E0-214 Applied Linear Algebra and Optimization

Assignment 5

1. Apply the k-means clustering (set $k = 2$) and spectral clustering algorithms (discussed in the class) to the two-dimensional dataset stored in 'Assignment5Dataset.csv'. Try using different values of ϵ while preparing the ϵ neighbourhood similarity graph for spectral clustering. Visualize the clusters obtained using each of the methods.

2. Consider the eight-dimensional dataset stored in the file 'Diabetes.csv' (first eight columns). After scaling the data, find the first two principal components using the correlation matrix. i) Draw the scatter plot of the data using the first two principal components. In the scatter plot, you can show the examples belonging to the two output classes (the ninth column in the dataset) using different colours. ii) Using the two dimensional representation (corresponding to the first two principal components) of the data and using 80% of the data, design a least squares classifier. Visualize the classifier decision function. Test the classifier performance on the left out 20% data. Ensure that class balance is maintained while doing the 80:20 split of the original data.

3. What is the minimum length least squares solution to the system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}?$$

4. Determine $\boldsymbol{A}^{100}$ where $\boldsymbol{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

5. Compute the singular value decomposition of

$$\boldsymbol{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

6. If $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, are $\boldsymbol{A}\boldsymbol{A}^T$ and $\boldsymbol{A}^T\boldsymbol{A}$ similar? Justify your answer.

7. Suppose that $\boldsymbol{A}$ and $\boldsymbol{B}$ are square matrices such that $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}$. Suppose that all eigenvalues of $\boldsymbol{A}$ are distinct. Show that there exists an invertible matrix $\boldsymbol{X}$ such that $\boldsymbol{A} = \boldsymbol{X}\Lambda_1\boldsymbol{X}^{-1}$ and $\boldsymbol{B} = \boldsymbol{X}\Lambda_2\boldsymbol{X}^{-1}$ where Λ_1 and Λ_2 are diagonal matrices.

8. For any matrix $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, by defining the 2-norm of a matrix as

$$\|\boldsymbol{A}\|_2 = \max_{\|\boldsymbol{x}\| = 1} \|\boldsymbol{A}\boldsymbol{x}\|_2$$

show that $\|\boldsymbol{A}\|_2 = \sigma_1$, where σ_1 is the largest singular value of the matrix $\boldsymbol{A}$.

9. Find the solution to the following problem by writing it as a constrained optimization problem.

$$\min_{\boldsymbol{x} \in \mathbb{R}^n} \frac{\boldsymbol{x}^T \boldsymbol{A} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}}$$

where $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

10. Generate a random 5×4 matrix $\boldsymbol{X}$ of rank 2 where every element of $\boldsymbol{X}$ is non-negative. Use Lee-Seung's multiplicative update algorithm to factorize this matrix as $\boldsymbol{U}\boldsymbol{V}$ where $\boldsymbol{U} \in \mathbb{R}_+^{5 \times 2}$ and $\boldsymbol{V} \in \mathbb{R}_+^{2 \times 4}$. Study the performance of the algorithm using different initializations of $\boldsymbol{U}$ and $\boldsymbol{V}$.