes with a large minimum distance and various dimensions which have been found based on the proposed construction. In addition, the quantum stabilizer codes with the largest minimum distance in comparison with other constructions have been noted.[31]

4. Conclusion

In this paper, a novel construction of quantum stabilizer codes has been discussed. We built up a framework to construct quantum codes with any code length and various dimensions. In addition, we investigated the parameters for quantum stabilizer codes from constraint vectors with a code length from six to twelve digits. As a consequence, the proposed quantum stabilizer codes provide various dimension for any length and demonstrate improved error correction in comparison with the other referenced quantum codes.

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