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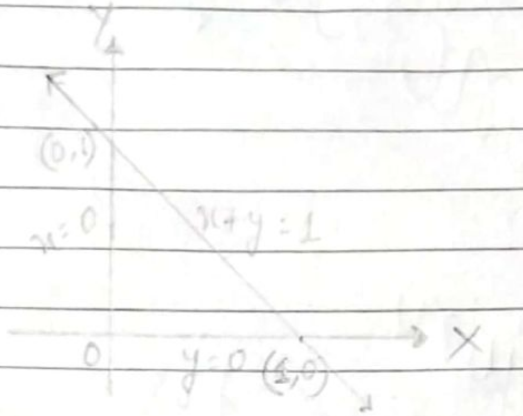
Q.1. Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$.

Soln:-



$$\int_0^1\int_0^{1-y} (x^2+y^2)\,dx\,dy \Rightarrow \int_0^1 \left(\frac{x^3}{3}+y^2x\right)\Big|_0^{1-y} dy$$

$$= \int_0^1 \left(\frac{(1-y)^3}{3} + y^2(1-y)\right) dy$$

$$= \int_0^1 \left(\frac{1+3y^2-3y-y^3}{3} + y^2 - y^3\right) dy$$

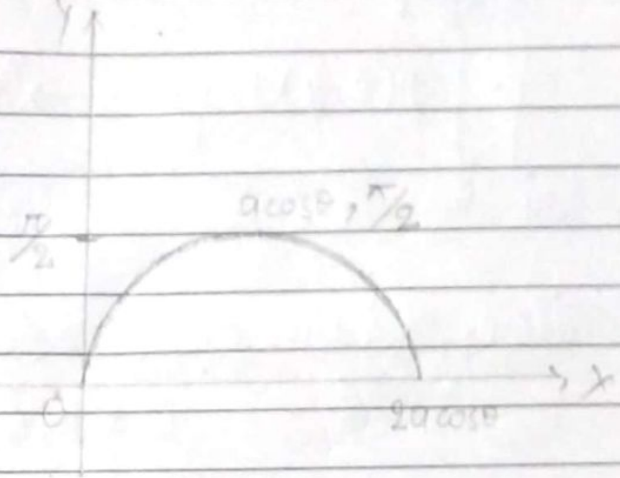
$$= \frac{1}{3}y + \frac{y^3}{3} - \frac{y^2}{2} - \frac{y^4}{12} + \frac{y^3}{3} - \frac{y^4}{4}\Big|_0^1$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{1}{2} - \frac{1}{12} + \frac{1}{3} - \frac{1}{4}$$

$$= \left(1 - \frac{5}{6}\right) = \frac{1}{6}$$ Ans.

Q.2. Show that $\iint_R r^2 \sin\theta \, dr \, d\theta = \frac{2a^3}{3}$, where R is the region bounded by the semi-circle $r = 2a\cos\theta$, above the initial line.

Soln:-



$$\int_0^{\pi/2}\int_0^{2a\cos\theta} (r^2 \sin\theta \, dr) \, d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3} \sin\theta \Big|_0^{2a\cos\theta} d\theta = \frac{8a^3}{3}\int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta$$

Let $\cos\theta = t$

$\Rightarrow \sin\theta \, d\theta = dt$

$\theta = 0 \rightarrow t = 1$

$\theta = \pi/2 \rightarrow t = 0$

$$\therefore \frac{8a^2}{3}\int_1^0 t^3 dt = \frac{8a^2}{3} \times \frac{t^4}{4}\Big|_1^0 = \frac{8a^2}{3}\left(0 - \frac{1}{4}\right) = \frac{-2a^2}{3}$$

In +ve quadrant it will be $\left|-\frac{2a^2}{3}\right| = \frac{2a^2}{3}$ Ans.

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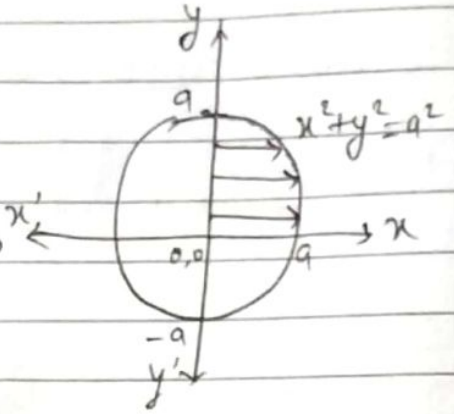
Q.3. Change the order of integration in

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy.$$

Soln:-

$$\int_{-a}^{a}\int_{0}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy \longrightarrow$$



Here x varies from 0 to $\sqrt{a^2-y^2}$

$$0 \le x \le \sqrt{a^2-y^2}$$

$$0 \le x^2 \le a^2-y^2 \longrightarrow (i)$$

in eqn (i); for $x^2 = a^2-y^2$

$$\therefore x^2+y^2 = a^2$$

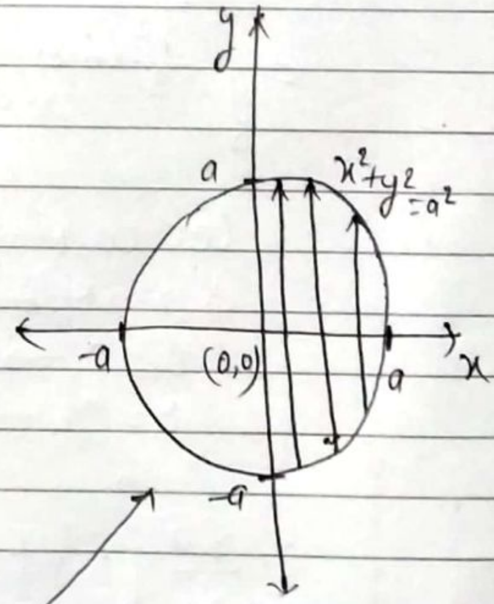
$$\Rightarrow y^2 = a^2-x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

in eqn (i); for $x=0$

$$y^2 = a^2 \Rightarrow y = \pm a$$

y varies from $-a \le y \le a$



$$\int_{0}^{a}\int_{-\sqrt{a^2-x^2}}^{+\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$$

Ans

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Q.4. Evaluate the following by changing to polar co-ordinate

$$\int_0^a \int_y^a \frac{x^2\,dx\,dy}{y\sqrt{x^2+y^2}}$$

Soln:- Changing to polar coordinate

$x = r\cos\theta$
$y = r\sin\theta$

Limit of $x \Rightarrow y \le x \le a$
Limit of $y \Rightarrow 0 \le y \le a$

$dx\,dy \rightarrow r\,dr\,d\theta$

$a = r\cos\theta$
$\therefore r = a\sec\theta \rightarrow (i)$
$r = 0$

Thus r varies $0 \le r \le a\sec\theta$

$y = r\sin\theta$
$\Rightarrow a = r\sin\theta \rightarrow (ii)$
$x = r\cos\theta$
$\Rightarrow a = r\cos\theta \rightarrow (iii)$

Dividing eqn. (ii) by (iii)

$\tan\theta = 1$
$\therefore \theta = \frac{\pi}{4}$

For $y = 0$

$$\frac{0}{a} = \tan\theta$$

$\therefore \tan\theta = 0$

$\therefore \theta = 0$

Thus $0 \le \theta \le \pi/4$

So the eqn. becomes:-

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\, dx\, dy = \int_0^{\pi/4} \int_0^{a\sec\theta} \frac{(r\cos\theta)^2\, r\, dr\, d\theta}{(r\cos\theta)^2 + (r\sin\theta)^2}$$

$$= \int_0^{\pi/4} \int_0^{a\sec\theta} r^2 \cos^2\theta\, dr\, d\theta$$

$$= \int_0^{\pi/4} \frac{r^3}{3}\Big|_0^{a\sec\theta} \cos^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \frac{a^3}{3} \sec^3\theta \cos^2\theta\, d\theta = \frac{a^3}{3} \int_0^{\pi/4} \sec\theta\, d\theta$$

$$= \frac{a^3}{3} \Big[\ln|\sec x + \tan x| \Big]_0^{\pi/4} + c = \frac{a^3}{3} \Big[\ln(\sqrt{2}+1) - \ln 1 \Big] + c$$

Ans.

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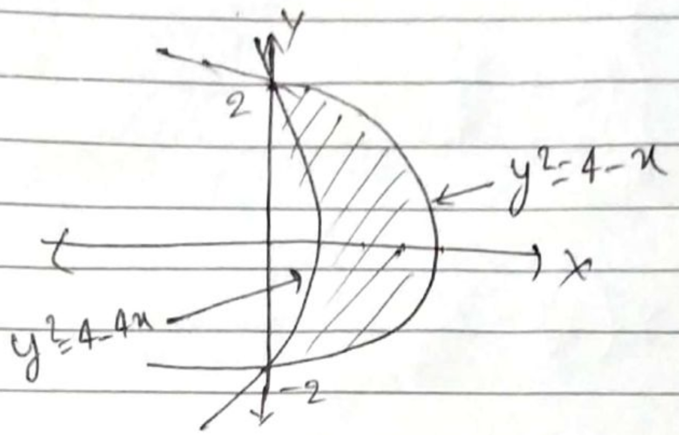
Q.5. Find the area bounded by the parabola
$y^2 = 4 - x$ & $y^2 = 4 - 4x$

Soln:



Here y ranges from -2 to 2 & x ranges from $(4-y^2)/4$ to $(4-y^2)$.

$$\therefore \int_{-2}^{2}\int_{\frac{4-y^2}{4}}^{4-y^2} dx\,dy$$

$$= \int_{-2}^{2} x \Big|_{\frac{4-y^2}{4}}^{4-y^2} dy = \int_{-2}^{2} 4-y^2-\left(\frac{4-y^2}{4}\right) dy$$

$$= 2\int_{0}^{2}\left(4-y^2-1+\frac{y^2}{4}\right) dy$$

$$= 2\int_0^2 \left(4 - 1 - \frac{3y^2}{4}\right) dy$$

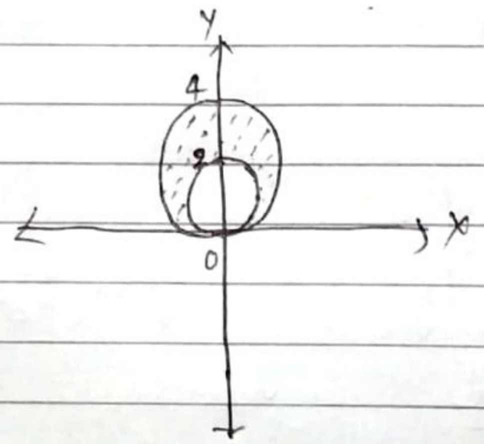
$$= 2\int_0^2 \left(3 - \frac{3y^2}{4}\right) dy$$

$$= 2\left(3y - \frac{3y^3}{12}\right)\Bigg|_0^2$$

$$= (6-2).2 \quad = \underline{8 \text{ Ans.}}$$

Q.6. Find the area bounded by the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

Soln:-



θ varies 0 to π

r varies $2\sin\theta$ to $4\sin\theta$

$$\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \frac{r^2}{2}\Bigg|_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_0^{\pi} \frac{(16-4)\sin^2\theta}{2}\, d\theta$$

$$= \int_0^{\pi} 6\sin^2\theta \, d\theta = \int_0^{\pi} 6\,\frac{(1-\cos 2\theta)}{2}\, d\theta$$

$$= 3\int_0^{\pi} (1-\cos 2\theta)\, d\theta$$

$$= 3\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi}$$

$$= 3(\pi)$$

$$= \underline{3\pi \text{ Ans.}}$$