7.13. SCALAR AND VECTOR POINT FUNCTIONS

Point Function. A variable quantity whose value at any point in a region of space depends upon the position of the point, is called a point function.

Point functions are of two types:

(*i*) Scalar Point Function (*ii*) Vector Point Function.

(*i*) **Scalar Point Function.** A function $\phi(x, y, z)$ is called a scalar point function if it associates a scalar with every point in region R of a space. Region R is called scalar field. The temperature distribution in a heated body, density of a body and potential due to gravity are examples of scalar point functions.

(ii) **Vector Point Function.** If a function $\vec{V}$ (x, y, z) defines a vector at every point of the region R of a space then $\vec{V}$ (x, y, z) is called a vector point function and R is called a vector field. Every vector $\vec{v}$ of the field is regarded as a localized vector attached to the corresponding point (x, y, z).

The velocity of a moving fluid at any instant, gravitational forces are examples of vector point function.

7.14. GRADIENT OF A SCALAR FIELD *(P.T.U., Jan. 2009)*

Let $\phi\ (x, y, z)$ be a function defining a scalar field, then the vector $\hat{i}\dfrac{\partial\phi}{\partial x}+\hat{j}\dfrac{\partial\phi}{\partial y}+\hat{k}\dfrac{\partial\phi}{\partial z}$ is called the gradient of the scalar field ϕ and is denoted by grad ϕ.

Thus $$\text{grad}\ \phi = \hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}$$

Grad ϕ is a vector quantity.

The gradient of scalar field ϕ is obtained by operating on ϕ the vector operator.

$$\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}$$

This operator is denoted by symbol ∇ (read as del) (also called nable)

Thus $$\text{grad}\ \phi = \nabla\phi.$$

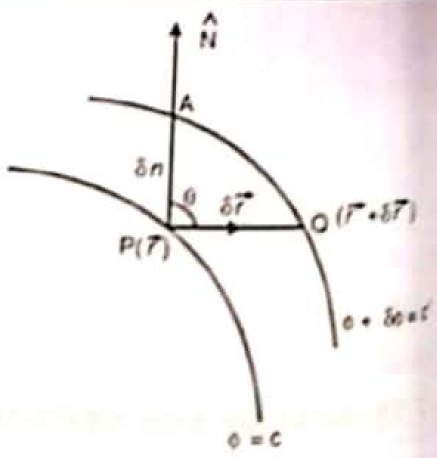
7.15. GEOMETRICAL INTERPRETATION OF GRADIENT

(P.T.U., May 2010, May 2012)

If a surface $\phi\ (x, y, z) = c$ is drawn through any point P such that at each point on the surface, the function has the same value as at P, then such a surface is called a **level surface** through P.

Through any point passes one and only one level surface. Also no two level surfaces can intersect.

Consider the level surface through the point P at which the function has value ϕ and let $\phi + \delta\phi$ be another level surface through the neighbouring point Q.



Let $\vec{r}$ and $\vec{r}+\delta\vec{r}$ be the position vectors of P and Q respectively then $\overrightarrow{PQ} = \delta\vec{r}$

Now, $$\nabla\phi\cdot\delta\vec{r} = \left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\left(\hat{i}\,\delta x+\hat{j}\,\delta y+\hat{k}\,\delta z\right)$$

$$= \frac{\partial\phi}{\partial x}\delta x+\frac{\partial\phi}{\partial y}\delta y+\frac{\partial\phi}{\partial z}\delta z = \delta\phi \quad \text{...(1)}$$

If Q lies on the same surface as P, then $\delta\phi = 0$

$\therefore$ (1) reduces to $\nabla\phi \cdot \delta\vec{r} = 0$

$\therefore$ $\nabla\phi$ is perpendicular to $\delta\vec{r}$ which is true for all values of r.

Hence $\nabla\phi$ is normal to the surface $\phi(x, y, z) = c$.

Let $\nabla\phi = |\nabla\phi|\,\hat{N}$, where $\hat{N}$ is a unit normal to $\phi = c$ at P. Let PA $= \delta n$ be the perpendicular distance between the two level surfaces $\phi = c$ and $\phi + \delta\phi = c'$. Then rate of change of ϕ in the direction of normal to the surface through P is $\dfrac{\partial\phi}{\partial n} = \underset{\delta n \to 0}{\mathrm{Lt}} \dfrac{\delta\phi}{\delta n} = \underset{\delta n \to 0}{\mathrm{Lt}} \dfrac{\nabla\phi \cdot \delta\vec{r}}{\delta n}$ [$\because$ of (1)]

$$= \underset{\delta n \to 0}{\mathrm{Lt}} \frac{|\nabla\phi|\,\hat{N} \cdot \delta\vec{r}}{\delta n}$$

Since, $$\hat{N} \cdot \delta\vec{r} = |\hat{N}|\,|\delta\vec{r}|\cos\theta = 1 \cdot PQ\cos\theta = \delta n$$

$$\therefore \quad \frac{\partial\phi}{\partial n} = \underset{\delta n \to 0}{\mathrm{Lt}} \frac{|\nabla\phi|\,\delta n}{\delta n} = |\nabla\phi|$$

$$\therefore \quad |\nabla\phi| = \frac{\partial\phi}{\partial n}.$$

Hence the gradient of a scalar field ϕ is a vector normal to the surface $\phi = c$ and has a magnitude equal to the rate of change of ϕ along the normal.

Cor 1. Equation of the tangent plane to a surface $\phi(x, y, z) = c$ at a point A (x_1, y_1, z_1) can be derived from the gradient vector at that point.

Since gradient vector at a point A (x_1, y_1, z_1) on the surface $\phi(x, y, z) = c$ represents normal to the surface at that point $\therefore$ if we take P(x, y, z) be any point on the tangent plane at (x_1, y_1, z_1) then the vector $(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}$ will be perpendicular to the normal vector at (x_1, y_1, z_1)

$$\therefore \quad (\nabla\phi)_{(x_1, y_1, z_1)} \cdot \left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] = 0$$

i.e., $$\left(\frac{\partial f}{\partial x}\right)_{(x_1, y_1, z_1)}(x - x_1) + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}(y - y_1) + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}(z - z_1) = 0$$

which is the equation of the tangent plane at (x_1, y_1, z_1) to $\phi(x, y, z) = c$.

Cor 2. Equation of the normal at A (x_1, y_1, z_1) to the surface $\phi(x, y, z) = c$.

Let P (x, y, z) be any variable point on the normal to the surface $\phi(x, y, z) = c$. Then $\overrightarrow{AP}$ is parallel to normal vector $\nabla\phi$ at (x_1, y_1, z_1)

$$\therefore \quad \overrightarrow{AP} \times (\nabla\phi)_{(x_1, y_1, z_1)} = \overline{0}$$

i.e., $$\left[(x - x_1)\hat{i} + (y - y_1)\hat{j} + (z - z_1)\hat{k}\right] \times \left[\left(\frac{\partial\phi}{\partial x}\right)_{(x_1, y_1, z_1)}\hat{i} + \left(\frac{\partial\phi}{\partial y}\right)_{(x_1, y_1, z_1)}\hat{j} + \left(\frac{\partial\phi}{\partial z}\right)_{(x_1, y_1, z_1)}\hat{k}\right] = \vec{0}$$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x - x_1 & y - y_1 & z - z_1 \\ \left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)} & \left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} & \left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} \end{vmatrix} = \vec{0}$$

$$\therefore \quad \sum \left[(y - y_1)\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} - (z - z_1)\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} \right] \hat{i} = \vec{0}$$

$$\therefore \quad (y - y_1)\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)} - (z - z_1)\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)} = 0; \quad \therefore \quad \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}} = \frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}}$$

Similarly, by equating components of $\hat{j}$ and $\hat{k}$ to zero, we get

$$\frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}} = \frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} \quad \text{and} \quad \frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} = \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}}$$

Combining all three, the equation of the normal at A (x_1, y_1, z_1) is

$$\frac{x - x_1}{\left(\frac{\partial \phi}{\partial x}\right)_{(x_1, y_1, z_1)}} = \frac{y - y_1}{\left(\frac{\partial \phi}{\partial y}\right)_{(x_1, y_1, z_1)}} = \frac{z - z_1}{\left(\frac{\partial \phi}{\partial z}\right)_{(x_1, y_1, z_1)}}.$$

7.16. DIRECTIONAL DERIVATIVE

(P.T.U., Jan. 2010)

Let PQ = δr then $\underset{\delta r \to 0}{\text{Lt}} \frac{\delta \phi}{\delta r} = \frac{\partial \phi}{\partial r}$ is called directional derivative of ϕ at P in the direction of PQ.

Let $\hat{N}'$ be a unit vector in the direction of PQ then $\hat{N} \cdot \hat{N}' = \cos \theta$

$$\delta r = \frac{\delta n}{\cos \theta} = \frac{\delta n}{\hat{N} \cdot \hat{N}'}$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \underset{\delta r \to 0}{\text{Lt}} \frac{\delta \phi}{\delta n} \hat{N} \cdot \hat{N}' = \hat{N} \cdot \hat{N}' \frac{\partial \phi}{\partial n}$$

$$\therefore \quad \frac{\partial \phi}{\partial r} = \hat{N} \cdot \hat{N}' |\nabla \phi| \qquad \therefore \quad |\nabla \phi| = \frac{\partial \phi}{\partial n} \text{ from art. 7·15}$$

$$= \hat{N}' \cdot |\nabla \phi| \hat{N} = \hat{N}' \cdot \nabla \phi \quad \because \quad \hat{N} |\nabla \phi| = \nabla \phi.$$

Thus the directional derivative $\frac{\partial\phi}{\partial r}$ *is the resolved part of* $\nabla\phi$ *in the direction of* $\hat{N}'$ i.e., $\vec{PQ}$

Since $$\frac{\partial\theta}{\partial r} = \hat{N}' \cdot \nabla\phi = |\nabla\phi| \cos\theta \le |\nabla\phi|$$

$\therefore$ **$\nabla\phi$ gives the maximum rate of change of ϕ and the magnitude of this maximum rate of change is $|\nabla\phi|$.**

7.17. PROPERTIES OF GRADIENT

(1) If ϕ is a constant scalar point function, then $\nabla\phi = \vec{0}$.

(2) If ϕ_1 and ϕ_2 are two scalar point functions, then

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(b) $\nabla(c_1\phi_1 + c_2\phi_2) = c_1\nabla\phi_1 + c_2\nabla\phi_2$, *where* c_1 *and* c_2 *are constants.*

(c) $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

(d) $\nabla\left(\frac{f_1}{f_2}\right) = \frac{f_2\nabla f_1 - \phi_1\nabla f_2}{f_2^{\,2}}$, $\phi_2 \neq 0$.

Proof. (1) $$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 0$$

[$\because$ ϕ is constant $\therefore \frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0$]

(2) (a) $$\nabla(\phi_1 \pm \phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\phi_1 \pm \phi_2)$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1 \pm \phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1 \pm \phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1 \pm \phi_2)$$

$$= \hat{i}\left[\frac{\partial\phi_1}{\partial x} \pm \frac{\partial\phi_2}{\partial x}\right] + \hat{j}\left[\frac{\partial\phi_1}{\partial y} \pm \frac{\partial\phi_2}{\partial y}\right] + \hat{k}\left[\frac{\partial\phi_1}{\partial z} \pm \frac{\partial\phi_2}{\partial z}\right]$$

$$\nabla(\phi_1 \pm \phi_2) = \left(\hat{i}\frac{\partial\phi_1}{\partial x} + \hat{j}\frac{\partial\phi_1}{\partial y} + \hat{k}\frac{\partial\phi_1}{\partial z}\right) \pm \left(\hat{i}\frac{\partial\phi_2}{\partial x} + \hat{j}\frac{\partial\phi_2}{\partial y} + \hat{k}\frac{\partial\phi_2}{\partial z}\right)$$

$$= \nabla\phi_1 \pm \nabla\phi_2$$

(b) Students can easily prove it.

(c) $$\nabla(\phi_1\phi_2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi_1\phi_2$$

$$= \hat{i}\frac{\partial}{\partial x}(\phi_1\phi_2) + \hat{j}\frac{\partial}{\partial y}(\phi_1\phi_2) + \hat{k}\frac{\partial}{\partial z}(\phi_1\phi_2)$$

$$= \hat{i}\left[\phi_1 \frac{\partial \phi_2}{\partial x} + \phi_2 \frac{\partial \phi_1}{\partial x}\right] + \hat{j}\left[\phi_1 \frac{\partial \phi_2}{\partial y} + \phi_2 \frac{\partial \phi_1}{\partial y}\right] + \hat{k}\left[\phi_1 \frac{\partial \phi_2}{\partial z} + \phi_2 \frac{\partial \phi_1}{\partial z}\right]$$

$$= \left(\hat{i}\,\phi_1 \frac{\partial \phi_2}{\partial x} + \hat{j}\,\phi_1 \frac{\partial \phi_2}{\partial y} + \hat{k}\,\phi_1 \frac{\partial \phi_2}{\partial z}\right) + \left(\hat{i}\,\phi_2 \frac{\partial \phi_1}{\partial x} + \hat{j}\,\phi_2 \frac{\partial \phi_1}{\partial y} + \hat{k}\,\phi_2 \frac{\partial \phi_1}{\partial z}\right)$$

$$= \phi_1 \nabla \phi_2 + \phi_2 \nabla \phi_1$$

(*d*) $$\nabla\left(\frac{\phi_1}{\phi_2}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi_1}{\phi_2}\right) = \hat{i}\frac{\partial}{\partial x}\left(\frac{\phi_1}{\phi_2}\right) + \hat{j}\frac{\partial}{\partial y}\left(\frac{\phi_1}{\phi_2}\right) + \hat{k}\frac{\partial}{\partial z}\left(\frac{\phi_1}{\phi_2}\right)$$

$$= \hat{i}\,\frac{\phi_2 \frac{\partial}{\partial x}\phi_1 - \phi_1 \frac{\partial}{\partial x}\phi_2}{\phi_2^2} + \hat{j}\,\frac{\phi_2 \frac{\partial}{\partial y}\phi_1 - \phi_1 \frac{\partial}{\partial y}\phi_2}{\phi_2^2} + \hat{k}\,\frac{\phi_2 \frac{\partial}{\partial z}\phi_1 - \phi_1 \frac{\partial}{\partial z}\phi_2}{\phi_2^2}$$

$$= \frac{1}{\phi_2^2}\left[\hat{i}\,\phi_2 \frac{\partial \phi_1}{\partial x} + \hat{j}\,\phi_2 \frac{\partial \phi_1}{\partial y} + \hat{k}\,\phi_2 \frac{\partial \phi_1}{\partial z}\right] - \left(\hat{i}\,\phi_1 \frac{\partial \phi_2}{\partial x} + \hat{j}\,\phi_1 \frac{\partial \phi_2}{\partial y} + \hat{k}\,\phi_1 \frac{\partial \phi_2}{\partial z}\right)$$

$$= \frac{1}{\phi_2^2}\left[\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2\right]$$

$$\therefore \quad \nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2 \nabla \phi_1 - \phi_1 \nabla \phi_2}{\phi_2^2}; \quad \phi_2 \neq 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Find gradient of the following functions*

(i) $\phi = y^2 - 4xy$ *at (1, 2)*

(ii) $\phi = x^3 + y^3 + 3xyz$ *at (1, –2, –1).* **(P.T.U., May 2008)**

Sol. (*i*) $\phi = y^2 - 4xy$

$$\text{grad. } \phi = \nabla \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(y^2 - 4xy\right)$$

$$= \hat{i}\,(-4y) + \hat{j}\,(2y - 4x)$$

At (1, 2); $\quad$ grad. $\phi = -8\hat{i} + 0\hat{j} = -8\hat{i}$

(*ii*) $$\phi = x^3 + y^3 + 3xyz$$

$$\text{grad. } \phi = \nabla \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^3 + y^3 + 3xyz\right)$$

$$= \hat{i}\left(3x^2 + 3yz\right) + \hat{j}\left(3y^2 + 3xz\right) + \hat{k}\,(3xy)$$

At (1, –2, –1) ; grad. $\phi = 9\hat{i} + 9\hat{j} - 6\hat{k} = 3\left(3\hat{i} + 3\hat{j} - 2\hat{k}\right).$

Example 2. *Find a unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at the point (1, 2, -1).*

(P.T.U., Dec. 2012)

Sol. Let $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$

We know that $\nabla\phi$ is a vector normal to the surface $\phi = c$

$$\therefore \quad \nabla\phi = \text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^3 + y^3 + 3xyz - 3\right)$$

$$= \hat{i}(3x^2 + 3yz) + \hat{j}(3y^2 + 3xz) + \hat{k}(3xy).$$

Normal vector at (1, 2, –1) is $-3\hat{i} + 9\hat{j} + 6\hat{k}$

Unit normal vector at (1, 2, –1)

$$= \frac{3(-\hat{i} + 3\hat{j} + 2\hat{k}}{3\sqrt{1+9+4}} = \frac{-\hat{i} + 3\hat{j} + 2\hat{k}}{\sqrt{14}}$$

Example 3. *Find the normal vector and the equation of the tangent plane to the surface $z = \sqrt{x^2 + y^2}$ at the point (3, 4, 5).*

(P.T.U., Jan. 2008)

Sol. Let $\phi(x, y, z) = \sqrt{x^2 + y^2} - z = 0$

$$\frac{\partial\phi}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \frac{\partial\phi}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \frac{\partial\phi}{\partial z} = -1$$

We know that $\nabla\phi$ is a vector normal to the surface $\phi = C$

$$\therefore \quad \nabla\phi = \text{grad } \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \frac{x}{\sqrt{x^2 + y^2}}\hat{i} + \frac{y}{\sqrt{x^2 + y^2}}\hat{j} - \hat{k}$$

Normal vector at (3, 4, 5) is

$$= \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j} - \hat{k}$$

Now, $\left(\frac{\partial\phi}{\partial x}\right)_{(3,4,5)} = \frac{3}{5}; \left(\frac{\partial\phi}{\partial y}\right)_{(3,4,5)} = \frac{4}{5}; \left(\frac{\partial\phi}{\partial z}\right)_{(3,4,5)} = -1$

Equation of the tangent plane at (3, 4, 5) is

$$(x-3)\frac{3}{5} + (y-4)\frac{4}{5} + (z-5)(-1) = 0$$

or

$$3x + 4y - z = 0.$$

Example 4. *If* $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, *show that*

(i) *grad* $r = \dfrac{\vec{r}}{r}$ (ii) *grad* $\left(\dfrac{1}{r}\right) = -\dfrac{\vec{r}}{r^3}$ **(P.T.U., Dec. 2003)**

(iii) $\nabla r^n = n\, r^{n-2}\vec{r}$

(iv) $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$, *where* $\vec{a}$ *is a constant vector.* **(P.T.U., Dec. 2002)**

Sol. (i) $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ or $r^2 = x^2 + y^2 + z^2$

Differentiate partially w.r.t. x, y and z respectively, we get $2r\dfrac{\partial r}{\partial x} = 2x$, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$.

Similarly, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Now, $$\text{grad } r = \nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(r)$$

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z} = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r} = \frac{1}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \frac{\vec{r}}{r}$$

Hence $$\text{grad } r = \frac{\vec{r}}{r}$$

(ii) $$\text{grad } \frac{1}{r} = \nabla\left(\frac{1}{r}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \hat{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \hat{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left(\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right) = -\frac{1}{r^3}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = -\frac{\vec{r}}{r^3}$$

(iii) $$\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^n$$

$$= \hat{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] = nr^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = nr^{n-2}\vec{r}$$

(iv) $\nabla(\vec{a}\cdot\vec{r})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, where a_1, a_2, a_3 are constants.

$$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z \qquad \text{Since } \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla(\vec{a}\cdot\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(a_1x + a_2y + a_3z)$$

$$= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z)$$

$$= \hat{i}\,a_1 + \hat{j}\,a_2 + \hat{k}\,a_3 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}$$

Hence $\nabla(\vec{a}\cdot\vec{r}) = \vec{a}$

Example 5. *What is the directional derivative of the function $xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?* **(P.T.U., Dec. 2011)**

Sol. Let $\phi(x, y, z) = xy^2 + yz^3$

Gradient of ϕ $= \nabla\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

$$= \hat{i}\,y^2 + \hat{j}(2xy + z^3) + \hat{k}(3yz^2)$$

$$\nabla\phi \text{ at } (2, -1, 1) = \hat{i} - 3\hat{j} - 3\hat{k}$$

If $\hat{n}$ is a unit vector in the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$, then $\hat{n} = \dfrac{\hat{i} + 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \dfrac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$

$\therefore$ Directional derivative of the given function ϕ at $(2, -1, 1)$ in the direction of

$$\hat{i} + 2\hat{j} + 2\hat{k} = [\nabla\phi \text{ at } (2, -1, 1)]\cdot\hat{n}$$

$$= (\hat{i} - 3\hat{j} - 3\hat{k})\cdot\frac{1}{3}(\hat{i} + 2\hat{j} + 2\hat{k}) = \frac{1-6-6}{3} = -\frac{11}{3}$$

Example 6. *Find all the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ, where Q is the point $(5, 0, 4)$.*

In what direction it will be maximum ? Find also the magnitude of this maximum.

Sol. Gradient of $f = \nabla f$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2z^2) = \hat{i}(2x) + \hat{j}\big((-2y) + \hat{k}(4z)\big)$$

$$\nabla f \text{ at } (1, 2, 3) = 2\hat{i} - 4\hat{j} + 12\hat{k}.$$

Now $\overrightarrow{PQ}$ = P.V of Q – P.V. of P $= 5\hat{i} + 4\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k}) = 4\hat{i} - 2\hat{j} + \hat{k}$

If $\hat{n}$ is a unit vector in the direction $\overrightarrow{PQ}$, then $\hat{n} = \dfrac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16+4+1}} = \dfrac{1}{\sqrt{21}}(4\hat{i} - 2\hat{j} + \hat{k})$

Direction derivative of f at (1, 2, 3) in the direction of $\overrightarrow{PQ} = \left[\nabla f \text{ at}(1,2,3)\right] \cdot \hat{n}$

$$= \left(2\hat{i} + 4\hat{j} + 12\hat{k}\right) \cdot \frac{1}{\sqrt{21}}\left(4\hat{i} + 2\hat{j} + \hat{k}\right) = \frac{1}{\sqrt{21}}(8 + 8 + 12) = \frac{28}{\sqrt{21}} = \frac{28}{21}\sqrt{21} = \frac{4}{3}\sqrt{21}$$

The directional derivative of f is maximum in the direction of the normal to the given surface *i.e.*, in the direction of $\nabla f = \left(2\hat{i} - 4\hat{j} + 12\hat{k}\right)$.

The maximum value of this directional derivative $= |\nabla f| = \sqrt{4 + 16 + 144}$

$$= \sqrt{164} = 2\sqrt{41}.$$

Example 7. *Find the directional derivative of $\phi = e^{2x} \cos yz$ at the origin in the direction of the tangent to the curve $x = a \sin t$, $y = a \cos t$, $z = at$ at $t = \frac{\pi}{4}$.*

Sol. Gradient of $\phi = \nabla \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(e^{2x} \cos yz\right)$

$$= \hat{i}\left(2e^{2x} \cos yz\right) + \hat{j}\left(-e^{2x} z \sin yz\right) + \hat{k}\left\{e^{2x}(-\sin yz)\, y\right\}$$

At the origin *i.e.*, when $x = 0, y = 0, z = 0.$

$$\nabla\phi = \hat{i}(2) = 2\hat{i}$$

Equation of the curve is $x = a \sin t,\ y = a \cos t,\ z = at$

Any point on the curve is $\vec{r} = \hat{i}(a \sin t) + \hat{j}(a \cos t) + \hat{k}(at)$

Direction of the tangent is given by $= \frac{d\vec{r}}{dt} = (a \cos t)\,\hat{i} - (a \sin t)\,\hat{j} + a\,\hat{k}$

At $t = \frac{\pi}{4}$, direction of tangent $= \frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}$

$\hat{n}$ = unit direction of the tangent

$$= \frac{\frac{a}{\sqrt{2}}\hat{i} - \frac{a}{\sqrt{2}}\hat{j} + a\hat{k}}{\sqrt{\frac{a^2}{2} + \frac{a^2}{2} + a^2}} = \frac{\frac{a}{\sqrt{2}}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right)}{\sqrt{2}\,a} = \frac{1}{2}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right)$$

Directional derivative of ϕ at (0, 0, 0) in the direction of tangent at $t = \frac{\pi}{4}$ is $= \nabla\phi \cdot \hat{n}$ at (0, 0, 0).

$$= 2\hat{i} \cdot \frac{1}{2}\left(\hat{i} - \hat{j} + \sqrt{2}\,\hat{k}\right) = 1.$$

Example 8. *Find the directional derivative of $\vec{\nabla}^2$, where $\vec{\nabla} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at the point (2, 0, 3) in the direction of outward normal to the sphere $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1).*

Sol.

$$\vec{\nabla} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$$

$$\vec{\nabla}^2 = \vec{\nabla}\cdot\vec{\nabla} = x^2y^4 + z^2y^4 + x^2z^4$$

$$\text{Gradient of } \vec{\nabla}^2 = \nabla(\nabla^2) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2y^4 + z^2y^4 + x^2z^4\right)$$

$$= \hat{i}\left(2xy^4 + 2xz^4\right) + \hat{j}\left(4x^2y^3 + 4z^2y^3\right) + \hat{k}\left(2zy^4 + 4x^2z^3\right)$$

Gradient of $\vec{\nabla}^2$ at (2, 0, 3) $= \hat{i}(324) + \hat{j}(0) + \hat{k}(432) = 108(3\hat{i} + 4\hat{k})$

Normal to the sphere $x^2 + y^2 + z^2 = 14$ is ∇f

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2 - 14\right) = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

Normal vector at (3, 2, 1) $= 6\hat{i} + 4\hat{j} + 2\hat{k}$

$$\hat{n} = \text{Unit Normal vector at } (3, 2, 1) = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{36+16+4}} = \frac{6\hat{i} + 4\hat{j} + 2\hat{k}}{2\sqrt{14}} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

Directional derivative of $\vec{\nabla}^2$ at (2, 0, 3) along the normal at (3, 2, 1)

$$= 108\left(3\hat{i} + 4\hat{k}\right)\cdot\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}$$

$$= \frac{108}{\sqrt{14}}(9+4) = \frac{(108)(13)}{\sqrt{14}} = \frac{1404}{\sqrt{14}}.$$

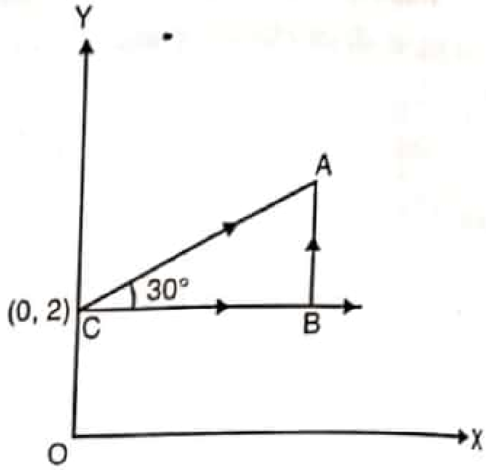
Example 9. *For the function $\phi(x, y) = \frac{x}{x^2 + y^2}$, find the magnitude of the directional derivative along a line making an angle 30° with the positive axis at (0, 2).*

Sol. Gradient of $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{x^2 + y^2}\right) = \hat{i}\left[\frac{\left(x^2 + y^2\right)\cdot 1 - x\cdot 2x}{\left(x^2 + y^2\right)^2}\right] + \hat{j}\left(\frac{-x\cdot 2y}{\left(x^2 + y^2\right)^2}\right)$$

$$= \frac{y^2 - x^2}{\left(x^2 + y^2\right)^2}\hat{i} - \frac{2xy}{\left(x^2 + y^2\right)^2}\hat{j}$$

Gradient of ϕ at $(0, 2) = \frac{4}{16}\hat{i} - 0 = \frac{\hat{i}}{4}$



Now, $\overrightarrow{CA} = \overrightarrow{CB} + \overrightarrow{BA}$

$\overrightarrow{CB} = CA\cos 30°\,\hat{i}$ $\because$ CB is $\parallel$ to X-axis

$\overrightarrow{BA} = CA\sin 30°\,\hat{j}$ $\because$ BA is $\parallel$ to Y-axis

$\therefore$ $\overrightarrow{CA} = CA\left\{\cos 30°\,\hat{i} + \sin 30°\,\hat{j}\right\}$

$$\frac{\overrightarrow{CA}}{CA} = \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} = \frac{\sqrt{3}\hat{i} + \hat{j}}{2}$$

$$\widehat{CA} = \frac{1}{2}\left(\sqrt{3}\,\hat{i} + \hat{j}\right)$$

Directional derivative of ϕ at (0, 2) in the direction of $\overrightarrow{CA} = \frac{\hat{i}}{4} \cdot \widehat{CA} = \frac{\hat{i}}{4} \cdot \frac{1}{2}\left(\sqrt{3}\hat{i} + \hat{j}\right) = \frac{\sqrt{3}}{8}$.

Example 10. *The temperature at any point in space is given by $T = xy + yz + zx$. Determine the directional derivative of T in the direction of the vector $3\hat{i} - 4\hat{k}$ at the point (1, 1, 1).*

Sol. $T = xy + yz + zx$

Gradient of $T = \nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx) = \hat{i}(y + z) + \hat{j}(z + x) + \hat{k}(x + y)$

Gradient of T at $(1, 1, 1) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

Directional derivative of T at (1, 1, 1) in the direction of $\left(3\hat{i} - 4\hat{k}\right)$

$$= \left(2\hat{i} + 2\hat{j} + 2\hat{k}\right) \cdot \left(\frac{3\hat{i} - 4\hat{k}}{\sqrt{9 + 16}}\right) = \frac{1}{5}(2 \cdot 3 - 2 \cdot 4) = \frac{-2}{5}.$$

Example 11. *(i) In what direction from (3, 1, – 2) is the directional derivative of $f = x^2 y^2 z^4$ maximum? Find also the magnitude of this maximum.*

(ii) Find the maximum value of directional derivative of $f = x^2 - 2y^2 + 4z^2$ at the point (1, 1, –1). **(P.T.U., May 2009)**

Sol. (*i*) Gradient of $\phi = \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 y^2 z^4)$

$$= \hat{i}\left(2xy^2z^4\right) + \hat{j}\left(2x^2yz^4\right) + \hat{k}\left(4x^2y^2z^3\right)$$

Gradient of ϕ at $(3, 1, -2) = 96\hat{i} + 288\hat{j} - 288\hat{k}$

Directional derivative is maximum in the direction given by $\Delta\phi$ at $(3, 1, -2)$ $= 96\hat{i} + 288\hat{j} - 288\hat{k}$

In any other direction the magnitude of the directional derivative will be less than its maximum value which is

$$= \sqrt{(96)^2 + (288)^2 + (288)^2} = 96\sqrt{1+9+9} = 96\sqrt{19}.$$

(ii) We know that the maximum value of directional derivative of $f = x^2 - 2y^2 + 4z^2$ at $(1, 1, -1)$ is obtained from the value of gradient of f

$$\therefore \quad \text{grad } f = \nabla f$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 - 2y^2 + 4z^2)$$

$$= 2x\hat{i} - 4y\hat{j} + 8z\hat{k}$$

Value of ∇f at $(1, 1, -1)$

$$= 2\hat{i} - 4\hat{j} - 8\hat{k}$$

$\therefore$ Maximum value of directional derivative of f at $(1, 1, -1)$

$$= \sqrt{4+16+64} = \sqrt{84} = 2\sqrt{21}$$

Example 12. *Let $f(x, y, z)$ and $\phi(x, y, z)$ be two scalar functions. Find an expression for $\nabla^2(fg)$ in terms of $\nabla^2 f$, $\nabla^2 g$, ∇f and ∇g.*

Sol.

$$\nabla(fg) = f\nabla(g) + g(\nabla f)$$

$$\nabla^2(fg) = \nabla[\nabla(fg)] = \nabla\{f(\nabla g)\} + \nabla\{g(\nabla f)\}$$

$$= f(\nabla^2 g) + (\nabla f)\cdot(\nabla g) + (\nabla g)\cdot(\nabla f) + g(\nabla^2 f)$$

$$= f(\nabla^2 g) + 2(\nabla f)\cdot(\nabla g) + g(\nabla^2 f).$$

Example 13. *Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$.*

Sol. Let

$$\phi_1 = x^2 + y^2 + z^2 - 9$$

and

$$\phi_2 = x^2 + y^2 - z - 3$$

$$\text{grad } \phi_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2 + y^2 + z^2 - 9) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

Similarly, $\text{grad } \phi_2 = 2x\hat{i} + 2y\hat{j} - \hat{k}$

Now, angle between the two surfaces at $(2, -1, 2)$ is the angle between their normals at the point.

$\therefore$ Let $\vec{n}_1$ and $\vec{n}_2$ be the normal vectors to ϕ_1 and ϕ_2 respectively at $(2, -1, 2)$.

Now,

$$\vec{n}_1 = \text{grad } \phi_1 = \nabla\phi_1 \quad \text{at} \quad (2, -1, 2) = 4\hat{i} - 2\hat{j} + 4\hat{k}.$$

$$\vec{n}_2 = \text{grad } \phi_2 = \nabla\phi_2 \quad \text{at} \quad (2, -1, 2) = 4\hat{i} - 2\hat{j} - \hat{k}$$

If θ be the angle between $\vec{n}_1$ and $\vec{n}_2$ then

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{4(4)-2(-2)+4(-1)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$\cos\theta = \frac{16+4-4}{6\sqrt{21}} = \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\therefore \qquad \theta = \cos^{-1}\frac{8}{3\sqrt{21}}.$$

TEST YOUR KNOWLEDGE

1. Find grad ϕ at $(1, -2, -1)$ if

   (i) $\phi = 3x^2y - y^3z^2$ (ii) $\phi = x^2 + yz$

   (iii) $\phi = \log(x^2 + y^2 + z^2)$

2. Find a unit vector normal to the following surfaces

   (i) $z^2 = x^2 + y^2$ at the point $(1, 0, -1)$

   (ii) $x\,y^3z^2 = 4$ at the point $(-1, -1, 2)$

   (iii) $x^2y + 2xz = 4$ at the point $(2, -2, 3)$

   (iv) $z = x^3 + y^2$ at $(1, -2, 5)$

   (v) $x^2 + 3y^2 + 2z^2 = 6$ at $(1, 0, 1)$ (P.T.U. May 2002)

3. If $r = |\vec{r}|$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

   (i) $\nabla f(r) = f'(r)\,\nabla(r)$ (ii) $\nabla(\log r) = \dfrac{\vec{r}}{r^2}$

   [Hint. S.E. 10 art 7.23]

   (iii) $\nabla\left(e^{r^2}\right) = 2e^{r^2}\vec{r}$ (iv) grad $|\vec{r}|^2 = 2\vec{r}$

4. (i) Find the directional derivative of $\phi = x^2 + y^2 + z^2$ at the point $(2, 2, 1)$ in the direction of $2\hat{i} + 2\hat{j} + \hat{k}$.

   (ii) What is the directional derivative of $2xy + z^2$ at the point $(1, -1, 3)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$?

5. Find the directional derivative of $\phi = 4xz^3 - 3x^2yz^2$ at the point $(2, -1, 2)$ along Z-axis. (P.T.U. May 2004)

6. Find the directional derivative of $f = 3e^{2x-y+z}$ at the point A $(1, 1, -1)$ in the direction $\overrightarrow{AB}$ where B is the point $(-3, 5, 6)$.

7. (i) Calculate the directional derivative of the function $\phi(x, y, z) = xy^2 + yz^3$ at the point $(1, -1, 1)$ in the direction of $(3, 1, -1)$.

   (ii) Find the directional derivative of $f(x, y, z) = x^2y^2z^2$ at the point $(1, 1, -1)$ in the direction of tangent to the curve $x = e^t$, $y = 2\sin t + 1$, $z = t - \cos t$ at $t = 0$.

   [Hint. See solved example 7]

8. Find the direction in which the directional derivative of $f(x, y) = (x^2 - y^2)\,xy$ at (1 [illegible] ero.

9. Find the directional derivative of the function $\phi = xy^2 + yz^3$

(i) In the direction of $\hat{i} + 2\hat{j} + 2\hat{k}$ at the point (2, –1, 1) ***(P.T.U., Dec. 2011)***

[**Hint.** See solved example 5]

(ii) In the direction of outward normal to the surface $x \log z - y^2 + 4 = 0$ at (–1, 2, 1)

[**Hint.** See solved example 8]

10. Find the directional derivative of the scalar function $f(x, y, z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point (3, 1, 3).

11. What is the greatest rate of increase of $u = x^2 + y z^2$ at the point (1, –1, 3) ?

12. If θ is the acute angle between the surfaces $xy^2z = 3x + z^2$ and $3x^2 - y^2 + 2z = 1$ at the point (1, –2, 1), show that $\cos q = \dfrac{3}{7\sqrt{6}}$.

13. Calculate the angle between the normals to the surface $xy = z^2$ at the point (4, 1, 2) and (3, 3, –3).

14. Find the angle between tangent planes to the surfaces $x \log z = y^2 - 1$ and $x^2 y = 2 - z$ at the point (1, 1, 1).

15. Find the values of constants a and b so that the surfaces $ax^2 - byz = (a + 2)x$ and $4x^2 y + z^3 = 4$ may intersect orthogonally at the point (1, –1, 2).

[**Hint.** The point P (1, –1, 2) lies on both the surfaces and ($\nabla\phi_1$ at P) • ($\nabla\phi_2$ at P) = 0]

16. The temperature at any point (x, y, z) in space is given by $T(x, y, z) = x^2 + y^2 - z$. A mosquito located at (1, 1, 2) desires to fly in such a direction that it will get warm as soon as possible. In which direction should it fly ?

17. If the directional derivative of $f(x, y, z) = axy + byz + czx$ at (1, 1, 1) has the maximum magnitude 4 in a direction parallel to x-axis then find the values of a, b, c.

18. Find the equation of the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at the point (1, –1, 2). Also find the equation of the normal at (1, –1, 2).

Answers

1. (i) $-12\hat{i} - 9\hat{j} - 16\hat{k}$ (ii) $2x\hat{i} + 2\hat{j} + y\hat{k}$ (iii) $\frac{1}{3}\left(\hat{i} - 2\hat{j} - \hat{k}\right)$

2. (i) $\frac{1}{\sqrt{2}}\left(\hat{i} + \hat{k}\right)$ (ii) $-\frac{1}{\sqrt{11}}\left(\hat{i} + 3\hat{j} - \hat{k}\right)$ (iii) $\frac{1}{3}\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$

(iv) $\dfrac{2\hat{i} - 4\hat{j} - \hat{k}}{\sqrt{21}}$ (v) $\dfrac{\hat{i} + 2\hat{k}}{\sqrt{5}}$

4. (i) 6 (ii) $\frac{14}{3}$

5. 144

6. $-\frac{5}{3}$

7. (i) $\frac{5}{\sqrt{11}}$ (ii) $\frac{2\sqrt{6}}{3}$

8. $\dfrac{\hat{i} + \hat{j}}{\sqrt{2}}$

9. (i) $-\frac{11}{3}$ (ii) $\frac{15}{\sqrt{17}}$

10. $\frac{27}{\sqrt{11}}$

11. $2\hat{i} + 9\hat{j} - 6\hat{k}$

13. $\cos^{-1}\sqrt{\frac{3}{62}}$

14. $\cos^{-1}\frac{1}{\sqrt{30}}$

15. $a = 2.5, b = 1$

16. $\frac{1}{3}\left(2\hat{i} + 2\hat{j} -\right.$

17. $a = 2, b = -2, c = 2$

18. $7x - 3y + 8z - 26 = 0; \dfrac{x-1}{7} = \dfrac{y+1}{-3} = \dfrac{z-2}{8}$.