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Class- B.Tech, ECE, A2

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Math Assignment - 3

Q.1. Find the fourier series to represent e^{ax} in the interval $-\pi < x < \pi$.

Soln: $F(x) = e^{ax} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} dx = \frac{1}{\pi} \frac{e^{ax}}{a} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{\pi a} e^{a\pi} - e^{-a\pi}$$

$$= \frac{2 \sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \cos nx \, dx$$

$$= \frac{1}{\pi} \frac{e^{ax}}{a^2+n^2} (a \cos nx + n \sin nx)_{-\pi}^{\pi}$$

$$= \frac{2a \cos n\pi}{\pi(a^2+n^2)} (e^{a\pi} - e^{-a\pi})$$

$$= \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{ax} \sin nx \, dx$$

$$= \frac{1}{\pi} \frac{e^{ax}}{a^2+n^2} (a \sin nx - n \cos nx)_{-\pi}^{\pi}$$

$$= \frac{2n(-1)^n\left(e^{a\pi}-e^{-a\pi}\right)}{\pi(a^2+n^2)}$$

$$= \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}$$

$$\therefore e^{ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \cos nx$$

$$+ \sum_{n=1}^{\infty} \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)} \sin nx$$

Q.2. Expand the function $\sin ax$, $-\pi < x < \pi$. as a fourier series given by $\sin ax$

$$= \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \dots\right)$$

Soln:- $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin ax \sin nx \, dx$

$$= \frac{2}{\pi} \times \frac{1}{2}\int_0^{\pi} [\cos(n-a)x - \cos(n+a)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n-a)x}{n-a} - \frac{\sin(n+a)x}{n+a}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-\sin a\pi)}{n-a} - \frac{(-1)^n \sin a\pi}{n+a}\right]$$

$$= \frac{(-1)^n}{\pi} \sin a\pi \left[\frac{1}{n-a} + \frac{1}{n+a}\right]$$

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$$= \frac{(-1)^{n+1}}{\pi} \sin a\pi \left(\frac{n+a+n-a}{n^2-a^2} \right)$$

$$= (-1)^{n+1} \frac{2n \sin a\pi}{\pi (n^2-a^2)}$$

$$\sin ax = \frac{2 \sin a\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin nx}{n^2-a^2}$$

$$\sin ax = \frac{2 \sin a\pi}{\pi} \left[\frac{\sin x}{1^2-a^2} - \frac{2 \sin 2x}{2^2-a^2} + \frac{3 \sin 3x}{3^2-a^2} - \cdots \right]$$

Q.3. Expand $\pi x - x^2$ in a half range sine series in the interval $(0, \pi)$ upto first three terms.

Soln:- $\pi x - x^2 = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} (\pi x - x^2) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right) \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-2\frac{\cos n\pi}{n^3} + \frac{2}{n^3}\right]$$

$$= \frac{4}{\pi n^3}\left[1-(-1)^n\right]$$

$$= \frac{8}{\pi n^3}\left[n \text{ is odd}\right]$$

$$= 0 \left[n \text{ is even}\right]$$

Q.4. If $f(x) = \begin{cases} \sin x & \text{for } x \quad 0 \le x \le \pi/4 \\ \cos x & \text{for } \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$

Expand $f(x)$ as half range sine series.

Soln:-

$$b_n = \frac{2\times 2}{\pi}\left[\int_0^{\pi/4} (\sin nx \sin nx)\,dx + \int_{\pi/4}^{\pi/2} \cos x \sin x\,dx\right.$$

$$= \frac{4}{\pi}\times\frac{1}{2}\left[\int_0^{\pi/4} (\cos(1-n)x - \cos(1+n)x)\,dx\right.$$

$$\left.+ \int_{\pi/4}^{\pi/2} \sin(1+n)x - \sin(1-n)x\,dx\right]$$

$$= \frac{2}{\pi}\left\{\left[\sin\frac{(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi/4}\right.$$

$$\left.+\left[\frac{-\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_{\pi/4}^{\pi/2}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\sin(1-n)\pi/4}{1-n} - \frac{\sin(1+n)\pi/4}{1+n} + \frac{\cos(1+n)\pi/4}{1+n} - \frac{\cos(1-n)\pi/4}{1-n}\right\}$$

Q.5. Find the area outside the circle $r=2$ & inside $r=4\sin\theta$

Soln:- For point of intersection,

$$2 = 4\sin\theta$$

$$\sin\theta = \frac{1}{2}\ ;\ \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

r varies 2 to $4\sin\theta$

θ varies $\frac{\pi}{6}$ to $\frac{5\pi}{6}$

$\therefore$ Required area $\Rightarrow$ $$\int_{\pi/6}^{5\pi/6}\int_{2}^{4\sin\theta} r\,dr\,d\theta = \frac{1}{2}\int_{\pi/6}^{5\pi/6}(16\sin^2\theta - 4)\,d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}\left(\cancel{16}^{8}\frac{(1-\cos 2\theta)}{\cancel{2}} - 4\right)d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6}(4 - 8\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\left[4\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) - \frac{8\sin 2\theta}{2}\Big|_{\pi/6}^{5\pi/6}\right]$$

$$= \frac{1}{2}\left[\frac{8\pi}{3} - 4\left(\sin\frac{10\pi}{6} - \sin\frac{\pi}{3}\right)\right]$$

$$= \frac{1}{2}\left[\frac{8\pi}{3} - 4\left(\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{4\pi}{3} \quad \underline{\text{Ans.}}$$

Q.6. Find the area outside the circle $r=2$ and inside cardiod $r = 2(1+\cos\theta)$.

Soln:- Point of intersection are

$$2 = 2(1+\cos\theta)$$

$$\therefore \cos\theta = 0 \quad \therefore \theta = \pi/2, -\pi/2$$

r varies a to $a(1+\cos\theta)$

$\therefore$ required area

$$= \int_{-\pi/2}^{\pi/2} \int_{a}^{a(1+\cos\theta)} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left[a^2(1+\cos\theta)^2 - a^2\right] d\theta$$

$$= a^2 \int_{0}^{\pi/2} (\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= a^2 \int_0^{\pi/2} \left[\frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$$

$$= a^2 \left[\frac{1}{2}\theta + \frac{\sin 2\theta}{4} + 2\sin\theta\right]_0^{\pi/2}$$

$$= a^2\left[\frac{\pi}{4} + 2\right] = \frac{a^2}{4}(8+\pi) \text{ Ans.}$$

Q.7. Evaluate $\int_0^{\pi}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos^2\theta \, dz\, dr\, d\theta$

Soln: $\int_0^{\pi}\int_0^{\sin\theta}\int_0^{r\sin\theta} r\cos^2\theta \, dz\, dr\, d\theta$

$$= \int_0^{\pi}\int_0^{\sin\theta} z\Big|_0^{r\sin\theta} r\cos^2\theta \, dr\, d\theta$$

$$= \int_0^{\pi}\int_0^{\sin\theta} r^2 \sin\theta\cos^2\theta \, dr\, d\theta$$

$$= \int_0^{\pi} \frac{r^3}{3}\Big|_0^{\sin\theta} \sin\theta\cos^2\theta \, d\theta$$

$$= \int_0^{\pi} \frac{\sin^3\theta}{3} \sin\theta\cos^2\theta \, d\theta = \int_0^{\pi} \frac{\sin^4\theta}{3}\cos^2\theta \, d\theta$$

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$$= \frac{1}{12}\int_0^{\pi} (\sin^2 2\theta)\sin^2\theta \, d\theta$$

$$= \frac{1}{12}\int_0^{\pi} \left(\frac{1-\cos 4\theta}{2}\right)\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{48}\int_0^{\pi} (1-\cos 2\theta - \cos 4\theta + \cos 4\theta \cos 2\theta)\, d\theta$$

$$= \frac{1}{48}\int_0^{\pi} 1-\cos 2\theta - \cos 4\theta + \frac{1}{2}[\cos 6\theta + \cos 2\theta] \, d\theta$$

$$= \frac{1}{48}\left[\theta - \frac{\sin 2\theta}{2} - \frac{\sin 4\theta}{4} + \frac{\sin 6\theta}{12} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \frac{\pi}{48} = \frac{\pi}{48} \quad \underline{\text{Ans}}$$

Q. 8. Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dx\, dy$ by changing the order of integration.

Soln:- y varies from 0 to $\sqrt{2x-x^2}$ and x varies from 0 to 2

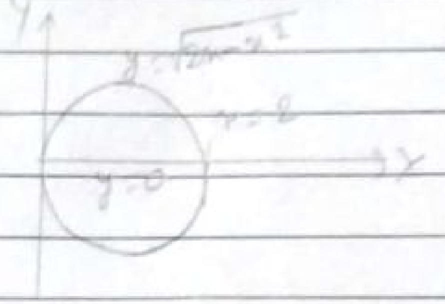
$$y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2$$

$$\Rightarrow x^2 + y^2 - 2x = 0$$

We have

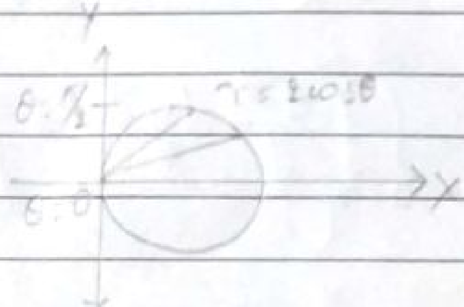
$r = 2\cos\theta$ and θ varies from 0 to $\pi/2$

replacing $x \to r\cos\theta$
$y \to r\sin\theta$
$dy\,dx \to r\,d\theta\,dr$



$$I = \int_0^{\pi/2}\int_0^{2\cos\theta} \frac{r\cos\theta \cdot r\,dr\,d\theta}{r}$$

$$I = \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^{2\cos\theta} \cos\theta\,d\theta$$



$$I = 2\int_0^{\pi/2} \cos^3\theta\,d\theta$$

$$= 2 \times \frac{2}{3} = \frac{4}{3} \text{ Ans}$$

Q.9. Evaluate by changing the order of integration

$$\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$$

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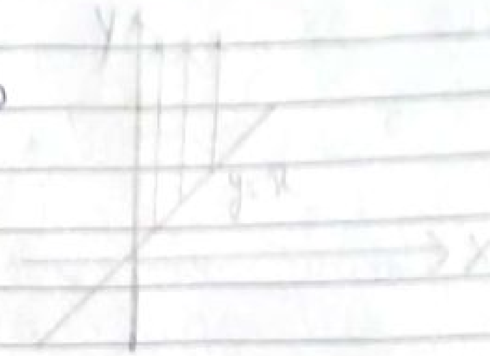
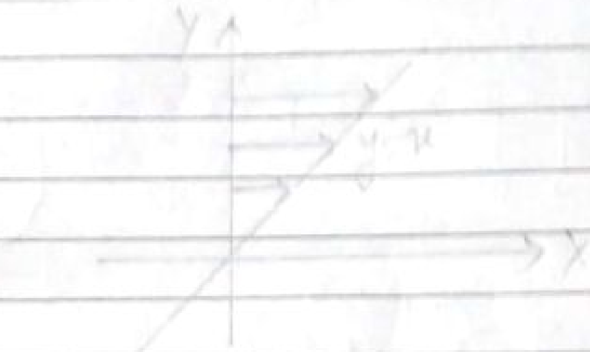
For horizontal strip
x varies 0 to y
y varies 0 to ∞





$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$\Rightarrow \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \frac{e^{-y}}{y} \cdot x \Big|_0^y dy = \int_0^\infty e^{-y}\, dy$$

$$= -e^{-y} \Big|_0^\infty$$

$$= 1 \text{ Ans.}$$

10 Evaluate $\iint_R \sin\pi(x^2+y^2)\,dx\,dy$ over the circle

$x^2+y^2 \le 1$ by the change of variable.

Soln: r varies from -1 to 1
θ varies from 0 to π

$$\therefore \int_0^\pi \int_{-1}^{1} \sin\pi r^2 \cdot r\,dr\,d\theta$$

$$= 2\int_0^\pi \left[\sin\pi \frac{r^2}{4}\right]_0^1 d\theta$$

$$= \frac{2}{4}\int_0^\pi \sin\pi\,d\theta$$

$$= \frac{1}{2} \cdot \frac{\cos\pi}{\pi}(\pi - 0)$$

$$= \frac{1}{2}$$ Ans.