Exercises Set 9

Dec. 21, 2015

Exercise 1: Lyapunov exponents

The *Hénon map*

$$\begin{aligned} x_1(n+1) &= 1 - ax_1^2(n) + bx_2(n) \\ x_2(n+1) &= x_1(n) \end{aligned}$$

generates for $a = 1.4$ and $b = 0.3$ the chaotic *Hénon attractor*.

a) Compute both Lyapunov exponents of the *Hénon attractor* using recursive QR-decomposition (use Python function numpy.linalg.qr and make sure that all diagonal elements of the triangular matrix are positive by introducing a diagonal matrix $D = \mathrm{diag}(\mathrm{sign}(R_{11}), \mathrm{sign}(R_{22}))$ such that $\hat{Q}\hat{R}$ with $\hat{Q} = QD$ and $\hat{R} = DR$, where Q and R are obtained with the Python function and $\hat{Q}$ and $\hat{R}$ are required for the computation of the Lyapunov exponents).

b) Show that the local contraction rate of the *Hénon map* is constant (hint: consider the Jacobian matrix). Since it is constant this contraction rate equals the sum of all Lyapunov exponents (giving the volume contraction rate - here: area contraction). Apply this relation between the Lyapunov exponents and the contraction rate to check the results of the numerical computation of both Lyapunov exponents of the *Hénon attractor*.

Exercise 2: Synchronization

a) Synchronization of coupled phase oscillators is given by the Adler equation

$$\frac{d\Delta\phi}{dt} = f(\Delta\phi) = \Delta\omega - \varepsilon \sin(\Delta\phi)$$

where $\Delta\phi$ denotes the phase difference of both oscillators and $\Delta\omega$ the difference of their natural frequencies. Perform a fixed point and a stability analysis and determine the type of local bifurcation which leads to synchronization ($\lim_{t\to\infty} \Delta\phi(t) = const.$)

b) Compute and plot the bifurcation diagram and the winding number (devil's staircase) of the circle map

$$\theta_{n+1} = f(\theta_n) = \theta_n + \Omega - \frac{K}{2\pi}\sin(2\pi\theta_n)$$

for $K = 0.9$ and Ω ranging from 0 to 2.

c) Compute and plot the Lyapunov exponent of the circle map for $K = 0.9$ and Ω ranging from 0 to 2 and interpret the result.