E0-214 Applied Linear Algebra and Optimization

Assignment 6

1. Consider the problem of finding the point on the parabola $x_2 = \frac{1}{5}(x_1 - 1)^2$ that is closest, in the Euclidean norm sense, to $(1, 2)^T$.

   (a) Use KKT conditions to solve this problem.

   (b) By eliminating the variable x_1, convert this problem into an unconstrained optimization problem and solve it.

2. Find the rectangle of a given perimeter that has greatest area by solving the first order necessary conditions. Verify that the second order sufficiency conditions are satisfied.

3. Consider the constraint set, $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 - (x_1 - 1)^2 \leqslant 0, x_1 \geqslant 0, x_2 \geqslant 0\}$. Is the point $(1, 0)^T$ feasible? Is LICQ satisfied at $(1, 0)^T$?

4. Maximize $14x_1 - x_1^2 + 6x_2 - x_2^2 + 7$ subject to $x_1 + x_2 \leqslant 2$ and $x_1 + 2x_2 \leqslant 3$.

5. Minimize $x_1^2 + x_2^2 + \frac{1}{4}x_3^2$ subject to $x_1 - x_3 + 1 = 0$ and $x_1^2 + x_2^2 = 2x_1$.

6. Solve the following problem:

$$\begin{array}{ll} \min & -2x_1 + x_2 \\ \text{s.t.} & (1 - x_1)^3 \geqslant x_2 \\ & x_2 + 0.25x_1^2 \geqslant 1 \end{array}$$

7. Find the minima of the function $f(\boldsymbol{x}) = x_1 x_2$ on the unit circle $x_1^2 + x_2^2 = 1$.

8. Consider the following problem:

$$\begin{array}{ll} \min & (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2 \\ \text{s.t.} & x_2 \geqslant x_1^2 \\ & x_1 + x_2 \leqslant 6 \\ & x_1, x_2 \geqslant 0 \end{array}$$

   (a) Write down the KKT optimality conditions and verify that these conditions are true at the point $\hat{\boldsymbol{x}} = (\frac{3}{2}, \frac{9}{4})^T$.

   (b) Interpret the KKT conditions at $\hat{\boldsymbol{x}}$ graphically.

   (c) Is $\hat{\boldsymbol{x}}$ the unique global optimal solution?

9. Find an optimal solution to the following problem:

$$\begin{array}{ll} \max & x_1^2 + x_2^2 + 4x_1x_2 \\ \text{s.t.} & \quad x_1^2 + x_2^2 = 1 \end{array}$$

10. Consider the following problem:

$$\begin{array}{ll} \max & \quad x_2 \\ \text{s.t.} & x_2 \geqslant 0 \\ & x_2 \leqslant x_1^2 \end{array}$$

(a) For $\hat{\boldsymbol{x}} = (0,0)^T$, draw $\mathcal{T}(\hat{\boldsymbol{x}})$ and $\tilde{F}(\hat{\boldsymbol{x}})$.

(b) Is LICQ satisfied at $\hat{\boldsymbol{x}}$?

(c) Are the first order KKT conditions satisfied at $\hat{\boldsymbol{x}}$?