Exercises Set 7

Dec. 7 2015

Exercise 1: Linear (2D) iterated map

Discuss the stability of the fixed point of the linear iterated map

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x_n}) = \mathbf{A}\mathbf{x_n}$$

with $\mathbf{x} \in \mathbb{R}^2$ by analyzing the Eigenvalues of $\mathbf{A}$. (You may neglect the degenerate case when A does not have two distinct Eigenvectors, i.e. you can assume that A is diagonalizable.) Which cases are possible if the system is supposed to be the linearization of the Poincaré map of a Hamiltonian system?

Exercise 2: Standard map

Study the standard map

$$x_{n+1} = x_n + p_n \pmod 1 \tag{1}$$

$$p_{n+1} = p_n + \frac{K}{2\pi}\sin(2\pi x_{n+1})\,. \tag{2}$$

(here x is scaled to the interval $[0,1]$ for convenience. For plotting, use periodic boundary conditions in p as well, e.g. in the interval $[-\frac{1}{2},\frac{1}{2}]$ or $[-1,1]$).

a.) Find the Eigenvalues of the linearized dynamics near the fixed point at $(x,p) = (0,0)$, and verify that the one is the inverse of the other as we know from the area preservation property.

b.) Find the corresponding Eigenvectors.

c.) Write a program that plots the stable and unstable manifolds of the fixed points at $(x,p) = (0,0)$. Plot them for several values of K and additionally plot them on top of a phase space portrait of the standard map at the same K values as well.

d.) The winding number for an arbitrary orbit can be defined by

$$\lim_{n\to\infty}\frac{x_{n+1}-x_0}{n},$$

where the x iterates are not taken modulo 1! Use this expression and for $K = 0.9716354$ try to find an unbroken original torus , i.e. an invariant torus (which is of course just a curve in our 2 dimensional phase space) that still spans the interval $[0,1]$ and thus is a barrier for transport in momentum direction. Do this by starting orbits at $x_0 = 0$ with varying p_0 and search for a torus with winding number of the golden mean $(\sqrt{5}-1)/2$. You will see that the winding number fluctuates strongly (and does not easily converge) in the chaotic layers but is smooth (and quickly converges) in periodic or quasi-periodic windows. Make a coarse grained plot of the winding number as a function of p_0 first and find a suitable interval of p_0 where you can search with a root finder or by interval sectioning to locate the golden torus (or at least a torus close by). Plot it in a phase space portrait.

[Remark: there are much more sophisticated methods to determine the winding number as the straight forward evaluation of the above expression, e.g. the algorithm termed "numerical analysis of fundamental frequencies" by Laskar, but we will content ourselves with the above in this exercise.]