Problem 1: Linear combinations of order statistics for uniformly distributed noise (10 points)
Consider the noise model $Y_i = x + Z_i$ for $i \in \{1, 2, \cdots, N\}$ for $N = 5$ where Z_i are independent and identically distributed according to a uniform distribution Unif$[-1, 1]$. Compute the order statistics filter coefficients $(\alpha_1, \alpha_2, \cdots, \alpha_N)$ that minimize the mean squared error between $\hat{X}$ and x where $\hat{X} = \sum_{i=1}^{N} \alpha_i Y_{(i)}$. (Ref: A. C. Bovik, T. S. Huang, and D. C. Munson, "A Generalization of Median Filtering Using Linear Combinations of Order Statistics," IEEE Transactions on Acoustics, Speech, and Signal Processing, vol.31, no.6, Dec. 1983).

→ Noise model

$Y_i = x + z_i \quad i \in \{1, 2, \cdots N\}$ for $N = 5$

z_i's are i.i.d and $z_i \sim U(-1, 1)$

$$f_{z_i}(z) = \begin{cases} 1/2 & -1 \le z \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_{z_i}(z) = \begin{cases} 0 & z < -1 \\ \frac{z+1}{2} & -1 \le z \le 1 \\ 1 & \text{otherwise} \end{cases}$$

→ ordered statistics filter

$$\hat{x} = \sum_{i=1}^{N} \alpha_i Y_{(i)}$$ where $Y_{(1)} \le Y_{(2)} \cdots \le Y_{(N)}$ and $Y_{(1)}, \cdots Y_{(N)}$ is a permutation of $Y_1, \cdots Y_N$.

Goal: Find α_i's $i \in \{1, \cdots N\}$ such that $E[(x - \hat{x})^2]$ is minimized

→ let $\alpha = [\alpha_1 \cdots \alpha_N]^T$ and $e = [1 \cdots 1]^T$

$H = [H_{ij}]_{N \times N} \qquad H_{ij} = E[z_{(i)} z_{(j)}]$

then we know that optimal $\alpha^* = \dfrac{H^{-1}e}{e^T H^{-1} e}$

→ $Y_{(1)}, \cdots Y_{(N)}$ are ordered values

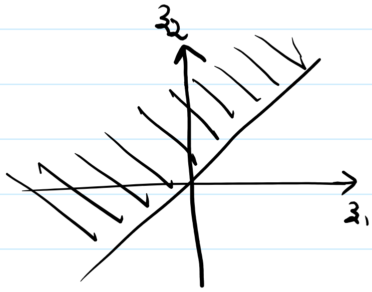
Support of $g_{z(i), z(j)}(z_1, z_2)$

→ $z_{(1)}, \cdots z_{(N)}$ are ordered values

z_2

$z_{(i)}, z_{(j)}$

→ $z_{(1)}, \dots z_{(N)}$ are ordered values



Joint pdf of $z_{(i)}, z_{(j)}$ $i<j$:

$$g_{z(i), z(j)}(z_1, z_2) = k_{ij} \; F_z^{i-1}(z_1) \cdot (F_z(z_2) - F_z(z_1))^{j-i-1} \cdot (1 - F_z(z_2))^{N-j} \; f_z(z_1) \; f_z(z_2)$$

where $k_{ij} = \dfrac{N!}{(i-1)! \, (j-i-1)! \, (N-j)!}$

$$g_{z(i), z(j)}(z_1, z_2) = k_{ij} \left(\frac{z_1+1}{2}\right)^{i-1} \left(\frac{z_2+1}{2} - \frac{z_1+1}{2}\right)^{j-i-1} \left(1 - \frac{z_2+1}{2}\right)^{N-j} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= \frac{1}{4} k_{ij} \left(\frac{z_1+1}{2}\right)^{i-1} \left(\frac{z_2 - z_1}{2}\right)^{j-i-1} \left(1 - \frac{z_2+1}{2}\right)^{N-j}$$

$$H_{ij} = E[z_{(i)} z_{(j)}] = \int_{z_1=-1}^{1} \int_{z_2=z_1}^{1} z_1 z_2 \, \frac{1}{4} k_{ij} \left(\frac{z_1+1}{2}\right)^{i-1} \left(\frac{z_2-z_1}{2}\right)^{j-i-1} \left(1 - \frac{z_2+1}{2}\right)^{N-j} dz_1 dz_2$$

Approximating integral using random numbers generated from uniform distribution.

$$I = \int_0^1 g(x) \, dx$$

let $U \sim \text{Uniform}(0,1)$ $\quad f_U(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$

→ let $Y = g(U)$

$$E[Y] = \int_{-\infty}^{\infty} g(u) \, f_U(u) \, du = \int_0^1 g(u) \, du = I$$

→ let $U_1, \dots U_n$ are independent uniform(0,1) distributions.

→ So $g(U_1), \dots g(U_n)$ are also independent random variables

By strong law of large numbers

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(U_i) = E[g(U_i)] = E[Y] = I$$

$$H_{ij} = E[Z_{(i)} Z_{(j)}] = \int_{z_1=-1}^{1} \int_{z_2=z_1}^{1} z_1 z_2 \frac{1}{4} K_{ij} \left(\frac{z_1+1}{2}\right)^{i-1} \left(\frac{z_2-z_1}{2}\right)^{j-i-1} \left(1-\frac{z_2+1}{2}\right)^{N-j} dz_1 dz_2$$

let $T(z_1, z_2) = z_1 z_2 \frac{1}{4} K_{ij} \left(\frac{z_1+1}{2}\right)^{i-1} \left(\frac{z_2-z_1}{2}\right)^{j-i-1} \left(1-\frac{z_2+1}{2}\right)^{N-j}$

$$H_{ij} = \int_{z_1=-1}^{1} \int_{z_2=z_1}^{1} T(z_1, z_2)\, dz_1 dz_2$$

$$= \int_{z_1=-1}^{1} \int_{z_2=z_1}^{1} I_{z_2}(z_1)\, T(z_1, z_2)\, dz_1 dz_2 \qquad I_{z_2}(z_1) = \begin{cases} 1 & \text{if } z_2 \geq z_1 \\ 0 & \text{otherwise} \end{cases}$$

let $x_1 = \frac{z_1+1}{2}$, $x_2 = \frac{z_2+1}{2}$

$dz_1 = 2dx_1$, $dz_2 = 2dx_2$

$$H_{ij} = 4\int_{x_1=0}^{1} \int_{x_2=0}^{1} I_{x_2}(x_1)\, T(2x_1-1, 2x_2-1)\, dx_1 dx_2 \qquad I_{x_2}(x_1) = \begin{cases} 1 & x_2 \geq x_1 \\ 0 & \text{otherwise} \end{cases}$$

let $S(x_1, x_2) = 4\, I_{x_2}(x_1)\, T(2x_1-1, 2x_2-1)$

$$H_{ij} = \int_{x_1=0}^{1} \int_{x_2=0}^{1} S(x_1, x_2)\, dx_1 dx_2$$

$H_{ij} = E[S(U_1, U_2)]$ where $U_1, U_2 \sim \text{Uniform}(0,1)$

$$\rightarrow H_{ij} = \lim_{k\to\infty} \sum_{i=1}^{k} \frac{S(U_1^i, U_2^i)}{k}$$

$\rightarrow H_{ij} = H_{ji}$

Pdf of $g_{z(i)}(z)$:

$$g_{z(i)}(z) = \frac{N!}{(i-1)!\,(N-i)!} F_z(z)^{i-1} \left(1-F_z(z)\right)^{N-i} f_z(z)$$

$$= \frac{N!}{(i-1)!\,(N-i)!} \left(\frac{z+1}{2}\right)^{i-1} \left(1-\frac{z+1}{2}\right)^{N-i} \cdot \frac{1}{2}$$

$$H_{ii} = \int_{-1}^{1} z^2 g_{z(i)}(z)\, dz$$

$$= \int_{-1}^{1} z^2 \cdot k \left(\frac{z+1}{2}\right)^{i-1} \left(1-\frac{z+1}{2}\right)^{N-i} \cdot \frac{1}{2} \quad dz \qquad k = \frac{N!}{(i-1)!\,(N-i)!}$$

let $T(z) = \frac{z^2 \cdot k}{2} \left(\frac{z+1}{2}\right)^{i-1} \left(1-\frac{z+1}{2}\right)^{N-i}$

$x = \frac{z+1}{2}$, $dz = 2dx$

$$H_{ii} = \int_0^1 T(2x-1)\, 2dx$$

let $2T(2x-1) = s(x)$

$$H_{ii} = \int_0^1 s(x)\, dx$$

→ All the above integrations are calculated by numerical approximation.

→ Related python code was also attached.

Final optimal a_i $i \in \{1, \ldots 5\}$

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array([[ 0.53213861],
       [-0.0023828 ],
       [-0.00533721],
       [ 0.00669734],
       [ 0.52755044]])
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→ As described in the paper for uniformly symmetric distributed noise will result in $a_1 = a_N = \frac{1}{2}$ and $a_i = 0$ $\forall i \in \{2, \ldots N-1\}$, we've got approximately the same results.