ASSIGNMENT 4

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Download all python codes from

https://github.com/manik2255/AI1103−PROBABILITY−AND−RANDOM−VARIABLES/blob/main/ASSIGNMENT_4/assignment_4.py

and latex-tikz codes from

https://github.com/manik2255/AI1103−PROBABILITY−AND−RANDOM−VARIABLES/blob/main/ASSIGNMENT_4/ASSIGNMENT_4.tex

1 GATE 2017 MA PROBLEM.49

Let X and Y be independent and identically distributed random variables with probability density function $f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases}$ Then $\Pr(max(X,Y) < 2)$ equals

2 SOLUTION

Given,

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & otherwise \end{cases} \tag{2.0.1}$$

We know,

$$\Pr(X < x) = \int_0^x f(x)\,dx \tag{2.0.2}$$

Using (2.0.1) in (2.0.2),

$$\Pr(X < x) = \int_0^x e^{-x}\,dx \tag{2.0.3}$$

We need to find $\Pr(max(X,Y) < 2)$,which is also can be written as

$$\Pr(max(X,Y) < 2) = \Pr(X < 2, Y < 2) \tag{2.0.4}$$

As X and Y be independent random variables,

$$\Pr(max(X,Y) < 2) = \Pr(X < 2)\Pr(Y < 2) \tag{2.0.5}$$

Using (2.0.3) in (2.0.5),

$$\Pr(max(X,Y) < 2) = (\int_0^2 e^{-x}\,dx)(\int_0^2 e^{-y}\,dy) \tag{2.0.6}$$

$$= [-e^{-x}]_0^2\,[-e^{-y}]_0^2 \tag{2.0.7}$$

$$= 0.748 \tag{2.0.8}$$

Finding the probability using uniform distribution,Let $F_X(x)$ be the cumulative distribution function of random variable X.

$$F_X(x) = \int_0^x f(x)\,dx \tag{2.0.9}$$

$F_X(x)$ can be obtained from the uniform distribution of a random variable U on (0,1) and let $U = e^{-x}$.

$$0 < U < 1 \tag{2.0.10}$$

As for random variable X also,

$$0 < F_X(x) < 1 \tag{2.0.11}$$

This similarity between U and $F_X(x)$ is used to generate the random variable X from U.

$$F_X(x) = \Pr(X < x) \tag{2.0.12}$$

$$= \Pr(-log_e U < x) \tag{2.0.13}$$

$$= \Pr(U > e^{-x}) \tag{2.0.14}$$

$$= 1 - \Pr(U < e^{-x}) \tag{2.0.15}$$

$$= 1 - F_U(e^{-x}) \tag{2.0.16}$$

From uniform distribution,

$$F_U(x) = x, 0 < x < 1 \tag{2.0.17}$$

In the figure 1,orange colour graph represents the pdf of the random variable X and blue colour graph represents the cdf of the random variable. Using (2.0.16) in (2.0.17), Cumulative distribution function (CDF) of random variable X is,

$$F_X(x) = \Pr(X < x) \tag{2.0.18}$$

$$= 1 - e^{-x} \tag{2.0.19}$$

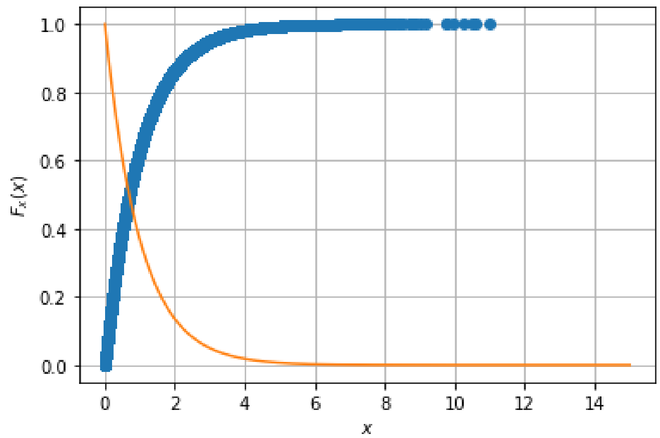

Fig. 1: CDF of random variable X

We need to find $\Pr(max(X, Y) < 2)$,which is also can be written as

$$\Pr(max(X, Y) < 2) = \Pr(X < 2, Y < 2) \quad (2.0.20)$$

As X and Y be independent random variables,

$$\Pr(max(X, Y) < 2) = \Pr(X < 2)\Pr(Y < 2) \quad (2.0.21)$$

using (2.0.19),

$$\Pr(max(X, Y) < 2) = (1 - e^{-2})(1 - e^{-2}) \quad (2.0.22)$$
$$= (1 - e^{-2})^2 \quad (2.0.23)$$
$$= 0.748. \quad (2.0.24)$$

We need to find $\Pr(max(X, Y) < 2)$,which is also can be written as

$$\Pr(max(X, Y) < 2) = \Pr(X < 2, Y < X) + \Pr(Y < 2, X \leq Y) \quad (2.0.25)$$

$\Pr(X < 2, Y < X)$ can also be written as,

$$\Pr(X < 2, Y < X) = \int_0^2 e^{-x}(\int_0^x e^{-y}\,dy)\,dx \quad (2.0.26)$$
$$= \int_0^2 e^{-x}(1 - e^{-x})\,dx \quad (2.0.27)$$
$$= \left[-e^{-x} + \frac{e^{-2x}}{2}\right]_0^2 \quad (2.0.28)$$
$$= 0.3738 \quad (2.0.29)$$

similarly, $\Pr(Y < 2, X \leq Y)$ can also be written as,

$$\Pr(Y < 2, X \leq Y) = \int_0^2 e^{-y}(\int_0^y e^{-x}\,dx)\,dy \quad (2.0.30)$$
$$= \int_0^2 e^{-y}(1 - e^{-y})\,dy \quad (2.0.31)$$
$$= \left[-e^{-y} + \frac{e^{-2y}}{2}\right]_0^2 \quad (2.0.32)$$
$$= 0.3738 \quad (2.0.33)$$

Using (2.0.29) and (2.0.33) in (2.0.25),

$$\Pr(max(X, Y) < 2) = 0.748 \quad (2.0.34)$$