ASSIGNMENT 3

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Download all python codes from

```
https://github.com/manik2255/AI1103-PROBABILITY-AND-RANDOM-VARIABLES/blob/main/ASSIGNMENT_3/assign_3.py
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and latex-tikz codes from

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https://github.com/manik2255/AI1103-PROBABILITY-AND-RANDOM-VARIABLES/blob/main/ASSIGNMENT_3/ASSIGNMENT_3.tex
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1 GATE 2017 MA PROBLEM.47

Let X and Y be independent and identically distributed random variables with probability mass function $p(n) = 2^{-n}$,$n = 1, 2, \dots$
Then $pr(X \geq 2Y)$ equals

2 Solution

given,

$$\Pr(X = x) = 2^{-x}, x = 1, 2, \dots \tag{2.0.1}$$
$$\Pr(Y = y) = 2^{-y}, y = 1, 2, \dots \tag{2.0.2}$$

We need to find $pr(X \geq 2Y)$,which is also can be written as

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} \Pr(X \geq 2y|Y = y) \tag{2.0.3}$$

as,X and Y are independent random variables

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} \Pr(X \geq 2y) \Pr(Y = y) \tag{2.0.4}$$
$$= \sum_{y=1}^{\infty} (1 - \Pr(X < 2y)) \Pr(Y = y) \tag{2.0.5}$$

using (2.0.1) and (2.0.2) in (2.0.5),

$$= \sum_{y=1}^{\infty} (1 - \sum_{i=1}^{2y-1} 2^{-i})(2^{-y}) \tag{2.0.6}$$
$$= \sum_{y=1}^{\infty} (1 - (1 - 2^{-(2y-1)}))(2^{-y}) \tag{2.0.7}$$
$$= \sum_{y=1}^{\infty} 2^{-(3y-1)} \tag{2.0.8}$$
$$= \frac{2}{7} \tag{2.0.9}$$

Finding the probability using cumulative distribution, Let $F_X(x)$ be the cumulative distribution function of random variable X.

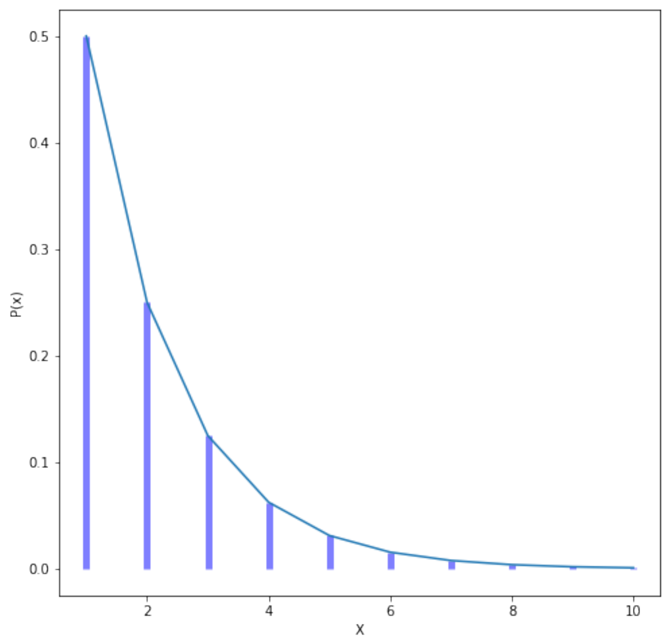

Fig. 1: Pmf of random variable X

$$F_X(x) = \Pr(X < x) \tag{2.0.10}$$

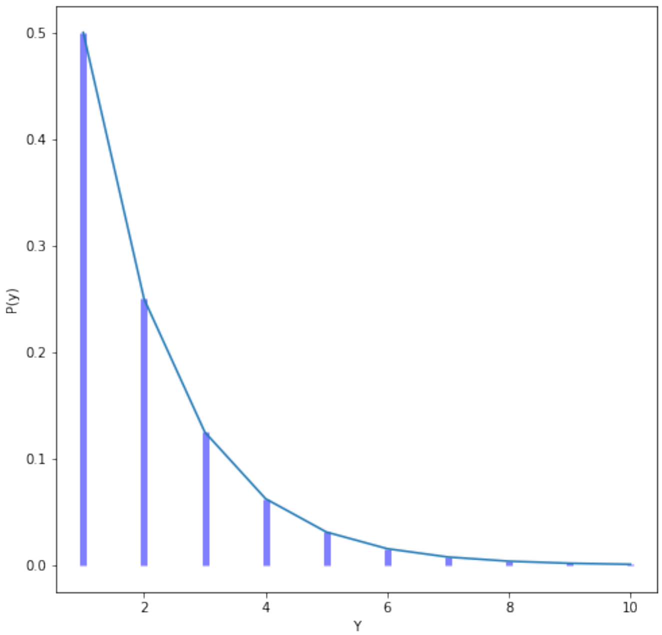

Fig. 2: Pmf of random variable Y

As it is a discrete probability distribution, $F_X(x)$ can be written as,

$$F_X(x) = \sum_{i=1}^{x-1} \Pr(X = x) \tag{2.0.11}$$

using (2.0.1) in (2)

$$F_X(x) = \sum_{i=1}^{x-1} 2^{-i} \tag{2.0.12}$$
$$= 1 - 2^{-(x-1)} \tag{2.0.13}$$

Cumulative distribution function (CDF) of random variable X is given by,

$$F_X(x) = 1 - 2^{-(x-1)}, x = 1, 2, \dots \tag{2.0.14}$$

We need to find $\Pr(X \geq 2Y)$,which is also can be written as

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} \Pr(X \geq 2y | Y = y) \tag{2.0.15}$$

As,X and Y are independent random variables

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} \Pr(X \geq 2y) \Pr(Y = y) \tag{2.0.16}$$
$$= \sum_{y=1}^{\infty} (1 - \Pr(X < 2y)) \Pr(Y = y) \tag{2.0.17}$$

using (2.0.2) and (2.0.14) in (2.0.17),

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} (1 - (1 - 2^{-(2y-1)}))(2^{-y}) \tag{2.0.18}$$
$$= \sum_{y=1}^{\infty} 2^{-(3y-1)} \tag{2.0.19}$$
$$= \frac{2}{7} \tag{2.0.20}$$

Alternative method, Let $F_Y(y)$ be the cumulative distribution function of random variable Y.

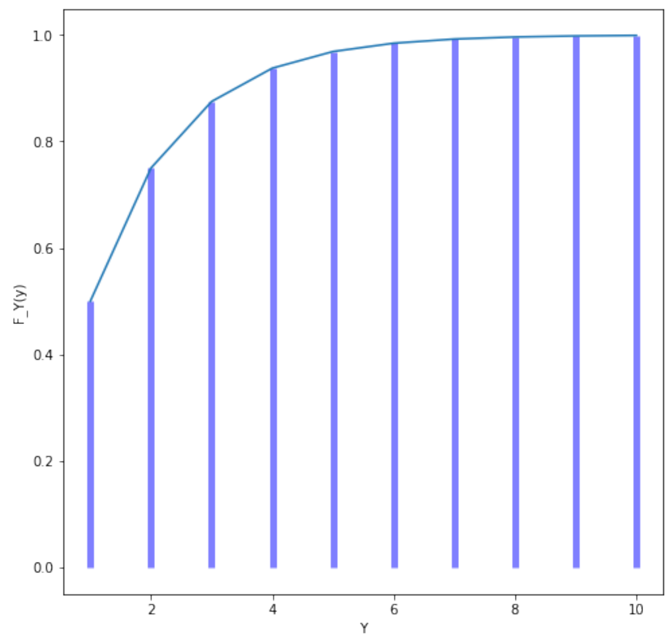

Fig. 3: cdf of random variable Y

$$F_Y(y) = \Pr(Y \leq y) \tag{2.0.21}$$

As it is a discrete probability distribution, $F_Y(y)$ can be written as,

$$F_Y(y) = \sum_{i=1}^{y} \Pr(Y = y) \tag{2.0.22}$$

using (2.0.1) in (2.0.22)

$$F_Y(y) = \sum_{i=1}^{y} 2^{-i} \tag{2.0.23}$$
$$= 1 - 2^{-(y)} \tag{2.0.24}$$

Cumulative distribution function (CDF) of random variable Y is given by,

$$F_Y(y) = 1 - 2^{-(y)}, y = 1, 2, \dots \tag{2.0.25}$$

We need to find $\Pr(X \geq 2Y)$,which is also can be written as

$$\Pr(X \geq 2Y) = \Pr\left(Y \leq \frac{X}{2}\right) \tag{2.0.26}$$

$$\Pr(X \geq 2Y) = \sum_{y=1}^{\infty} \Pr\left(Y \leq \frac{t}{2}\right) \Pr(X = t) \tag{2.0.27}$$

As t can be either an even number or an odd number,equation (2.0.27) can be written as,

$$\Pr(X \geq 2Y) = \sum_{n_1=1}^{\infty} \Pr\left(Y \leq \frac{2n_1}{2}\right) \Pr(X = 2n_1) + \sum_{n_2=1}^{\infty} \Pr\left(Y \leq \frac{2n_2 - 1}{2}\right) \Pr(X = 2n_2 - 1) \tag{2.0.28}$$

$$= \sum_{n_1=1}^{\infty} \Pr(Y \leq n_1) \Pr(X = 2n_1) + \sum_{n_2=1}^{\infty} \Pr\left(Y \leq n_2 - \frac{1}{2}\right) \Pr(X = 2n_2 - 1) \tag{2.0.29}$$

as,

$$\Pr\left(Y \leq 2n_2 - \frac{1}{2}\right) = \Pr(Y \leq n_2 - 1) + \Pr\left(n_2 - 1 < Y \leq n_2 - \frac{1}{2}\right) \tag{2.0.30}$$

as there is no integer in (n_2 -1 ,$n_2 - \frac{1}{2}$],

$$\Pr\left(n_2 - 1 < Y \leq n_2 - \frac{1}{2}\right) = 0 \tag{2.0.31}$$

using (2.0.31) in (2.0.30),

$$\Pr\left(Y \leq n_2 - \frac{1}{2}\right) = \Pr(Y \leq n_2 - 1) \tag{2.0.32}$$

using (2.0.32) in (2.0.29),

$$\Pr(X \geq 2Y) = \sum_{n_1=1}^{\infty} \Pr(Y \leq n_1) \Pr(X = 2n_1) + \sum_{n_2=1}^{\infty} \Pr(Y \leq n_2 - 1) \Pr(X = 2n_2 - 1) \tag{2.0.33}$$

using (2.0.25) and (2.0.1) in (2.0.33),

$$\Pr(X \geq 2Y) = \sum_{n_1=1}^{\infty} (1 - 2^{-n_1})(2^{-2n_1}) + \sum_{n_2=1}^{\infty} (1 - 2^{-(n_2-1)})(2^{-(2n_2-1)}) \tag{2.0.34}$$

$$= \sum_{n_1=1}^{\infty} (2^{-2n_1} - 2^{-3n_1}) + \sum_{n_2=1}^{\infty} (2^{-(2n_2-1)} - 2^{-(3n_2-2)}) \tag{2.0.35}$$

$$= \frac{1}{3} - \frac{1}{7} + \frac{2}{3} - \frac{4}{7} \tag{2.0.36}$$

$$= \frac{2}{7} \tag{2.0.37}$$