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Chapter :- Vector Calculus

Ques: Find the directional derivative of $F = 2xy + Z^2$ at point $(1, -1, 3)$ in the directio of $\hat{i} + 2\hat{j} + 3\hat{k}$.

Solu: Given $F = 2xy + z^2$, point $P(x,y,z) = (1,-1,3)$

$$\vec{N} = \hat{i} + 2\hat{j} + 3\hat{k}$$

Direction derivative $= (\text{grad} F)_p \cdot \frac{\vec{N}}{|\vec{N}|}$

$$\text{grad} F = \nabla F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$$

$$\Rightarrow \hat{i}\frac{\partial(2xy+z^2)}{\partial x} + \hat{j}\frac{\partial(2xy+z^2)}{\partial y} + \hat{k}\frac{\partial(2xy+z^2)}{\partial z}$$

$$\nabla F = \hat{i}(2y) + \hat{j}(2x) + \hat{k}(2z)$$

$$(\text{grad} f)_p = i(2(-1)) + \hat{j}(2(1)) + \hat{k}(2\times 3)$$

$$= -2\hat{i} + 2\hat{j} + 6\hat{k}$$

Directional dirivative $= \frac{(-2\hat{i}+2\hat{j}+6\hat{k}) \cdot (\hat{i}+2\hat{j}+3\hat{k})}{|\hat{i}+2\hat{j}+3\hat{k}|}$

$$= \frac{(-2+4+18)}{\sqrt{1^2+2^2+3^2}} \Rightarrow \frac{20}{\sqrt{1+13}}$$

$$\Rightarrow \frac{20}{\sqrt{14}}$$

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Q→4) Find unit normal vector of surface $x^2y + 2xz = 4$ at $(2,-2,3)$

Sol→ Given Surface $f = x^2y + 2xz$

Now normal vector to $f = \nabla f = \text{grad} f$

$$\text{grad} f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\frac{\partial(x^2y+2xz)}{\partial x} + \hat{j}\frac{\partial(x^2y+2xz)}{\partial y} + \hat{k}\frac{\partial(x^2y+2xz)}{\partial z}$$

$$= \hat{i}(2xy+2z) + \hat{j}(x^2) + \hat{k}(2x)$$

Normal vector at point $(2,-2,3) = (\text{grad} f)_{(2,-2,3)}$

$$= \hat{i}(2\times 2\times -2 + 2\times 3) + \hat{j}(2^2) + \hat{k}(2\times 2)$$

$$= \hat{i}(-8+6) + \hat{j}(4) + \hat{k}(4)$$

$$= -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Unit normal vector $= \frac{\text{grad} f}{|\text{grad} f|}$

$$= \frac{-2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{(-2)^2+(4)^2+4^2}}$$

$$= \frac{-2\hat{i}+4\hat{j}+4\hat{k}}{\sqrt{4+16+16}} \Rightarrow \frac{2(-\hat{i}+2\hat{j}+2\hat{k})}{\sqrt{36}}$$

$$\Rightarrow \frac{-i+2j+2k}{3}$$

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Q35) Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ at $(1,-1,1)$

Sol→ $\text{Div } \vec{F} = \nabla \cdot \vec{F}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(2x^2yz) - \frac{\partial}{\partial z}(3yz^2)$$

$$= y^2 + 2x^2z - 6yz$$

$$\Rightarrow (\text{div } \vec{F})_{(1,-1,1)} = (-1)^2 + 2(1)^2\times 1 - 6(-1)\times 1$$

$$= 1 + 2 + 6$$

$$= 9 \text{ Ans}$$

$$\text{Curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-3yz^2) - \frac{\partial}{\partial z}(2x^2yz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-3yz^2) - \frac{\partial}{\partial z}(xy^2)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(2x^2yz) - \frac{\partial}{\partial y}(xy^2)\right]$$

$$= \hat{i}\left((-3z^2) - (2x^2y)\right) - \hat{j}(0) + \hat{k}\left((4xyz) - (2xy)\right)$$

$$= \hat{i}(-2x^2y - 3z^2) + \hat{k}(4xyz - 2xy)$$

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$$(\text{Curl } \vec{F})_{(1,-1,1)} = \hat{i}(-2\times1\times1 - 3\times1) + \hat{k}(4\times\cancel{1}\times -1\times1 - 2\times1\times1$$

$$= \hat{i}(-1) + \hat{k}(-2)$$

$$= -\hat{i} - 2\hat{k}$$

Q:21 Find the magnitude of gradient of a function

$$u = \frac{x^2}{2} + \frac{y^2}{3} \quad \text{at } (1,3)$$

Sol: gradint of $u = \nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}x\hat{i} + \frac{1}{3} 2y\hat{j}$$

$$= x\hat{i} + \frac{2}{3}y\hat{j}$$

$$|\nabla u| = \text{magnitude} = \sqrt{x^2 + \left(\frac{2}{3}y\right)^2}$$

$$\text{magnitude at point } (1,3) = \sqrt{1^2 + \left(\frac{2}{\cancel{3}}\cdot\cancel{3}\right)^2}$$

$$= \sqrt{1+2^2} = \sqrt{5} \quad \text{Ans}$$

Q:6 Evaluate $\iint_S \vec{F}\cdot\vec{n}\, ds$ where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$ and S is closed surface bounded by plane $z=0$ and $z=1$ and cylinder $x^2+y^2=4$. Also Verify Gauss divergence theorem

Sol: A vector normal to the surface S is:

$$\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$$

$\hat{n}$ .. a unit vector normal to S.

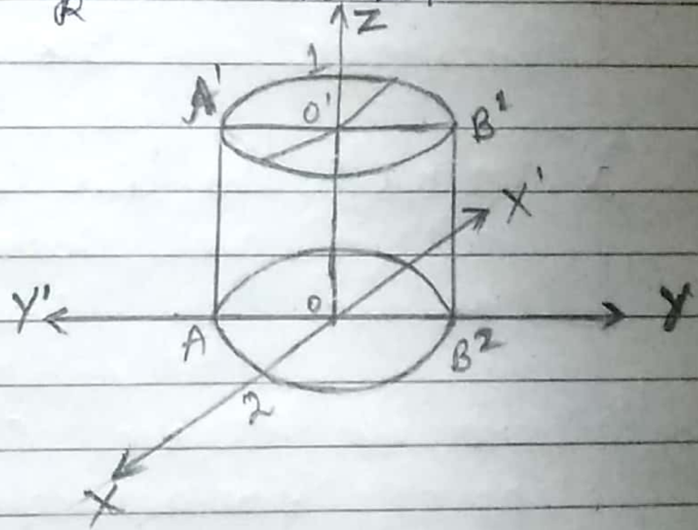
$$= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{2x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{4}}$$

$$= \frac{x\hat{i}+y\hat{j}}{2}$$

Let R be the projection of S on YZ plane

then $\iint_S \vec{F}.\hat{n}\,ds = \iint_R \vec{F}.\hat{n}\,\frac{dy\,dz}{|\hat{i}.\hat{n}|}$

where R is ABB'A' which is enclosed by $y=-2$ to $y=2$ and $z=0$ to $z=1$



$$\hat{i}.\hat{n} = \hat{i}.\frac{x\hat{i}+y\hat{j}}{2}$$

$$= \frac{x}{2}$$

$$\vec{F}.\vec{n} = (x\hat{i} - y\hat{j} + (z^2-1)\vec{k}).\frac{x\hat{i}+y\hat{j}}{2}$$

$$= \frac{x^2-y^2}{2} \Rightarrow \frac{x^2-y^2}{2}$$

$$\therefore \iint_S \vec{F}.\vec{n}\,ds = \iint_R \vec{F}.\vec{n}\,\frac{dy\,dz}{|\hat{i}.\vec{n}|}$$

$$= \iint_R \left(\frac{x^2-y^2}{2}\right).\frac{dy\,dz}{\frac{x}{2}} = \iint_R \frac{x^2-y^2}{x}\,dy\,dz$$

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$$= \int_0^1 \int_{-2}^{2} \frac{x^2 - y^2}{x} \, dy \, dz = \int_0^1 \left[x.y - \frac{y^3}{3x} \right]_{-2}^{2} dz$$

$$= \int_0^1 \left[\left(x.2 - \frac{8}{3x} \right) - \left(x.-2 \right) - \frac{(-8)}{3x} \right) \right] dz$$

$$= \int_0^1 \left(4x - \frac{16}{3x} \right) dz = \left[4x.z = \frac{16}{3x} .z \right]_0^1$$

$$= 4x - \frac{16}{3x} - (0-0) = 4x - \frac{16}{3x} \quad \underline{\text{Ans}}$$

Q:> 3) Evaluate the line integeral $\int_c (x^2+xy)dx + (x^2+y^2)dy$ where c is the square formed by lines $x = \pm 1$, $y = \pm 1$

Sol:> Here in square c, x varies from -1 to 1 and y varies from -1 to 1.

$$\Rightarrow \int_c (x^2+xy)dx + (x^2+y^2)dy$$

$$\Rightarrow \int_{-1}^{1} (x^2+xy)dx + \int_{-1}^{1} (x^2+y^2)\,dy$$

$$= \left[\frac{x^3}{3} + y.\frac{x^2}{2} \right]_{-1}^{1} + \left[x^2.y + \frac{y^3}{3} \right]_{-1}^{1}$$

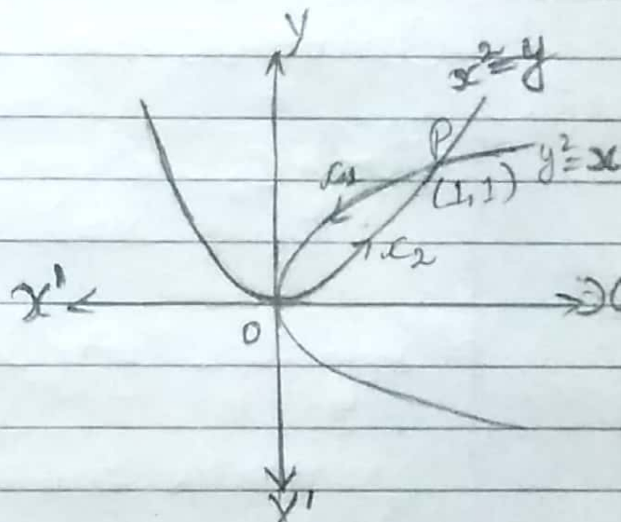
$$= \left[\frac{1}{3} + \frac{y}{2} \right] - \left(\frac{-1}{3} + \frac{y}{2} \right) + \left(x^2 + \frac{1}{3} \right) - \left(-x^2 + \frac{-1}{3} \right)$$

$$\Rightarrow \frac{1}{3} + \frac{y}{2} + \frac{1}{3} - \frac{y}{2} + x^2 + \frac{1}{3} + x^2 + \frac{1}{3}$$

$$= 2x^2 + \frac{4}{3} \quad \underline{\text{Ans}}$$

Q57) Verify Green's theorem in the plane for $\int_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where c is bounded by the region $y = \sqrt{x}$ and $y = x^2$

Sol: Here C is bounded by 2 parabolas $y^2 = x$ & $x^2 = y$



By Green's theorem

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = 3x^2 - 8y^2$; $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad ; \quad \frac{\partial N}{\partial x} = -6y$$

$$\int_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \iint_R (-6y + 16y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\,dy\,dx = \int_0^1 5y^2 \Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (5x - 5x)\,dx$$

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$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\frac{3}{10} = \frac{3}{2}$$

$\therefore$ By green's theorem $\oint_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$$= \frac{3}{2} \text{ ——— (i)}$$

Now, lineintegral $\oint_C (3x^2 - 8y^2) + (4y - 6xy)dy$

Curve C consists of two parts C_2 and C_1

Along C_2; $x^2 = y$ $\therefore dy = 2x\,dx$ and x varies from 0 to 1.

Along C_1, $y^2 = x$; $x = y^2$ $\therefore dx = 2y\,dy$ and y varies from 1 to 0.

$\therefore$ Direction of C_1 is from P to O.

$$\oint_C (3x^2 - 8y^2) + (4y - 6xy)\,dy$$

$$= \oint_{C_2} (3x^2 - 8y^2)dx + (4y - 6xy)dy + \oint_{C_1} (3x^2 - 8y^2)dx + (4y - 6xy)\,dy$$

$$= \oint_0^1 (3x^2 - 8x)dx + (4x^2 - 6x^3)2x\,dx + \int_1^0 (3y^4 - 8y^2)(2y\,dy) + (4y - 6y^3)dy$$

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$$\Rightarrow \int_0^1 (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx + \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= x^3 - 20\frac{x^5}{5} + \frac{8x^4}{4}\Big|_0^1 + \left\{\frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2}\right\}_1^0$$

$$= 1 - 4 + 2 - \left[1 - \frac{11}{2} + 2\right]$$

$$= -1 - 3 + \frac{11}{2} = \frac{3}{2} \longrightarrow \text{(ii)}$$

$\because$ (i) & (ii) are same.
So green's theorem verified.

Q-38) Prove that $\nabla^2 r^n = n(n+1) r^{n-2}$

Sol-> $\nabla^2 r^n = \nabla.\nabla r^n = \text{div grad } r^n$

$$\text{grad } r^n = \hat{i}\frac{\partial}{\partial x} r^n + \hat{j}\frac{\partial}{\partial y} r^n + \hat{k}\frac{\partial}{\partial z} r^n$$

$$= \hat{i}.\,nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left\{\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right\}$$

$$= nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

Now $\nabla^2 r^n = \text{div}\left[(nr^{n-2})\vec{r}\right]$ which is of the $\text{div}(\phi.\vec{a})$

$$= (nr^{n-2})\,\text{div}\,\vec{r} + \vec{r}.\text{grad}(nr^{n-2})$$

$$= (nr^{n-2})3 + \vec{r}\cdot n\,\text{grad}\, r^{n-2}$$

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$= 3n r^{n-2} + n\vec{r} \cdot (n-2) r^{n-4} \vec{r}$

$= 3n r^{n-2} + n(n-2) r^{n-4} (\vec{r} \cdot \vec{r})$

$= 3n r^{n-2} + n(n-2) r^{n-4} \cdot r^2$

$= 3n r^{n-2} + n(n-2) r^{n-2}$

$= (3n + n^2 - 2n) r^{n-2}$

$= (n + n^2) r^{n-2} = n(n+1) r^{n-2}$ **Ans**

Q-3(10) If $\vec{V} = \dfrac{\vec{r}}{r^3}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

prove that $\vec{V}$ is a solenoid vector.

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$\therefore \vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

then div $\vec{V} = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right) \cdot \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$= \dfrac{\partial}{\partial x}\left[\dfrac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \dfrac{\partial}{\partial y}\left[\dfrac{y}{(x^2+y^2+z^2)^{3/2}}\right]$

$+ \dfrac{\partial}{\partial z}\left[\dfrac{z}{(x^2+y^2+z^2)^{3/2}}\right]$

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$$= \sum \frac{(x^2+y^2+z^2)^{3/2} \cdot 1 - x\frac{3}{2}(x^2+y^2+z^2)^{1/2} \cdot 2x}{(x^2+y^2+z^2)^3}$$

$$= \sum \frac{(x^2+y^2+z^2)^{1/2}\left[x^2+y^2+z^2-3x^2\right]}{(x^2+y^2+z^2)^3}$$

$$= \sum \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\therefore \vec{V} = \frac{\vec{r}}{r^3}$ is a solenoid vector. Ans

Q-11) Let $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{a}$ be a constant vector, find the value div $\frac{\vec{a}\times\vec{r}}{r^n}$

Sol→ Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}(a_2z-a_3y)$$

$$+\hat{j}(a_3x-a_1z)+\hat{k}(a_1y-a_2x)$$

div $\left[\frac{\vec{a}\times\vec{r}}{r^n}\right]$ = div $\left[\frac{1}{r^n}(\vec{a}\times\vec{r})\right]$ is of the type div $(\phi\vec{A})$, where $\phi = \frac{1}{r^n}$ and $\vec{A} = \vec{a}\times\vec{r}$

we know that div $(\phi\vec{A})$ = ϕ div $\vec{A}$ + (grad ϕ)$\cdot\vec{A}$

$$\text{div}\left[\frac{1}{r^n}\vec{a}\times\vec{r}\right] = \frac{1}{r^n}\,\text{div}(\vec{a}\times\vec{r}) + \left[\text{grad}\,\frac{1}{r^n}\right]\cdot\vec{a}\times\vec{r} \quad \text{—(i)}$$

Now, $\text{div}(\vec{a}\times\vec{r}) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]$

$$\left[\hat{i}(a_2 z - a_3 y) + \hat{j}(a_3 x - a_1 z) + \hat{k}(a_1 y - a_2 x)\right]$$

$$= \hat{i}.0 + \hat{j}\cdot 0 + \hat{k}.0 = 0$$

$$\text{grad}\left[\frac{1}{r^n}\right] = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](x^2+y^2+z^2)^{-n/2}$$

$$= \sum \hat{i}\left[\frac{-n}{2}\right](x^2+y^2+z^2)^{-n/2}\cdot 2x$$

$$= \sum \hat{i}(-nx)r^{2\left[\frac{-n+2}{2}\right]} = -n\sum \hat{i}\,x\,r^{-(n+2)}$$

$$= \frac{-n}{r^{n+2}}(x\hat{i} + y\hat{j} + z\hat{k}) = \frac{-n\vec{r}}{r^{n+2}}$$

from (i) $\text{div}\,\frac{\vec{a}\times\vec{r}}{r^n} = \frac{1}{r^n}\cdot 0 - \frac{n}{r^{n+2}}(\vec{r}.\vec{a}\times\vec{r})$

$$= \frac{1}{r^n}\cdot 0 - \frac{n}{r^{n+2}}\cdot 0 = 0$$

$\therefore\ \text{div}\,\frac{\vec{a}\times\vec{r}}{r^n} = 0$ **Ans**