Intertemporal Analysis using Symbolic Regression

Manila Devaraja

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Abstract

Humans deal with decision-making between choices every day that eventually have outcomes over the period. Studies on choice behavior have shown that humans are more attracted to immediate than delayed rewards. This myopic view of humans choosing immediate rewards is referred to as discounting. We articulate a study to understand intertemporal choices and formulate a function that accounts for the discounting model of human decision-making over time. The study includes a survey that accesses the discounting structural thinking. Mathematical formulas can naturally describe many real-world problems. We use regression to find a symbolic expression from the survey data that matches the data from an unknown function.

1 Introduction

Human decision-making is hard to decode and to be generalized. Several machine learning algorithms have been developed to mimic human decision-making, serve as a decision aid, and automate many routine human decisions. However, the key to developing such algorithms is understanding how complex or simple the human decision is. For instance, people use simple heuristics to reach a decision, or their decisions may be biased. The factors that go behind a decision are numerous. But the idea is to identify the possibility of having a standard set of factors that humans think of when making a decision. The task of finding formulas from a set of observed inputs and outputs is called symbolic regression. Mathematical models help prove a theory logically and also, can be interpreted easily. Through this project, we intend to get an expression for discounting behavior in humans by using symbolic regression.

1.1 Literature Survey

The mathematical representation of intertemporal discounting has helped psychologists and economists give a definite meaning to human decision making over time[2]. Exponential discount function which was long considered to satisfy the consistent preferences over time, failed to integrate human behavior to discount high in shorter run than in long run.
Studies[3] on choice behavior have shown that humans are attracted to immediate rewards

than delayed rewards in the short run than in the longer run, which led to incorporating hyperbolic behavior to discount function. Arguing that human preferences are dynamically inconsistent over time, Strotz[3] defined the quasi-hyperbolic discount function as below:

$$f(x) = \begin{cases} 1, & \text{if} \tau < 0 \\ \beta.\delta(\tau), & \text{if} \tau = 1, 2, 3 \end{cases}$$

The discount factor is the inverse of the continuously compounded discount rate $\rho(\tau)$. Exponential discount functions fail to match several empirical regularities(discount functions decline at a higher rate in the short run than in the long run).

When $0 < \beta < 1$ and $0 < \delta < 1$ the quasi-hyperbolic discounting function has a high short-run discount rate and a relatively low long-run discount rate. Quasi-hyperbolic time preferences are also called present-biased and quasi-geometric.

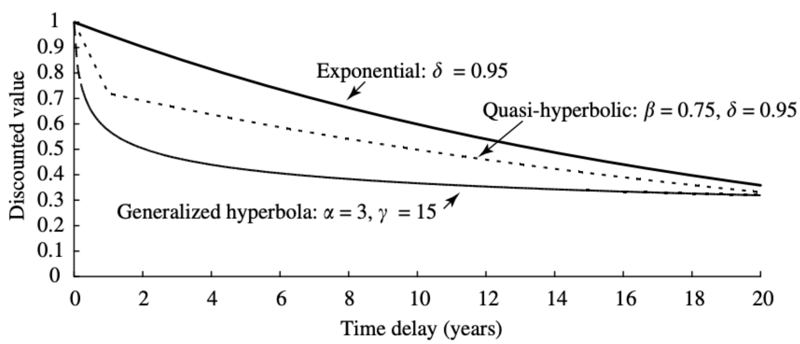


Figure 1: Three calibrated discount functions [3]

The discount function is believed to be the function of time according to [3]. In an attempt to generalize temporal discounting with other factors, a brain-inspired mathematical model[8] breaks down the understanding of the decision-making to *at least* two systems- a valuation system and a control system.

$$D_\tau = \omega\delta_1^\tau + (1 - \omega)\delta_2^\tau$$

Here, ω indicates the impact each system has on the decision. Increasing activity in the valuation system would increase selection of smaller, sooner outcomes, and increasing activity in the control system would increase selection of larger, later outcomes.
Although the above work[8] justifies the contextual influences on discounting and attempts to generalize the discounting function, there is still no work considering other factors like the future amount, current amount, profit/loss frame, near/far time frames, etc., which can potentially contribute to the decisions made by humans.
Machine learning methods have long been used to interpret the function the variables fit to. The methods search over a space of functions to check which function fits well for

the data provided. Symbolic regression, however, searches over the space of symbols to fit the variables[5]. This results in an interpretable and justifiable prediction. In this way, the models logically relate the features and give a mathematical meaning. Traditionally handled as a part of genetic programming, symbolic regression today has found an increasing interest in deep learning in search of better interpretable models. In this project, we intend to use symbolic regression to understand how the features contribute to human intertemporal behavior.

2 Data

The crucial aspect of the project was data collection. The data for this project was collected through a survey. To conduct the study, I had to go through the IRB training for Social/Behavioral Research. The survey was conducted through the Qualtrics platform. The process of formulating the questionnaire, data cleaning, and analysis are discussed in the following sections.

2.1 Survey

The survey had a total of seven questions. The three main features we wanted to capture are the future amount, the current amount, and the time gap between. Each question had two features given to them, and the user had to answer the other one. There were three gain questions, three loss questions, and one gain question with a different time gap. With respect to demographic information, we collected age and gender. These features could help analyze how different gender or age groups make decisions differently. However, for this project, we do not use those features.

The questions were formulated to extract respective features as follows:

- Q1(Future gain amount): When you are given a current reward and asked to input the desirable future gain reward for the given period.
- Q2(Gain time from *now*): When you are given a current and future reward, you are asked to input the time you are willing to wait to get that.
- Q3(Gain Acceleration or the Current Amount): You are told you have a reward, and if you wait for a given time, you receive a future amount, but if you want it *now*, you are asked to input the desired amount.
- Q4(Future Loss Amount): When you are given a current penalty and asked to input the desirable future penalty you would like to pay for the given period.
- Q5(Loss time from *now*): When given a current and future penalty amount, you are asked to input the time you are willing to wait to pay that.

- Q6(Loss Acceleration or Current Penalty): You are told you have a penalty, and if you wait for a given time, you will have to pay a given amount, but if you want to pay *now*, you are asked to input the desired amount.

- Q7(Future gain amount from *later*): You are told that you can get a given reward after a given time and asked to input the desirable future gain reward if you want to wait for another period.

The theory that the behavior changes for smaller and shorted rewards(SS) as compared to larger and longer(LL) rewards have been previously researched [7]. The data range posed to the user as rewards/penalties were varied to capture behavioral changes mentioned in [7]. The sampling of the data ranges for each question was another task.

The survey was conducted as a part of the David Eccles Business School Coursework during Spring 2023. A total of 79 participants took part, and as each answered seven questions, we had a total of 553 data points.

2.2 Data cleaning

Data cleaning and analysis are crucial to get the data in the desired format and also analyze the statistics of the data. To start with, Qualtrics collects data with object datatype, and converting them into integers was necessary.

2.3 Feature extraction

Having the features in a desired format is helpful in analysis. The questions which asked the user to input time were collected in the format of years, months, and days. To maintain consistency, all the features in time were converted into days and compressed into one column. The formula used to convert are

$$Days = timeinyears * 365 + \frac{timeinyear}{4}$$
$$Days = timeinmonth * 30$$

For this purpose, we divided the data into the gain and the loss section to handle the data individually.

2.4 Wide to long format

Qualtrics collects the data in the wide format which means the features and answers collected from the users are stored horizontally. For our analysis we would need the data points in the rows. For this purpose, we converted

We consider a single feature in each question, whether asked by the user or given by us and put them in an array with blocks of data from each question. Similarly, we create three arrays, each containing the values of future amount, current amount, and later time in the order of questions Q1, Q2, Q3, Q7, Q4, Q5, Q6. The reason for reordering Q7 is to keep all gain questions together. Combining all three arrays into three columns in a dataframe will give us the long format of the same data.

2.5 New features- Loss/Gain, Elicitation, Time gap

Adding new features is crucial when it can potentially influence the output. The features we chose to add explicitly are:

- Loss/Gain: This feature indicates if the question was a Loss/Gain question
- Elicitation: This feature indicates "what" was asked of the user
- Time gap: This feature is the time gap between the current and future amounts.

2.6 Statistics and outlier removal

Understanding the data distribution helps in avoiding any overfitting or underfitting during modeling. The data distribution of three main features after data cleaning is as shown in 2.

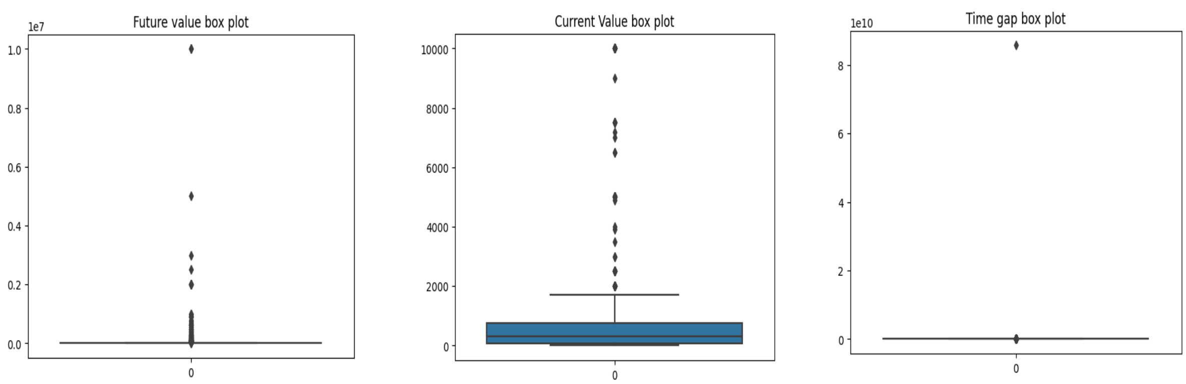
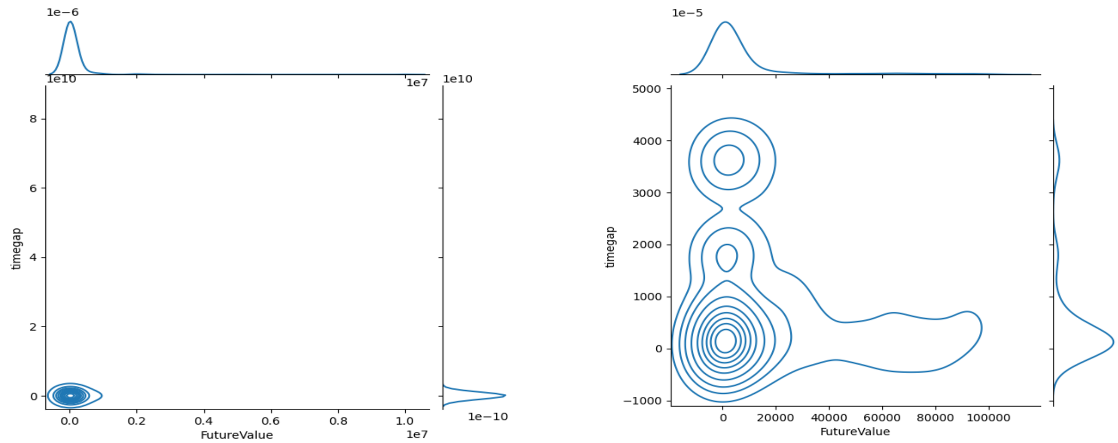


Figure 2: Data distribution of future amount, current amount and the time gap

We can clearly say there are outliers in future amounts and time gaps, which skewed the data. I identified ten values in future amounts and three in time gaps much larger than anticipated. Some were user inputted, and some were software errors in the survey when sampling the data. Joint plots help understand the relationship between the variables and the distribution of the data[9]. The distribution and density variation can be seen improving once the outliers are removed, as shown in 2.



(a) With Outliers

(b) Without Outliers

Figure 3: Comparision of density plots before and after outlier removal

3 Modelling

Once the data is cleaned and analyzed, we can start with modeling. In this section, we try to understand the relationship between the features and the ultimate meaning of *how* it influences the discounting behavior in humans.

3.1 Experiment 1-Individual predictor models

In this experiment, we divide the data into three sections based on the elicitation. We will consider the Loss/gain and main three features for this experiment. The model will predict the feature that the user provided.

3.1.1 Linear Symbolic Regression

Linear regression is the type of modeling where we try to fit the data with a line where we assume the features have a linear relationship. In linear regression, the target variable is the linear relationship of feature attributes given by :

$$f(x) = \sum_{j} \theta_i f_j(x)$$

Here, θ_j is the weight coefficient, and f_j is the unary operation of the function class. With linear regression, we reduce the complexity of symbolic regression by redefining the model to be linear, and thereby, we have to learn the parameters.
Linear model symbolic models are as follows:

- Linear model on Future Amount prediction Linear model generates

$$FutureAmount = 1.827CurrentValue + 4.544timegap + 1884.808L/G - 2788.842 \quad (1)$$

- Linear model on Current Amount prediction

$$CurrentValue = 0.6421FutureValue - 28.505L/G - 0.008614timegap + 18.7601 \quad (2)$$

- Linear model on time-gap prediction

$$Timegap = -0.4987CurrentValue - 0.00123FutureValue - 59.403L/G + 499.0776 \quad (3)$$

The performance of the linear model can be accessed by rearranging the equations 2 and 3 to get an equation for futureamount. Rewriting 1 along with rearranged equations of 2 and 3, we get:

$$FutureAmount = 1.827CurrentValue + 4.544timegap + 1884.808L/G - 2788.842$$

$$FutureAmount = 1.557CurrentValue + 0.01341timegap + 44.39L/G + 29.2167$$

$$FutureAmount = -405.44CurrentValue - 813.008timegap - 48295.12L/G + 405753.6585$$

We do see that the signs of the coefficients are the same in the first two equations. But the signs of the coefficients in the time-gap equation are wrong and might result in a negative time-gap.

3.1.2 Polynomial Symbolic Regression

In polynomial modeling, we do not assume a linear relationship between the input and output. We consider the polynomial variations of the features along with linear features. Although this seems like an improvement, there is always a chance of overfitting the training data as we increase the degree. As a part of [6], we use the function PolynomialFeature to compute the features of the degree provided. For our purpose, we are restricting to the second degree to avoid any overfitting. With degree two, for feature a, b, c, we compute $1, a, b, c, a^2, b^2, c^2, ab, bc, ca$. We then run a linear regression on these features instead of just linear features and compute the weight coefficients of the polynomial features. Here, FV is FutureValue, CV is current value, T is the timegap and L/G is Loss/Gain.

Polynomial symbolic models are as follows:

- Polynomial model on Future Amount prediction

$$FutureValue = 1.7889CV - 5.3226T - 77.883L/G - 0.00013CV^2 + 0.0016T^2 + 0.0477L/G^2 + 0.00126CV * T + 3.953T * L/G - 77.883L/G * CV + 178.644$$

- Polynomial model on Current Amount prediction

$$CurrentValue = 1.092FV + 119.691L/G - 0.0225T - 0.00006FV^2 - 0.02398L/G^2 + 0.000097T^2 + 119.69FV * L/G - 0.2507L/G * T + 0.000003T * FV - 254.6154$$

- Polynomial model on time-gap prediction

$$Timegap = -1.2223CV + 0.0039FV - 63.7721L/G + 0.000634CV^2 + 0.0000063FV^2 - 0.481L/G^2 - 0.00000008CV * FV + 0.0039FV * L/G - 63.77L/G * CV + 507.507$$

The expressions get more complex with polynomial features.

3.1.3 Genetic Programming - gplearn

Genetic Programming(GP) is the non-linear symbolic regression that searches over the space of mathematical symbols to find the solution to the relationship between input and output. Mathematical expressions can be visualized in the form of a tree. GP starts with a generation of individuals(trees), which is randomly generated. GP evolutionary operations are of three kinds:

- Mutation - Random variations to an individual by replacing a subtree with a randomly generated one
- crossover - Exchanging subtrees between two individuals
- Selection - Select individuals from current population to persist onto next one

gplearn package[1] an API that allows us to implement genetic programming. It starts with a randomly chosen set of available functions and variables. But with the hyperparameters provided by the modeler, it gradually improves with generation. Some of the important parameters to highlight are:

- function_set: The mathematical functions to be considered. We used (add, sub, mul, div, neg, inv).
- generations: The maximum number of generations to run to stop the program. We keep it to 40.
- stopping_criteria: The score on reaching stops the program
- p_crossover: The percentage of a random subtree of the winner to be replaced for the next generation. Since we would want this to be higher, we keep it 70%
- p_subtree_mutation: The percentage of a random subtree from the winner to be replaced.

- ‘p_hoist_mutation‘: The percentage of selection of a part of a random subtree from the winner.
- ‘p_point_mutation‘: The percentage to select random *nodes* from winner to be replaced.

gplearn symbolic models are as follows:

- gplearn model on Future Amount prediction

$$FutureValue = CV + timegap - \frac{1}{(\frac{1}{(timegap-L/G)(2*CV-1.315+L/G)}) + \frac{1}{timegap}}$$

- gplearn model on Current Amount prediction
  Current Value =

$$\frac{-FutureValuetimegap\left(FutureValue+timegap\right)+0.671FutureValue-\frac{0.733+\frac{0.0789755290780142}{FutureValue}}{0.671FutureValue-timegap+\frac{FutureValue+1.223}{FutureValue^2-0.735}+0.787}+ -0.912925170068027FutureValue-timegap+\tilde{\infty}\left(0.671FutureValue+0.552\right)+\frac{5.57600089216014Gain}{FutureValue^2Loss(-timegap-0.485)}}{-2timegap-1.424+\frac{4.69483568075117timegap}{FutureValue}}$$

- gplearn model on time-gap prediction
  Time gap =

$$\frac{22.7272727272727}{0.446+\frac{CurrentValue+\frac{Loss}{Gain}}{(0.788CurrentValue+0.197)\left(-\frac{16.1290322580645CurrentValueLoss^2}{Gain^2}+\frac{1}{\frac{1}{-CurrentValue-0.262-\frac{Loss}{Gain}}+\frac{1}{CurrentValue}}-\frac{0.748+\frac{Loss}{Gain\left(0.505+\frac{Loss}{CurrentValueFutureValueGain}\right)}+\frac{FutureValueGain}{CurrentValue^2Loss}}{FutureValue^2(CurrentValue-FutureValue)\left(-CurrentValue-0.280088-\frac{0.628}{-0.446-\frac{1}{-FutureValue-0.165-\frac{Loss}{FutureValueGain}}-\frac{Loss}{FutureValueGain}+\frac{Loss}{CurrentValueGain}}-\frac{Loss}{Gain}+\frac{1}{CurrentValue}\right)}\right)}+\frac{Loss}{FutureValueGain}}$$

3.1.4 Performance of models

Root Mean Squared Error(RMSE) is the metric we will be using to validate the performance of the models. RMSE is calculated by:

$$RMSE = \sqrt{\frac{\sum_i^n (y_i - \hat{y})^2}{n}}$$

here y_i is the true value and $\hat{y}$ is the predicted value.
The comparison of model performance on each predictor is shown in 4. We do see that polynomial and linear models tend to overfit the data. Although GP model does not perform the best, the GP model can be seen performing much better on the current amount prediction with test RMSE much lesser than the train RMSE.

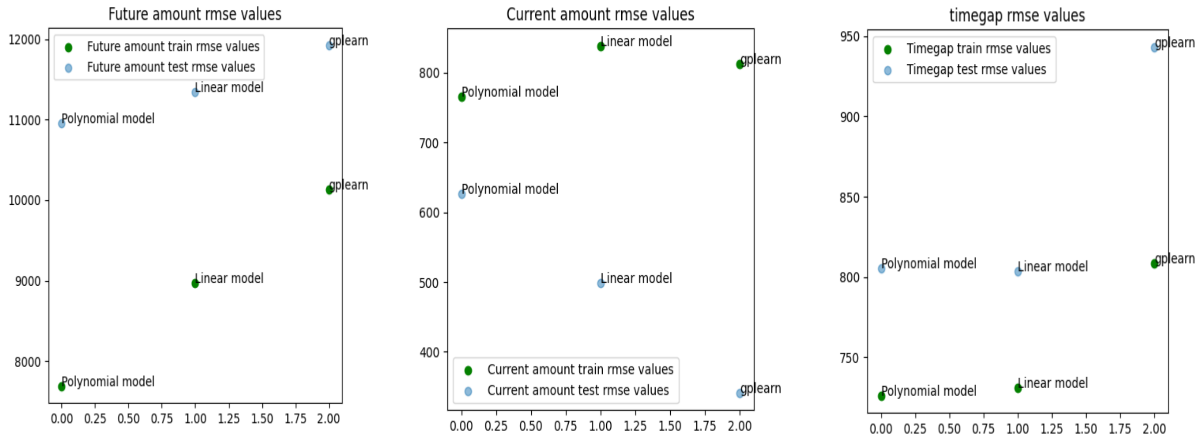


Figure 4: Validation of models using RMSE

3.2 Experiment 2- Model performance on full data

For this experiment, we consider all of the data points irrespective of the elicitation and check the performance of the models trained in 3.1.

The plot of the full data based on the loss/gain feature is as shown in 3.3

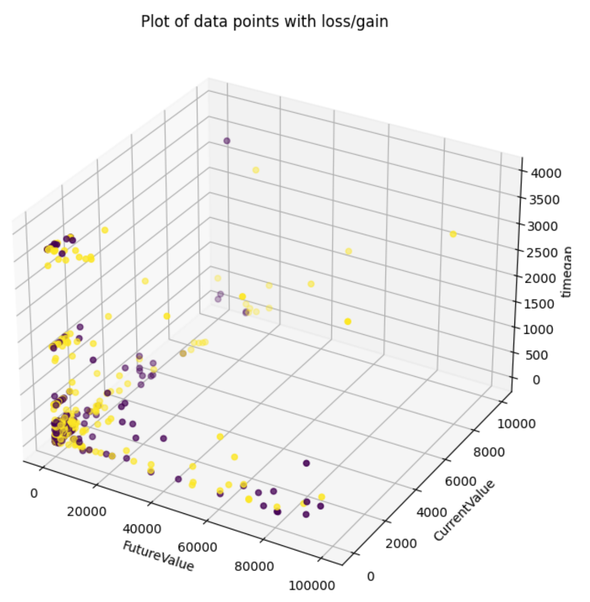


Figure 5: Data points based on Loss/Gain feature

The performance of each model on full data is shown in 6. Although the performance cannot be evaluated much through the 3D plot presented in 6 shows an interesting observation with axes values. We see a significant difference in the time gap axis with performance on linear models and the current value axis with performance on polynomial models.

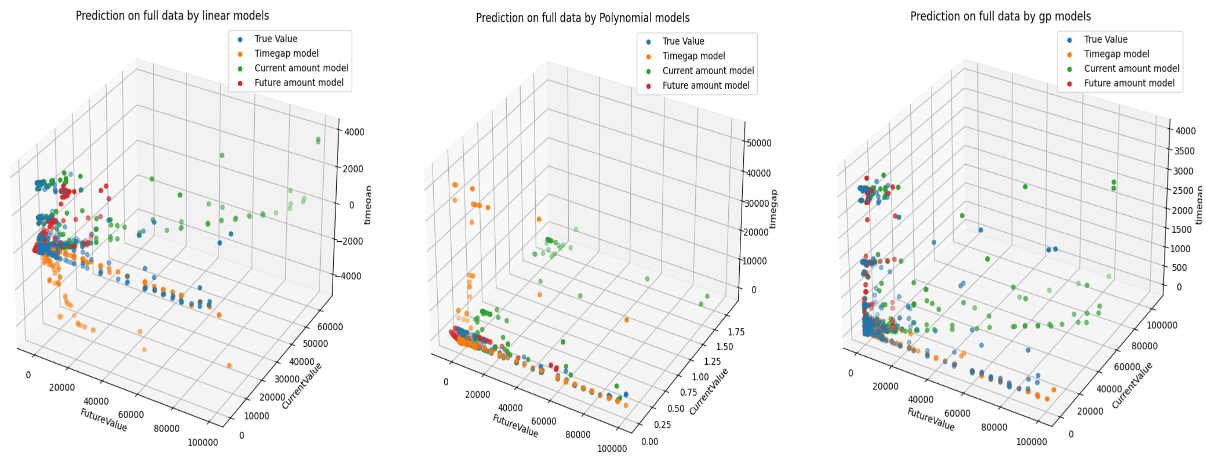
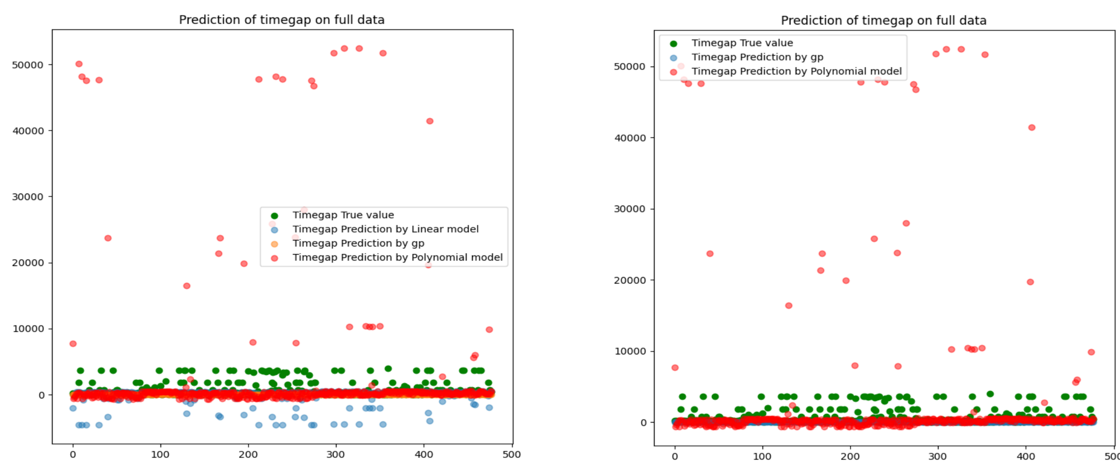


Figure 6: Performance of models on full data

To know the actual performance, we plot just the prediction of the time gap using all models and plot of prediction using all models except linear. This is shown in 7. We see that the linear model predicts a negative time gap which is not realistic, and also, considering that there were no negative values in the time gap in data, the linear model does not fit well for this prediction. As I mentioned during the linear model analysis, because of the negative signs of coefficients, we see a negative time gap value.

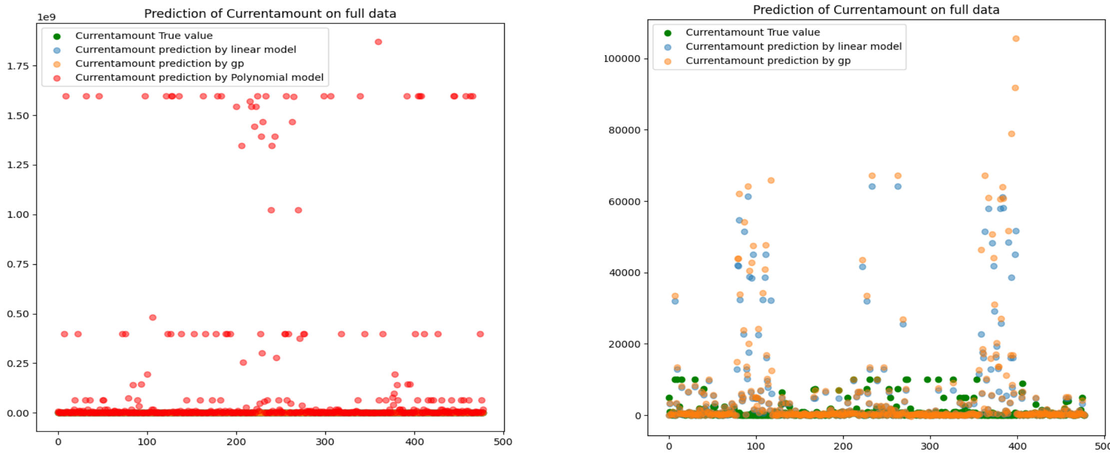


(a) With Linear model

(b) Without Linear model

Figure 7: Comparison of Time gap prediction with and without linear model

Similarly, a plot of the prediction of the current amount using all models and the prediction using all models except polynomial as shown in 8. It can be seen that the polynomial model has overfitted the training data.



(a) With Polynomial model (b) Without Polynomial model

Figure 8: Comparison of Current amount prediction with and without Polynomial model

Out of the three predictions, future data performance is better than the other two, as shown in 9.

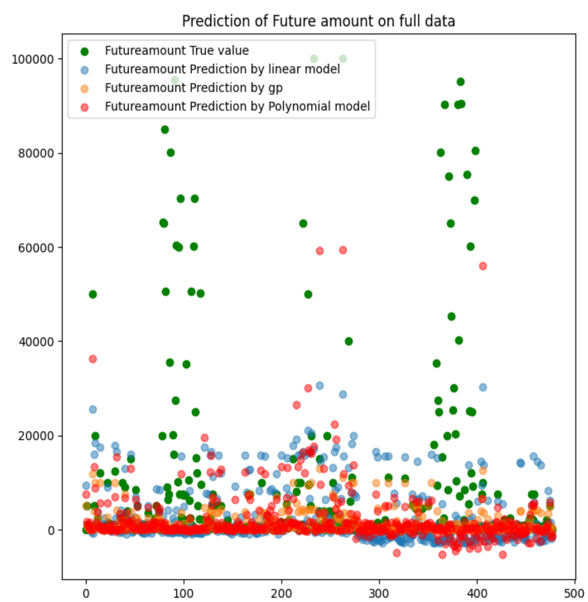


Figure 9: Future amount prediction

Although the plots are good to visualize, there are not the right measure of validation. The validation of the models using RMSE on the full data is as shown in 10.

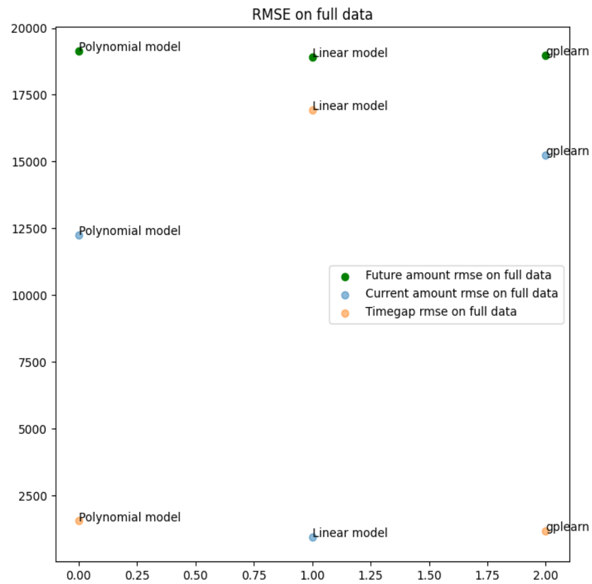


Figure 10: Validation of models on full data using RMSE

3.3 Experiment 3- without using Loss/Gain feature

The complexity of the symbolic models can increase with features. As an experiment, I removed the Loss/Gain feature and repeated 3.1 and 3.2. The generated expressions are simpler than when Loss/Gain was considered.
The linear symbolic models without Loss/Gain are:

- Linear model on Future Amount prediction Linear model generates

$$FutureAmount = 1.8311 CurrentValue + 4.6808 Timegap + 1573.119475$$

- Linear model on Current Amount prediction

$$CurrentValue = 0.6421 FutureValue - 0.009742 timegap + 5.033$$

- Linear model on time-gap prediction

$$Timegap = -0.4976 CurrentValue - 0.0011 FutureValue + 445.223$$

The performance of the linear models continues to be the same with or without the Loss/Gain feature. As we can see with the time-gap prediction, the model continues to predict negative values because of the negative signs.
The Polynomial symbolic models without Loss/Gain are:

- Polynomial model on Future Amount prediction

$$
\begin{aligned}
FutureValue = 1.7171CV - 1.8836timegap - \\
0.000121CV^2 + 0.001689T^2 + 0.00119CV * T + 26.31493582
\end{aligned}
$$

- Polynomial model on Current Amount prediction

$$
\begin{aligned}
CurrentValue = 1.0374FV - 0.16033T - 0.000056FV^2 + 0.00010T^2 \\
- 0.000003FV * T - 116.5419
\end{aligned}
$$

- Polynomial model on time-gap prediction

$$
\begin{aligned}
Timegap = -1.507CV + 0.00635FV + 0.00106CV^2 + 0.0000025FV^2 \\
- 0.00000008CV * FV + 445.2229
\end{aligned}
$$

Although some parameters are close to zero coefficients, the polynomials do not imply useful expressions. Along similar lines, polynomial regression overfits the training data and performs poorly on the full data.
gplearn symbolic models without Loss/Gain feature is:

- gplearn model on Future amount prediction: The expression could not be simplified
- gplearn model on Current amount prediction

$$
\begin{aligned}
CurrentValue = 0.79432955FutureValue + timegap * 0.067522FutureValue \\
0.032613126timegap
\end{aligned}
$$

- gplean model on timegap prediction
  Timegap=

$$
\frac{1}{-0.371CurrentValueFutureValue\left(-\frac{1.2987012987013CurrentValue(0.378CurrentValue-0.378FutureValue)(FutureValue-0.808742112482853)}{FutureValue(0.951-CurrentValue)\left(-2CurrentValue+FutureValue+0.984-\frac{0.149}{CurrentValue}\right)}+0.563\right)\left(\frac{0.202}{FutureValue+0.72}+\frac{1}{CurrentValue(FutureValue+0.88)}\right)+\frac{1}{CurrentValue^2}}
$$

With two of the predictions being complicated, the expressions could not do any better than with the Loss/Gain feature.
Although the expressions seem simpler, the RMSE comparison of the three models remains the same or somewhat worse than with the Loss/Gain feature. The only difference was that the current amount prediction got better compared to 8. This is shown in 11.
This experiment may not be as valuable as loss/gain might be an essential feature for decision-making.

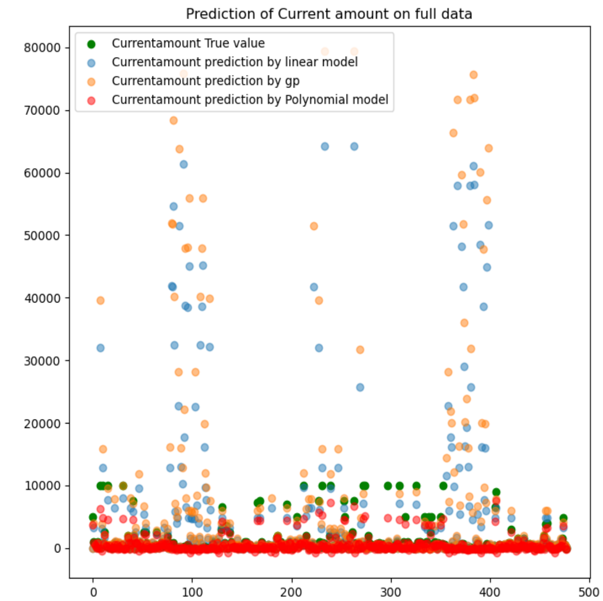


Figure 11: Polynomial model performance on current data

4 Conclusion and Future work

Making any definite conclusions is challenging with the data available, and even if attempted, the conclusions might *not* be accurate. The end-to-end process of the project is achieved by collecting the data, cleaning the data, analyzing the data, fitting the data, and performance validation. Capturing human behavior mathematically is a significant challenge, and this is just the first step to using better regression models to understand human discounting behavior. For future work, the first step would be to obtain more data and continue the same experiments to make conclusions. Secondly, we could analyze how base time matters to the choice by keeping basetime and timegap in the analysis.

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