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The fog on: Generalized teleportation by means of discrete-time quantum walks on N-lines and N-cycles

Duc Manh Nguyen* and Sunghwan Kim†
School of Electrical Engineering, University of Ulsan, Ulsan 44610, Korea
**nguyenmanhduc18@gmail.com*
†*sungkim@ulsan.ac.kr*

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The recent paper entitled "Generalized teleportation by means of discrete-time quantum walks on N-lines and N-cycles" by Yang *et al.* [*Mod. Phys. Lett. B* **33**(6) (2019) 1950069] proposed the quantum teleportation by means of discrete-time quantum walks on N-lines and N-cycles. However, further investigation shows that the quantum walk over the one-dimensional infinite line can be based over the N-cycles and cannot be based on N-lines. The proofs of our claims on quantum walks based on finite lines are also provided in detail.

Keywords: Quantum information; quantum walks; quantum teleportation.

1. Introduction

Quantum information processing is proved to have the possibility to solve the difficult tasks which are hard to solve by the classical information processing such as in factoring the number to find the primer numbers,[1] in searching from unordered sets,[2] and in the security of cryptography protocol.[3] The most important physical phenomenon lying on quantum information is the quantum entanglement[4] which shares the spatial proximity in ways such that the quantum state of each particle cannot be described independently even when the particles are separated by a large distance. Therefore, many researches use quantum entanglement as the key points, such as quantum algorithm, quantum walk, quantum dialogue, and quantum error correction code.[5] In 2018, Zidan proposed a novel technique to use one of the entanglement measures that is called concurrence as a key step to solve some intractable quantum problems.[6] This technique uses the principles of quantum superposition phenomena and concurrence to find the solution of the problem at hands based on

†Corresponding author.

the degree of entanglement between two extra qubits. Recently, Zidan's technique has been used to propose novel quantum algorithms for testing junta variables[7] and to measure the Hamming distance between Boolean functions.[8]

As a quantum analogue of classical random walks, in recent time quantum walks (QWs) have been well-studied processes. The same motivation as classical random walks, QWs are devised as the mathematical basis to develop sophisticated algorithms. Unlike random walks transformed by the probability transition matrices on the probability distribution, QWs are transformed by unitary revolutions. The probability distributions of quantum states are defined as the sum of squares of the norms of amplitudes. Due to the quantum interference effects, there exists a nonlinearity map between the quantum state and the probability distribution. As a result, QWs have been shown to outperform random walks at certain computational tasks. Moreover, there exist both discrete and continuous QWs as the universal models of quantum computation. Both discrete and continuous QWs could be run on graphs, but their evolution is different. We use Schrödinger equation and Laplacian matrix for the time evolution of continuous QWs. On the other hand, the unitary operators which are applied in discrete time steps describe the time evolution of discrete QWs.

As for discrete QWs, an extra qubit register is defined to be the direction in which the walker unitary moves from a node to its neighbor nodes. This model of QWs is called coin-driven QWs, is composed of two quantum systems: (1) a walker, which is a Hilbert space of finite or infinite H_P and (2) a coin, which is quantum system living in a two-dimensional Hilbert space H_C. Coin-driven QWs have been studied some detail by the scientific community such as the application to be perfect state transfer (PST),[9] to do the teleportation,[10] to apply in quantum search algorithm,[11] and many mathematical, statistical and computational properties remain to be discovered and explored.[12]

Yang *et al.*[13] propose the quantum teleportation by means of discrete-time quantum walks on N-lines and N-cycles. However, the further investigation shows that in the one-dimensional quantum walk model, the quantum walks over the one-dimensional infinite line can be based on N-cycles and cannot be based on N-lines. This paper is organized as follows. In Sec. 2, we will consider QW system over the three cases of one-dimensional lines: infinite line, N-lines, and N-cycles. In addition, the proofs of our claims are also provided in detail. In Sec. 3, it will be conclusion and open the discussion.

2. Discrete Time QWs on Lines

A discrete-time QWs over the line involves in two Hilbert spaces. The total space of the walk is given as

$$H = H_P \otimes H_C \,, \tag{1}$$

where H_P is the position space, which is spanned by $\{|n\rangle$ where $n \in \mathrm{Z}\}$, and H_C is the coin space, which is spanned by $\{|0\rangle, |1\rangle\}$.

The coin operator $\mathbf{C} \in H_C$ is the unitary matrix. We can choose $\mathbf{C}$ as Hadamard ($\mathbf{H}$), Identify ($\mathbf{I}$), or flip-bit ($\mathbf{X}$). The shift operator changes the location $|n\rangle$ to $|n+1\rangle$ if the value of coin is $|0\rangle$. In contrast, it will change the location $|n-1\rangle$ to $|n\rangle$ if the value of coin is $|1\rangle$. Hence, the shift operator can be explained as

$$\mathbf{S} = \sum_{-\infty}^{+\infty} |n+1\rangle\langle n| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n-1\rangle\langle n| \otimes |1\rangle\langle 1|\,, \tag{2}$$

where $\langle n|$ is the transpose of $|n\rangle$.

The QW system at step t is as follows:

$$\psi(t) = \mathbf{U}\psi(t-1) = \mathbf{U}^t\psi(0)\,, \tag{3}$$

where $\mathbf{U} = \mathbf{S}(\mathbf{I} \otimes \mathbf{C})$ and $\psi(0)$ is initial state. In addition, $\mathbf{U}$, $\mathbf{S}$ and $\mathbf{C}$ must be all unitary matrices.

Before going to next section, we recall the following preliminaries for matrices $\mathbf{A}$ and $\mathbf{B}$, we assume the size of matrices $\mathbf{A}$ and $\mathbf{B}$ are suitable for the product operator.

- $(\mathbf{A} \otimes \mathbf{B})^T = \mathbf{A}^T \otimes \mathbf{B}^T$.
- $(\mathbf{A} * \mathbf{B})^T = \mathbf{B}^T * \mathbf{A}^T \rightarrow (|x\rangle\langle y|)^T = |y\rangle\langle x|$.
- $\langle 0||0\rangle = 1$, $\langle 1||0\rangle = 0, \langle 1||1\rangle = 1$ for $|0\rangle, |1\rangle$ are the basis states in the space H_C.
- $\langle x||x\rangle = 1, \langle y||x\rangle = 0$ for $|x\rangle \neq |y\rangle$ and $|x\rangle$ and $|y\rangle$ are the basis states in the space H_P.

2.1. *One-dimensional QW*

The one-dimensional QW is proposed by Ambainis,[14] where the author consider the Hadamard QW over infinite lines and it is given in Fig. 1.

Since the shifted operator of the system is given as Eq. (2), its transpose operator is calculated as follows:

$$\begin{aligned} \mathbf{S}^T &= \left[\sum_{-\infty}^{+\infty} |n+1\rangle\langle n| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n-1\rangle\langle n| \otimes |1\rangle\langle 1|\right]^T \\ &= \sum_{-\infty}^{+\infty} |n\rangle\langle n+1| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n\rangle\langle n-1| \otimes |1\rangle\langle 1|\,. \end{aligned} \tag{4}$$

Fig. 1. The QW over infinite line with starting point at 0 position.

To consider the unitary property of the shifted operator, we need to consider the product of the shifted operator and its transpose. This product is given as follows:

$$\begin{aligned}
\mathbf{S}^T * \mathbf{S} &= \left[\sum_{-\infty}^{+\infty} |n\rangle\langle n+1| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n\rangle\langle n-1| \otimes |1\rangle\langle 1|\right] \\
&\quad * \left[\sum_{-\infty}^{+\infty} |n+1\rangle\langle n| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n-1\rangle\langle n| \otimes |1\rangle\langle 1|\right] \\
&= \sum_{n=-\infty}^{n=+\infty} |n\rangle\langle n| \otimes |0\rangle\langle 0| + \sum_{n=-\infty}^{n=+\infty} |n\rangle\langle n| \otimes |1\rangle\langle 1| \\
&= \sum_{t=-\infty}^{t=+\infty} [|t\rangle\langle t| \otimes |0\rangle\langle 0| + |t\rangle\langle t| \otimes |1\rangle\langle 1|], \\
\mathbf{S} * \mathbf{S}^T &= \left[\sum_{-\infty}^{+\infty} |n+1\rangle\langle n| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n-1\rangle\langle n| \otimes |1\rangle\langle 1|\right] \\
&\quad * \left[\sum_{-\infty}^{+\infty} |n\rangle\langle n+1| \otimes |0\rangle\langle 0| + \sum_{-\infty}^{+\infty} |n\rangle\langle n-1| \otimes |1\rangle\langle 1|\right] \\
&= \sum_{n=-\infty}^{n=+\infty} |n+1\rangle\langle n+1| \otimes |0\rangle\langle 0| + \sum_{n=-\infty}^{n=+\infty} |n-1\rangle\langle n-1| \otimes |1\rangle\langle 1| \\
&= \sum_{t=-\infty}^{t=+\infty} [|t\rangle\langle t| \otimes |0\rangle\langle 0| + |t\rangle\langle t| \otimes |1\rangle\langle 1|]\,. \qquad (5)
\end{aligned}$$

Since $\mathbf{S}$ does not contain the imaginary part, the product finally will be as follows:

$$\begin{aligned}
\mathbf{S} * \mathbf{S}^\dagger &= \mathbf{S} * \mathbf{S}^T = \mathrm{I}\,, \\
\mathbf{S}^\dagger * \mathbf{S} &= \mathbf{S}^T * \mathbf{S} = \mathrm{I}\,,
\end{aligned} \qquad (6)$$

where "$\dagger$" denotes the transpose and conjugate operator.

2.2. *On N-lines*

The QW over N-line is the consideration of the one-dimensional QW over the boundary condition. The system in the boundary location are reflected to itself as in Fig. 2.

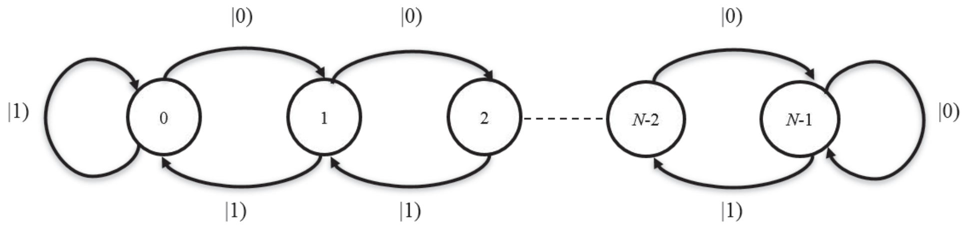

Fig. 2. The QW over the N-line.

The shifted operator according to the step of QW system in Fig. 2 is expressed as follows:

$$\begin{aligned}\mathbf{S}_1 &= \sum_{t=0}^{N-2} |t+1\rangle\langle t| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\ &\quad + \sum_{t=1}^{N-1} |t-1\rangle\langle t| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|\,. \end{aligned} \tag{7}$$

Since the shifted operator is given as Eq. (7), its transpose operator is calculated as follows:

$$\begin{aligned}\mathbf{S}_1{}^T &= \begin{bmatrix} \sum_{t=0}^{N-2} |t+1\rangle\langle t| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\ \quad + \sum_{t=1}^{N-1} |t-1\rangle\langle t| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \end{bmatrix}^T \\ &= \sum_{t=0}^{N-2} |t\rangle\langle t+1| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\ &\quad + \sum_{t=1}^{N-1} |t\rangle\langle t-1| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1|\,. \end{aligned} \tag{8}$$

To consider the unitary property of the shifted operator, we need to consider the product of the shifted operator and its transpose. The product is given as follows:

$$\begin{aligned}\mathbf{S}_1 * \mathbf{S}_1{}^T &= \begin{bmatrix} \sum_{t=0}^{N-2} |t+1\rangle\langle t| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\ \quad + \sum_{t=1}^{N-1} |t-1\rangle\langle t| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \end{bmatrix} \\ &\quad * \begin{bmatrix} \sum_{t=0}^{N-2} |t\rangle\langle t+1| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\ \quad + \sum_{t=1}^{N-1} |t\rangle\langle t-1| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{t=0}^{N-2} |t+1\rangle\langle t+1| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\
&\quad + \sum_{t=1}^{N-1} |t-1\rangle\langle t-1| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \\
&= \sum_{k=1}^{N-1} |k\rangle\langle k| \otimes |0\rangle\langle 0| + |N-1\rangle\langle N-1| \otimes |0\rangle\langle 0| \\
&\quad + \sum_{k=0}^{N-2} |k\rangle\langle k| \otimes |1\rangle\langle 1| + |0\rangle\langle 0| \otimes |1\rangle\langle 1| \,. \qquad (9)
\end{aligned}
$$

It is clear from Eq. (9) that

- there are two times $|N-1\rangle\langle N-1| \otimes |0\rangle\langle 0|$;
- there are two times $|0\rangle\langle 0| \otimes |1\rangle\langle 1|$;
- there are not $|0\rangle\langle 0| \otimes |0\rangle\langle 0|$ and $|N-1\rangle\langle N-1| \otimes |1\rangle\langle 1|$.

Then, the product finally will be the matrix in the following form:

$$
\mathbf{S}_1 * \mathbf{S}_1^T = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix} \neq \mathbf{I} \,. \qquad (10)
$$

In addition, $\mathbf{S}_1$ has no imaginary part. Hence, we have proved that

$$
\mathbf{S_1} * \mathbf{S}_1^\dagger = \mathbf{S}_1 * \mathbf{S}_1^T \neq \mathbf{I} \,. \qquad (11)
$$

2.3. *On N-cycles*

Another consideration of the one-dimensional QW over the boundary condition is N-cycle, where the system in the boundary locations is connected to each other, as in Fig. 3.

The shifted operator according to the step of QW system in Fig. 3 is expressed as follows:

$$
\mathbf{S}_2 = \sum_{t=0}^{N-2} |t \oplus 1\rangle\langle t| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t \ominus 1\rangle\langle t| \otimes |1\rangle\langle 1| \,, \qquad (12)
$$

where $\oplus$ and $\ominus$ are plus and minus operators modulo N.

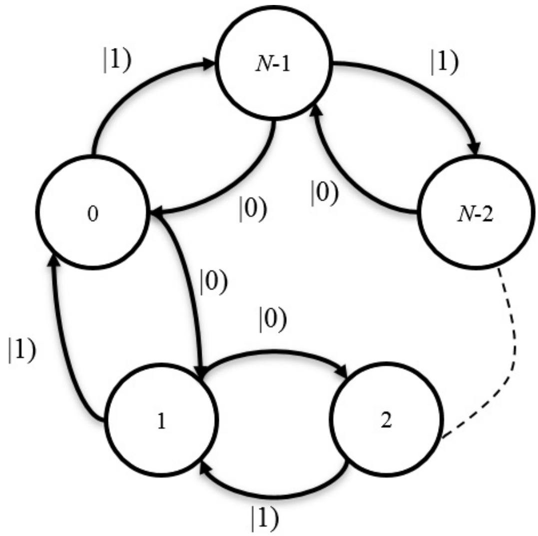

Fig. 3. The QW over N-cycle.

Since the shifted operator is given as Eq. (12), its transpose operator is calculated as follows:

$$\begin{aligned}
\mathbf{S}_2^T &= \left[\sum_{t=0}^{N-2} |t\oplus 1\rangle\langle t| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\ominus 1\rangle\langle t| \otimes |1\rangle\langle 1|\right]^T \\
&= \sum_{t=0}^{N-2} |t\rangle\langle t\oplus 1| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\rangle\langle t\ominus 1| \otimes |1\rangle\langle 1| \,. \qquad (13)
\end{aligned}$$

To consider the unitary property of the shifted operator, we need to consider the product of the shifted operator and its transpose. The product is given as follows:

$$\begin{aligned}
\mathbf{S}_2^T * \mathbf{S}_2 &= \left[\sum_{t=0}^{N-2} |t\rangle\langle t\oplus 1| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\rangle\langle t\ominus 1| \otimes |1\rangle\langle 1|\right] \\
&\quad * \left[\sum_{t=0}^{N-2} |t\oplus 1\rangle\langle t| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\ominus 1\rangle\langle t| \otimes |1\rangle\langle 1|\right] \\
&= \sum_{t=0}^{t=N-2} |t\rangle\langle t| \otimes |0\rangle\langle 0| + \sum_{t=1}^{t=N-1} |t\rangle\langle t| \otimes |1\rangle\langle 1| \,, \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\mathbf{S}_2 * \mathbf{S}_2{}^T &= \left[\sum_{t=0}^{N-2} |t\oplus 1\rangle\langle t| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\ominus 1\rangle\langle t| \otimes |1\rangle\langle 1|\right] \\
&\quad * \left[\sum_{t=0}^{N-2} |t\rangle\langle t\oplus 1| \otimes |0\rangle\langle 0| + \sum_{t=1}^{N-1} |t\rangle\langle t\ominus 1| \otimes |1\rangle\langle 1|\right]
\end{aligned}$$

$$= \sum_{t=0}^{t=N-2} |t \oplus 1\rangle\langle t \oplus 1| \otimes |0\rangle\langle 0| + \sum_{t=1}^{t=N-1} |t \ominus 1\rangle\langle t \ominus 1| \otimes |1\rangle\langle 1|$$

$$= \sum_{k=1}^{k=N-1} |k\rangle\langle k| \otimes |0\rangle\langle 0| + \sum_{k=0}^{k=N-2} |k\rangle\langle k| \otimes |1\rangle\langle 1| .$$

Since $\mathbf{S}_2$ has no imaginary part, the product finally will be as follows:

$$\begin{aligned} \mathbf{S}_2 * \mathbf{S}_2^\dagger = \mathbf{S}_2 * \mathbf{S}_2^T = \mathbf{I} , \\ \mathbf{S}_2^\dagger * \mathbf{S}_2 = \mathbf{S}_2^T * \mathbf{S}_2 = \mathbf{I} . \end{aligned} \tag{15}$$

3. Discussion and Conclusion

Unlike random walks transformed by the probability transition matrices, QWs are transformed by unitary revolutions. Hence, the shifted operators must be the unitary matrix. As the proofs in Eqs. (6), (10) and (15), we conclude that $\mathbf{S}$ and $\mathbf{S}_2$ are unitary matrices and $\mathbf{S}_1$ is not unitary matrix. Then, we can use the QW models over the graph N-cycles and we cannot use the QW model over the graph N-lines.

This paper has investigated the QW system over the finite lines, and one question for the finite graphs is opened. Some finite graphs which can be used in random walks by the transition matrices of the probability distribution cannot be used for QW system since the quantum evolution with unitary transformation is requested.

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