E0-214 Applied Linear Algebra and Optimization

Assignment 2

1. Which of the following sets are convex?

   - $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2^2 \geq 1\}$
   - $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2^2 = 1\}$
   - $\{\boldsymbol{x} \in \mathbb{R}^n : \max_{i=1,\ldots,n} x_i \geq 1\}$
   - $\{\boldsymbol{x} \in \mathbb{R}^n : \min_{i=1,\ldots,n} x_i \leq 1\}$

2. Let $C_1 = \{(x, y) : x^2 + y^2 \leq 1\}$ and $C_2 = \{(x, y) : (x - 2)^2 + y^2 \leq 1\}$.

   - Show that C_1 and C_2 are closed convex subsets of $\mathbb{R}^2$.
   - Are C_1 and C_2 strictly separable? Are C_1 and C_2 strongly separable?
   - Draw the convex hull of $C_1 \cup C2$.

3. Is it true that the union of two convex sets is a convex set? If not, find a counterexample.

4. Let $S_1 = \{(x, y) : y \leq 0\}$ and $S_2 = \{(x, y) : y \geq \frac{1}{x}, x \neq 0\}$. Find a hyperplane which separates S_1 and S_2. Is this a strictly separating hyperplane?

5. Find all the supporting hyperplanes of the set $C = \{(x, y) : x \geq 0\} \cap \{(x, y) : y \geq 0\} \cap \{(x, y) : x^2 + y^2 = 1\}$.

6. Discuss the convexity and concavity of the following functions:

   (a) $f(\boldsymbol{x}) = \log(\sum_{i=1}^n e^{x_i})$

   (b) $f(\boldsymbol{x}) = e^{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}$ where $\boldsymbol{A}$ is a positive definite matrix.

   (c) $f(\boldsymbol{x}) = \|\boldsymbol{x}\|_2$

   (d) $f(\boldsymbol{x}) = -x_1^2 - 4x_2^2 - 9x_3^2 + 2x_1x_2 + 3x_1x_3 + 6x_2x_3$

   (e) $f(x) = \sqrt{1 - x^2}, -1 \leq x \leq 1$

   (f) $f(\boldsymbol{x}) = \log(x_1^a x_2^a \ldots x_a^n)$ where $x_i > 0 \; \forall \; i$ and $a > 0$

7. Prove that $2e^{x+y} \leq e^{2x} + e^{2y}$ for all $x, y \in \mathbb{R}$.

8. Are the following problems convex programming problems? Justify your answer.

   (1)

$$\begin{array}{ll} \max & \log(1+x_1)+x_2 \\ \text{s.t.} & 2x_1+x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

   (2)

$$\begin{array}{ll} \min & |x-1|+|x-4| \\ \text{s.t.} & 0 \leq x \leq 5 \end{array}$$

9. Solve the problem:

$$\begin{array}{ll} \min & (x_1+2x_2-3)^2 \\ \text{s.t.} & x_1, x_2 \in \mathbb{R} \end{array}$$

10. If f and h are convex functions, then show that the function $\max\{f, h\}$ is also convex.

11. Show that a convex function defined on a closed real interval attains its maximum at one of the endpoints of the interval.

12. Sketch the contours of the functions,

    (a) $f(\boldsymbol{x}) = x_1^2 + 4x_2^2 - 4x_1 - 8x_2$

    (b) $f(\boldsymbol{x}) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$

    Deduce the value $\boldsymbol{x}^*$ which minimizes/maximizes f.

13. Find the point (x, y) on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$. Formulate this as a constrained optimization problem.

14. Let $S = \{\boldsymbol{x} : x_1^2 + x_2^2 \leq 1, x_1 - x_2^2 \geq 0\}$ and $\boldsymbol{y} = (\frac{1}{2}, 5)^T$. We want to find a point $\boldsymbol{x}^*$ in the set S which is "closest" to $\boldsymbol{y}$. Formulate this as a constrained minimization problem. Also, solve this problem graphically.