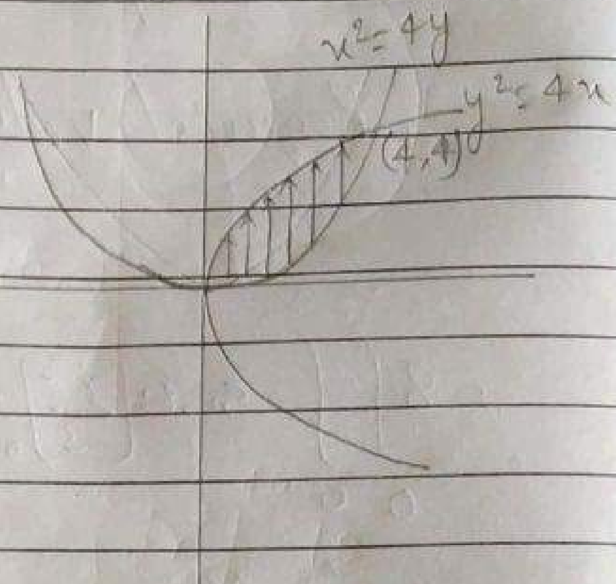
Q1. Evaluate $\iint_R y\,dx\,dy$, where R is a region bounded by parabalas $x^2 = 4y$ and $y^2 = 4x$.

Soln:- $x^2 = 4y$

$y^2 = 4x$

Intersecting points $= (4,4)$



$\therefore$ x varies from 0 to 4

y varies from $x^2/4$ to $\sqrt{4x}$

$$\therefore \iint_R y\,dx\,dy = \int_0^4 \int_{x^2/4}^{\sqrt{4x}} y\,dy\,dx$$

$$= \int_0^4 \left.\frac{y^2}{2}\right|_{x^2/4}^{\sqrt{4x}} dx = \int_0^4 \left(\frac{4x}{2} - \frac{x^4}{32}\right) dx$$

$$= \int_0^4 \left(2x - \frac{x^4}{32}\right) dx = \left[\cancel{2}\frac{x^2}{\cancel{2}} - \frac{x^5}{32\cdot 5}\right]_0^4$$

$$= \left[16 - \frac{32}{5}\right] = \frac{80-32}{5} = \frac{48}{5} \text{ Ans.}$$

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Q2. Evaluate $\int_0^1\int_0^x e^{y/x}\,dy\,dx$.

Soln: $\int_0^1\int_0^x e^{y/x}\,dy\,dx$

put $y/x = t$

$\frac{1}{x}\,dy = dt$

$dy = x\,dt$

$\therefore \int e^{y/x}\,dy = x\int e^t dt = x.e^t = x.e^{y/x}$

$\therefore \int_0^1\int_0^x e^{y/x}\,dy\,dx = \int_0^1 x.e^{y/x}\Big|_0^x dx$

$= \int_0^1 (xe - x)\,dx = \int_0^1 x(e-1)\,dx$

$= (e-1)\frac{x^2}{2}\Big|_0^1 = \frac{1}{2}(e-1)$ Ans.

Q.3. Evaluate $\iint r^3\,dr\,d\theta$ over the area bounded by the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.

Soln:- $\rightarrow r = 2\cos\theta$

Here centre $= (\frac{2}{2}, 0) = (1, 0)$ and radius $= 1$

$\rightarrow r = 4\cos\theta$

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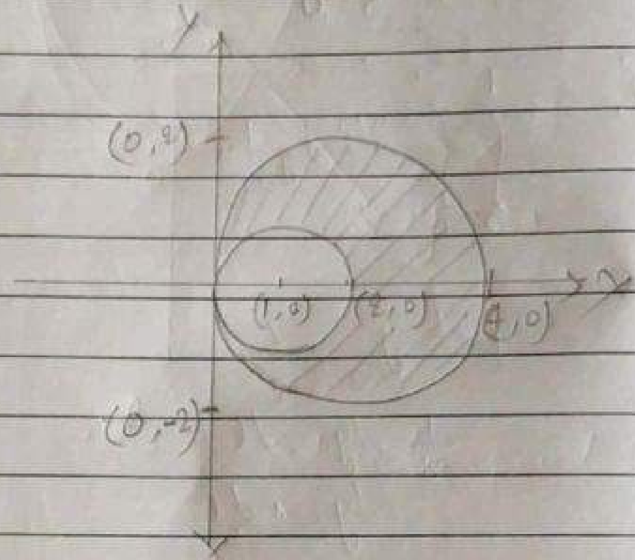
Here centre $= \left(\frac{4}{2}, 0\right) = (2,0)$ & radius $= 2$

Here area is shaded part.



Here r varies from $2\cos\theta$ to $4\cos\theta$ and θ varies from $-\pi/2$ to $\pi/2$.

$$\therefore \text{Area} = \int_{-\pi/2}^{\pi/2}\int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big|_{2\cos\theta}^{4\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left[\frac{(4\cos\theta)^4}{4} - \frac{(2\cos\theta)^4}{4}\right] d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 60\cos^4\theta\,d\theta = 60\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta$$

$$= 60\left[\frac{\sin\theta\cos^3\theta}{4} + \frac{3}{8}(\cos\theta\sin\theta) + \frac{3}{8}\theta\right]_{-\pi/2}^{\pi/2}$$

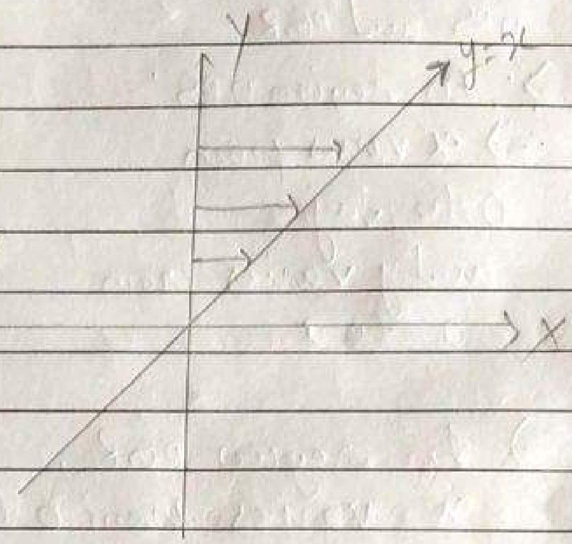
$$= 60\left(\frac{3}{8}\cdot\frac{\pi}{2}\right) - 60\left(\frac{3}{8}\cdot\frac{-\pi}{2}\right) = \frac{90\pi}{8} + \frac{90\pi}{8}$$

$$= \frac{45}{2}\pi \text{ Ans.}$$

Q.4. Changing order of integration, evaluate the following integrals.

(i) $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx$

Soln:- By changing the order of integration,



x varies from 0 to y
& y varies from 0 to ∞

$$\therefore \int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\, dy\, dx = \int_0^{\infty}\int_0^{y} \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^{\infty} \frac{e^{-y}}{y}[x]_0^y\, dy = \int_0^{\infty} e^{-y}\, dy = -e^{-y}\Big|_0^{\infty}$$

$$= 0 + 1 = 1 \text{ Ans.}$$

(ii) $\int_0^{a}\int_{x^2/a}^{2a-x} xy\, dy\, dx$

Soln:- Here $y = x^2/a \Rightarrow x^2 = ya$
& $y = 2a - x \Rightarrow x + y = 2a$

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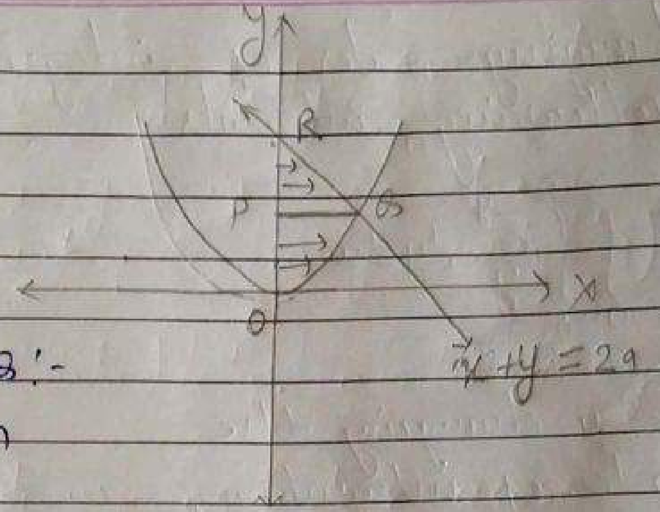
Now Region is divided into two parts i.e OPQ and PQR



⟹ ∴ In region OPQ :-
⟹ x varies from 0 to $\sqrt{ay}$
and y varies from 0 to a (∵ Intersection point = (a,a))

⟹ In region PQR,
x varies from 0 to $2a-y$
& y varies from a to $2a$

$$\therefore \int_0^a\int_{x^2/a}^{2a-x} xy\,dy\,dx = \int_0^a\int_0^{\sqrt{ay}} xy\,dx\,dy + \int_a^{2a}\int_0^{(2a-y)} xy\,\cancel{dx}\,dy$$

$$= \int_0^a y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy + \int_a^{2a} y\cdot\frac{x^2}{2}\Big|_0^{2a-y} dy$$

$$= \int_0^a \frac{y\cdot ay}{2}\,dy + \int_a^{2a} \frac{y\cdot(2a-y)^2}{2}\,dy$$

$$= \frac{a}{2}\int_0^a y^2\,dy + \int_a^{2a} \frac{(4a^2y + y^3 - 4ay^2)}{2}\,dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^a + \left[2a^2\cdot\frac{y^2}{2} + \frac{y^4}{8} - 2a\cdot\frac{y^3}{3}\right]_0^{2a}$$

$$= \frac{a}{2}\cdot\frac{a^3}{3} + \left[\cancel{2}a^2\cdot\frac{4a^2}{\cancel{2}} + \frac{{}^{2}\cancel{16}a^4}{\cancel{8}} - \frac{2a\cdot 8a^3}{3}\right]$$

$$= \frac{a^4}{6} + 4a^4 + 2a^4 - \frac{16}{3}a^4$$

$$= \frac{a^4+24a^4+12a^4-32a^4}{6} = \frac{5a^4}{6}$$ Ans.

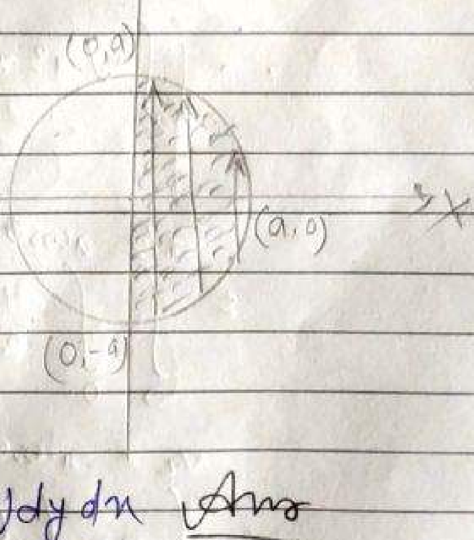
(iii) $\displaystyle\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

Soln:- Here x varies from 0 to $\sqrt{a^2-y^2}$

$\Rightarrow x = \sqrt{a^2-y^2} \Rightarrow x^2 = a^2-y^2 \Rightarrow x^2+y^2=a^2$

y varies from $-a$ to a.

So, region is shaded part.



Here x varies from 0 to a
and y varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

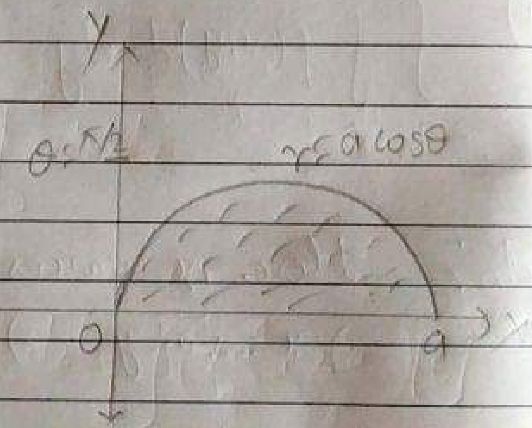
$$\therefore \int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy = \int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$ Ans

Q.5. Evaluate $\iint \sqrt{a^2-x^2-y^2}\,dx\,dy$ over the semi circle $x^2+y^2=ax$ in the +ve quadrant. (change variables).

Soln:- Changing into polar co-ordinate system,

$x^2+y^2=ax$ transforms into $r=a\cos\theta$

For the region of integration R,
r varies from 0 to $a\cos\theta$
& θ varies from 0 to $\pi/2$



$$\therefore \iint_R \sqrt{a^2-x^2-y^2}\,dx\,dy = \int_0^{\pi/2}\int_0^{a\cos\theta} \sqrt{a^2-r^2}\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\int_0^{a\cos\theta} -\frac{1}{2}(a^2-r^2)^{1/2}\cdot(-2r)\,dr\,d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}\int_0^{a\cos\theta} t^{1/2}\,dt\,d\theta \qquad (\text{putting } a^2-r^2=t)$$

$$= \int_0^{\pi/2} -\frac{1}{2}\left[\frac{t^{3/2}}{3/2}\right]_0^{a\cos\theta} d\theta$$

$$= \int_0^{\pi/2} -\frac{\cancel{2}}{\cancel{2}\cdot 3}\left[a^2-r^2\right]^{3/2}\Big|_0^{a\cos\theta} d\theta$$

$$= -\frac{1}{3}\int_0^{\pi/2} (a^3\sin^3\theta - a^3)\,d\theta = -\frac{a^3}{3}\int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta$$

$$= -\frac{a^3}{3}\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{a^3}{3}\left(\frac{\pi}{2}-\frac{2}{3}\right) \text{ Ans}$$

$$\left\{ \because \int_0^{\pi/2}(\sin^3\theta - 1)\,d\theta = \left(\frac{1}{3}\cos^3\theta - \cos\theta - \theta\right)\Big|_0^{\pi/2} = \left[\frac{2}{3}-\frac{\pi}{2}\right] \right\}$$

Q.6. Evaluate $\iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$ through the volume of the cylinder $x^2+y^2=a^2$ intercepted by the plane $z=0$ & $z=h$.

Soln:- $x^2+y^2=a^2=r^2$

$\therefore r=a$

putting the value of $x=r\cos\theta$ & $y=r\sin\theta$ and replacing $dx\,dy\,dz = r\,dr\,d\theta\,dz$

Here r varies from 0 to a
θ varies from 0 to 2π
z varies from 0 to h

$$\therefore \iiint z(x^2+y^2+z^2)\,dx\,dy\,dz$$

$$= \int_0^h \int_0^{2\pi} \int_0^a z(r^2+z^2)\,r\,dr\,d\theta\,dz$$

$$= \int_0^h \int_0^{2\pi} \int_0^a (zr^3+z^3r)\,dr\,d\theta\,dz$$

$$= \int_0^h \int_0^{2\pi} z\frac{r^4}{4}+z^3\frac{r^2}{2}\Big|_0^a d\theta\,dz$$

$$= \int_0^h \int_0^{2\pi} \left(\frac{za^4}{4}+\frac{z^3a^2}{2}\right)d\theta\,dz = \int_0^h \left(\frac{za^4}{4}+\frac{z^3a^2}{2}\right)[\theta]_0^{2\pi}\,dz$$

$$= \int_0^h 2\pi\left[\frac{a^4}{4}z+\frac{a^2}{2}z^3\right]dz$$

$$= \left[\frac{\cancel{2}\pi a^4}{4}\cdot\frac{z^2}{\cancel{2}} + \frac{\cancel{2}\pi a^2}{\cancel{2}}\cdot\frac{z^4}{4}\right]_0^h$$

$$= \frac{\pi a^4h^2}{4}+\frac{\pi a^2h^4}{4} = \frac{\pi a^2h^2}{4}(a^2+h^2) \text{ Ans.}$$

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Q.7. Evaluate through volume of the sphere $x^2+y^2+z^2=1$, by changing it into spherical polar co-ordinates;

the integration is $\iiint z^2 dx dy dz$

Soln: Changing to spherical polar co-ordinates by putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2+y^2+z^2 = r^2 \rightarrow (i)$ (polar co-ordinates)

$\Rightarrow x^2+y^2+z^2=1 \rightarrow (ii)$ (cartesian co-ordinates)

comparing (i) & (ii)

$r^2 = 1 \Rightarrow r = 1$

On account of symmetry, the required volume is 8 times the volume of the sphere in the positive octate for which r varies from 0 to 1, θ varies from 0 to $\pi/2$ and ϕ varies from 0 to $\pi/2$.

$\therefore$ Required volume $= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^2\sin\theta . z^2 \, dr\, d\theta\, d\phi$

$= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^4\sin\theta . \cos^2\theta \, dr\, d\theta\, d\phi$

$= 8\int_0^{\pi/2}\int_0^{\pi/2}\left[\sin\theta\cos^2\theta \frac{r^5}{5}\right]_0^1 d\theta\, d\phi = 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{\cos^2\theta\sin\theta}{5} d\theta\, d\phi$

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Let $\cos\theta = t$

$$-\sin\theta \, d\theta = dt$$

$$d\theta = \frac{dt}{-\sin\theta}$$

$$\therefore 8\int_0^{\pi/2} \frac{-1}{5} \int_0^{\pi/2} t^2 \, dt \, dz = 8\int_0^{\pi/2} \frac{-1}{5} \cdot \frac{t^3}{3} \Big|_0^{\pi/2} dz$$

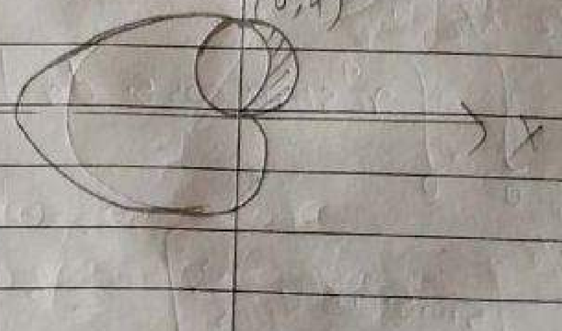
$$= 8\int_0^{\pi/2} \frac{-\cos^3\theta}{15} \Big|_0^{\pi/2} d\phi$$

$$= 8\int_0^{\pi/2} \frac{1}{15} \, d\phi \Rightarrow \frac{8}{15}\left[\phi\right]_0^{\pi/2}$$

$$= \frac{8}{15} \cdot \frac{\pi}{2} = \frac{4\pi}{15} \text{ Ans.}$$

Q.8. Find the area lying inside the circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$.

Soln:-



Eqn. of circle $r = a\sin\theta$

$\therefore$ centre of circle $= \left(0, \frac{a}{2}\right)$ and radius $= \frac{a}{2}$

cardiald $r = a(1-\cos\theta)$

$\Rightarrow \quad r = a\sin\theta \rightarrow (i)$

$r = a(1-\cos\theta) \rightarrow (ii)$

$\Rightarrow \; a\sin\theta = a(1-\cos\theta)$

$\sin\theta + \cos\theta = a$

Squiring both side

$1 + 2\sin\theta\cos\theta = 1$

$\Rightarrow \sin 2\theta = 0$

$2\theta = \sin^{-1} 0$

$2\theta = 0$ or π

$\theta = 0$ or $\pi/2$

$\therefore$ r varies from $a(1-\cos\theta)$ to $a\sin\theta$

& θ varies from 0 to $\pi/2$

$$\therefore \text{Area} = \int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^2}{2}\Bigg|_{a(1-\cos\theta)}^{a\sin\theta} d\theta = \int_0^{\pi/2} \frac{a^2\sin^2\theta - a^2(1-\cos\theta)^2}{2}\, d\theta$$

$$= \int_0^{\pi/2} \frac{a^2}{2}\left(\sin^2\theta - 1 - \cos^2\theta + 2\cos\theta\right) d\theta$$

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$$= \frac{a^2}{2}\int_0^{\pi/2}\left(-2\cos^2\theta + 2\cos\theta\right)\Big|_0^{\pi/2} d\theta$$

$$= a^2\int_0^{\pi/2}\left(-\cos^2\theta + \cos\theta\right) d\theta$$

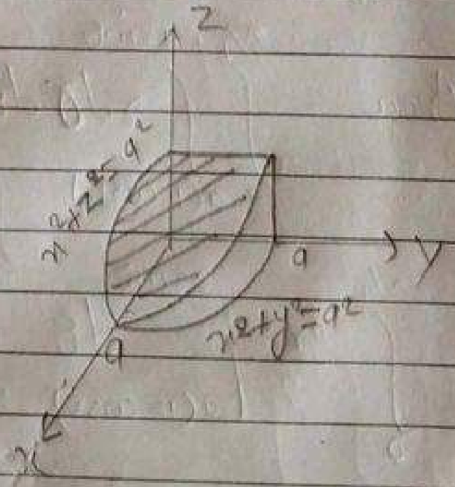
$$= a^2\left[-\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \sin\theta\right]_0^{\pi/2}$$

$$= a^2\left[-\frac{\pi}{4} + 1\right]$$

$$= a^2\left[1 - \frac{\pi}{4}\right]$$ Ans

Q.9. Find the volume common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$.

Soln:- Total volume
$= 8 \times$ (volume of positive quadrant)



Here x varies from 0 to a
y varies from 0 to $\sqrt{a^2-x^2}$

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$\therefore$ Required volume $= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} z \, dy \, dx$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} \, dy \, dx$$

$$= 8\int_0^a \sqrt{a^2-x^2} \left[y\right]_0^{\sqrt{a^2-x^2}} dx = 8\int_0^a (a^2-x^2) \, dx$$

$$= 8\left[a^2x - \frac{x^3}{3}\right]_0^a = 8\left[a^3 - \frac{a^3}{3}\right]$$

$$= 8\left[a^3 - \frac{a^3}{3}\right] = \frac{24a^3 - 8a^3}{3} = \frac{16a^3}{3} \text{ Ans.}$$

Q.10. Find the volume of sphere of radius a by triple integration.

Soln:- Here r varies from 0 to a
θ varies from 0 to $\pi/2$ $\Big\}$ In positive quadrant.
ϕ varies from 0 to $\pi/2$

total volume of sphere $= 8 \times$ (volume of positive quadrant)

$\therefore$ Required volume $= 8\int_0^{\pi/2}\int_0^{\pi/2}\int_0^a r^2 \sin\theta \, dr \, d\theta \, d\phi$

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$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{r^3}{3}\right]_0^a d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\int_0^{\pi/2} \frac{a^3}{3}\sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\cdot\frac{a^3}{3}\int_0^{\pi/2}\sin\theta\, d\theta\, d\phi$$

$$= 8\int_0^{\pi/2}\frac{a^3}{3}\cdot\left[-\cos\theta\right]_0^{\pi/2}$$

$$= 8\int_0^{\pi/2}\frac{a^3}{3}\, d\phi$$

$$= \frac{8a^3}{3}\left[\phi\right]_0^{\pi/2} = \frac{\cancel{8}^{4}a^3}{3}\cdot\frac{\pi}{\cancel{2}} = \frac{4a^3}{3}\pi \text{ Ans}$$

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