7.18. DIVERGENCE OF A VECTOR FUNCTION

The divergence of a continuously differentiable vector point function $\vec{V}$ is denoted by div $\vec{V}$ and is defined as

$$\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot \vec{V} = \hat{i}\cdot\frac{\partial \vec{V}}{\partial x} + \hat{j}\cdot\frac{\partial \vec{V}}{\partial y} + \hat{k}\cdot\frac{\partial \vec{V}}{\partial z}$$

Clearly, divergence of a vector point function is a scalar point function.

If $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

then $\text{div } \vec{V} = \nabla \cdot \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$

For example, if $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then $\text{div } \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$

7.19. PHYSICAL INTERPRETATION OF DIVERGENCE

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to X-axis, Y-axis and Z-axis respectively.

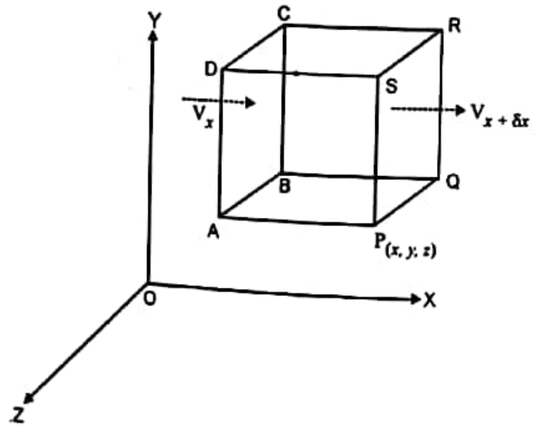
Let $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be the velocity of fluid at P (x, y, z) where (V_x, V_y, V_z) are components of $\vec{V}$ parallel to X-axis, Y-axis, Z-axis respectively.

Mass of the fluid flowing in through the face ABCD per unit time = Velocity × area of the face = $V_x\,(dy\,dz)$

∴ Mass of the fluid flowing out across the face PQRS per unit time = $V_{x+\delta x}\,(dy\,dz)$

$$V_{x+\delta x} = V_x + \delta x\,\frac{\partial V_x}{\partial x} + \cdots \text{ by Taylor's theorem.}$$

$$\therefore \quad V_{x+\delta x}\,(dydz) = \left(V_x + \frac{\partial V_x}{\partial x}\,dx\right)(dydz)$$



Decrease in mass of fluid in the parallelopiped corresponding to the flow along X-axis per unit time

$$= V_x\,dy\,dz - \left(V_x + \frac{\partial V_x}{\partial x}dx\right)dy\,dz = -\frac{\partial V_x}{\partial x}dx\,dy\,dz \quad \text{(–ve sign shows decrease)}$$

Similarly, the decrease in mass of fluid to the flow along Y-axis $= \dfrac{\partial V_y}{\partial y}dx\,dy\,dz$ and decrease in mass along Z-axis $= \dfrac{\partial V_z}{\partial z}dx\,dy\,dz$.

Total decrease in mass of fluid per unit time $= \left(\dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}\right)dx\,dy\,dz$

The rate of loss of fluid per unit volume $= \dfrac{\partial V_x}{\partial x} + \dfrac{\partial V_y}{\partial y} + \dfrac{\partial V_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}V_x + \hat{j}V_y + \hat{k}V_z\right) = \nabla\cdot\vec{V} = \operatorname{div}\vec{V} \quad \ldots(1)$$

$\therefore$ div $\vec{V}$ is the rate at which the fluid is flowing at a point per unit volume. If the flux entering any element of the space is the same as that leaving it *i.e.*, div $\vec{V} = 0$ everywhere then such a point function is called a **Solenoidal Vector Function.**

Equation (1) is also called the equation of continuity or conservation of mass.

Note. For details of the solenoidal vector function consult chapter 8 art 8.2.

7.20. CURL OF A VECTOR POINT FUNCTION *(P.T.U., May 2007)*

The curl of a continuously differentiable vector point function $\vec{V}$ is defined by the equation

$$\operatorname{Curl}\vec{V} = \nabla\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\vec{V}\right)$$

$$\operatorname{Curl}\vec{V} = \hat{i}\times\frac{\partial\vec{V}}{\partial x} + \hat{j}\times\frac{\partial\vec{V}}{\partial y} + \hat{k}\times\frac{\partial\vec{V}}{\partial z}$$

Clearly the curl of a vector point function is a vector point function.

If $\quad \vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Then $\quad \operatorname{curl}\vec{V} = \left(\hat{i}\dfrac{\partial}{\partial x} + \hat{j}\dfrac{\partial}{\partial y} + \hat{k}\dfrac{\partial}{\partial z}\right)\times\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Note. Curl of a vector point function is also called rotation of a vector point function.

7.21. PHYSICAL INTERPRETATION OF CURL OF A VECTOR POINT FUNCTION

(*P.T.U., Dec. 2003, May 2010, Dec. 2011*)

Consider a rigid body rotating about a fixed axis through the point O with uniform angular velocity ω. If $\vec{V}$ be the Linear Velocity and $\vec{r}$ be the position vector of any point on the rotating body.

then $\vec{V} = \vec{\omega} \times \vec{r}$

then $\text{curl } \vec{V} = \nabla \times \vec{V} = \nabla \times (\vec{\omega} \times \vec{r})$

Let $\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

then $\text{curl } \vec{V} = \nabla \times \left\{ \left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \right) \times \left(x\hat{i} + y\hat{j} + z\hat{k} \right) \right\}$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times \left\{ \hat{i}\,(\omega_2 z - \omega_3 y) + \hat{j}\,(\omega_3 x - \omega_1 z) + \hat{k}\,(\omega_1 y - \omega_2 x) \right\}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left\{ (\omega_2 z - \omega_3 y)\,\hat{i} + (\omega_3 x - \omega_1 z)\,\hat{j} + (\omega_1 y - \omega_2 x)\,\hat{k} \right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}\left\{ \frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_3 x - \omega_1 z) \right\} + \hat{j}\left\{ \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) - \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) \right\}$$

$$+ \hat{k}\left\{ \frac{\partial}{\partial x}(\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y) \right\}$$

$$= \hat{i}\,(\omega_1 + \omega_1) + \hat{j}\,(\omega_2 + \omega_2) + \hat{k}\,(\omega_3 + \omega_3) = \hat{i}\,2\omega_1 + \hat{j}\,2\omega_2 + \hat{k}\,2\omega_3$$

$$= 2\left(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k} \right) = 2\vec{\omega}$$

Curl $\vec{V} = 2\vec{\omega}$ shows that curl of a vector field is connected with rotational properties of the vector field and justifies the name *rotation* used for *curl*.

Hence the angular velocity at any point is equal to half the curl of linear velocity at that point of the body.

Cor. *If curl $\vec{V} = \vec{0}$, then $\vec{V}$ is called irrotational vector and the field V is termed irrotational.*

Note. For details of the irrotational vectors consult chapter 8 art 8.3 and 8.4.

7.22. PROPERTIES OF DIVERGENCE AND CURL

1. For a constant vector $\vec{a}$, div $\vec{a} = 0$, curl $\vec{a} = \vec{0}$

2. div $(\vec{A}+\vec{B}) =$ div $\vec{A}$ + div $\vec{B}$ or $\nabla\cdot(\vec{A}+\vec{B}) = \nabla\cdot\vec{A} + \nabla\cdot\vec{B}$

Proof.
$$\text{div}(\vec{A}+\vec{B}) = \nabla\cdot(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot(\vec{A}+\vec{B})$$
$$= \hat{i}\cdot\frac{\partial}{\partial x}(\vec{A}+\vec{B}) + \hat{j}\cdot\frac{\partial}{\partial y}(\vec{A}+\vec{B}) + \hat{k}\cdot\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$
$$= \hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)+\hat{j}\cdot\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right)+\hat{k}\cdot\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(i\cdot\frac{\partial\vec{A}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{A}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{A}}{\partial z}\right)+\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}+\hat{j}\cdot\frac{\partial\vec{B}}{\partial y}+\hat{k}\cdot\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{A}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\vec{B}$$
$$= \nabla\cdot\vec{A}+\nabla\cdot\vec{B}$$
$$= \text{div}\vec{A}+\text{div}\vec{B}$$

3. Curl $(\vec{A}+\vec{B}) =$ curl $\vec{A}$ + curl $\vec{B}$ or $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}$

Proof. Curl $(\vec{A}+\vec{B}) = \nabla\times(\vec{A}+\vec{B})$
$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times(\vec{A}+\vec{B})$$
$$= \hat{i}\times\frac{\partial}{\partial x}(\vec{A}+\vec{B})+\hat{j}\times\frac{\partial}{\partial y}(\vec{A}+\vec{B})+\hat{k}\times\frac{\partial}{\partial z}(\vec{A}+\vec{B})$$
$$= \hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}+\frac{\partial\vec{B}}{\partial x}\right)+\hat{j}\times\left(\frac{\partial\vec{A}}{\partial y}+\frac{\partial\vec{B}}{\partial y}\right)+\hat{k}\times\left(\frac{\partial\vec{A}}{\partial z}+\frac{\partial\vec{B}}{\partial z}\right)$$
$$= \left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}+\hat{j}\frac{\partial\vec{A}}{\partial y}+\hat{k}\frac{\partial\vec{A}}{\partial z}\right)+\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}+\hat{j}\frac{\partial\vec{B}}{\partial y}+\hat{k}\frac{\partial\vec{B}}{\partial z}\right)$$
$$\text{Curl}(\vec{A}+\vec{B}) = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{A}+\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{B}$$
$$= \nabla\times\vec{A}+\nabla\times\vec{B} = \text{curl}\,\vec{A}+\text{curl}\,\vec{B}$$

Hence curl $(\vec{A}+\vec{B}) =$ curl $\vec{A}$ + curl $\vec{B}$

4. If $\vec{A}$ is a vector function and ϕ is a scalar function, then

$$\text{div}\,(\phi\vec{A}) = \phi(\text{div}\,\vec{A}) + (\text{grad}\,\phi)\cdot\vec{A}$$

or
$$\nabla\cdot(\phi\vec{A}) = \phi(\nabla\cdot\vec{A}) + (\nabla\phi)\cdot\vec{A}$$

Proof. $\text{div}(\phi\vec{A}) = \nabla\cdot(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(\phi\vec{A})$

$$= \hat{i}\cdot\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\cdot\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\cdot\frac{\partial}{\partial z}(\phi\vec{A}) = \Sigma\hat{i}\cdot\left(\phi\frac{\partial}{\partial x}\vec{A} + \frac{\partial\phi}{\partial x}\vec{A}\right)$$

$$= \phi\Sigma\hat{i}\cdot\frac{\partial}{\partial x}\vec{A} + \Sigma\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A}$$

$$= \phi\left\{\hat{i}\cdot\frac{\partial}{\partial x}\vec{A} + \hat{j}\cdot\frac{\partial}{\partial y}\vec{A} + \hat{k}\cdot\frac{\partial}{\partial z}\vec{A}\right\} + \left\{\left(\hat{i}\frac{\partial\phi}{\partial x}\right)\cdot\vec{A} + \left(\hat{j}\frac{\partial\phi}{\partial y}\right)\cdot\vec{A} + \left(\hat{k}\frac{\partial\phi}{\partial z}\right)\cdot\vec{A}\right\}$$

$$= \phi\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\cdot\vec{A} + \left\{\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right\}\cdot\vec{A}$$

$$= \phi(\nabla\cdot\vec{A}) + (\nabla\phi)\cdot\vec{A}$$

$$\therefore\ \text{div}\,(\phi\vec{A}) = \phi(\text{div}\,\vec{A}) + (\text{grad}\,\phi)\cdot\vec{A}$$

5. If $\vec{A}$ is a vector function and ϕ is a scalar function then *(P.T.U., Jan. 2010)*

$$\text{Curl}\,(\phi\vec{A}) = (\text{grad}\,\phi)\times\vec{A} + \phi\,\text{curl}\,\vec{A}$$

or
$$\nabla\times(\phi\vec{A}) = (\nabla\phi)\times\vec{A} + \phi(\nabla\times\vec{A}).$$

Proof. $\text{Curl}\,(\phi\vec{A}) = \nabla\times(\phi\vec{A}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times(\phi\vec{A})$

$$= \hat{i}\times\frac{\partial}{\partial x}(\phi\vec{A}) + \hat{j}\times\frac{\partial}{\partial y}(\phi\vec{A}) + \hat{k}\times\frac{\partial}{\partial z}(\phi\vec{A})$$

$$= \hat{i}\times\left\{\frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}\right\} + \hat{j}\times\left\{\frac{\partial\phi}{\partial y}\vec{A} + \phi\frac{\partial\vec{A}}{\partial y}\right\} + \hat{k}\times\left\{\frac{\partial\phi}{\partial z}\vec{A} + \phi\frac{\partial\vec{A}}{\partial z}\right\}$$

$$= \left\{\hat{i}\times\left(\frac{\partial\phi}{\partial x}\vec{A}\right) + \hat{j}\times\left(\frac{\partial\phi}{\partial y}\vec{A}\right) + \hat{k}\times\left(\frac{\partial\phi}{\partial z}\vec{A}\right)\right\} + \phi\left\{\hat{i}\times\frac{\partial\vec{A}}{\partial x} + \hat{j}\times\frac{\partial\vec{A}}{\partial y} + \hat{k}\times\frac{\partial\vec{A}}{\partial z}\right\}$$

$$= \left\{\frac{\partial\phi}{\partial x}\hat{i}\times\vec{A} + \frac{\partial\phi}{\partial y}\hat{j}\times\vec{A} + \frac{\partial\phi}{\partial z}\hat{k}\times\vec{A}\right\} + \phi\left\{\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right\}\times\vec{A}$$

$$= \left\{\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\phi\right\}\times\vec{A}+\phi\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}\times\vec{A}$$

$$= (\nabla\phi)\times\vec{A}+\phi\left(\nabla\times\vec{A}\right) = (\text{grad}\ \phi)\times\vec{A}+\phi\left(\text{curl}\ \vec{A}\right)$$

6. $\nabla\left(\vec{A}\cdot\vec{B}\right)=\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)$

Proof. $\nabla\left(\vec{A}\cdot\vec{B}\right)=\Sigma\,\hat{i}\,\frac{\partial}{\partial x}\left(\vec{A}\cdot\vec{B}\right)=\Sigma\,i\left\{A\,\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\cdot\vec{B}\right\}=\Sigma\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}+\Sigma\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}$

Now, we know that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(\vec{a}\cdot\vec{c})\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{c}$

$\therefore$ $$\left(\vec{a}\cdot\vec{b}\right)\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-\vec{a}\times\left(\vec{b}\times\vec{c}\right)$$

$\therefore$ $$\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}=\left(\vec{A}\cdot\hat{i}\right)\frac{\partial\vec{B}}{\partial x}-\vec{A}\times\left(\frac{\partial\vec{B}}{\partial x}\times\hat{i}\right)=\left(\vec{A}\cdot\hat{i}\right)\frac{\partial\vec{B}}{\partial x}+\vec{A}\times\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$\therefore$ $$\Sigma\left(\vec{A}\cdot\frac{\partial\vec{B}}{\partial x}\right)\hat{i}=\left(\vec{A}\cdot\Sigma\,\hat{i}\,\frac{\partial}{\partial x}\right)\vec{B}+\vec{A}\times\Sigma\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)=\left(\vec{A}\cdot\nabla\right)\vec{B}+\vec{A}\times\left(\nabla\times\vec{B}\right)$$

Similarly, $$\Sigma\left(\vec{B}\cdot\frac{\partial\vec{A}}{\partial x}\right)\hat{i}=\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{B}\times\left(\nabla\times\vec{A}\right)$$ **(By interchanging A and B)**

$\therefore$ $$\nabla\left(\vec{A}\cdot\vec{B}\right)=\left(\vec{A}\cdot\nabla\right)\vec{B}+\vec{A}\times\left(\nabla\times\vec{B}\right)+\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{B}\times\left(\nabla\times\vec{A}\right)$$

$$=\left(\vec{A}\cdot\nabla\right)\vec{B}+\left(\vec{B}\cdot\nabla\right)\vec{A}+\vec{A}\times\left(\nabla\times\vec{B}\right)+\vec{B}\times\left(\nabla\times\vec{A}\right)$$

7. $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$ ***(P.T.U., Jan. 2010)***

Proof. $\nabla\cdot\left(\vec{A}\times\vec{B}\right)=\Sigma\,\hat{i}\cdot\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)=\Sigma\,\hat{i}\cdot\left\{\vec{A}\times\frac{\partial\vec{B}}{\partial x}+\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right\}$

$$=\Sigma\left\{\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)\right\}+\Sigma\left\{\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)\right\}=\Sigma\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)-\Sigma\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)$$

$$=\Sigma\left(\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}-\Sigma\left(\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}=\left(\nabla\times\vec{A}\right)\cdot\vec{B}-\left(\nabla\times\vec{B}\right)\cdot\vec{A}$$

$$=\vec{B}\cdot\left(\nabla\times\vec{A}\right)-\vec{A}\cdot\left(\nabla\times\vec{B}\right)$$

8. $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$ *(P.T.U., May 2010, May 2012)*

Proof.
$$\nabla \times (\vec{A} \times \vec{B}) = \Sigma \hat{i} \times \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) = \Sigma \hat{i} \times \left\{\frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x}\right\}$$
$$= \Sigma \hat{i} \times \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) + \Sigma \hat{i} \times \left(\vec{A} \times \frac{\partial \vec{B}}{\partial x}\right)$$
$$= \Sigma\left\{(\hat{i} \cdot \vec{B})\frac{\partial \vec{A}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right)\vec{B}\right\} + \Sigma\left\{\left(\hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right)\vec{A} - (\hat{i} \cdot \vec{A})\frac{\partial \vec{B}}{\partial x}\right\}$$
$$= \Sigma(\vec{B} \cdot \hat{i})\frac{\partial \vec{A}}{\partial x} - \left\{\Sigma \hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right\}\vec{B} + \left\{\Sigma \hat{i} \cdot \frac{\partial \vec{B}}{\partial x}\right\}\vec{A} - \Sigma(\vec{A} \cdot \hat{i})\frac{\partial \vec{B}}{\partial x}$$
$$= \left(\vec{B} \cdot \Sigma \hat{i} \frac{\partial}{\partial x}\right)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - \left(\vec{A} \cdot \Sigma \hat{i} \frac{\partial}{\partial x}\right)\vec{B}$$
$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$
$$= (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

7.23. REPEATED OPERATIONS BY ∇

Before starting with the repeated operations by ∇, students are advised to note the following:

If $\phi(x, y, z)$ and $\vec{V}(x, y, z)$ be scalar and vector point functions respectively, then

(*i*) Since ϕ is scalar we can take its gradient only.

(*ii*) Since grad ϕ and $\vec{V}$ are both vector functions we can take their divergence as well as curl.

(*iii*) Since div $\vec{V}$ is a scalar function we can take its gradient only.

1. Div (grad ϕ) = $\nabla^2 \phi$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Proof. Div (grad ϕ) = $\nabla \cdot (\nabla \phi)$
$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2 \phi$$

Note 1. ∇^2 is called Laplacian Operator and $\nabla^2 \phi = 0$ is called **Laplace Equation.**

Note 2. A function satisfying Laplace Equation is called **Harmonic Function** *i.e.*, ϕ is Harmonic Function.

2. Curl (grad ϕ) = $\nabla \times (\nabla \phi) = \vec{0}$

Proof. Curl (grad ϕ) = $\nabla \times (\nabla \phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \Sigma \hat{i} \left\{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial y}\right)\right\} = \Sigma \hat{i} \left\{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right\} = \vec{0} \qquad \left(\because \frac{\partial^2 \phi}{\partial y \partial z} = \frac{\partial^2 \phi}{\partial z \partial y}\right)$$

Hence curl (grad ϕ) = $\vec{0}$

Note. Curl (grad ϕ) = $\vec{0}$ implies that gradient field describes an irrotational motion.

3. Div. $\left(\text{Curl } \vec{V}\right) = \nabla \cdot \left(\nabla \times \vec{V}\right) = 0$

Proof. Div $\left(\text{Curl } \vec{V}\right) = \nabla \cdot \left(\nabla \times \vec{V}\right)$

Let $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\therefore \qquad \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \Sigma \hat{i} \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)$$

$$\therefore \quad \text{Div}\left(\text{Curl } \vec{V}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\Sigma \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\right)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left\{\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left\{\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right\}$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y} = 0$$

Hence Div $\cdot \left(\text{Curl } \vec{V}\right) = 0$

Note. Div $\left(\text{curl } \vec{V}\right) = 0$ implies that curl $\vec{V}$ is a solenoidal vector point function.

4. Curl $\left(\text{Curl } \vec{V}\right)$ = grad div $\vec{V} - \nabla^2 \vec{V}$ *(P.T.U., June 2003, Dec. 2005)*

or Curl $\left(\text{Curl } \vec{V}\right) = \nabla\left(\nabla \cdot \vec{V}\right) - (\nabla \cdot \nabla)\vec{V}$

Proof. Let $\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

$$\text{Curl } \vec{V} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$\text{Curl Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} & \frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} & \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \end{vmatrix}$$

$$= \Sigma \hat{i}\left\{\frac{\partial}{\partial y}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right)\right\} = \Sigma \hat{i}\left\{\frac{\partial^2 V_2}{\partial y \partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2} + \frac{\partial^2 V_3}{\partial z\, \partial x}\right\}$$

$$= \Sigma \hat{i}\left\{\frac{\partial^2 V_2}{\partial y\, \partial x} + \frac{\partial^2 V_3}{\partial z\, \partial x} - \frac{\partial^2 V_1}{\partial y^2} - \frac{\partial^2 V_1}{\partial z^2}\right\}$$

Add and subtract $\dfrac{\partial^2 V_1}{\partial x^2}$

$$= \Sigma \hat{i}\left\{\left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_2}{\partial x\, \partial y} + \frac{\partial^2 V_3}{\partial x\, \partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right\}$$

$$= \Sigma \hat{i}\left[\frac{\partial}{\partial x}\left(\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\right) - \left(\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}\right)\right]$$

$$= \Sigma \hat{i}\left\{\frac{\partial}{\partial x}\left(\nabla \cdot \vec{V}\right) - \nabla^2 V_1\right\} = \Sigma \hat{i}\frac{\partial}{\partial x}\left(\nabla \cdot \vec{V}\right) - \nabla^2 \Sigma \hat{i}\, V_1$$

$$= \nabla\left(\nabla \cdot \vec{V}\right) - \nabla^2 \vec{V} = \text{grad }(\text{div } \vec{V}) - \nabla^2 \vec{V}$$

Cor. From above result we can also deduce

$$\text{grad }\left(\text{div } \vec{V}\right) = \text{Curl}\left(\text{Curl } \vec{V}\right) + \nabla^2 \vec{V}$$ ***(P.T.U. Dec. 2011)***

or
$$\nabla\left(\nabla \cdot \vec{V}\right) = \nabla \times \left(\nabla \times \vec{V}\right) + \nabla^2 \vec{V}$$

Note. For application in questions, the results of repeated application of ∇ can easily be written down (treating ∇ as a vector)

(i) $\nabla \cdot \nabla \phi = \nabla^2 \phi$ $\quad\because\ \vec{a} \cdot \vec{a} = a^2$

(ii) $\nabla \times \nabla \phi = \vec{0}$ $\quad\because\ \vec{a} \times \vec{a} = \vec{0}$

(iii) $\nabla \cdot \left(\nabla \times \vec{V}\right) = 0$ $\quad\because$ in scalar triple product $\vec{a} \cdot \left(\vec{a} \times \vec{b}\right) = 0$

(iv) $\nabla \times (\nabla \times \vec{V}) = (\nabla \cdot \vec{V})\, \nabla - \nabla^2 V$ $\quad\because\ \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\, \vec{b} - (\vec{a} \cdot \vec{b})\, \vec{c}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate the following:*

(i) *Div*$\left[(xy\sin z)\hat{i} + \left(y^2 \sin x\right)\hat{j} + \left(z^2 \sin xy\right)\hat{k}\right]$ *at the point* $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$.

(ii) *Curl Curl of* $\bar{V} = \left(2xz^2\right)\hat{i} - yz\,\hat{j} + \left(3xz^3\right)\hat{k}$ *at (1, 1, 1)* **(P.T.U., May 2006)**

Sol. (i) Div $\left[(xy\sin z)\hat{i} + \left(y^2 \sin x\right)\hat{j} + \left(z^2 \sin xy\right)\hat{k}\right]$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[(xy\sin z)\hat{i} + \left(y^2 \sin x\right)\hat{j} + \left(z^2 \sin xy\right)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xy\sin z) + \frac{\partial}{\partial y}\left(y^2 \sin x\right) + \frac{\partial}{\partial z}\left(z^2 \sin xy\right)$$

$$= y \sin z + 2y \sin x + 2z \sin xy$$

At the point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\text{div}\left\{(xy\sin z)\hat{i} + \left(y^2 \sin x\right)\hat{j} + \left(z^2 \sin xy\right)\hat{k}\right\} = \frac{\pi}{2} + 0 + 0 = \frac{\pi}{2}$$

(ii) $$\vec{V} = \left(2xz^2\right)\hat{i} - yz\,\hat{j} + \left(3xz^3\right)\hat{k}$$

$$\text{Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}\left(3xz^3\right) - \frac{\partial}{\partial z}(-yz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}\left(3xz^3\right) - \frac{\partial}{\partial z}\left(2xz^2\right)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial y}\left(2xz^2\right)\right\}$$

$$= \hat{i}\,\{y\} - \hat{j}\left\{3z^3 - 4xz\right\} + \hat{k}\,\{0\} = y\hat{i} + \left(-3z^3 + 4xz\right)\hat{j}$$

$$\text{Curl Curl } \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -3z^3 + 4xz & 0 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(-3z^3 + 4xz\right)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(y)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}\left(-3z^3 + 4xz\right) - \frac{\partial}{\partial y}(y)\right\}$$

$$= -\hat{i}\left(-9z^2 + 4x\right) + \hat{j}(0) + \hat{k}\{4z - 1\}$$

$$= \left(9z^2 - 4x\right)\hat{i} + (4z - 1)\hat{k}$$

At (1, 1, 1) curl curl $\vec{V} = 5\hat{i} + 3\hat{k}$.

Example 2. *Find div $\vec{F}$ and curl $\vec{F}$, where $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$.*

Sol. $\vec{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^3 + y^3 + z^3 - 3xyz\right)$

$$= \hat{i}\left(3x^2 - 3yz\right) + \hat{j}\left(3y^2 - 3zx\right) + \hat{k}\left(3z^2 - 3xy\right)$$

Now, $\text{div}\,\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left\{3\left(x^2 - yz\right)\hat{i} + 3\left(y^2 - zx\right)\hat{j} + 3\left(z^2 - xy\right)\hat{k}\right\}$

$$= 3\left\{\frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right)\right\}$$

$$= 3\,\{2x + 2y + 2z\} = 6\,(x + y + z)$$

Curl $\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left\{3\left(x^2 - yz\right)\hat{i} + 3\left(y^2 - zx\right)\hat{j} + 3\left(z^2 - xy\right)\hat{k}\right\}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix} = 3\Sigma\hat{i}\left\{\frac{\partial}{\partial y}\left(z^2 - xy\right) - \frac{\partial}{\partial z}\left(y^2 - zx\right)\right\} = 3\Sigma\hat{i}\,\{-x + x\}$$

$$= 3\,\Sigma\,\hat{i}\,.\,0 = \vec{0}.$$

Example 3. *If $u = x^2 + y^2 + z^2$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then find $\text{div}(u\vec{r})$ in terms of u.*

(P.T.U., Dec. 2005)

Sol. $\text{div}\,(u\vec{r}) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(u\vec{r})$

Now, $(u\vec{r}) = (x^2 + y^2 + z^2)\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$

$$= x\left(x^2 + y^2 + z^2\right)\hat{i} + y\left(x^2 + y^2 + z^2\right)\hat{j} + z\left(x^2 + y^2 + z^2\right)\hat{k}$$

$$\frac{\partial}{\partial x}(u\vec{r}) = \left(3x^2 + y^2 + z^2\right)\hat{i} + (2xy)\,\hat{j} + (2xz)\,\hat{k}$$

$$\frac{\partial}{\partial y}(u\vec{r}) = (2xy)\hat{i} + \left(x^2 + z^2 + 3y^2\right)\hat{j} + (2yz)\hat{k}$$

$$\frac{\partial}{\partial z}(u\vec{r}) = (2xz)\hat{i} + (2yz)\hat{j} + \left(x^2 + y^2 + 3z^2\right)\hat{k}$$

$$\therefore \quad \text{div}\,(u\vec{r}) = \hat{i}\cdot\frac{\partial}{\partial x}(u\vec{r}) + \hat{j}\cdot\frac{\partial}{\partial y}(u\vec{r}) + \hat{k}\cdot\frac{\partial}{\partial z}(u\vec{r})$$

$$= (3x^2 + y^2 + z^2) + (x^2 + z^2 + 3y^2) + (x^2 + y^2 + 3z^2)$$

$$= 5x^2 + 5y^2 + 5z^2 = 5(x^2 + y^2 + z^2) = 5u.$$

Example 4. *Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{a}$ be a constant vector, find the value of div $\dfrac{\vec{a}\times\vec{r}}{r^n}$.*

Sol. Let $\quad \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\vec{a}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = \hat{i}(a_2z - a_3y) + \hat{j}(a_3x - a_1z) + \hat{k}(a_1y - a_2x)$$

$$\frac{\vec{a}\times\vec{r}}{r^n} = \frac{\Sigma\hat{i}\,(a_2z - a_3y)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}$$

$$\text{div}\left(\frac{\vec{a}\times\vec{r}}{r^n}\right) = \nabla\cdot\frac{\vec{a}\times\vec{r}}{r^n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\frac{(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}$$

$$= \frac{\partial}{\partial x}\frac{(a_2z - a_3y)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial y}\frac{(a_3x - a_1z)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}} + \frac{\partial}{\partial z}\frac{(a_1y - a_2x)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}}}$$

$$= \frac{(a_2z - a_3y)\left(-\frac{n}{2}\right)(2x)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}+1}} + \frac{(a_3x - a_1z)\left(-\frac{n}{2}\right)(2y)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}+1}} + \frac{(a_1y - a_2x)\left(-\frac{n}{2}\right)(2z)}{\left(x^2+y^2+z^2\right)^{\frac{n}{2}+1}}$$

$$= \frac{-n}{\left(x^2+y^2+z^2\right)^{\frac{n+2}{2}}}\left[x(a_2z - a_3y) + y(a_3x - a_1z) + z(a_1y - a_2x)\right]$$

$$= \frac{-n}{\left(x^2+y^2+z^2\right)^{\frac{n+2}{2}}}\cdot 0 = 0$$

Hence $\text{div}\left(\dfrac{\vec{a}\times\vec{r}}{r^n}\right) = 0.$

Example 5. *Given the vector field* $\vec{V} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$ *find Curl* $\vec{V}$. *Show that the vectors given by Curl* $\vec{V}$ *at P (1, 2, – 3) and Q (2, 3, 12) are orthogonal.*

Sol. $\vec{V} = (x^2 - y^2 + 2xz)\hat{i} + (xz - xy + yz)\hat{j} + (z^2 + x^2)\hat{k}$

Curl $\vec{V} = \nabla \times \vec{V}$.

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 + 2xz & xz - xy + yz & z^2 + x^2 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(z^2 + x^2) - \frac{\partial}{\partial z}(xz - xy + yz)\right\} + \hat{j}\left\{\frac{\partial}{\partial z}(x^2 - y^2 + 2xz) - \frac{\partial}{\partial x}(z^2 + x^2)\right\}$$

$$+ \hat{k}\left\{\frac{\partial}{\partial x}(xz - xy + yz) - \frac{\partial}{\partial y}(x^2 - y^2 + 2xz)\right\}$$

$$= \hat{i}\{-(x + y)\} + \hat{j}\{2x - 2x\} + \hat{k}\{z - y + 2y\} = -(x + y)\hat{i} + (y + z)\hat{k}$$

Curl $\vec{V}$ at P (1, 2, – 3) $= -3\hat{i} - \hat{k}$

Curl $\vec{V}$ at Q (2, 3, 12) $= -5\hat{i} + 15\hat{k}$.

Curl $\vec{V}$ at P, Q will be orthogonal if their dot product is zero.

i.e., $\quad (-3\hat{i} - \hat{k}) \cdot (-5\hat{i} + 15\hat{k}) = 15 - 15 = 0.$

Hence curl vectors at P and Q are orthogonal.

Example 6. *If* $u\vec{F} = \nabla v$, *where u, v are scalars fields and* $\vec{F}$ *is a vector field, show that* $\vec{F} \cdot \text{curl}\, \vec{F} = 0$.

Sol. $\quad$ Curl $\vec{F}$ = Curl $\left(\frac{1}{u}\nabla v\right) = \nabla \times \left(\frac{1}{u}\nabla v\right)$

We know that $\nabla \times (\phi\vec{A}) = \nabla\phi \times \vec{A} + \phi\nabla \times \vec{A}$

$\therefore \quad$ Curl $\vec{F} = \left(\nabla \frac{1}{u}\right) \times (\nabla v) + \frac{1}{u}\nabla \times (\nabla v) = \left(\nabla \frac{1}{u}\right) \times (\nabla v) \qquad (\because \nabla \times (\nabla v) = \vec{0})$

$\vec{F} \cdot$ Curl $\vec{F} = \left(\frac{1}{u}\nabla v\right) \cdot \left(\nabla \frac{1}{u}\right) \times (\nabla v) = \frac{1}{u}\left\{\nabla v \cdot \nabla \frac{1}{u} \times \nabla v\right\}$

$$= \frac{1}{u}\left[\nabla v, \nabla \frac{1}{u}, \nabla v\right] = \frac{1}{u} \bullet 0 = 0$$

[$\because$ In scalar triple product two vectors are equal]

Hence $\vec{F} \bullet \text{Curl } \vec{F} = 0$

Example 7. *If* $\vec{A} = \nabla \times (\phi \hat{i})$ *where* $\nabla^2 \phi = 0$, *show that* $\vec{A} \bullet \nabla \times \vec{A} = \frac{\partial \phi}{\partial z} \bullet \frac{\partial^2 \phi}{\partial y\, \partial x} - \frac{\partial \phi}{\partial y} \bullet \frac{\partial^2 \phi}{\partial z\, \partial x}$.

Sol. $\vec{A} = \nabla \times (\phi \hat{i}) = \nabla \phi \times \hat{i}$

$$\nabla \times \vec{A} = \nabla \times (\nabla \phi \times \hat{i}) = (\nabla \bullet \hat{i})\nabla \phi - (\nabla \bullet \nabla \phi)\hat{i} \text{ using } \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \bullet \vec{b})\vec{b} - (\vec{a} \bullet \vec{b})\vec{c}$$

$$\nabla \bullet \hat{i} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \bullet \hat{i} = \frac{\partial}{\partial x}$$

$$\therefore\ \nabla \times \vec{A} = \frac{\partial}{\partial x}(\nabla \phi) - (\nabla^2 \phi)\hat{i} = \frac{\partial}{\partial x}\left[\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right] - \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i}$$

$$= \hat{i}\frac{\partial^2 \phi}{\partial x^2} + \hat{j}\frac{\partial^2 \phi}{\partial x\, \partial y} + \hat{k}\frac{\partial^2 \phi}{\partial x\, \partial z} - \frac{\partial^2 \phi}{\partial x^2}\hat{i} - \frac{\partial^2 \phi}{\partial y^2}\hat{i} - \frac{\partial^2 \phi}{\partial z^2}\hat{i}$$

$$= -\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}$$

$$\vec{A} \bullet \nabla \times \vec{A} = (\nabla \times \phi \hat{i}) \bullet \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= (\nabla \phi \times \hat{i}) \bullet \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= \left[\left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \times \hat{i}\right] \bullet \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$= \left[-\frac{\partial \phi}{\partial y}\hat{k} + \frac{\partial \phi}{\partial z}\hat{j}\right] \bullet \left[-\left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right)\hat{i} + \frac{\partial^2 \phi}{\partial x\, \partial y}\hat{j} + \frac{\partial^2 \phi}{\partial x\, \partial z}\hat{k}\right]$$

$$\left(\because \hat{i} \times \hat{i} = \vec{0},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{i} = \hat{j}\right)$$

$$= -\frac{\partial \phi}{\partial y} \bullet \frac{\partial^2 \phi}{\partial x\, \partial z} + \frac{\partial \phi}{\partial z} \bullet \frac{\partial^2 \phi}{\partial x\, \partial y} = \frac{\partial \phi}{\partial z} \bullet \frac{\partial^2 \phi}{\partial y\, \partial x} - \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial z\, \partial x}$$

Example 8. *By taking $\vec{F} = u \nabla v$, where u and v are scalars, prove that*

$\nabla \cdot \vec{F} = u \nabla^2 v + \nabla u \cdot \nabla v.$

Sol.
$$\vec{F} = u \nabla v = u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$\nabla \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot u\left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

Differentiate by product rule.

$$= \left(\hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$+ u\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(\hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}\right)$$

$$= \nabla u \cdot \nabla v + u\left[\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial z}\right)\right]$$

$$= \nabla u \cdot \nabla v + u\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] = \nabla u \cdot \nabla v + u\nabla^2 v$$

$$= u \nabla^2 v + \nabla u \cdot \nabla v$$

Example 9. *If r is the distance of a point (x, y, z) from the origin, prove that* $curl\left(\hat{k} \times grad\,\frac{1}{r}\right) + grad\left(\hat{k} \cdot grad\,\frac{1}{r}\right) = 0$, *where $\hat{k}$ is a unit vector in the direction of z.*

Sol.
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \frac{1}{r} = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$$

$$\text{grad}\,\frac{1}{r} = \nabla\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-\frac{3}{2}}\left[2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right]$$

$$= \frac{-1}{\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}}}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Now $\hat{k}\times \text{grad}\,\frac{1}{r} = \hat{k}\times\left\{\frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right\}\left\{x\hat{i}+y\hat{j}+z\hat{k}\right\} = \frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left\{x\hat{j}-y\hat{i}\right\}$

$$\text{Curl}\left(\hat{k}\times \text{grad}\frac{1}{r}\right) = \nabla\times\left[\frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right]\left[x\hat{j}-y\hat{i}\right]$$

$$= \nabla\times\left\{\frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\hat{j}+\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\hat{i}\right\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} & \frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}} & 0 \end{vmatrix}$$

$$= \hat{i}\left\{0-\frac{\partial}{\partial z}\frac{-x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right\}+\hat{j}\left\{\frac{\partial}{\partial z}\left(\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)-0\right\}$$

$$+\hat{k}\left\{-\frac{\partial}{\partial x}\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}-\frac{\partial}{\partial y}\frac{y}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right\}$$

$$= \hat{i}\left\{\frac{-3xz}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\right\}+\hat{j}\left\{\frac{-3yz}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\right\}$$

$$+\hat{k}\left\{\frac{3x^2-x^2-y^2-z^2+3y^2-x^2-y^2-z^2}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}\right\}$$

$$\therefore\quad \text{Curl}\left(\hat{k}\times \text{grad}\,\frac{1}{r}\right) = \frac{\hat{i}(-3xz)+\hat{j}(-3yz)+\hat{k}\left(x^2+y^2-2z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \qquad \ldots(1)$$

Now, $\hat{k}\bullet\text{grad}\,\frac{1}{r} = \hat{k}\bullet\frac{-1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\left(x\hat{i}+y\hat{j}+z\hat{k}\right) = -\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}z$

$$\text{grad}\left(\hat{k}\cdot\text{grad}\frac{1}{r}\right)=\nabla\left\{-\frac{z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right\}=\left\{\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right\}\left(\frac{-z}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)$$

$$=\left[\frac{\hat{i}(-z)\left(-\frac{3}{2}\right)(2x)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}+\hat{j}\frac{(-z)\left(-\frac{3}{2}\right)(2y)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}+\hat{k}\left\{\frac{(-z)\left(-\frac{3}{2}\right)(2z)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}-\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right\}\right]$$

$$=\frac{\hat{i}\,(3xz)+\hat{j}(3yz)+\hat{k}\left(3z^2-x^2-y^2-z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}}$$

$$=\frac{\hat{i}\,(3xz)+\hat{j}(3yz)+\hat{k}\left(-x^2-y^2+2z^2\right)}{\left(x^2+y^2+z^2\right)^{\frac{5}{2}}} \qquad \text{...(2)}$$

Adding (1) and (2), we get $Curl\left(\hat{k}\times grad\frac{1}{r}\right)+grad\left(\hat{k}\cdot grad\frac{1}{r}\right)=0.$

Example 10. *Prove that (i)* $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r).$

(ii) $\nabla^2(r^n)=n(n+1)r^{n-2}$ **(P.T.U., May 2007, May 2008)**

Sol. (i) $\nabla^2 f(r)=\nabla\cdot\{\nabla f(r)\}=\text{div}\,\{\text{grad}\,f(r)\}$

$$\text{grad}\,f(r)=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)f(r)=\hat{i}\frac{\partial}{\partial x}f(r)+\hat{j}\frac{\partial}{\partial y}f(r)+\hat{k}\frac{\partial}{\partial z}f(r)$$

$$=\hat{i}f'(r)\frac{\partial r}{\partial x}+\hat{j}f'(r)\frac{\partial r}{\partial y}+\hat{k}f'(r)\frac{\partial r}{\partial z}=f'(r)\left[\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}\right]$$

$$=f'(r)\,\text{grad}\,r \qquad \text{...(1)}$$

$\because \qquad \nabla^2 f(r)=\text{div}\,[f'(r)\,\text{grad}\,r]$

We know that $\text{grad}\,r=\frac{\vec{r}}{r}$ (See S.E. 4 art 7.17)

$$=\text{div}\left[f'(r)\frac{\vec{r}}{r}\right]$$

$$=\text{div}\left[\frac{f'(r)}{r}\vec{r}\right]$$

which is of the type $\text{div}(\phi\vec{a})$ where $\phi=\frac{f'(r)}{r}$ and $\vec{a}=\vec{r}$

$$\therefore \quad \operatorname{div}\left[\frac{f'(r)}{r}\vec{r}\right] = \frac{f'(r)}{r}\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\frac{f'(r)}{r}$$

Since $\operatorname{div}\vec{r} = 3$ and from (1) grad $f(r) = f'(r)$ grad r; Replace $f(r)$ by $\frac{f'(r)}{r}$, we get

$$\operatorname{grad}\frac{f'(r)}{r} = \frac{d}{dr}\left[\frac{f'(r)}{r}\right]\operatorname{grad} r$$

$$= \frac{f'(r)}{r}3 + \vec{r}\cdot\left[\frac{d}{dr}\left(\frac{f'(r)}{r}\right)\operatorname{grad} r\right]$$

$$= \frac{3f'(r)}{r} + \vec{r}\cdot\left\{\frac{rf''(r)-f'(r)}{r^2}\right\}\cdot\frac{\vec{r}}{r}$$

$$= \frac{3f'(r)}{r} + \left\{\frac{f''(r)}{r^2} - \frac{f'(r)}{r^3}\right\}\vec{r}\cdot\vec{r} = \frac{3f'(r)}{r} + \left\{f''(r) - \frac{f'(r)}{r}\right\}\frac{r^2}{r^2}$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = f''(r) + \frac{2}{r}f'(r)$$

(ii) $\nabla^2 r^n = \nabla\cdot\nabla r^n = \operatorname{div}\operatorname{grad} r^n$

$$\operatorname{grad} r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n$$

$$= \hat{i}\cdot nr^{n-1}\frac{\partial r}{\partial x} + \hat{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \hat{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left\{\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right\} = nr^{n-1}\frac{\vec{r}}{r} = nr^{n-2}\vec{r}$$

Now $\quad \nabla^2 r^n = \operatorname{div}\left[\left(nr^{n-2}\right)\vec{r}\right]$ which is of the type $\operatorname{div}(\phi\vec{a})$

$$= \left(nr^{n-2}\right)\operatorname{div}\vec{r} + \vec{r}\cdot\operatorname{grad}\left(nr^{n-2}\right)$$

$$= \left(nr^{n-2}\right)3 + \vec{r}\;n\operatorname{grad} r^{n-2}$$

$$= 3nr^{n-2} + n\vec{r}\cdot(n-2)r^{n-4}\vec{r}$$

$\because$ grad $r^n = nr^{n-2}\vec{r}$; change n to $n-2$

grad $r^{n-2} = (n-2)r^{n-4}\vec{r}$

$$= 3n\,r^{n-2} + n\,(n-2)\,r^{n-4}\,(\vec{r}\cdot\vec{r})$$
$$= 3n\,r^{n-2} + n\,(n-2)\,r^{n-4}\cdot r^2$$
$$= 3n\,r^{n-2} + n\,(n-2)\,r^{n-2}$$
$$= (3n + n^2 - 2n)\,r^{n-2}$$
$$= (n + n^2)\,r^{n-2} = n\,(n+1)\,r^{n-2}.$$

Example 11. *Find directional derivative of div $(\vec{u})$ at the point (1, 2, 2) in the direction of the outer normal of the sphere $x^2+y^2+z^2=9$ for $\vec{u} = x^4\,\hat{i} + y^4\,\hat{j} + z^4\,\hat{k}$.*

Sol. div $(\vec{u}) = \nabla\cdot\vec{u}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^4\,\hat{i} + y^4\,\hat{j} + z^4\,\hat{k}\right) = 4\,(x^3+y^3+z^3)$$

Outer normal to the sphere $= \nabla(x^2+y^2+z^2-9)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2+y^2+z^2-9\right)$$
$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}\,(2z) = 2\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

Outer normal at the point (1, 2, 2) $= 2\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$

Gradient of div $\vec{u} = \nabla\,(4\,x^3 + 4\,y^3 + 4\,z^3)$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(4x^3+4y^3+4z^3\right) = 12\left(x^2\,\hat{i} + y^2\,\hat{j} + z^2\hat{k}\right)$$

Gradient of div $\vec{u}$ at (1, 2, 2) $= 12\left(\hat{i} + 4\hat{j} + 4\hat{k}\right)$

Directional derivative of div $\vec{u}$ in the direction of outer normal

$$= 12\left(\hat{i} + 4\hat{j} + 4\hat{k}\right)\cdot\frac{\left(2\hat{i} + 4\hat{j} + 4\hat{k}\right)}{\sqrt{4+16+16}}$$
$$= \frac{12}{6}\,(1.2 + 4.4 + 4.4) = 2\,(2 + 16 + 16) = 68.$$

TEST YOUR KNOWLEDGE

1. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, show that div $\vec{r} = 3$, curl $\vec{r} = \vec{0}$. *(P.T.U., Dec. 2006)*
2. (*a*) Find divergence and Curl of the vector $\vec{V} = (xyz)\,\hat{i} + \left(3x^2y\right)\hat{j} + \left(xz^2 - y^2z\right)\hat{k}$ at the point (2, –1, 1).

   (*b*) If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\,\hat{k}$, find $\Delta\cdot\vec{A}$ at the point (1, –1, 1).
3. If $\vec{F} = (x+y+1)\,\hat{i} + \hat{j} - (x+y)\,\hat{k}$, show that $\vec{F}\cdot\text{curl}\,\vec{F} = 0$ *(P.T.U., Dec. 2012)*

4. If $\vec{A} = \left(3xz^2\right)\hat{i} - (yz)\,\hat{j} + (x+2z)\,\hat{k}$, find Curl (Curl $\vec{A}$).
5. If $\vec{V} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}}$, show that $\nabla\cdot\vec{V} = \dfrac{2}{\sqrt{x^2+y^2+z^2}}$ and $\nabla\times\vec{V} = \vec{0}$ or If $\vec{V} = \dfrac{\vec{r}}{r}$, show that divergence of $\vec{V} = \dfrac{2}{r}$ and Curl $\vec{V} = \vec{0}$.
6. If $\vec{V}_1$ and $\vec{V}_2$ be the vectors joining the fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) to a variable point (x, y, z) prove that

   (*i*) div$\left(\vec{V}_1\times\vec{V}_2\right) = 0$ (*ii*) grad$\left(\vec{V}_1\cdot\vec{V}_2\right) = \vec{V}_1 + \vec{V}_2$

   (*iii*) Curl $\left(\vec{V}_1\times\vec{V}_2\right) = 2\left(\vec{V}_1 - \vec{V}_2\right)$.
7. If $\vec{a}$ is a constant vector and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ prove that

   (*i*) div $(\vec{a}\times\vec{r}) = 0$ (*ii*) Curl$\left[(\vec{a}\cdot\vec{r})\vec{r}\right] = \vec{a}\times\vec{r}$

   (*iii*) $\nabla\,(\vec{a}\cdot\vec{a}) = 2\,(\vec{a}\cdot\nabla)\,\vec{a} + 2\vec{a}\times(\nabla\times\vec{a})$.

   (*iv*) Curl $(\vec{a}\times\vec{r}) = 2\vec{a}$ *(P.T.U., May 2010)*

   (*v*) grad $(\vec{a}\cdot\vec{r}) = \vec{a}$
8. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, prove that

   (*i*) $\nabla^2\left(\dfrac{1}{r}\right) = 0$ *(P.T.U., June 2003)* (*ii*) $\nabla^2\left(r^n\right)\vec{r} = n\,(n+1)\,r^{n-2}\,\vec{r}$ *(P.T.U., Dec. 2003)*

   (*iii*) $\nabla\cdot\left\{r\,\nabla\left(\dfrac{1}{r^3}\right)\right\} = \dfrac{3}{r^4}$ (*iv*) $\nabla^2\left\{\nabla\cdot\left(\dfrac{\vec{r}}{r^2}\right)\right\} = 2r^{-4}$.
9. Find the directional derivative of $\nabla\cdot(\nabla\phi)$ at the point (1, –2, 1) in the direction of outer normal to the surface $xy^2z = 3x + z^2$ where, $\phi = 2x^3\,y^2\,z^4$.

Answers

2. (*a*) $14, 2\hat{i} - 3\hat{j} - 14\hat{k}$ (*b*) –3

4. $-6x\,\hat{i} + (6z-1)\,\hat{k}$

9. $\dfrac{1724}{\sqrt{21}}$.

7.24. INTEGRATION OF VECTORS

Definition. Let $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ be two vectors functions of a scalar variable t such that $\dfrac{d}{dt}\overrightarrow{g(t)}$ $= \overrightarrow{f(t)}$ then $\overrightarrow{g(t)}$ is called an integral of $\overrightarrow{f(t)}$ with respect to t and we write