Practical-4

Solution of Differential Equation by Variation of Parameter Method

Question 1: Solve second order differential equation
d^2y/dx^2=9y=sec(3x)
by variation of parameter method

Solution:

```
Sol = DSolve[y''[x] + 9 y[x] == 0, y[x], x]
sol1 = Evaluate[y[x] /. sol[[1]] /. {C[1] → 1, C[2] → 0}]
sol2 = y[x] /. sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {sol1, sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[-sol2 Sec[3 x], x]) / wd
u2 = (integrate[sol1 Sec[3 x], x]) / wd
yc = DSolve[y''[x] + 9 y[x] == 0, y[x], x]
yp = Evaluate[y[x] /. sol[[1]] /. {C[1] → u1, C[2] → u2}]
yg = yc + yp
```

{{y[x] → C[1] Cos[3 x] + C[2] Sin[3 x]}}

Cos[3 x]

Sin[3 x]

{Cos[3 x], Sin[3 x]}

$$\begin{pmatrix} \text{Cos}[3\,x] & \text{Sin}[3\,x] \\ -3\,\text{Sin}[3\,x] & 3\,\text{Cos}[3\,x] \end{pmatrix}$$

3

$$\frac{1}{9}\,\text{Log}[\text{Cos}[3\,x]]$$

$$\frac{1}{3}\,\text{integrate}[1,\,x]$$

{{y[x] → C[1] Cos[3 x] + C[2] Sin[3 x]}}

$$\frac{1}{9}\,\text{Cos}[3\,x]\,\text{Log}[\text{Cos}[3\,x]] + \frac{1}{3}\,\text{integrate}[1,\,x]\,\text{Sin}[3\,x]$$

$$\left\{\left\{\frac{1}{9}\,\text{Cos}[3\,x]\,\text{Log}[\text{Cos}[3\,x]] + (y[x] \to C[1]\,\text{Cos}[3\,x] + C[2]\,\text{Sin}[3\,x]) + \frac{1}{3}\,\text{integrate}[1,\,x]\,\text{Sin}[3\,x]\right\}\right\}$$

Question 2: Solve third order differential equation d^3y/dx^3+4dy/dx=sec(2x) by variation of parameter method

Solution:

```
sol = DSolve[y'''[x] + 4 y'[x] == 0, y[x], x]
sol1 =
 Evaluate[y[x] /. sol[[1]] /. {C[1] → 2, C[2] → 0, C[3] → 0}]
sol2 = y[x] /. sol[[1]] /. {C[1] → 0, C[2] → -2, C[3] → 0}
sol3 = y[x] /. sol[[1]] /. {C[1] → 0, C[2] → 0, C[3] → 1}
fs = {sol1, sol2, sol3}
wm = {fs, D[fs, x], D[fs, {x, 2}]}; MatrixForm[wm]
wd = Simplify[Det[wm]]
a = (1 / wd (Det[{{0, sol2, sol3},
       {0, D[sol2, x], D[sol3, x]}, {Sec[2 x],
        D[sol2, {x, 2}], D[sol3, {x, 2}]}}])) // Simplify
u1 = Integrate[a, x]
b = (1 / wd (Det[{{sol1, 0, sol3},
       {D[sol1, x], 0, D[sol3, x]}, {D[sol1, {x, 2}],
        Sec[2 x], D[sol3, {x, 2}]}}])) // Simplify
u2 = Integrate[b, x]
c = (1 / wd (Det[{{sol1, sol2, 0},
       {D[sol1, x], D[sol2, x], 0}, {D[sol1, {x, 2}],
        D[sol2, {x, 2}], Sec[2 x]}}])) // simplify
u3 = Integrate[c, x]
yc = Evaluate[y[x] /. sol[[1]]]
yp = Evaluate[
  y[x] /. sol[[1]] /. {C[1] → u1, C[2] → u2, C[3] → u3}]
yg =
 yc +
  yp
```

$$\left\{\left\{y[x] \to C[3] - \frac{1}{2} C[2]\, \text{Cos}[2\,x] + \frac{1}{2} C[1]\, \text{Sin}[2\,x]\right\}\right\}$$

Sin[2 x]

Cos[2 x]

$$1$$

$$\{\text{Sin}[2\,x],\ \text{Cos}[2\,x],\ 1\}$$

$$\begin{pmatrix} \text{Sin}[2\,x] & \text{Cos}[2\,x] & 1 \\ 2\,\text{Cos}[2\,x] & -2\,\text{Sin}[2\,x] & 0 \\ -4\,\text{Sin}[2\,x] & -4\,\text{Cos}[2\,x] & 0 \end{pmatrix}$$

$$-8$$

$$-\frac{1}{4}\,\text{Tan}[2\,x]$$

$$\frac{1}{8}\,\text{Log}[\text{Cos}[2\,x]]$$

$$-\frac{1}{4}$$

$$-\frac{x}{4}$$

$$\text{simplify}\left[-\frac{1}{8}\,\text{Sec}[2\,x]\,\left(-2\,\text{Cos}[2\,x]^2-2\,\text{Sin}[2\,x]^2\right)\right]$$

$$\int \text{simplify}\left[-\frac{1}{8}\,\text{Sec}[2\,x]\,\left(-2\,\text{Cos}[2\,x]^2-2\,\text{Sin}[2\,x]^2\right)\right] dx$$

$$C[3]-\frac{1}{2}\,C[2]\,\text{Cos}[2\,x]+\frac{1}{2}\,C[1]\,\text{Sin}[2\,x]$$

$$\frac{1}{8}\,x\,\text{Cos}[2\,x]+ \int \text{simplify}\left[-\frac{1}{8}\,\text{Sec}[2\,x]\,\left(-2\,\text{Cos}[2\,x]^2-2\,\text{Sin}[2\,x]^2\right)\right] dx + \frac{1}{16}\,\text{Log}[\text{Cos}[2\,x]]\,\text{Sin}[2\,x]$$

$$C[3] + \frac{1}{8} x \operatorname{Cos}[2 x] - \frac{1}{2} C[2] \operatorname{Cos}[2 x] + \int \text{simplify}\left[-\frac{1}{8} \operatorname{Sec}[2 x] \left(-2 \operatorname{Cos}[2 x]^2 - 2 \operatorname{Sin}[2 x]^2\right)\right] dx + \frac{1}{2} C[1] \operatorname{Sin}[2 x] + \frac{1}{16} \operatorname{Log}[\operatorname{Cos}[2 x]] \operatorname{Sin}[2 x]$$

Question 3: Solve second order differential equation
d^2y/dx^2+t=Tan(x)
by variation of parameter method

Solution:

```
Sol = DSolve[y''[x] + y[x] == 0, y[x], x]
Sol1 = Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0}]
Sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {Sol1, Sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[-Sol2 Tan[x], x]) / wd
u2 = (Integrate[Sol1 Tan[x], x]) / wd
yc = DSolve[y''[x] + y[x] == 0, y[x], x]
yp = Simplify[
  Evaluate[y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2}]]
yg =
 yc +
  yp
```

{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

Cos[x]

Sin[x]

{Cos[x], Sin[x]}

$$\begin{pmatrix} \text{Cos}[x] & \text{Sin}[x] \\ -\text{Sin}[x] & \text{Cos}[x] \end{pmatrix}$$

1

$$\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right] + \text{Sin}[x]$$

-Cos[x]

{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

$$\text{Cos}[x] \left(\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right]\right)$$

$$\left\{\left\{\text{Cos}[x] \left(\text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] - \text{Sin}\left[\frac{x}{2}\right]\right] - \text{Log}\left[\text{Cos}\left[\frac{x}{2}\right] + \text{Sin}\left[\frac{x}{2}\right]\right]\right) + (y[x] \to C[1]\, \text{Cos}[x] + C[2]\, \text{Sin}[x])\right\}\right\}$$

Question 4: Solve third order differential equation
$d^\wedge 3\, y/\text{dx}^\wedge 3 - 6\, d^\wedge 2\, y/\text{dx}^\wedge 2 + 11\, \text{dy}/\text{dx} - 6\, y = e^x$
by variation of parameter method

Solution:

```
Sol = DSolve[y'''[x] - 6 y''[x] + 11 y'[x] - 6 y[x] == 0, y[x], x]
Sol1 =
 Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0, C[3] → 0}]
Sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1, C[3] → 0}
Sol3 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 0, C[3] → 1}
fs = {Sol1, Sol2, Sol3}
wm = {fs, D[fs, x], D[fs, {x, 2}]}; MatrixForm[wm]
wd = Simplify[Det[wm]]
a = (1 / wd (Det[
      {{0, Sol2, Sol3}, {0, D[Sol2, x], D[Sol3, x]}, {Exp[x],
        D[Sol2, {x, 2}], D[Sol3, {x, 2}]}}])) // Simplify
u1 = Integrate[a, x]
b = (1 / wd (Det[{{Sol1, 0, Sol3},
       {D[Sol1, x], 0, D[Sol3, x]}, {D[Sol1, {x, 2}],
        Exp[x], D[Sol3, {x, 2}]}}])) // Simplify
u2 = Integrate[b, x]
c = (1 / wd (Det[{{Sol1, Sol2, 0},
       {D[Sol1, x], D[Sol2, x], 0}, {D[Sol1, {x, 2}],
        D[Sol2, {x, 2}], Exp[x]}}])) // Simplify
u3 = Integrate[c, x]
yc = Evaluate[y[x] /. Sol[[1]]]
yp = Evaluate[
  y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2, C[3] → u3}]
yg =
 yc +
  yp
```

$\{\{y[x] \to e^{x} C[1] + e^{2x} C[2] + e^{3x} C[3]\}\}$

e^{x}

e^{2x}

e^{3x}

$\{e^{x}, e^{2x}, e^{3x}\}$

$$\begin{pmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2 e^{2x} & 3 e^{3x} \\ e^{x} & 4 e^{2x} & 9 e^{3x} \end{pmatrix}$$

$2 e^{6x}$

$$\frac{1}{2}$$

$$\frac{x}{2}$$

$$-e^{-x}$$

$$e^{-x}$$

$$\frac{e^{-2x}}{2}$$

$$-\frac{1}{4}e^{-2x}$$

$$e^{x}C[1]+e^{2x}C[2]+e^{3x}C[3]$$

$$\frac{3e^{x}}{4}+\frac{e^{x}x}{2}$$

$$\frac{3e^{x}}{4}+\frac{e^{x}x}{2}+e^{x}C[1]+e^{2x}C[2]+e^{3x}C[3]$$

Question 5: Solve second order differential equation
d^2y/dx^2-2y=4x-8
given condition is y[1]=4,y’[1]=-1 by variation of parameter method

Solution:

```
Sol = DSolve[x^2 y''[x] - 2 y[x] == 0, y[x], x]
sol1 = Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0}]
sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {sol1, sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[-sol2 (4 x - 8), x]) / wd
u2 = (Integrate[sol1 (4 x - 8), x]) / wd
yc = DSolve[x^2 y''[x] - 2 y[x] == 0, y[x], x]
yp = Simplify[
  Evaluate[y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2}]]
yg =
 yc +
  yp
```

$$\left\{\left\{y[x] \to x^2\, C[1] + \frac{C[2]}{x}\right\}\right\}$$

$$x^2$$

$$\frac{1}{x}$$

$$\left\{x^2,\ \frac{1}{x}\right\}$$

$$\begin{pmatrix} x^2 & \frac{1}{x} \\ 2\,x & -\frac{1}{x^2} \end{pmatrix}$$

$$-3$$

$$\frac{1}{3}\,(4\,x - 8\,\mathrm{Log}[x])$$

$$-\frac{4}{3}\left(-\frac{2\,x^3}{3} + \frac{x^4}{4}\right)$$

$$\left\{\left\{y[x] \to x^2\, C[1] + \frac{C[2]}{x}\right\}\right\}$$

$$\frac{1}{9}\,x^2\,(8 + 9\,x - 24\,\mathrm{Log}[x])$$

$$\left\{\left\{\frac{1}{9}\,x^2\,(8 + 9\,x - 24\,\mathrm{Log}[x]) + \left(y[x] \to x^2\, C[1] + \frac{C[2]}{x}\right)\right\}\right\}$$

Question 6: Solve second order differential equation

$d\,^\wedge 2\, y/\mathrm{dx}\,^\wedge 2 + y = \mathrm{Tan}(x)\,\mathrm{Sec}(x)$

by variation of parameter method

Solution:

```
Sol = DSolve[y''[x] + y[x] == 0, y[x], x]
Sol1 = Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0}]
Sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {Sol1, Sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[-Sol2 Tan[x] Sec[x], x]) / wd
u2 = (Integrate[Sol1 Tan[x] Sec[x], x]) / wd
yc = DSolve[y''[x] + y[x] == 0, y[x], x]
yp = Simplify[
  Evaluate[y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2}]]
yg =
 yc +
  yp
```

```
{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

Cos[x]

Sin[x]

{Cos[x], Sin[x]}
```

$$\begin{pmatrix} \mathrm{Cos}[x] & \mathrm{Sin}[x] \\ -\mathrm{Sin}[x] & \mathrm{Cos}[x] \end{pmatrix}$$

```
1

x - Tan[x]

-Log[Cos[x]]

{{y[x] → C[1] Cos[x] + C[2] Sin[x]}}

x Cos[x] - (1 + Log[Cos[x]]) Sin[x]

{{x Cos[x] + (y[x] → C[1] Cos[x] + C[2] Sin[x]) -
   (1 + Log[Cos[x]]) Sin[x]}}
```

Question 7: Solve second order differential equation $d^2 y / dx^2 - 2\, dy/dx + 5\, y = e^x \mathrm{Tan}(2x)$ by variation of parameter method

Solution:

```
Sol = DSolve[y''[x] - 2 y'[x] + 5 y[x] == 0, y[x], x]
sol1 = Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0}]
sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {sol1, sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[-sol2 Exp[x] Tan[2 x], x]) / wd
u2 = (Integrate[sol1 Exp[x] Tan[2 x], x]) / wd
yc = DSolve[y''[x] - 2 y'[x] + 5 y[x] == 0, y[x], x]
yp = Simplify[
  Evaluate[y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2}]]
yg =
 yc +
  yp
```

$\{\{y[x] \to e^x C[2] \mathrm{Cos}[2x] + e^x C[1] \mathrm{Sin}[2x]\}\}$

$e^x \mathrm{Sin}[2x]$

$e^x \mathrm{Cos}[2x]$

$\{e^x \mathrm{Sin}[2x], e^x \mathrm{Cos}[2x]\}$

$$\begin{pmatrix} e^x \mathrm{Sin}[2x] & e^x \mathrm{Cos}[2x] \\ 2 e^x \mathrm{Cos}[2x] + e^x \mathrm{Sin}[2x] & e^x \mathrm{Cos}[2x] - 2 e^x \mathrm{Sin}[2x] \end{pmatrix}$$

$-2 e^{2x}$

$$\frac{1}{8}(-\mathrm{Cos}[2x] + \mathrm{Sin}[2x])$$

$$\frac{1}{8}\Big(\mathrm{Cos}[2x] - (2 - 2i)\, e^{2ix}\, \mathrm{Hypergeometric2F1}\Big[\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, -e^{4ix}\Big] + \mathrm{Sin}[2x]\Big)$$

$\{\{y[x] \to e^x C[2] \mathrm{Cos}[2x] + e^x C[1] \mathrm{Sin}[2x]\}\}$

$$\frac{1}{8} e^{x} \left(1 - (1 - i) \left(1 + e^{4 i x}\right) \text{Hypergeometric2F1}\left[\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, -e^{4 i x}\right]\right)$$

$$\left\{\left\{\frac{1}{8} e^{x} \left(1 - (1 - i) \left(1 + e^{4 i x}\right) \text{Hypergeometric2F1}\left[\frac{1}{2} - \frac{i}{2}, 1, \frac{3}{2} - \frac{i}{2}, -e^{4 i x}\right]\right) + (y[x] \to e^{x} C[2] \text{Cos}[2 x] + e^{x} C[1] \text{Sin}[2 x])\right\}\right\}$$

Questin 8: Solve second order differential equation
d ^ 2 y / dx ^ 2 + 3 dy / dx + 3 y = e^x/x
by variation of parameter method

Solution:

```
Sol = DSolve[y''[x] + 3 y'[x] + 2 y[x] == 0, y[x], x]
sol1 = Evaluate[y[x] /. Sol[[1]] /. {C[1] → 1, C[2] → 0}]
sol2 = y[x] /. Sol[[1]] /. {C[1] → 0, C[2] → 1}
fs = {sol1, sol2}
wm = {fs, D[fs, x]}; wm // MatrixForm
wd = Simplify[Det[wm]]
u1 = (Integrate[sol2 Exp[-x] / x, x]) / wd
u2 = (Integrate[sol1 Exp[-x] / x, x]) / wd
yc = DSolve[y''[x] + 3 y'[x] + 2 y[x] == 0, y[x], x]
yp = Simplify[
  Evaluate[y[x] /. Sol[[1]] /. {C[1] → u1, C[2] → u2}]]
yg =
 yc +
  yp
```

$\{\{y[x] \to e^{-2x} C[1] + e^{-x} C[2]\}\}$

e^{-2x}

e^{-x}

$\{e^{-2x}, e^{-x}\}$

$\begin{pmatrix} e^{-2x} & e^{-x} \\ -2 e^{-2x} & -e^{-x} \end{pmatrix}$

e^{-3x}

$e^{3x}\,\text{ExpIntegralEi}[-2x]$

$e^{3x}\,\text{ExpIntegralEi}[-3x]$

$\{\{y[x] \to e^{-2x} C[1] + e^{-x} C[2]\}\}$

$e^{x} (e^{x}\,\text{ExpIntegralEi}[-3x] + \text{ExpIntegralEi}[-2x])$

$\{\{e^{x} (e^{x}\,\text{ExpIntegralEi}[-3x] + \text{ExpIntegralEi}[-2x]) + (y[x] \to e^{-2x} C[1] + e^{-x} C[2])\}\}$