Computational Methods in Physics (PHY 365)

FA23

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Lab 17

Differential equations

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- A homogeneous* DE is one in which the source function is zero.

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- A DE does not, in general, determine a unique solution function.
- A DE is usually accompanied by auxiliary conditions.
- These conditions, together with the DE, specify the unknown function precisely.

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- An ODE is called implicit when the derivative of the dependent variable can not be isolated and moved to the other side of the equal sign.

$$y^{(1)}(x) = f\left(x, y(x), y^{(1)}(x)\right).$$

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- We often want a numerical solution to a differential equation because
  - → the closed form solution may be very complicated and difficult to evaluate, or
  - → no closed-form solution can be found.

Euler's method

- This numerical method does not have the utmost generality, but it is natural and capable of high precision.
- Its principle is to represent the solution of a differential equation locally by a few terms of its Taylor series.

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2!}x''(t) + \ldots + \frac{h^m}{m!}x^{(m)}(t) + \ldots \tag{1}$$

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- For numerical purposes, the Taylor series truncated after $m+1$ terms enables us to compute $x(t+h)$ rather accurately
  - → if h is small, and
  - → if the m derivatives are known.

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- The Taylor series method of order 1 is known as Euler's method.
- To find approximate values of the solutions to the initial-value problem

$$\begin{cases} x' = f(t, x(t)) \\ x(a) = x_a \end{cases} \tag{2}$$

over the interval $[a, b]$, the first two terms in the Taylor series (1) are used.

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over the interval $[a, b]$, the first two terms in the Taylor series (1) are used.

$$x(t+h) \approx x(t) + hx'(t). \tag{3}$$

Euler's method

- Hence, the formula

$$x(t+h) = x(t) + hf(t, x(t)). \tag{4}$$

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- Since only two terms of the Taylor series are used in Euler's method, it is not very accurate.

- In solving an IVP, it is useful to distinguish two types of errors
  - local truncation error
  - global truncation error

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- For Euler's method, the local error is simply the remainder of Taylor's approximation, i.e., $O\left(h^2\right)$.
- Since at each step of Euler's method an additional truncation error is introduced, the accumulation of these errors is called the global truncation error.

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Problem: Using Euler's method, compute an approximate value for $x(1)$ for the following ODE.

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- The interval, step size, and initial condition

```
a = 0;

b = 1;

no_steps = 1;

step_size = ( b - a ) / no_steps;

x_0 = 1;
```

Euler's method

- The function

```
f = @ (t , x) x;
```

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- Applying Euler's method

```
for n = 1 : no_steps
    x = x_0 + step_size * f(a , x_0);
    x_0 = x;
    a = a + step_size;
end
```

Euler's method

- Displaying the result

```
disp ( ['The approximate value is ', num2str(x) ] )
```

References

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