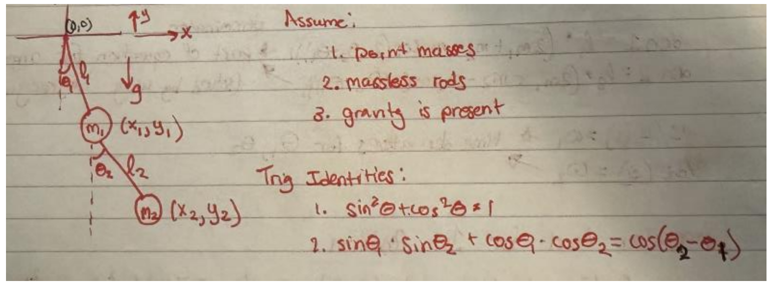
Kinematic Constants



$$x_1 = l_1 \sin\theta_1$$
$$y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

Velocities (time derivative)

$$\dot{x}_1 = \left(l_1 \cos\theta_1\right)\dot{\theta}_1$$
$$\dot{y}_1 = \left(l_1 \sin\theta_1\right)\dot{\theta}_1$$
$$\dot{x}_2 = \left(l_1 \cos\theta_1\right)\dot{\theta}_1 + \left(l_2 \cos\theta_2\right)\dot{\theta}_2$$
$$\dot{y}_2 = \left(l_1 \sin\theta_1\right)\dot{\theta}_1 + \left(l_2 \sin\theta_2\right)\dot{\theta}_2$$

Potential Energy (V)

$$V = \sum_i mgh_i$$
$$V = m_1 g y_1 + m_2 g y_2$$
$$V = m_1 g\left(-l_1 \cos\theta_1\right) + m_2 g\left(-l_1 \cos\theta_1 + -l_2 \cos\theta_2\right)$$
$$V = -m_1 g \cdot l_1 \cos\theta_1 - m_2 g \cdot l_1 \cos\theta_1 - m_2 g \cdot l_2 \cos\theta_2$$
$$V = -\left(m_1 + m_2\right) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

Kinetic Energy (T)

$$T = \sum_i \left(\frac{1}{2} m v_i^2\right)$$
$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2 = \frac{1}{2} m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$
$$T = \frac{1}{2} m_1\left(l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1\right) + \frac{1}{2} m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$
$$T = \frac{1}{2} m_1\left(l_1^2 \dot{\theta}_1^2\left(\cos^2\theta_1 + \sin^2\theta_1\right)\right) + \frac{1}{2} m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$
$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)$$

$$\dot{x}_2^2 = \left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)\left(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2\right)$$
$$\dot{x}_2^2 = l_1^2\dot{\theta}_1^2\cos^2\theta_1 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + l_2^2\dot{\theta}_2^2\cos^2\theta_2$$

$$\dot{y}_2^2 = \left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)\left(l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2\right)$$
$$\dot{y}_2^2 = l_1^2\dot{\theta}_1^2\sin^2\theta_1 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + l_2^2\dot{\theta}_2^2\sin^2\theta_2$$

$$\dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\theta}_1^2\left(\cos^2\theta_1+\sin^2\theta_1\right) + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right) + l_2^2\dot{\theta}_2^2\left(\cos^2\theta_1+\sin^2\theta_1\right)$$
$$\dot{x}_2^2+\dot{y}_2^2 = l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\left(\theta_2-\theta_1\right)\right) + l_2^2\dot{\theta}_2^2$$

$$T = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\left(\theta_2-\theta_1\right)\right) + l_2^2\dot{\theta}_2^2\right)$$

Total Energy (L)

$$L = T + V$$
$$L = \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\left(\theta_2-\theta_1\right)\right) + l_2^2\dot{\theta}_2^2\right) + \left(m_1+m_2\right)gl_1\cos\theta_1 + m_2gl_2\cos\theta_2$$

Lagrange Equation of Motion

$$Q_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = 0$$
$$\frac{\partial L}{\partial\dot{\theta}_1} = m_1l_1^2\dot{\theta}_1 + m_2l_1^2\dot{\theta}_1 + m_2l_1l_2\dot{\theta}_2\cos\left(\theta_2-\theta_1\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) = m_1l_1^2\ddot{\theta}_1 + m_2l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2\cos\left(\theta_2-\theta_1\right) - m_2l_1l_2\dot{\theta}_2\sin\left(\theta_2-\theta_1\right)\left(\dot{\theta}_2-\dot{\theta}_1\right)$$
$$\frac{\partial L}{\partial\theta_1} = m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_2-\theta_1\right) - \left(m_1+m_2\right)gl_1\sin\theta_1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = m_1l_1^2\ddot{\theta}_1 + m_2l_1^2\ddot{\theta}_1 + m_2l_1l_2\ddot{\theta}_2\cos\left(\theta_2-\theta_1\right) - m_2l_1l_2\dot{\theta}_2^2\sin\left(\theta_2-\theta_1\right)$$
$$+ m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_2-\theta_1\right) - m_2l_1l_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_2-\theta_1\right) + \left(m_1+m_2\right)gl_1\sin\theta_1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right) - \frac{\partial L}{\partial\theta_1} = \left(m_1+m_2\right)l_1\ddot{\theta}_1 + m_2l_2\ddot{\theta}_2\cos\left(\theta_2-\theta_1\right) - m_2l_2\dot{\theta}_2^2\sin\left(\theta_2-\theta_1\right) + \left(m_1+m_2\right)g\sin\theta_1 = 0$$

$$Q_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l_2^2 \dot{\theta}_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)$$

$$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) - m_2 g l_2 \sin\theta_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)$$
$$+ m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1) + m_2 g l_2 \sin\theta_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + m_2 g l_2 \sin\theta_2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) + g \sin\theta_2 = 0$$

System of 4 Equations (Solved for angular accelerations in MATLAB)

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) = m_2 l_2 \dot{\theta}_2^2 \sin(\theta_2 - \theta_1) - (m_1 + m_2) g \sin\theta_1$$

$$l_1 \ddot{\theta}_1 \cos(\theta_2 - \theta_1) + l_2 \ddot{\theta}_2 = -l_1 \dot{\theta}_1^2 \sin(\theta_2 - \theta_1) - g \sin\theta_2$$

$$\dot{\theta}_1 = w_1$$

$$\dot{w}_1 = \ddot{\theta}_1$$

$$\dot{\theta}_2 = w_2$$

$$\dot{w}_2 = \ddot{\theta}_2$$