Exercises Set 8

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Exercise 1: Fractal Sets

Let $\mathcal{A}$ be a fractal set which is (similar to the Cantor set) iteratively constructed as follows:

- start set: interval [0,1]
- first step ($n = 1$): remove an interval of length 1/4 in the middle such that the intervals $[0, \frac{3}{8}]$ and $[\frac{5}{8}, 1]$ remain
- second step ($n = 2$): remove 1/16 from the middle of each of the two remaining sets of step one such that the following intervals remain: $[0, \frac{5}{32}]$, $[\frac{7}{32}, \frac{3}{8}]$, $[\frac{5}{8}, \frac{25}{32}]$, and $[\frac{27}{32}, 1]$
- in the following steps ($n = 3, 4, ...$): remove 2^{n-1} subintervals of lengths $1/2^{2n}$ from the middle of the remaining intervals (from the previous step)

a) Compute the "length" (Lebesgue measure) of the limit set (obtained with $n \to \infty$) (Hint: Compute the total length L_n of the remaining intervals in construction step n).

b) Compute the Capacity Dimension of the limit set. (Hint: Cover the set in step n with intervals of length $\varepsilon_n = 2^{-2n-1}$ and use the result of (a).)

Exercise 2: Information Dimension

Implement a box counting based algorithm for estimating the Information dimension d_I of two-dimensional point sets $\mathcal{A} = \{\mathbf{x}_n\}_{n=1}^N$ (cover the set with boxes of size ε, count the number of points in each box, estimate probability p_k to find a point of the set $\mathcal{A}$ in box number k, compute information (Shannon entropy) $I(\varepsilon)$, plot I vs $\log(1/\varepsilon)$, identify linear scaling region and estimate slope). Apply this algorithm to the example data set *example_data.txt* that can be downloaded from stud.ip (ASCII data, two columns with coordinates $(x_1(n), x_2(n))$ of $N = 100000$ data points (rows)). How does the graph (scaling region) changes if not all data points but subsets of size 10000, 20000, etc. are used?

Exercise 3: Lyapunov exponents

a) Implement the Hénon map

$$x_1(n+1) = 1 - a x_1^2(n) + b x_2(n) \quad (1)$$

$$x_2(n+1) = x_1(n) \quad (2)$$

and compute the largest Lyapunov exponent λ_1 of the chaotic attractor occurring for $a = 1.4$ and $b = 0.3$. (Hint: Simulate the (linearized) dynamics of a perturbation $\mathbf{y}(0)$ and compute $\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{\|\mathbf{y}(n)\|}{\|\mathbf{y}(0)\|}\right)$.)

b) Let

$$\mathbf{x}(n+1) = f(\mathbf{x}(n)) \tag{3}$$

be a d-dimensional discrete dynamical system with a fixed point $\mathbf{x}_F \in \mathbb{R}^d$. Let μ_i be the eigenvalues of the Jacobian matrix $Df(\mathbf{x}_F)$. Derive the relation between eigenvalues and the largest Lyapunov exponent (hint: Consider the temporal evolution of eigenvectors).

c) Let

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{4}$$

be a d-dimensional continuous dynamical system with a fixed point $\mathbf{x}_F \in \mathbb{R}^d$. Let μ_i be the eigenvalues of the Jacobian matrix $Df(\mathbf{x}_F)$. Derive the relation between eigenvalues and the largest Lyapunov exponent.