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# Logarithmic strain, logarithmic spin and logarithmic rate

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**Summary.** Two yet undiscovered relations between the Eulerian logarithmic strain ln $V$ and two fundamental mechanical quantities, the stretching and the Cauchy stress, are disclosed. A new spin tensor and a new objective tensor-rate are accordingly introduced. Further, new rate-form constitutive models based on this objective tensor-rate are established. It is proved that

(i). an objective corotational rate of the logarithmic strain ln $V$ can be exactly identical with the stretching and in all strain measures only ln $V$ enjoys this property, and
(ii). ln $V$ and the Cauchy stress $\sigma$ form a work-conjugate pair of strain and stress.

These properties of ln $V$ are shown to determine a unique smooth spin tensor called logarithmic spin and by virtue of this spin a new tensor-rate called logarithmic rate is proposed. In all possible rate-form constitutive models relating the same kind of objective corotational rates of an Eulerian stress measure and an Eulerian strain measure, it is proved that the logarithmic rate is the only choice and the strain measure must be the logarithmic strain ln $V$ if the stretching, as is commonly assumed, is used to measure the rate of change of deformation. As an illustration, it is shown that all finite deformation responses of the grade-zero hypoelastic model based on the logarithmic rate are completely in agreement with those of a finite deformation elastic model and moreover this simplest rate-form constitutive model based on the logarithmic rate can predict the phenomenon of the known hypoelastic yield at simple shear.

## 1 Introduction

The logarithmic strain, introduced by Hencky [15], has long been popular and enjoyed favoured treatment in solid mechanics, materials science and metallurgy due to its remarkable properties, such as its simple additive decomposition into bulk strain and distortion, etc. Sometimes it is referred to as the *true* or *natural* strain. Earlier, the logarithmic strain was used by Richter [32] to formulate the constitutive equation of isotropic materials. The work of Hill [16], [17] shows that the logarithmic strain measures have inherent advantages in certain constitutive inequalities and, furthermore, his later work [18] treats the logarithmic strain and its rate and conjugate stress as basic measures for strains, rates of strains, and stresses, respectively. Rice [31], Stören and Rice [34], Hutchinson and Neale [21], and Neale [29] find the logarithmic strain to be useful in the formulation of the deformation theory of plasticity. In the past two decades, various expressions for the rates and conjugate stresses of the Lagrangean and Eulerian logarithmic strains were derived (cf. Fitzgerald [9], Gurtin and Spear [12], Hoger [19], [20], Lehmann and Liang [25], Man and Guo [27], and Xiao [37], etc.). Recently, Heiduschke [14] has established a logarithmic strain space description.

Impressed by the above noticeable facts, one may seek the root and conjecture that the logarithmic strain should possess certain intrinsic far-reaching properties to establish its favoured position in all possible strain measures, which remain unknown. It is indeed true. In this paper, we first disclose two hidden relations between the Eulerian logarithmic strain $\ln V$ and two fundamental mechanical quantities, the stretching and the Cauchy stress, and then we show that by virtue of the two relations the logarithmic strain $\ln V$ indeed can establish its favoured and even unequalled position in the formulation of rate-form constitutive models.

Throughout, we employ direct tensor notation. The main results are as follows. In Section 2, we prove that a corotational rate of the Eulerian logarithmic strain $\ln V$ can be exactly identical with the stretching and moreover in all possible strain measures only $\ln V$ enjoys this property, i.e. any corotational rate of any other strain measure cannot be identical with the stretching. In Section 3, based on the notion of observers and the objectivity of the stress power, we show that Hill's work-conjugacy notion of Lagrangean strain and stress measures can be naturally extended to Eulerian strain and stress measures, and then we prove that $\ln V$ and the Cauchy stress form a work-conjugate pair of strain and stress. In Section 4 we indicate that the unique properties of $\ln V$ disclosed in Sections 2 and 3 determine a unique smooth spin tensor called logarithmic spin and accordingly determine a new objective corotational rate called logarithmic rate, and, moreover, we derive an explicit basis-free expression for the logarithmic spin in terms of the left Cauchy-Green tensor, the stretching and the vorticity tensor. In Section 5, we prove that in all possible rate-form constitutive models relating the same kind of objective rates of an Eulerian stress measure and an Eulerian strain measure, the logarithmic rate of the stress measure is the only choice and the strain measure must be $\ln V$ if the stretching, as is commonly assumed, is used to measure the rate of change of deformation. By virtue of this fact, we establish new rate-form constitutive models based on the logarithmic rate. Finally, as an example, we consider all possible finite deformation responses of a grade-zero hypoelastic constitutive model based on the logarithmic rate to illustrate our results and moreover we show that this simplest constitutive model based on the logarithmic rate can predict the phenomenon of the hypoelastic yield at the simple shear deformation.

In the rest of this Section, we outline some facts that will be used. Consider a solid body experiencing continued finite deformation. Let $\boldsymbol{F}$ be the deformation gradient. Then the following decomposition holds:

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}, \tag{1}$$

$$\boldsymbol{V}^2 = \boldsymbol{F}\boldsymbol{F}^T = \boldsymbol{B}. \tag{2}$$

Here, $\boldsymbol{V}$, $\boldsymbol{R}$ and $\boldsymbol{B}$ are the left stretch tensor, the rotation tensor and the left Cauchy-Green tensor, respectively. $\boldsymbol{V}$ and $\boldsymbol{B}$ offer two Eulerian strain measures. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the principal stretches, i.e. the eigenvalues of $\boldsymbol{V}$, and $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$ be three corresponding subordinate orthonormal eigenvectors of $\boldsymbol{V}$. Then a general class of Eulerian strain measures, called Eulerian Hill's strain measures, can be defined as follows (cf. Hill [16]−[18] for the definition of Lagrangean strain measures):

$$\boldsymbol{e} = f(\boldsymbol{V}) = \sum_{i=1}^{3} f(\lambda_i)\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \tag{3}$$

where the scale function $f(\lambda)$ is required to be a smooth monotone function fulfilling the following conditions:

$$f(1) = 0, \quad f'(1) = 1. \tag{4}$$

By taking the scale functions $f(\lambda) = \frac{1}{m}(\lambda^m - 1)$ indexed by integers $m$, one can infer that all commonly-known Eulerian strain measures are available. In particular, for the logarithmic function $f(\lambda) = \ln \lambda$ (as $m \to 0$), (3) offers the Eulerian logarithmic strain

$$\ln \boldsymbol{V} = \sum_{i=1}^{3} (\ln \lambda_i)\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i, \tag{5}$$

which here will be of particular interest.

On the other hand, let $\boldsymbol{L}$ be the velocity gradient. Then

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \tag{6}$$

and the following unique additive decomposition holds:

$$\boldsymbol{L} = \boldsymbol{D} + \boldsymbol{W}, \tag{7}$$

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T), \quad \boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T). \tag{8}$$

Here, $\boldsymbol{D}$, the symmetric part of $\boldsymbol{L}$, is the stretching, and $\boldsymbol{W}$, the antisymmetric part of $\boldsymbol{L}$, is the vorticity tensor. Here and hereafter both the notation $(\dot{\ })$ and $(\ )^{\cdot}$, such as $\dot{\boldsymbol{F}}$ and $(\ln \boldsymbol{V})^{\cdot}$ etc., are used to denote the material time derivative.

Let $\boldsymbol{\Omega}^*$ be a spin tensor, i.e. a skew-symmetric tensor varying with the time, which defines a rotating frame relative to a fixed background frame. Then the corotational rate of an objective or Eulerian symmetric second order tensor $\boldsymbol{G}$ is defined by

$$\mathring{\boldsymbol{G}}^* = \dot{\boldsymbol{G}} + \boldsymbol{G}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{G}. \tag{9}$$

Let further $\boldsymbol{Q}^*$ be a rotation tensor defining the spin $\boldsymbol{\Omega}^*$, i.e.

$$\boldsymbol{\Omega}^* = \dot{\boldsymbol{Q}}^{*T}\boldsymbol{Q}^*. \tag{10}$$

Then, an observer in a rotating frame defined by the spin $\boldsymbol{\Omega}^*$ will observe that the tensor $\boldsymbol{G}$ becomes $\boldsymbol{Q}^*\boldsymbol{G}\boldsymbol{Q}^{*T}$ and therefore its time rate is given by

$$(\boldsymbol{Q}^*\boldsymbol{G}\boldsymbol{Q}^{*T})^{\cdot} = \boldsymbol{Q}^*\mathring{\boldsymbol{G}}^*\boldsymbol{Q}^{*T}. \tag{11}$$

The physical meaning of the corotational rate defined by (9) is thus displayed. A corotational rate is not necessarily objective. The Jaumann rate

$$\mathring{\boldsymbol{G}}^J = \dot{\boldsymbol{G}} + \boldsymbol{G}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{G} \tag{12}$$

is a well-known example of objective rates, which is obtained by taking $\boldsymbol{\Omega}^* = \boldsymbol{W}$ in (9). Another important example is provided by taking $\boldsymbol{\Omega}^* = \dot{\boldsymbol{R}}^T\boldsymbol{R}$ in (9) (cf. Dienes [5]). In reality, there are infinite kinds of objective corotational rates (cf. Guo [10], Haupt and Tsakmakis [13], Xiao, Bruhns and Meyers [38], etc.).

Finally, let $\boldsymbol{\sigma}$ and $\rho$ be the Cauchy stress and the mass density in the current configuration. Then the stress power per unit mass density $\dot{w}$ is given by

$$\rho\dot{w} = \mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{D}). \tag{13}$$

Since both $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ are objective, the stress power (specific rate of work) $\dot{w}$ is an objective scalar.

## 2 A direct relation between ln $V$ and the stretching $D$

As a fundamental kinematic quantity, the stretching $\boldsymbol{D}$ offers a direct natural measure of the rate of length change of any line element and the rate of change of the angle between any two intersecting line elements in a deforming body. In view of this and the fact that a strain is a measure of change of the length of any line element and change of the angle between any two intersecting line elements, accordingly the stretching is also referred to as the Eulerian strain-rate, the rate of deformation tensor, or simply as the strain-rate, the deformation rate, etc. However, by now the stretching has been known simply as a symmetric part of the velocity gradient, as mentioned before, and it has not been known whether or not it is really a rate of the change of a strain measure. Gurtin and Spear [12] and Hoger [19] prove that certain corotational rates of the logarithmic strain ln $V$, such as the Jaumann rate etc., can equal the stretching for certain particular $V$. Inspired by these works, in this Section we prove that a corotational rate of ln $V$ can be exactly identical with the stretching for all $V$ and furthermore we prove that in all strain measures only ln $V$ enjoys this property, i.e. any corotational rate of any other strain measure cannot be identical with the stretching.

Since the material time rate of a Lagrangean strain measure is not an Eulerian quantity, in the following we only need to consider the Eulerian strain measures. Let $\boldsymbol{e}$ (cf. (3)) be any given Eulerian strain measure and $\boldsymbol{\Omega}^*$ be any given spin tensor. Under the double arbitrary choices we consider the following tensor equation:

$$\mathring{\boldsymbol{e}}^* = \dot{\boldsymbol{e}} + \boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e} = \boldsymbol{D}. \tag{14}$$

The above equation just states the fact that the corotational rate $\mathring{\boldsymbol{e}}^*$ of $\boldsymbol{e}$ is identical with the stretching. Equation (14) may be recast in

$$\boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e} = \boldsymbol{D} - \dot{\boldsymbol{e}}. \tag{15}$$

Here and hereafter we denote all the distinct eigenvalues of $V$ by $\{\lambda_1, \cdots, \lambda_m\}$ and the corresponding subordinate eigenprojections of $V$ by $\{\boldsymbol{P}_1, \cdots, \boldsymbol{P}_m\}$. From the analysis for a similar tensor equation for the skew-symmetric tensor $\boldsymbol{\Omega}^*$ in Xiao [37], we infer that (15) has a solution for $\boldsymbol{\Omega}^*$, iff

$$\boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\sigma = \boldsymbol{0}, \quad \sigma = 1, \cdots, m. \tag{16}$$

In reality, the condition (16) is necessary for the consistency of the tensor Eq. (15). Reformulating (3) as

$$\boldsymbol{e} = \sum_{\sigma=1}^{m} f(\lambda_\sigma)\,\boldsymbol{P}_\sigma \tag{17}$$

and applying the identitites for the eigenprojections

$$\boldsymbol{P}_\sigma\boldsymbol{P}_\tau = \begin{cases} \boldsymbol{P}_\sigma, & \sigma = \tau, \\ \boldsymbol{0}, & \sigma \neq \tau, \end{cases} \tag{18}$$

we infer

$$\boldsymbol{P}_\sigma\boldsymbol{e} = \boldsymbol{e}\boldsymbol{P}_\sigma = f(\lambda_\sigma)\,\boldsymbol{P}_\sigma, \quad \sigma = 1, \cdots, m, \tag{19}$$

and therefore

$$\boldsymbol{P}_\sigma(\boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e})\,\boldsymbol{P}_\sigma = (\boldsymbol{P}_\sigma\boldsymbol{e})\,\boldsymbol{\Omega}^*\boldsymbol{P}_\sigma - \boldsymbol{P}_\sigma\boldsymbol{\Omega}^*(\boldsymbol{e}\boldsymbol{P}_\sigma)$$
$$= f(\lambda_\sigma)\,\boldsymbol{P}_\sigma\boldsymbol{\Omega}^*\boldsymbol{P}_\sigma - f(\lambda_\sigma)\,\boldsymbol{P}_\sigma\boldsymbol{\Omega}^*\boldsymbol{P}_\sigma = \boldsymbol{0}$$

for each $\sigma = 1, \cdots, m$.

From this and (15) we can derive (16) immediately.

Moreover, we prove that the condition (16) is sufficient for the consistency of the tensor equation (15). To this end, we prove that if (16) is fulfilled by $\boldsymbol{e}$, then

$$\boldsymbol{\Omega}_0{}^* = \sum_{\sigma=1}^{m} \boldsymbol{P}_\sigma\boldsymbol{W}\boldsymbol{P}_\sigma + \sum_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{m} \left(f(\lambda_\sigma) - f(\lambda_\tau)\right)^{-1} \boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\tau \tag{20}$$

satisfies the tensor equation (15). Here and hereafter the second summation shown above is assumed to be zero for $m = 1$. In reality, utilizing (17) and (19) and denoting $f_{\sigma\tau} = f(\lambda_\sigma) - f(\lambda_\tau)$, we derive sequentially

$$\begin{aligned}\boldsymbol{e}\boldsymbol{\Omega}_0{}^* - \boldsymbol{\Omega}_0{}^*\boldsymbol{e} &= \sum_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{m} f_{\sigma\tau}^{-1}[(\boldsymbol{e}\boldsymbol{P}_\sigma)(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\tau - \boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,(\boldsymbol{P}_\tau\boldsymbol{e})]\\ &= \sum_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{m} f_{\sigma\tau}^{-1}(f(\lambda_\sigma) - f(\lambda_\tau))\,\boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\tau\\ &= \sum_{\substack{\sigma,\tau=1\\ \sigma\neq\tau}}^{m} \boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\tau.\end{aligned} \tag{21}$$

From the latter and the condition (16) as well as the identity

$$\sum_{\sigma=1}^{m} \boldsymbol{P}_\sigma = \sum_{\tau=1}^{m} \boldsymbol{P}_\tau = \boldsymbol{I}, \tag{22}$$

where $\boldsymbol{I}$ is the identity tensor, we derive

$$\boldsymbol{e}\boldsymbol{\Omega}_0{}^* - \boldsymbol{\Omega}_0{}^*\boldsymbol{e} = \sum_{\sigma,\tau=1}^{m} \boldsymbol{P}_\sigma(\boldsymbol{D} - \dot{\boldsymbol{e}})\,\boldsymbol{P}_\tau = \boldsymbol{D} - \dot{\boldsymbol{e}}. \tag{23}$$

The latter indicates that (20) is a solution of the tensor equation (15) if the condition (16) is fulfilled.

Now we are in a position to find out what the condition (16) implies. According to Xiao [37] (see also Carlson and Hoger [2]),

$$\dot{\boldsymbol{e}} = \sum_{\sigma,\tau=1}^{m} \frac{f(\lambda_\sigma) - f(\lambda_\tau)}{\lambda_\sigma - \lambda_\tau}\,\boldsymbol{P}_\sigma\dot{\boldsymbol{V}}\boldsymbol{P}_\tau, \tag{24}$$

where the limiting process $\lim\limits_{\lambda_\sigma\to\lambda_\tau} \dfrac{f(\lambda_\sigma) - f(\lambda_\tau)}{\lambda_\sigma - \lambda_\tau} = f'(\lambda_\sigma)$ is understood for $\sigma = \tau$. By using (24) and (18) we derive

$$\boldsymbol{P}_\sigma\dot{\boldsymbol{e}}\boldsymbol{P}_\sigma = f'(\lambda_\sigma)\,\boldsymbol{P}_\sigma\dot{\boldsymbol{V}}\boldsymbol{P}_\sigma, \quad \sigma = 1, \cdots, m. \tag{25}$$

To establish the relationship between $\dot{\boldsymbol{V}}$ and $\boldsymbol{D}$ and $\boldsymbol{V}$, we need the following tensor equation for $\dot{\boldsymbol{V}}$:

$$\boldsymbol{V}\dot{\boldsymbol{V}} + \dot{\boldsymbol{V}}\boldsymbol{V} = \boldsymbol{D}\boldsymbol{V}^2 + \boldsymbol{V}^2\boldsymbol{D} + \boldsymbol{W}\boldsymbol{V}^2 - \boldsymbol{V}^2\boldsymbol{W}, \tag{26}$$

which can be derived by differentiating (2.1) and using (6)–(8). By using the equalities for any integer $k$:

$$V^k P_\sigma = P_\sigma V^k = \lambda_\sigma{}^k P_\sigma, \quad \sigma = 1, \cdots, m, \tag{27}$$

from the tensor equation (26) we derive

$$P_\sigma \dot{V} P_\sigma = \lambda_\sigma P_\sigma D P_\sigma. \tag{28}$$

Then, utilizing (25) and (28), we infer that the condition (16) is equivalent to

$$(\lambda_\sigma f'(\lambda_\sigma) - 1)\, P_\sigma D P_\sigma = \mathbf{0}, \quad \delta = 1, \cdots, m. \tag{29}$$

From the arbitrariness of $D$, we know that (29) holds iff

$$f'(\lambda_\sigma) = \frac{1}{\lambda_\sigma}, \quad \sigma = 1, \cdots, m$$

i.e.

$$f(\lambda) = \ln \lambda, \tag{30}$$

where the condition (4) has been used.

Thus, we conclude that under the double arbitrary choices of $e$ and $\Omega^*$, the tensor equation (14), i.e. (15) is consistent iff the scale function $f(\lambda)$ is the logarithmic function $\ln \lambda$, i.e. iff the strain measure $e$ is the Eulerian logarithmic strain $\ln V$. The main result of this Section is thus proved.

The above surprising fact arising from (14) reveals a unique intrinsic property of the Eulerian logarithmic strain $\ln V$, which implies far-reaching results, as will be shown in the succeeding Sections.

## 3 A direct relation between ln $V$ and the Cauchy stress $\sigma$

The Cauchy stress, introduced by Cauchy's fundamental stress principle, offers a direct natural measure of the contact force on any current plane element in a deforming body. Its introduction is recognized as one of the most far-reaching discoveries in continuum mechnaics (cf. Gurtin [11]) and referred to as the *true* stress (cf. Ogden [30]). In solid mechanics, other stress measures, such as the Jaumann stress, the second Piola-Kirchhoff stress etc., are used. According to Hill [16]–[18] and Macvean [26], a useful relation between Lagrangean strain and stress measures can be established with the help of the work-conjugacy notion (see also Ogden [30]). However, the work-conjugacy notion for Lagrangean strain and stress measures can not apply to Eulerian strain and stress measures, except for some particular cases (cf. Macvean [26], Ogden [30], Hoger [20], etc.). Owing to the lack of an appropriate work-conjugacy notion, until now there has been little result on the work-conjugacy relation between Eulerian strain and stress measures, except for the work of Lehmann and Liang [25]. Consequently, by now it has not been known whether or not the Cauchy stress $\sigma$ is conjugate to an Eulerian strain measure. In this Section, we indicate that the answer is positive and, moreover, that it is the logarithmic strain ln $V$ that enjoys the just-mentioned property.

To establish the work-conjugacy relation between Eulerian strain and stress measures, it is necessary to extend Hill's work-conjugacy notion for Lagrangean strain and stress measures. In what follows, basing on the notion of observers and the objectivity of stress power, we state a natural extension of Hill's work-conjugacy notion.

Consider a pair of Eulerian stress and strain measures, $(\boldsymbol{\Pi}, \boldsymbol{e})$. Let $\boldsymbol{\Omega}^*$ be a spin tensor. In a rotating frame defined by the spin $\boldsymbol{\Omega}^*$ relative to a fixed background frame, the pair $(\boldsymbol{\Pi}, \boldsymbol{e})$ becomes $(\boldsymbol{Q}^*\boldsymbol{\Pi}\boldsymbol{Q}^{*T}, \boldsymbol{Q}^*\boldsymbol{e}\boldsymbol{Q}^{*T})$, where $\boldsymbol{Q}^*$ is a rotation tensor defining the spin $\boldsymbol{\Omega}^*$ (cf. (10)). Hence an observer (*) in the just-mentioned rotating frame observes that the time rate of $\boldsymbol{e}$ is given by $(\boldsymbol{Q}^*\boldsymbol{e}\boldsymbol{Q}^{*T})^{\cdot}$. Now the observer (*) forms the inner product $\mathrm{Tr}\big((\boldsymbol{Q}^*\boldsymbol{\Pi}\boldsymbol{Q}^{*T})\,(\boldsymbol{Q}^*\boldsymbol{e}\boldsymbol{Q}^{*T})^{\cdot}\big)$, just as an observer in a fixed background frame does for a pair of Lagrangean stress and strain measures. Then, basing upon the same argument used by the observers in a fixed background frame for the pairs of Lagrangean strain and stress measures, the observer (*) judges that the pair $(\boldsymbol{Q}^*\boldsymbol{\Pi}\boldsymbol{Q}^{*T}, \boldsymbol{Q}^*\boldsymbol{e}\boldsymbol{Q}^{*T})$, i.e. $(\boldsymbol{\Pi}, \boldsymbol{e})$, is a work-conjugate pair iff the aforementioned inner product furnishes the stress power, i.e.

$$\rho\dot{w} = \mathrm{Tr}\big((\boldsymbol{Q}^*\boldsymbol{\Pi}\boldsymbol{Q}^{*T})\,(\boldsymbol{Q}^*\boldsymbol{e}\boldsymbol{Q}^{*T})^{\cdot}\big). \tag{31}$$

By using (11), the above formula defining the work-conjugacy relation can be rewritten into

$$\rho\dot{w} = \mathrm{Tr}(\boldsymbol{\Pi}\mathring{\boldsymbol{e}}^*), \tag{32}$$

where the corotational rate $\mathring{\boldsymbol{e}}^*$ is defined by (9) by taking $\boldsymbol{G} = \boldsymbol{e}$. Lehmann and Liang [25] were the first to use (32) to define the work-conjugacy relation between Eulerian stress and strain measures. The physical motivation for (32) can be found in thermodynamical considerations and the introduction of dual variational principles (e.g. see Macvean [26], Lehmann [24], and Haupt and Tsakmakis [13], etc.).

Since both $\rho\dot{w}$ and $\boldsymbol{\Pi}$ are objective, from (32) one can see that the spin $\boldsymbol{\Omega}^*$ must be such that the corotational rate $\mathring{\boldsymbol{e}}^*$ is an objective one. Moreover, one can see that the above work-conjugacy notion for Eulerian strain and stress measures is broader than that for Lagrangean stress measures. In reality, the latter allows only one rate, i.e. the material time rate, to be considered, while the former permits using different kinds of objective corotational rates. Owing to the latter fact, in the sense of the above work-conjugacy, a given Eulerian strain measure may be associated with different Eulerian stress measures by different kinds of observers (cf. Lehmann and Liang [25]), while a given Eulerian stress measure either can be associated with a unique Eulerian strain measure or cannot be associated with any Eulerian strain measure. For details, see Xiao, Bruhns and Meyers [38]. Here we are mainly concerned with the Cauchy stress $\boldsymbol{\sigma}$. Suppose that there is an Eulerian strain measure $\boldsymbol{e}$ (cf. (3)) which is associated with the Cauchy stress $\boldsymbol{\sigma}$ by an observer in a rotating frame with the spin $\boldsymbol{\Omega}^*$ in the sense of the work-conjugacy defined by (31). Then by means of (32) and (13) we infer

$$\mathrm{Tr}(\boldsymbol{\sigma}\mathring{\boldsymbol{e}}^*) = \mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{D}). \tag{33}$$

From the arbitrariness of $\boldsymbol{\sigma}$ we derive (14) again. Thus from this and the fact proved in Section 2 we conclude that the Cauchy stress $\boldsymbol{\sigma}$ and the logarithmic strain $\ln \boldsymbol{V}$ can form a work-conjugate pair of stress and strain. The rotating frame fulfilling the work-conjugacy relation of $\boldsymbol{\sigma}$ and $\ln \boldsymbol{V}$ will be determined in the next Section.

## 4 Logarithmic spin and logarithmic rate

In Section 2, we have proved that the linear tensor equation (15) for the spin $\boldsymbol{\Omega}^*$ has a solution iff the strain measure $\boldsymbol{e}$ is the logarithmic strain $\ln \boldsymbol{V}$, i.e. $\boldsymbol{e} = \ln \boldsymbol{V}$. Let $\boldsymbol{\Omega}^{\log}$ denote this solution. Then its defining equation is (see (35))

$$(\ln \mathring{\boldsymbol{V}})^{\log} = \boldsymbol{D}, \quad \text{i.e.} \quad (\ln \boldsymbol{V})\,\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log} \ln \boldsymbol{V} = \boldsymbol{D} - (\ln \boldsymbol{V})^{\cdot}. \tag{34}$$

We refer to the spin $\boldsymbol{\Omega}^{\log}$ as the logarithmic spin, or simply the log-spin. Physically, an observer in a rotating frame defined by the log-spin $\boldsymbol{\Omega}^{\log}$ will observe that the rate of change of the Eulerian logarithmic strain $\ln \boldsymbol{V}$ is exactly identical with the stretching $\boldsymbol{D}$ and that $\ln \boldsymbol{V}$ and the Cauchy stress form a work-conjugate pair of strain and stress. Furthermore, the log-spin $\boldsymbol{\Omega}^{\log}$ is the only spin enjoying these properties. Precisely, the observers in the rotating frames defined by the other spins except the log-spin $\boldsymbol{\Omega}^{\log}$ will observe that the rate of any strain measure cannot equal the stretching and the Cauchy stress cannot be conjugate to any strain measure. In view of the striking properties of the log-spin $\boldsymbol{\Omega}^{\log}$, for any Eulerian symmetric tensor $\boldsymbol{G}$, we define the following corotational rate of $\boldsymbol{G}$:

$$\mathring{G}^{\log} = \dot{\boldsymbol{G}} + \boldsymbol{G}\boldsymbol{\Omega}^{\log} - \boldsymbol{\Omega}^{\log}\boldsymbol{G}. \tag{35}$$

We refer to $\mathring{\boldsymbol{G}}^{\log}$ as the logarithmic rate of $\boldsymbol{G}$,or simply the log-rate of $\boldsymbol{G}$. It will be seen that the log-rate of an Eulerian tensor is an objective rate. Mathematically, the main facts disclosed in Section 2 and 3 can be expressed as

$$\mathring{\boldsymbol{e}}^* = \dot{\boldsymbol{e}} + \boldsymbol{e}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{e} = \boldsymbol{D} \Leftrightarrow \boldsymbol{e} = \ln \boldsymbol{V} \quad \text{and} \quad \boldsymbol{\Omega}^* = \boldsymbol{\Omega}^{\log} \tag{36}$$

$$\rho \dot{w} = \mathrm{Tr}(\boldsymbol{\sigma}\boldsymbol{D}) = \mathrm{Tr}\big(\boldsymbol{\sigma}(\ln \overset{\circ}{\boldsymbol{V}})^{\log}\big). \tag{37}$$

In the following, we derive an explicit basis-free expression for the log-spin $\boldsymbol{\Omega}^{\log}$ in terms of $\boldsymbol{D}$, $\boldsymbol{W}$ and the left Cauchy-Green tensor $\boldsymbol{B}$.

According to Xiao [37], the linear tensor equation (34) has a unique solution for the case when $\boldsymbol{e}$, i.e. $\boldsymbol{V}$ has three distinct eigenvalues. However, (34) has infinitely many solutions for the cases when the eigenvalues of $\ln \boldsymbol{V}$ are doubly, triply, coalescent, respectively. For an observer defined by the log-spin $\boldsymbol{\Omega}^{\log}$, it is reasonable to require that the log-spin $\boldsymbol{\Omega}^{\log}$ is continuous or smooth when the eigenvalues of $\boldsymbol{V}$ become repeated. Under this requirement the linear tensor equation (34) provides a unique smooth solution, which will be given below.

From the proof for the sufficiency of the condition (16) in Section 2, we know that (20) is a solution of the tensor equation (15) if the condition (16) is fulfilled. Since the condition (16) is equivalent to $\boldsymbol{e} = \ln \boldsymbol{V}$, we infer

$$\boldsymbol{\Omega}^{\log} = \sum_{\sigma=1}^{m} \boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\sigma + \sum_{\substack{\sigma,\tau=1 \\ \sigma\neq\tau}}^{m} [(\ln \lambda_\sigma - \ln \lambda_\tau)^{-1} \boldsymbol{P}_\sigma(\boldsymbol{D} - (\ln \boldsymbol{V})^{\boldsymbol{\cdot}})\, \boldsymbol{P}_\tau] \tag{38}$$

is a solution of the linear tensor equation (34). By applying (24) for $\boldsymbol{e} = \ln \boldsymbol{V}$ as well as (18) we may reformulate (38) as

$$\boldsymbol{\Omega}^{\log} = \sum_{\sigma=1}^{m} \boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\sigma + \sum_{\substack{\sigma,\tau=1 \\ \sigma\neq\tau}}^{m} [(\ln \lambda_\sigma - \ln \lambda_\tau)^{-1} \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau - (\lambda_\sigma - \lambda_\tau)^{-1} \boldsymbol{P}_\sigma \dot{\boldsymbol{V}} \boldsymbol{P}_\tau]. \tag{39}$$

On the other hand, by means of (27) and $\sum_{\theta=1}^{m} \boldsymbol{P}_\theta = \boldsymbol{I}$, from the tensor equation (26) we derive sequentially

$$\begin{aligned}
\dot{\boldsymbol{V}} &= \sum_{\sigma,\tau=1}^{m} [(\lambda_\sigma + \lambda_\tau)^{-1} \boldsymbol{P}_\sigma(\boldsymbol{V}\dot{\boldsymbol{V}} + \dot{\boldsymbol{V}}\boldsymbol{V})\, \boldsymbol{P}_\tau] \\
&= \sum_{\sigma,\tau=1}^{m} [(\lambda_\sigma + \lambda_\tau)^{-1} \boldsymbol{P}_\sigma(\boldsymbol{D}\boldsymbol{V}^2 + \boldsymbol{V}^2\boldsymbol{D} + \boldsymbol{W}\boldsymbol{V}^2 - \boldsymbol{V}^2\boldsymbol{W})\, \boldsymbol{P}_\tau] \\
&= \sum_{\sigma,\tau=1}^{m} \left[ \frac{\lambda_\sigma^2 + \lambda_\tau^2}{\lambda_\sigma + \lambda_\tau} \boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau - (\lambda_\sigma - \lambda_\tau)\, \boldsymbol{P}_\sigma \boldsymbol{W} \boldsymbol{P}_\tau \right].
\end{aligned} \tag{40}$$

Then, substituting (40) into (39) and using (18) and (22), we obtain

$$\boldsymbol{\Omega}^{\log} = \boldsymbol{W} + \sum_{\substack{\sigma,\tau=1 \\ \sigma\neq\tau}}^{m} \left[\left(\frac{1+(b_\sigma/b_\tau)}{1-(b_\sigma/b_\tau)} + \frac{2}{\ln(b_\sigma/b_\tau)}\right)\boldsymbol{P}_\sigma \boldsymbol{D} \boldsymbol{P}_\tau\right], \tag{41}$$

where $b_\sigma = \lambda_\sigma{}^2$ are the eigenvalues of the left Cauchy-Green tensor $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$. By noticing the fact that the eigenprojection $\boldsymbol{P}_\sigma$ subordinate to eigenvalue $\lambda_\sigma > 0$ of $\boldsymbol{V}$ is also eigenprojection subordinate to the eigenvalue $b_\sigma$ of $\boldsymbol{B}$, we know that the following Sylvester formula holds:

$$\boldsymbol{P}_\sigma = \prod_{\substack{\theta=1 \\ \theta\neq\sigma}}^{m} \frac{\boldsymbol{B} - b_\theta \boldsymbol{I}}{b_\sigma - b_\theta} \quad \text{for} \quad m \geqq 2; \quad \boldsymbol{P}_\sigma = \boldsymbol{I} \quad \text{for} \quad m = 1, \tag{42}$$

and hence for $b_1 \neq b_2 \neq b_3 \neq b_1$,

$$\begin{aligned}
\boldsymbol{P}_1 &= \frac{b_3 - b_2}{\Delta}(\boldsymbol{B}^2 - (b_3 + b_2)\boldsymbol{B} + b_3 b_2 \boldsymbol{I}), \\
\boldsymbol{P}_2 &= \frac{b_1 - b_3}{\Delta}(\boldsymbol{B}^2 - (b_1 + b_3)\boldsymbol{B} + b_1 b_3 \boldsymbol{I}), \\
\boldsymbol{P}_3 &= \frac{b_2 - b_1}{\Delta}(\boldsymbol{B}^2 - (b_2 + b_1)\boldsymbol{B} + b_2 b_1 \boldsymbol{I}), \\
\Delta &= (b_1 - b_2)(b_2 - b_3)(b_3 - b_1);
\end{aligned} \tag{43}$$

and for $b_1 \neq b_2 = b_3$,

$$\begin{aligned}
\boldsymbol{P}_1 &= \frac{1}{b_1 - b_2}(\boldsymbol{B} - b_2 \boldsymbol{I}), \\
\boldsymbol{P}_2 &= \frac{1}{b_2 - b_1}(\boldsymbol{B} - b_1 \boldsymbol{I}).
\end{aligned} \tag{44}$$

Substituting the above formulae into (41) we finally obtain an explicit basis-free expression of the log-spin $\boldsymbol{\Omega}^{\log}$ in terms of $\boldsymbol{D}$, $\boldsymbol{W}$ and $\boldsymbol{B}$ as follows:

$$\boldsymbol{\Omega}^{\log} = \boldsymbol{W} + \boldsymbol{N}^{\log}, \tag{45}$$

$$\boldsymbol{N}^{\log} = \begin{cases} \boldsymbol{0}, & b_1 = b_2 = b_3, \\ \nu[\boldsymbol{B}\boldsymbol{D}], & b_1 \neq b_2 = b_3, \\ \nu_1[\boldsymbol{B}\boldsymbol{D}] + \nu_2[\boldsymbol{B}^2\boldsymbol{D}] + \nu_3[\boldsymbol{B}^2\boldsymbol{D}\boldsymbol{B}], & b_1 \neq b_2 \neq b_3 \neq b_1, \end{cases} \tag{46}$$

where the coefficients are isotropic invariants of $\boldsymbol{B}$, given by

$$\nu = \frac{1}{b_1 - b_2}\left(\frac{1 + (b_1/b_2)}{1 - (b_1/b_2)} + \frac{2}{\ln(b_1/b_2)}\right); \tag{47}$$

$$\begin{aligned}
\nu_k &= \frac{1}{\Delta}\sum_{i=1}^{3}(-b_i)^{3-k}\left(\frac{1+\varepsilon_i}{1-\varepsilon_i} + \frac{2}{\ln \varepsilon_i}\right) \quad \varkappa = 1, 2, 3, \\
\varepsilon_1 &= b_2/b_3, \quad \varepsilon_2 = b_3/b_1, \quad \varepsilon_3 = b_1/b_2,
\end{aligned} \tag{48}$$

and the following notation is used:

$$[\boldsymbol{B}^r\boldsymbol{D}\boldsymbol{B}^s] = \boldsymbol{B}^r\boldsymbol{D}\boldsymbol{B}^s - \boldsymbol{B}^s\boldsymbol{D}\boldsymbol{B}^r. \tag{49}$$

It should be pointed out that the antisymmetric tensor $\boldsymbol{N}^{\log}$ is a particular form of the general skewsymmetric tensor derived by Dafalias [4] in the investigation of the plastic spin by applying the representation theorem for skew-symmetric tensor-valued isotropic functions of two symmetric tensors.

The three eigenvalues of $\boldsymbol{B}$ (possibly repeated) can be determined by the principal invariants of $\boldsymbol{B}$. In reality, $b_1$, $b_2$ and $b_3$ are the three roots of the characteristic equations of $\boldsymbol{B}$,

$$\varkappa^3 - I_{\boldsymbol{B}}\varkappa^2 + II_{\boldsymbol{B}}\varkappa - III_{\boldsymbol{B}} = 0, \tag{50}$$

and hence

$$\begin{aligned} b_i &= \frac{1}{3}\left[ I_{\boldsymbol{B}} + 2\sqrt{I_{\boldsymbol{B}}^2 - 3II_{\boldsymbol{B}}}\cos\left(\frac{1}{3}(\varphi - 2\pi i)\right)\right], \quad i = 1, 2, 3, \\ \varphi &= \arccos\left[\frac{2I_{\boldsymbol{B}}^3 - 9I_{\boldsymbol{B}}II_{\boldsymbol{B}} + 27III_{\boldsymbol{B}}}{2(I_{\boldsymbol{B}}^2 - 3II_{\boldsymbol{B}})^{3/2}}\right], \end{aligned} \tag{51}$$

where the three principal invariants of $\boldsymbol{B}$ are given by

$$\begin{aligned} I_{\boldsymbol{B}} &= \mathrm{Tr}\boldsymbol{B}, \\ II_{\boldsymbol{B}} &= \frac{1}{2}\left((\mathrm{Tr}\boldsymbol{B})^2 - \mathrm{Tr}\boldsymbol{B}^2\right), \\ III_{\boldsymbol{B}} &= \det\boldsymbol{B} = \frac{1}{6}(\mathrm{Tr}\boldsymbol{B})^3 - \frac{1}{2}(\mathrm{Tr}\boldsymbol{B})(\mathrm{Tr}\boldsymbol{B}^2) + \frac{1}{3}\mathrm{Tr}\boldsymbol{B}^3. \end{aligned} \tag{52}$$

From the above, we see that the explicit basis-free expressions (45)–(52) enable us to determine the log-spin $\boldsymbol{\Omega}^{\log}$ directly using the deformation gradient $\boldsymbol{F}$ given under any coordinate system.

The continuity of $\boldsymbol{\Omega}^{\log}$, i.e. $\boldsymbol{N}^{\log}$ can be inferred from (41) for the case: $(b_1, b_2, b_3) \to (b_1, b_2, b_2)$ and $b_1 \neq b_2$ by using the facts

$$\begin{aligned} &\lim_{b_3 \to b_2}\left(\frac{1+\varepsilon_1}{1-\varepsilon_1} + \frac{2}{\ln\varepsilon_1}\right) = \lim_{\varepsilon_1 \to 1}\left(\frac{1+\varepsilon_1}{1-\varepsilon_1} + \frac{2}{\ln\varepsilon_1}\right) = 0, \\ &\lim_{b_3 \to b_2}\left(\frac{1+\varepsilon_2}{1-\varepsilon_2} + \frac{2}{\ln\varepsilon_2}\right) = -\left(\frac{1+\varepsilon_3}{1-\varepsilon_3} + \frac{2}{\ln\varepsilon_3}\right), \end{aligned}$$

and for the case $(b_1, b_2, b_2) \to (b_1, b_1, b_1)$ by using the fact

$$\lim_{b_2 \to b_1}\left(\frac{1+\varepsilon_3}{1-\varepsilon_3} + \frac{2}{\ln\varepsilon_3}\right) = \lim_{\varepsilon_3 \to 1}\left(\frac{1+\varepsilon_3}{1-\varepsilon_3} + \frac{2}{\ln\varepsilon_3}\right) = 0.$$

Finally, by means of (12) and (45) and (35) we establish the relationship between the log-rate and the Jaumann rate as follows:

$$\overset{\circ}{\boldsymbol{G}}{}^{\log} = \overset{\circ}{\boldsymbol{G}}{}^{J} + \boldsymbol{G}\boldsymbol{N}^{\log} - \boldsymbol{N}^{\log}\boldsymbol{G}. \tag{53}$$

## 5 On rate-form constitutive models

Since the inappropriateness of the Jaumann stress rate to rate-form constitutive models was recognized (cf. Lehmann [23], Dienes [5], Nagtegaal and de Jong [28], Lee et al. [22],Dafalias [3], etc.), considerable efforts have been made to deal with the problem of choosing an appropriate objective stress rate for rate-form constitutive models (e.g. see Dienes [5], [6], Neale [29], Dafalias [3], [4], Dubey [7], Aifantis [1], Stickforth and Wegener [33], Xia and Ellyin [40]; see also the respective lists of references in [5], [40]). Several stress rates, such as the corotational rate defined by the spin $\boldsymbol{\Omega}^* = \dot{\boldsymbol{R}}\boldsymbol{R}^T$ and the spin of the Eulerian triad $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$, etc. (cf. Dienes [5], Dubey [7] and Xia and Ellyin [40]), have been suggested and shown to be plausible alternatives by means of the reasonable simple shear solutions of the constitutive models based on these rates (see Section 6 for details).

In this Section, we show that in all possible rate-form constitutive models relating the same kind of objective rates of an Eulerian stress measure and an Eulerian strain measure, the logarithmic stress rate is the only choice if the stretching, as is commonly assumed, is used to measure the rate of change of deformation.

Let a pair of Eulerian stress and strain measures $(\boldsymbol{\pi}, \boldsymbol{e})$ be used to measure the states of stress and deformation. Moreover, for inelastic deformations, a set of internal variables, $Z$, may be introduced to characterize the change of interior states of the material. It has been known that the following rate-form constitutive model relative to a fixed background frame:

$$\dot{\boldsymbol{\pi}} = \Phi(\dot{\boldsymbol{e}}, \boldsymbol{\pi}, Z)$$

cannot satisfy the principle of objectivity, since the material time rates $\dot{\boldsymbol{\pi}}$ and $\dot{\boldsymbol{e}}$ are not objective, which do not vanish under rigid spin. In view of this, the following rate-form constitutive model relative to a rotating frame is considered:

$$\mathring{\boldsymbol{\pi}}^* = \varphi(\mathring{\boldsymbol{e}}^*, \boldsymbol{\pi}, Z), \tag{54}$$

where $\mathring{\boldsymbol{\pi}}^*$ and $\mathring{\boldsymbol{e}}^*$ are the corotational rates defined by the same spin $\boldsymbol{\Omega}^*$ (cf. (9)) and the latter should be such that the tensor-rate defined by it is objective. The following particular forms of the constitutive model (54) are also considered:

$$\mathring{\boldsymbol{\pi}}^* = \varphi(\mathring{\boldsymbol{e}}^*, \boldsymbol{\pi}), \tag{55}$$

$$\mathring{\boldsymbol{\pi}}^* = \boldsymbol{L} : \mathring{\boldsymbol{e}}^*, \tag{56}$$

$$\mathring{\boldsymbol{\pi}}^* = \boldsymbol{H}(\pi) : \mathring{\boldsymbol{e}}^*. \tag{57}$$

The moduli $\boldsymbol{L}$ in the second model may be path-dependent, and the moduli $\boldsymbol{H}$ in the latter merely depend on the stress measure.

As mentioned before, there are infinitely many kinds of objective tensor-rates. In the constitutive model (54), the stress and strain measures $\boldsymbol{\pi}$ and $\boldsymbol{e}$ as well as their objective rates can be chosen arbitrarily. However, if the stretching $\boldsymbol{D}$, as is commonly assumed, is used to measure the rate of change of deformation, then the log-rate is the only choice and the strain measure $\boldsymbol{e}$ must be ln $\boldsymbol{V}$. In reality, let the rate of the strain measure $\boldsymbol{e}$ in (54) be replaced by the stretching $\boldsymbol{D}$, i.e.

$$\mathring{\boldsymbol{e}}^* = \boldsymbol{D}. \tag{58}$$

Then from Sections 2 and 4 we conclude that the rates $\mathring{e}^*$ and $\mathring{\pi}^*$ must be the log-rates and the strain measure $\boldsymbol{e}$ must be $\ln \boldsymbol{V}$. Thus the constitutive model (54) must be of the form

$$\mathring{\pi}^{\log} = \varphi\big((\ln^\circ \boldsymbol{V})^{\log}, \boldsymbol{\pi}, Z\big) = \varphi(\boldsymbol{D}, \boldsymbol{\pi}, Z). \tag{59}$$

The requirement (58) is assumed mainly due to the two facts: First, the stretching $\boldsymbol{D}$ is a direct natural measure of the rate of change of deformation, and second, the stretching $\boldsymbol{D}$ is the simplest in all objective corotational rates $\mathring{e}^*$ for all Eulerian strain measures $\boldsymbol{e}$ (cf. Xiao, Bruhns and Meyers [39]).

Furthermore, since the Cauchy stress $\boldsymbol{\sigma}$ is the true stress, it is natural and beneficial to assume it as the stress measure in (59), i.e.

$$\mathring{\boldsymbol{\sigma}}^{\log} = \varphi\big((\ln^\circ \boldsymbol{V})^{\log}, \boldsymbol{\sigma}, Z\big) = \varphi(\boldsymbol{D}, \boldsymbol{\sigma}, Z), \tag{60}$$

and perhaps more essentially, as such the stress-strain pair ($\boldsymbol{\sigma}$, $\ln \boldsymbol{V}$) entering the constitutive model (60) is a work-conjugate one (cf. Section 3). In particular, we propose the following constitutive models:

$$\mathring{\boldsymbol{\sigma}}^{\log} = \varphi\big((\ln^\circ \boldsymbol{V})^{\log},\ \boldsymbol{\sigma}\big) = \varphi(\boldsymbol{D}, \boldsymbol{\sigma}), \tag{61}$$

$$\mathring{\boldsymbol{\sigma}}^{\log} = \boldsymbol{H} : (\ln^\circ \boldsymbol{V})^{\log} = \boldsymbol{H} : \boldsymbol{D}. \tag{62}$$

For the former, $\varphi$ is regarded as nonlinear in $\boldsymbol{D}$ and hence (61) can serve as a rate-dependent constitutive model. The latter is linear in $\boldsymbol{D}$ and hence can serve as a rate-independent constitutive model, where the moduli $\boldsymbol{H}$ may be path-dependent. In particular, if $\boldsymbol{H}$ merely depends on the Cauchy stress $\boldsymbol{\sigma}$, i.e. $\boldsymbol{H} = \boldsymbol{H}(\boldsymbol{\sigma})$, (62) becomes

$$\mathring{\boldsymbol{\sigma}}^{\log} = \boldsymbol{H}(\boldsymbol{\sigma}) : (\ln^\circ \boldsymbol{V})^{\log} = \boldsymbol{H}(\boldsymbol{\sigma}) : \boldsymbol{D}. \tag{63}$$

The latter is the very hypoelastic constitutive model (cf. Truesdell and Noll [36] and Eringen [8]).

In the Introduction of this paper, the favoured position of the logarithmic strain is indicated. Here, it can be seen that the Eulerian logarithmic strain $\ln \boldsymbol{V}$ indeed establishes its favoured, even unequalled position in all possible strain tensor measures by virtue of its own inherent and unique properties, although its final appearance in the constitutive models (60)–(63) is not so direct.

In the past, other stress rates were used in (60), i.e. the following rate-form constitutive model was considered:

$$\mathring{\boldsymbol{\sigma}}^* = \varphi(\boldsymbol{D}, \boldsymbol{\sigma}, Z), \quad \mathring{\boldsymbol{\sigma}}^* \neq \mathring{\boldsymbol{\sigma}}^{\log}. \tag{64}$$

By means of (34.1) the above model can be converted into

$$\mathring{\boldsymbol{\sigma}}^* = \varphi\big((\ln^\circ \boldsymbol{V})^{\log}, \boldsymbol{\sigma}, Z\big) = \varphi(\boldsymbol{D}, \boldsymbol{\sigma}, Z), \quad \mathring{\boldsymbol{\sigma}}^* \neq \mathring{\boldsymbol{\sigma}}^{\log}.$$

The latter indicates that the inconsistency of the stress rate and the strain rate is incurred. From (34) one can see that such inconsistency cannot be removed, no matter what strain measure is chosen, if any other tensor-rate except the log-rate is assumed in the constitutive model (54).

According to Aifantis [1] and Stickforth and Wegener [33], the log-rate should be modified for crystalline media. As such, in the expressions for the log-spin $\boldsymbol{\Omega}^{\log}$ (cf. (45)–(49)), the vorticity tensor $\boldsymbol{W}$, the stretching $\boldsymbol{D}$ and the left Cauchy-Green tensor $\boldsymbol{B}$ should be replaced by their respective elastic parts.

## 6 Examples

Consider the grade-zero hypoelastic model

$$\mathring{\boldsymbol{\sigma}}^{\log} = \lambda(\mathrm{Tr}\boldsymbol{D})\,\boldsymbol{I} + 2\mu\boldsymbol{D}, \tag{65}$$

which is the simplest rate-form constitutive model based on the log-rate (refer to [41] for a study of the general hypoelastic model (63)). Here, we do not confine ourselves to investigate the simple shear response only, as usually done. We investigate all possible finite deformation responses.

Let $\boldsymbol{\Omega}^*$ be a smooth rotation tensor defining the log-spin $\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^{\log}$ (cf. (10)). Then we recast (65) in

$$\boldsymbol{Q}^*\mathring{\boldsymbol{\sigma}}^{\log}\boldsymbol{Q}^{*T} = \lambda(\mathrm{Tr}\boldsymbol{D})\,\boldsymbol{I} + 2\mu\boldsymbol{Q}^*\boldsymbol{D}\boldsymbol{Q}^{*T}. \tag{66}$$

By applying (34.1) and (11) as well as the identity

$$\mathrm{Tr}(\mathring{\boldsymbol{G}}^*) = \mathrm{Tr}\dot{\boldsymbol{G}} = (\mathrm{Tr}\boldsymbol{G})^{\boldsymbol{\cdot}} \tag{67}$$

we further rewrite (66) into

$$[\boldsymbol{Q}^*\boldsymbol{\sigma}\boldsymbol{Q}^{*T}]^{\boldsymbol{\cdot}} = [\lambda(\mathrm{Tr}(\ln \boldsymbol{V}))\,\boldsymbol{I} + 2\mu\boldsymbol{Q}^*(\ln \boldsymbol{V})\,\boldsymbol{Q}^{*T}]^{\boldsymbol{\cdot}}. \tag{68}$$

Let the initial state of the body under consideration be a natural one, i.e.

$$\boldsymbol{\sigma}|_{t=0} = \boldsymbol{0}, \quad \boldsymbol{V}|_{t=0} = \boldsymbol{I}. \tag{69}$$

Then from (68) we derive

$$\boldsymbol{Q}^*\boldsymbol{\sigma}\boldsymbol{Q}^{*T} = \lambda(\mathrm{Tr}(\ln \boldsymbol{V}))\,\boldsymbol{I} + 2\mu\boldsymbol{Q}^*(\ln \boldsymbol{V})\,\boldsymbol{Q}^{*T}, \tag{70}$$

i.e.

$$\boldsymbol{\sigma} = \lambda(\mathrm{Tr}(\ln \boldsymbol{V}))\,\boldsymbol{I} + 2\mu\ln \boldsymbol{V}. \tag{71}$$

The latter is just a finite deformation isotropic elastic model. We mention that it is a natural generalization of the Hooke's law, where the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ is replaced by the logarithmic strain $\ln \boldsymbol{V}$.

The simple shear responses of two hypoelastic constitutive models defined by

$$\mathring{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{\sigma} = \lambda(\mathrm{Tr}\boldsymbol{D})\,\boldsymbol{I} + 2\mu\boldsymbol{D} \tag{72}$$

have been considered by Dienes [5], where the stress rate $\mathring{\boldsymbol{\sigma}}^*$ is assumed as the Jaumann rate $\mathring{\boldsymbol{\sigma}}^J$ and the rate defined by $\boldsymbol{\Omega}^* = \dot{\boldsymbol{R}}^T\boldsymbol{R}$, respectively. Dienes' results reveal that the simple shear solution of the model (72) based on the Jaumann rate predicts the unstable shear oscillatory phenomenon, while the simple shear response of the model (72) based on the second rate is stable and therefore reasonable. However, neither of the two solutions agrees with the simple shear solution of a finite deformation isotropic elastic model. Basing on a methodology of choosing tensor-rates, Dubey [7] reconsiders the simple shear problems of both a finite deformation elastic model and a hypoelastic model based on the rate defined by the spin of the Eulerian triad $\{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3\}$. The two solutions reported in Dubey [7] coincide.

We have shown that not only the simple shear response but all possible responses of the hypoelastic model (65) based on the log-rate agree with those of the finite deformation isotropic elastic model defined by (71). And there is more. The simplest hypoelastic model (65) based on the log-rate can predict the phenomenon of the known hypoelastic yield at the simple shear deformation (cf. Truesdell [35] and Truesdell and Noll [36]), while the three grade-zero hypoelastic models based on the rate defined by the spin $\boldsymbol{\Omega}^* = \dot{\boldsymbol{R}}^T\boldsymbol{R}$ and the convected-rate as well as a rate proposed recently predict that the shear stress increases to infinity with the increasing shear strain (cf. Truesdell [35], Dienes [5], Xia and Ellyin [40], etc.; the former treats the more general case, i.e. the grade-one hypoelastic model, in which the grade-zero model is referred to as the neutral case).

To explain the latter fact regarding the simple shear response of the model (65) based on the log-rate, in the following we provide the complete analysis. Consider the simple shear deformation

$$\boldsymbol{x} = (X_1 + \gamma X_2)\,\boldsymbol{e}_1{}^0 + X_2\boldsymbol{e}_2{}^0 + X_3\boldsymbol{e}_3{}^0, \tag{73}$$

where $(\boldsymbol{e}_1{}^0, \boldsymbol{e}_2{}^0, \boldsymbol{e}_3{}^0)$ is a fixed orthonormal basis; $\boldsymbol{X} = \sum_{i=1}^{3} X_i\boldsymbol{e}_i{}^0$ and $\boldsymbol{x} = \sum_{i=1}^{3} x_i\boldsymbol{e}_i{}^0$ are the initial position vector and the current position vector of a material particle, respectively.

The left Cauchy-Green tensor $\boldsymbol{B}$ is of the form (cf. Dienes [5], Dafalias [3]):

$$\boldsymbol{B} = (1+\gamma^2)\,\boldsymbol{e}_1{}^0 \otimes \boldsymbol{e}_1{}^0 + \gamma(\boldsymbol{e}_1{}^0 \otimes \boldsymbol{e}_2{}^0 + \boldsymbol{e}_2{}^0 \otimes \boldsymbol{e}_1{}^0) + \boldsymbol{e}_2{}^0 \otimes \boldsymbol{e}_2{}^0 + \boldsymbol{e}_3{}^0 \otimes \boldsymbol{e}_3{}^0. \tag{74}$$

The eigenvalues of $\boldsymbol{B}$ are given by (cf. Eringen [8])

$$\begin{aligned} b_1 &= \lambda_1{}^2 = \left(2+\gamma^2+\gamma\sqrt{4+\gamma^2}\right)/2, \\ b_2 &= \lambda_2{}^2 = \left(2+\gamma^2-\gamma\sqrt{4+\gamma^2}\right)/2 = b_1{}^{-1}, \\ b_3 &= \lambda_3{}^2 = 1, \end{aligned} \tag{75}$$

and the eigenprojections subordinate to $b_1$ and $b_2$ are given by

$$\begin{aligned} \boldsymbol{P}_1 &= \frac{1}{b_1-b_2}\,(\boldsymbol{B} - b_2\bar{\boldsymbol{I}} - \boldsymbol{e}_3{}^0 \otimes \boldsymbol{e}_3{}^0), \\ \boldsymbol{P}_2 &= \frac{1}{b_2-b_1}\,(\boldsymbol{B} - b_1\bar{\boldsymbol{I}} - \boldsymbol{e}_3{}^0 \otimes \boldsymbol{e}_3{}^0), \\ \bar{\boldsymbol{I}} &= \boldsymbol{I} - \boldsymbol{e}_3{}^0 \otimes \boldsymbol{e}_3{}^0. \end{aligned} \tag{76}$$

Hence the logarithmic strain $\ln \boldsymbol{V}$ is of the form

$$\ln \boldsymbol{V} = \frac{1}{2}\,[(\ln b_1)\,\boldsymbol{P}_1 + (\ln b_2)\,\boldsymbol{P}_2] = \frac{\ln b_1 - \ln b_2}{2(b_1-b_2)}\,(\boldsymbol{B} - \boldsymbol{e}_3{}^0 \otimes \boldsymbol{e}_3{}^0) + \frac{b_1 \ln b_2 - b_2 \ln b_1}{2(b_1-b_2)}\,\bar{\boldsymbol{I}}. \tag{77}$$

From the latter and (71) we obtain the expression of the shear stress $\sigma_{12}$ in terms of the shear strain $\gamma$ as follows:

$$\sigma_{12} = \sigma_{12}(\gamma) = \frac{\ln b_1 - \ln b_2}{b_1 - b_2}\,\mu\gamma = \frac{2\mu}{\sqrt{4+\gamma^2}}\ln\left(1 + \frac{\gamma^2}{2} + \gamma\sqrt{1+\frac{\gamma^2}{4}}\right). \tag{78}$$

Hence,

$$\sigma'_{12}(\gamma) = \frac{d\sigma_{12}}{d\gamma} = 2\mu \frac{\gamma}{(4+\gamma^2)^{3/2}} f(\gamma),$$
$$f(\gamma) = \frac{2\sqrt{4+\gamma^2}}{\gamma} - \ln\left(1 + \frac{\gamma^2}{2} + \gamma\sqrt{1+\frac{\gamma^2}{4}}\right). \tag{79}$$

The root of $\sigma'_{12}(\gamma) = 0$ is determined by $f(\gamma) = 0$. Since

$$f'(\gamma) = -\frac{2\sqrt{4+\gamma^2}}{\gamma^2} < 0 \quad \text{for all} \quad \gamma > 0,$$
$$\lim_{\gamma\to 0^+} f(\gamma) = +\infty, \qquad \lim_{\gamma\to+\infty} f(\gamma) = -\infty, \tag{80}$$

we know that $f(\gamma) = 0$, i.e. $\sigma'_{12}(\gamma) = 0$ has one and only one root, denoted by $\gamma_m$ and determined by

$$2\sqrt{4+\gamma_m^2} = \gamma_m \ln\left(1 + \frac{\gamma_m^2}{2} + \gamma_m\sqrt{1+\frac{\gamma_m^2}{4}}\right). \tag{81}$$

From the above we infer that $\sigma'_{12}(\gamma) > 0$ for $\gamma < \gamma_m$ and $\sigma'_{12}(\gamma) < 0$ for $\gamma > \gamma_m$. Thus the grade-zero hypoelastic model (65) based on the log-rate predicts that there is an upper bound $\tau_m = \sigma_{12}(\gamma_m)$ for the shear stress $\sigma_{12}(\gamma)$ and therefore that the material reaches its hypoelastic yield at the upper bound $\tau_m$, since the shear stress $\sigma_{12}(\gamma)$ invariably decreases with the increasing shear strain $\gamma$ after the upper bound $\tau_m$ is attained.

From Eq. (81) we obtain

$$\gamma_m = 3.0177171 \quad \text{and} \quad \tau_m = \frac{4\mu}{\gamma_m} = 1.32548684\mu. \tag{82}$$

Results are plotted in Fig. 1.

The above example and the facts mentioned before show the rationality and efficacy of the suggested constitutive models based on the logarithmic stress rate.

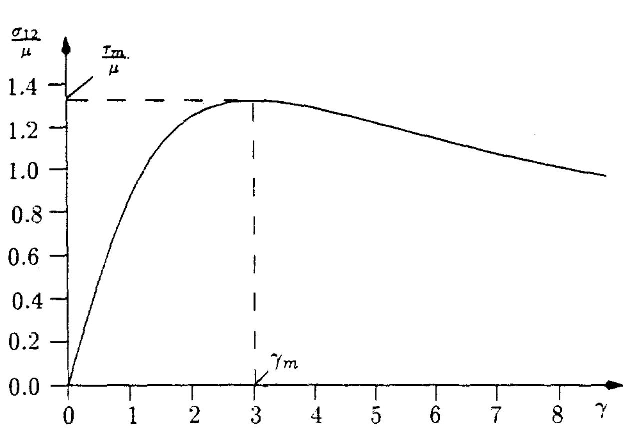


**Fig. 1.** Stress $\delta_2$ versus strain $\gamma$ at simple shear

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