E0-214 Applied Linear Algebra and Optimization

Assignment 3

1. Consider the following problems:

   (a) Minimize x^3 subject to $0 < x \leq 1$

   (b) Minimize $\log x$ subject to $x \geq 1$

   For every problem, check if an optimal solution exists using graphical method. Justify your answer.

2. The function $f(x) = -5x^5 + 4x^4 - 12x^3 + 11x^2 - 2x + 1$ is unimodal in the interval $[-0.5, 0.5]$. Use bisection search to find the minimizer of f with the length of final interval of uncertainty less that 10^{-5}.

3. Find the maximum value of $f(x) = 2\sin x - \frac{x^2}{10}$ using Newton method (with $x^0 = 2.5$).

4. Apply Newton's method to find a minimum of the function

$$f(x) = \frac{11}{546}x^6 - \frac{38}{364}x^4 + \frac{1}{2}x^2$$

   with $x^0 = 1.01$. Verify that f^k is a monotonically decreasing sequence. Does $\{x^k\}$ converge to a local minimum?

5. Prove the following inequality for all $x > 0$ using optimality conditions:

$$\frac{1}{x} + x \geq 2.$$

6. Investigate the stationary points of the following functions:

   (a) $f(\boldsymbol{x}) = x_1^2x_2^2 - 4x_1^2x_2 + 4x_1^2 + 2x_1x_2^2 + x_2^2 - 8x_1x_2 + 8x_1 - 4x_2$

   (b) $f(\boldsymbol{x}) = 2x_1^3 - 3x_1^2 - 6x_1x_2(x_1 - x_2 - 1)$

   (c) $f(\boldsymbol{x}) = 2x_1^2 + x_2^2 - 2x_1x_2 + 2x_1^3 + x_1^4$

7. Let $f(\boldsymbol{x}) = (x_1 + x_2^2)^2$. Verify that $d = (-1, 1)^T$ is a descent direction of f at the point $(1, 0)^T$. Use exact line search method to determine all permissible step lengths.

8. Consider the problem,

$$\min \quad 10x_1^2 + x_2^2$$

Apply the following methods to solve this problem, using $\boldsymbol{x}^0 = (1/10, 1)^T$.

(a) Steepest descent method with exact line search

(b) Classical Newton Algorithm

(c) Conjugate Gradient method of Fletcher-Reeves with exact line search

Sketch the contours of the function and also the sequences generated by each algorithm.

9. Use the coordinate descent method with exact line search to minimize

$$f(\boldsymbol{x}) = 4x_1^2 + x_2^2 - 2x_1x_2$$

and compare the iterates with those given by Conjugate Gradient (Fletcher-Reeves) method with exact line search. Sketch the contours of the function and also the sequences $\{x^k\}$ generated by the two methods. Use the same $\boldsymbol{x}^0$ for both the methods.

10. Solve the system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

11. Find the minimum of $x^2 + e^{-x}$ by each of the following methods:

- Newton method
- Bisection method