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Math Assignment

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Vector Calculus

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Q.1. Find the magnitude of gradient of a function $u = \frac{x^2}{2} + \frac{y^2}{3}$ at $(1,3)$

Soln:- gradient of $u = \nabla u = \frac{\partial u}{\partial x}\hat{i} + \frac{\partial u}{\partial y}\hat{j} + \frac{\partial u}{\partial z}\hat{k}$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}x\hat{i} + \frac{1}{3} \cdot 2y\hat{j}$

$= x\hat{i} + \frac{2}{3}y\hat{j}$

$|\Delta u|$ = magnitude $= \sqrt{x^2 + \left(\frac{2}{3}y\right)^2}$

magnitude at point $(1,3) = \sqrt{1^2 + \left(\frac{2}{\cancel{3}} \cdot \cancel{3}\right)^2}$

$= \sqrt{1+2^2} = \sqrt{5}$ Ans.

Q.2. Find the directional derivative of $F = 2xy + z^2$ at point $(1,-1,3)$ in the direction of $\hat{i} + 2\hat{j} + 3\hat{k}$.

Soln:- $F = 2xy + z^2$

Gradiant of $F = \nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$

$= 2y\hat{i} + 2x\hat{j} + 2z\hat{k}$

∇F at point $(1,-1,3) = -2\hat{i} + 2\hat{j} + 6\hat{k}$

If $\hat{n}$ is a unit vector in the direction of $\hat{i}+2\hat{j}+3\hat{k}$

then $\hat{n} = \dfrac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1+4+9}} = \dfrac{1}{\sqrt{14}}(\hat{i}+2\hat{j}+3\hat{k})$

$\therefore$ Directional Derivative of the function F at $(1,-1,3)$ in the direction of $\hat{i}+2\hat{j}+3\hat{k}$

$= [\nabla \vec{F} \text{ at } (1,-1,3)] \cdot \hat{n}$

$= (-2\hat{i}+2\hat{j}+6\hat{k}) \cdot \dfrac{1}{\sqrt{14}}(\hat{i}+2\hat{j}+3\hat{k})$

$= \dfrac{-2}{\sqrt{14}} + \dfrac{4}{\sqrt{14}} + \dfrac{18}{\sqrt{14}} = \underline{\dfrac{20}{\sqrt{14}}}$ Ans.

Q.3. Evaluate the line integral $\int_C (x^2+xy)dx + (x^2+y^2)dy$ where C is the square formed by line $x=\pm 1$, $y=\pm 1$.

Soln:- Here in square C, x varies from -1 to 1 and y varies from -1 to 1.

$\Rightarrow \int_C (x^2+xy)dx + (x^2+y^2)dy$

$= \int_{-1}^{1} (x^2+xy)dx + \int_{-1}^{1} (x^2+y^2)dy$

$$= \left[\frac{x^3}{3} + y\cdot\frac{x^2}{2}\right]_{-1}^{1} + \left[x^2 y + \frac{y^3}{3}\right]_{-1}^{1}$$

$$= \left(\frac{1}{3} + \frac{y}{2}\right) - \left(\frac{-1}{3} + \frac{y}{2}\right) + \left(x^2 + \frac{1}{3}\right) - \left(-x^2 + \frac{-1}{3}\right)$$

$$= \frac{1}{3} + \frac{y}{2} + \frac{1}{3} - \frac{y}{2} + x^2 + \frac{1}{3} + x^2 + \frac{1}{3}$$

$$= 2x^2 + \frac{4}{3} \text{ Ans.}$$

Q.4. Find unit normal vector of surface $x^2y + 2xz = 4$ at $(2,-2,3)$.

Soln: Let $A(x,y,z) = x^2y + 2xz - 4$

We know that ∇A is a vector normal to the surface $A = x^2y + 2xz - 4$

$$\therefore \nabla A = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\cdot\left(x^2y + 2xz - 4\right)$$

$$= \hat{i}(2xy + 2z) + \hat{j}(x^2) + \hat{k}(2x)$$

Normal vector at $(2,-2,3)$ is $-2\hat{i} + 4\hat{j} + 4\hat{k}$

Unit normal vector at $(2,-2,3)$

$$= \frac{2(-\hat{i} + 2\hat{j} + 2\hat{k})}{2\sqrt{1+4+4}} = \frac{-\hat{i} + 2\hat{j} + 2\hat{k}}{3} \text{ Ans}$$

Q5. Find div $\vec{F}$ and curl $\vec{F}$ where $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ at $(1,-1,1)$.

Soln:- $\vec{F} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$

$$\text{div } \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}\right)$$

$$= \frac{\partial(xy^2)}{\partial x} + \frac{\partial(2x^2yz)}{\partial y} + \frac{\partial(-3yz^2)}{\partial z}$$

$$= y^2 + 2x^2z - 6yz$$

div $\vec{F}$ at point $(1,-1,1) = 1 + 2 + 6 = \underline{9}$ Ans.

$$\text{Curl } \vec{F} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \times \left[xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}\right]$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & (-3yz^2) \end{vmatrix} = \hat{i}\left(-3z^2 - 2x^2y\right) - \hat{j}(0) + \hat{k}\left(4xyz - 2xy\right)$$

curl $\vec{F}$ at point $(1,-1,1) = \underline{-\hat{i} - 2\hat{k}}$ Ans.

Q.6. Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = x\hat{i} - y\hat{j} + (z^2-1)\hat{k}$

and S is closed surface bounded by plane $z=0$ and $z=1$ and cylinder $x^2+y^2=4$.

Soln: A vector normal to the surface S is

$\nabla(x^2+y^2) = 2x\hat{i} + 2y\hat{j}$

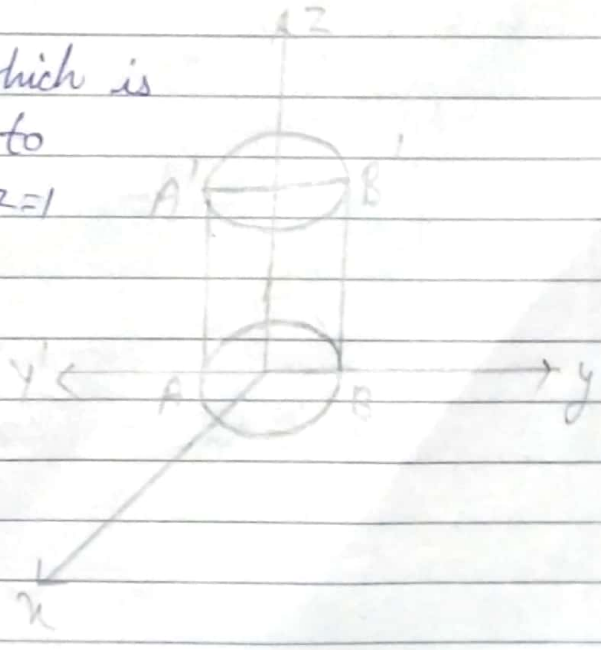
$\hat{n}$, a unit vector normal to S

$$= \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{\sqrt{4}} = \frac{x\hat{i}+y\hat{j}}{2}$$

Let R be the projection of S on YZ plane

then $\iint_S \vec{F}\cdot\hat{n}\, dS = \iint_R \vec{F}\cdot\hat{n}\, \frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$

Where R is ABB'A' which is enclosed by $y=-2$ to $y=2$ and $z=0$ to $z=1$



$$\hat{i}\cdot\hat{n} = \hat{i}\cdot\frac{x\hat{i}+y\hat{j}}{2} = \frac{x}{2}$$

$$\vec{F}\cdot\hat{n} = (x\hat{i}-y\hat{j}+(z^2-1)\hat{k})\cdot\frac{x\hat{i}+y\hat{j}}{2}$$

$$= \frac{x^2-y^2}{2}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{|\hat{i}\cdot\hat{n}|}$$

$$= \iint_R \frac{x^2-y^2}{\cancel{2}}\cdot\frac{dy\,dz}{\frac{x}{\cancel{2}}} = \iint_R \frac{x^2-y^2}{x}\,dy\,dz$$

$$= \int_0^1\int_{-2}^{2} \frac{x^2-y^2}{x}\,dy\,dz = \int_0^1 \left[x.y - \frac{y^3}{3x}\right]_{-2}^{2} dz$$

$$= \int_0^1 \left[\left(2x - \frac{8}{3x}\right) - \left(-2x - \frac{(-8)}{3x}\right)\right] dz$$

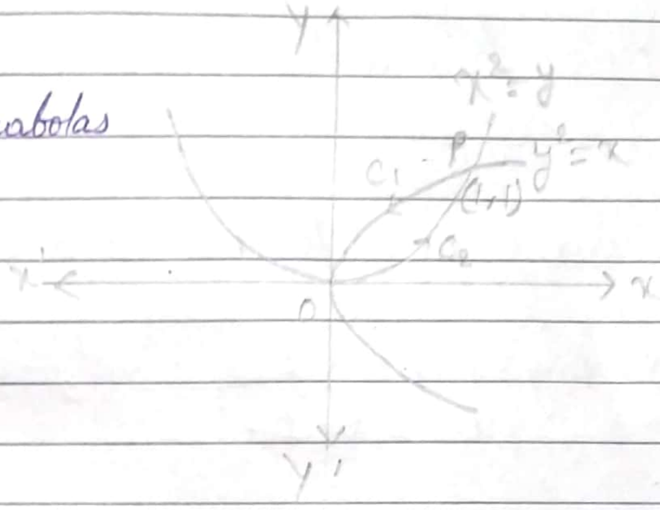
$$= \int_0^1 \left(4x - \frac{16}{3x}\right) dz = \left[4xz - \frac{16}{3x}\cdot z\right]_0^1 = 4x - \frac{16}{3x} \text{ Ans.}$$

Q7. Verify Green's theorem in the plane for

$$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

where C is bounded by the region $y=\sqrt{x}$ and $y=x^2$

Soln:- Here C is bounded by 2 parabolas $y^2=x$ & $x^2=y$



By Green's theorem

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Here $M = 3x^2-8y^2$; $N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y$; $\frac{\partial N}{\partial x} = -6y$

$$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy = \iint_R (-6y+16y)\,dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\,dy\,dx = \int_0^1 5y^2\Big|_{x^2}^{\sqrt{x}} dx = \int_0^1 (5x-5x^4)\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5\left[\frac{1}{2} - \frac{1}{5}\right] = 5\,\frac{3}{10} = \frac{3}{2}$$

$\therefore$ By Green's theorem $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy = \frac{3}{2}$ $\rightarrow$ (i)

Now, line integral $\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$

Curve C consists of two parts C_2 and C_1

Along C_2, $x^2 = y$ $\therefore$ $dy = 2x\,dx$ and x varies from 0 to 1.

Along C_1, $y^2 = x$; $x = y^2$ $\therefore$ $dx = 2y\,dy$ and y varies from 1 to 0

$\therefore$ Direction of C_1 is from P to O.

$$\oint_C (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= \int_{C_2} (3x^2-8y^2)dx + (4y-6xy)dy + \int_{C_1} (3x^2-8y^2)dx + (4y-6xy)dy$$

$$= \int_0^1 (3x^2-8x^4)dx + (4x^2-6x^3)2x\,dx + \int_1^0 (3y^4-8y^2)(2y\,dy) + (4y-6y^3)dy$$

$$= \int_0^1 (3x^2-8x^4+8x^3-12x^4)dx + \int_1^0 (6y^5-16y^3+4y-6y^3)dy$$

$$= x^3 - 20\frac{x^5}{5} + 8\frac{x^4}{4}\Big|_0^1 + \left\{\frac{6y^6}{6} - 22\frac{y^4}{4} + \frac{4y^2}{2}\right\}_1^0$$

$$= 1-4+2-\left[1-\frac{11}{2}+2\right] = -1-3+\frac{11}{2} = \frac{3}{2} \rightarrow (ii)$$

$\because$ (i) & (ii) are same.

So Green's theorem verified.

Q.8. Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$

Soln:- $\nabla^2 r^n = \nabla \cdot \nabla r^n = \text{div grad } r^n$

$$\text{grad } r^n = \hat{i}\frac{\partial}{\partial x} r^n + \hat{j}\frac{\partial}{\partial y} r^n + \hat{k}\frac{\partial}{\partial z} r^n$$

$$= \hat{i} \cdot n r^{n-1}\frac{\partial r}{\partial x} + \hat{j} n r^{n-1}\frac{\partial r}{\partial y} + \hat{k} n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left\{\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right\} = n r^{n-1}\frac{\vec{r}}{r} = n r^{n-2}\vec{r}$$

Now $\nabla^2 r^n = \text{div}[(n r^{n-2})\vec{r}]$ which is of the

$\text{div}(\phi\vec{a})$

$$= (n r^{n-2})\,\text{div}\,\vec{r} + \vec{r} \cdot \text{grad}(n r^{n-2})$$

$$= (n r^{n-2})3 + \vec{r} \cdot n\,\text{grad}\, r^{n-2}$$

$$= 3n r^{n-2} + n\vec{r} \cdot (n-2) r^{n-4}\vec{r}$$

$$= 3n r^{n-2} + n(n-2) r^{n-4}(\vec{r} \cdot \vec{r})$$

$= 3nr^{n-2} + n(n-2)r^{n-4} \cdot r^2$

$= 3n\,r^{n-2} + n(n-2)\,r^{n-2}$

$= (3n + n^2 - 2n)\, r^{n-2}$

$= (n + n^2) r^{n-2} = \underline{n(n+1)\, r^{n-2}}$ Ans.

Q.9. Evaluate $\iint_S \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = xy\,\hat{i} - x^2\hat{j} + (x+z)\hat{k}$

S is that portion of the plane $2x + 2y + z = 6$ included in the first octant and $\hat{n}$ is unit normal to S.

Soln:- A vector normal to the surface S is given by

$$\nabla(2x + 2y + z - 6) = 2\hat{i} + 2\hat{j} + \hat{k}$$

$\hat{n}$ = a unit vector normal to surface S

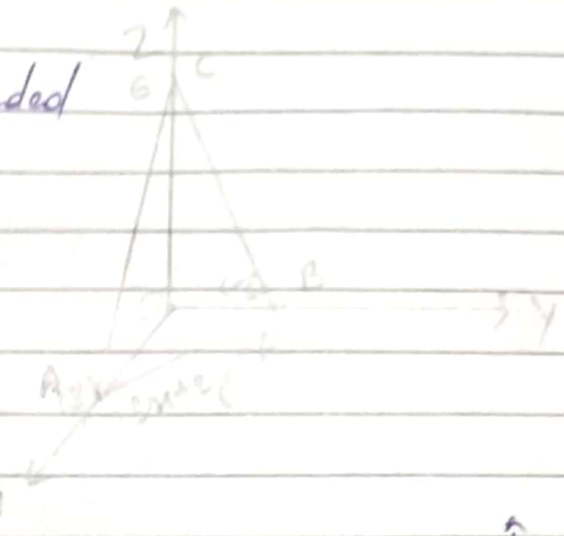
$$= \frac{2\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{4+4+1}} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{9}$$

$$\hat{k} \cdot \hat{n} = \frac{1}{3}$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \vec{F} \cdot \hat{n} \frac{dx\,dy}{|\hat{k} \cdot \hat{n}|}$$

Where R is the projection of S on xy-plane where S is ΔABC.

The region R (i.e ΔABO) is bounded by x-axis, y axis and the line $2x+2y=6$, $z=0$



Now $\vec{F}\cdot\hat{n} = (xy\,\hat{i} - x^2\hat{j} + (x+z)\hat{k}) \cdot \dfrac{2\hat{i}+2\hat{j}+\hat{k}}{3}$

$$= \frac{2xy - 2x^2 + x + z}{3}$$

Substituting the value of $z = 6-2x-2y$

$$\vec{F}\cdot\hat{n} = \frac{2xy - 2x^2 + x + 6 - 2x - 2y}{3} = \frac{-2x^2+2xy-x-2y+6}{3}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dS = \iint_R \left(\frac{-2x^2+2xy-x-2y+6}{\cancel{3}}\right)\frac{dx\,dy}{1/\cancel{3}}$$

Here y varies from 0 to $\dfrac{6-2x}{2} = 3-x$

& x varies from 0 to 3

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,dS = \int_0^3\int_0^{(3-x)} (-2x^2+2xy-x-2y+6)\,dy\,dx$$

$$= \int_0^3 \left[-2x^2y + 2x\frac{y^2}{2} - xy - 2\frac{y^2}{2} + 6y\right]_0^{(3-x)} dx$$

$$= \int_0^3 \left[-2x^2(3-x) + x(3-x)^2 - x(3-x) - (3-x)^2 + 6(3-x)\right] dx$$

$$= \int_0^3 \left(-6x^2 + 2x^3 + 9x + x^3 - 6x^2 - 3x + x^2 - 9 - x^2 + 6x + 18 - 6x\right) dx$$

$$= \int_0^3 \left(-12x^2 + 3x^3 + 6x + 9\right) dx$$

$$= \left[-12\frac{x^3}{3} + 3\frac{x^4}{4} + 6\frac{x^2}{2} + 9x\right]_0^3$$

$$= \left[-4x^3 + \frac{3}{4}x + 3x^2 + 9x\right]_0^3$$

$$= -4 \times 27 + \frac{3}{4} \times 3 + 3 \times 9 + 9 \times 3$$

$$= -108 + \frac{9}{4} + 27 + 27 = \frac{9}{4} - 54 = \frac{9 - 4 \times 54}{4}$$

$$= \frac{9 - 216}{4} = \frac{-207}{4} \text{ Ans.}$$

Q.10. If $\vec{V} = \frac{\vec{r}}{r^3}$ where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$, prove that $\vec{V}$ is a solenoidal vector.

Soln:- $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$ $\quad \therefore \vec{V} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

then $\text{div}\,\vec{V} = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] \cdot \frac{x\hat{i}+y\hat{j}+z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$

$$= \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$+ \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$= \sum \frac{(x^2+y^2+z^2)^{3/2}\cdot 1 - x\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x}{(x^2+y^2+z^2)^3}$$

$$= \sum \frac{(x^2+y^2+z^2)^{1/2}\left[x^2+y^2+z^2-3x^2\right]}{(x^2+y^2+z^2)^3}$$

$$= \sum \frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{5/2}}$$

$$= \frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\therefore \left[\vec{V} = \frac{\vec{r}}{r^3} \text{ is a solenoidal vector}\right]$ Ans.

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Q.11. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of div $\frac{\vec{a}\times\vec{r}}{r^n}$.

Soln:- Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}(a_2 z - a_3 y) + \hat{j}(a_3 x - a_1 z)$$
$$+ \hat{k}(a_1 y - a_2 x$$

div$\left[\frac{\vec{a}\times\vec{r}}{r^n}\right]$ = div$\left[\frac{1}{r^n}(\vec{a}\times\vec{r})\right]$ is of the ~~the~~ type div$(\phi\vec{A})$, where $\phi = \frac{1}{r^n}$ and $\vec{A} = \vec{a}\times\vec{r}$

We know that div$(\phi\vec{A}) = \phi$ div $\vec{A}$ + (grad ϕ)$\cdot\vec{A}$

$$\text{div}\left[\frac{1}{r^n}\vec{a}\times\vec{r}\right] = \frac{1}{r^n}\text{div}(\vec{a}\times\vec{r}) + \left[\text{grad}\frac{1}{r^n}\right]\cdot\vec{a}\times\vec{r} \quad \rightarrow (i)$$

Now, div $(\vec{a}\times\vec{r}) = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\cdot$

$$\left[\hat{i}(a_2 z - a_3 y) + \hat{j}(a_3 x - a_1 z) + \hat{k}(a_1 y - a_2 x)\right]$$
$$= \hat{i}\cdot 0 + j\cdot 0 + \hat{k}\cdot 0 = 0$$

$$\text{grad}\left[\frac{1}{r^n}\right] = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](x^2+y^2+z^2)^{-n/2}$$

$$= \sum \hat{i}\left[-\frac{n}{2}\right](x^2+y^2+z^2)^{-n/2-1} \cdot 2x$$

$$= \sum \hat{i}(-nx)\, r^{2\left[\frac{-n+2}{2}\right]} = -n \sum \hat{i}\, x\, r^{-(n+2)}$$

$$= -\frac{n}{r^{n+2}}(x\hat{i}+y\hat{j}+z\hat{k}) = -\frac{n\vec{r}}{r^{n+2}}$$

From (i) $\quad div\, \frac{\vec{a}\times\vec{r}}{r^n} = \frac{1}{r^n}\cdot 0 - \frac{n}{r^{n+2}}(\vec{r}\cdot\vec{a}\times\vec{r})$

$$= \frac{1}{r^n}\cdot 0 - \frac{n}{r^{n+2}}\cdot 0 = 0$$

$\therefore \underbrace{div\, \frac{\vec{a}\times\vec{r}}{r^n} = 0}$ Ans.