Problem 2: Bit allocation for uniform scalar quantization of independent sources (10 points):
Consider a pair of uniformly distributed continuous independent sources X_1 and X_2, both with mean zero and variances 5 and 10 respectively. Let the rate of the uniform scalar quantizers used for X_1 and X_2 be R_1 and R_2 respectively where $R_i = \log_2 K_i$ for $i \in \{1, 2\}$ and K_i denotes the number of quantization points. Given a budget sum rate constraint of $R_1 + R_2 = 3$, compute the rate pair (R_1, R_2) that minimizes the sum of squared error in the respective reconstructions $\mathbb{E}\left[(X_1 - \hat{X}_1)^2\right] + \mathbb{E}\left[(X_2 - \hat{X}_2)^2\right]$. Comment on how you would allocate bits among different DCT coefficients based on this result.

→ Given X_1 follows uniform distribution with mean $=0$, variance $=5$

$$\therefore X_1 \sim U[-\sqrt{15}, \sqrt{15}]$$

→ Given X_2 follows uniform distribution with mean $=0$, variance $=10$

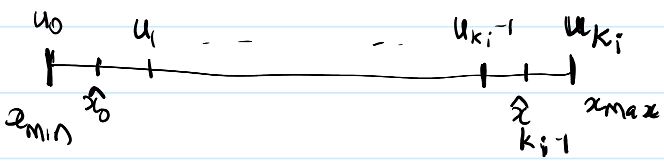
$$\therefore X_2 \sim U[-\sqrt{30}, \sqrt{30}]$$

→ The rate of uniform scalar quantizers of X_1, X_2 are R_1, R_2

$$R_i = \log_2 K_i \qquad i = 1, 2$$

where K_i is number of quantization points

→ Since the ranges of source signals are in finite range, we can use uniform scalar quantization with finite support



$$Q_i(x) = \hat{x}_j \qquad x \in [u_j, u_{j+1}]$$

$$u_{j+1} - u_j = \Delta, \quad \hat{x}_j = \frac{u_j + u_{j+1}}{2}, \quad \Delta = \frac{x_{max} - x_{min}}{k}$$

For the source X_1

$$\text{Distortion} = E[(X_1 - \hat{X}_1)^2] = \sum^{K_1-1} \int^{u_{i+1}} (x_1 - \hat{x}_i)^2 \, P_{X_1}(x_1) \, dx_1$$

for the source X_1

Distortion $= E\left[(X_1 - \hat{X}_1)^2\right] = \sum_{i=0}^{K_1-1} \int_{u_i}^{u_{i+1}} (x_1 - \hat{x}_i)^2 \, P_{X_1}(x_1) \, dx_1$

$P_{X_1}(x) = \begin{cases} \frac{1}{A} & -\sqrt{15} \le x \le \sqrt{15} \\ 0 & \text{otherwise} \end{cases}$

$= \sum_{i=0}^{K_1-1} \int_{u_i}^{u_{i+1}} \frac{1}{A} \left(x_1 - \left(x_{min} + \frac{i\Delta}{2}\right)^2\right) dx_1$

where $A = 2\sqrt{15}$

$x_{min} = -\sqrt{15}$

$= \sum_{i=0}^{K_1-1} \int_{u_i}^{u_{i+1}} \frac{1}{A} \left(x_1 - \left(x_{min} + \frac{iA}{2K_1}\right)\right)^2 dx_1$

$= \frac{A^2}{12} 2^{-2R_1} = \sigma_1^2 2^{-2R_1}$

Similarly $E\left[(X_2 - \hat{X}_2)^2\right] = \sigma_2^2 2^{-2R_2}$ $\quad \sigma_1^2 = 5, \quad \sigma_2^2 = 10$

$\min \quad \sigma_1^2 2^{-2R_1} + \sigma_2^2 2^{-2R_2}$

s.t $\quad R_1 + R_2 = 3$

$\rightarrow \min \quad 5\, 2^{-2\log_2 k_1} + 10\, 2^{-2\log_2 k_2}$

s.t $\quad \log_2 k_1 + \log_2 k_3 = 3$

$\rightarrow \log_2 k_1 + \log_2 k_2 = 3 \implies k_1 k_2 = 8$

$\rightarrow$ k_1, k_2 takes only natural numbers.

$\rightarrow$ Posibble values for k_1, k_2 are $(1,8), (2,4), (4,2), (8,1)$

$\rightarrow$ Among them $k_1, k_2 = (2,4)$ minimizes $5 * 2^{-2\log k_1} + 10\, 2^{-2\log_2 k_2}$

$\therefore (R_1, R_2) = (1, 2)$

Bit allocation:

→ If variance of the input source is high, then the distortion increases. So as to keep the distortion lower we allocate more bits to that DCT coefficient which is evident from the distortion formulae $D = \sigma^2 2^{-2R}$.

→ Here variance of X_2 > Variance of X_1

$\therefore R_2 > R_1$