Analysis for Engineers

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These notes are based on the Analysis books by Terence Tao. This is one of the best books on mathematical analysis if you want to understand why things are being done the way they are. An engineer needs to be aware of the way of analysis, if they want to remove the ever present lingering doubts about the mathematics they use so often. This might offend a pure mathematics student, but as an engineer, interested in applications, the most important thing is to understand the construct, the derivative arguments to the final results, at the expense of details of the proofs. Ultimately, the big picture is what you need to carry with you at the end, to the field of applications. You will undergo four main steps in this journey.

- Understand set theory and construct the real line (0-6).
- Do limits, continuity, differentiation and integration on real line (7-11).
- Generalize to Metric spaces to study limits, continuity and differentiability (12-17).
- Introduce Lebesgue measure to generalize integration (18-19).

It is essential to grasp the basics fully, even at the expense of being shallow on specialized topics. A 3-read process with some problem solving (read the material to get the idea, re-read to understand the details, solve problems, and re-read again focusing on foundational and important results) should be enough to get a solid grounding in this material. This course is the building block to almost all the mathematics used at the advanced level.

Contents

0.1 The basics of mathematical logic

There is difference between being convinced and being sure. Writing logically is essential to communicate in mathematics. Because logic is innate the laws of logic that, what you learn should make sense and should not be memorized. Mathematicians have made a careful study of logic, and the main result *the completeness theorem for first-order predicate calculus*, shows that the logical reasoning we are going to use in this book, if used correctly, is both sound and incapable of being improved.

0.1.1 Mathematical statements

Any mathematical argument proceeds in a sequence of mathematical statements. Well-formed statements are either true or false, but not both. Ill-formed or ill-defined statements are considered to be neither true or false, e.g. $0/0 = 1$. A logical argument should not contain any ill-formed statements and only well-formed ones. Being true is different from being useful or efficient. In mathematics we only concern ourselves with true statements; the reason is that truth is objective.

Statements are different form expressions. Statements are true or false; expressions are a sequence of mathematical symbols which produce some mathematical object like a number, matrix, function etc. $2 + 3 * 5$ is an expression while $2+3*5 = 17$ is a statement. An expression can be ill-formed as well. One can make statements out of expressions by using relations ($=, <, \geq, \in, \subset$, etc.) or by using properties ('is prime', 'is continuous', etc.). One can make compound statement from more primitive statements by using logical connectives such as and, or, not, if-then, if-and-only-if etc.

- Conjunction: If X is a statement and Y is a statement, the statement "X *and* Y" (represented as $X \wedge Y$) is true if X and Y are both true, and is false otherwise.
- Disjunction: If X is a statement and Y is a statement, the statement "X *or* Y" (represented as $X \vee Y$) is true if either X or Y is true, or both. The worse 'or' in mathematical logic defaults to inclusive or.
- Negation: The statement "X is not true" or "X is false" is called the negation of X (and represented by $\neg X$), and is true if and only if X if false, and if false if an only if X is true. Negation converts "and" into "or". Similarly, negation converts "or" into "and".
- if and only if (iff): If X is a statement, and Y is a statement, we say that "X is true if and only if Y is true", whenever X is true, Y has to be also, and whenever Y is true, X has to be also. Other way of saying the same thing are "X and Y are equivalent", or "X is true iff Y is true", or "$X \leftrightarrow Y$".

0.1.2 Implication

This is the least intuitive of the logical connectives - implication. If X is a statement, and Y is a statement, then "if X, then Y" is the implication from X to Y (represented by $X \implies Y$). When X is true, the statement "if X, then Y" implies that Y is true. But when X is false, the statement "if X, then Y" offers no information about whether Y is true or not; the statement is true, but vacuous. The falsity of the hypothesis does not destroy the truth of an implication. The only way to disprove an implication is to show that the hypothesis is true while the conclusion is false.

The statement "If X, then Y" is not the same as "If Y, then X"; they are in fact converses of each other, and don't imply each other. Similarly, the statement "If X is true, then Y is true" is not the same as "If X is false, then Y is false"; they are in fact inverse of each other, and don't imply each other. The statement "If Y is false, then X is false" is known as the contrapositive of "If X, then Y" and both statements are equally true. In particular, if you know that X implies something which is known to be false, then X itself must be false. This is the idea behind proof by contradiction.

0.1.3 The structure of the proofs

To show an implication there are several ways to proceed: you can work forward from the hypothesis (start from the hypothesis and work towards a conclusion). You can also work backward from the conclusion (working backwards from the conclusion and seeing what it would take to imply it). Note that this proof is different from the incorrect approach of starting with the conclusion and seeing what it would imply; instead, we start with the conclusion and see what would imply it. One could do any combination of moving forward from the hypothesis and backwards from the conclusion. Or you can divide the cases in the hope to split the problem into several easier sub-problems. Another is to argue by contradiction. In all proofs it helps to keep track of which statements are know and which statements are desired.

0.1.4 Variables and quantifiers

Propositional logic or Boolean logic works by staring with primitives statements, then forming compound statements using logical connectives, and then using various laws of logic to pass form hypotheses to conclusions. Mathematical logic is same as propositional logic but with the additional ingredient of variables added. A variable is a symbol, which denotes a certain type of mathematical object, which is usually declared. One can form expressions and statements involving variables. The truth of a statement involving a variable may depend on the context of the statement; this is different form propositional logic where all statements have a definite truth value.

Sometimes we don't set a variable to anything, calling it a free variable. Statements with free variables might not have a definite truth value. At other times, we set a variable to equal a fixed value, calling it a bound variable. These statements do have a definite truth value. One can also turn a free variable into a bound variable by using the quantifiers "for all" or "for some".

- **Universal quantifiers** $\forall$: "$P(x)$ is true for all x of type T" is same as "$P(x)$ is true $\forall x \in T$". Producing a single counterexample can disprove the statement. But producing a single example does not prove the statement. It occasionally happens that there are in fact no variables x of type T. In that case the statement is *vacuously* true.
- **Existential quantifiers** $\exists$: "$P(x)$ is true for some x of type T" is same as "$\exists x \in T : P(x)$ is true". Producing a single example can prove the statement in this case.

One can nest two or more quantifiers together. Negating a universal statement produces an existential statement. Similarly, negating an existential statement produces a universal statement. Swapping two "for all" quantifiers makes no difference; similarly swapping two "there exists" quantifiers makes no difference. However, swapping a "for all" with a "there exists" makes a lot of difference, in general because the inner variables may possibly depend on the outer variables, but not vice versa.

0.1.5 Equality

The most important relation is equality. It is a relationship linking two objects x and y of the same type T. Given two objects x and y, the statement $x = y$ may or may not be true depending on the value of x and y and also how equality is defined for the class T of objects under consideration. For the purpose of logic we require that equality obeys the following four **axioms of equality**:

- Reflexive axiom Given any object x, we have $x = x$.
- Symmetry axiom Given any two objects x and y of the same type, if $x = y$, then $y = x$.
- Transitive axiom Given any three objects x, y, z of the same type, if $x = y$ and $y = z$, then $x = z$.
- Substitution axiom Given any two objects x and y of the same type, if $x = y$, then $f(x) = f(y)$ for all functions or operations f.

Similarly for any property $P(x)$ depending on x, if $x = y$, then $P(x)$ and $P(y)$ are equivalent statements. The first three axioms assert that equality is an **equivalence relation**.

0.1.6 Lemma, proposition, theorem and corollary

Lemma, proposition, theorem, or corollary - they all need proofs. The terms are used to suggest different level of importance and difficulty. A lemma is an easily proved claim which is helpful in proving other proposition and theorems, but is usually not particularly interesting in its own right. A proposition is a statement which is interesting in its own right, while a theorem is a more important statement than a proposition which says something definitive on the subject, and often takes more effort to prove than a proposition or lemma. A corollary is a quick consequence of a proposition or a theorem that was proven recently.

0.2 The decimal system

Natural numbers are simply postulated to exist, and obey five axioms; the integers arise via differences of the natural numbers; the rationals then came from quotients of the integers, and the real then come form limits of the rationals. The well familiar Hindu-Arabic base 10 system *decimal system* itself is not essential to mathematics. We study it here as it is the choice of representation.

Definition 0.2.1. *(Digits) A digit is any one of the ten symbols 0,1,...,9. We equate these digits with natural numbers by the formula* $0 \equiv 0$, $1 \equiv 0++$, $2 \equiv 1++$, *etc. all the way up to* $9 \equiv 8++$. *We also define the number ten by the formula ten* $\equiv 9++$.

Definition 0.2.2. *(Positive integer decimals) Any string* $a_n a_{n-1} \ldots x_0$ *of digits is a positive integer decimal, where* $n \geq 0$ *is a natural number, and the first digit* a_n *is non-zero. We equate each positive integer decimal with a positive integer by the formula* $a_n a_{n-1} \ldots x_0 \equiv \sum_{i=0}^{n} a_i \times ten^i$.

This definition implies that $10 = ten$; and a single digit integer decimal is exactly equal to that digit itself.

Theorem 0.1. *(Uniqueness and existence of decimal representations) Every positive integer* m *is equal to exactly one positive integer decimal. We refer to the decimal given as the decimal representation of* m.

We can now derive the usual laws of long addition and long multiplication to connect the decimal representation of $x + y$ and $x \times y$ to that of x or y. Once we have decimal representation of positive integers, we can represent negative integers decimally as well by using the $-$ sign. Finally, we let 0 be a decimal as well. This gives decimal representation of all integers. Every rational is then the ratio of two decimals.

For decimal representation of real numbers we need a new symbol: the *decimal point* ".".

Definition 0.2.3. *(Real decimals) A real decimal is any sequence of digits, and a decimal point, arranged as*

$$\pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots$$

which is finite to the left of the decimal point (so n *is a natural number), but infinite to the right of the decimal point, where* $\pm$ *is either* $+$ *or* $-$, *and* $a_n \ldots a_0$ *is a natural number decimal. This decimal is equated to the real number*

$$\pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots \equiv \pm 1 \times \sum_{i=-\infty}^{n} a_i \times 10^i.$$

The series is always convergent.

Theorem 0.2. *(Existence of decimal representations) Every real number* x *has at least one decimal representation* $a = \pm a_n \ldots a_0 . a_{-1} a_{-2} \ldots$.

Proposition 0.2.1. *(Failure of uniqueness of decimal representations) The number* 1 *has two different decimal representations:* $1.000\ldots$ *and* $0.999\ldots$.

It turns out these two are the only two decimal representations of 1. In fact, all real numbers have either one or two decimal representations - two if the real is a terminating decimal, and one otherwise.

1 Introduction

Real analysis is the theoretical foundation which underlies calculus, which is the collection of computational algorithms which one uses to manipulate functions. Real analysis is related to complex analysis, harmonic analysis, functional analysis etc.

- $1 \times 0 = 2 \times 0 \implies 1 = 2$ if we cancel 0 on both sides, which does not work. It is because there is Division by zero involved. Why is that prohibited?

- For a series $S = 1 + \frac{1}{2} + \frac{1}{4} + \ldots$, we can use the trick of multiplying by 2 to get $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots = 2 + s$, which can be used to solve for $S = 2$. If we apply the same trick to $S = 1 + 2 + 4 + \ldots$, we get $2S = 2 + 4 + 8 + \ldots = S - 1 \implies S = -1$, which is nonsensical. Why can we trust the first but not the second manipulation which involved divergent series?

- For x a real number, for the limit $L = \lim_{n\to\infty} x^n$, we could write $n = m + 1$ and hence $L = \lim_{m+1\to\infty} x^{m+1} = x \lim_{m\to\infty} x^m$, since $m + 1 \to \infty$ implies $m \to \infty$. Hence, $xL = L$. We can then say $\lim_{n\to\infty} x^n = 0$ for all $x \neq 1$. But this conclusion is absurd for say $x = 2$ or $x = -1$. What is the problem with the above argument?

There are many questions like these which can be resolved by analysis, thus separating the useful applications of these rules from the nonsense. It further deepens and develops an "analytical way of thinking".

2 The natural numbers

Why do rules of algebra work at all? We will use mathematical induction a lot in our endeavor. In increasing order of sophistication, we have natural numbers $\mathbf{N}$, the integers $\mathbf{Z}$, the rationals $\mathbf{Q}$, the real numbers $\mathbf{R}$, and the complex numbers $\mathbf{C}$. How do we define the natural numbers? We will assume nothing, not even the decimal system.

2.1 The Peano axioms

Informally a natural number is any element of the set $\mathbf{N} := \{0, 1, 2, 3, \ldots\}$, which is the set of all the numbers created by starting with 0 and then counting forward indefinitely. We call $\mathbf{N}$ the set of natural numbers. But that is unsatisfactory as it requires definition of set and mathematical operations. In fact the most fundamental operation is **incrementing**. So, it seems like we want to say that $\mathbf{N}$ consists of 0 and everything which can be obtained from 0 by incrementing.

Axiom 2.1. *0 is a natural number.*

Axiom 2.2. *If n is a natural number, then $n{++}$ is also a natural number.*

Definition 2.1.1. *We define 1 to be the number $0{++}$, 2 to be the number $(0{++}){++}$, etc.*

It may seem that this is enough to describe the natural numbers, but there can be a 'wrap-around issue'. For example consider a number system which consists of the number 0,1,2,3, in which the increment operation wraps back from 3 to 0, i.e. $3{++} = 0$ and also equal to 4 by definition. To prevent it we need the following axiom.

Axiom 2.3. *0 is not the successor of any natural number, i.e. we have $n{++} \neq 0$ for every natural number n.*

However, even with our new axiom, it is still possible that our number system behaves in other pathological ways. For example, consider a number system consisting of five numbers 0,1,2,3,4, in which the increment operation hits a 'ceiling' of 4, i.e. $4{++} = 4$, and also equal to 5 by definition. Another number system with a similar problem is one in which increment wraps around, but not to zero, for example suppose that $4{++} = 1$. This does not contradict the three Axioms. To exclude those we require the next Axiom.

Axiom 2.4. *Different natural numbers must have different successors; i.e. if n, m are natural numbers and $n \neq m$, then $n{++} \neq m{++}$.*

There is however still one more problem. There is the problem of excluding 'rogue' elements in our number system which are not of this form, e.g. $\mathbf{N} := \{0, 0.5, 1, 1.5, 2, \ldots\}$, i.e. we need to only include whole numbers in $\mathbf{N}$. There is an ingenious solution to capture this fact:

Axiom 2.5. *(Principle of mathematical induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n{++})$ is also true. Then $P(n)$ is true for every natural number n.*

Apply Axiom 2.5 to the property $P(n) = n$ 'is not a half integer'. Then $P(0)$ is true, and if P(n) is true, then $P(n\,{+}{+})$ is true. Thus $P(n)$ is true for all natural numbers n.

Axioms 2.1-2.5 are known as the **Peano axioms** for the natural numbers. They are all very plausible, and so we shall make the following assumption.

Assumption 2.1. *There exists a number system* $\mathbf{N}$*, whose elements we will call natural numbers, for which Axioms 2.1-2.5 are true.*

All the seemingly different natural number systems, like Hindu-Arabic or the Roman number system are isomorphic due to one-to-one correspondence. A remarkable accomplishment of modern analysis is that just by starting from these five very primitive axioms, and some additional axioms from set theory, we can build all the other number systems, create functions, and do all the algebra and calculus that we are used to. One interesting feature about the natural numbers is that while each individual natural number is finite, the set of natural numbers is infinite. Also note that our definition of the natural numbers is axiomatic rather than constructive. This is because, historically, each of these advancements - the number zero, negative numbers, irrational numbers and complex numbers - have created problems for the then famous constructive view. One consequence of the axioms is that we can now define sequences recursively.

Proposition 2.1.1. *Suppose for each natural number* n*, we have some function* $f_n : \mathbf{N} \to \mathbf{N}$ *from the natural numbers to the natural numbers. Let* c *be a natural number. Then we can assign a unique natural number* a_n *to each natural number* n*, such that* $a_0 = c$ *and* $a_{n++} = f_n(a_n)$ *for each natural number* n*.*

Proof: We use induction. We first observe that this procedure gives a single value to a_0, namely c. None of the other definitions $a_{n++} := f_n(a_n)$ will redefine the value of a_0, because of Axiom 2.3. Now suppose inductively that the procedure gives a single value to a_n. Then it gives a single value to a_{n++}, namely $a_{n++} := f_n(a_n)$. None of the other definitions $a_{m++} : f_m(a_m)$ will redefine the value of a_{n++}, because of Axiom 2.4. This completes the induction, and via Axiom 2.5, a_n is defined for each natural number n, with a single value assigned to each a_n. $\square$

All the axioms have to be used to prove the above preposition. In a system which has some sort of wrap-around, recursive definitions would not work because some elements of the sequence would constantly be redefined.

2.2 Addition

Definition 2.2.1. *(Addition of natural numbers) Let* m *be a natural number. To add zero to* m*, we define* $0 + m := m$*. Now suppose inductively that we have defined how to add* n *to* m*. Then we can add* $n\,{+}{+}$ *to* m *by defining* $(n\,{+}{+}) + m := (n + m)\,{+}{+}$

The above definition is a specialization of the recursive setting where $a_n = n + m$ and $f_n(a_n) = a_n\,{+}{+}$. We can now build on the usual mathematics of addition based on the current constructs.

- For any natural number n, $n + 0 = n$.
- For any natural number n and m, $n + (m\,{+}{+}) = (n + m)\,{+}{+}$.
- $n\,{+}{+} = n + 1$.
- Addition is commutative: For any natural numbers n and m, $n + m = m + n$
- Addition is associative: For any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.
- Cancellation law: Let a, b, c be natural numbers such that $a + b = a + c$. Then $b = c$.

We now discuss how addition interacts with positivity.

Definition 2.2.2. *(positive natural numbers) A natural number* n *is said to be positive iff it is not equal to 0.*

- *If* a *is positive and* b *is a natural number, then* $a + b$ *is positive.*

- *If a and b are natural numbers such that $a + b = 0$, then $a = 0$ and $b = 0$.*

- *Let a be a positive number. Then there exists exactly one natural number b such that $b{+\!+} = a$.*

Once we have a notion of addition, we can begin defining a notion of order.

Definition 2.2.3. *(Ordering of the natural numbers) Let n and m be natural numbers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

Note that $n{+\!+} > n$ for any n; thus there is no largest natural number n, because the next number $n{+\!+}$ is always larger still.

Proposition 2.2.1. *(Basic properties of order for natural numbers) Let a, b, c be natural numbers. Then*

1. *Order is reflexive: $a \geq a$.*

2. *Order is transitive: If $a \geq b$ and $b \geq c$, then $a \geq c$.*

3. *Order is anti-symmetric: If $a \geq b$ and $b \geq a$, then $a = b$.*

4. *Addition preserves order: $a \geq b$ iff $a + c \geq b + c$.*

5. *$a < b$ iff $a{+\!+} \leq b$.*

6. *$a < b$ iff $b = a + d$ for some positive number d.*

Proposition 2.2.2. *(Trichotomy of order for natural numbers) Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.*

The properties of order allow one to obtain a stronger version of the principle of induction.

Proposition 2.2.3. *(Stronger principle of induction) Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

Proof: Let $Q(n)$ be the property that $P(m)$ is true for all $m_0 \leq m < n$. Hence, for each $m \geq m_0$, if $Q(m)$ is true then $P(m)$ is also true. We use induction.

$Q(0)$ says that $P(m)$ is true for all $m_0 \leq m < 0$, which is vacuously true since there is no natural number strictly less than 0. Now, suppose inductively that we have shown $Q(n)$ is true, i.e. $P(m)$ is true for $m_0 \leq m < n$. We want to show $Q(n{+\!+})$ is true, i.e. $P(m)$ is true for $m_0 \leq m < n{+\!+}$.

Since m is a natural number, $m < n{+\!+}$ implies $m{+\!+} \leq n{+\!+}$ by Proposition 2.2.1(5). Which further means $m \leq n$ via Proposition 2.2.1(4). This means either $m < n$ or $m = n$. This is to say, $m_0 \leq m < n{+\!+}$ could either mean $m_0 \leq m < n$ or $m = n$. For the first case $P(m)$ is already true due to the induction assumption.

We need to show that $P(m)$ is true in the case of $m = n$, i.e. $P(n)$ is true. Since, $m_0 \leq m < n{+\!+}$, by Proposition 2.2.1(2) $m_0 < n{+\!+}$, so $m_0{+\!+} \leq n{+\!+}$ by Proposition 2.2.1(5), implying $m_0 \leq n$ by Proposition 2.2.1(4). Since $m_0 \leq n$ we can use the hypothesis for P, which says that if $Q(n)$ is true $P(n)$ is true for each $n \geq m_0$. Since $Q(n)$ is true by inductive hypothesis, $P(n)$ is also true. This closes the induction.

We have shown that $Q(n)$ is true for all natural number n. Let $m \geq m_0$. Then we know that $Q(m{+\!+})$ is true. Since $m_0 \leq m < m{+\!+}$, we see that $P(m)$ is true for all $m \geq m_0$, as desired. $\square$

In applications we usually use this principle with $m_0 = 0$ or $m_0 = 1$.

Example 2.2.1. *(Principle of backward induction) Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+}{+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$.*

Proof: Let $Q(n)$ be the statement: if $P(n)$ is true, then $P(m)$ is true for all $m \leq n$. We use induction. As the first step we need to show $Q(0)$ is true. Suppose $P(0)$ is true, then we have to show that $P(m)$ is true for all $m \leq 0$. The only possible value of m is 0 and we know $P(0)$ is true, hence $Q(0)$ is true.

Now we assume $Q(n)$ is true. This means $P(m)$ is true for all $m \leq n$. The task is to prove $Q(n{+}{+})$ is true. We assume $P(n{+}{+})$ is true and we need to show $P(m)$ is true for all $m \leq n{+}{+}$. By the property of P since $P(n{+}{+})$ is true, it implies that $P(n)$ is also true. By the induction hypothesis we then have $P(m)$ true for all $m \leq n$. For $m \neq n{+}{+}$ and $m \leq n{+}{+}$, by definition of strict order, it means $m < n{+}{+}$. By Proposition 2.2.1(5), we have $m{+}{+} \leq n{+}{+}$ and this leads to $m \leq n$ by Proposition 2.2.1(4). Hence, what we have shown is that for $m \leq n{+}{+}$, either $m \leq n$ or $m = n{+}{+}$. We have $P(m)$ true under both conditions. This closes the induction.

Now let n be a natural number and suppose $P(n)$ is true. Since $Q(n)$ it true, it implies that $P(m)$ is true for all natural numbers $m \leq n$. This proves the result.

2.3 Multiplication

Just as addition is the iterated increment operation, multiplication is iterated addition.

Definition 2.3.1. *(Multiplication of natural numbers) Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$. We abbreviate $n \times m$ as nm and multiplication takes preference over addition.*

- Commutative Law: Let n, m be natural numbers, then $n \times m = m \times n$.
- Positive natural numbers have no zero divisors: Let n, m be natural numbers. Then $n \times m = 0$ iff at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.
- Distributive Law: For any natural numbers a, b, c, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.
- Associative Law:For any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.
- Order preservation: If a, b are natural numbers such that $a < b$, and c is positive, then $ac < bc$.
- Cancellation Law: Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero, then $a = b$.

Proposition 2.3.1. *(Euclidean algorithm) Let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

Proof: We fix q and induct on n. For $n = 0$ we need to show that there exists natural numbers m, r such that $0 \leq r < q$ and $0 = mq + r$. We can choose $m := 0$ and $r := 0$ and indeed we have $0 \leq 0 < q$ since q is positive; and $0 = 0 \times q + 0$. Next we suppose inductively that there are natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$. We must now show that there are natural numbers m', r' such that $0 \leq r' < q$ and $n + 1 = m'q + r'$.

We have two cases, $r + 1 = q$ and $r + 1 \neq q$. If $r + 1 = q$, we can take $m' := m + 1$ and $r' := 0$. Then $0 \leq r' = 0 < q$ since q is positive; and $m'q + r' = (m + 1) \times q + 0$ by definition of m' and r', which equals $mq + q$ by distributive law, which equals $mq + r + 1$ by using $q = r + 1$, which equals $n + 1$ by the inductive hypothesis $n = mq + r$. Thus $m'q + r' = n + 1$ as required.

Now suppose $r + 1 \neq q$. From the inductive hypothesis we have $r < q$, which by definition means $r \leq q$, but the assumption $r + 1 \neq q$ means $r + 1 < q$. Here we can choose $r' := r + 1$ and $m' := m$. Now $0' < q$ as we just showed; and $m'q + r' = mq + r + 1 = n + 1$ by definition of m', r' and the inductive hypothesis $n = mq + r$. This closes the induction and proves Euclidean algorithm.

This algorithm marks the beginning of number theory.

Definition 2.3.2. *(Exponentiation for natural numbers) Let m be a natural number. To raise m to the power 0, we define $m^0 := 1$, in particular, we define $0^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n, then we define $m^{n+\!\!+} := m^n \times m$.*

3 Set theory

We present elementary aspects of axiomatic set theory. We introduce the 10 Zermelo-Fraenkel axioms of set theory. In a later chapter we will add the famous *axiom of choice* giving rise to the *Zermelo-Fraenkel-Choice (ZFC) axioms of set theory.*

3.1 Fundamentals

Definition 3.1.1. *We define a set A to be any unordered collection of objects. If x is an object, we say x is an element of A or $x \in A$ if x lies in the collection; otherwise we say that $x \notin A$.*

We next define some axioms to explain various things we can do with sets.

Definition 3.1.2. *(Equality of sets) Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice versa.*

This notion of set equality is reflexive, symmetric, and transitive. The relation $\in$ obeys the axiom of substitution.

Axiom 3.1. *(Basic set axioms)*

1. *Sets are objects: If A is a set, then A is also an object. Given two sets A and B, it is meaningful to ask whether A is also an element of B.*

2. *Empty set: There exists a set , known as the empty set, which contains no elements, i.e., for every object x we have $x \notin \emptyset$.*

3. *Singleton and pair sets: If a is an object, then there exists a set $\{a\}$ whose element is a, i.e., for every object y we have $y \in \{a\}$ iff $y = a$; we refer to $\{a\}$ as the singleton set whose element is a. Furthermore, if a and b are objects, then there exists a set $\{a, b\}$ whose only elements are a and b, i.e., for every object y, we have $y \in \{a, b\}$ iff $y = a$ or $y = b$; we refer to this set as the pair set formed by a and b.*

4. *Pairwise union: Given any two sets A and B, there exists a set $A \cup B$, called the union $A \cup B$ of A and B, whose elements consists of all the elements which belong to A or B or both. In other words, for any object x, $x \in A \cup B \equiv (x \in A$ or $x \in B)$.*

Proposition 3.1.1. *(Single choice) Let A be a non-empty set. Then there exists an object x such that $x \in A$.*

The above proposition asserts that given any non-empty set A we are allowed to choose an element x of A. Later on we will show that given any finite number of non-empty sets, say $A_1, \dots, A_n$, it is possible to choose one element $x_1, \dots, x_n$ from each set $A_1, \dots, A_n$; this is known as 'finite choice'. However, in order to choose elements from an infinite number of sets, we need an additional axiom, the axiom of choice, which we shall discuss later.

Proposition 3.1.2. *If a and b are objects, then $\{a, b\} = \{a\} \cup \{b\}$. If A, B, C are sets, then the union operation is commutative, i.e., $A \cup B = B \cup A$ and associative, i.e., $(A) \cup C = A \cup (B \cup C)$. Also, we have $A \cup A = A \cup \emptyset = \emptyset \cup A = A$.*

Definition 3.1.3. *(Subset) Let A, B be sets. We say that A is a subset of B, denoted $A \subseteq B$, iff every element of A is also an element of B, i.e. for any object x, $x \in A \implies x \in B$. We say that A is a proper subset of B, denoted A if $A \subseteq B$ and $A \neq B$.*

Proposition 3.1.3. *(Sets are partially ordered by set inclusion) Let A, B, C be sets. If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$. Finally if A and $B \subset C$ then $A \subset C$.*

Notice that $2 \in \{1, 2, 3\}$, but $2 \nsubseteq \{1, 2, 3\}$. Indeed, 2 is not even a set. Conversely, while $\{2\} \subseteq \{1, 2, 3\}$ but $\{2\} \notin \{1, 2, 3\}$. Number 2 and set $\{2\}$ are distinct objects.

Axiom 3.2. *(Axiom of specification/separation) Let A be a set, and fore each $x \in A$, let $P(x)$ be a property pertaining to x, i.e., $P(x)$ is either a true statement or a false statement. Then there exists a set, called $\{x \in A : P(x)\}$, whose elements are precisely the elements x in A for which $P(x)$ is true. In other words, for any object y, $y \in \{x \in A : P(x)\} \iff (y \in A \text{ and } P(y))$*

Definition 3.1.4. *(Intersection) The intersection $S_1 \cap S_2$ of two sets is defined to be the set $\{x \in S_1 : x \in S_2\}$.*

Two sets A, B are said to be *disjoint* if $A \cap B = \emptyset$. This is different from being *distinct* meaning $A \neq B$.

Definition 3.1.5. *(Difference sets) Given two sets A and B, we define the set $A - B$ or $A \backslash B$ to be the set A with any elements of B removed: $A \backslash B := \{x \in A : x \notin B\}$.*

Proposition 3.1.4. *(**Laws of Boolean algebra**) Let A, B, C be set, and let X be the set containing A, B, C as subsets.*

1. *Minimal element: We have $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.*
2. *Maximal element: We have $A \cup X = X$ and $A \cap X = A$.*
3. *Identity: We have $A \cap A = A$ and $A \cup A = A$.*
4. *Commutativity: We have $A \cup B = B \cup A$ and $A \cap B = B \cap A$.*
5. *Associativity: We have $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.*
6. *Distributivity: We have $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*
7. *Partition: We have $A \cup (X \backslash A) = X$ and $A \cap (X \backslash A) = \emptyset$.*
8. *De Morgan laws: We have $X \backslash (A \cup B) = (X \backslash A) \cap (X \backslash B)$ and $X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$.*

Proof:(De Morgan Laws) Suppose $x \in X \backslash (A \cup B)$. Then $x \in X$ and $x \notin A \cup B$. The second statement means that $x \notin A$ and $x \notin B$. But since $x \in X$ and $x \notin A$, which means $x \in X \backslash A$; we also have both $x \in X$ and $x \notin B$, so $x \in X \backslash B$. Thus we have $x \in (X \backslash A) \cap (X \backslash B)$ as required. This mean $X \backslash (A \cup B) \subseteq (X \backslash A) \cap (X \backslash B)$. Conversely, suppose $x \in (X \backslash A) \cap (X \backslash B)$. Thus we have both $x \in X \backslash A$ and $x \in X \backslash B$. The first of this means $x \in X$ and $x \notin A$, and the second means $x \in X$ and $x \notin B$. Now, $x \notin A$ and $x \notin B$ means $x \notin A \cup B$. Since we know $x \in X$, this means that $x \in X \backslash (A \cup B)$. This mean $(X \backslash A) \cap (X \backslash B) \subseteq X \backslash (A \cup B)$. This proves the equality.

There is certain symmetry between $\cup$ and $\cap$, and between X and $\emptyset$. This is an example of *duality* - two distinct properties or objects being dual to each other. Here the duality is manifested by the complementation relation $A \mapsto X \backslash A$.

Axiom 3.3. *(Axiom of replacement) Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y)$ is true for some $x \in A\}$, such that for any object z,*

$$x \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, y) \text{ is true for some } x \in A.$$

Axiom 3.4. *(Axiom of Infinity) There exists a set $\boldsymbol{N}$, whose elements are called natural numbers, as well as an object 0 in $\boldsymbol{N}$, and an object $n{+}{+}$ assigned to every natural number $n \in \boldsymbol{N}$, such that the Peano axioms hold.*

3.2 Russell's paradox

The axiom of comprehension or universal specification goes as follows. Suppose for every object x we have a property $P(x)$ pertaining to x (so for every x, $P(x)$ is either true or false). Then there exists a set $\{x : P(x)\}$ such that for every object y, $y \in \{x : P(x)\} \iff P(y)$ is true. It asserts that every property corresponds to a set; if we assumed this axiom, most of the axioms in previous section are implied by it. Unfortunately, this creates a logical contradiction known as Russell's paradox.

Let $P(x)$ be the statement $P(x) \iff$ "x is a set, and $x \notin x$", i.e. $P(x)$ is true only when x is a set which does not contain itself. If we set S be a set of all sets, which exists from the axiom of universal specification, then since S is itself a set, it is an element of S, and so $P(S)$ is false. Now, using the axiom of universal specification we can create the set $\Omega := \{x : P(x) \text{ is true}\} = \{x : x \text{ is a set and } x \notin x\}$, i.e., the sets of all sets which do not contain themselves. Now we inquire, does Ω contain itself, i.e. $\Omega \in \Omega$? If Ω did contain itself, then by definition of the set Ω this means $P(\Omega)$ is true, i.e. Ω is a set and $\Omega \notin \Omega$. On the other hand, if Ω did not contain itself, then the property $P(\Omega)$ would be true, i.e., $\Omega \in \Omega$. Thus in either case we have both $\Omega \in \Omega$ and $\Omega \notin \Omega$, which is absurd.

The problem with the above axiom is that it *creates sets which are far too "large"*, for example we can use the axiom to talk about the set of all objects. One way to informally resolve this issue is to arrange sets in a hierarchy. At the bottom of the hierarchy are *primitive objects* (objects that are not sets). Then on the next level are sets whose elements consist only of primitive objects or the empty set , called *primitive sets*. Then there are sets whose elements consist only of primitive objects and primitive sets; and so forth. At each stage we only see sets whose elements consist of objects at the lower stages of the hierarchy, and so at no stage do we ever construct a set which contains itself. We simply postulate an axiom which precludes the possibility of Russell's paradox.

Axiom 3.5. *(Axiom of foundation) If A is a non-empty set, then there is at least one element x of A which is either not a set, or is disjoint from A.*

The point of this axiom is that it is asserting that at least one of the elements of A is so low on the hierarchy of objects that it does not contains any of the other elements of A. One particular consequence of this axiom is that sets are no longer allowed to contain themselves.

3.3 Functions

We need the definition of a function before we can lay out the last two axioms of Zermelo-Fraenkel axioms of set theory.

Definition 3.3.1. *(Functions) Let X, Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$, such that for every $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true (called the vertical line test). Then we define the function $f : X \to Y$ defined by P on the domain X and range Y to be the object which, given an input $x \in X$, assigns an output $f(x) \in Y$, defined to be the unique object $f(x)$ for which $P(x, f(x))$ is true. Thus, for any $x \in X$ and $y \in Y$, $y = f(x) \iff P(x, y)$ is true.*

Functions are also referred to as *maps* or *transformations* and are form of *morphisms*. One common way to define a function is simply to specify its domain, its range, and how one generates the output $f(x)$ from each input; this is known as an *explicit* definition of a function. In other cases we only define a function f by specifying what property $P(x, y)$ links the input x with the output $f(x)$; this is an *implicit* definition of a function. Functions obey the axiom of substitution: if $x = x'$, then $f(x) = f(x')$. On the other hand, unequal inputs do not necessarily ensure unequal outputs. Functions are not sets, and sets are not functions. But it is possible to start with a function $f : X \to Y$ and construct its *graph* $\{(x, f(x)) : x \in X\}$, which describes the function completely.

Definition 3.3.2. *(Equality of functions) Two functions $f : x \to Y$, $g : X \to Y$ with the same domain and range are said to be equal, $f = g$, if and only if $f(x) = g(x)$ for all $x \in X$.*

The concept of equality of functions can depend on the choice of domain, e.g., $x \mapsto x$ and $x \mapsto |x|$ are equal on the positive real axis. For each set X, there is only one function from $\emptyset$ to X, called the *empty function* $f : \emptyset \to X$, and all functions from $\emptyset$ to X are equal.

Definition 3.3.3. *(Composition) Let $f : X \to Y$ and $g : Y \to Z$ be two functions, such that the range of f is the same set as the domain of g. We then define that composition $g \circ f : X \to Z$ of the two functions g and f to be the function defined explicitly by the formula $(g \circ f)(x) := g(f(x))$. If the range of f does not match the domain of g, we leave the composition $g \circ f$ undefined.*

Composition obeys the axiom of substitution. Composition is not commutative $f \circ g \neq g \circ f$.

Lemma 3.3.1. *(Composition is associative) Let $f : Z \to W$, $g : Y \to Z$, and $h : X \to Y$ be functions. Then $f \circ (g \circ h) = (f \circ g) \circ h$.*

Definition 3.3.4. *(One-to-one functions) A function f is one-to-one (or **injective**) if different elements map to different elements: $x \neq x' \implies f(x) \neq f(x')$. Equivalently, a function is one-to-one if $f(x) = f(x') \implies x = x'$.*

The notion of a one-to-one function depends no just on what the function does, but also what its domain is.

Definition 3.3.5. *(Onto functions) A function f is onto (or **surjective**) if $f(X) = Y$, i.e., every element in Y comes from applying f to some element in X: for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$.*

The notion of an onto function depends not just on what the function does, but also what its range is. The concept of injectivity and surjectivity are in many ways dual to each other.

Definition 3.3.6. *(Invertible functions) Functions $f : X \to Y$ which are both one-to-one and onto are also called **bijective** or invertible.*

$f : \boldsymbol{N} \to \boldsymbol{N} \backslash \{0\}$ defined by $f(n) := n{+}{+}$ is a bijection. But the function $g : \boldsymbol{N} \to \boldsymbol{N}$ defined by same $g(n) := n{+}{+}$ is not a bijection. Hence, the notion of bijective function depends not just on what the function does, but also what its range and domain are. A bijective function is also called a *perfect matching* or a *one-to-one correspondence*. If f is bijective, then for every $y \in Y$, there is exactly one x such that $f(x) = y$. This value of x is denoted by $f^{-1}(y)$; thus f^{-1} is a function from Y to X. We call f^{-1} the inverse of f.

3.4 Images and inverse images

Definition 3.4.1. *(Images of sets) If $f : X \to Y$ is a function from X to Y, and S is a set in X, we define $f(S)$ to be the set $f(S) := \{f(x) : x \in S\}$; this set is a subset of Y, and is called the image of S under the map f. We call $f(S)$ the forward image of S.*

Notice that $f(x) \in f(S) \nRightarrow x \in S$, instead $y \in f(S) \iff y = f(x)$ for some $x \in S$.

Definition 3.4.2. *(Inverse images) If U is a subset of Y, we define the set $f^{-1}(U)$ to be the set $f^{-1}(U) := \{x \in X : f(x) \in U\}$. In other words $f^{-1}(U)$ consists of all the elements of X which map into U: $f(x) \in U \iff x \in f^{-1}(U)$. We call $f^{-1}(U)$ the inverse image of U.*

Forward and inverse images don't necessarily invert each other.

Functions are not sets but a type of object. So we should be able to consider a set of functions, in particular, a set of all functions from X to Y.

Axiom 3.6. *(Power set axiom) Let X and Y be sets. Then there exists a set, denoted Y^X, which consists of all the functions from X to Y, thus*

$$f \in Y^X \iff f \text{ is a function with domain } X \text{ and range } Y.$$

One consequence of this axiom is: let X be a set. Then the set $\{Y : Y \text{ is a subset of } X\}$ is a set. This set is known as the power set of X and is denoted by 2^X.

Axiom 3.7. *(Axiom of union) Let A be a set, all of whose elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are precisely those objects which are elements of the elements of A, thus for all objects x*

$$x \in \bigcup A \iff x \in S \text{ for some } S \in A$$

The axiom of union, combined with the axiom of pair set, implies the axiom of pairwise union. Further, if one has a set I, and for every element $\alpha \in I$ we have some set A_α, then we can form the union set $\bigcup_{\alpha \in I} A_\alpha$ by defining $\bigcup_{\alpha \in I} A_\alpha := \bigcup\{A_\alpha : \alpha \in I\}$, which is a set (due to axiom of replacement and union). More generally, we see that for any object y,

$$y \in \bigcup_{\alpha \in I} A_\alpha \iff y \in A_\alpha \text{ for some } \alpha \in I.$$

We refer to I as an index set, and the elements α of this index set as labels; the sets A_α are then called a family of sets, and are indexed by the labels $\alpha \in I$. When I is empty, then $\bigcup_{\alpha \in I} A_\alpha$ would automatically also be empty. We can similarly from intersections of families of sets, as long as the index set is non-empty. For any object y,

$$y \in \bigcap_{\alpha \in I} A_\alpha \iff y \in A_{alpha} \text{ for all } \alpha \in I.$$

3.5 Cartesian products

Definition 3.5.1. *(Ordered pair) If x and y are any objects, we define the **ordered pair** (x, y) to be a new object, consisting of x as its first component and y as its second component. Two ordered pairs (x, y) and (x', y') are considered equal if and only if both their components match.*

Definition 3.5.2. *(Cartesian products) If X and Y are sets, then we define the **Cartesian product** $X \times Y$ to be the collection of ordered pairs, whose first component lies in X and second component lies in Y, thus $X \times Y = \{(x, y) : x \in X, y \in Y\}$.*

We can re-interpret the addition operation $+$ on the natural numbers as a function $+ : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{N}$, defined by $(x, y) \mapsto x + y$.

Definition 3.5.3. *(Ordered n-tuple and n-fold Cartesian product) Let n be a natural number. An ordered n-tuple $(x_i)_{i \leq i \leq n} \equiv (x_1, \ldots, x_n)$ is a collection of objects x_i, one for every natural number i between 1 and n; we refer to x_i as the ith component of the ordered n-tuple. Two ordered n-tuples $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are said to be equal iff $x_i = y_i$ for all $1 \leq i \leq n$. If $(X_i)_{1 \leq i \leq n}$ is an ordered n-tuple of sets, we defin their Cartesian product $\prod_{1 \leq i \leq n} X_i \equiv X_1 \times \ldots \times X_n$ by*

$$\prod_{1 \leq i \leq n} X_i := \{(x_i)_{1 \leq i \leq n} : x_i \in X_i \textit{ for all } 1 \leq i \leq n\}.$$

The empty Cartesian product $\prod_{i \leq i \leq n} X_i$ given not the empty set $\{\}$, but rather the singleton set $\{()\}$ whose only element is the 0-tuple $()$, also known as the empty tuple. If n is a natural number, we often write X^n as shorthand for the n-fold Cartesian product $X^n := \prod_{i \leq i \leq n} X$.

Lemma 3.5.1. *(Finite choice) Let $n \geq 1$ be a natural number, and for each natural number $1 \leq i \leq n$, let X_i be a non-empty set. Then there exists an n-tuple $(x_i)_{1 \leq i \leq n}$ such that $x_i \in X_i$ for all $1 \leq i \leq n$.*

It may seem that we can extend this lemma for infinite number of choices, but this cannot be done automatically; it requires an additional axiom; the *axiom of choice.*

3.6 Cardinality of sets

The Peano axiom approach treats natural numbers more like ordinals, and are used to order a sequence of objects. This is philosophically different from one of the main conceptualizations of natural numbers - that of cardinality, or measuring how many elements there are in a set. We now address the issue of whether natural numbers can be used to count sets. We will not that we indeed can use natural numbers to count the cardinality of sets, as long as the set is finite.

Definition 3.6.1. *(Equal cardinality) We say two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$ from X to Y.*

This definition is valid whether X is finite or infinite. The fact that two sets have equal cardinality does not preclude one of the sets from containing the other. For instance, $\boldsymbol{N}$ and set of even natural numbers, have same cardinality because of the presence of the bijection $f(n) := 2n$. The notion of having equal cardinality is an equivalence relation, i.e. it follows reflexive, symmetric and transitive axioms.

Definition 3.6.2. *Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with $\{i \in \mathbf{N} : i < n\}$. We also say that X has n elements iff it has cardinality n.*

Proposition 3.6.1. *(Uniqueness of cardinality) Let X be a set with some cardinality n. Then X cannot have any other cardinality, i.e., X cannot have cardinality m for any $m \neq n$.*

Definition 3.6.3. *(Finite sets) A set is finite iff it has cardinality n for some natural number n; otherwise, the set is called infinite. We use $\#(X)$ to denote cardinality of a finite set X.*

Theorem 3.1. *A set of natural numbers $\mathbf{N}$ is infinite.*

Proposition 3.6.2. *(Cardinal arithmetic)*

1. *Let X be a finite set, and let x be an object which is not an element of X. Then $X \cup \{x\}$ is finite and $\#(X \cup \{x\}) = \#(X) + 1$.*
2. *Let X and Y be finite sets. Then $X \cup Y$ is finite and $\#(X \cup Y) \leq \#(X) + \#(Y)$. If in addition X and Y are disjoint, i.e., $X \cap Y = \emptyset$, then $\#(X \cap Y) = \#(X) + \#(Y)$.*
3. *Let X be finite set, and let Y be a subset of X. Then Y is finite, and $\#(Y) \leq \#(X)$. If in addition $Y \neq X$, then we have $\#(Y) < \#(X)$.*
4. *If X is a finite set, and $f : X \to Y$ is a function, then $f(X)$ is a finite set with $\#(f(X)) \leq \#(X)$. If in addition f is one-to-one, then $\#(f(X)) = \#(X)$.*
5. *Let X and Y be finite sets. Then Cartesian product $X \times Y$ is finite and $\#(X \times Y) = \#(X) \times \#(Y)$.*
6. *Let X and Y be finite sets. Then the sets Y^X is finite and $\#(Y^X) = (Y)^{\#(X)}$.*

This proposition suggests that there is another way to define the arithmetic operations of natural numbers; not defined recursively as previously, but instead using the notions of union, Cartesian product, and power set. This is the bases of cardinal arithmetic, which is an alternative foundation to arithmetic than the Peano arithmetic we developed earlier.

4 Integers and rationals

4.1 The integer

Definition 4.1.1. *(Integer) An integer is an expression of the form a—b, where a and b are natural numbers. Two integers are considered to be equal, a—b=c—d, iff $a + d = c + b$. We let $\mathbf{Z}$ denote the set of all integers.*

Definition 4.1.2. *The sum of two integers, (a—b)+(c—d), is defined by the formula (a—b)+(c—d):=(a+c)—(b+d). The product of two integers, (a—b)×(c—d), is defined by (a—b)×(c—d):= $(ac + bd)$—$(ad + bc)$*

We may identify the natural numbers with integers by setting $n \equiv n$—0. We can also define incrementation on the integers by defining $x +\!\!+ := x + 1$ for any integer x.

Definition 4.1.3. *(Negation of integers) If (a—b) is an integer, we define the negation -(a—b) to be the integer (b—a). In particular if n=n—0 is a positive natural number, we can define its negation -n=0—n.*

Lemma 4.1.1. *(Trichotomy of integers) Let x be an integer. Then exactly one of the following three statements is true: x is zero; x is equal to a positive natural number n; or x is the negation $-n$ of a positive natural number n.*

Proposition 4.1.1. *(Laws of algebra for integers: integers form a* ***commutative ring****) Let x, y, z be integers. Then we have*

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $x + 0 = 0 + x = x$
4. $x + (-x) = (-x) + x = 0$
5. $xy = yx$
6. $(xy)z = x(yz)$
7. $x1 = 1x = x$
8. $x(y + z) = xy + xz$
9. $(y + z)x = yx + zx$

If one deleted the identity $xy = yx$, then they would only assert that the integers form a **ring**. We can now define the operation of *subtraction* $x - y$ of two integers by the formula $x - y := x + (-y)$. One can now verify that $a - b$ =a—b, and hence we can discard the — notation from here on.

Proposition 4.1.2. *(Integers have no zeros divisors) Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$, or both.*

we can use this proposition to state the **Cancellation law of integers**. If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Definition 4.1.4. *(Ordering of the integer) Let n and m be integers. We say that n is greater than or equal to m, and write $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. We say that n is strictly greater than m, and write $n > m$ or $m < n$, iff $n \geq m$ and $n \neq m$.*

Lemma 4.1.2. *(Properties of order) Let a, b, c be integers.*

1. *$a > b$ iff $a - b$ is a positive natural number.*
2. *Addition preserves order: If $a > b$, then $a + c > b + c$.*
3. *Positive multiplication preserves order: If $a > b$ and c is positive, then $ac > bc$.*
4. *Negation reverses order: If $a > b$, then $-a < -b$.*
5. *Order is transitive: If $a > b$ and $b > c$, then $a > c$.*
6. *Order trichotomy: Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.*

4.2 The rationals

We create a new symbol $//$, which will eventually be superceded by division, and define

Definition 4.2.1. *(Rational number) A rational number is an expression of the form $a//b$, where a and b are integers and b is non-zero. $a//0$ is not considered to be a rational number. Two rational numbers are considered to be equal, $a//b = c//d$, iff $ad = cb$. The set of all rational numbers is denoted* **Q**.

Definition 4.2.2. *If $a//b$ and $c//d$ are rational numbers, we define their sum $(a//b) + (c//d) := (ad + bc)//db$; their product $(a//b) * (c//d) := (ac)//(bd)$; and the negation $-(a//b) = (-a)//b$. The sum, product of two rational numbers remains a rational number.*

We note that the rational numbers $a//1$ behave in a manner identical to the integer a. Hence, the arithmetic of the integers is consistent with the arithmetic of the rationals. Thus just as we embedded the natural numbers inside the integers, we embed the integers inside the rational numbers. We now define a new operation on the rational: reciprocal x^{-1} of a rational number $x = a//b$ to be a rational number $x^{-1} := b//a$.

Proposition 4.2.1. *(Laws of algebra for rationals: rationals form a* ***field****) Let x, y, z be rationals. Then the following laws of algebra hold:*

1. $x + y = y + x$
2. $(x + y) + z = x + (y + z)$
3. $x + 0 = 0 + x = x$
4. $x + (-x) = (-x) + x = 0$
5. $xy = yx$
6. $(xy)z = x(yz)$
7. $x1 = 1x = x$
8. $x(y + z) = xy + xz$
9. $(y + z)x = yx + zx$
10. *If x is non-zero, $xx^{-1} = x^{-1}x = 1$.*

We can now define the quotient x/y of two rational numbers x and y, provided that y is non-zero, by the formula $x/y := x \times y^{-1}$. Thus we can now discard the $//$ notation. In a similar spirit, we define subtraction on the rationals by the formula $x - y := x + (-y)$.

Definition 4.2.3. *A rational number x is said to be positive iff we have $x = a/b$ for some positive integers a and b. It is said to be negative iff we have $x = -y$ for some positive rational y.*

Lemma 4.2.1. *(Trichotomy of rationals) Let x be a rational number. Then exactly one of the following three statements is true: x is equal to 0; x is a positive rational number; or x is a negative rational number.*

Definition 4.2.4. *(Ordering of the rationals) Let x and y be rational numbers. We say that $x > y$ iff $x - y$ is a positive rational number, and $x < y$ iff $x - y$ is a negative rational number. We write $x \geq y$ iff either $x > y$ or $x = y$, and similarly define $x \leq y$.*

Proposition 4.2.2. *(Properties of order on the rationals) Let x, y, z be rational numbers. Then the following properties hold.*

1. *Order trichotomy: Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.*
2. *Order is anti-symmetric: One has $x < y$ iff $y > x$.*
3. *Order is transitive: If $x < y$ and $y < z$, then $x < z$.*
4. *Addition preserve order: If $x < y$, then $x + z < y + z$.*
5. *Positive multiplication preserves order: If $x < y$ and z is positive, then $xz < yz$.*

The above five properties, combined with the 10 field axioms on rationals have a name: they assert that the rationals $\mathbf{Q}$ form an **ordered field**.

4.3 Absolute value and exponentiation

Definition 4.3.1. *(Absolute value) If x is a rational number, the absolute value $|x|$ of x is defined as follows. If x is positive, then $|x| := x$. If x is negative, then $|x| := -x$. If x is zero, then $|x| := 0$.*

Definition 4.3.2. *(Distance) Let x and y be rational numbers. The quantity $|x - y|$ is called the distance between x and y and is sometimes denoted $d(x, y)$, thus $d(x, y) := |x - y|$.*

Proposition 4.3.1. *(Basic properties of absolute value and distance) Let x, y, z be rational numbers.*

1. *Non-degeneracy of absolute value: We have $| \geq 0$. Also, $|x| = 0$ iff x is 0.*
2. *Triangle inequality for absolute value: We have $|x + y| \leq |x| + |y|$.*
3. *We have the inequalities $-y \leq x \leq y$ iff $y \geq |x|$. In particular, we have $-|x| \leq x \leq |x|$.*
4. *Multiplicativity of absolute value: We have $|xy| = |x||y|$. In particular, $| - x| = |x|$.*
5. *Non-degeneracy of distance: We have $d(x, y) \geq 0$. Also, $d(x, y) = 0$ iff $x = y$.*
6. *Symmetry of distance: $d(x, y) = d(y, x)$.*
7. *Triangle inequality for distance: $d(x, z) \leq d(x, y) + d(y, z)$.*

Absolute value is useful for measuring how "close" two numbers are. As an artificial definition we define ε-closeness. Let $\varepsilon > 0$ be a rational number, and let x, y be rational numbers. We say that y is **ε-close** to x iff we have $d(x, y) \leq \varepsilon$.

Proposition 4.3.2. *Let x, y, z, w be rational numbers.*

1. *If $x = y$, then x is ε-close to y for every $\varepsilon > 0$. Conversely, if x is ε-close to y for every $\varepsilon > 0$, then we have $x = y$.*
2. *Let $\varepsilon > 0$. If x is ε-close to y, then y is ε-close to x.*
3. *Let $\varepsilon, \delta > 0$. If x is ε-close to y, and y is δ-close to z, then x and z are $(\varepsilon + \delta)$-close.*
4. *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then $x + z$ and $y + w$ are $(\varepsilon + \delta)$-close, and $x - z$ and $y - w$ are also $(\varepsilon + \delta)$-close.*
5. *Let $\varepsilon > 0$. If x and y are ε-close, they are also ε'-close for every $\varepsilon' > \varepsilon$.*
6. *Let $\varepsilon > 0$. If y and z are both ε-close to x, and w is between y and z ($y \leq w \leq z$ or $z \leq w \leq y$), then w is also ε-close to x.*
7. *Let $\varepsilon > 0$. If x and y are ε-close, and z is non-zero, then xz and yz are $\varepsilon|z|$-close.*
8. *Let $\varepsilon, \delta > 0$. If x and y are ε-close, and z and w are δ-close, then xz and yw are $(\varepsilon|z| + \delta|x| + \varepsilon\delta)$-close.*

The first three propositions when compared against the reflexive, symmetric, and transitive axioms of equality, gives us a notion of "ε-close" as an approximation for equality in analysis.

Definition 4.3.3. *(Exponentiation to a natural number) Let x be a rational number. To raise x to the power or 0, we define $x^0 := 1$; in particular we define $0^0 := 1$. Now suppose inductively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.*

Proposition 4.3.3. *(Properties of exponentiation I) Let x, y be rational numbers, and let n, m be natural numbers.*

1. *We have $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, and $(xy)^n = x^n y^n$.*
2. *Suppose $n > 0$. Then we have $x^n = 0$ iff $x = 0$.*
3. *If $x \geq y \geq 0$, then $x^n \geq y^n \geq 0$. If $x > y \geq 0$ and $n > 0$, then $x^n > y^n \geq 0$.*

4. *We have* $|x^n| = |x|^n$.

Definition 4.3.4. *(Exponentiation to a negative number) Let* x *be a non-zero rational number. Then for any negative integer* $-n$*, we define* $x^{-n} := 1/x^n$.

Proposition 4.3.4. *(Properties of exponentiation II) Let* x, y *be non-zero rational numbers, and let* n, m *be integers.*

1. *We have* $x^n x^m = x^{n+m}$, $(x^n)^m = x^{nm}$, *and* $(xy)^n = x^n y^n$.

2. *If* $x \geq y > 0$, *then* $x^n \geq y^n > 0$ *if* n *is positive, and* $0 < x^n \leq y^n$ *if* n *is negative.*

3. *If* $x, y > 0$, $n \neq 0$, *and* $x^n = y^n$, *then* $x = y$.

4. *We have* $|x^n| = |x|^n$.

4.4 Gaps in the rational numbers

Proposition 4.4.1. *(Interspersing of integers by rationals) Let* x *be a rational number. Then there exists a unique integer* n *such that* $n \leq x < n+1$, *denoted by* $\lfloor x \rfloor$. *In particular, there exists a natural number* N *such that* $N > x$.

Proof: Suppose $x \geq 0$. Then we can write $x = a/b$, where a is a natural number and b is a positive integer. By Euclidean algorithm there exist natural number m, r such that $a = mb + r$ and $0 \leq r < b$. Since $0 \leq r < b$, we have $mb \leq mb + r < mb + b = (m+1)b$. Since b is positive, so is $1/b$, and we can multiply through by $1/b$ to obtain $m \leq a/b < m + 1$. Thus we can take $n := m$ to be the integer we are looking for.

Now suppose x is negative. Then $-x$ is positive and using what we just showed, we have $m \leq -x < m+1$ for some integer m. Thus, we have $-(m+1) < x \leq -m$. We have two cases now. For $x \neq -m$, we can take $n := -(m+1)$ to be the required integer. Which implies $n + 1 = -m$ and hence $n \leq x < n+1$. On the other hand if $x = -m$ then we can take $n := -m$ to be the required integer. We then have $n = x < n+1$ as required.

Next, we show uniqueness. Suppose we did have distinct integers n, m such that $n \leq x < n+1$ and $m \leq x < m+1$. Since $n \neq m$ we have either $n < m$ or $n > m$. We can assume $n < m$, since if $n > m$ we can just repeat the proof by swapping n and m. So we have $n < m \leq x < n+1$, which is simplified to $n < m < n+1$. By definition of order, we have $m = n + d$ and $n + 1 = m + d'$ for positive integers d, d'. So we have $n + 1 = n + d + d'$, which means $1 = d + d'$. Since d is positive there is a natural number k such that $d = k+1$. Thus we have $1 = k+1+d'$ or $0 = k+d'$. This means that $d' = 0$, a contradiction. This contradiction shows that we must have $n = m$.

To find a natural number N such that $N > x$, we can split into two cases. If x is positive or zero, we have an integer n such that $n \leq x < n+1$ as we showed above. So we can take $N := n + 1$. Then $0 \leq x < N$, and $N = n+1$ is a natural number. On the other hand, if x is negative, we can take $N := 0$. We have $x < 0 = N$ as required. □

Proposition 4.4.2. *(Interspersing of rationals by rationals) If* x *and* y *are two rationals such that* $x < y$, *then there exists a third rational* z *such that* $x < z < y$.

Proof: We set $z := (x+y)/2$. Since $x < y$, and $1/2$ is positive, we have that $x/2 < y/2$. If we add $y/2$ to both sides we obtain $x/2 + y/2 < y/2 + y/2$, i.e., $z < y$. If we instead add $x/2$ to both sides we obtain $x/2 + x/2 < y/2 + x/2$, i.e., $x < z$. Thus $x < z < y$ as desired. □

Despite the rationals having this denseness property, they are still incomplete; there are still an infinite number of "gaps" between the rationals.

Proposition 4.4.3. *There does not exist any rational number* x *for which* $x^2 = 2$.

Proof: Suppose for the sake of contradiction that we had a rational number x for which $x^2 = 2$. Clearly

x is not zero. We may assume x is positive, for if x were negative then we could just replace x by $-x$, since $x^2 = (-x)^2$. Thus $x = p/q$ for some positive integers p, q, so $(p/q)^2 = 2$, which we can rearrange as $p^2 = 2q^2$. Define a natural number p to be even if $p = 2k$ for some natural number k, and odd if $p = 2k+1$ for some natural number k. Every natural number is either even or odd but not both.

Aside: every natural number is either even or odd, but not both: We use induction to first show that every natural number is even or odd. For the base case we see that $0 = 2 \times 0 + 0$ so 0 is even. Now suppose inductively that n is even or odd. We have to show that $n+1$ is even or odd. If n is even, then $n = 2k$ for some natural number k, by definition. Thus $n + 1 = 2k + 1$ or $n + 1$ is odd, by definition. On the other hand, if n is odd, then $n = 2k + 1$ for some natural number k. Thus $n + 1 = 2k + 2 = 2(k + 1)$ is even. In either case, we have show that $n + 1$ is even or odd, closing the induction. Now we show that a natural number cannot be both even and odd. Suppose for the sake of contradiction that there is some natural number n which is both even and odd. Thus we have $n = 2k$ for some natural number k and $n = 2r + 1$ for some natural number r. So $2k = 2r + 1$. Now we have two cases, $r \geq k$ or $r < k$. Suppose first that $r \geq k$. Then $2k = 2r + 1$ implies $0 = 2(r - k) + 1$. Since $r \geq k$, we see that $r - k$ is a natural number, and since 0 is not a successor of any natural number we have reached a contradiction. Now suppose that $r < k$. Then $r + 1 \leq k$ by definition. So, $1 \leq k - r$ and since $k - r$ is positive, $0 < k - r$ so we can add $k - r$ on both sides to get $k - r < 2(k - r)$. This means we have $1 \leq k - r < 2(k - r) = 1$, resulting in $1 < 1$, a contradiction. In either case we arrive at a contradiction, which means our assumption that n is both even and odd was false.

Aside: if p is odd, then p^2 is also odd: If p is odd we can write $p = 2k + 1$ for some natural number k. This means that $p^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Thus for $r = 2k^2 + 2k$ we have $p^2 = 2r + 1$, which shows that p^2 is odd as well.

If p is odd, then p^2 is also odd. This contradicts $p^2 = 2q^2$. Thus p is even, i.e. $p = 2k$ for some natural number k. Since p is positive, k must also be positive. Inserting $p = 2k$ into $p^2 = 2q^2$ we obtain $4k^2 = 2q^2$, so that $q^2 = 2k^2$. We started with a pair (p, q) of positive integers such that $p^2 = 2q^2$, and ended up with a pair (q, k) of positive integers such that $q^2 = 2k^2$. Since $p^2 = 2q^2$, we have $q < p$.

Aside: If $p^2 = 2q^2$ with p, q positive integers, we have $q < p$: Suppose for the sake of contradiction that $q \geq p$. Then since p, q are positive, we can multiply on both sides by p to get $pq \geq p^2$ and multiply on both sides by q to get $q^2 \geq pq$, so that $q^2 \geq p^2$. Also since q is positive, so is q^2, so $q^2 > 0$, which means that we can add q^2 to both sides to get $2q^2 > q^2$. Putting these two inequalities together, we have $2q^2 > q^2 \geq p^2$, which implies that $2q^2 \neq p^2$, a contradiction.

If we rewrite $p' := q$ and $q' := k$, we thus can pass from one solution (p, q) to the equation $p^2 = 2q^2$ to a new solution (p', q') to the same equation which has a smaller value of p. But then we can repeat this procedure again and again, obtaining a sequence (p'', q''), (p''', q'''), etc. of solutions to $p^2 = 2q^2$, each one with a smaller value of p than the previous, and each one consisting of positive integers.

Aside: principle of infinite descent: A sequence $a_0, a_1, a_2, \ldots$ of numbers (natural, integer, rationals or reals) is said to be in infinite descent if we have $a_n > a_{n+1}$ for all natural numbers n, i.e., $a_0 > a_1 > a_2 > \ldots$. We want to prove that it is not possible to have a sequence of natural numbers which is in infinite descent. Suppose for the sake of contradiction that $a_0 > a_1 > a_2 > \ldots$ is a sequence of natural numbers which is in infinite descent. We will use induction on k to show that $a_n \geq k$ for all $k \in \mathbf{N}$ and all $n \in \mathbf{N}$. This is same as the statement $P(k)$: for each $n \in \mathbf{N}$, we have $a_n \geq k$. We want to prove $P(k)$ to be true for each natural number k. For the base case $k = 0$, we see that $a_n \geq 0$ for all n, since $a_0, a_1, a_2, \ldots$ are all natural numbers. Now suppose inductively that we have $a_n \geq k$ for all n. We must show that $a_n \geq k+1$ for all n. Let n be given. Then by inductive hypothesis, we know that $a_n \geq k$ for all n, so in particular for $n + 1$ we have $a_{n+1} \geq k$. We also know that $a_n > a_{n+1}$ since the sequence is in infinite descent. Thus we have $a_n > k$ by transitivity of order, which means $a_n \geq k + 1$. This closes the induction. Now we obtain a contradiction. For $n = 0$ and $k = a_0 + 1$ in particular, we must have $a_0 \geq a_0 + 1$. This means $0 \geq 1$, a contradiction.

This solution we constructed contradicts the principle of infinite descent. This contradiction shows that we could not have had a rational x for which $x^2 = 2$. □

On the other hand, we can get rational numbers which are arbitrarily close to a square root of 2.

Proposition 4.4.4. *For every rational number $\varepsilon > 0$, there exists a non-negative rational number x such that $x^2 < 2 < (x+\varepsilon)^2$.*

Proof: Let $\varepsilon > 0$ be rational. Suppose for the sake of contradiction that there is no non-negative rational number x for which $x^2 < 2 < (x+\varepsilon)^2$. This means that whenever x is non-negative and $x^2 < 2$, we must also have $(x+\varepsilon)^2 < 2$, since $(x+\varepsilon)^2$ cannot be equal to 2, by previous proposition. Since $0^2 < 2$, we thus have $\varepsilon^2 < 2$, which then implies $(2\varepsilon)^2 < 2$, and indeed a simple induction shows that $(n\varepsilon)^2 < 2$ for every natural number n. But by proposition 4.4.1 we can find an integer n such that $n > 2/\varepsilon$, which implies that $n\varepsilon > 2$, which implies that $(n\varepsilon)^2 > 4 > 2$, contradicting the claim that $(n\varepsilon)^2 < 2$ for all natural number n. This contradiction gives the proof. □

We see that while the set **Q** of rationals does not actually have $\sqrt{2}$ as a member, we can get as close as we wish to $\sqrt{2}$. Thus it seems that we can create a square root of 2 by taking a "limit" of a sequence of rationals. That is what do do next.

5 The real numbers

We defined natural number system (**N**) using the five Peano axioms, and postulated that such a number system existed. This is plausible, since the natural numbers correspond to the very intuitive and fundamental notion of sequential counting. Using that number system one could then recursively define addition and multiplication, and verify that they obeyed the usual laws of algebra. We then constructed the integers (**Z**) by taking formal differences of the natural numbers, a—b. We then constructed the rationals (**Q**) by taking formal quotients of the integers, $a//b$, although we need to exclude division by zero in order to keep the laws of algebra reasonable. The rational system is sufficient to do a lot of mathematics but for geometry we need to introduce irrational numbers like $\sqrt{2}$ and transcendental numbers like π or $cos(1)$. The real numbers construction from rationals is not as straightforward as before since we are passing from "discrete" system to "continuous" one - we require the notion of *limit*. We shall now fill the "gaps" in the rational numbers using limits to create the real numbers (**R**) using the procedure of *completing* one metric space to form another. The real number system will end up being lot like the rational numbers, but will have some new operations - notably that of *supremum*, which can then be used to define limits and thence to everything else that calculus needs.

5.1 Cauchy sequences

Definition 5.1.1. *(Sequences) Let m be an integer. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from the set $\{n \in \mathbf{Z} : n \geq m\}$ to $\mathbf{Q}$, i.e., a mapping which assigns to each integer n greater than or equal to m, a rational number a_n.*

We want to define the real numbers as the limits of sequences of rational numbers. To do so, we have to distinguish which sequences of rationals are convergent and which ones are not. Recall that two rational numbers x, y are ε-close if $d(x, y) = |x - y| \leq \varepsilon$.

Definition 5.1.2. *(ε-steadiness) Let $\varepsilon > 0$. A sequence $(a_n)_{n=m}^{\infty}$ is said to be **ε-steady** iff each pair a_j, a_k of sequence elements is ε-close for every natural number j, k.*

ε-steadiness does not really capture the limiting behavior of a sequence, because it is too sensitive to the initial members of the sequence. So we need a more robust notion of steadiness that does not care about the initial members of a sequence.

Definition 5.1.3. *(Eventual ε-steadiness) Let $\varepsilon > 0$. A sequence $(a_n)_{n=0}^{\infty}$ is said to be **eventually ε-steady** iff the sequence $a_N, a_{N+1}, a_{N+2}, \ldots$ is ε-steady for some natural number $N \geq 0$, i.e. iff there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$.*

Definition 5.1.4. *(**Cauchy sequences**) A sequence $(a_n)_{n=0}^{\infty}$ of rational numbers is said to be a Cauchy sequence iff for every rational $\varepsilon > 0$, the sequence $(a_n)_{n=0}^{\infty}$ if eventually ε-steady, i.e., for every $\varepsilon > 0$, there exists an $N \geq 0$ such that $d(a_j, a_k) \leq \varepsilon$ for all $j, k \geq N$.*

Definition 5.1.5. *(Bounded sequences) Let $M \geq 0$ be rational. A finite sequence $a_1, a_2, \ldots, a_n$ is bounded by M iff $|a_i| \leq M$ for all $1 \leq i \leq n$. An infinite sequence $(a_n)_{n=1}^{\infty}$ is bounded by M iff $|a_i| \leq M$ for all $i \geq 1$. A sequence is said to be bounded iff it is bounded by M for some rational $M \geq 0$.*

Every finite sequence $a_1, a_2, \ldots, a_n$ is bounded. But, not all infinite sequences are bounded; infinity is not a natural number.

Lemma 5.1.1. *(Cauchy sequences are bounded) Every Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded.*

Proof: Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rational numbers. By the definition of Cauchy sequences this sequence is eventually ε-steady for every rational $\varepsilon > 0$, so in particular for $\varepsilon = 1$ it is eventually 1-steady. This means that there exists $N \geq 1$ such that $|a_j - a_k| \leq 1$ for all $j, k \geq N$. Since $N \geq N$, we may take k to be N. Thus for all $j \geq N$ we have $|a_j - a_N| \leq 1$. By Triangle inequality for absolute values we have $|(a_j - a_N) + a_N| \leq |a_j - a_N| + |a_N|$, i.e. $|a_j| - |a_N| \leq |a_j - a_N| \leq 1$. Thus we have $|a_j| \leq 1 + |a_N|$ for all $j \geq N$, so we have bounded the infinite tail of the sequence. WE already established that the finite sequence $a_1, \ldots, a_{N-1}$ is bounded by some number $M \geq 0$. So now we can take the bound over the entire sequence to be $M' := M + 1 + |a_N|$. If $1 \leq n < N$, then $|a_n| \leq M \leq M'$, and if $n \geq N$ then $|a_n| \leq 1 + |a_N| \leq M'$. This proves the result.

5.2 Equivalent Cauchy sequences

Definition 5.2.1. *(ε-close sequences) Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is ε-close to $(b_n)_{n=0}^{\infty}$ iff a_n is ε-close to b_n for each $n \in \mathbf{N}$, i.e. if $|a_n - b_n| \leq \varepsilon$ for all $n = 0, 1, 2, \ldots$.*

Definition 5.2.2. *(Eventually ε-close sequences) Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be two sequences, and let $\varepsilon > 0$. We say that the sequence $(a_n)_{n=0}^{\infty}$ is eventually ε-close to $(b_n)_{n=0}^{\infty}$ iff there exists an $N \geq 0$ such that the sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are ε-close, i.e. iff there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.*

Definition 5.2.3. *(Equivalent sequences) Two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are equivalent iff for each rational $\varepsilon > 0$, the two sequences are eventually ε-close, i.e., there exists an $N \geq 0$ such that $|a_n - b_n| \leq \varepsilon$ for all $n \geq N$.*

5.3 The construction of the real numbers

Definition 5.3.1. *(Real numbers) A real number is defined to be an object of the form $LIM_{n\to\infty} a_n$, where $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence of rational numbers. Two real numbers $LIM_{n\to\infty} a_n$ and $LIM_{n\to\infty} b_n$ are said to be equal iff $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. The set of all real numbers is denoted by $\boldsymbol{R}$.*

We will refer to $LIM_{n\to\infty} a_n$ are the formal limit of the sequence $(a_n)_{n=1}^{\infty}$. Later on we will show it is equivalent to the genuine notion of limit.

Proposition 5.3.1. *(Formal limits are well-defined) Let $x = LIM_{n\to\infty} a_n$, $y = LIM_{n\to\infty} b_n$, and $z = LIM_{n\to\infty} c_n$ be real numbers. Then, with the above definition of equality for real numbers, we have $x = x$. Also, if $x = y$, then $y = x$. Finally, if $x = y$ and $y = z$, then $x = z$.*

Definition 5.3.2. *(Addition of reals) Let $x = LIM_{n\to\infty} a_n$ and $y = LIM_{n\to\infty} b_n$ be real numbers. Then we define the sum $x + y$ to be $x + y := LIM_{n\to\infty}(a_n + b_n)$.*

Lemma 5.3.1. *(Sum of Cauchy sequences is Cauchy) Let $x = LIM_{n\to\infty} a_n$ and $y = LIM_{n\to\infty} b_n$ be real numbers. Then $x + y$ is also a real number, i.e. $(a_n + b_n)_{n=1}^{\infty}$ is a Cauchy sequence of rationals.*

Lemma 5.3.2. *(Sums of equivalent Cauchy sequences are equivalent) Let $x = LIM_{n\infty} a_n$, $y = LIM_{n\to\infty} b_n$, and $x' = LIM_{n\to\infty} a'_n$ be real numbers. Suppose that $x = x'$. Then we have $x + y = x' + y$.*

Definition 5.3.3. *(Multiplication of reals) Let $x = LIM_{n\to\infty} a_n$ and $y = LIM_{n\to\infty} b_n$ be real numbers THen we define the product $xy := LIM_{n\to\infty} a_n b_n$.*

Proposition 5.3.2. *(Multiplication is well-defined) Let $x = LIM_{n\to\infty} a_n$, $y = LIM_{n\to\infty} b_n$, and $x' = LIM_{n\to\infty} a'_n$ be real numbers Then xy is also a real number. Furthermore, if $x = x'$, then $xy = x'y$.*

We can now embed the rationals back into the reals, by equating every rational number q with the real number $LIM_{n\to\infty}q$. This embedding is consistent with our definitions of addition, multiplication and equality. We can now easily define negation $-x$ for real numbers x by the formula $-x := (-1)\times x$. Thereafter we can define subtraction as usual by $x-y := x+(-y)$. Now we can show that real numbers obey all the usual rules of algebra except perhaps for the laws involving division.

Proposition 5.3.3. *All the laws of algebra hold not only for the integers, but for the reals as well.*

The last basic arithmetic operation we need to define is reciprocation $x \to x^{-1}$. We should only allow the operation of reciprocal when x is non-zero otherwise the sequence diverges and is not a real number. To get around these problems we need to keep our Cauchy sequence away from zero.

Definition 5.3.4. *(Sequence bounded away from zero) A sequence $(a_n)_{n=1}^{\infty}$ of rational numbers is said to be bounded away from zero iff there exists a rational number $c > 0$ such that $|a_n| \geq c$ for all $n \geq 1$.*

Lemma 5.3.3. *Let x be a non-zero real number. Then $x = LIM_{n\to\infty}a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is bounded away from zero.*

Lemma 5.3.4. *Suppose that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence which is bounded away from zero. Then the sequence $(a_n^{-1})_{n=1}^{\infty}$ is also a Cauchy sequence.*

Definition 5.3.5. *(Reciprocals of real numbers) Let x be non-zero real number. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence bounded away from zero such that $x = LIM_{n\to\infty}a_n$. Then we define the reciprocal $x^{-1} := LIM_{n\to\infty}a_n^{-1}$.*

Lemma 5.3.5. *(Reciprocation is well defined) Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two Cauchy sequences bounded away from zero such that $LIM_{n\to\infty}a_n = LIM_{n\to\infty}b_n$. Then $LIM_{n\to\infty}a_n^{-1} = LIM_{n\to\infty}b_n^{-1}$.*

It is clear from the definition that $x^{-1}x = 1$ for any non-zero real number x. Thus all the field axioms apply to the reals as well as to the rationals. Once one have reciprocal, one can define division x/y of two real numbers x, y, provided y is non-zero by $x/h := x \times y^{-1}$, just as we did for rationals. In particular, we have the cancellation law: if x, y, z are real numbers such that $xz = yz$, and z is non-zero, then by dividing by z we conclude that $x = y$. Note that this cancellation law does not work when z is zero.

5.4 Ordering the reals

Definition 5.4.1. *Let $(a_n)_{n=1}^{\infty}$ be a sequence of rationals. We say that this sequence is positively bounded away from zero iff we have a positive rational $c > 0$ such that $a_n \geq c$ for all $n \geq 1$. The sequence is negatively bound away from zero iff we have a negative rational $-c < 0$ such that $a_n \leq -c$ for all $n \geq 1$.*

Definition 5.4.2. *A real number x is said to be positive iff it can be written as $x = LIM_{n\to\infty}a_n$ for some Cauchy sequence $(a_n)_{n=1}^{\infty}$ which is positively bounded away from zero. x is said to be negative iff it can be written as $x = LIM_{n\to\infty}a_n$ for some sequence $(a_n)_{n=1}^{\infty}$ which is negatively bounded away from zero.*

Proposition 5.4.1. *(Basic properties of positive reals) For every real number x, exactly one of the following three statements is true: (a) x is zero; (b) x is positive; (c) x is negative. A real number x is negative if and only if $-x$ is positive. If x and y are positive, then so are $x+y$ and xy.*

Definition 5.4.3. *(Absolute value) Let x be a real number. We define the absolute value $|x|$ of x to equal x if x is positive, $-x$ when x is negative, and 0 when x is zero.*

Definition 5.4.4. *(Ordering of the real numbers) Let x and y be real numbers. We say that x is greater than y, and write $x > y$ if $x-y$ is a positive real number, and $x < y$ iff $x-y$ is a negative real number. We define $x \geq y$ iff $x > y$ or $x = y$, and similarly define $x \leq y$.*

All the properties of order on the rationals hold for real as well. Hence, real $\boldsymbol{R}$ form an ordered field. We have seen that the formal limit of positive rationals need not be positive; it could be zero. Set of positive reals is open. However, the formal limits of non-negative rationals is non-negative. Set of non-negative reals is closed.

Proposition 5.4.2. *(The non-negative reals are closed) Let $a_1, a_2, \ldots$ be a Cauchy sequence of non-negative rational numbers. Then $LIM_{n\to\infty}a_n$ is a non-negative real number.*

Proof: We argue by contradiction, and suppose the real number $x := LIM_{n\to\infty}a_n$ is a negative number. Then by definition of negative real number, we have $x = LIM_{n\to\infty}b_n$ for some b_n which is negatively bounded away from zero, i.e. there is a negative rational $-c < 0$ such that $b_n \le -c$ for all $n \ge 1$. On the other hand, we have $a_n \ge 0$ for all $n \ge 1$, by hypothesis. Thus the numbers a_n and b_n are never $c/2$-close, since $c/2 < c$. Thus the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not eventually $c/2$-close. Since $c/2 > 0$, this implies that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equivalent. But this contradicts the fact that both these sequences have x as their formal limit.

Lemma 5.4.1. *Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be Cauchy sequences of rationals such that $a_n \ge b_n$ for all $n \ge 1$. Then $LIM_{n\to\infty}a_n \ge LIM_{n\to\infty}b_n$.*

Note that the above Lemma does not work if the $\ge$ signs are replaced by $>$. We now define the distance $d(x, y) := |x - y|$ just as we did for the rationals.

Proposition 5.4.3. *(Bounding of reals by rationals) Let x be a positive real number. Then there exists a positive rational number q such that $q \le x$, and there exists a positive integer N such that $x \le N$.*

Proposition 5.4.4. *(Archimedean property) Let x and ε be any positive real numbers. Then there exists a positive integer M such that $M\varepsilon > x$.*

This property says that no matter how large x is and how small ε is, if one keeps adding ε to itself, one will eventually overtake x.

Proposition 5.4.5. *Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.*

The real number system contains the rationals, and has almost everything that the rational number system has: the arithmetic operations, the laws of algebra, the laws of order. It is at least as good as the rational number system. We now show that it can go beyond it, e.g. we can take square roots.

5.5 The least upper bound property

One of the most basic advantages of the real number over the rationals is that one can take the *least upper bound* $\sup(E)$ of any subset E of the real numbers $\boldsymbol{R}$.

Definition 5.5.1. *(Upper bound) Let E be a subset of $\boldsymbol{R}$, and let M be a real number. We say that M is an upper bound for E, iff we have $x \le M$ for every element x in E.*

$\boldsymbol{R}^+$ has no upper bounds. Every number M is an upper bound for $\emptyset$. If M is an upper bound of E, then any larger number $M' \ge M$ is also an upper bound of E.

Definition 5.5.2. *(Least upper bound) Let E be a subset of $\boldsymbol{R}$, and M be a real number. We say that M is the least upper bound for E iff M is an upper bound for E, and also any other upper bound M' for E must be larger than or equal to M.*

The empty set does not have a least upper bound.

Proposition 5.5.1. *(Uniqueness of the least upper bound) Let E be a subset of $\boldsymbol{R}$. Then E can have at most one least upper bound.*

Proof: Let E be a subset of $\boldsymbol{R}$ and let M_1 and M_2 be two least upper bounds. Since M_1 is a least upper bound and M_2 is an upper bound, then by definition of least upper bound we have $M_2 \ge M_1$. Since M_2 is a least upper bound and M_1 is an upper bound, we similarly have $M_1 \ge M_2$. Thus $M_1 = M_2$. Thus there is at most one least upper bound. □

Theorem 5.1. *(Existence of least upper bound) Let E be a non-empty subset of $\boldsymbol{R}$. If E has an upper bound, then it must have exactly one least upper bound.*

Proof: Let E be a non-empty subset of $\boldsymbol{R}$ then there is at most one least upper bound, say M. We now have to show that E has at least one least upper bound. Since E is non-empty, we can choose some element x_0 in E. Let $n \geq 1$ be a positive integer. By Archimedean property, we can find an integer K such that $K/n \geq M$, and hence K/n is also an upper bound for E. By the Archimedean property again, there exists another integer L such that $L/n < x_0$. Since x_0 lies in E, we see that L/n is not an upper bound for E. Since K/n is an upper bound but L/n is not, we see that $K \geq L$.

We can find a unique integer $L < m_n \leq K$ with the property that m_n/n is an upper bound for E, but $(m_n - 1)/n$ is not. **Aside**: Suppose for the sake of contradiction that there is no such m_n such that $L < m_n \leq K$ and where m_n/n is an upper bound for E and $(m_n - 1)/n$ is not an upper bound for E. Let $P(d)$ be the statement "for all integers K, L, if $K - L = d$, K/n is an upper bound for E , and L/n is not an upper bound for E, then there exists an integer $L < m_n \leq K$ such that m_n/n is an upper bound for E and $(m_n - 1)/n$ is not an upper bound for E". We will use induction to show it is true for all $d \geq 1$. For the base case we have $d = 1$. Let K, L be two integers such that $K - L = 1$, and such that K/n is an upper bound for E and L/n is not an upper bound for E. We can choose $m_n := K$ and see that indeed $L < m_n \leq K$. Also, $m_n/n = K/n$ is an upper bound for E, while $(m_n - 1)/n = L/n$ is not an upper bound for E. Thus P(1) is true. Now suppose inductively that $P(d)$ is true. We want to show that $P(d+1)$ is also true. Let K, L be two integers such that $K - L = d + 1$, and where K/n be an upper bound for E and L/n is not an upper bound for E. We have two cases. (a) Suppose first that $(L + 1)/n$ is not an upper bound for E. Then $K - (L + 1) = d$ so we may use the inductive hypothesis to conclude that there exists an integer $L+1 < m_n \leq K$ such that m_n/n is an upper bound for E and $(m_n - 1)/n$ is not an upper bound for E. But $L < L + 1$, so we have $L < m_n \leq K$, which means this same m_n with work. (b) Now suppose that $(L + 1)/n$ is an upper bound for E. Then we may take $m_n : L + 1$. Indeed we have $L < L + 1 = m_n$ and also $K - L = d + 1$ so $K = L + 1 = d$, which means $K \geq L + 1 = m_n$. Thus we have $L < m_n \leq K$. Also $m_n/n = (L + 1)/n$ is an upper bound for E while $(m_n - 1)/n = L/n$ is not. In both cases we have shown that such an integer m_n exists, so this closes the induction. Finally, let $L < K$ be integers. Then $K - L > 0$ is a positive integer, so P(K-L) is true. Thus we are given an integer m_n with the properties we want. **Aside**: To show uniqueness let's assume integers m and m' with properties that m/n and m'/n are upper bounds for E, but $(m - 1)/n$ and $(m' - 1)/n$ are not upper bounds for E. Hence, there exists some $x \in E$ such that $(m - 1)/n < x$ and there exists some $y \in E$ such that $(m' - 1)/n < y$. Thus we have $(m - 1)/n < x \leq m'/n$ and $(m' - 1)/n < y \leq m/n$. This implies $m - 1 < m'$ and $m' - a < m$. And $m \leq m'$ and $m' \leq m$, which means $m' = m$.

This gives a well defined and unique sequence $m_1, m_2, m_3, \ldots$ of integers, with each of m_n/n being upper bounds and each of the $(m_n - 1)/n$ not being upper bounds. Now let $N \geq 1$ be a positive integer, and let $n, n' \geq N$ be integers. Since m_n/n is an upper bound for E and $(m_{n'} - 1)/n'$ is not, we must have $m_n/n > (m_{n'} - 1)/n'$. Thus we have $\frac{m_n}{n} - \frac{m_{n'}}{n'} > -\frac{1}{n'} \geq -\frac{1}{N}$. Similarly, since $m_{n'}/n'$ is an upper bound for E and $(m_n - 1)/n$ is not, we have $m_{n'}/n' > (m_n - 1)/n$, and hence $\frac{m_n}{n} - \frac{m_{n'}}{n'} < \frac{1}{n} \leq \frac{1}{N}$. Putting these two bounds together, we see that $|\frac{m_n}{n} - \frac{m_{n'}}{n'}| \leq \frac{1}{N}$ for all $n, n' \geq N \geq 1$. **Aside**: Let $\varepsilon > 0$ be a rational number. Now, we can always find a positive integer N such that $\varepsilon > 1/N > 0$. Thus we can say $|\frac{m_n}{n} - \frac{m_{n'}}{n'}| \leq \frac{1}{N} < \varepsilon$. This shows that it is a Cauchy sequence. Since the m_n/n are rational numbers, we can now define the real number $S := LIM_{n\to\infty} m_n/n$. This converges to the zero sequence so is equivalent to $S = LIM_{n\to\infty}(m_n - 1)/n$.

To finish the proof of the theorem, we need to show that S is the least upper bound for E. First we show that it is an upper bound. Let x be any element of E. Then since m_n/n is an upper bound for E, we have $x \leq m_n/n$ for all $n \geq 1$. Hence, $x \leq LIM_{n\to\infty} m_n/n = S$. Thus S is indeed an upper bound for E. Now we show it is a least upper bound. Suppose y is an upper bound for E. Since $(m_n - 1)/n$ is not an upper bound, we conclude that $y \geq (m_n - 1)/n$ for all $n \geq 1$. Then we conclude that $y \geq LIM_{n\to\infty}(m_n - 1)/n = S$. Thus the upper bound S is less than or equal to every upper bound of E, and S is thus a least upper bound of E. □

Definition 5.5.3. *(Supremum) Let E be a subset of the real numbers. If E is non-empty and has some upper bound, we define the supremum of E, $\sup(E)$ to be the least upper bound of E. We introduce two additional symbols, $+\infty$ and $-\infty$. If E is non-empty and has no upper bound, we set up $\sup(E) := +\infty$; if E is empty, we set $\sup(E) := -\infty$.*

Proposition 5.5.2. *There exists a positive real number x such that $x^2 = 2$.*

Proof: Let E be the set $\{y \in \boldsymbol{R} : y \geq 0 \text{ and } y^2 < 2\}$; thus E is the set of all non-negative real numbers whose square is less than 2. Observe that E has an upper bound or 2, because $y > 2$, then $y^2 > 4 > 2$ and hence $y \notin E$. Also, E is non-empty, for instance, 1 is an element of E. Thus by the least upper bound property, we have a real number $x := sup(E)$ which is the least upper bound of E. Then x is greater than or equal to 1 and less than or equal to 2. So x is positive. Now we show that $x^2 = 2$.

We argue this by contradiction. We show that both $x^2 > 2$ and $x^2 < 2$ lead to contradiction. First suppose that $x^2 < 2$. Let $0 < \varepsilon < 1$ be a small number; then we have $(x+\varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 \leq x^2 + 2\varepsilon + \varepsilon x^2 + 5\varepsilon$, since $x \leq 2$ and $\varepsilon^2 \leq \varepsilon$. Since $x^2 < 2$, we see that we can choose an $0 < \varepsilon < 1$ such that $x^2 + 5\varepsilon < 2$, thus $(x+\varepsilon)^2 < 2$. By construction of E, this means that $x + \varepsilon \in E$; but this contradicts the fact that x is an upper bound of E.

Now suppose that $x^2 > 2$. Let $0 < \varepsilon < 1$ be a small number; then we have $(x-\varepsilon)^2 = x^2 - 2\varepsilon x + \varepsilon^2 \geq x^2 - 2\varepsilon x \geq x^2 - 4\varepsilon$, since $x \leq 2$ and $\varepsilon^2 \geq 0$. Since $x^2 > 2$, we can choose $0 < \varepsilon < 1$ such that $x^2 - 4\varepsilon > 2$, and thus $(x-\varepsilon)^2 > 2$. But then this implies that $x - \varepsilon \geq y$ for all $y \in E$, because if $x - \varepsilon < y$ then $(x-\varepsilon)^2 < y^2 \leq 2$, a contradiction. Thus $x - \varepsilon$ is an upper bound for E, which contradicts the fact that x is the least upper bound of E. From these two contradictions we see that $x^2 = 2$, as desired. □

We see that the least upper bound property allows us to construct the square root. Rationals $\boldsymbol{Q}$ do not obey the least upper bound property. Similarly, the greatest lower bounds, of sets E also known as the infimum of E and denoted by $\inf(E)$ can be defined.

Example 5.5.1. Let E be a subset of the real number $\boldsymbol{R}$, and suppose that E has the least upper bound M which is a real number, i.e. $M = \sup(E)$. Let $-E$ be the set $\{-x : x \in E\}$. Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.

Proof: First we show that $-M$ is a lower bound of $-E$. Let $y \in -E$ be an element of $-E$. Thus $y = -x$ for some element $x \in E$. Since M is an upper bound of E, we have $x \leq M$. Thus we have $-M \leq -x$, i.e. $-M \leq y$. Since y was arbitrary, this shows that $-M$ is a lower bound of $-E$.

Next we show that $-M$ is the greatest lower bound. Let L be any lower bound of $-E$, i.e. $L \leq y$ for all $y \in -E$. We will show that $-L$ is an upper bound of E. Let $x \in E$. Then $-x \in -E$ so $L \leq -x$. This means $x \leq -L$. Since $x \in E$ was arbitrary, this means $-L$ is an upper bound of E. By hypothesis, M is the least upper bound of E, so $M \leq -L$, which means $L \leq -M$. This means $-M$ is greater than or equal to any lower bound of E, which is what we wanted to show. □

5.6 Real exponentiation I

Definition 5.6.1. *(Exponentiating a real by a natural number) Let x be a real number. To raise x to the power 0, we define $x^0 := 1$. Now suppose recursively that x^n has been defined for some natural number n, then we define $x^{n+1} := x^n \times x$.*

Definition 5.6.2. *(Exponentiating a real by an integer) Let x be a non-zero real number. Then for any negative integer $-n$, we define $x^{-n} := 1/x^n$.*

All the properties of exponentiation remain valid if x and y are assumed to be real numbers instead of rational numbers. This is because these exponentiation rely on the laws of algebra and laws of order which continue to hold for real numbers as well as rationals. Exponentiating to exponents which are not integers needs careful construction.

Definition 5.6.3. *Let $x \geq 0$ be a non-negative real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$, also known as the nth root of x, by the formula $x^{1/n} := \sup\{y \in \boldsymbol{R} : y \geq 0 \text{ and } y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a real number.*

Lemma 5.6.1. *Let $x, y \geq 0$ be non-negative reals, and let $n, m \geq 1$ be positive integers.*

1. *If $y = x^{1/n}$, then $y^n = x$.*
2. *Conversely, if $y^n = x$, then $y = x^{1/n}$.*

3. $x^{1/n}$ *is a positive real number.*

4. *We have* $x > y$ *iff* $x^{1/n} > y^{1/n}$.

5. *If* $x > 1$, *then* $x^{1/k}$ *is a decreasing function of* k, *if* $x < 1$ *it is an increasing function of* k. *If* $x = 1$, *then* $x^{1/k} = 1$ *for all* k.

6. *we have* $(xy)^{1/n} = x^{1/n}y^{1/n}$.

7. *We have* $(x^{1/n})^{1/m} = x^{1/nm}$.

Definition 5.6.4. *Let* $x > 0$ *be a positive real number, and let* q *be a rational number. To define* x^q, *we write* $q = a/b$ *for some integer* a *and positive integer* b, *and define* $x^q := (x^{1/b})^a$.

Lemma 5.6.2. *Let* a *and* a' *be integers and* b, b' *be positive integers such that* $a/b = a'/b'$, *and let* x *be a positive real number. Then we have* $(x^{1/b'})^{a'} = (a^{1/b})^a$.

Lemma 5.6.3. *Let* $x, y > 0$ *be positive reals, and let* q, r *be rationals.*

1. x^q *is a positive real.*

2. $x^{q+r} = x^q x^r$ *and* $(x^q)^r = x^{qr}$.

3. $x^{-q} = 1/x^q$.

4. *If* $q > 0$, *then* $x > y$ *iff* $x^q > y^q$.

5. *If* $X > 1$, *then* $x^q > x^r$ *iff* $q > r$. *If* $x < 1$, *then* $x^q > x^r$ *iff* $q < r$.

To show real exponentiation x^y, where y is a real number, we need to formalize the concept of limit.

6 Limits of sequences

6.1 Convergence and limit laws

We begin by repeating much of the machinery of ε-close sequences, etc. again - but this time, we do it for sequences of real numbers, not rational numbers.

Definition 6.1.1. *(Distance between two real numbers) Given two real numbers* x *and* y, *we define their distance* $d(x, y) := |x - y|$.

Definition 6.1.2. *(ε-close real numbers) Let* $\varepsilon > 0$ *be a real number. We say that two real numbers* x, y *are* ε*-close iff we have* $d(y, x) \leq \varepsilon$.

Definition 6.1.3. *(Cauchy sequences of reals) Let* $\varepsilon > 0$ *be a real number. A sequence* $(a_n)_{n=N}^{\infty}$ *of real numbers starting at some integer index* N *is said to be* ε*-steady iff* a_j *and* a_k *are* ε*-close for every* $j, k \geq N$. *A sequence* $(a_n)_{n=m}^{\infty}$ *starting at some integer index* m *is said to be eventually* ε*-steady iff there exists an* $N \geq m$ *such that* $(a_n)_{n=N}^{\infty}$ *is* ε*-steady. We say that* $(a_n)_{n=m}^{\infty}$ *is a Cauchy sequence iff it is eventually* ε*-steady for every* $\varepsilon > 0$.

In other words, a sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence if, for every real $\varepsilon > 0$, there exists an $N \geq m$ such that $|a_n - a_{n'}| \leq \varepsilon$ for all $n, n' \geq N$. We will now view the concept of a Cauchy sequence as a single unified concept.

Definition 6.1.4. *(Convergence of sequences) Let* $\varepsilon > 0$ *be a real number, and let* L *be a real number. A sequence* $(a_n)_{n=N}^{\infty}$ *of real numbers is said to be* ε*-close to* L *iff* a_n *is* ε*-close to* L *for every* $n \geq N$, *i.e., we have* $|a_n - L| \leq \varepsilon$ *for every* $n \geq N$. *We say that a sequence* $(a_n)_{n=m}^{\infty}$ *is eventually* ε*-close to* L *iff there exists an* $N \geq m$ *such that* $(a_n)_{n=N}^{\infty}$ *is* ε*-close to* L. *We say that a sequence* $(a_n)_{n=m}^{\infty}$ *converges to* L *iff it is eventually* ε*-close to* L *for every real* $\varepsilon > 0$.

In other words, $(a_n)_{n=m}^{\infty}$ converges to L iff, given any real $\varepsilon > 0$, one can find an $N \geq m$ such that $|a_n - L| \leq \varepsilon$ for all $n \geq N$.

Proposition 6.1.1. *(Uniqueness of limits) Let $(a_n)_{n=m}^{\infty}$ be a real sequence starting at some integer index m, and let $L \neq L'$ be two distinct real numbers. Then it is not possible for $(a_n)_{n=m}^{\infty}$ to converge to L while also converging to L'.*

Definition 6.1.5. *(Limit of sequences) If a sequence $(a_n)_{n=m}^{\infty}$ converges to some real number L, we say the sequence is convergent and that its limit is L; we write $L = \lim_{n\to\infty} a_n$ to denote this fact. If a sequence $(a_n)_{n=m}^{\infty}$ is not converging to any real number L, we say that the sequence is divergent and we leave $\lim_{n\to\infty} a_n$ undefined.*

We sometimes use the phrase "$a_n \to x$ as $n \to \infty$" as an alternative way of writing the statement "$(a_n)_{n=m}^{\infty}$ converges to x".

Proposition 6.1.2. *(Convergent sequences are Cauchy) Suppose that $(a_n)_{n=m}^{\infty}$ is a convergent sequence of real numbers. Then $(a_n)_{n=m}^{\infty}$ is also a Cauchy sequence.*

Now we show that formal limits can be superceded by actual limits, just as formal subtraction was superceded by actual subtraction when constructing the integer, and formal division was superceded by actual division when constructing the rational numbers.

Proposition 6.1.3. *Suppose that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence of rational numbers. Then $(a_n)_{n=m}^{\infty}$ converges to $LIM_{n\to\infty} a_n$, i.e. $LIM_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.*

Definition 6.1.6. *(Bounded sequences) A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is bounded by a real number M iff we have $|a_n| \leq M$ for all $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is bounded iff is it bounded by M for some real number $M > 0$.*

Lemma 6.1.1. *Every convergent sequence of real number is bounded.*

Theorem 6.1. *(Limit laws) Let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}\infty$ be convergent sequences of real numbers, and let x, y be the real numbers $x := \lim_{n\to\infty} a_n$ and $y := \lim_{n\to\infty} b_n$,*

1. *The sequence $(a_n + b_n)_{n=m}^{\infty}$ converges to $x + y$, i.e. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.*
2. *The sequence $(a_n b_n)_{n=m}^{\infty}$ converges to xy, i.e. $\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n$.*
3. *For any real number c, the sequence $(ca_n)_{n=m}^{\infty}$ converges to cx, i.e. $\lim_{n\to\infty}(ca_n) = c \lim_{n\to\infty} a_n$.*
4. *The sequence $(a_n - b_n)_{n=m}^{\infty}$ converges to $x - y$, i.e. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$.*
5. *Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(b_n^{-1})_{n=m}^{\infty}$ converges to y^{-1}; i.e. $\lim_{n\to\infty} b_n^{-1} = \left(\lim_{n\to\infty} b_n\right)^{-1}$.*
6. *Suppose that $y \neq 0$, and that $b_n \neq 0$ for all $n \geq m$. Then the sequence $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y; i.e. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.*
7. *The sequence $(max(a_n, b_n))_{n=m}^{\infty}$ converges to $max(x, y)$;*
   i.e. $\lim_{n\to\infty} max(a_n, b_n) = max(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$.
8. *The sequence $(min(a_n, b_n))_{n=m}^{\infty}$ converges to $min(x, y)$;*
   i.e. $\lim_{n\to\infty} min(a_n, b_n) = min(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n)$.

6.2 The extended real number system

To make the discussion of non converging sequences precise we need to talk about something called the extended real number system.

Definition 6.2.1. *(Extended real number system) The extended real number system $\boldsymbol{R}^*$ is the real line $\boldsymbol{R}$ with two additional elements attached, called $+\infty$ and $-\infty$. These elements are distinct from each other and also distinct from every real number. An extended real number x is called finite iff it is a real number, and infinite iff it is equal to $+\infty$ or $-\infty$.*

Definition 6.2.2. *(Negation of extended reals) The operation of negation $x \mapsto -x$ on $\boldsymbol{R}$, we now extend to $\boldsymbol{R}^*$ by defining $-(+\infty) := -\infty$ and $-(-\infty) := +\infty$.*

Definition 6.2.3. *(Ordering of extended reals) Let x and y be extended real numbers. We say that $x \leq y$, i.e. x is less than or equal to y, iff one of the following three statements is true:*

1. *x and y are real numbers, and $x \leq y$ as real numbers.*
2. *$y = +\infty$.*
3. *$x = -\infty$.*

Proposition 6.2.1. *Let x, y, and z be extended real numbers. Then the following statements are true:*

1. *Reflexivity: $x \leq x$.*
2. *Trichotomy: Exactly on of the statements $x < y$, $x = y$, or $x > y$ is true.*
3. *Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$.*
4. *Negation reverse order: If $x \leq y$, then $-y \leq x$.*

Adding other operations on the extended real number system is dangerous, as the cancellation laws begin to break down once we try to involve infinity.

Definition 6.2.4. *(Supremum of sets of extended reals) Let E be a subset of $\boldsymbol{R}^*$. Then we define the supremum* $\sup(E)$ *or least upper bound of E by the following rule.*

- *if E is contained in $\boldsymbol{R}$, i.e., $+\infty$ and $-\infty$ are not elements of E, then we let* $\sup(E)$ *be as supremum on $\boldsymbol{R}$ as the least upper bound of E.*
- *If E contains $+\infty$, then we set* $\sup(E) := +\infty$.
- *If E does not contain $+\infty$ but does contain $-\infty$, then we set* $\sup(E) := \sup(E \backslash -\infty)$.

We also define the infimum $\inf(E)$ of E, also known as the greatest lower bound of E by the formula $\inf(E) := -\sup(-E)$, where $-E$ is the set $-E := \{-x : x \in E\}$. Infimum and Supremum need not belong to the set E, but in some sense 'touch' it. If E is an empty set then $\sup(E) = -\infty$ and $\inf(E) = +\infty$. This is the only case in which the supremum can be less than the infimum.

Theorem 6.2. *Let E be a subset of $\boldsymbol{R}^*$. Then the followings statements are true.*

- *For every $x \in E$ we have $x \leq \sup(E)$ and $x \geq \inf(E)$.*
- *Suppose that $M \in \boldsymbol{R}^*$ is an upper bound for E, i.e., $x \leq M$ for all $x \in E$. Then we have $\sup(E) \leq M$.*
- *Suppose that $M \in \boldsymbol{R}^*$ is a lower bound for E, i.e., $x \geq M$ for all $x \in E$. Then we have $\inf(E) \geq M$.*

6.3 Suprema and Infima of sequences

Definition 6.3.1. *(Sup and inf of sequences) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then we define $\sup(a_n)_{n=m}^{\infty}$ to be the supremum of the set $\{a_n : n \geq m\}$, and $\inf(a_n)_{n=m}^{\infty}$ to the infimum of the same set.*

Proposition 6.3.1. *(Least upper bound property) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be the extended real number $x := \sup(a_n)_{n=m}^{\infty}$. Then we have $a_n \leq x$ for all $n \geq m$. Also, whenever $M \in \boldsymbol{R}^*$ is an upper bound for a_n, i.e., $a_n \leq M$ for all $n \geq m$, we have $x \leq M$. Finally, for every extended real number y for which $y < x$, there exists at least one $n \geq m$ for which $y < a_n \leq x$.*

We saw that all convergent sequences are bounded. The converse is not true. However, all bounded sequences which are *monotone* must converge.

Proposition 6.3.2. *(Monotone bounded sequences converge) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which has some finite upper bound $M \in \boldsymbol{R}$, and which is also increasing, i.e. $a_{n+1} \geq a_n$ for all $n \geq m$. Then $(a_n)_{n=m}^{\infty}$ is convergent, and $\lim_{n\to\infty} a_n = \sup(a_n)_{n=m}^{\infty} \leq M$.*

One can similarly prove that if a sequence $(a_n)_{n=m}^{\infty}$ is bounded below and decreasing, then it is convergent, and that the limit is equal to the infimum. A sequence is said to be monotone if it is either increasing or decreasing. We see that a monotone sequence converges if and only if it is bounded.

Example 6.3.1. *Let $0 < x < 1$. Then we have $\lim_{n\to\infty} x^n = 0$.*

Proof: Since $0 < x < 1$, the sequence $(x^n)_{n=1}^{\infty}$ is a decreasing sequence. On the other hand, the sequence $(a^n)_{n=1}^{\infty}$ has a lower bound of 0. Thus by monotone bounded convergence proposition the sequence $(x^n)_{n=1}^{\infty}$ converges to some limit L. Since $x^{n+1} = x \times x^n$, we thus see from the limit laws that $(x^{n+1})_{n=1}^{\infty}$ converges to xL. But the sequence $(x^{n+1})_{n=1}^{\infty}$ is just the sequence $(x^n)_{n=2}^{\infty}$ shifted by one, and so they must have the same limits. So $xL = L$. Since $x \neq 1$, we can solve for L to obtain $L = 0$. Thus $(x^n)_{n=1}^{\infty}$ converges to 0. $\square$

6.4 Limsup, Liminf, and limit points

Definition 6.4.1. *(Limit points) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let x be a real number, and let $\epsilon > 0$ be a real number. We say that x is ε-adherent to $(a_n)_{n=m}^{\infty}$ iff there exists an $n \geq m$ such that a_n is ε-close to x. We say that x is continually ε-adherent to $(a_n)_{n=m}^{\infty}$ iff it is ε-adherent to $(a_n)_{n=m}^{\infty}$ for every $N \geq m$. We say that x is a limit point or adherent point of $(a_n)_{n=m}^{\infty}$ iff it is continually ε-adherent to $(a_n)_{n=m}^{\infty}$ for every $\varepsilon > 0$.*

Unwrapping, x is a limit point of $(a_n)_{n=m}^{\infty}$ if, fore every $\varepsilon > 0$ and every $N \geq m$, there exists an $n \geq N$ such that $|a_n - x| \leq \varepsilon$. Limits are of course a special case of limit points.

Proposition 6.4.1. *(Limits are limit points) Let $(a_n)_{n=m}^{\infty}$ be a sequence which converges to a real number c. Then c is a limit point of $(a_n)_{n=m}^{\infty}$, and in fact it is the only limit point of $(a_n)_{n=m}^{\infty}$.*

Definition 6.4.2. *(Limit superior and limit inferior) Suppose that $(a_n)_{n=m}^{\infty}$ is a sequence. We define a new sequence $(a_N^+)_{N=m}^{\infty}$ by $a_N^+ := \sup(a_n)_{n=N}^{\infty}$, i.e. a_N^+ is the supremum of all the elements in the sequence from a_N onwards. We define the limit superiorof the sequence $(a_n)_{n=m}^{\infty}$, denoted $\limsup_{n\to\infty} a_n := \inf(a_n^+)_{N=m}^{\infty}$. Similarly, we define $a_N^- := \inf(a_n)_{n=N}^{\infty}$ and define the limit inferior of the sequence $(a_n)_{n=m}^{\infty}$, denoted $\liminf_{n\to\infty} a_n := \sup(a_N^-)_{N=m}^{\infty}$.*

Proposition 6.4.2. *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let L^+ be the limit superior of this sequence, and let L^- be the limit inferior of this sequence.*

1. *For every $x > L^+$, there exists an $N \geq m$ such that $a_n < x$ for all $n \geq N$, i.e. for every $x > L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ are eventually less than x. Similarly, for every $y < L^-$ there exists an $N \geq m$ such that $a_n > y$ for all $n \geq N$.*

2. *For every $x < L^+$, and every $N \geq m$, there exists an $n \geq N$ such that $a_n > x$, i.e. for every $x < L^+$, the elements of the sequence $(a_n)_{n=m}^{\infty}$ exceed x infinitely often. Similarly, for every $y > L^-$ and every $N \geq m$, there exists an $n \geq N$ such that $a_n < y$.*

3. *We have $\inf(a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup(a_n)_{n=m}^{\infty}$.*

4. *If c is any limit point of $(a_n)_{n=m}^{\infty}$, then we have $L^- \leq c \leq L^+$.*

5. *If L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$. Similarly, if L^- is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$.*

6. *Let c be a real number. If $(a_n)_{n=m}^{\infty}$ converges to c, then we must have $L^+ = L^- = c$. Conversely, if $L^+ = L^- = c$, then $(a_n)_{n=m}^{\infty}$ converges to c.*

Lemma 6.4.1. *(Comparison principle) Suppose that $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are two sequences of real numbers such that $a_n \leq b_n$ for all $n \geq m$. Then we have the inequalities*

- $\sup(a_n)_{n=m}^{\infty} \leq \sup(b_n)_{n=m}^{\infty}$.
- $\inf(a_n)_{n=m}^{\infty} \leq \inf(b_n)_{n=m}^{\infty}$.
- $\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n$.
- $\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n$.

Lemma 6.4.2. *(Squeeze test) Let $(a_n)_{n=m}^{\infty}$, $(b_n)_{n=m}^{\infty}$, and $(c_n)_{n=m}^{\infty}$ be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \geq m$. Suppose also that $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ both converge to the same limit L Then $(b_n)_{n=m}^{\infty}$ is also convergent to L.*

The squeeze test, combined with the limit laws and the principle that monotone bounded sequence always have limits, allows to compute a large number of limits.

Lemma 6.4.3. *(Zero test for sequences) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. Then the limit $\lim_{n\to\infty} a_n$ exists and is equal to zero iff the limit $\lim_{n\to\infty} |a_n|$ exists and is equal to zero.*

Theorem 6.3. *(Completeness of the reals) A sequence $(a_n)_{n=m}^{\infty}$ of real numbers is a Cauchy sequence iff it is convergent.*

In the language of metric spaces it means that the real numbers are a *complete* metric space - that they do not contain "holes" the same way the rationals do. This property is closely related to the least upper bound property, and is one of the principal characteristics which make the real numbers superior to the rational numbers for the purpose of doing analysis - taking limits, derivatives and integrals.

6.5 Subsequences

Definition 6.5.1. *Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences or real numbers. We say that $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$ iff there exists a function $f : \boldsymbol{N} \to \boldsymbol{N}$ which is strictly increasing, i.e. $f(n+1) > f(n)$ for all $n \in \boldsymbol{N}$, such that $b_n = a_{f(n)}$ for all $n \in \boldsymbol{N}$.*

The function f is not necessarily bijective, but is necessarily injective. The property of being a subsequence is reflexive and transitive, though not symmetric.

Lemma 6.5.1. *Let $(a_n)_{n=0}^{\infty}$, $(b_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ be sequences of real numbers. Then $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$. Furthermore, if $(b_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, and $(c_n)_{n=0}^{\infty}$ is a subsequence of $(b_n)_{n=0}^{\infty}$, then $(c_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.*

Proposition 6.5.1. *(Subsequences related to limits) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent:*

1. *The sequence $(a_n)_{n=0}^{\infty}$ converges to L.*
2. *Every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.*

Proposition 6.5.2. *(Subsequences related to limit points) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let L be a real number. Then the following two statements are logically equivalent*

- *L is a limit point of $(a_n)_{n=0}^{\infty}$.*
- *There exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.*

Theorem 6.4. *(Bolzano-Weierstrass theorem) Let $(a_n)_{n=0}^{\infty}$ be a bounded sequence, i.e. there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \boldsymbol{N}$. Then there is at least one subsequence of $(a_n)_{n=0}^{\infty}$ which converges.*

Proof: Let L be the limit superior of the sequence $(a_n)_{n=0}^{\infty}$. Since we have $-M \leq a_n \leq M$ for all natural numbers n, then according to comparison principle $-M \leq L \leq M$, i.e. L is a real number and not $+\infty$ or $-\infty$. Then by proposition on Limit superior, L is a limit point of $(a_n)_{n=0}^{\infty}$. Thus by proposition of subsequence related to limit points, there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges, in fact to L. □

The theorem states that if a sequence is bounded, then eventually it has no choice but to converge in some places; it has no room to spread out and stop itself form acquiring limit points. In the language of topology, this means that the interval $\{x \in \boldsymbol{R} : -M \leq x \leq M\}$ is compact, whereas an unbounded set such as the real line $\boldsymbol{R}$ is not compact.

6.6 Real exponentiation, part II

We finally define the value of x^α where both x and α are real numbers.

Lemma 6.6.1. *(Continuity of exponentiation) Let* $x > 0$*, and let* α *be a real number. Let* $(q_n)_{n=1}^\infty$ *be any sequence of rational numbers converging to* α*. Then* $(x^{q_n})_{n=1}^\infty$ *is also a convergent sequence. Furthermore, if* $(q'_n)_{n=1}^\infty$ *is any other sequence of rational numbers converging to* α*, then* $(x^{q'_n})_{n=1}^\infty$ *has the same limits are* $(x^{q_n})_{n=1}^\infty$*.*

Definition 6.6.1. *(Exponentiation to a real exponent) Let* $x > 0$ *be real, and let* α *be a real number. We define the quantity* x^α *by the formula* $x^\alpha = \lim_{n\to\infty} x^{q_n}$*, where* $(q_n)_{n=1}^\infty$ *is any sequence of rational numbers converging to* α*.*

All the results of exponentiation of rational lemma which held for rational number q and r, continue to hold for real numbers q and r.

7 Series

7.1 Finite series

Definition 7.1.1. *(Finite series) Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive, $m \le n$. Then we define the finite sum, or finite series $\sum\limits_{i=m}^{n} a_i$ by the recursive formula*

$$\sum_{i=m}^{n} a_i := 0 \quad \textit{whenever } n < m$$
$$\sum_{i=m}^{n+1} a_i := \left(\sum_{i=m}^{n} a_i\right) + a_{n+1} \; n \ge m-1$$

This implies that $\sum\limits_{i=m}^{m-2} a_i = 0$, $\sum\limits_{i=m}^{m-1} a_i = 0$, $\sum\limits_{i=m}^{m} a_i = a_m$. The variable i called the index is a dummy variable and the summation does not depend on any quantity named i. One can replace the index of summation i with any other symbol, i.e. $\sum\limits_{i=m}^{n} a_i = \sum\limits_{j=m}^{n} a_j$.

Lemma 7.1.1. *Some basic properties of summation.*

1. *Let $m \le n < p$ be integers, and let a_i be a real number assigned to each integer $m \le i \le p$. Then $\sum\limits_{i=m}^{n} a_i + \sum\limits_{i=n+1}^{p} a_i = \sum\limits_{i=m}^{p} a_i$.*
2. *Let $m \le n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \le i \le n$. Then $\sum\limits_{i=m}^{n} a_i = \sum\limits_{j=m+k}^{n+k} a_{j-k}$.*
3. *Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Then $\sum\limits_{i=m}^{n} (a_i+b_i) = \sum\limits_{i=m}^{n} a_i + \sum\limits_{i=m}^{n} b_i$.*
4. *Let $m \le n$ be integers, and let a_i be real number assigned to each integer $m \le i \le n$, and let c be another real number. Then $\sum\limits_{i=m}^{n} (ca_i) = c\sum\limits_{i=m}^{n} a_i$.*
5. *(Triangle inequality for finite series) Let $m \le n$ be integers, and let a_i be a real number assigned to each integer $m \le i \le n$. Then $|\sum\limits_{i=m}^{n} a_i| \le \sum\limits_{i=m}^{n} |a_i|$.*
6. *(Comparison test for finite series) Let $m \le n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \le i \le n$. Suppose that $a_i \le b_i$ for all $m \le i \le n$. Then $\sum\limits_{i=m}^{n} a_i \le \sum\limits_{i=m}^{n} b_i$.*

Definition 7.1.2. *(Summation over finite sets) Let X be a finite set with $n \in \boldsymbol{N}$ elements, and let $f : X \to \boldsymbol{R}$ be a function from X to the real numbers. Then we can define the finite sum $\sum\limits_{x\in X} f(x)$ by first selecting any bijection g from $\{i \in \boldsymbol{N} : 1 \le i \le n\}$ to X (such a bijection exists since X is assumed to have n elements); and then defining $\sum\limits_{x\in X} f(s) := \sum\limits_{i=1}^{n} f(g(i))$.*

To verify that this definition actually does give a single, well-defined value to $\sum\limits_{x\in X} f(x)$, we have to check that different bijections g from $\{i \in \boldsymbol{N} : 1 \le i \le n\}$ to X give the same sum.

Proposition 7.1.1. *(Finite summations are well-defined) Let X be a finite set with $n \in \boldsymbol{N}$ elements, let $f : X \to \boldsymbol{R}$ be a function, and let $g : \{i \in \boldsymbol{N} : 1 \le i \le n\} \to X$ and $h : \{i \in \boldsymbol{N} : 1 \le i \le n\} \to X$ be bijections. Then $\sum\limits_{i=1}^{n} f(g(i)) = \sum\limits_{i=1}^{n} f(h(i)$.*

Proposition 7.1.2. *(Basic properties of summation over finite sets)*

1. *If X is empty, and $f : X \to \mathbf{R}$ is the empty function, we have* $\sum_{x\in X} f(x) = 0$.
2. *If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbf{R}$ is a function, we have* $\sum_{x\in X} f(x) = f(x_0)$.
3. *(substitution, part I) If X is a finite set, $f : X \to \mathbf{R}$ is a function, and $g : Y \to X$ is a bijection, then* $\sum_{x\in X} f(x) = \sum_{y\in Y} f(g(y))$.
4. *(substitution, part II) Let $n \leq m$ be integers, and let X be the set $X := \{i \in \mathbf{Z} : n \leq i \leq m\}$. If a_i is a real number assigned to each integer $i \in X$, then we have* $\sum_{i=n}^{m} a_i = \sum_{i\in X} a_i$.
5. *Let X, Y be disjoint finite sets, so $X \cap Y = \emptyset$, and $f : X \cup Y \to \mathbf{R}$ is a function. Then we have* $\sum_{z\in X\cup Y} f(z) = \sum_{x\in X} f(x) + \sum_{y\in Y} f(y)$.
6. *(Linearity, part I) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Then* $\sum_{x\in X} (f(x) + g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} f(x)$.
7. *(Linearity, part II) Let X be a finite set, let $f : X \to \mathbf{R}$ be a function, and let c be a real number. Then* $\sum_{x\in X} cf(x) = x \sum_{x\in X} f(x)$.
8. *(Monotonicity) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions such that $f(x) \leq g(x)$ for all $x \in X$. Then we have* $\sum_{x\in X} f(x) \leq \sum_{x\in X} g(x)$.
9. *(Triangle inequality) Let X be a finite set, and let $f : X \to \mathbf{R}$ be a function, then* $|\sum_{x\in X} f(x)| \leq \sum_{x\in X} |f(x)|$.

Informally, we can rearrange the elements of a finite sequence at will and still obtain the same value. Now we look at double finite series - finite series of finite series - and how they connect with Cartesian products.

Lemma 7.1.2. *Let X, Y be finite sets, and let $f : X \times Y \to \mathbf{R}$ be a function then* $\sum_{x\in X} \left(\sum_{y\in Y} f(x, y)\right) = \sum_{(x,y)\in X\times Y} f(x, y)$.

Theorem 7.1. *(Fubini's theorem for finite series) Let X, Y be finite sets, and let $f : X \times Y \to \mathbf{R}$ be a function. Then* $\sum_{x\in X} \left(\sum_{y\in Y} f(x, y)\right) = \sum_{y\in Y} \left(\sum_{x\in X} f(x, y)\right)$

7.2 Infinite series

Definition 7.2.1. *(Formal infinite series) A infinite series is any expression of the form $\sum_{n=m}^{\infty} a_n$, where m is an integer, and a_n is a real number for any integer $n \geq m$.*

To rigorously define what the series actually sums to, we need another definition.

Definition 7.2.2. *(Convergence of series) Let $\sum_{n=m}^{\infty} a_n$ be an infinite series. For any integer $N \geq m$, we define the Nth partial sum S_N of this series to be $S_n := \sum_{n=m}^{N} a_n$; hence S_N is a real number. If the sequence $(S_N)_{N=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is convergent, and converges to L; we also write $L = \sum_{n=m}^{\infty} a_n$, and say that L is the sum of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is divergent, and we do not assign any real number value to that series.*

If the series converges, then it converges to a unique value and hence the sum is unique too.

Proposition 7.2.1. *Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges iff, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that $|\sum_{n=p}^{q} a_n| \leq \varepsilon$ for all $p, q \geq N$.*

Lemma 7.2.1. *(Zero test) Let $\sum_{n=m}^{\infty} a_n$ be a convergent series of real numbers. Then we must have $\lim_{n\to\infty} a_n = 0$, i.e. if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.*

Convergence of a series is a different notion than convergence of a sequence. For example, the sequence $a_n := 1$ is convergent but the series $\sum_{n=1}^{\infty} a_n$ is divergent, because the sequence does not converge to zero. If a sequence $(a_n)_{n=m}^{\infty}$ does converge to zero, then the series $\sum_{n=m}^{\infty} a_n$ may or may not be convergent; it depends on the series. For example, series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent despite the fact that $\frac{1}{n}$ converges to 0 as $n \to \infty$.

Definition 7.2.3. *(Absolute convergence) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers. We say that this series is absolutely convergent iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent.*

In order to distinguish convergence from absolute convergence, we sometimes refer to the former as conditional convergence.

Proposition 7.2.2. *(Absolute convergence test) Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality $|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n|$.*

The converse to this proposition is not true; there exist series which are conditionally convergent but not absolutely convergent. For example, the sequence $(1/n)_{n=1}^{\infty}$ is non-negative, decreasing and converges to zero. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent; but it is not absolutely convergent because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Thus lack of absolute convergence does not imply lack of conditional convergence, even though absolute convergence implies conditional convergence.

Proposition 7.2.3. *(Alternating series test) Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent iff the sequence a_n converges to 0 as $n \to \infty$.*

Proof: From the zero test, we know that if $\sum_{n=m}^{\infty} (-1)^n a_n$ is a convergent series, then the sequence $(-1)^n a_n$ converges to 0, which implies that a_n also converges to 0, since $(-1)^n a_n$ and a_n have the same distance from 0.

Now suppose conversely that a_n converges to 0. For each N, let S_N be the partial sum $S_N := \sum_{n=m}^{N} (-1)^n a_n$; we need to show that S_N converges. Notice that $S_{N+2} = S_N + (-1)^{N+1}(a_{N+1} - a_{N+2})$. Since $a_{N+1} - a_{N+2}$ is non-negative, we have $S_{N+2} \geq S_N$ when N is odd and $S_{N+2 \leq S_N}$ when N is even.

Suppose N is even. To prove $S_{N+2k} \leq S_N$, the assertion for $k = 0$ and $k = 1$ is easy to see. Assuming the induction step we see that $S_{N+2(k+1)} = S_{N+2k} + (-1)^{N+2k+1}(a_{N+2k+1} - a_{N+2k+2}) \leq S_{N+2k}$ since N is even (so $N + 2k + 1$ is odd) and the sequence is non-negative and decreasing so $a_{N+2k+1} - a_{N+2k+2}$ is positive. Using the induction step we get $S_{N+2(k+1)} \leq S_N$, closing the induction and proving $S_{N+2k} \leq S_N$ for all natural number k. Similarly, to prove $S_{N+2k+1} \geq S_{N+1}$ for N being even we note that $k = 0$ is trivially satisfied. For $k = 1$, we have $S_{N+3} = S_{N+1} + (a_{N+2} - a_{N+3}) > S_{N+1}$, since the other term is non-negative.Assuming the induction step we see that $S_{N+2(k+1)+1} = S_{N+2k+1} + (a_{N+2k+2} - a_{N+2k+3}) \geq S_{N+2k+1} \geq S_{N+1}$ closing the induction.

Finally, we have $S_{N+2k+1} = S_{N+2k} - a_{N+2k+1} \leq S_{N+2k}$, thus we have $S_{N+1} = S_N - a_{N+1} \leq S_{N+2k+1} \leq S_{N+2k} \leq S_N$ for all k. In particular, we have $S_N - a_{N+1} \leq S_n \leq S_N$ for all $n \geq N$ (for $N + 2k = n$ or $N + 2k + 1 = n$).

In particular, the sequence S_n is eventually a_{N+1}-steady. But the sequence a_N converges to 0 as $N \to \infty$, thus this implies that S_n is eventually ε-steady for every $\varepsilon > 0$. Thus S_n converges, and so the series $\sum\limits_{n=m}^{\infty} (-1)^n a_n$ is convergent. $\square$

Proposition 7.2.4. *(Series laws)*

1. *If $\sum\limits_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum\limits_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum\limits_{n=m}^{\infty} (a_n + b_n)$ is also a convergent series, and converges to $x + y$, i.e. $\sum\limits_{n=m}^{\infty} (a_n + b_n) = \sum\limits_{n=m}^{\infty} a_n + \sum\limits_{n=m}^{\infty} b_n$.*

2. *If $\sum\limits_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum\limits_{n=m}^{\infty} (ca_n)$ is also a convergent series, and converges to cx. In particular, we have $\sum\limits_{n=m}^{\infty} ca_n = c \sum\limits_{n=m}^{\infty} a_n$.*

3. *Let $\sum\limits_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k > 0$ be an integer. If one of the two series $\sum\limits_{n=m}^{\infty} a_n$ and $\sum\limits_{n=m+k}^{\infty} a_n$ are convergent, then the other is also and we have the identity $\sum\limits_{n=m}^{\infty} a_n = \sum\limits_{n=m}^{m+k-1} a_n + \sum\limits_{n=m+k}^{\infty} a_n$.*

4. *Let $\sum\limits_{n=m}^{\infty} a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum\limits_{n=m+k}^{\infty} a_{n-k}$ also converges to x.*

Lemma 7.2.2. *(Telescoping series) Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which converge to 0, i.e. $\lim\limits_{n\to\infty} a_n =$ 0. Then the series $\sum\limits_{n=0}^{\infty} (a_n - a_{n+1})$ converges to a_0.*

Proof: The partial series $S_N = \sum\limits_{n=0}^{N} (a_n - a_{n+1}) = (a_0 - a_1) + (a_1 + a_2) + \ldots + (a_{N-1} - a_N) + (a_N - a_{N+1}) = a_0 - a_{N+1}$. Now $\lim\limits_{N\to\infty} S_N = \lim\limits_{N\to\infty} (a_0 - a_{N+1}) = \lim\limits_{N\to\infty} a_0 - \lim\limits_{N\to\infty} a_{N+1} = a_0 - \lim\limits_{M\to\infty} a_M$, where $M = N + 1$. Hence, $\lim\limits_{N\to\infty} S_N = a_0$. $\square$

Telescoping series are easy to sum.

7.3 Sums of non-negative numbers

When all the terms in the series $\sum\limits_{n=m}^{\infty} a_n$ are non-negative, there is no difference between conditional and absolute convergence.

Proposition 7.3.1. *Let $\sum\limits_{n=m}^{\infty} a_n$ be a formal series of non-negative real numbers. Then this series is convergent iff there is a real number M such that $\sum_{n=m}^{\infty} a_n \leq M$ for all integers $N \geq m$.*

Lemma 7.3.1. *(Comparison test) Let $\sum\limits_{n=m}^{\infty} a_n$ and $\sum\limits_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum\limits_{n=m}^{\infty} b_n$ is convergent, then $\sum\limits_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact $|\sum\limits_{n=m}^{\infty} a_n| \leq \sum\limits_{n=m}^{\infty} |a_n| \leq \sum\limits_{n=m}^{\infty} b_n$.*

The contrapositive of comparison test also holds: if we have $|a_n| \leq b_n$ for all $n \geq m$, and $\sum\limits_{n=m}^{\infty} a_n$ is not absolutely convergent, then $\sum\limits_{n=m}^{\infty} b_n$ is not conditionally convergent.

Lemma 7.3.2. *(Geometric series) Let x be a real number. If $|x| > 1$, then the series $\sum\limits_{n=0}^{\infty} x^n$ is divergent. If $|x| < 1$, then the series is absolutely convergent and $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$.*

Proposition 7.3.2. *(Cauchy criterion) Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence of non-negative real numbers. Then the series $\sum_{n=0}^{\infty} a_n$ is convergent iff the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.*

Proof: Let $S_N := \sum_{n=1}^{N} a_n$ be the partial sums of $\sum_{n=1}^{\infty} a_n$, and let $T_K := \sum_{k=0}^{K} 2^k a_{2^k}$ be the partial sumps of $\sum_{k=0}^{\infty} 2^k a_{2^k}$. We need to show that the sequence $(S_N)_{N=1}^{\infty}$ is bounded iff the sequence $(T_K)_{K=0}^{\infty}$ is bounded. We later prove that $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$ for any natural number K. Thus, if $(S_N)_{N=1}^{\infty}$ is bounded, then $(S_{2^K})_{K=0}^{\infty}$ is bounded, and hence $(T_K)_{K=0}^{\infty}$ is bounded. Conversely, if $(T_K)_{K=0}^{\infty}$ is bounded, then the claim implies that $S_{2^{K+1}-1}$ is bounded, i.e., there is an M such that $S_{2^{K+1}-1} \leq M$ for all natural number K. And since, $2^{K+1} - 1 \geq K + 1$, as shown below, we have $S_{K+1} \leq M$ for all natural number K, hence $(S_N)_{N=1}^{\infty}$ is bounded.

Aside: We now show that for any natural number K, we have $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$ using induction. For $K = 0$ we have to show $S_1 \leq T_0 \leq 2S_1$, i.e. $a_1 \leq a_1 \leq 2a_1$, which is apparent since a_1 is non-negative. We assume induction and prove for $K+1$: $S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}$. Noting that a_n is a decreasing sequence, we have $S_{2^{K+1}} = S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} \geq S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_{2^{K+1}} = S_{2^K} + 2^K a_{2^{K+1}}$. Similarly we have $S_{2^{K+2}-1} = S_{2^{K+1}-1} + \sum_{n=2^K+1}^{2^{K+2}-1} a_n \leq S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_{2^{K+1}} = S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}$. We also have $T_{K+1} = T_K + 2^{K+1} a_{2^{K+1}}$. Hence using $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$ we obtain $S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}$ as desired, closing the induction.

Aside: We now show that $2^{K+1} - 1 \geq K + 1$ for any natural number K. This is apparent for $K = 0$ and $K = 1$. We assume the induction step and then show $2^{K+2} - 1 = 2\ 2^{K+1} - 1 = 2\ (2^{K+1} - 1) + 1 \geq 2(K+1) + 1 = 2K + 3 \geq K + 2$ as required, closing the induction. □

Lemma 7.3.3. *Let $q > 0$ be a rational number. Then the series $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q > 1$ and divergent when $q \leq 1$.*

Proof: The sequence $(\frac{1}{n^q})_{n=1}^{\infty}$ is non-negative and decreasing and so the Cauchy criterion applies. Thus this series is convergent iff $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q} = \sum_{k=0}^{\infty} (2^{1-q})^k$ is convergent. Which is same as saying it converges iff $|2^{1-q}| < 1$, this happens iff $q > 1$. □

In particular, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as the **harmonic series** and is divergent. However, the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. The quantity $\sum_{n=1}^{\infty} \frac{1}{n^q}$, when convergent, is called $\zeta(q)$, the **Riemann-zeta function** of q, which plays a key role in distribution of primes. In fact $\zeta(2) = \frac{\pi^2}{6}$.

7.4 Rearrangement of series

Proposition 7.4.1. *Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of non-negative real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also convergent, and has the same sum.*

Proposition 7.4.2. *(Rearrangement of series) Let $\sum_{n=0}^{\infty}$ be an absolutely convergent series of real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also absolutely convergent, and has the same sum.*

Surprisingly, when the series is not absolutely convergent, then the rearrangements are very badly behaved. There is in fact a surprising result of Riemann, which shows that a series which is conditionally convergent but not absolutely convergent can in fact be rearranged to converge to any value, or rearranged to diverge. To summarize, rearranging series is safe when the series is absolutely convergent, but is somewhat dangerous otherwise.

7.5 The root and ratio tests

Theorem 7.2. *(Root test) Let $\sum\limits_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha := \limsup\limits_{n\to\infty} |a_n|^{1/n}$.*

1. *If $\alpha < 1$, then the series $\sum\limits_{n=m}^{\infty} a_n$ is absolutely convergent and hence conditionally convergent.*
2. *If $\alpha > 1$, then the series $\sum\limits_{n=m}^{\alpha} a_n$ is not conditionally convergent and hence cannot be absolutely convergent either.*
3. *If $\alpha = 1$, we cannot assert any conclusion.*

Proof: Let's first suppose $\alpha < 1$. We must have $\alpha \geq 0$, since $|\alpha_n|^{1/n} \geq 0$ for every n. Hence we can find an $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 1$. Since α is a limit superior of the sequences, there exists an $N \geq m$ such that $|a_n|^{1/n} \leq \alpha + \varepsilon$ for all $n \geq N$, i.e. we have $|a_n| \leq (\alpha + \varepsilon)^n$ for all $n \geq N$. But since $0 < \alpha + \varepsilon < 1$, by geometric series we have $\sum\limits_{n=N}^{\infty} (\alpha + \varepsilon)^n$ is absolutely convergent. Thus, by comparison test we see that $\sum\limits_{n=N}^{\infty} a_n$ is absolutely convergent, and thus $\sum\limits_{n=m}^{\infty} a_n$ is absolutely convergent, as the convergence of the series does not depend on the first few elements.

Now suppose $\alpha > 1$. Then for every $N \geq m$ there exists an $n \geq N$ such that $|a_n|^{1/n} \geq 1$, and hence that $|a_n| \geq 1$. In particular, $(a_n)_{n=N}^{\infty}$ is not 1-close to 0 for any N, and hence $(a_n)_{n=m}^{\infty}$ is not eventually 1-close to 0. In particular $(a_n)_{n=m}^{\infty}$ does not converge to zero. Thus by zero test, $\sum\limits_{n=m}^{\infty}$ is not conditionally convergent. $\square$

The root test can be phrased using limit as well, when the limit exists. But it is sometimes difficult to use, so we develop the more useful ratio test here.

Lemma 7.5.1. *Let $(c_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then we have*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Theorem 7.3. *(Ratio test) Let $\sum\limits_{n=m}^{\infty} a_n$ be a series of non-zero numbers. Then*

- *If $_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum\limits_{n=m}^{\infty}$ is absolutely convergent, hence conditionally convergent.*
- *If $_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum\limits_{n=m}^{\infty}$ is not conditionally convergent, hence cannot be absolutely convergent.*
- *In the remaining cases, we cannot assert any conclusion.*

Proposition 7.5.1. *We have $\lim\limits_{n\infty} n^{1/n} = 1$*

Proof: Using the previous lemma we have $\limsup\limits_{n\to\infty} n^{1/n} \leq \limsup\limits_{n\to\infty} \frac{n+1}{n} = \limsup\limits_{n\to\infty} 1 + \frac{1}{n} = 1$ using limit laws. Similarly we have $\liminf\limits_{n\to\infty} n^{1/n} \geq \liminf\limits_{n\to\infty} \frac{n+1}{n} = \liminf\limits_{n\to\infty} 1 + \frac{1}{n} = 1$. Hence, the claim follows. $\square$

8 Infinite sets

We now return to the study of cardinality of sets which are infinite.

8.1 Countability

We know that the set $\boldsymbol{N}$ of natural numbers in infinite. The set $\boldsymbol{N} - \{0\}$ is also infinite, and is a proper subset of $\boldsymbol{N}$. However, the set $\boldsymbol{N} - \{0\}$, despite being smaller than $\boldsymbol{N}$, still has the same cardinality as $\boldsymbol{N}$, because the function $f : \boldsymbol{N} \to \boldsymbol{N} - \{0\}$ defined by $f(n) := n + 1$, is a bijection from $\boldsymbol{N}$ to $\boldsymbol{N} - \{0\}$. This is one characteristic of infinite sets.

Definition 8.1.1. *(Countable sets) A set X is said to be countably infinite (or just countable) iff is has equal cardinality with the natural number N. A set X is said to be at most countable iff it is either countable or finite. We say that a set is uncountable if it is infinite but not countable.*

Countably infinite set are also called denumerable sets. We note that $\boldsymbol{N}$ is countable, and so is $\boldsymbol{N} - \{0\}$. If X be a countable set then there exists a bijection $f : \boldsymbol{N}X$, i.e. a countable set can be arranged in a sequence. Countable sets are infinite however all infinite sets are countable.

Proposition 8.1.1. *(Well ordering principle) Let X be a non-empty subset of the natural numbers $\boldsymbol{N}$. Then there exists exactly one element $n \in X$ such that $n \leq m$ for all $m \in X$. In other words, every non-empty set of natural numbers has a minimum element.*

We refer to the elements n as the minimum of X and written as $min(X)$, which is clearly the same as the infimum of X.

Proposition 8.1.2. *Let X be an infinite subset of the natural numbers $\boldsymbol{N}$. Then there exists a unique bijection $f : N \to X$ which is increasing, in the sense that $f(n+1) > f(n)$ for all $n \in \boldsymbol{N}$. In particular, X has equal cardinality with $\boldsymbol{N}$ and is hence countable.*

All subsets of the natural numbers are at most countable. If X is an at most countable set, and Y is a subset of X, then Y is at most countable.

Proposition 8.1.3. *Let Y be a set, and let $f : \boldsymbol{N} \to Y$ be a function. Then $f(\boldsymbol{N})$ is at most countable.*

Let X be a countable set, and let $f : X \to Y$ be a function. Then $f(X)$ is at most countable.

Proposition 8.1.4. *Let X be a countable set, and let Y be a countable set. Then $X \cup Y$ is a countable set.*

Corollary 8.1.1. *The integers $\boldsymbol{Z}$ are countable.*

Lemma 8.1.1. *The set $A := \{(n, m) \in \boldsymbol{N} \times \boldsymbol{N} : 0 \leq m \leq n\}$ is countable.*

The set $\boldsymbol{N} \times \boldsymbol{N}$ is countable. If X and Y are countable, then $X \times Y$ is countable.

Corollary 8.1.2. *The rationals $\boldsymbol{Q}$ are countable.*

The above is true because $f(\boldsymbol{Z} \times (\boldsymbol{Z} - \{0\})) = \boldsymbol{Q}$. Because the rationals are countable, we know in principle that it is possible to arrange the rational numbers as a sequence.

8.2 Summation on infinite sets

Definition 8.2.1. *(Series on countable sets) Let X be a countable set, and let $f : X \to \boldsymbol{R}$ be a function. We say that the series $\sum_{x \in X} f(x)$ is absolutely convergent iff for some bijection $g : \boldsymbol{N} \to X$, the sum $\sum_{n=0}^{\infty} f(g(n))$ is absolutely convergent. We then define the sum of $\sum_{x \in X} f(x)$ by the formula $\sum_{x \in X} f(x) = \sum_{n=0}^{\infty} f(g(n))$. These definitions do not depend on the choice of g, and so are well defined.*

Theorem 8.1. *(Fubini's theorem for infinite sums) let $f : \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{R}$ be a function such that $\sum_{(n,m) \in \boldsymbol{N} \times \boldsymbol{N}} f(n, m)$ is absolutely convergent. Then we have*

$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} f(n, m) \right) = \sum_{(n,m) \in \boldsymbol{N} \times \boldsymbol{N}} f(n, m) = \sum_{(m,n) \in \boldsymbol{N} \times \boldsymbol{N}} f(n, m) = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} f(n, m) \right)$$

In other words, we can switch the order of infinite sums provided the entire sum is absolutely convergent.

Proof: The second equality is proved by using the bijection $(n, m) \to (m, n)$. To prove the first equality we consider the case of $f(n, m)$ begin non-negative to begin with. Let $L := \sum_{(n,m) \in \boldsymbol{N} \times \boldsymbol{N}} f(n, m)$, then we want to show that $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} f(n, m)$ converges to L.

For all finite sets $X \subset \boldsymbol{N} \times \boldsymbol{N}$, $\sum_{(n,m)\in X} f(n,m)$ is bounded and hence $\leq L$. In particular, for every $n \in \boldsymbol{N}$ and $M \in \boldsymbol{N}$ we have $\sum_{m=0}^{M} f(n,m) \leq L$. Since monotone bounded sequences converge we see that $\sum_{m=0}^{\infty} f(n,m)$ is convergent for each n. Similarly, for any $N \in \boldsymbol{N}$ and $M \in \boldsymbol{N}$ we have by Fubini's theorem for finite series $\sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) \leq \sum_{(n,m)\in X} f(n,m) \leq L$ where X is the set $\{(n,m) \in \boldsymbol{N}\times\boldsymbol{N} : n \leq N, m \leq M\}$ which is finite by cardinal arithmetic rules and hence bounded by L. Now for finite N applying monotone convergence theorem on the increasing sequence $\sum_{m=0}^{M} f(n,m)$, and using the limit laws, we get $\sum_{n=0}^{N}\sum_{m=0}^{\infty} f(n,m) \leq L$. Finally applying monotone convergence theorem on the increasing and bounded sequence $\sum_{n=0}^{N}\sum_{m=0}^{\infty} f(n,m)$ we get the convergence $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \leq L$.

Now let $\varepsilon > 0$. By definition of L, we can find a finite set $X \subseteq \boldsymbol{N}\times\boldsymbol{N}$ such that $\sum_{(n,m)\in X} f(n,m) \geq L-\varepsilon$. This set, being finite, must be contained in some set of the form $Y := \{(n,m) \in \boldsymbol{N}\times\boldsymbol{N} : n \leq N; m \leq M\}$. Thus using Fubini's theorem for finite double series we get $\sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) = \sum_{(n,m)\in Y} f(n,m) \geq \sum_{(n,m)\in X} f(n.m) \geq L-\varepsilon$ and hence using the fact that $f(n,m)$ is positive $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \geq \sum_{n=0}^{N}\sum_{m=0}^{\infty} f(n,m) \geq \sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) \geq L-\varepsilon$. Now since this is true for a general ε we have show that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \geq L$.

This proves the claim for all non-negative $f(n,m)$. A similar argument works when the $f(n,m)$ are all non-positive by applying the results on the function $-f(n,m)$ and then use limit laws to remove the $-$. For the general case, not that any function $f(n,m)$ can be written as $f_+(n,m) - f_-(n,m)$, where $f_+(n,m)$ is the positive part of $f(n,m)$, i.e. it equals $f(n,m)$ when $f(n,m)$ is positive, and 0 otherwise. Similarly, $f_-(n,m)$ is the negative part. It is trivial to show that if $\sum_{(n,m)\in\boldsymbol{N}\times\boldsymbol{N}} f(n,m)$ is absolutely convergent, then so are $\sum_{(n,m)\in\boldsymbol{N}\times\boldsymbol{N}} f_+(n,m)$ and $\sum_{(n,m)\in\boldsymbol{N}\times\boldsymbol{N}} f_-(n,m)$. So now one applies the results just obtained to f_+ and f_- and adds them together using limit laws to obtain the results for a general f. □

Lemma 8.2.1. *Let X be an at most countable set, and let $f : X \to \boldsymbol{R}$ be a function. Then the series $\sum_{x\in X} f(x)$ is absolutely convergent iff $\sup\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite }\} < \infty$.*

We can now define the concept of an absolutely convergent series even when the set X could be uncountable.

Definition 8.2.2. *Let X be a set, which could be uncountable, and let $f : X \to \boldsymbol{R}$ be a function. We say that the series $\sum_{x\in X} f(x)$ is absolutely convergent iff $\sup\{\sum_{x\in A} |f(x)| : A \subseteq X, A \text{ finite }\} < \infty$.*

Lemma 8.2.2. *Let X be a set, which could be uncountable, and let $f : X \to \boldsymbol{R}$ be a function such that the series $\sum_{x\in X} f(x)$ is absolutely convergent. Then the set $\{x \in X : f(x) \neq 0\}$ is at most countable.*

This results requires the axiom of choice which we will introduce shortly. Because of this we can define the value of $\sum_{x\in X} f(x)$ for any absolutely convergent series on an uncountable set X by the formula $\sum_{x\in X} f(x) := \sum_{x\in X: f(x)\neq 0} f(x)$, since we have replaced a sum on an uncountable set X by a sum on the countable set $\{x \in X : f(x) \neq 0\}$.

Proposition 8.2.1. *(Absolutely convergent series laws) Let X be an arbitrary set, possibly uncountable, and let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions such that the series $\sum_{x\in X} f(x)$ and $\sum_{x\in X} g(x)$ are both absolutely convergent.*

1. *The series $\sum_{x\in X} (f(x) + g(x)) = \sum_{x\in X} f(x) + \sum_{x\in X} g(x)$ is absolutely convergent.*
2. *If c is a real number, then $\sum_{x\in X} cf(x) = c\sum_{x\in X} f(x)$ is absolutely convergent.*

3. *If $X = X_1 \cup X_2$ for some disjoint sets X_1 and X_2, then $\sum_{x \in X_1} f(x) + \sum_{x \in X_2} f(x) = \sum_{x \in X_1 \cup X_2} f(x)$ are absolutely convergent. Conversely, if $h : X \to \boldsymbol{R}$ is such that $\sum_{x \in X_1} h(x)$ and $\sum_{x \in X_2} h(x)$ are absolutely convergent, then $\sum_{x \in X_1 \cup X_2} h(x) = \sum_{x \in X_1} h(x) + \sum_{x \in X_2} h(x)$ is also absolutely convergent.*
4. *If Y is another set, and $\phi : Y \to X$ is a bijection then $\sum_{y \in Y} f(\phi(y))$ is absolutely convergent, and $\sum_{y \in Y} f(\phi(y)) = \sum_{x \in X} f(x)$. This result requires the axiom of choice when X is uncountable.*

If a series is conditionally convergent, but not absolutely convergent, then its behaviour with respect to rearrangements is bad.

Lemma 8.2.3. *Let $\sum_{n=0}^{\infty} a_n$ be a series of real numbers with is conditionally convergent, but not absolutely convergent. Define the sets $A_+ := \{n \in \boldsymbol{N} : a_n \geq 0\}$ and $A_- := \{n \in \boldsymbol{N} : a_n < 0\}$, thus $A_+ \cup A_- = \boldsymbol{N}$ and $A_+ \cap A_- = \emptyset$. Then both of the series $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$ are not conditionally convergent and thus not absolutely convergent.*

The remarkable theorem of Georg Riemann, which asserts that a series which converges conditionally but not absolutely can be rearranged to converge to any value on pleases, is now presented .

Theorem 8.2. *Let $\sum_{n=0}^{\infty} a_n$ be a series which is conditionally convergent, but not absolutely convergent, and let L be any real number. Then there exists a bijection $f : \boldsymbol{N} \to \boldsymbol{N}$ such that $\sum_{m=0}^{\infty} a_{f(m)}$ converges conditionally to L.*

8.3 Uncountable sets

Recall that if X is a set, then the power set of X, denoted $2^X := \{A : A \subseteq X\}$, is the set of all subsets of X.

Theorem 8.3. *(Cantor's theorem) Let X be an arbitrary set, finite of infinite. Then the sets X and 2^X cannot have equal cardinality.*

Proof: Suppose for sake of contradiction that the sets X and 2^X had equal cardinality. Then there exists a bijection $f : X \to 2^X$ between X and the power set of X. Now the set $A := \{x \in X : x \notin f(x)\}$ is well defined since $f(x)$ is an element of 2^X and is hence of subset of X. Clearly A is a subset of X hence is an element of 2^X. Since f is a bijection there must therefore exist $x \in X$ such that $f(x) = A$. If $x \in A$, then by definition of A we ave $x \notin f(x)$, hence $x \notin A$, a contradiction. But if $x \notin A$, then $x \notin f(x)$, hence by definition of A we have $x \in A$, a contradiction. This proves the result. □

The proof can be compared against Russell's paradox. The point is that a bijection between X and 2^X would come dangerously close to the concept of a set X containing itself. By this theorem $\boldsymbol{N}$ and $2^{\boldsymbol{N}}$ cannot have equal cardinality, hence $2^{\boldsymbol{N}}$ is either uncountable or finite. However, $2^{\boldsymbol{N}}$ contains as a subset the set of singletons $\{\{n\} : n \in \boldsymbol{N}\}$, which is clearly bijective to $\boldsymbol{N}$ and hence countably infinite. Thus $2^{\boldsymbol{N}}$ cannot be finite and hence uncountable.

Corollary 8.3.1. *$\boldsymbol{R}$ is uncountable.*

Reals have strictly larger cardinality than the natural numbers. The Continuum Hypothesis asserts that no sets exist that have strictly larger cardinality than the natural numbers, but strictly smaller cardinality than the reals. This Hypothesis is independent of the other axioms of set theory.

8.4 The axiom of choice

The final axiom of the standard Zermelo-Fraenkel-Choice system of set theory, is the axiom of choice. Logician Kurt Godel showed that a result proven using the axiom of choice will never contradict a result proven without the axiom of choice, unless all the other axioms of set theory are themselves inconsistent, which is highly unlikely. This means that axiom of choice is undecidable, i.e. can neither be proved or disproved from the other axioms of set theory, so long as those axioms are themselves consistent. Thus one can view the axiom of choice as a convenient and safe labour-saving device in analysis.

Definition 8.4.1. *(Infinite Cartesian products) Let I be a set, possibly infinite, and for each $\alpha \in I$ let X_α be a set. We then define the Cartesian product $\prod_{\alpha\in I} X_\alpha = \{(x_\alpha)_{\alpha\in I} \in (\bigcup_{\beta\in I} X_\beta)^I : x_\alpha \in X_\alpha,\ \forall\alpha \in I\}$, where $(\bigcup_{\alpha\in I} X_\alpha)^I$ is the set of all functions $(x_\alpha)_{\alpha\in I}$ which assign an element $x_\alpha \in \bigcup_{\beta\in I} X_\beta$ to each $\alpha \in I$. Thus $\prod_{\alpha\in I} X_\alpha$ is a subset of that set of functions, consisting instead of those functions $(x_\alpha)_{\alpha\in I}$ which assign an element $x_\alpha \in X_\alpha$ to each $\alpha \in I$.*

For any sets I and X, we have $\prod_{\alpha\in I} X = X^I$. If I is a set of the form $I := \{i \in \boldsymbol{N} : 1 \leq i \leq n\}$, then $\prod_{\alpha\in I} X_\alpha$ is the same set as the set $\prod_{1\leq i\leq N} X_i = \{(x_i)_{1\leq i\leq n} : x_i \in X_i,\ \forall 1 \leq i \leq n\}$, where $(x_i)_{i\leq i\leq n}$ is an n-ordered tuple.

If $X_1, X_2, \ldots, X_n$ were any finite collection of non-empty sets, then the finite Cartesian product $\prod_{1\leq i\leq n} X_i$ was also non-empty. The axiom of choice asserts that this statement is also true for infinite Cartesian products:

Axiom 8.1 (**Axiom of Choice**)**.** *Let I be a set, and for each $\alpha \in I$, let X_α be a non-empty set. Then $\prod_{\alpha\in I} X_\alpha$ is also non-empty. In other words, there exists a function $(x_\alpha)_{\alpha\in I}$ which assigns to each $\alpha \in I$ an element $x_\alpha \in X_\alpha$.*

Given a collection of non-empty sets X_α, one should be able to choose a single element x_α from each one, and then form the possibly infinite tuple $(x_\alpha)_{\alpha\in I}$ from all the choices one has made. Axiom of choice can lead to proofs which are non-constructive - demonstrating existence of an object without actually constructing the object explicitly. However, as long as one is aware of the distinction between a non-constructive existence statement, and a constructive existence statement, with the latter being preferred, there is no difficulty here, except on a philosophical level. In analysis one often only need a weaker version of axiom of choice - *axiom of countable choice*, witch the index set I restricted to be at most countable.

Lemma 8.4.1. *Let E be a non-empty subset of the real line with $\sup(E) < \infty$. Then there exists a sequence $(a_n)_{n=1}^{\infty}$ whose elements a_n all lie in E, such that $\lim_{n\to\infty} a_n = \sup(E)$.*

Proposition 8.4.1. *Let X and Y be sets, and let $P(x, y)$ be a property pertaining to an object $x \in X$ and an object $y \in Y$ such that for every $x \in X$ there is at least on $y \in Y$ such that $P(x, y)$ is true. Then there exists a function $f : X \to Y$ such that $P(x, f(x))$ is true for all $x \in X$.*

8.5 Ordered sets

Axiom of choice is intimately connected to the theory of ordered sets. We explore three types here.

Definition 8.5.1. *(Partially ordered sets) A partially ordered set (poset) is a set X, together with a relation $\leq_X$ on X. Thus for any two objects $x, y \in X$, the statement $x \leq_X y$ is either a true statement or a false statement. Furthermore, this relation is assumed to obey the following three properties*

- *Reflexivity: For any $x \in X$, we have $x \leq_X x$.*
- *Anti-symmetry: If $x, y \in X$ are such that $x \leq_X y$ and $y \leq_X x$, then $x = y$.*
- *Transitivity: If $x, y, z \in X$ are such that $x \leq_X y$ and $y \leq_X z$, then $x \leq_X z$.*

We refer to $\leq_X$ as the ordering relation. We write $x <_X y$ if $x \leq_X y$ and $x \neq y$.

The natural numbers $\boldsymbol{N}$ together with the usual less-than-or-equal-to relation $\leq$ forms a partially ordered set. Similarly, integers $\boldsymbol{Z}$, the rationals $\boldsymbol{Q}$, the reals $\boldsymbol{R}$, and the extended reals $\boldsymbol{R}^*$ are also partially ordered sets. Meanwhile, if X is a collection of sets, and one uses the relation of is-a-subset-of $\subseteq$ for the ordering relation $\leq_X$, then X is also partially ordered.

Definition 8.5.2. *(Totally ordered sets) Let X be a partially ordered set with some order relation $\leq_X$. A subset Y of X is said to be totally ordered if, given any two $y, y' \in Y$, we either have $y \leq_X y'$ or $y' \leq_X y$ or both. If X itself is totally ordered, we say that X is totally ordered set (chain) with the order relation $\leq_X$.*

The natural numbers $\boldsymbol{N}$, the integers $\boldsymbol{Z}$, the rational $\boldsymbol{Q}$, reals $\boldsymbol{R}$, and the extended reals $\boldsymbol{R}^*$, all with the usual ordering relation $\leq$, are totally ordered. Also, any subset of a totally ordered set is again totally ordered. On the other hand, a collection of set with $\subseteq$ relation is usually not totally ordered. For instance, if X is the set $\{\{1,2\},\{2\},\{2,3\},\{2,3,4\},\{5\}\}$, ordered by the set inclusion relation $\subseteq$, then the elements $\{1,2\}$ and $\{2,3\}$ of X are not comparable to each other, i.e., $\{1,2\} \not\subseteq \{2,3\}$ and $\{2,3\} \not\subseteq \{1,2\}$.

Definition 8.5.3. *(Maximal and minimal elements) Let X be a partially ordered set, and let Y be a subset of X. We say that y is a minimal element of Y if $y \in Y$ and there is no element $y' \in Y$ such that $y' < y$. We say that y is a maximal element of Y if $y \in Y$ and there is no element $y; \in Y$ such that $y < y'$.*

From the previous example, $\{2\}$ is a minimal element, $\{1,2\}$ and $\{2,3,4\}$ are maximal elements, $\{5\}$ is both a minimal and a maximal element, and $\{2,3\}$ is neither a minimal nor a maximal element. Hence, a partially ordered set can have multiple maxima and minima; however, a totally ordered set cannot. The natural numbers $\boldsymbol{N}$ ordered by $\leq$ has a minimal element, namely 0, but no maximal element. The set of integers $\boldsymbol{Z}$ has no maximal and no minimal element.

Definition 8.5.4. *(Well ordered sets) Let X be a partially ordered set, and let Y be a totally ordered subset of X. We say that Y is well-ordered if every non-empty subset of Y has a minimal element $min(Y)$.*

The natural numbers $\boldsymbol{N}$ are well-ordered. However, the integers $\boldsymbol{Z}$, the rationals $\boldsymbol{Q}$, and the real numbers $\boldsymbol{R}$ are not. Every finite totally ordered set is well-ordered. Every subset of a well-ordered set is again well-ordered.

Well-ordered sets automatically obey principle of strong induction.

Proposition 8.5.1. *(Principle of strong induction) Let X be a well-ordered set with an ordering relation $\leq$, and let $P(n)$ be a property pertaining to an element $n \in X$, i.e. for each $n \in X$, $P(n)$ is either a true or false statement. Suppose that for every $n \in X$, we have the following implication: if $P(m)$ is true for all $m \in X$ with $m <_X n$, then $P(n)$ is also true for all $n \in X$.*

The axiom of choice plays a role once we introduce the notion of an upper bound and a strict upper bound.

Definition 8.5.5. *(Upper bounds and strict upper bounds) Let X be a partially ordered set with ordering relation $\leq$, and let Y be a subset of X. If $x \in X$, we say that x is an upper bound for Y iff $y \leq x$ for all $y \in Y$. If in addition $x \notin Y$, we say that x is a strict upper bound for Y. Equivalently, x is a strict upper bound for Y iff $y < x$ for all $y \in Y$.*

For a real number system $\boldsymbol{R}$ with the usual order $\leq$, 2 is an upper bound for the set $\{x \in \boldsymbol{R} : 1 \leq x \leq 2\}$ but is not a strict upper bound. The number 3 is a strict upper bound for this set.

Lemma 8.5.1. *Let X be a partially ordered set with ordering relation $\leq$, and let x_0 be an element of X. Then there is a well-ordered subset Y of X which has x_0 as its minimal element, and which has no strict upper bound.*

The intuition behind this lemma is that one is trying to perform the following algorithm: we initialize $Y := \{x_0\}$. If Y has no strict upper bound, then we are done; otherwise, we choose a strict upper bound and add it to Y. Then we look again to see if Y has a strict upper bound or not. If not, we are done; otherwise we choose another strict upper bound and add it to Y. We continue this algorithm infinitely often until we exhaust all the strict upper bounds; the axiom of choice comes in because infinitely many choices are involved.

Lemma 8.5.2. *(Zorn's lemma or principle of transfinite induction) Let X be a non-empty partially ordered set, with the property that every totally ordered subset Y of X has an upper bound. Then X contains at least one maximal element.*

9 Continuous functions on R

Sequences can be seen as functions from $\boldsymbol{N}$ to $\boldsymbol{R}$ and we did things like take their limit at infinity, form suprema, infima, etc., or computed the sum of all elements in the sequence. Now we look at the functions on a continuum such as $\boldsymbol{R}$ or an interval of $\boldsymbol{R}$ and perform operations like taking limits, computing derivatives, evaluate integrals.

9.1 Subsets of the real line

Definition 9.1.1. *(Intervals) Let $a, b \in \boldsymbol{R}^*$ be extended real numbers. We define the closed interval $[a,b] := \{x \in \boldsymbol{R}^* : a \leq x \leq b\}$, the half-open intervals $[a,b) := \{x \in \boldsymbol{R}^* : a \leq x < b\}$, $(a,b] = \{x \in \boldsymbol{R}^* : a < x \leq b\}$, and the open interval $(a,b) := \{x \in \boldsymbol{R}^* : a < x < b\}$. We call a the left endpoint and b the right endpoint of these intervals.*

One can form additional sets using such operations as union and intersection.

Definition 9.1.2. *(ε-adherent points) Let X be a subset of $\boldsymbol{R}$, let $\varepsilon > 0$, and let $x \in \boldsymbol{R}$. We say that x is ε-adherent to X iff there exists a $y \in X$ which is ε-close to x, i.e., $|x - y| \leq \varepsilon$.*

Definition 9.1.3. *(Adherent points) Let X be a subset of $\boldsymbol{R}$, and let $x \in \boldsymbol{R}$. We say that x is an adherent point of X iff it is ε-adherent to X for every $\varepsilon > 0$.*

Definition 9.1.4. *(Closure) Let X be a subset of $\boldsymbol{R}$. The closure of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of X.*

Let X and Y be arbitrary subsets of $\boldsymbol{R}$. Then $X \subseteq \overline{X}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, and $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$. If $X \subseteq Y$, then $\overline{X} \subseteq \overline{Y}$.

Lemma 9.1.1. *(Closures of intervals) Let $a < b$ be real numbers, and let I be any one of the four intervals $(a,b), (a,b], [a,b), [a,b]$. Then the closure of I is $[a,b]$. Similarly, the closure of (a,∞) or $[a,\infty)$ is $[a,\infty)$, while the closure of $(-\infty,a)$ or $(-\infty,a]$ is $(-\infty,a]$. Finally, the closure of $(-\infty,\infty)$ is $(-\infty,\infty)$.*

The closure of $\boldsymbol{N}$ is $\boldsymbol{N}$. The closure of $\boldsymbol{Z}$ is $\boldsymbol{Z}$. The closure of $\boldsymbol{Q}$ is $\boldsymbol{R}$, and the closure of $\boldsymbol{R}$ is $\boldsymbol{R}$. The closure of the empty set $\emptyset$ is $\emptyset$.

Lemma 9.1.2. *Let X be a subset of $\boldsymbol{R}$, and let $x \in \boldsymbol{R}$. Then x is an adherent point of X iff there exits a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X which converges to x.*

Definition 9.1.5. *A subset $E \subseteq \boldsymbol{R}$ is said to be closed if $\overline{E} = E$, or in other words that E contains all of its adherent points.*

Corollary 9.1.1. *Let X be a subset of $\boldsymbol{R}$. If X is closed, and $(a_n)_{n=0}^{\infty}$ is a convergent sequence consisting of elements in X, then $\lim_{n\to\infty} a_n$ also lies in X. Conversely, if it is true that every convergent sequence $(a_n)_{n=0}^{\infty}$ of elements in X has its limits in X as well, then X is necessarily closed.*

Definition 9.1.6. *(Limit points) Let X be a subset of the real line. We say that x is a limit point, or a cluster point, of X iff it is an adherent point of $X \backslash \{x\}$. We say that x is an isolated point of X if $x \in X$ and there exists some $\varepsilon > 0$ such that $|x - y| > \varepsilon$ for all $y \in X \backslash \{x\}$.*

For $X = (1,2) \cup \{3\}$, 3 is an adherent point of X, but not a limit point, instead an isolated point of X. 2 is still a limit point of X but not an isolated point. It turns out that the set of adherent points slits into the set of limit points and the set of isolated points.

Lemma 9.1.3. *Let I be a non-empty interval, possibly infinite. Then every element of I is a limit point of I.*

Definition 9.1.7. *(Bounded sets) A subset X of the real line is said to be bounded if we have $X \subset [-M, M]$ for some real number $M > 0$.*

Theorem 9.1. *(Heine-Borel theorem for the line) Let X be a subset of $\boldsymbol{R}$. Then the following two statements are equivalent:*

- *X is closed and bounded.*
- *Given any sequence $(a_n)_{n=0}^{\infty}$ of real numbers which takes values in X, there exists a subsequence $(a_{n_j})_{j=0}^{\infty}$ of the original sequence, which converges to some number L in X.*

This theorem asserts that every subset of the real line which is closed and bounded, is also compact.

9.2 The algebra of real-valued functions

We can take a function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined on all $\boldsymbol{R}$, and restrict the domain to a smaller set $X \subseteq \boldsymbol{R}$, creating a new function, sometimes called $f|_X$, from X to $\boldsymbol{R}$. Thus $f|_X(x) := f(x)$ when $x \in X$, and $f|_X(x)$ is undefined when $x \notin X$. One could also restrict the range from $\boldsymbol{R}$ to some smaller subset Y of $\boldsymbol{R}$, provided of course that all the values of $f(x)$ lie inside $\boldsymbol{Y}$. If X is a subset of $\boldsymbol{R}$, and $f : X \to \boldsymbol{R}$ is a function, we can form the graph $\{(x, f(x)) : x \in X\}$ of the function f; this is a subset of $X \times \boldsymbol{R}$, and hence a subset of the Euclidean plane $\boldsymbol{R}^2 = \boldsymbol{R} \times \boldsymbol{R}$. We pursue a more analytic approach, in which we rely instead on the properties of the real numbers to analyze these functions. The two approaches are complementary.

Definition 9.2.1. *(Arithmetic operations on functions) Given two functions $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$, we can define their*

- *sum $f + g : X \to \boldsymbol{R}$ by the formula $(f + g)(x) := f(x) + g(x)$,*
- *difference $f - g : X \to \boldsymbol{R}$ by the formula $(f - g)(x) := f(x) - g(x)$,*
- *maximum $max(f, g) : X \to \boldsymbol{R}$ by $max(f, g)(x) := max(f(x), g(x))$,*
- *minimum $min(f, g) : X \to \boldsymbol{R}$ by $min(f, g)(x) := min(f(x), g(x))$,*
- *product $fg : X \to \boldsymbol{R}$ by formula $(fg)(x) := f(x)g(x)$,*
- *the quotient $f/g : X \to \boldsymbol{R}$ by the formula $(f/g)(x) := f(x)/g(x)$, provided that $g(x) \neq 0$ for all $x \in X$*
- *if c is a real number, we can define the function $cf : X \to \boldsymbol{R}$ by the formula $(cf)(x) := c \times f(x)$.*

9.3 Limiting values of functions

Definition 9.3.1. *(ε-closeness) Let x be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a function, let L be a real number, and let $\varepsilon > 0$ be a real number. We say that the function f is ε-close to L iff $f(x)$ is ε-close to L for every $x \in X$.*

Definition 9.3.2. *(Local ε-closeness) Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a function, let L be a real number, x_0 be an adherent point of X, and $\varepsilon > 0$ be a real number. We say that f is ε-close to L near x_0 iff there exists a $\delta > 0$ such that f becomes ε-close to L when restricted to the set $\{x \in X : |x - x_0| < \delta\}$.*

Definition 9.3.3. *(Convergence of functions at a point) Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a real number. We say that f converges to L at x_0 in E, and write $\lim_{x \to x_0 : x \in E} f(x) = L$, iff f, after restricting to E, is ε-close to L near x_0 for every $\varepsilon > 0$. If f does not converge to any number L at x_0, we say that f diverges at x_0, and leave $\lim_{x \to x_0 ; x \in E} f(x)$ undefined.*

That is, we have $\lim_{x \to x_0 ; x \in E} f(x) = L$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$. For the example of $f : \boldsymbol{R} \to \boldsymbol{R}$ defining a function $f(x) = 1$ when $x = 0$ and $f(x) = 0$ when $x \neq 0$, then $\lim_{x0 ; x \in \boldsymbol{R} \setminus \{x_0\}} f(x) = 0$ but $\lim_{x0 ; x \in \boldsymbol{R}} f(x)$ is undefined.

Proposition 9.3.1. *Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are logically equivalent:*

- *f converges to L at x_0 in E.*
- *For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E and converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to L.*

This yields the following: if $\lim_{x \to x_0 ; x \in E} f(x) = L$, and $\lim_{n \to \infty} a_n = x_0$, then $\lim_{n \to \infty} f(a_n) = L$. It is only worth defining the limit of a function when x_0 is an adherent point of E.

Corollary 9.3.1. *Let X be a subset or $\boldsymbol{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \boldsymbol{R}$ be a function. Then f can have at most one limit at x_0 in E.*

This can be proved by invoking the uniqueness of limits of sequences.

Proposition 9.3.2. *(Limit laws for functions) Let X be a subset of $\boldsymbol{R}$, let E by a subset of X, let x_0 be an adherent point of E, and let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions. Suppose that f has a limit L at x_0 in E, and g has a limit M at x_0 in E. Then,*

- *$f + g$ has a limit $L + M$ at x_0 in E,*
- *$f - g$ has a limit $L - M$ at x_0 in E,*
- *$max(f, g)$ has a limit $max(L, M)$ at x_0 in E,*
- *$min(f, g)$ has a limit $min(L, M)$ at x_0 in E,*
- *fg has a limit LM at x_0 in E.*
- *If c is a real number, then cf has a limit cL at x_0 in E.*
- *If g is non-zero on E and M is non-zero, then f/g has a limit L/M at x_0 in E.*

For the function $sgn(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ we have $\lim_{x \to 0; x \in (0,\infty)} sgn(x) = 1$, and $\lim_{x \to 0; x \in (-\infty,0)} sgn(x) = -1$, and $\lim_{x \to 0; x \in \boldsymbol{R}} sgn(x)$ is undefined. Thus, it is sometimes dangerous to drop the set X from the notation of limit. The set E should always be included in the limit, unless it is obvious. The function $f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$ has $\lim_{x \to 0; x \in \boldsymbol{R} - \{0\}} f(x) = 0$, but $\lim_{x \to 0; x \in \boldsymbol{R}} f(s)$ is undefined. When this happens, we say f has a **removable singularity** at 0. It is sometimes the convention when writing $\lim_{x \to x_0} f(x)$ to automatically exclude x_0 from the set. The limit at x_0 should only depend on the values of the function near x_0; the value away from x_0 are not relevant.

Proposition 9.3.3. *(Limits are local) Let X be a subset of $\boldsymbol{R}$, let E be a subset of X, let x_0 be an adherent point of E, let $f : X \to \boldsymbol{R}$ be a function, and let L be a real number. Let $\delta > 0$. Then we have $\lim_{x \to x_0; x \in E} f(x) = L$ iff $\lim_{x \to x_0; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L$.*

Consider the functions $f : \boldsymbol{R} \to \boldsymbol{R}$ and $g : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) := x + 2$ and $g(x) := x + 1$. Then $\lim_{x \to 2; x \in \boldsymbol{R}} f(x) = 4$ and $\lim_{x \to 2; x \in \boldsymbol{R}} g(x) = 3$. We want to calculate $\lim_{x \to 2; x \in \boldsymbol{R}} f(x)/g(x)$. Strictly speaking we can't use the limit laws as $g(x) = 0$ at $x = -1$ so $f(x)/g(x)$ is not defined. We can resolve it by restricting the domain of f and g from $\boldsymbol{R}$ to $\boldsymbol{R} - \{1\}$. The the limit laws apply and we have $\lim_{x \to 2; x\boldsymbol{R} - \{1\}} \frac{x+2}{x+1} = \frac{4}{3}$.

For the function $f : \boldsymbol{R} - \{1\} \to \boldsymbol{R}$ defined by $f(x) := \frac{x^2-1}{x-1}$ which is defined for every real number except 1 ($f(1)$ is undefined), 1 is still an adherent point of $\boldsymbol{R} - \{1\}$, and the limit $\lim_{x \to 1; x \in \boldsymbol{R} - \{1\}} f(x)$ is still defined. This is because on the domain $\boldsymbol{R} - \{1\}$ we have $\frac{x^2-1}{x-1} = x + 1$, and hence $\lim_{x \to 1; x \in \boldsymbol{R} - \{1\}} x + 1 = 2$.

Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) := \begin{cases} 1 & \text{if } x \in \boldsymbol{Q} \\ 0 & \text{if } x \notin \boldsymbol{Q} \end{cases}$. This function does not have a limit at 0 in $\boldsymbol{R}$.

9.4 Continuous functions

Definition 9.4.1. *(Continuity) Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. Let x_0 be an element of X. We say that f is continuous at x_0 iff we have $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$, i.e. the limit of $f(x)$ as x converges to x_0 in X exists and is equal to $f(x_0)$. We say that f is continuous on X iff f is continuous at x_0 for every $x_0 \in X$. We say that f is discontinuous at x_0 iff it is not continuous at x_0.*

The signum function defined on $\boldsymbol{R} \to \boldsymbol{R}$ is continuous at every non-zero value of x. On the other hand, it is not continuous at 0, since the limit $\lim_{x\to 0; x\in\boldsymbol{R}} sgn(x)$ does not exist. For $f : \boldsymbol{Q} \to \boldsymbol{Q}$ where $f(x) := \begin{cases} 1 & x \in \boldsymbol{Q} \\ 0 & x \notin \boldsymbol{Q} \end{cases}$, is not continuous at any real number x_0. The function $f : \boldsymbol{R} \to \boldsymbol{R}$ with $f(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$, is continuous at every non-zero real number, but is not continuous at 0. However, if we restrict f to the right-hand line $[0, \infty)$, then the resulting function $f|_{[0,\infty)}$ now becomes continuous everywhere in its domain, including 0. Thus restricting the domain of a function can make a discontinuous function continuous again.

Proposition 9.4.1. *(Equivalent formulations of continuity) Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a function, and let x_0 be an element of X. Then the following four statements are logically equivalent:*

- *f is continuous at x_0.*
- *For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.*
- *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$ with $|x - x_0| < \delta$.*
- *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for all $x \in X$ with $|x - x_0| \leq \delta$.*

Proposition 9.4.2. *(Arithmetic preserves continuity) Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions. Let $x_0 \in X$. Then if f and g are both continuous at x_0, then the functions $f + g$, $f - g$, $max(f, g)$ and $min(f, g)$ and fg are also continuous at x_0.*

If g is non-zero on X, then f/g is also continuous at x_0. Let $a > 0$ be a positive real number, then the function $f : \boldsymbol{R} \to \boldsymbol{R}$ define by $f(x) := a^x$ is continuous. Let p be a real number. Then the function $f : (0, \infty) \to \boldsymbol{R}$ defined by $f(x) := x^p$ is continuous. Thus the exponentiation is jointly continuous in both the exponent and the base. Finally the function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) := |x|$ is continuous.

Proposition 9.4.3. *(Composition preserves continuity) Let X and Y be subsets of $\boldsymbol{R}$, and let $f : X \to Y$ and $g : Y \to \boldsymbol{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \boldsymbol{R}$ is continuous at x_0.*

9.5 Left and right limits

Definition 9.5.1. *(Left and right limits) Let X be a subset of $\boldsymbol{R}$, $f : X \to \boldsymbol{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the right limit $f(x_0+)$ of f at x_0 by the formula $f(x_0+) := \lim_{x\to x_0; x\in X\cap(x_0,\infty)} f(x)$, provided the limits exists. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the left limit $f(x_0-)$ of f at x_0 by the formula $f(x_0-) := \lim_{x\to x_0; x\in X\cap(-\infty,x_0)} .f(x)$, provided the limit exists. We use the shorthand to write $\lim_{x\to x_0+} f(x)$ and $\lim_{x\to x_0-} f(x)$ for the right and left limits respectively, when the domain X of f is clear from context.*

For a signum function $sng : \boldsymbol{R} \to \boldsymbol{R}$, we have $sgn(0+) = 1$, $sgn(0-) = -1$ and $sgn(0) = 0$. f does not necessarily have to be defined at x_0 in order for $f(x_0+)$ or $f(x_0-)$ to be defined. For instance, if $f : \boldsymbol{R} - \{0\} \to \boldsymbol{R}$ is a function $f(x) := x/|x|$, then $f(0+) = 1$ and $f(0-) = -1$, even though $f(0)$ is undefined.

Proposition 9.5.1. *Let X be a subset of $\boldsymbol{R}$ containing a real number x_0, and suppose that x_0 is an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. Let $f : X \to \boldsymbol{R}$ be a function. If $f(x_0+)$ and $f(x_0-)$ both exist and are both equal to $f(x_0)$, then f if continuous at x_0.*

It is possible for the right and left limits of a function f at a point x_0 to both exist, but not be equal to each other - *jump discontinuity* at x_0, e.g., $f(x) := \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Also, it is possible for left and right limit to exist and be equal to each other but not be equal to $f(x_0)$ - *removable discontinuity* at x_0, e.g. $f(x) := \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Another way for a function to be discontinuous is to go to infinity, e.g, $f : \boldsymbol{R} - \{0\} \to \boldsymbol{R}$ define by $f(x) := \frac{1}{x}$

has a discontinuity at 0 - *asymptotic discontinuity*. When the function remains bounded but still does not have a limit near x_0, it is called *oscillatory discontinuity*, e.g., for $f : \boldsymbol{R} \to \boldsymbol{R}$ define by $f(x) := \begin{cases} 1 & x \in \boldsymbol{Q} \\ 0 & x \notin \boldsymbol{Q} \end{cases}$. This function has an oscillatory discontinuity at any real number because the function does not have left or right limits at any point, despite the fact that the function is bounded.

9.6 The maximum principle

We now show that continuous functions enjoy a number of other useful properties, especially if their domain is closed interval, realizing the full power of Heine-Borel theorem.

Definition 9.6.1. *Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. We say that f is bounded from above if there exists a real number M such that $f(x) \leq M$ for all $x \in X$. We say that f is bounded from below if there exists a real number M such that $f(x) \geq -M$ for all $x \in X$. We say that f is bounded if there exists a real number M such that $|f(x)| \leq M$ for all $x \in X$.*

Lemma 9.6.1. *Let $a < b$ be real numbers, and let $f : [a, b] \to \boldsymbol{R}$ be a function continuous on $[a, b]$. Then f is a bounded function.*

Proof: Suppose for the sake of contradiction that f is not bounded. Thus for every real number M there exists an element $x \in [a, b]$ such that $|f(x)| \geq M$. That is, for every natural number n, the set $\{x \in [a, b] : |f(x)| \geq n\}$ is non-empty. We choose a sequence $(x_n)_{n=0}^{\infty}$ in $[a, b]$ such that $|f(x_n)| \geq n$ for all n. This sequence lies in $[a, b]$, and so by Heine-Borel theorem there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ which converges to some limit $L \in [a, b]$, where $n_0 < n_1 < n_2 < \ldots$ is an increasing sequence of natural numbers. In particular, we see that $n_j \geq j$ for all $j \in \boldsymbol{N}$. Since f is continuous on $[a, b]$, it is continuous on L, and in particular we see that $\lim\limits_{j\to\infty} f(a_{n_j}) = f(L)$. Thus the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is convergent, and hence is is bounded. On the other hand, we know the the construction that $|f(x_{n_j})| \geq n_j \geq j$ for all j, and hence the sequence $(f(x_{n_j}))_{j=0}^{\infty}$ is not bonded, a contradiction. $\square$

Definition 9.6.2. *(Maxima and Minima) Let $f : X \to \boldsymbol{R}$ be a function, and let $x_0 \in X$. We say that f attains its maximum at x_0 if we have $f(x_0) \geq f(x)$ for all $x \in X$, i.e. the value of f at the point x_0 is larger than or equal to the value of f at any other point in X. We say that f attains its minimum at x_0 if we have $f(x_0) \leq f(x)$.*

If a function attains its maximum somewhere, then it must be bounded from above and if it attains its minimum somewhere, then it must be bounded from below. These are global maxima and minima.

Proposition 9.6.1. *(Maximum principle) Let $a < b$ be real numbers, and let $f : [a, b] \to \boldsymbol{R}$ be a function continuous on $[a, b]$. Then f attains its maximum at some point $x_{max} \in [a, b]$, and also attains its minimum at some point $x_{min} \in [a, b]$*

Note that the maximum principle does not prevent a function from attaining its maximum or minimum at more than one point. Let us write $\sup\limits_{x\in[a,b]} f(x)$ as short-hand for $\sup\{f(x) : x \in [a, b]\}$, and similarly define $\inf\limits_{x\in[a,b]} f(x)$. The maximum principle thus asserts that $m := \sup\limits_{x\in[a,b]} f(x)$ is a real number and is the maximum value of f on $[a, b]$, i.e. there is at least one point x_{max} in $[a, b]$ for which $f(x_{max}) = m$, and for every other $x \in [a, b]$, $f(x)$ is less than or equal to m. Similarly $\inf\limits_{x\in[a,b]} f(x)$ is the minimum value of f on $[a, b]$.

9.7 The intermediate value theorem

Theorem 9.2. *(Intermediate value theorem) Let $a < b$, and let $f : [a, b] \to \boldsymbol{R}$ be a continuous function on $[a, b]$. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in [a, b]$ such that $f(c) = y$.*

Proof: We assume $f(a) < y < f(b)$, as other cases are easy to prove or similar in nature. Let $E := \{x \in [a, b] : f(x) < y\}$. Clearly E is a subset of $[a, b]$ and hence bounded. Also, since $f(a) < y$, $a \in E$ and hence E is non-empty. By the least upper bound principle, the supremum $c := sup(E)$ is thus finite. Since E is bounded by b, we know that $c \leq b$; since E contains a we know that $c \geq a$ and thus $c \in [a, b]$.

Let $n \geq 1$ be an integer. The number $c - \frac{1}{n}$ is less than $c = sup(E)$ and hence cannot be an upper bound for E. Thus there exists a point x_n which lies in E and which is greater than $c - \frac{1}{n}$. Also $x_n \leq c$ since c is an upper bound for E. Thus, $c - \frac{1}{n} \leq x_n \leq c$. By squeeze test we thus have $\lim_{n\to\infty} x_n = c$. Since f is continuous at c, this implies that $\lim_{n\to\infty} f(x_n) = f(c)$. But since x_n lies in E for every n, we have $f(x_n) < y$ for every n. By the comparison principle we thus have $f(c) \leq y$. Since $f(b) > f(c)$, we conclude $c \neq b$.

Since $c \neq b$ and $c \in [a, b]$, we must have $c < b$. In particular there is an $N > 0$ such that $c + \frac{1}{n} < b$ for all $n > N$, since $c + \frac{1}{n}$ converges to c as $n \to \infty$. Since c is the supremum of E and $c + \frac{1}{n} > c$, we thus have $c + \frac{1}{n} \notin E$ for all $n > N$. Since $c + \frac{1}{n} \in [a, b]$, we thus have $f(c + \frac{1}{n}) \geq y$ for all $n \geq N$. But $c + \frac{1}{n}$ converges to c, and f is continuous at c, thus $f(c) \geq y$. But we already know that $f(c) \leq y$, thus $f(c) = y$, as desired. $\square$

The intermediate value theorem says that if f takes the values $f(a)$ and $f(b)$, then it must also take all the values in between. For discontinuous function intermediate value theorem does not apply as those functions can 'jump' past intermediate values; however continuous functions cannot do so. A continuous function may take an intermediate value multiple times. This theorem is another way to show that one can take nth roots of a number.

Corollary 9.7.1. *(Images of continuous functions) Let $a < b$, and let $f : [a, b] \to \boldsymbol{R}$ be a continuous function on $[a, b]$. Let $M := \sum_{x\in[a,b]} f(x)$ be the maximum value of f, and let $m := \inf_{x\in[a,b]} f(x)$ be the minimum value. Let y be a real number between m and M. Then there exists a $c \in [a, b]$ such that $f(c) = y$. Furthermore, we have $f([a, b]) = [m, M]$.*

9.8 Monotonic functions

Definition 9.8.1. *(Monotonic functions) Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. We say that f is monotone increasing iff $f(y) \geq f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is strictly monotone increasing iff $f(y) > f(x)$ whenever $x, y \in X$ and $y > x$. Similarly, we say f is monotone decreasing iff $f(y) \leq f(x)$ whenever $x, y \in X$ and $y > x$, and strictly monotone decreasing iff $f(y) < f(x)$ whenever $x, y \in X$ and $y > x$. We say that f is monotone if it is monotone increasing or monotone decreasing, and strictly monotone if it is strictly monotone increasing or strictly monotone decreasing.*

Continuous functions are not necessarily monotone and monotone functions are not necessarily continuous. Monotone functions obey the maximum principle but not the intermediate value principle. If a function is both strictly monotone and continuous, then it is invertible.

Proposition 9.8.1. *Let $a < b$ be real numbers, and let $f : [a, b] \to \boldsymbol{R}$ be a function which is both continuous and strictly monotone increasing. Then f is a bijection from $[a, b]$ to $[f(a), f(b)]$, and the inverse $f^{-1} : [f(a), f(b)] \to [a, b]$ is also continuous and strictly monotone increasing.*

There is a similar Proposition for functions that are strictly monotone decreasing.

9.9 Uniform continuity

A continuous function on a closed interval $[a, b]$ remains bounded. However, if we replace the closed interval by an open interval, then continuous functions need not be bounded any more. $f : (0, 2) \to \boldsymbol{R}$ defined by $f(x) := 1/x$ is continuous everywhere in $(0, 2)$ but is not bounded as $x \to 0$, or becomes 'less and less' continuous as one approaches the endpoint 0.

We know that if $f : X \to \boldsymbol{R}$ is continuous at a point x_0, then for every $\varepsilon > 0$ there exists a δ such that $f(x)$ will be ε-close to $f(x_0)$ whenever $x \in X$ is δ-close to x_0. In other words, around every point x_0 there is an 'island of stability' $(x_0 - \delta, x_0 + \delta)$, where the function $f(x)$ doesn't stray by more than ε from $f(x_0)$. For the function $f(x) := 1/x$ for 0.1-closeness of $f(x)$ to $f(x_0)$, at $x_0 = 1$ the δ required is $1/11$, while at $x_0 = 0.1$ the δ required is $1/1010$. Thus the island of stability is much narrower around 0.1 than around 1. On the other hand, for function $g : (0, 2) \to \boldsymbol{R}$ defined by $g(x) := 2x$ the island of stability for $\varepsilon = 0.1$ is $\delta = 0.05$ on the full domain.

Definition 9.9.1. *(Uniform continuity) Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. We say that f is uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ and $f(x_0)$ are ε-close whenever $x, x_0 \in X$ are two points in X which are δ-close.*

The difference between uniform continuity and continuity is that in uniform continuity one can take a single δ which works for all $x_0 \in X$; for ordinary continuity, each $x_0 \in X$ might use a different δ. Thus every uniformly continuous function is continuous, but not conversely. The function$f : (0,2) \to \boldsymbol{R}$ given by $f(x) := 1/x$ is continuous but not uniformly continuous on $(0,2)$.

Definition 9.9.2. *(Equivalent sequences) Let m be an integer, let $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ be two sequences of real numbers, and let $\varepsilon > 0$ be given. We say that $(a_n)_{n=m}^{\infty}$ is ε-close to $(b_n)_{n=m}^{\infty}$ iff a_n is ε-close to b_n for each $n \geq m$. We say that $(a_n)_{n=m}^{\infty}$ is eventually ε-close to $(b_n)_{n=m}^{\infty}$ iff there exists an $N \geq m$ such that the sequences $(a_n)_{n=N}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are ε-close. Two sequences $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are equivalent iff for each $\varepsilon > 0$, they are eventually ε-close.*

Lemma 9.9.1. *Let $(a_n)_{N=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be sequences of real numbers, not necessarily bounded or convergent. Then $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent iff $\lim_{n\to\infty}(a_n - b_n) = 0$.*

Proposition 9.9.1. *Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. Then the following two statements are logically equivalent:*

- *f is uniformly continuous on X.*
- *Whenever $(a_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$ are two equivalent sequences consisting of elements of X, the sequences $(f(x_n))_{n=0}^{\infty}$ and $(f(y_n))_{n=0}^{\infty}$ are also equivalent.*

If f is continuous, then f maps convergent sequences to convergent sequences. In contract, this Proposition asserts that if f is uniformly continuous, then f maps equivalent pairs of sequences to equivalent pairs of sequences. This is connected by the fact that $(x_n)_{n=0}^{\infty}$ will converge to x_* iff the sequences $(x_n)_{n=0}^{\infty}$ and $(x_*)_{n=0}^{\infty}$ are equivalent.

Consider the function $f : (0,2) \to \boldsymbol{R}$ defined by $f(x) := \frac{1}{x}$. We see that the sequences $(\frac{1}{n})_{n=1}^{\infty}$ and $(\frac{1}{2n})_{n=1}^{\infty}$ are equivalent sequences on $(0,2)$. However the sequences $(f(\frac{1}{n}))_{n=1}^{\infty}$ and $(f(\frac{1}{2n}))_{n=1}^{\infty}$ are not equivalent, so f is not uniformly continuous. Similar logic can be applied to show $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) := x^2$ is not uniformly continuous, using the sequences $(n)_{n=1}^{\infty}$ and $(n + \frac{1}{n})_{n=1}^{\infty}$.

Proposition 9.9.2. *Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a uniformly continuous function. Let $(x_n)_{n=0}^{\infty}$ be a Cauchy sequence consisting entirely of elements in X. Then $(f(x_n))_{n=0}^{\infty}$ is also a Cauchy sequence.*

Again, $f : (0,2) \to \boldsymbol{R}$ given by $f(x) := \frac{1}{x}$ is not uniformly continuous. The sequence $(\frac{1}{n})_{n=1}^{\infty}$ is a Cauchy sequence in $(0,,2)$, but the sequence $(f(\frac{1}{n}))_{n=1}^{\infty}$ is not a Cauchy sequence. Thus, f is not uniformly continuous.

Corollary 9.9.1. *Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a uniformly continuous function, and let x_0 be an adherent point of X. Then the limit $\lim_{x\to x_0; x\in X} f(x)$ exists and is a real number.*

Proposition 9.9.3. *Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ be a uniformly continuous function. Suppose that E is a bounded subset of X. Then $f(E)$ is also bounded.*

Theorem 9.3. *Let $a < b$ be real numbers, and let $f : [a,b] \to \boldsymbol{R}$ be a function which is continuous on $[a,b]$. Then f is also uniformly continuous.*

9.10 Limits at infinity

Definition 9.10.1. *(Infinite adherent points) Let X be a subset of $\boldsymbol{R}$. We say that $+\infty$ is adherent to X iff for every $M \in bmR$ there exists an $x \in X$ such that $x > M$; we say that $-\infty$ is adherent to X iff for every $M \in \boldsymbol{R}$ there exists an $x \in X$ such that $x < M$.*

In other words, $+\infty$ is adherent to X iff X has no upper bound, or equivalently, iff $\sup(X) = +\infty$. Similarly $-\infty$ is adherent to X iff X has no lower bound, of iff $\inf(X) = -\infty$. Thus a set is bounded if an only if $+\infty$ and $-\infty$ are not adherent points.

Definition 9.10.2. *(Limits at infinity) Let X be a subset of $\boldsymbol{R}$ with $+\infty$ as an adherent point, and let $f : X \to \boldsymbol{R}$ be a function. We say that $f(x)$ converges to L as $x \to +\infty$ in X, and write $\lim_{x\to+\infty;x\in X} f(x) = L$, iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (M, +\infty)$, i.e., $|f(x) - L| \leq \varepsilon$ for all $x \in X$ such that $x > M$. Similarly we say that $f(x)$ converges to L as $x \to -\infty$ iff for every $\varepsilon > 0$ there exists an M such that f is ε-close to L on $X \cap (-\infty, M)$.*

All the limit laws continue to hold and this notion is consistent with the notion of a limit $\lim_{n\to\infty} a_n$ of a sequence.

10 Differentiation of functions

10.1 Basic definitions

Definition 10.1.1. *(Differentiability at a point) Let X be a subset of $\boldsymbol{R}$, and let $x_0 \in X$ be an element of X which is also a limit point of X. Let $f : X \to \boldsymbol{R}$ be a function. If the limit $\lim_{x\to x_0;x\in X-\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0}$ converges to some real number L, then we say that f is differentiable at x_0 on X with derivative L, and write $f'(x_0) := L$. If the limit does not exist, or if x_0 is not an element of X or not a limit point of X, we leave $f'(x_0)$ undefined, and say that f is not differentiable at x_0 on X.*

Note that we need x_0 be be a limit point in order for x_0 to be adherent to $X - \{x_0\}$, otherwise the above limit would automatically be undefined. In particular, we don not define the derivative of a function at an isolated point. The function $f : \boldsymbol{R} \to R$ defined by $f(x) := |x|$ is not differentiable at $x_0 = 0$. However, if one restricts f to $[0, \infty)$, then the restricted function $f|_{[0,\infty)}$ is differentiable at 0 on $[0, \infty)$, with derivative 1: $\lim_{x\to0;x\in[0,\infty)-\{0\}} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0;x\in(0,\infty)} \frac{|x|}{x} = 1$. Thus, even when a function is not differentiable, it is sometimes possible to restore the differentiability by restricting the domain of the function.

Proposition 10.1.1. *(Newton's approximation) Let X be a subset of $\boldsymbol{R}$, let $x_0 \in X$ be a limit point of X, let $f : X \to \boldsymbol{R}$ be a function, and let L be a real number. Then the following statements are logically equivalent:*

- *f is differentiable at x_0 on X with derivative L.*
- *For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is $\varepsilon|x - x_0|$-close to $f(x_0) + L(x - x_0)$ whenever $x \in X$ is δ-close to x_0, i.e., we have $|f(x) - (f(x_0) + L(x - x_0))| \leq \varepsilon|x - x_0|$ whenever $x \in X$ and $|x - x_0| \leq \delta$.*

More informally, if f is differentiable at x_0, then one has the approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$, and conversely.

Proposition 10.1.2. *(Differentiability implies continuity) Let X be a subset of $\boldsymbol{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \boldsymbol{R}$ be a function. If f is differentiable at x_0, then f is also continuous at x_0.*

Definition 10.1.2. *(Differentiability on a domain) Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. We say that f is differentiable on X if, for every limit point $x_0 \in X$, the function f is differentiable at x_0 on X.*

Corollary 10.1.1. *Let X be a subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function which is differentiable on X. Then f is also continuous on X.*

Theorem 10.1. *(**Differential calculus**) Let X be a subset of $\boldsymbol{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X\boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions.*

1. *If f is a constant function, i.e., there exists a real number c such that $f(x) = c$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 0$.*
2. *If f is the identity function, i.e., $f(x) = x$ for all $x \in X$, then f is differentiable at x_0 and $f'(x_0) = 1$.*
3. *Sum rule: If f and g are differentiable at x_0, then $f + g$ is also differentiable at x_0, and $(f + g)'(x_0) = f'(x_0) + g'(x_0)$.*
4. *Product rule: If f and g are differentiable at x_0, then fg is also differentiable at x_0, and $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.*

5. *If f is differentiable at x_0 and c is a real number, then cf is also differentiable at x_0, and $(cf)'(x_0) = cf'(x_0)$.*
6. *Difference rule: If f and g are differentiable at x_0, then $f-g$ is also differentiable at x_0, and $(f-g)'(x_0) = f'(x_0) - g'(x_0)$.*
7. *If g is differentiable at x_0, and g is non-zero on X, i.e., $g(x) \neq 0$ for all $x \in X$, then $1/g$ is also differentiable at x_0, and $(\frac{1}{g})'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}$.*
8. *Quotient rule: If f and g are differentiable at x_0, and g is non-zero on X, then f/g is also differentiable at x_0, and $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{g(x_0)^2}$*

The product rule is also known as the Leibniz rule, who was the other founder of differential and integral calculus besides Newton.

Theorem 10.2. *(Chain rule) Let X, Y be subsets of $\boldsymbol{R}$, let $x_0 \in X$ be a limit point of X, and let $y_0 \in Y$ be a limit point of Y. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$, and such that f is differentiable at x_0. Suppose that $g : Y \to \boldsymbol{R}$ is a function which is differentiable at y_0. Then the function $g \circ f : X \to \boldsymbol{R}$ is differentiable at x_0, and $(g \circ f)'(x_0) = g'(y_0)f'(x_0)$.*

10.2 Local maxima, local minima, and derivatives

Definition 10.2.1. *(Local maxima and minima) Let $f : X \to \boldsymbol{R}$ be a function, and let $x_0 \in X$. We say that f attains a local maximum at x_0 iff there exists a $\delta > 0$ such that the restriction $f|_{X\cap(x_0-\delta,x_0+\delta)}$ of f to $X \cap (x_0-\delta, x_0+\delta)$ attains a maximum at x_0. We say that f attains a local minimum at x_0 iff there exists $\delta > 0$ such that the restriction $f|_{X\cap(x_0-\delta,x_0+\delta)}$ of f to $X \cap (x_0 - \delta, x_0 + \delta)$ attains a minimum at x_0.*

Let $f : \boldsymbol{Z} \to \boldsymbol{R}$ be a function $f(x) = x$, defined on the integers only. Then f has no global maximum or global minimum, but attains both a local maximum and local minimum at every integer n. If $f : X \to \boldsymbol{R}$ attains a local maximum at a point x_0 in X, and $Y \subset X$ is a subset of X which contains x_0, then the restriction $f|_Y : Y \to \boldsymbol{R}$ also attains a local maximum at x_0. Similarly for minima.

Proposition 10.2.1. *(Local extrema are stationary) Let $a < b$ be real numbers, and let $f : (a, b) \to \boldsymbol{R}$ be a function. If $x_0 \in (a, b)$, f is differentiable at x_0, and f attains either a local maximum or local minimum at x_0, then $f'(x_0) = 0$.*

Note that f must be differentiable for this proposition to work. Also, this proposition does not work if the open interval (a, b) is replaced by closed interval $[a, b]$. For example, the function $f : [1, 2] \to \boldsymbol{R}$ defined by $f(x) := x$ has a local maximum at $x_0 = 2$ and a local minimum at $x_0 = 1$, but at both points the derivatives are not 0. Thus the endpoints of an interval can be local maxima or minima even if the derivative is not zero there. Finally, the converse of this proposition is not true.

Theorem 10.3. *(Rolle's Theorem) Let $a < b$ be real numbers, and let $g : [a, b] \to \boldsymbol{R}$ be a continuous function which is differentiable on (a, b). Suppose also that $g(a) = g(b)$. Then there exists an $x \in (a, b)$ such that $g'(x) = 0$.*

Corollary 10.2.1. *(Mean value theorem) Let $a < b$ be real numbers, and let $f : [a, b] \to \boldsymbol{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists an $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.*

10.3 Monotone functions and derivatives

Proposition 10.3.1. *Let X be a subset of $\boldsymbol{R}$, let $x_0 \in X$ be a limit point of X, and let $f : X \to \boldsymbol{R}$ be a function. If f is monotone increasing and f is differentiable at x_0, then $f'(x_0) \geq 0$. If f is monotone decreasing and f is differentiable at x_0, then $f'(x_0) \leq 0$.*

One might naively guess that if f were strictly monotone increasing, and f was differentiable at x_0, then the derivative $f'(x_0)$ would be strictly positive instead of merely non-negative. Unfortunately, this is not always the case. The converse is true, though.

Proposition 10.3.2. *Let $a < b$, and let $f : [a, b] \to \boldsymbol{R}$ be a differentiable function. If $f'(x) > 0$ for all $x \in [a, b]$, then f is strictly monotone increasing. If $f'(x) < 0$ for all $x \in [a, b]$, then f is strictly monotone decreasing. If $f'(x) = 0$ for all $x \in [a, b]$, then f is a constant function.*

10.4 Inverse functions and derivatives

Lemma 10.4.1. *Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $y_0 = f(x_0)$, i.e. $x_0 = f^{-1}(y_0)$. If f is differentiable at x_0, and f^{-1} is differentiable at y_0, then $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.*

Theorem 10.4. *(Inverse function theorem) Let $f : X \to Y$ be an invertible function, with inverse $f^{-1} : Y \to X$. Suppose that $x_0 \in X$ and $y_0 \in Y$ are such that $f(x_0) = y_0$. If f is differentiable at x_0, f^{-1} is continuous at y_0, and $f'(x_0) \neq 0$, then f^{-1} is differentiable at y_0 and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.*

10.5 L'Hopital's rule

Proposition 10.5.1. *(L'Hopital's rule I) Let X be a subset of $\boldsymbol{R}$, let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions, and let $x_0 \in X$ be a limit point of X. Suppose that $f(x_0) = g(x_0) = 0$, that f and g are both differentiable at x_0, but $g'(x_0) \neq 0$. Then there exists a $\delta > 0$ such that $g(x) > 0$ for all $x \in (X \cap (x_0 - \delta, x_0 + \delta)) - \{x_0\}$, and*

$$\lim_{x \to x_0; x \in (X \cap (x_0 - \delta, x_0 + \delta)) - \{x_0\}} \frac{f(x_0)}{g(x_0)} = \frac{f'(x_0)}{f'(x_0)}.$$

Proposition 10.5.2. *(L'Hopital's rule II) Let $a < b$ be real numbers, let $f : [a, b] \to \boldsymbol{R}$ and $g : [a, b] \to \boldsymbol{R}$ be functions which are differentiable on $[a, b]$. Suppose that $f(a) = g(a) = 0$, that g' is non-zero on $[a, b]$, i.e., $g'(x) \neq 0$ for all $x \in [a, b]$, and $\lim_{x \to a; x \in (a, b]} \frac{f'(x)}{g'(x)}$ exists and equals L. Then $g(x) \neq 0$ for all $x \in (a, b]$, and $\lim_{x \to a; x \in (a, b]} \frac{f(x)}{g(x)}$ exists and equals L.*

11 The Riemann integral

11.1 Partitions

Definition 11.1.1. *Let X be a subset of $\boldsymbol{R}$. We say that X is connected iff the following property if true: whenever x, y are elements in X such that $x < y$, the bounded interval $[x, y]$ is a subset of X, i.e., every number between x and y is also in X.*

The empty set, as well as singleton sets are connected, so is the real line.

Lemma 11.1.1. *Let X be a subset of the real line. Then the following two statements are logically equivalent:*

- *X is bounded and connected.*
- *X is bounded interval.*

If I and J are bounded intervals, then the intersection $I \cap J$ is also a bounded interval.

Definition 11.1.2. *(Length of intervals) If I is a bounded interval, we define the length of I, denoted $|I|$ as follows. If I is one of the intervals $[a, b], (a, b), (a, b], [a, b)$ for some real numbers $a < b$, then we define $|I| := b - a$. Otherwise, if I is a point of the empty set, we define $|I| = 0$.*

Definition 11.1.3. *(partitions) Let I be a bounded interval. A partition of I is a finite set $\boldsymbol{P}$ of bounded intervals contained in I, such that every x in I lies in exactly one of the bounded intervals J in $\boldsymbol{P}$.*

Partition is a set of intervals, while each interval is itself a set of real numbers. Thus a partition is a set consisting of other sets. Not that $\{(1, 3), (3, 5)\}$, nor $\{[0, 3), [3, 5)\}$ are not a partitions of $(1, 5)$.

Theorem 11.1. *(Length is finitely additive) Let I be a bounded interval, n be a natural number, and let $\boldsymbol{P}$ be a partition of I of cardinality n. Then $|I| = \sum_{j \in \boldsymbol{P}} |J|$.*

Definition 11.1.4. *(Finer and coarser partitions) Let I be a bounded interval, and let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be two partitions of I. We say that $\boldsymbol{P}'$ is finer than $\boldsymbol{P}$, or equivalently, that $\boldsymbol{P}$ is coarser than $\boldsymbol{P}'$, if for every J in $\boldsymbol{P}'$, there exists a K in $\boldsymbol{P}$ such that $J \subseteq K$.*

The coarsest possible partition of a set X is $\{X\}$ itself. There is no finest partition.

Definition 11.1.5. *(Common refinement) Let I be a bounded interval, and let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be two partitions of I. We define the common refinement $\boldsymbol{P}\#\boldsymbol{P}'$ of $\boldsymbol{P}$ and $\boldsymbol{P}'$ to be the set $\boldsymbol{P}\#\boldsymbol{P}' := \{K \cap J : K \in \boldsymbol{P}$ and $J \in \boldsymbol{P}'\}$.*

Lemma 11.1.2. *Let I be a bounded interval, and let $\boldsymbol{P}$ and $\boldsymbol{P}'$ be two partitions of I. Then $\boldsymbol{P}\#\boldsymbol{P}'$ is also a partition of I, and is both finer than $\boldsymbol{P}$ and finer than $\boldsymbol{P}'$.*

11.2 Piecewise constant functions

Definition 11.2.1. *(Constant functions) Let X be subset of $\boldsymbol{R}$, and let $f : X \to \boldsymbol{R}$ be a function. We say that f is constant iff there exists a real number c such that $f(x) = c$ for all $x \in X$. If E is a subset of X, we say that f is constant on E if the restriction $f|_E$ of f to E is constant, in other words there exists a real number c such that $f(x) = c$ for all $x \in E$. We refer to c as the constant value of f on E.*

If E is empty, every real number c is a constant value for f on E. If E is non-empty set, then a function f which is constant on E can have only one constant value.

Definition 11.2.2. *(Piecewise constant functions I) Let I be a bounded interval, let $f : I \to \boldsymbol{R}$ be a function, and let $\boldsymbol{P}$ be a partition of I. We say that f is piecewise constant with respect to $\boldsymbol{P}$ if for every $J \in \boldsymbol{P}$, f is constant on J.*

Definition 11.2.3. *(Piecewise constant functions II) Let I be a bounded interval, let $f : I \to \boldsymbol{R}$ be a function. We say that f is piecewise constant on I if there exists a partition $\boldsymbol{P}$ of I such that f is piecewise constant with respect to $\boldsymbol{P}$.*

Lemma 11.2.1. *Let I be a bounded interval, let $\boldsymbol{P}$ be a partition of I, and let $f : I \to \boldsymbol{R}$ be a function which is piecewise constant with respect to $\boldsymbol{P}$. Let $\boldsymbol{P}'$ be a partition of I which is finer than $\boldsymbol{P}$. Then f is also piecewise constant with respect to $\boldsymbol{P}'$.*

Lemma 11.2.2. *Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$ be piecewise constant functions on I. Then the functions $f + g$, $f - g$, $max(f, g)$ and fg are also piecewise constant functions on I, where $max(f, g)(x) := max(f(x), g(x))$. If g does not vanish anywhere on I, i.e., $g(x) \neq 0$ for all $x \in I$ then f/g is also a piecewise constant function on I.*

Definition 11.2.4. *(Piecewise constant integral I) Let I be a bounded interval, let $\boldsymbol{P}$ be a partition of I. Let $f : I \to \boldsymbol{R}$ be a function which is piecewise constant with respect to $\boldsymbol{P}$. Then we define the piecewise constant integral p.c. $\int_{[\boldsymbol{P}]} f$ of f with respect to the partition $\boldsymbol{P}$ by the formula*

$$p.c. \int_{[\boldsymbol{P}]} f; = \sum_{j \in \boldsymbol{P}} c_J |J|,$$

where for each J in $\boldsymbol{P}$, we let c_J be the constant value of f on J.

Due to $\boldsymbol{P}$ being finite, the above sum is always well defined, i.e., never diverges.

Proposition 11.2.1. *(Piecewise constant integral is independent of partition) Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ be a function. Suppose that $\boldsymbol{P}$ and $\boldsymbol{P}'$ are partitions of I such that f is piecewise constant both with respect to $\boldsymbol{P}$ and with respect to $\boldsymbol{P}'$. Then p.c. $\int_{[\boldsymbol{P}]} f =$ p.c. $\int_{[\boldsymbol{P}']} f$.*

Definition 11.2.5. *(Piecewise constant integral II) Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ be a piecewise constant function on I. We define the piecewise constant integral p.c $\int_I f$ by the formula p.c. $\int_I f :=$ p.c. $\int_{[\boldsymbol{P}]} f$, where $\boldsymbol{P}$ is any partition of I with respect to which f is piecewise constant.*

Theorem 11.2. *(Laws of piecewise constant integration) Let I be a bounded interval and let $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$ be piecewise constant functions on I.*

- *We have p.c. $\int_I (f + g) =$ p.c. $\int_I f +$ p.c. $\int_I g$.*

- *For any real number c, we have* $p.c.\int_I (cf) = c(p.c.\int_I f)$.
- *We have* $p.c.\int_I (f-g) = p.c.\int_I f - p.c.\int_I g$.
- *If $f(x) \geq 0$ for all $x \in I$, then* $p.c.\int_I f \geq 0$.
- *If $f(x) \geq g(x)$ for all $x \in I$, then* $p.c.\int_I f \geq p.c.\int_I g$.
- *If f is the constant function $f(x) = c$ for all $x \in I$, then* $p.c.\int_I f = c|I|$.
- *Let J be a bounded interval containing $I \subseteq J$, and let $F : J \to \boldsymbol{R}$ be the function* $F(x) := \begin{cases} f(x) & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$.

  Then F is piecewise constant on J, and $p.c.\int_J F = p.c.\int_I f$.
- *Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the functions $f|_J : J \to \boldsymbol{R}$ and $f|_K : K \to \boldsymbol{R}$ are piecewise constant on J and K respectively, and we have* $p.c.\int_I f = p.c.\int_J f|_J + p.c.\int_K f|_K$.

11.3 Upper and lower Riemann integrals

For a bounded function $f : I \to \boldsymbol{R}$ on the bounded interval I, we want to define the Riemann integral $\int_I f$. To do this, we define the upper and lower Riemann integrals like we defined lim sup and lim inf of a sequence.

Definition 11.3.1. *(Majorization of functions) Let $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$. We say that g majorizes f on I if we have $g(x) \geq f(x)$ for all $x \in I$, and that g minorizes f on I if $g(x) \leq f(x)$ for all $x \in I$.*

The idea of the Riemann integral is to try to integrate a function by first majorizing or minorizing that function by a piecewise constant function, which we already know how to integrate.

Definition 11.3.2. *(Upper and lower Riemann integrals) Let $f : I \to \boldsymbol{R}$ be a bounded function defined on a bounded interval I. We define the upper Riemann integral $\overline{\int}_I f$ by the formula*

$$\overline{\int}_I f := \inf \left\{ p.c. \int_I g : g \textit{ is a p.c. function on } I \textit{ which majorizes } f \right\}$$

and the lower Riemann integral $\underline{\int}_I f$ by the formula

$$\underline{\int}_I f := \sup \left\{ p.c. \int_I g : g \textit{ is a p.c. function on } I \textit{ which minorizes } f \right\}$$

Lemma 11.3.1. *Let $f : I \to \boldsymbol{R}$ be a function on a bounded interval I with is bounded by some real number M, i.e., $-M \leq f(x) \leq M$ for all $x \in I$. Then we have $-M|I| \leq \underline{\int}_I f \leq \overline{\int}_I f \leq M|I|$. In particular, both the lower and upper Riemann integrals are real numbers.*

Definition 11.3.3. *(Riemann integral) Let $f : I \to \boldsymbol{R}$ be bounded function on a bounded interval I. If $\underline{\int}_I f = \overline{\int}_I f$, then we say that f is Riemann integrable on I and define $\int_I f := \underline{\int}_I f = \overline{\int}_I f$.*

If the upper and lower Riemann integrals are unequal, we say that f is not Riemann integrable. This is similar to the proposition that lim sum is always greater than or equal to the lim inf, but only equal when the sequence converges. We do not consider unbounded functions to be Riemann integrable; an integral involving such functions is known as an improper integral.

Lemma 11.3.2. *Let $f : I \to \boldsymbol{R}$ be a piecewise constant function on a bounded interval I. Then f is Riemann integrable, and $\int_I f = p.c.\int_I f$.*

Definition 11.3.4. *(Riemann sums) Let $f : I \to \boldsymbol{R}$ be a bounded function on a bounded interval I, and let $\boldsymbol{P}$ be a partition of I. We define the upper Riemann sum $U(f, \boldsymbol{P})$ and the lower Riemann sum $L(f, \boldsymbol{P})$ by $U(f, \boldsymbol{P}) := \sum_{J \in \boldsymbol{P} : J \neq \emptyset} (\sup_{x \in J} f(x))|J|$ and $L(f, \boldsymbol{P}) := \sum_{J \in \boldsymbol{P} : J \neq \emptyset} (\inf_{x \in J} f(x))|J|$.*

Lemma 11.3.3. *Let $f : I \to \boldsymbol{R}$ be a bounded function on a bounded interval I, and let g be a function which majorizes f and which is piecewise constant with respect to some partition $\boldsymbol{P}$ of I. Then p.c. $\int_I g \geq U(f, \boldsymbol{P})$. Similarly, if h is a function which minorizes f and is piecewise constant with respect to $\boldsymbol{P}$, then p.c. $\int_I h \leq L(f\boldsymbol{P})$.*

Proposition 11.3.1. *Let $f : I \to \boldsymbol{R}$ be a bounded function on a bounded interval I. Then $\overline{\int}_I f = \inf\{U(f, \boldsymbol{P}) : \boldsymbol{P}$ is a partition of $I\}$ and $\underline{\int}_I f = \sum\{L(f, \boldsymbol{P}) : \boldsymbol{P}$ is a partition of $I\}$.*

11.4 Basic properties of the Riemann integral

Theorem 11.3. *(Laws of Riemann integration) Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$ be Riemann integrable functions on I.*

- *The function $f + g$ is Riemann integrable, and we have $\int_I (f + g) = \int_I f + \int_I g$.*
- *For any real number c, the function cf is Riemann integrable, and we have $\int_I (cf) = c \int_I f$.*
- *The function $f - g$ is Riemann integrable, and we have $\int_I (f - g) = \int_I f - \int_I g$.*
- *If $f(x) \geq 0$ for all $x \in I$, then $\int_I f \geq 0$.*
- *If $f(x) \geq g(x)$ for all $x \in I$, then $\int_I f \geq \int_I g$.*
- *If f is the constant function $f(x) = c$ for all x in I, then $\int_I f = c|I|$.*
- *Let J be a bounded interval containing I, i.e., $I \subseteq J$, and let $F : J \to \boldsymbol{R}$ be a function $F(x) := \begin{cases} f(x) & x \in I \\ 0 & x \notin I \end{cases}$. Then F is Riemann integrable on J, and $\int_J F = \int_I f$.*
- *Suppose that $\{J, K\}$ is a partition of I into two intervals J and K. Then the functions $f|_J : J \to \boldsymbol{R}$ and $f|_K : K \to \boldsymbol{R}$ are Riemann integrable on J and K respectively, and we have $\int_I f = \int_J f|_J + \int_K f|_K$.*

Theorem 11.4. *(Max and min preserve integrability) Let I by a bounded interval, and let $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$ be a Riemann integrable function. Then the functions $max(f, g) : I \to \boldsymbol{R}$ and $min(f, g) : I \to \boldsymbol{R}$ defined by $max(f, g)(x) := max(f(x), g(x))$ and $min(f, g)(x) := min(f(x), g(x))$ are also Riemann integrable.*

Corollary 11.4.1. *(Absolute values preserve Riemann integrability) Let I be a bounded interval. If $f : I \to \boldsymbol{R}$ is a Riemann integrable function, then the positive part $f_+ := max(f, 0)$ and the negative part $f_- := min(f, 0)$ are also Riemann integrable on I. Also, the absolute value $|f| = f_+ - f_-$ is also Riemann integrable on I.*

Theorem 11.5. *(Products preserve Riemann integrability) Let I be a bounded interval. If $f : I \to \boldsymbol{R}$ and $g : I \to \boldsymbol{R}$ are Riemann integrable, then $fg : I \to \boldsymbol{R}$ is also Riemann integrable.*

11.5 Riemann integrability of continuous functions

Theorem 11.6. *Let I be a bounded interval, and let f be a function which is uniformly continuous on I. Then f is Riemann integrable.*

Corollary 11.5.1. *Let $[a, b]$ be a closed interval, and let $f : [a, b] \to \boldsymbol{R}$ be continuous. Then f is Riemann integrable.*

Proposition 11.5.1. *Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ be both continuous and bounded. Then f is Riemann integrable on I.*

Definition 11.5.1. *Let I be bounded interval, and let $f : I\boldsymbol{R}$. We say that f is piecewise continuous on I if there exists a partition $\boldsymbol{P}$ of I such that $f|_J$ is continuous on J for all $J \in \boldsymbol{P}$.*

Proposition 11.5.2. *Let I be bounded interval, and let $f : I \to \boldsymbol{R}$ be both piecewise continuous and bounded. Then f is Riemann integrable.*

11.6 Riemann integrability of monotone functions

Proposition 11.6.1. *Let $[a, b]$ be a closed and bounded interval and let $f : [a, b] \to \boldsymbol{R}$ be a monotone function. Then f is Riemann integrable on $[a, b]$.*

Let I be a bounded interval, and $f : I \to \boldsymbol{R}$ be both monotone and bounded. Then f is Riemann integrable on I.

Proposition 11.6.2. *(Integral test) Let $f : [0, \infty) \to \boldsymbol{R}$ be a monotone decreasing function which is non-negative, i.e $f(x) \geq 0$ for all $x \geq 0$. Then the $\sum_{n=0}^{\infty} f(n)$ is convergent iff $\sup_{N>0} \int_{[0,N]} f$ is finite.*

Let p be a real number. Then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges absolutely when $p > 1$ and diverges when $p \leq 1$.

11.7 A non-Riemann integrable function

There do exist bounded functions which are not Riemann integrable. Let $f : [0, 1] \to \boldsymbol{R}$ be the discontinuous function $f(x) := \begin{cases} 1 & x \in \boldsymbol{Q} \\ 0 & x \notin \boldsymbol{Q} \end{cases}$. Then f is bounded but not Riemann integrable. Let $\boldsymbol{P}$ be any partition of $[0, 1]$. For any $J \in \boldsymbol{P}$, observe that if J is not a point or empty set, then $\sup_{x \in J} f(x) = 1$. In particular we have $\left(\sup_{x \in J} f(x)\right) |J| = |J|$. In particular we see that $I(f, \boldsymbol{P}) = \sum_{j \in \boldsymbol{P}; J \neq \emptyset} |J| = 1$; note that the empty set does not contribute anything to the total integral. In particular we have $\overline{\int}_{[0,1]} f = 1$. A similar argument gives that $\inf_{x \in J} f(x) = 0$ for all J, and so $L(f, \boldsymbol{P}) = \sum_{j \in \boldsymbol{P}; j \neq \emptyset} 0 = 0$. In particular we have $\underline{\int}_{[0,1]} f = 0$. Thus the upper and lower Riemann integrals do not match, and so this function is not Riemann integrable. As you can see, it is only rather 'artificial' bounded functions which are not Riemann integrable. Because of this, the Riemann integral is good enough for a large majority of cases. The improvement on this is called Lebesgue integral which we discuss in later sections.

11.8 The Riemann-Stieltjes integral

There is a generalization of the Riemann integration, called the *Riemann-Stieltjes integral.* Instead of taking the length $|J|$ for the intervals J, we take the α-length $\alpha[J]$, where for a bounded interval I, $\alpha : I \to \boldsymbol{R}$ be a monotonic increasing function. If J is a point or the empty set, then $\alpha[J] := 0$. If J is an interval of the form $[a, b]$, (a, b), $(a, b]$ or $[a, b)$, then $\alpha[J] := \alpha(b) - \alpha(b)$. If α is the identity function then $\alpha[J]$ is same as $|J|$. Much of the above theory, work out by simple replacement.

Definition 11.8.1. *(α-length) Let I be a bonded interval, and let $\alpha : X \to \boldsymbol{R}$ be a function defined on some domain X which contains I. Then we define the α-length $\alpha[I]$ of I as follows. If I is a point or the empty set, we set $\alpha[I] = 0$. If I is an interval of the form $[a, b]$, (a, b), $(a, b]$ or $[a, b)$ for some $b > a$, then we set $\alpha[I] = \alpha(b) - \alpha(a)$.*

The notion of length is a special case of the notion of α-length. We denote $\alpha[[a, b]]$ by $\alpha|_a^b$.

Lemma 11.8.1. *Let I be a bounded interval, let $\alpha : X \to \boldsymbol{R}$ be a function defined on some domain X which contains I, and let $\boldsymbol{P}$ be a partition of I. Then we have $\alpha[I] = \sum_{j \in \boldsymbol{P}} \alpha[J]$.*

Definition 11.8.2. *(p.c. Riemann-Stieltjes integral) Let I be a bonded interval, and let $\boldsymbol{P}$ be a partition of I. Let $\alpha : X \to \boldsymbol{R}$ be a function defined on some domain X which contains I, and let $f : I \to \boldsymbol{R}$ be a function which is piecewise constant with respect to $\boldsymbol{P}$. Then we define p.c. $\int_{[\boldsymbol{P}]} f d\alpha := \sum_{j \in \boldsymbol{P}} c_J \alpha[J]$ where c_J is the constant value of f on J.*

We can thus define *p.c.* $\int_I f d\alpha$ for any piecewise constant function $f : I \to \boldsymbol{R}$ and any $\alpha : X \to \boldsymbol{R}$ defined on a domain containing I. Using the property that α is monotone increasing, i.e., $\alpha(y) \geq \alpha(x)$ whenever $x, y \in X$ and $y \geq x$. Thus $\alpha(I) \geq 0$ for all intervals of X. We an then define upper and lower Riemann-Stieltjes integrals

$\overline{\int}_I f d\alpha$ and $\underline{\int}_I f d\alpha$ whenever $f : I \to \boldsymbol{R}$ is bounded and α is defined on a domain containing I, using similar formulation as before. We then say that f is Riemann-Stieltjes integrable on I with respect to α if the upper and lower Riemann-Stieltjes integrals match, in which case we set $\int_I f d\alpha = \overline{\int}_I f d\alpha = \underline{\int}_I f d\alpha$. Riemann-Stieltjes integral is a generalization of the Riemann integral. Some of the original results break if α is discontinuous at key places, e.g., if f and α are both discontinuous at the same point, then $\int_I f d\alpha$ is unlikely to be defined.

11.9 The two fundamental theorems of calculus

We now connect integration and differentiation via the fundamental theorem of calculus

Theorem 11.7. *(First fundamental Theorem of Calculus) Let $a < b$ be real numbers, and let $f : [a,b] \to \boldsymbol{R}$ be a Riemann integrable function. Let $F : [a,b] \to \boldsymbol{R}$ be the function $F(x) := \int\limits_{[a,x]} f$. Then F is continuous. Furthermore, if $x_0 \in [a,b]$ and f is continuous at x_0, then F is differentiable at x_0, and $F'(x_0) = f(x_0)$.*

Definition 11.9.1. *(Antiderivatives) Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ be a function. We say that a function $F : I \to \boldsymbol{R}$ is an antiderivative of f if F if differentiable on I and $F'(x) = f(x)$ for all $x \in I$.*

Theorem 11.8. *(Second Fundamental Theorem of Calculus) Let $a < b$ be real numbers, and let $f : [a,b] \to \boldsymbol{R}$ be a Riemann integrable function. If $F : [a,b] \to \boldsymbol{R}$ is an antiderivative of f, then $\int\limits_{[a,b]} f = F(b) - F(a)$.*

Lemma 11.9.1. *Let I be a bounded interval, and let $f : I \to \boldsymbol{R}$ be a function. Let $F : I \to \boldsymbol{R}$ and $G : I \to \boldsymbol{R}$ be two antiderivatives of f. Then there exists a real number C such that $F(x) = G(x) + C$ for all $x \in I$.*

11.10 Consequences of the fundamental theorems

Proposition 11.10.1. *(Integration by parts formula) Let $I = [a,b]$, and let $F : [a,b] \to \boldsymbol{R}$ and $G : [a,b] \to \boldsymbol{R}$ be differentiable functions on $[a,b]$ such that F' and G' are Riemann integrable on I. Then we have*

$$\int_{[a,b]} FG' = F(b)G(b) - F(a)G(a) - \int_{[a,b]} F'G.$$

Next, we show that under certain circumstances, one can write a Riemann-Stieltjes integral as a Riemann integral. Starting with piece-wise constant functions:

Theorem 11.9. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a,b]$, with α' being Riemann integrable. Let $f : [a,b] \to \boldsymbol{R}$ be a piecewise constant function on $[a,b]$. Then $f\alpha'$ is Riemann integrable on $[a,b]$ and $\int_{[a,b]} f d\alpha = \int_{[a,b]} f\alpha'$.*

Corollary 11.10.1. *Let $\alpha : [a,b] \to \boldsymbol{R}$ be a monotone increasing function, and suppose that α is also differentiable on $[a,b]$, with α' being Riemann integrable. Let $f : [a,b] \to \boldsymbol{R}$ be a function which is Riemann-Stieltjes integrable with respect to α in $[a,b]$. Then $f\alpha'$ is Riemann integrable on $[a,b]$, and $\int_{[a,b]} f d\alpha = \int_{[a,b]} f\alpha'$.*

Informally, this means that $f d\alpha$ is equivalent to $f \frac{d\alpha}{dx} dx$, when α is differentiable. However, the advantage of the Riemann-Stieltjes integral is that it still makes sense even when α is not differentiable.

Lemma 11.10.1. *(Change of variables formula I) Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \boldsymbol{R}$ be a **piecewise constant** function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a,b] \to \boldsymbol{R}$ is also piecewise constant on $[a,b]$, and $\int\limits_{[a,b]} f \circ \phi \; d\phi = \int\limits_{[\phi(a),\phi(b)]} f$.*

Proposition 11.10.2. *(Change of variables formula II) Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a continuous monotone increasing function. Let $f : [\phi(a), \phi(b)] \to \boldsymbol{R}$ be a **Riemann integrable** function on $[\phi(a), \phi(b)]$. Then $f \circ \phi : [a,b] \to \boldsymbol{R}$ is Riemann-Stieltjes integrable with respect to ϕ on $[a,b]$, and $\int\limits_{[a,b]} f \circ \phi \; d\phi = \int\limits_{[\phi(a),\phi(b)]} f$.*

Proposition 11.10.3. *(Change of variables formula III) Let $[a,b]$ be a closed interval, and let $\phi : [a,b] \to [\phi(a), \phi(b)]$ be a **differentiable** monotone increasing function such that ϕ' is Riemann integrable. Let $f : [\phi(a), \phi(b)] \to \boldsymbol{R}$ be a Riemann integrable function on $[\phi(a), \phi(b)]$. Then $(f \circ \phi)\phi' : [a,b] \to \boldsymbol{R}$ is Riemann integrable on $[a,b]$, and $\int\limits_{[a,b]} (f \circ \phi) \; \phi' = \int\limits_{[\phi(a),\phi(b)]} f$.*

12 Metric spaces

12.1 Definitions and examples

Let $(x_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let x be another real number. Then $(x_n)_{n=m}^{\infty}$ converges to x iff $\lim_{n\to\infty} d(x_n, x) = 0$. To generalize this notion of convergence, we work abstractly, defining a very general class of spaces and the notion of convergence on this. There are two useful classes of spaces which do the job - metric spaces and topological spaces.

Definition 12.1.1. *(Metric spaces) A metric space (X, d) is a space X of objects, called points, together with a distance function or metric $d : X \times X \to [0, \infty)$, which associates to each pair x, y of points in X a non-negative real number $d(x, y) \geq 0$ and satisfies the four axioms:*

1. *Identity: For any $x \in X$, we have $d(x, x) = 0$.*
2. *Positivity: For any distinct $x, y \in X$, we have $d(x, y) > 0$.*
3. *Symmetry: For any $x, y \in X$, we have $d(x, y) = d(y, x)$.*
4. *Triangle inequality: For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) = d(y, z)$.*

Let $\boldsymbol{R}$ be the real numbers, and let $d : \boldsymbol{R} \times \boldsymbol{R} \to [0, \infty)$ be the metric $d(x, y) := |x - y|$. Then $(\boldsymbol{R}, d)$ is a metric space, referred to as **standard metric** in $\boldsymbol{R}$.

Let (X, d) be any metric space, and let Y be a subset of X. Then we can restrict the metric function $d : X \times X \to [0, +\infty)$ to the subset $Y \times Y$ or $X \times X$ to create a restricted metric function $d|_{Y\times Y} : Y \times Y \to [0, \infty)$ or Y; this is known as the metric on Y **induced** by the metric d on X. The pair $(Y, d|_{Y\times Y})$ is a metric space and is known as the subspace of (X, d) induced by Y. For example, the standard metric on $\boldsymbol{R}$ induces a metric space structure on any subset of the reals, like $\boldsymbol{Z}$ or an interval $[a, b]$, etc.

Let $n \geq 1$ be a natural number, and let $\boldsymbol{R}^n$ be the space of n-tuple of real numbers: $\boldsymbol{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in \boldsymbol{R}\}$. We define the **Euclidean metric**, also called the l^2 **metric** $d_{l^2} : \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}$ by $d_{l^2}((x_1, , x_n), (y_1, \ldots, y_n)) := \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$. We refer to $(\boldsymbol{R}^n, d_{l^2})$ as the **Euclidean space of dimension** n.

Let $n \geq 1$, and let $\boldsymbol{R}^n$ be the space of n-tuples of real numbers as before. We define **taxicab metric** or l^1 **metric** by $d_{l^1}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sum_{i=1}^{n} |x_i - y_i|$. $(\boldsymbol{R}^n, d_{l^1})$ is also a metric space. We have $d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y)$ for all x, y.

Let $n \geq 1$, and let $\boldsymbol{R}^n$ be the space of n-tuples of real numbers as before. We define **sup norm metric** or l^∞ **metric** by $d_{l^\infty}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}$. The space $(\boldsymbol{R}^n, d_{l^\infty})$ is also a metric space. We have $\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y)$ for all x, y. These are special cases of the more general l^p **metrics**, where $p \in [1, +\infty]$.

Let X be an arbitrary set, finite or infinite, and define the **discrete metric** by $d_{disc}(x, y) := 0$ when $x = y$ and $d_{disc}(x, y) := 1$ if $x \neq y$. In this metric, all points are equally far apart. The space (X, d_{disc}) is a metric space. Thus every set X has at least one metric on it.

Definition 12.1.2. *(Convergence of sequence in metric spaces) Let m be an integer, (X, d) be a metric space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X, i.e., for every natural number $n \geq m$, we assume that $x^{(n)}$ is an element of X. Let x be a point in X. We say that $x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the metric d, iff the limit $\lim_{n\to\infty} d(x^{(n)}, x)$ exists and is equal to 0. In other words, $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(n)}, x) \leq \varepsilon$ for all $n \geq N$.*

Convergence of a sequence can depend on what metric one uses. For example $x^{(n)} := (1/n, 1/n)$ converges to 0 in Euclidean space $\boldsymbol{R}^2$ but fails to converge in the discrete metric space. The spaces l^1, l^2, l^∞ are equivalent in $\boldsymbol{R}^n$ in the sense that convergence in one implies convergence in other. This is not true in infinite-dimensional versions of these. For the discrete metric, convergence is much rarer: the sequence must be eventually constant in order to converge, i.e. there exists an $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.

Proposition 12.1.1. *(Uniqueness of limits) Let (X, d) be a metric space, and let $x^{(n)})_{n=m}^{\infty}$ be a sequence in X. Suppose that there are two points $x, x' \in X$ such that $x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d and $x^{(n)})_{n=m}^{\infty}$ also converges to x' with respect to d. Then we have $x = x'$.*

It is possible for a sequence to converge to one point using one metric, and another point using a different metric. Thus changing the metric on a space can greatly affect the nature of convergence (also called the topology) on that space.

12.2 Some point-set topology of metric spaces

The study of notions like open set, closed set, interior, exterior, boundary and adherent point is known as point-set topology.

Definition 12.2.1. *(Metric balls) Let (X, d) be a metric space, let x_0 be a point in X, and let $r > 0$. We define the ball $B_{(X,d)(x_0,r)}$ in X, centered at x_0, and with radius r, in the metric d, to be the set $B_{(X,d)(x_0,r)} := \{x \in X : d(x, x_0) < r\}$. When the metric space is apparent we denote it by $B(x_0, r)$.*

Balls of zero or negative radius are empty sets.

Definition 12.2.2. *(Interior, exterior, boundary) Let (x, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an interior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an exterior point of E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \emptyset$. We say that x_0 is a boundary point of E if it is neither an interior or exterior point of E.*

$int(E)$ denote the set of all interior of E and $ext(E)$ denote the set of all exterior of E. Finally ∂E denote the set of all boundary of E.

Definition 12.2.3. *(Closure) Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X We say that x_0 is an adherent point of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is called the closure of E and is denoted $\overline{E}$.*

Proposition 12.2.1. *Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are logically equivalent*

- *x_0 is an adherent point of E.*
- *x_0 is either an interior point or a boundary point of E.*
- *There exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d.*

We can easily show that $\overline{E} = int(E) \cup \partial E = X \backslash ext(E)$.

Definition 12.2.4. *(Open and closed sets) Let (X, d) be a metric space, and let E be a subset of X. We say that E is closed if it contains all of its boundary points. We say that E is open if it contains non of its boundary points. If E contains some of its boundary points but not others, then it is neither open nor closed.*

It is possible for a set to be simultaneously open and closed, if it has no boundary. For instance, in the metric space (X, d), the whole space X has no boundary, and so X is both open and closed. The empty set $\emptyset$ also has no boundary, and so it is both open and closed. Using the discrete metric, every set is both open and closed. Thus the notion of being open and being closed are not negations of each other.

Proposition 12.2.2. *(Basic properties of open and closed sets) Let (X, d) be a metric space and let E be a subset of X.*

1. *E is open iff $E = int(E)$, i.e., for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.*
2. *E is closed iff E contains all its adherent points, i.e. for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n \to} x_n$ of that sequence also lies in E.*
3. *For any $x_0 \in X$ and $r > 0$, then the ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set, called the closed ball of radius r centered at x_0.*

4. *Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.*

5. *If E is a subset of X, then E is open iff the complement $X \backslash E := \{x \in X : x \notin E\}$ is closed.*

6. *If $E_1, \ldots, E_n$ are finite collection of open sets in X, then $E_1 \cap E_2 \cap \ldots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is also closed.*

7. *If $\{E_{\alpha \alpha \in I}\}$ is a collection of open sets in X, where the index set I could be finite, countable or uncountable, then the union $\cup_{\alpha \in I} E_\alpha := \{x \in X : x \in E_\alpha$ for some $\alpha \in I\}$ is also open. If $\{F_\alpha\}_{\alpha \in I}$ is a collection of closed sets in X, then the intersection $\cap_{\alpha \in I} F_\alpha := \{x \in X : x \in F_\alpha$ for all $\alpha \in I\}$ is also closed.*

8. *If E is any subset of X, then $int(E)$ is the largest open set which is contained in E, i.e. $int(E)$ is open, and given any other open set $V \subseteq E$, we have $V \subseteq int(E)$. Similarly $\overline{E}$ is the smallest closed set which contains E, i.e. $\overline{E}$ is closed, and given any other closed set $K \supset E$, $K \supset \overline{E}$.*

12.3 Relative topology

The choice of ambient space X also affects whether a set is open or not. For example, a segment on x axis when viewed as a subset of $\boldsymbol{R}^2$ it is not an open set, but as viewed as a subset of $\boldsymbol{R}$ it is an open set. Similarly, the set $[0, 1)$ is not closed on $\boldsymbol{R}$, but is closed on the interval $X := (-1, 1)$.

Definition 12.3.1. *(Relative topology) Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is relatively open with respect to Y if it is open in the metric subspace $(Y, d|_{Y \times Y}$. Similarly, we say that E is relatively closed with respect to Y if it is closed in the metric space $(Y, d|_{Y \times Y})$.*

Proposition 12.3.1. *Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.*

- *E is relatively open with respect to Y iff $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.*
- *E is relatively closed with respect to Y iff $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.*

12.4 Cauchy sequences and complete metric spaces

Definition 12.4.1. *(Subsequences) Suppose that $(x^{(n)})_{n=m}^\infty$ is a sequence of points in a metric space (X, d). Suppose that $n_1, n_2, \ldots$ is an increasing sequence of integers which are at least as large as m, thus $m \leq n_1 < n_2 < n_3 < \ldots$. Then we call the sequence $(x^{(n_j)})_{j=1}^\infty$ a subsequence of the original sequence $(x^{(n)})_{n=m}^\infty$.*

Lemma 12.4.1. *Let $(x^{(n)})_{n=m}^\infty$ be a sequence in (X, d) which converges to some limit x_0. Then every subsequence $(x^{(n_j)})_{j=1}^\infty$ of that sequence also converges to x_0.*

On the other hand, it is possible for a subsequence to be convergent without the sequence as a whole being convergent.

Definition 12.4.2. *(Limit points) Suppose that $(x^{(n)})_{n=m}^\infty$ is a sequence of points in a metric space (X, d), and let $L \in X$. We say that L is a limit point of $(x^{(n)})_{n=m}^\infty$ iff for every $N \geq m$ and $\varepsilon > 0$ there exists an $n \geq N$ such that $d(x^{(n)}, L) \leq \varepsilon$.*

Proposition 12.4.1. *Let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in a metric space (X, d), and let $L \in X$. Then the following are equivalent:*

- *L is a limit point of $(x^{(n)})_{n=m}^\infty$.*
- *There exists a subsequence $(x^{(n_j)})_{j=1}^\infty$ of the original sequence $(x^{(n)})_{n=m}^\infty$ which converges to L.*

Definition 12.4.3. *(Cauchy sequences) Let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in a metric space (X, d). We say that this sequence is a Cauchy sequence iff for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon$ for all $j, k \geq N$.*

Lemma 12.4.2. *(Convergent sequences are Cauchy sequences) Let $(x^{(n)})_{n=m}^\infty$ be a sequence in (X, d) which converges to some limit x_0. Then $(x^{(n)})_{n=m}^\infty$ is also a Cauchy sequence.*

Subsequences of a Cauchy sequence is also a Cauchy sequence. However, not every Cauchy sequence converges. For example the sequence $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ in the metric space $(\boldsymbol{Q}, d)$ is not convergent, while is convergent in the metric space $(\boldsymbol{R}, d)$.

Lemma 12.4.3. *Let $(x^{(n)})_{n=m}^{\infty}$ be a Cauchy sequence in (X, d). Suppose that there is some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of this sequence which converges to a limit x_0 in X. Then the original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to x_0.*

Definition 12.4.4. *(Complete metric spaces) A metric space (X, d) is said to be complete iff every Cauchy sequence in (X, d) is in fact convergent in (X, d).*

Complete metric spaces like $(\boldsymbol{R}, d)$ are intrinsically closed: no matter what space one places them in, they are always closed sets.

Proposition 12.4.2. *Let (X, d) be a metric space, and let $(Y, d|_{Y\times Y})$ be a subspace of (X, d). If $(Y, d|_{Y\times Y})$ is complete, then Y must be closed in X. Conversely, suppose that (X, d) is a complete metric space, and Y is closed subset of X. Then the subspace $(Y, d|_{Y\times Y})$ is also complete.*

An incomplete metric space such as $(\boldsymbol{Q}, d)$ may be considered closed in some spaces (for instance $\boldsymbol{Q}$ is closed in $\boldsymbol{Q}$) but not in others (for instance, $\boldsymbol{Q}$ is not closed in $\boldsymbol{R}$). For any given incomplete metric space (X, d), there exists a completion $(\overline{X}, \overline{d})$, which is a larger metric space containing (X, d), which is complete an such that X is not closed in $\overline{X}$. For instance, one possible completion of $\boldsymbol{Q}$ is $\boldsymbol{R}$.

12.5 Compact metric spaces

Heine-Borel theorem asserted that every sequence in a closed and bounded subset X of the real line $\boldsymbol{R}$ had a convergent subsequence whole limit was also in X. Conversely, only the closed and bounded sets have this property.

Definition 12.5.1. *(Compactness) A metric space (X, d) is said to be compact iff every sequence in (X, d) has at least one convergent subsequence. A subset Y of a metric space X is said to be compact if the subspace $(Y, d|_{Y\times Y})$ is compact.*

The notion of a set Y being compact is intrinsic, in the sense that it depends on the metric function $d|_{Y\times Y}$ restricted to Y, and not on the choice of the ambient space X. The notion of completeness and boundedness, which follows, are also intrinsic, but the notion of open and closed are not. Now we investigate how Heine-Borel extends to other metric spaces.

Definition 12.5.2. *(Bounded sets) Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is bounded iff there exists a ball $B(x, r)$ in X which contains Y.*

Proposition 12.5.1. *Let (X, d) be a compact metric space. Then (X, d) is both complete and bounded.*

Let (X, d) be a metric space, and let Y be a compact subset of X. Then Y is closed and bounded.

Theorem 12.1. *(Heine-Borel theorem) Let $(\boldsymbol{R}^n, d)$ be a Euclidean space with either the Euclidean metric, the taxicab metric, or the sup norm metric. Let E be a subset of $\boldsymbol{R}^n$. Then E is compact iff it is closed and bounded.*

However, the Heine-Borel theorem is not true for more general metrics, e.g. $\boldsymbol{Z}$. However with a stronger notion of total boundedness Heine-Borel theorem can be make to work.

One can characterize compactness topologically via the statement - every open cover of a compact set has a finite subcover.

Theorem 12.2. *Let (x, d) be a metric space, and let Y be a compact subset of X. Let $(V_\alpha)_{\alpha\in I}$ be a collection of open set in X, and suppose that $Y \subseteq \bigcup_{\alpha\in I} V_\alpha$, i.e., the collection $(V_\alpha)_{\alpha\in I}$ covers Y). Then there exists a finite subset F of I such that $Y \subseteq \bigcup_{\alpha\in F} V_\alpha$.*

The converse: if Y has the property that every open cover has a finite sub-cover, then it is compact - is true. This property is often considered more fundamental than the sequence-based one.

Corollary 12.5.1. *Let (X, d) be a metric space, and let $K_1, K_2, K_3, \dots$ be a sequence of non-empty compact subsets of X such that $K_1 \supset K_2 \supset K_3 \supset \dots$. Then the intersection $\bigcap_{n=1}^{\infty} K_n$ is non-empty.*

Theorem 12.3. *Let (X, d) be a metric space.*

- *If Y is a compact subset of X, and $Z \subseteq Y$, then Z is compact iff Z is closed.*
- *If $Y_1, \dots, Y_n$ are a finite collection of compact subsets of X, then their union $Y_1 \cup \dots, \cup Y_n$ is also compact.*
- *Every finite subset of X, including the empty set, is compact.*

13 Continuous functions on metric spaces

13.1 Continuous functions

Definition 13.1.1. *(Continuous functions or continuous maps) Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is continuous at x_0 iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is continuous iff it is continuous at every point $x \in X$.*

Theorem 13.1. *(Continuity preserves convergence) Suppose that (X, d_X) and (Y, d_Y) are metric spaces. Let $f : X \to Y$ be a function, and let $x_0 \in X$ be a point in X. Then the following three statements are logically equivalent:*

- *f is continuous at x_0.*
- *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*
- *For every open set $V \subset Y$ that contains $f(x_0)$, there exists an open set $U \subset X$ containing x_0 such that $f(U) \subseteq V$.*

Theorem 13.2. *Let (X, d_X) be a metric space, and let (Y, d_Y) be another metric space. Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

- *f is continuous.*
- *Whenever $(x^{(n)})_{n=1}^{\infty}$ is a sequence in X which converges to some point $x_0 \in X$ with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to the metric d_Y.*
- *Whenever V is an open set in Y, the set $f^{-1}(V) := \{x \in X : f(x) \in V\}$ is an open set in X.*
- *Whenever F is a closed set in Y, the set $f^{-1}(F) := \{x \in X : f(x) \in F\}$ is a closed set in X.*

Continuity ensures that the inverse image of an open set is open. The forward image of an open set may not be open, though.

Corollary 13.1.1. *(Continuity preserved by composition) Let (X, d_X), (Y, d_Y) and (z, d_Z) bet metric spaces.*

- *If $f : X \to Y$ is continuous at a point $x_0 \in X$, and $g : Y \to Z$ is continuous at $f(x_0)$, then the composition $g \circ f : X \to Z$, defined by $(g \circ f)(x) := g(f(x))$, is continuous at x_0.*
- *If $f : X \to Y$ is continuous, and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is also continuous.*

13.2 Continuity and product spaces

Lemma 13.2.1. *Let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions, and let $f \oplus g : X \to \boldsymbol{R}^2$ be their direct sum. We give $\boldsymbol{R}^2$ the Euclidean metric.*

- *If $x_0 \in X$, then f and g are both continuous at x_0 iff $f \oplus g$ is continuous at x_0.*
- *f and g are both continuous iff $f \oplus g$ is continuous.*

The addition function $(x, y) \mapsto x + y$, the subtraction function $(x, y) \mapsto x - y$, the multiplication function $(x, y) \mapsto xy$, the maximum function $(x, y) \mapsto max(x, y)$, and the minimum function $(x, y) \mapsto min(x, y)$, are all continuous functions from $\boldsymbol{R}^2$ to $\boldsymbol{R}$. The division function $(x, y) \mapsto x/y$ is a continuous function from $\boldsymbol{R} \times (\boldsymbol{R} \backslash \{0\}) = \{(x, y) \in R^2 : y \neq 0\}$ to $\boldsymbol{R}$. For any real number c, the function $x \mapsto cx$ is a continuous function from $\boldsymbol{R}$ to $\boldsymbol{R}$.

Corollary 13.2.1. *Let (X, d) be a metric space, let $f : X \to \boldsymbol{R}$ and $g : X \to \boldsymbol{R}$ be functions. Let c be a real number.*

- *If $x_0 \in X$ and f and g are continuous at x_0, then the functions $f + g : X \to \boldsymbol{R}$, $f - g : X \to \boldsymbol{R}$, $fg : X \to \boldsymbol{R}$, $max(f, g) \to \boldsymbol{R}$, $min(f, g) \to \boldsymbol{R}$, and $cf : X \to \boldsymbol{R}$ are also continuous at x_0. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \boldsymbol{R}$ is also continuous at x_0.*
- *If f and g are continuous, then the functions $f+g : X \to \boldsymbol{R}$, $f-g : X \to \boldsymbol{R}$, $fg : X \to \boldsymbol{R}$, $max(f, g) \to \boldsymbol{R}$, $min(f, g) \to \boldsymbol{R}$, and $cf : X \to \boldsymbol{R}$ are also continuous. If $g(x) \neq 0$ for all $x \in X$, then $f/g : X \to \boldsymbol{R}$ is also continuous.*

13.3 Continuity and compactness

Theorem 13.3. *(Continuous map preserve compactness) Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (y, d_Y). Let $K \subseteq X$ be any compact subset of X. Then the image $f(K) := \{f(x) : x \in K\}$ of K is also compact.*

Proposition 13.3.1. *(Maximum principle) Let (X, d) be compact metric space, and let $f : X \to \boldsymbol{R}$ be a continuous function. Then f is bounded. Furthermore, f attains its maximum at some point $x_{max} \in X$, and also attains its minimum at some point $x_{min} \in X$.*

Definition 13.3.1. *(Uniform continuity) Let $f : X \to Y$ be a map from one metric space (X, d_X) to another (Y, d_Y). We say that f is uniformly continuous if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $x, x' \in X$ are such that $d_X(x, x') < delta$.*

Theorem 13.4. *Let (X, d_X) and (Y, d_Y) be metric spaces, and suppose that (X, d_X) is compact. If $f : X \to Y$ is function, then f is continuous iff it is uniformly continuous.*

13.4 Continuity and connectedness

Definition 13.4.1. *(Connected spaces) Let (X, d) be a metric space. We say that X is disconnected iff there exists disjoint non-empty open sets V and W in X such that $V \cup W = X$. Equivalently, X is disconnected iff X contains a non-empty subset which is simultaneously closed and open. We say that X is connected iff it is non-empty and not disconnected.*

The empty set is neither connected nor disconnected.

Definition 13.4.2. *(Connected sets) Let (X, d) be a metric space, and let Y be a subset of X. We say that Y is connected iff the metric space $(Y, d|_{Y \times Y})$ is connected, and we say that Y is disconnected iff the metric space $(Y, d|_{YtimesY})$ is disconnected.*

Theorem 13.5. *Let X be a subset of the real line $\boldsymbol{R}$. Then the following statements are equivalent.*

- *X is connected.*

- *Whenever $x, y \in X$ and $x < y$, the interval $[x, y]$ is also contained in X.*

- *X is an interval.*

Theorem 13.6. *(Continuity preserves connectedness) Let $f : X \to Y$ be a continuous map from one metric space (X, d_X) to another (Y, d_Y). Let E be any connected set of X. Then $f(E)$ is also connected.*

Corollary 13.4.1. *(Intermediate value theorem) Let $f : X \to \mathbf{R}$ be a continuous map from one metric space (X, d_X) to the real line. Let E be any connected subset of X, and let a, b be any two elements of E. Let y be a real number between $f(a)$ and $f(b)$, i.e., either $f(a) \leq y \leq f(b)$ or $f(a) \geq y \geq f(b)$. Then there exists $c \in E$ such that $f(c) = y$.*

13.5 Topological spaces

The concept of a metric space can be generalized to that of a topological space, where there is no metric at all but only collection of open sets. Starting from open sets, one cannot necessarily reconstruct a usable notion of a ball or metric, but not all topological spaces will be metric spaces, but remarkably on can still define many of the concepts in the preceding sections.

Definition 13.5.1. *(Topological spaces) A topological space is a pair $(X, \mathcal{F})$, where X is a set, and $\mathcal{F} \subset 2^X$ is a collection of subsets of X, whose elements are referred to as open sets, with the collection $\mathcal{F}$ obeying the following three properties:*

1. *The empty set $\emptyset$ and the whole set X are open, i.e $\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$.*

2. *Any finite intersection of open sets is open, i.e. if $V_1, \ldots, V_n$ are elements of $\mathcal{F}$, then $V_1 \cap \ldots \cap V_n$ is also in $\mathcal{F}$.*

3. *Any arbitrary union of open sets is open, including infinite unions, i.e., if $(V_\alpha)_{\alpha \in I}$ is a family of sets in $\mathcal{F}$, then $\bigcup_{\alpha \in I} V_\alpha$ is also in $\mathcal{F}$.*

Every metric space (X, d) is automatically also a topological space, if we set $\mathcal{F}$ equal to the collection of sets which are open in (X, d). However, there do exist topological spaces which do not arise from metric spaces. The notion of ball is replaced by the notion of a neighbourhood.

Definition 13.5.2. *(Neighbourhoods) Let $(X, \mathcal{F})$ be a topological space, and let $x \in X$. A neighbourhood of x is defined to be any open set in $\mathcal{F}$ which contains x.*

Definition 13.5.3. *(Topological convergence) Let m be an integer, $(X, \mathcal{F})$ be a topological space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X. Let x be a point in X. We say that $(x^{(n)})_{n=m}^{\infty}$ converges to x iff, for every neighbourhood V of x, there exists an $N \geq m$ such that $x^{(n)} \in V$ for all $n \geq N$.*

If the topological space has an additional property - Hausdorff property: for every pair of points in the set, there are disjoint open sets that contain the points - it also has the property of uniqueness of limits.

Definition 13.5.4. *(Interior, exterior, boundary) Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an interior point of E if there exists a neighbourhood V of x_0 such that $V \subseteq E$. We say that x_0 is an exterior point of E if there exists a neighbourhood V of x_0 such that $V \cap E = \emptyset$. We say that x_0 is a boundary point of E if it is neither an interior point nor an exterior point of E.*

Definition 13.5.5. *(Closure) Let $(X, \mathcal{F})$ be a topological space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an adherent point of E if every neighbourhood V of x_0 has a non-empty intersection with E. The set of all adherent points of E is called the closure of E and is denoted by $\overline{E}$.*

We define a set K in a topological space $(X, \mathcal{F})$ to be closed iff its complement $X \backslash K$ is open.

Definition 13.5.6. *(Relative topology) Let $(X, \mathcal{F})$ be a topological space, and Y be a subset of X. Then we define $\mathcal{F}_Y := \{V \cap Y : V \in \mathcal{F}\}$, and refer this as the topology on Y induced by $(X, \mathcal{F})$. We call $(Y, \mathcal{F}_Y)$ a topological subspace of (X, F). This is indeed a topological space.*

Definition 13.5.7. *(Continuous functions) Let $(X, \mathcal{F}_X)$ and $(Y, \mathcal{F}_Y)$ be topological spaces, and let $f : X \to Y$ be a function. If $x_0 \in X$, we say that f is continuous at x_0 iff for every neighbourhood V of $f(x_0)$, there exists a neighbourhood U of x_0 such that $f(U) \subseteq V$. We say that f is continuous iff it is continuous at every point $x \in X$.*

There is no notion of a Cauchy sequence, a complete space, or a bounded space, for topological spaces. Neither is there any notion of uniform continuity in topological space.

Definition 13.5.8. *(Compact topological spaces) Let $(X, \mathcal{F})$ be a topological space. We say that this space is compact if every open cover of X has a finite subcover. If Y is a subset of X, we say that Y is compact if the topological space on Y induced by $(X, \mathcal{F})$ is compact.*

The notion of connectedness holds verbatim as before.

14 Uniform convergence

14.1 Limiting values of functions

Definition 14.1.1. *(Limiting value of a function) Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. If $x_0 \in X$ is an adherent point of E, and $L \in Y$, we say that $f(x)$ converges to L in Y as x converges to x_0 in E, or write $\lim_{x \to x_0; x \in E} f(x) = L$, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(x), L) < \varepsilon$ for all $x \in E$ such that $d_X(x, x_0) < \delta$.*

Thus f is continuous at x_0 iff $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$ and f is continuous on X iff we have $\lim_{x \to x_0; x \in X} f(x) = f(x_0)$ for all $x_0 \in X$. We can rephrase it in terms of sequences as well.

Proposition 14.1.1. *Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, and let $f : X \to Y$ be a function. Let $x_0 \in X$ be an adherent point of E and $L \in Y$. Then the following four statements are logically equivalent:*

- $\lim_{x \to x_0; x \in E} f(x) = L$.
- *For every sequence $(x^{(n)})_{n=1}^\infty$ in E which converges to x_0 with respect to the metric d_X, the sequence $(f(x^{(n)}))_{n=1}^\infty$ converges to L with respect to the metric d_Y.*
- *For every open set $V \subset Y$ which contains L, there exists an open set $U \subset X$ containing x_0 such that $f(U \cap E) \subseteq V$.*
- *If one defines the function $g : E \cup \{x_0\} \to Y$ by defining $g(x_0) := L$, and $g(x) := f(x)$ for $x \in E \backslash \{x_0\}$ then g is continuous at x_0. Furthermore, if $x_0 \in E$, then $f(x_0) = L$.*

14.2 Pointwise and uniform convergence

Definition 14.2.1. *(Pointwise convergence) Let $(f^{(n)})_{n=1}^\infty$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^\infty$ converges pointwise to f on X if we have $\lim_{n \to} f^{(n)}(x) = f(x)$ for all $x \in X$, i.e. $\lim_{n \to \infty} d_Y(f^{(n)}(x), f(x)) = 0$. We call the function f the pointwise limit of the functions $f^{(n)}$.*

We do not use the fact that (X, d_X) is a metric space for this definition. Pointwise convergence does not conserve continuity, derivatives, limits, or integrals. For example the functions $f^{(n)} L[0,1] \to \boldsymbol{R}$ defined by $f^{(n)}(x) = x^n$ converge to $f : [0,1] \to \boldsymbol{R}$ with $f(x) := \begin{cases} 1 & x = 1 \\ 0 & 0 \le x < 1 \end{cases}$. However the functions $f^{(n)}$ are continuous while the function $f(x)$ is not. It also does not preserve limits, in fact $\lim_{n \to \infty} \lim_{x \to x_0; x_0 \in X} f^{(n)}(x) \neq \lim_{x \to x_0; x \in X} \lim_{n \to \infty} f^{(n)}(x)$, e.g. for $x_0 = 1$. Further for $[a,b] := [0,1]$ and $f^{(n)}(x) := \begin{cases} 2n & x \in [1/2n, 1/n] \\ 0 & otherwise \end{cases}$, then $f^{(n)}$ converges pointwise to the zero function $f(x) := 0$. On the other hand $\int_{[0,1]} f^{(n)} = 1$ for every n, while $\int_{[0,1]} f = 0$. Thus,

$\lim\limits_{n\to\infty} \int\limits_{[a,b]} f^{(n)} \neq \int\limits_{[0,1]} \lim\limits_{n\to\infty} f^{(n)}$.

The examples show that pointwise convergence is too weak a concept to be of much use. While $f^{(n)}$ converges to $f(x)$ for each x, the rate of that convergence varies substantially with x. To put in another way, the convergence of $f^{(n)}$ to f is not uniform in x - the N that one needs to get $f^{(n)}(x)$ within ε of f depends on x as well as on ε.

Definition 14.2.2. *(Uniform convergence) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function. We say that $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f on X if for every $\varepsilon > 0$ there exists $N > 0$ such that $d_Y(f^{(n)}, f(x)) < \varepsilon$ for every $n > N$ and $x \in X$. We call the function f the uniform limit of the function $f^{(n)}$.*

Notice that N no longer depends on x. If $f^{(n)}$ converges uniformly to f on X, then it also converges pointwise to the same function f. If a sequence $f^{(n)} : X \to Y$ of functions converges pointwise (or uniformly) to a function $f : X \to Y$, then the restrictions $f^{(n)}|_E : E \to Y$ of $f^{(n)}$ to some subset E of X will also converge pointwise (or uniformly) to $f|_Y$.

14.3 Uniform convergence and continuity

Theorem 14.1. *(Uniform limits preserves continuity) Suppose $(f^{(n)})_{n=1}^{\infty}$ is a sequence of functions from one metric space (X, d_X) to another (Y, d_Y) and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let x_0 be a point in X. If the functions $f^{(n)}$ are continuous at x_0 for each n, then the limiting function f is also continuous at x_0. Further, if the functions $f^{(n)}$ are continuous on X for each n, then the limiting function f is also continuous on X.*

Proposition 14.3.1. *(Interchange of limits and uniform limits) Let (X, d_X) and (Y, d_Y) be metric spaces, with Y complete, and let E be a subset of X. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from E to Y, and suppose that this sequence converges uniformly in E to some function $f : E \to Y$. Let $x_0 \in X$ be an adherent point of E, and suppose that for each n the limit $\lim\limits_{x\to x_0; x\in E} f^{(n)}(x)$ exists. Then the limit $\lim\limits_{x\to x_0; x\in E} f(x)$ also exists, and is equal to the limit fo the sequence $(\lim\limits_{x\to x_0; x\in E} f^{(n)}(x))_{n=1}^{\infty}$, i.e. we have the interchange of limits*

$$\lim_{n\to\infty} \lim_{x\to x_0; x\in E} f^{(n)}(x) = \lim_{x\to x_0; x\in E} \lim_{n\to\infty} f^{(n)}(x)$$

Proposition 14.3.2. *Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of continuous functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. Let $x^{(n)}$ be a sequence of points in X which converges to some limit x. Then $f^{(n)}(x^{(n)})$ converges in Y to $f(x)$.*

Definition 14.3.1. *(Bounded functions) A function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y) is bounded if $f(X)$ is a bounded set, i.e. there exists are ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$.*

Proposition 14.3.3. *(Uniform limits preserve boundedness) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and suppose that this sequence converges uniformly to another function $f : X \to Y$. If the functions $f^{(n)}$ are bounded on X for each n, then the limiting function f is also bounded on X.*

14.4 The metric of uniform convergence

The four notions of limits (1) $\lim\limits_{n\to\infty} x^{(n)}$ of sequences of points in a metric space, (2) $\lim\limits_{x\to x_0; x\in E} f(x)$ of functions at a point, (3) pointwise limit of f of functions $f^{(n)}$, and (4) uniform limits f of functions $f^{(n)}$, can be unified using the notion of convergence in topological spaces.

Definition 14.4.1. *(Metric space of bounded functions) Suppose (X, d_X) and (Y, d_Y) are metric spaces. We let $B(X \to Y)$ denote the space of bounded functions from X to Y: $B(X \to Y) := \{f | f : X \to Y$ is a bounded function We define the metric $d_\infty : B(X \to Y) \times B(X \to Y) \to \mathbf{R}^+$ by defining $d_\infty(f, g) := \sup\limits_{x\in X} d_Y(f(x), g(x)) =$ $\sup\{d_Y(f(x), g(x)) : x \in X\}$ for all $f, g \in B(X \to Y)$. This metric is sometimes know as the sup norm metric or the L^∞ metric, also denoted by $d_{B(X\to Y)}$.*

Proposition 14.4.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions in $B(X \to Y)$, and let f be another function in $B(X \to Y)$. Then $(f^{(n)})_{n=1}^{\infty}$ converges to f in the metric $d_{B(X\to Y)}$ iff $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f.*

Theorem 14.2. *(The space of continuous functions is complete) Let (X, d_X) is a metric space, and let (Y, d_Y) be a complete metric space. Let $C(X \to Y)$ be the space of bounded continuous functions form X to Y, i.e., $C(X \to Y) := \{f \in B(X \to Y) | f$ is continuous$\}$. The space $(C(X \to Y), d_{B(X\to Y)})$ is a complete subspace of $(B(X \to Y), d_{B(X\to Y)})$. In other words, every Cauchy sequence of functions in $C(X \to Y)$ converges to a function in $C(X \to Y)$.*

14.5 Series of functions; the Weierstrass M-test

Functions whose range is $\boldsymbol{R}$ are called real-valued functions. Given any finite collection $f^{(1)}, \dots, f^{(N)}$ of functions from X to $\boldsymbol{R}$, we can define the finite sum $\sum_{i=1}^{N} f^{(i)} : X \to \boldsymbol{R}$ by $\left(\sum_{i=1}^{N} f^{(i)}\right)(x) := \sum_{i=1}^{N} f^{(i)}(x)$. Finite sums of bounded functions are bounded, and finite sums of continuous functions are continuous.

Definition 14.5.1. *(Infinite series) Let (X, d_X) be a metric space. Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from X to $\boldsymbol{R}$, and let f be another function from X to $\boldsymbol{R}$. If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converge pointwise to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ converges pointwise to f, and write $f = \sum_{n=1}^{\infty} f^{(n)}$. If the partial sums $\sum_{n=1}^{N} f^{(n)}$ converge uniformly to f on X as $N \to \infty$, we say that the infinite series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to f, and write $f = \sum_{n=1}^{\infty} f^{(n)}$.*

For the sequence of functions $f^{(n)} : (-1, 1) \to \boldsymbol{R}$ $f^{(n)}(x) := x^n$, the sum $\sum_{n=1}^{\infty} x^n$ converges pointwise, but not uniformly to the function $x/(1-x)$.

Definition 14.5.2. *(Sup norm) If $f : X \to \boldsymbol{R}$ is abounded real-valued function, we define the sup norm $\|f\|_\infty$ of f to be the number $\|f\|_\infty := \sup\{|f(x)| : x \in X\} = d_\infty(f, 0)$.*

Theorem 14.3. *(Weierstrass M-test) Let (X, d) be a metric space, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of bounded real-valued continuous functions on X such that the series $\sum_{n=1}^{\infty} \|f^{(n)}\|_\infty$ is convergent. Then the series $\sum_{n=1}^{\infty} f^{(n)}$ converges uniformly to some function f on X, and that function f is also continuous.*

Succinctly, absolute convergence of sup norms implies uniform convergence of functions. The series $\sum_{n=1}^{\infty} x^n$ is pointwise convergent, but not uniformly convergent on $(-1, 1)$, but is uniformly convergent on the small interval $[-r, r]$ for any $0 < r < 1$.

14.6 Uniform convergence and integration

Theorem 14.4. *Let $[a, b]$ be an interval, and for each integer $n \geq 1$, let $f^{(n)} : [a, b] \to \boldsymbol{R}$ be a Riemann-integrable function. Suppose $f^{(n)}$ converges uniformly on $[a, b]$ to a function $f : [a, b] \to \boldsymbol{R}$. Then f is also Riemann integrable, and $\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} f$.*

This implies we can rearrange limits and integrals on compact intervals $[a, b]$, $\lim_{n\to\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \lim_{n\to\infty} f^{(n)}$, provided that the convergence is uniform.

Corollary 14.6.1. *Let $[a, b]$ be an interval, and let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a, b]$ such that the series $\sum_{n=1}^{\infty} f^{(n)}$ is uniformly convergent. Then we have $\sum_{n=1}^{\infty} \int_{[a,b]} f^{(n)} = \int_{[a,b]} \sum_{n=1}^{\infty} f^{(n)}$*

We have the geometric series identity $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ for $x \in (-1, 1)$, and the convergence is uniform by the Weierstrass M-test. Adding 1 on both sides and integrating on $[0, r]$ and using the last corollary we obtain $\sum_{n=0}^{\infty} \int_{[0,r]} x^n dx = \int_{[0,r]} \frac{1}{1-x} dx$. This gives the expression $-\log(1-r) = \sum_{n=0}^{\infty} \frac{r^{n+1}}{n+1}$ for all $0 < r < 1$.

14.7 Uniform convergence and derivatives

We can ask: if f_n converges uniformly to f, and the functions f_n are differentiable, does this imply that f is also differentiable? And does f_n' also converge to f'? The answer to both, unfortunately, is no. For example $f_n : [0, 2\pi] \to \boldsymbol{R}$ defined by $f_n(x) = \frac{sin(nx)}{\sqrt{n}}$ uniformly converges to 0. On the other hand, $f_n'(x) = \sqrt{n}cos(nx)$ and so f_n' does not converge pointwise to f', and so in particular does not converge uniformly either. In particular we have $\frac{d}{dx}\lim\limits_{n\to\infty} \neq \lim\limits_{n\to\infty} \frac{d}{dx}f_n(x)$. Uniform convergence of the functions f_n says nothing about the convergence of the derivative f_n'. However, the converse is true, as long as f_n converges at at least one point.

Theorem 14.5. *Let $[a, b]$ be an integral, and for every integer $n \geq 1$, let $f_n : [a, b] \to \boldsymbol{R}$ be a differentiable function whose derivative $f_n' : [a, b] \to \boldsymbol{R}$ is continuous. Suppose that the derivatives f_n' converge uniformly to a function $g : [a, b] \to \boldsymbol{R}$. Suppose also that there exists a point x_0 such that the limit $\lim\limits_{n\to\infty} f_n(x_0)$ exists. Then the functions f_n converge uniformly to a differentiable function f, and the derivative of f equals g.*

That is, if f_n' converges uniformly, and $f_n(x_0)$ converges for some x_0, then f_n also converges uniformly, and $\frac{d}{dx}\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{d}{dx}f_n(x)$.

Corollary 14.7.1. *Let $[a, b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a, b] \to \boldsymbol{R}$ be a differentiable function whose derivative $f_n' : [a, b] \to \boldsymbol{R}$ is continuous. Suppose that the series $\sum\limits_{n=1}^{\infty} \|f_n'\|_\infty$ is absolutely convergent, where $\|f_n'\|_\infty := \sup\limits_{x\in[a,b]} |f_n'(x)|$ is the sup norm of f_n'. Suppose also that the series $\sum\limits_{n=1}^{\infty} f_n(x_0)$ is convergent for some $x_0 \in [a, b]$. Then the series $\sum\limits_{n=}^{\infty} f_n$ converges uniformly on $[a, b]$ to a differentiable function, and in fact $\frac{d}{dx}\sum\limits_{n=1}^{\infty} f_n(x) = \sum\limits_{n=1}^{\infty} \frac{d}{dx}f_n(x)$ for all $x \in [a, b]$.*

Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be the function $f(x) := \sum\limits_{n=1}^{\infty} 4^{-n}cos(32^n\pi x)$. This series is uniformly convergent and so is the function f. However it is not differentiable nowhere!

14.8 Uniform approximation by polynomials

Fortunately, while most continuous functions are not as well behaved as polynomials, they can always be uniformly approximated by polynomials. This is known as **Weierstrass approximation theorem**.

Definition 14.8.1. *Let $[a, b]$ be an interval. A polynomial on $[a, b]$ is a function $f : [a, b] \to \boldsymbol{R}$ of the form $f(x) := \sum\limits_{j=0}^{n} c_jx^j$, where $n \geq 0$ is an integer and $c_0, \dots, c_n$ are real numbers. If $c_n \neq 0$, then n is called the degree of f.*

Theorem 14.6. *(Weierstrass approximation theorem) If $[a, b]$ is an interval, $f : [a, b] \to \boldsymbol{R}$ is a continuous function, and $\varepsilon > 0$, then there exists a polynomial P on $[a, b]$ such that $d_\infty(P, f) \leq \varepsilon$, i.e. $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$.*

This implies that the closure of the space of polynomials is the space of continuous functions: $\overline{P([a, b] \to \boldsymbol{R})} = C([a, b] \to \boldsymbol{R})$. In other words the space of polynomials is dense in the space of continuous functions, in the uniform topology.

Definition 14.8.2. *(Compactly supported functions) Let $[a, b]$ be an interval. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is said to be supported on $[a, b]$ iff $f(x) = 0$ for all $x \notin [a, b]$. We say that f is compactly supported iff it is supported on some interval $[a, b]$. If f is continuous and supported on $[a, b]$, we define the improper integral $\int\limits_{-\infty}^{\infty} f$ to be $\int\limits_{-\infty}^{\infty} f := \int\limits_{[a,b]} f$.*

Lemma 14.8.1. *If $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous and supported on an interval $[a, b]$, and is also supported on another interval $[c, d]$, then $\int\limits_{[a,b]} f = \int\limits_{[c,d]} f$.*

Definition 14.8.3. *(Approximation to the identity) Let $\varepsilon > 0$ and $0 < \delta < 1$. A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is said to be an (ε, δ)-approximation to the identity if it obeys the following three properties:*

1. *f is supported on $[-1, 1]$, and $f(x) \geq 0$ for all $-1 \leq x \leq 1$.*
2. *f is continuous, and $\int\limits_{-\infty}^{\infty} f = 1$.*
3. *$|f(x)| \leq \varepsilon$ for all $\delta \leq |x| \leq 1$.*

Approximations to the identity are ways to approximation **Dirac delta** function by a continuous function.

Lemma 14.8.2. *(Polynomials can approximate the identity). For every $\varepsilon > 0$ and $0 < \delta < 1$ there exists an (ε, δ)-approximation to the identity which is a polynomial P on $[-1, 1]$.*

Definition 14.8.4. *(Convolution) Let $f : \boldsymbol{R} \to \boldsymbol{R}$ and $g : \boldsymbol{R} \to \boldsymbol{R}$ be continuous, compactly supported functions. We define the convolution $f * g : \boldsymbol{R} \to \boldsymbol{R}$ of f and g to be the function $(f * g)(x) := \int\limits_{-\infty}^{\infty} f(y)g(x - y)dy$.*

Proposition 14.8.1. *(Basic properties of convolution) Let $f : \boldsymbol{R} \to \boldsymbol{R}$, $g : \boldsymbol{R} \to \boldsymbol{R}$, and $h : \boldsymbol{R} \to \boldsymbol{R}$ be continuous, compactly supported functions. Then the following statements are true.*

- *The convolution $f * g$ is also a continuous, compactly supported function.*
- *(Convolution is commutative) We have $f * g = g * f$; in other words $f * g(x) = \int\limits_{-\infty}^{\infty} f(y)g(x - y)dy = \int\limits_{-\infty}^{\infty} g(y)f(x - y)dy = g * f(x)$.*
- *(Convolution is linear) We have $f * (g + h) = f * g + f * h$. Also, for any real number c, we have $f * (cg) = (cf) * g = c(f * g)$.*

Convolution is associative $(f * g) * h = f * (g * h)$, and it commutes with derivatives, $(f * g)' = g' * g = f * g'$, when f and g are differentiable. The Dirac delta function δ is an identity for convolution $f * \delta = \delta * f = f$.

Lemma 14.8.3. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function supported on $[0, 1]$, and let $g : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function supported on $[-1, 1]$ which is a polynomial on $[-1, 1]$. Then $f * g$ is a polynomial on $[0, 1]$. It may be a non-polynomial outside of $[0, 1]$.*

Lemma 14.8.4. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function supported on $[0, 1]$, which is bounded by some $M > 0$, i.e., $|f(x)| \leq M$ for all $x \in \boldsymbol{R}$, and let $\varepsilon > 0$ and $0 < \delta < 1$ be such that one has $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \boldsymbol{R}$ and $|x - y| < \delta$. Let g be any (ε, δ)-approximation to the identity. Then we have $|f * g(x) - f(x)| \leq (1 + 4M)\varepsilon$ for all $x \in [0, 1]$.*

Corollary 14.8.1. *(Weierstrass approximation theorem I) Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a continuous function supported on $[0, 1]$. Then for every $\varepsilon > 0$, there exists a function $P : \boldsymbol{R} \to \boldsymbol{R}$ which is polynomial on $[0, 1]$ and such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Lemma 14.8.5. *Let $f : [0, 1] \to \boldsymbol{R}$ be a continuous function which equals 0 on the boundary of $[0, 1]$, i.e. $f(0) = f(1) = 0$. Let $F : \boldsymbol{R} \to \boldsymbol{R}$ be the function defined by setting $F(x) := f(x)$ for $x \in [0, 1]$ and $F(x) := 0$ for $x \notin [0, 1]$. Then F is also continuous. This is known as* ***extension of f by zero****.*

Corollary 14.8.2. *(Weierstrass approximation theorem II) Let $f : [0, 1] \to \boldsymbol{R}$ be a continuous function supported on $[0, 1]$ such that $f(0) = f(1) = 0$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \boldsymbol{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Corollary 14.8.3. *(Weierstrass approximation theorem III) Let $f : [0, 1] \to \boldsymbol{R}$ be a continuous function supported on $[0, 1]$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0, 1] \to \boldsymbol{R}$ such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0, 1]$.*

Weierstrass approximation theorem only works on bounded intervals $[a, b]$; continuous functions on $\boldsymbol{R}$ cannot be uniformly approximated by polynomials. There is a generalization of the Weierstrass approximation theorem to higher dimensions: if K is any compact subset of $\boldsymbol{R}^n$, with the Euclidean metric d_{l^2}, and $f : K \to \boldsymbol{R}$ is a continuous function, then for every $\varepsilon > 0$ there exists a polynomial $P : K\boldsymbol{R}$ on n variables $x_1, \ldots, x_n$ such that $d_\infty(f, P) < \varepsilon$. An even more general version of this theorem applicable to an arbitrary metric space, is known as *Stone-Weierstrass theorem.*

15 Power series

15.1 Formal power series

Definition 15.1.1. *(Formal power series) Let a be a real number. A formal power series centered at a is any series of the form $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ where $c_0, c_1, \ldots$ is a sequence of real numbers, not depending on x. We refer to c_n as the nth coefficient of this series.*

These series automatically converge for $x = a$. In general, the closer x gets to a, the easier it is for this series to converge.

Definition 15.1.2. *(Radius of convergence) Let $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series. We define the radius of convergence R of this series to be the quantity $R := (\limsup\limits_{n\to\infty} |c_n|^{1/n})^{-1}$ where we adopt the convention that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$.*

Theorem 15.1. *Let $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence.*

- *Divergence outside of the radius of convergence: If $x \in \boldsymbol{R}$ is such that $|x-a| > R$, then the series is divergent for that value of x.*

- *Convergence inside the radius of convergence: If $x \in \boldsymbol{R}$ is such that $|x-a| < R$, then the series is absolutely convergent for that value of x.*

Let $R > 0$ and $f : (a-R, a+R)\boldsymbol{R}$ be the function $f(x) := \sum\limits_{n=0}^{\infty} c_n(x-a)^n$; this function is guaranteed to exist by (b).

- *Uniform convergence on compact sets: For any $0 < r < R$, the series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly to f on the compact interval $[a-r, a+r]$. In particular, f is continuous on $(a-R, a+R)$.*

- *Differentiation of power series: The function f is differentiable on $(a-R, a+R)$, and for any $0 < r < R$, the series $\sum\limits_{n=1}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to f' on the interval $[a-r, a+r]$.*

- *Integration of power series: For any closed interval $[y, z]$ contained in $(a-R, a+R)$, we have $\int\limits_{[y,z]} f = \sum\limits_{n=0}^{\infty} c_n \frac{1}{n+1}((z-a)^{n+1} - (y-a)^{n+1})$.*

The theorem assures us that the power series will converge pointwise on the interval $(a-R, a+R)$, it need not converge uniformly on that interval. However, part (c) assures us that the power series will converge on any smaller interval $[a-r, a+r]$.

15.2 Real analytic functions

Definition 15.2.1. *(Real analytic functions) Let E be a subset of $\boldsymbol{R}$, and let $f : E \to \boldsymbol{R}$ be a function. If a is an interior point of E, we say that f is real analytic at a if there exists an open interval $(a-r, a+r)$ in E for some $r > 0$ such that there exists an open interval $(a-r, a+r)$ in E for some $r > 0$ such that there exists a power series $\sum\limits_{n=0}^{\infty} c_n(x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a-r, a+r)$. If E is an open set, and f is real analytic at every point a of E, we say that f is real analytic on E.*

Definition 15.2.2. *(k-times differentiability) Let E be a subset of $\boldsymbol{R}$. We say a function $f : E \to \boldsymbol{R}$ is once differentiable on E iff it is differentiable. More generally, for any $k \geq 2$ we say that $f : E \to \boldsymbol{R}$ is k times differentiable on E, or just k times differentiable, iff f is differentiable, and f' is $k-1$ times differentiable. If f is k times differentiable, we define the kth derivative $f^{(k)} : E \to \boldsymbol{R}$ by the recursive rule $f^{(1)} := f'$, and $f^{(k)} := (f^{(k-1)})'$ for all $k \geq 2$. We also define $f^{(0)} := f$, and we allow every function to be zero times differentiable. A function is said to be infinitely differentiable, or smooth iff it is k times differentiable for every $k \geq 0$.*

Proposition 15.2.1. *(Real analytic functions are k-times differentiable) Let E be a subset of $\boldsymbol{R}$, let a be an interior point of E, and let f be a function which is real analytic at a, thus there is an $r > 0$ for which we have the power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-r, a+r)$. Then for every $k \geq 0$, the function f is k-times differentiable on $(a-r, a+r)$, and for each $k \geq 0$ the kth derivative is given by $f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\ldots(n+k)(x-a)^n = \sum_{n=0}^{\infty} c_{n+k}\frac{(n+k)!}{n!}(x-a)^n$ for all $x \in (a-r, a+r)$.*

Corollary 15.2.1. *(Real analytical functions are infinitely differentiable) Let E be an open subset of $\boldsymbol{R}$, and let $f : E \to \boldsymbol{R}$ be a real analytic function on E. Then f is infinitely differentiable on E. Also, all derivatives of f are also real analytic on E.*

The converse of this is not true. There are infinitely differentiable functions which are not real analytic.

Corollary 15.2.2. *(Taylor's formula) Let E be a subset of $\boldsymbol{R}$, let a be an interior point of E, and let $f : E \to \boldsymbol{R}$ be a function which is real analytic at a and has the power series expansion $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all $x \in (a-r, a+r)$ and some $r > 0$. Then for any integer $k \geq 0$, we have $f^{(k)}(a) = k!c_k$, where $k! = 1 \times 2 \times \ldots \times k$, and we adopt the convention that $0! = 1$. In particular we have Taylor's formula $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ for all x in $(a-r, a+r)$.*

Corollary 15.2.3. *(Uniqueness of power series) Let E be subset of $\boldsymbol{R}$, let a be an interior point of E, and let $f : E \to \boldsymbol{R}$ be a function which is real analytic at a. Suppose that f has two power series expansions $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and*

$$f(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$$

centered at a, each with a non-zero radius of convergence. Then $c_n = d_n$ for all $n \geq 0$.

Real analytic function can certainly have different power series at different points.

15.3 Abel's theorem

Theorem 15.2. *(Abel's formula) Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a with radius of convergence $0 < R < \infty$. If the power series converges at $a+R$, then f is continuous at $a+R$, i.e. $\lim_{x \to a+R; x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n$. Similarly, if the power series converges at $a-R$, then f is continuous at $a-R$, i.e. $\lim_{x \to a-R; x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n(-R)^n$*

Lemma 15.3.1. *(Summation by parts formula) Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers which converge to limits A and B respectively, i.e., $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose that the sum $\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n$ is convergent. Then the sum $\sum_{n=0}^{\infty} a_{n+1}(b_{n+1}-b_n)$ is also convergent, and $\sum_{n=0}^{\infty}(a_{n+1}-a_n)b_n = AB - a_0b_0 - \sum_{n=0}^{\infty} a_{n+1}(b_{n+1}-b_n)$.*

This should be compared with the more well known integration by parts formula.

15.4 Multiplication of power series

Theorem 15.3. *Let $f : (a-r, a+r) \to \boldsymbol{R}$ and $g : (a-r, a+r) \to \boldsymbol{R}$ be functions analytic on $(a-r, a+r)$, with power series expansions $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ and $g(x) = \sum_{n=0}^{\infty} d_n(x-a)^n$ respectively. Then $fg : (a-r, a+r) \to \boldsymbol{R}$ is also analytic on $(a-r, a+r)$, with power series expansion $f(x)g(x) = \sum_{n=0}^{\infty} e_n(x-a)^n$ where $e_n := \sum_{m=0}^{n} e_m d_{n-m}$.*

$(e_n)_{n=0}^{\infty}$ is referred to as the convolution of the sequences $(c_n)_{n=0}^{\infty}$ and $(d_n)_{n=0}^{\infty}$.

15.5 The exponential and logarithm functions

Definition 15.5.1. *(Exponential function) For every real number x, we define the exponential function $exp(x)$ to be the real number* $\exp x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Theorem 15.4. *(Basic properties of exponential)*

1. *For every real number x, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. In particular, $exp(x)$ exists and is real for every $x \in \boldsymbol{R}$, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence, and exp is a real analytic function on $(-\infty, \infty)$.*
2. *exp is differentiable on $\boldsymbol{R}$, and for every $x \in \boldsymbol{R}$, $exp'(x) = exp(x)$.*
3. *exp is continuous on $\boldsymbol{R}$, and for every interval $[a, b]$, we have $\int_{[a,b]} exp(x)dx = exp(b) - exp(a)$.*
4. *For every $x, y \in \boldsymbol{R}$, we have* $\exp(x + y) = exp(x)exp(y)$.
5. *We have $exp(0) = 1$. Also, for every $x \in \boldsymbol{R}$, $exp(x)$ is positive, and $exp(-x) = \frac{1}{exp(x)}$.*
6. *exp is strictly monotone increasing: in other words, if x, y are real numbers, then we have $exp(y) > exp(x)$ iff $y > x$.*

One can write the exponential function in a more compact form, introducing famous **Euler's number** $e = 2.71828183\ldots = exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$, also known as the base of the natural logarithm. So that for every real number x we have $exp(x) = e^x$.

Definition 15.5.2. *(Logarithm) We define the natural logarithm function $log : (0, \infty) \to \boldsymbol{R}$, also called ln, to be the inverse of the exponential function. Thus $exp(log(x)) = x$ and $log(exp(x)) = x$.*

Theorem 15.5. *(Logarithm properties)*

1. *For every $x \in (0, \infty)$, we have $ln'(x) = \frac{1}{x}$. In particular, by the fundamental theorem of calculus, we have $\int_{[a,b]} \frac{1}{x}dx = ln(b) - ln(a)$ for any interval $[a, b]$ in $(0, \infty)$.*
2. *We have $ln(xy) = ln(x) + ln(y)$ for all $x, y \in (0, \infty)$.*
3. *We have $ln(1) = 0$ and $ln(1/x) = -ln(x)$ for all $x \in (0, \infty)$.*
4. *For any $x \in (0, \infty)$ and $y \in \boldsymbol{R}$, we have $ln(x^y) = yln(x)$.*
5. *For any $x \in (-1, 1)$, we have $ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$. In particular, ln is analytic at 1, with the power series expansion $ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$ for $x \in (0, 2)$, with radius of convergence 1.*

15.6 A digression on complex numbers

Informally, complex numbers $\boldsymbol{C}$ are the set of all numbers of the form $a + ib$, where a, b are real numbers, and i is a square root of -1, $i^2 = -1$. To be rigorous, we introduce and develop it formally.

Definition 15.6.1. *A complex number is any pair of the form (a, b), where a, b are real numbers. Two complex numbers $(a, b), (c, d)$ are said to be equal iff $a = c$ and $b = d$. The set of all complex numbers is denoted by $\boldsymbol{C}$.*

At this stage one can think of complex number system $\boldsymbol{C}$ as the Cartesian plane $\boldsymbol{R}^2$ equipped with a number of additional structures.

Definition 15.6.2. *(Complex addition, negation, and zero) If $z = (a, b)$ and $w = (c, d)$ are two complex numbers, we define their sum $z + w$ to be the complex number $z + w := (a + c, b + d)$. We also define the negation $-z$ of z to be the complex number $-z := (-a, -b)$. The complex zero $0_{\boldsymbol{C}}$ is the complex number $(0, 0)$.*

Lemma 15.6.1. *(The complex numbers are an additive group) If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1 + z_2 = z_2 + z_1$, the associative property $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$, the identity property $z_1 + 0_{\boldsymbol{C}} = 0_{\boldsymbol{C}} + z_1 = z_1$, and the inverse property $z_1 + (-z_1) = (-z_1) + z_1 = 0_{\boldsymbol{C}}$.*

Definition 15.6.3. *(Complex multiplication) If $z = (a, b)$ and $w = (c, d)$ are complex numbers, then we define their product zw to be the complex number $zw := (ac - bd, ad + bc)$. The complex multiplicative identity is $1_{\boldsymbol{C}} = (1, 0)$.*

Lemma 15.6.2. *If z_1, z_2, z_3 are complex numbers, then we have the commutative property $z_1 z_2 = z_2 z_1$, the associative property $(z_1 z_2) z_3 = z_1 (z_2 z_3)$, the identity property $z_1 1_{\boldsymbol{C}} = 1_{\boldsymbol{C}} z_1 = z_1$, and the distributivity property $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ and $(z_1 + z_2) z_3 = z_1 z_3 + z_1 z_2$.*

The above lemma asserts that $\boldsymbol{C}$ is a commutative ring. We can now identify the real numbers $\boldsymbol{R}$ with a subset of the complex numbers $\boldsymbol{C}$ as $x \equiv (x, 0)$ We now define i to be the complex number $i := (0, 1)$.

Lemma 15.6.3. *Every complex number $z \in \boldsymbol{C}$ can be written as $z = a + bi$ for exactly one pair a, b of real numbers. Also, we have $i^2 = -1$, and $-z = (-1)z$.*

Definition 15.6.4. *(Real and imaginary parts) IF z is a complex number with the representation $z = a + bi$ for some real numbers a, b, we shall call a the real part of z and denote $\Re(z) := a$, and call b the imaginary part of z and denote $\Im(z) := b$. In general, $z = \Re(z) + i\Im(z)$. We define complex conjugate $\overline{z}$ of z to be the complex number $\overline{z} = \Re z - i\Im z$.*

Lemma 15.6.4. *(Complex conjugation is an involution) Let z, w be complex numbers, then $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{-z} = -\overline{z}$, and $\overline{zw} = \overline{z}\,\overline{w}$. Also $\overline{\overline{z}} = z$. Finally, we have $\overline{z} = \overline{w}$ iff $z = w$ and $\overline{z} = z$ iff z if real.*

Definition 15.6.5. *(Complex absolute value) It $z = a + bi$ is a complex number, we define the absolute value $|z|$ or z to be the real number $|z| := \sqrt{a^2 + b^2}$.*

Lemma 15.6.5. *(Properties of complex absolute value) Let z, w be complex numbers. Then $|z|$ is a non-negative real number, and $|z| = 0$ iff $z = 0$. Also we have the identity $z\overline{z} = |z|^2$, and so $|z| = \sqrt{z\overline{z}}$. As a consequence we have $|zw| = |z||w|$ and $|\overline{z}| = |z|$. Finally, we have the inequalities $-|z| \leq \Re(z) \leq |z|$, $-|z| \leq \Im(z) \leq |z|$, $|z| \leq |\Re(z)| + |\Im(z)|$, as well as the triangle inequality $|z + w| \leq |z| + |w|$.*

Definition 15.6.6. *(Complex reciprocal) If z is a non-zero complex number, we define the reciprocal z^{-1} or z to be the complex number $z^{-1} := |z|^{-2}\overline{z}$. The reciprocal of $z = 0$ is undefined. We see that $zz^{-1} = 1$. The complex numbers can be given a distance by defining $d(z, w) = |z - w|$.*

Lemma 15.6.6. *The complex numbers $\boldsymbol{C}$ with the distant d forms a metric space. If $(z_n)_{n=1}^{\infty}$ is a sequence of complex numbers, and z is another complex number, then we have $\lim_{n\to\infty} z_n = z$ in this metric space iff $\lim_{n\to\infty} \Re(z) = \Re(z)$ and $\lim_{n\to\infty} \Im(z) = \Im(z)$.*

This metric space is complete, connected, but not compact.

Lemma 15.6.7. *(Complex limit laws) Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be convergent sequence of complex numbers, and let c be a complex number. Then the sequences $(z_n + w_n)_{n=1}^{\infty}$, $(z_n - w_n)_{n=1}^{\infty}$, $(cz_n)_{n=1}^{\infty}$, $(z_n w_n)_{n=1}^{\infty}$, and $(\overline{z_n})_{n=1}^{\infty}$ are also convergent, with $\lim_{n\to\infty} z_n + w_n = \lim_{n\to\infty} z_n + \lim_{n\to\infty} w_n$, $\lim_{n\to\infty} z_n - w_n = \lim_{n\to\infty} z_n - \lim_{n\to\infty} w_n$, $\lim_{n\to\infty} cz_n = c \lim_{n\to\infty} z_n$, $\lim_{n\to\infty} z_n w_n = \left(\lim_{n\to\infty} z_n\right)\left(\lim_{n\to\infty} w_n\right)$, $\lim_{n\to\infty} \overline{z_n} = \overline{\lim_{n\to\infty} z_n}$. Also, if the w_n are all non-zero and $\lim_{n\to\infty} w_n$ is also non-zero, then $(z_n/w_n)_{n=1}^{\infty}$ is also a convergent sequence, with $\lim_{n\to\infty} z_n/w_n = \left(\lim_{n\to\infty} z_n\right) / \left(\lim_{n\to\infty} w_n\right)$.*

The theory of pointwise and uniform convergence or the theory of power series extends without any difficulty to complex valued functions. In particular, we can define the complex exponential function in exactly the same manner as for real numbers:

Definition 15.6.7. *(Complex exponential) If z is a complex number, we define the function e^z by the formula $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$.*

e^z converges for every z. We also have $e^{z+w} = e^z e^w$ and $\overline{e^z} = e^{\overline{z}}$. The complex logarithm is more subtle, because exponential is no longer invertible, and because the various power series for the logarithm only have a finite radius of convergence, unlike exponential which has and infinite radius of convergence.

15.7 Trignometric functions

Definition 15.7.1. *(Trigonometric functions) If z is a complex number, then we define $cos(z) := \frac{1}{2}(e^{iz} + e^{-iz})$, and $siz(z) := \frac{1}{2}(e^{iz} - e^{-iz})$. We refer to cos and sin as the cosine and sine functions respectively.*

From power series definition we have $cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$ and $sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$. In particular $cos(x)$ and $sin(x)$ are always real whenever x is real. The power series is also convergent for every x, thus $sin(x)$ and $cos(x)$ are real analytic at 0 with an infinite radius of convergence. In particular the sine and cosine functions are continuous and differentiable.

Theorem 15.6. *(Trignometric identities) Let x, y be real numbers.*

- *We have $sin^2 x + cos^2 x = 1$. In particular we have $sin(x) \in [-1, 1]$ and $cos(x) \in [-1, 1]$ for all $x \in \boldsymbol{R}$.*
- *We have $sin'(x) = cos(x)$ and $cos'(x) = -sin(x)$.*
- *We have $sin(-x) = -sin(x)$ and $cos(-x) = cos(x)$.*
- *We have $cos(x + y) = cos(x)cos(y) - sin(x)sin(y)$ and $sin(x + y) = sin(x)cos(x) + cos(x)sin(y)$.*
- *We have $sin(0) = 0$ and $cos(0) = 1$.*
- *We have $e^{ix} = cos(x) + isin(x)$ and $e^{-ix} = cos(x) - isin(x)$. In particular $cos(x) = \Re(e^{ix})$ and $sin(x) = \Im(e^{ix})$.*

Lemma 15.7.1. *There exists a positive number x such that sin(x) is equal to 0.*

Definition 15.7.2. *We define π to be the number $\pi := \inf\{x \in (0, \infty) : sin(x) = 0\}$*

This leads to Euler's famous formula $e^{\pi i} = cos(\pi) + isin(\pi) = -1$.

Theorem 15.7. *(Periodicity of trigonometric functions) Let x be a real number.*

1. *We have $cos(+\pi) = -cos(x)$ and $sin(x + \pi) = -sin(x)$. In particular we have $cos(x + 2\pi) = cos(x)$ and $sin(x + 2\pi) = sin(x)$, i.e. sin and cos are periodic with period 2π.*
2. *We have $sin(x) = 0$ iff x/π is an integer.*
3. *We have $cos(x) = 0$ iff x/π is an integer plus $1/2$.*

16 Fourier series

Instead of analyzing compactly supported functions, Fourier series analyzes periodic functions; instead of decomposing it into polynomials, it decomposes into trigonometric polynomials.

16.1 Periodic functions

Definition 16.1.1. *Let $L > 0$ be a real number. A function $f : \boldsymbol{R} \to \boldsymbol{C}$ is a periodic with period L, or L-periodic, if we have $f(x + L) = f(x)$ for every real number x.*

If a function f is L-periodic, then we have $f(x + kL) = f(x)$ for every integer k. If a function is 1-periodic it is called $\boldsymbol{Z}$-periodic, e.g., $cos(2\pi nx)$ and $sin(2\pi nx)$ are $\boldsymbol{Z}$-periodic. Another example of a $\boldsymbol{Z}$-periodic function is the function $f : \boldsymbol{R} \to \boldsymbol{C}$ defined by $f(x) := 1$ when $x \in [n, n + \frac{1}{2})$ for some integer n, and $f(x) := 0$ when $x \in [n + \frac{1}{2}, n + 1)$ for some integer n. This function is an example of a square wave. We only deal with functions which are $\boldsymbol{Z}$-periodic with specification on $[0, 1)$, since this will determine any periodic function with values of f everywhere else. We then say f is extended periodically to all of $\boldsymbol{R}$.

The space of complex-valued continuous $\boldsymbol{Z}$-periodic functions is denoted $C(\boldsymbol{R}/\boldsymbol{Z}; \boldsymbol{C})$. Thus, the functions $sin(2\pi nx)$, $cos(2\pi nx)$, $e^{2\pi inx}$, and constant functions are all elements of $C(\boldsymbol{R}/\boldsymbol{Z}; \boldsymbol{C})$ but the square wave function described earlier is not because it is not continuous. Neither are $sin(x)$ and $cos(x)$ because they are not $\boldsymbol{Z}$-period.

Lemma 16.1.1. *(Basic properties of $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$)*

- *Boundedness: If $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, then f is bounded, i.e., there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \boldsymbol{R}$.*
- *Vector space and algebra properties: If $f, g \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, then the functions $f+g$, $f-g$, and fg are also in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$. Also if c is any complex number, then the function cf is also in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*
- *Closure under uniform limits: If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ which converges uniformly to another function $f : \boldsymbol{R} \to \boldsymbol{C}$, then f is also in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*

One can make $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ into a metric space by introducing the sup-norm metric $d_\infty(f,g) = \sup\limits_{x\in\boldsymbol{R}} |f(x)-g(x)| = \sup\limits_{x\in[0,1)} |f(x)-g(x)|$ of uniform convergence.

16.2 Inner products on periodic functions

Definition 16.2.1. *(Inner product) If $f, g \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, we define the inner product $\langle f, g\rangle$ to be the quantity $\langle f, g\rangle = \int\limits_{[0,1]} f(x)\overline{g(x)}dx$.*

In order to integrate a complex-valued function, we simply integrate the real and imaginary part separately. All the standard rules of calculus hold. Also, note that $f(x)\overline{g(x)}$ will be Riemann integrable since both functions are bounded and continuous. Roughly speaking, the inner product $\langle f, g\rangle$ is to the space $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ what the dot product $x \cdot y$ is to Euclidean space such as $\boldsymbol{R}^n$.

Lemma 16.2.1. *Let $f, g, h \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*

- *Hermitian property: We have $\langle g, g\rangle = \overline{\langle f, g\rangle}$.*
- *Positivity: We have $\langle f, f\rangle \geq 0$. Furthermore, we have $\langle f, g\rangle = 0$ iff $f = 0$.*
- *Linearity in the first variable: We have $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$. For any complex number c, we have $\langle cf, g\rangle = c\langle f, g\rangle$.*
- *Antilinearity in the second variable: We have $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$. For any complex number c, we have $\langle f, cg\rangle = \overline{c}\langle f, g\rangle$.*

From the positivity property, it makes sense to define the L^2 norm $\|f\|_2$ of a function $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ by the formula $\|f\|_2 = \sqrt{\langle f, f\rangle}$, also called the **root mean square** of f. L^2 norm is related to L^∞ norm by $0 \leq \|f\|_2 \leq \|f\|_\infty$.

Lemma 16.2.2. *Let $f, g \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*

- *Non-degeneracy: we have $\|f\|_2 = 0$ iff $f = 0$.*
- *Cauchy-Schwarz inequality: We have $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$.*
- *Triangle inequality: We have $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.*
- *Pythagoras' theorem: If $\langle f, g\rangle = 0$, then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$.*
- *Homogeneity: W have $\|cf\|_2 = |c|\|f\|_2$ for all $c \in \boldsymbol{C}$.*

We say f and g are orthogonal if $\langle f, g\rangle = 0$. We define the L^2 metric d_{L^2} on $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ by defining $d_{L^2}(f,g) := \|f-g\|_2$. d_{L^2} is indeed a metric. L^2 metric is very similar to l^2 metric on Euclidean spaces $\boldsymbol{R}^n$. Note that a sequence f_n of functions in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ will converge in the L^2 metric to $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ if $d_{L^2}(f_n, f) \to 0$ as $n \to \infty$, i.e., $\lim\limits_{n\to\infty} \int\limits_{[0,1]} |f_n(x) - f(x)|^2 dx = 0$. The notion of convergence in L^2 metric is different from that of uniform or pointwise convergence. The L^2 metric is not as well-behaved as the L^∞ metric. For example, the space $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ is not complete in L^2 metric, despite being complete in L^∞ metric.

16.3 Trignometric polynomials

Definition 16.3.1. *(Characters) For every integer n, we let $e_n \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ denote the function $e_n(x) := e^{2\pi i n x}$, referred to as the character with frequency n.*

Definition 16.3.2. *(Trignometric polynomials) A function f in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ is said to be a trigonometric polynomial if we can write $f = \sum_{n=-N}^{N} c_n e_n$ for some integer $N \geq 0$ and some complex numbers $(c_n)_{n=-N}^{N}$.*

For example, $cos(2\pi nx) = \frac{1}{2}e_{-n} + \frac{1}{2}^n$ and $sin(2\pi nx) = -\frac{1}{2i}e_{-n} + \frac{1}{2i}^n$ are trigonometric polynomials. Any linear combinations of sines and cosines is also a trigonometric polynomial. The Fourier theorem will allow us to write any function in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ as a Fourier series, which is to trigonometric polynomials what power series is to polynomials.

Lemma 16.3.1. *(Characters are an orthonormal system) For any integers n and m, we have $\langle e_n, e_m\rangle = 1$ when $n = m$ and $\langle e_n, e_m\rangle = 0$ when $n \neq m$. Also, we have $\|e + n\| = 1$.*

Corollary 16.3.1. *Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then we have the formula $c_n = \langle f, e_n\rangle$ for all integers $-N \leq n \leq N$. Also, we have $0 = \langle f, e_n\rangle$ whenever $n > N$ or $n < -N$. Also, we have the identity $\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2$.*

Definition 16.3.3. *(Fourier transform) For any function $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, and any integer $n \in \boldsymbol{Z}$, we define the nth Fourier coefficient of f, denoted by $\hat{f}(n)$, by the formula $\hat{f}(n) := \langle f, e_n\rangle = \int_{[0,1]} f(x)e^{-2\pi i n x}dx$. The function $\hat{f} : \boldsymbol{Z} \to \boldsymbol{C}$ is called the Fourier transform of f.*

We see that whenever $f = \sum_{n=-N}^{N} c_n e_n$ is a trigonometric polynomial, we have $f = \sum_{n=-N}^{N} \langle f, e_n\rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n\rangle e_n$ and in particular we have the Fourier inversion formula $f = \sum_{n=\infty}^{\infty} \hat{f}(n)e_n$ or in other words $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi i n x}$. The right-hand side is referred to as the **Fourier series** of f. Also, we can infer the **Plancherel formula** $\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$. We have only proven the Fourier inversion and Plancherel formulae in the case where f is a trigonometric polynomial, in which case most of the Fourier coefficients $\hat{f}(n)$ are 0 and only non-zero if $-N \leq n \leq N$. The above series converge pointwise, uniformly, and in L^2 metric, since they are just finite sums. We now extend the Fourier inversion and Plancherel formulae to general function in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, not just trigonometric polynomials. It is also possible to extend the formula to discontinuous functions such as the square wave. To do this we will need a version of Weierstrass approximation theorem, this time requiring that a continuous periodic function be approximated uniformly by trigonometric polynomials. We will the notion of convolution tailored for period functions.

16.4 Periodic convolutions

We want to prove that any continuous periodic function can be uniformly approximated by trigonometric polynomials, i.e., if we let $P(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ denote the space of all trigonometric polynomials, then the closure of $P(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ in the L^∞ metric is $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$. This can be proved directly from Weierstrass approximation theorem for polynomials, and is a special case of Stone-Weierstrass theorem.

Definition 16.4.1. *(Periodic convolution) Let $f, g \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$. Then we define the periodic convolution $f * g : \boldsymbol{R} \to \boldsymbol{C}$ of f and g by the formula $f * g(x) := \int_{[0,1]} f(y)g(x-y)dy$.*

Lemma 16.4.1. *(Basic properties of periodic convolution) Let $f, g, h \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*

- *Closure: The convolution $f * g$ is continuous and $\boldsymbol{Z}$-periodic. In other words, $f * g \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$.*
- *Commutativity: We have $f * g = g * f$.*

- *Bilinearity: We have $f * (g+h) = f*g + f*h$ and $(f+g)*h = f*h + g*h$. For any complex number c, we have $c(f*g) = (cf)*g = f*(cg)$.*

For any $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ and any integer n, we have $f * e_n = \hat{f}(n)e_n$. We can see it by following $f * e_n(x) = \int_{[0,1]} f(y)e^{2\pi i n(x-y)}dy = e^{2\pi i n x}\int_{[0,1]} f(y)e^{-2\pi i n y}dy = \hat{f}(n)e^{2\pi i n x} = \hat{f}(n)e_n$. More generally, we see that for any trigonometric polynomial $P = \sum_{n=-N}^{n=N} c_n(f * e_n) = \sum_{n=-N}^{n=N} \hat{f}(n)c_n e_n$. Thus the periodic convolution of any function in $C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ with a trigonometric polynomial, is again a trigonometric polynomial.

Definition 16.4.2. *(Periodic approximation to the identity) Let $\varepsilon > 0$ and $0 < \delta < 1/2$. A function $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$ is said to be a periodic (ε,δ) approximation to the identity if the following properties are true:*

- *$f(x) \geq 0$ for all $x \in \boldsymbol{R}$, and $\int_{[0,1]} f = 1$.*
- *We have $f(x) < \varepsilon$ for all $\delta \leq |x|1-\delta$.*

Lemma 16.4.2. *For every $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a trigonometric polynomial P which is an (ε,δ) approximation to the identity.*

Theorem 16.1. *(Weierstrass approximation theorem) Let $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, and let $\varepsilon > 0$. Then there exists a trigonometric polynomial P such that $\|f - P\|_\infty \leq \varepsilon$.*

16.5 The Fourier and Plancherel theorems

Theorem 16.2. *(Fourier theorem) For any $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, the series $\sum_{-\infty}^{\infty} \hat{f}(n)e_n$ converges in L^2 metric to f. In other words, we have*

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 = 0.$$

The convergence is not in uniform or pointwise sense. However, if one assumes that the function f is not only continuous but also continuously differentiable, then one can recover pointwise convergence, if one assumes continuously twice differentiable, then one gets uniform convergence as well.

Theorem 16.3. *Let $f \in C(\boldsymbol{R}/;\boldsymbol{C})$, and suppose that the series $\sum_{n=\infty}^{\infty} |\hat{f}(n)e_n|$ is absolutely convergent. Then the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges uniformly to f. In other words, we have $\lim N \to \infty \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0$.*

Theorem 16.4. *(Plancherel theorem) For any $f \in C(\boldsymbol{R}/\boldsymbol{Z};\boldsymbol{C})$, then series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ is absolutely convergent, and*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

17 Several variable differential calculus

17.1 Linear transformations

Definition 17.1.1. *(Row vectors) Let $n \geq 1$ be an integer. We refer to elements of $\boldsymbol{R}^n$ as n-dimensional row vectors. A typical n-dimensional row vector may take the form $x = (x_1, x_2, \ldots, x_n)$, which we abbreviate as $(x_i)_{1\leq i\leq n}$; the quantities $(x_1, x_2, \ldots, x_n)$, which we abbreviate as $(x_i)_{1\leq i\leq n}$ and $(y_i)_{1|leqi\leq n}$ are n-dimensional row vectors, we can define their vector sum by $(x_i)_{1\leq i\leq n} + (y_i)_{1\leq i\leq n} = (x_i + y_i)_{1\leq i\leq n}$, and also if $c \in \boldsymbol{R}$ is any scalar, we can define the scalar product $c(x_i)_{1\leq i\leq n}$ by $c(x_i)_{1\leq i\leq n} := (cx_i)_{1\leq i\leq n}$.*

Lemma 17.1.1. *($\boldsymbol{R}^n$ is a vector space) Let x, y, z be vectors in $\boldsymbol{R}^n$, and let c, d be real numbers. Then we have the commutative property $x + y = y + x$, the additive associativity property $(x+y)+z = x+(y+z)$, the additive property $x + 0 = 0 + x$, the additive inverse property $x + (-x) = (-x) + x = 0$, the multiplicative associativity property $(cd)x = c(dx)$, the distributivity properties $c(x+y) = cx + cy$ and $(c+d)x = cx + dx$, and the multiplicative identity property $1x = x$.*

Definition 17.1.2. *(Transpose) If $(x_i)_{1\leq i\leq n} = (x_1, \ldots, x_n)$ is an n-dimensional row vector, we can define its transpose $(x_i)^T_{1\leq i\leq n}$ by $(x_i)^T_{1\leq i\leq n} = (x_1, \ldots, x_n)^T = \begin{bmatrix} x_1 \\ \\ x_n \end{bmatrix}$. We refer to objects such as $(x_i)^T_{1\leq i\leq n}$ as n-dimensional column vectors.*

Definition 17.1.3. *(Standard basis row vectors) We identify n special vectors in $\boldsymbol{R}^n$, the standard basis row vectors $e_1, \ldots, e_n$. For each $1 \leq j \leq n$, e_j is the vector which has 0 in all entries except the jth entry, which is equal to 1.*

In $\boldsymbol{R}^n$ we can write any vector $(x_i)_{1\leq i\leq n}$ a $x = \sum_{j=1}^{n} x_j e_j$.

Definition 17.1.4. *(Linear transformations) A linear transformation $T\colon \boldsymbol{R}^n \to \boldsymbol{R}^m$ is any function from one Euclidean space $\boldsymbol{R}^n$ to another $\boldsymbol{R}^m$ which obeys the following two axioms:*

- *Additivity: For ever $x, x' \in \boldsymbol{R}^n$, we have $T(x + x') = Tx + Tx'$.*
- *Homogeneity: For every $x \in \boldsymbol{R}^n$ and every $c \in \boldsymbol{R}$, we have $T(cx) = cTx$.*

Some examples are dilation, rotation, projection, inclusion and identity operators.

Definition 17.1.5. *(Matrices) An $m \times n$ matrix is an object A of the form $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{bmatrix}$; we shall abbreviate this as $A = (a_{ij})_{1\leq i\leq m; 1\leq j\leq n}$.*

Definition 17.1.6. *(Matrix product) Given an $m\times n$ matrix A and an $n\times p$ matrix B, we can define the matrix product AB to be the $m \times p$ matrix defined as $(a_{ij})_{1\leq i\leq m;1\leq j\leq n}(b_{jk})_{1\leq j\leq n;1\leq k\leq p} := \left(\sum_{j=1}^{n} a_{ij}b_{jk}\right)_{1\leq i\leq m;1\leq k\leq p}$. In particular, if $x^T = (x_j)^T_{1\leq j\leq n}$ is an n-dimensional column vector, and $A = (a_{ij})_{1\leq i\leq m;1\leq j\leq n}$ is an $m \times n$ matrix, then Ax^T is an m-dimensional column vector: $Ax^T = \left(\sum_{j=1}^{n} a_{ij}x_j\right)^T_{1\leq i\leq m}$.*

If A is an $m \times n$ matrix, we can define the transformation $L_A : \boldsymbol{R}^n \to \boldsymbol{R}^m$ by the formula $(L_A x)^T := Ax^T$ with $L_a(x_j)_{1\leq j\leq n} = \left(\sum_{j=1}^{n} a_{ij}x_j\right)_{1\leq i\leq m}$. For any $m \times n$ matrix A, the transformation L_A is automatically linear, i.e $L_A(x + y) = L_A x + L_A y$ and $L_A(cx) = c(L_A x)$ for any n-dimensional row vectors x, y and any scalar c. The converse is also true.

Lemma 17.1.2. *Let $T : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a linear transformation. Then there exists exactly one $m \times n$ matrix A such that $T = L_A$.*

Lemma 17.1.3. *Let A be an $m \times n$ matrix and let B be an $n \times p$ matrix. Then $L_A L_B = L_{AB}$.*

17.2 Derivatives in several variable calculus

Lemma 17.2.1. *Let E be a subset of $\boldsymbol{R}$, $f : E \to \boldsymbol{R}$ be a function, $x_0 \in E$, and $L \in \boldsymbol{R}$. Then the following two statements are equivalent.*

- *f is differentiable at x_0, and $f'(x_0) = L$.*
- *We have $\lim\limits_{x\to x_0; x\in E-\{x_0\}} \frac{|f(x)-(f(x_0)+L(x-x_0))|}{|x-x_0|} = 0$*

In other words, the derivative is the quantity L such that we have the approximation $f(x) - f(x_0) \approx L(x - x_0)$.

Definition 17.2.1. *(Differentiability) Let E be a subset of $\boldsymbol{R}^n$, $f : E \to \boldsymbol{R}^m$ be a function, $x_0 \in E$ be a point, and let $L : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be a linear transformation. We say that f is differentiable at x_0 with derivative L if we have $\lim\limits_{x \to x_0; x \in E - \{x_0\}} \frac{\|f(x)-(f(x_0)+L(x-x_0))\|}{\|x-x_0\|} = 0$. Here $\|x\|$ is the length of x as measured in the l^2 metric: $\|(x_1, x_2, \ldots, x_n)\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$.*

Lemma 17.2.2. *(Uniqueness of derivatives) Let E be a subset of $\boldsymbol{R}^n$, $f : E \to \boldsymbol{R}^m$ be a function, $x_0 \in E$ be an interior point of E, and let $L_1 : \boldsymbol{R}^n \to \boldsymbol{R}^m$ and $L_2 : \boldsymbol{R}^n \to \boldsymbol{R}^m$ be linear transformations. Suppose that f is differentiable at x_0 with derivative L_1, and also differentiable at x_0 with derivative L_2. Then $L_1 = L_2$.*

Thus, we can write the Newton's approximation $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. $f'(x)$ is called the total derivative. For example, let $f : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ be the map $f(x, y) := (x^2, y^2)$, then $L : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ is $L(x, y) := (2x, 4y)$, at $x_0 := (1, 2)$.

17.3 Partial and directional derivatives

Definition 17.3.1. *(Directional derivative) Let E be a subset of $\boldsymbol{R}^n$, $f : E \to \boldsymbol{R}^m$ be a function, let x_0 be an interior point of E, and let v be a vector in $\boldsymbol{R}^n$. If the limit $\lim\limits_{t \to 0; t>0, x_0+tv \in E} \frac{f(x_0+tv)-f(x_0)}{t}$ exists, we say that f is differentiable in the direction v at x_0, and we denote the above limit by $D_v f(x_0)$.*

For example for the function $f : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ defined by $f(x, y) := (x^2, y^2)$ at $x_0 := (1, 2)$ and $v := (3, 4)$ we have $D_v f(x_0) = (6, 16)$.

Lemma 17.3.1. *Let E be a subset of $\boldsymbol{R}^n$, $f : E \to \boldsymbol{R}^m$ be a function, x_0 be an interior point of E, and let v be a vector in $\boldsymbol{R}^n$. If f is differentiable at x_0, then f is also differentiable in the direction v at x_0, and $D_v f(x_0) = f'(x_0)v$.*

Total differentiability implies directional differentiability, but not the other way round.

Definition 17.3.2. *(Partial derivative) Let E be a subset of $\boldsymbol{R}^n$, let $f : E \to \boldsymbol{R}^m$ be a function, let x_0 be an interior point of E, and let $1 \leq j \leq n$. Then the partial derivative of f with respect to the x_j variable at x_0 is define by $\frac{\partial f}{\partial x_j}(x_0) := \lim\limits_{t \to 0; t \neq 0, x_0+te_j \in E} \frac{f(x_0+te_j)-f(x_0)}{t} = \frac{d}{dt} f(x_0 + te_j)|_{t=0}$ provided of course that the limit exists.*

Notice that if f takes values in $\boldsymbol{R}^m$, i.e $f = (f_1, \ldots, f_m)$, then so will $\frac{\partial f}{\partial x_j}(x_0) = (\frac{\partial f_1}{\partial x_j}(x_0), \ldots, \frac{\partial f_m}{\partial x_j}(x_0))$.

Theorem 17.1. *Let E be a subset of $\boldsymbol{R}^n$, $f : E \to \boldsymbol{R}^m$ be a function, F be a subset of E, and x_0 be an interior point of F. If all the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on F and are continuous at x_0, then f is differentiable at x_0, and the linear transformation $f'(x_0) : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is defined by $f'(x_0)(v_j)_{1 \leq j \leq n} = \sum\limits_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0)$.*

If the partial derivatives of a function $f : E \to \boldsymbol{R}^m$ exists and are continuous on some set F, then all the directional derivatives also exist at every interior point x_0 of F, and we have the formula

$$D_{(v_1, \ldots, v_n)} f(x_0) = \sum_{j=1}^{n} v_j \frac{\partial f}{\partial x_j}(x_0).$$

In particular, if $f : E \to \boldsymbol{R}$ is a real-valued function, and we define the gradient $\nabla f(x_0)$ of f at x_0 to be the n-dimensional row vector $\nabla f(x_0) := (\frac{\partial f}{\partial x_1}(x_0), \ldots, \frac{\partial f}{\partial x_n}(x_0))$, then we have the familiar formula

$$D_v f(x_0) = v.\nabla f(x_0)$$

where x_0 is in the interior of the region where the gradient exists and is continuous. More generally, if $f : E \to \boldsymbol{R}^m$ is a function taking values in $\boldsymbol{R}^m$ with $f = (f_1, \ldots, f_m)$, and x_0 is in the interior of the region where the partial derivatives of f exist and are continuous, then we have

$$Df(x_0) := \left(\frac{\partial f_i}{\partial x_j}(x_0)\right)_{i \leq m; 1 \leq j \leq n} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x_0) & \frac{\partial f_1}{\partial x_2}(x_0) & \ldots & \frac{\partial f_1}{\partial x_n}(x_0) \\ \frac{\partial f_2}{\partial x_1}(x_0) & \frac{\partial f_2}{\partial x_2}(x_0) & \ldots & \frac{\partial f_2}{\partial x_n}(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x_0) & \frac{\partial f_m}{\partial x_2}(x_0) & \ldots & \frac{\partial f_m}{\partial x_n}(x_0) \end{bmatrix}.$$

17.4 The several variable calculus chain rule

Theorem 17.2. *(Several variable calculus chain rule) Let E be a subset of $\boldsymbol{R}^n$, and let F be a subset of $\boldsymbol{R}^m$. Let $f : E \to F$ be a function, and let $g : F \to \boldsymbol{R}^p$ be another function. Let x_0 be a point in the interior of E. Suppose that f is differentiable at x_0, and that $f(x_0)$ is in the interior of F. Suppose also that g is differentiable at $f(x_0)$. Then $g \circ f : E \to \boldsymbol{R}^p$ is also differentiable at x_0, and we have the formula $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.*

As a corollary we see that $D(g \circ f)(x_0) = Dg(f(x_0))Df(x_0)$. Further, we have $\nabla(fg) = f\nabla g + f\nabla g$. If $T : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is a linear transformation, then for any differentiable function $f : E \to \boldsymbol{R}^n$, we see that $Tf : E \to \boldsymbol{R}^m$ is also differentiable, and by chain rule $(Tf)'(x_0) = T(f'(x_0))$. Finally, if $f : \boldsymbol{R}^n \to \boldsymbol{R}^m$ is some differentiable function, and $x_j : \boldsymbol{R} \to \boldsymbol{R}$ are differentiable functions for each $j = 1, \ldots, n$, then $\frac{d}{dt} f(x_1(t), \ldots, x_n(t)) = \sum_{j=1}^{n} x_j'(t) \frac{\partial f}{\partial x_j}(x_1(t), \ldots, x_n(t))$.

17.5 Double derivatives and Clairaut's theorem

Definition 17.5.1. *(Twice continuous differentiability, C^2) Let E be an open subset of $\boldsymbol{R}^n$, and let $f : E \to \boldsymbol{R}^m$ be a function. We say that f is continuously differentiable if the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ exist an are continuous on E. We say that f is twice continuously differentiable if it is continuously differentiable, and the partial derivatives $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ are themselves continuously differentiable.*

Theorem 17.3. *(**Clairaut's theorem**) Let E be an open subset of $\boldsymbol{R}^n$, and let $f : E \to \boldsymbol{R}^m$ be a twice continuously differentiable function on E. Then we have $\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_i}(x_0) = \frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}(x_0)$ for all $1 \le i, j \le n$.*

17.6 The contraction mapping theorem

Definition 17.6.1. *(Contraction) Let (X, d) be a metric space, and let $f : X \to X$ be a map. We say that f is contraction if we have $d(f(x), f(y)) \le d(x, y)$ for all $x, y \in X$. We say that f is a strict contraction if there exists a constant $0 < c < 1$ such that $d(f(x), f(y)) \le cd(x, y)$ for all $x, y \in X$; we call c the contraction constant of f.*

Definition 17.6.2. *(Fixed points) Let $f : X \to X$ be a map, and $x \in X$. We say that x is a fixed point of f if $f(x) = x$.*

Theorem 17.4. *(Contraction mapping theorem) Let (X, d) be a metric space, and let $f : X \to X$ be a strict contraction. Then f can have at most one fixed point. Moreover, if we also assume that X is non-empty and complete, then f has exactly one fixed point.*

Lemma 17.6.1. *Let $B(0, r)$ be a ball in $\boldsymbol{R}^n$ centered at the origin, and let $g : B(0, r) \to \boldsymbol{R}^n$ be a map such that $g(0) = 0$ and $\|g(x) - g(y)\| \le \frac{1}{2}\|x - y\|$ for all $x, y \in B(0, r)$, here $\|x\|$ denotes the length of x in $\boldsymbol{R}^n$. Then the function $f : B(0, r) \to \boldsymbol{R}^n$ defined by $f(x) := x + g(x)$ is one to one, and furthermore the image $f(B(0, r))$ of this map contains the ball $B(0, r/2)$.*

This means that for any map f on a ball which is a small perturbation of the identity map, remains one-to-one and cannot create any internal hols in the ball.

17.7 The inverse function theorem in several variable calculus

The inverse function theorem of single variable calculus asserts that if a function $f : \boldsymbol{R} \to \boldsymbol{R}$ is invertible, differentiable, and $f'(x_0)$ is non-zero, then f^{-1} is differentiable at $f(x_0)$, and $(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)}$. We now develop this theorem for $f : \boldsymbol{R}^n \to \boldsymbol{R}^n$.

Lemma 17.7.1. *Let $T : \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a linear transformation which is also invertible. Then the inverse transformation $T^{-1} : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is also linear.*

Theorem 17.5. *(Inverse function theorem) Let E be an open subset of $\boldsymbol{R}^n$, and let $f : E \to \boldsymbol{R}^n$ be a function which is continuously differentiable on E. Suppose $x_0 \in E$ is such that the linear transformation $f'(x_0)) : \boldsymbol{R}^n \to \boldsymbol{R}^n$ is invertible. Then there exists an open set U in E containing x_0, and an open set V in $\boldsymbol{R}^n$ containing $f(x_0)$, such that f is a bijection from U to V. In particular, there is an inverse map $f^{-1} : V \to U$. Furthermore, this inverse map is differentiable at $f(x_0)$, and $(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}$.*

17.8 The implicit function theorem

Any function $g : \boldsymbol{R}^n \to \boldsymbol{R}$ gives rise to a graph $\{(x, g(x)) : x \in \boldsymbol{R}^n\}$ in $\boldsymbol{R}^{n+1}$ called hypersurface. To answer the question with hypersurfaces are actually graphs of some function, and whether that function is continuous or differentiable, we use implicit function theorem.

Theorem 17.6. *(Implicit function theorem) Let E be an open subset of $\boldsymbol{R}^n$, let $f : E \to \boldsymbol{R}$ be continuously differentiable, and let $y = (y_1, \ldots, y_n)$ be a point in E such that $f(y) = 0$ and $\frac{\partial f}{\partial x_n}(y) \neq 0$. Then there exists an open subset U of $\boldsymbol{R}^{n-1}$ containing $(y_1, \ldots, y_{n-1})$, an open subset V of E containing y, and a function $g : U \to \boldsymbol{R}$ such that $g(y_1, \ldots, y_{n-1}) = y_n$, and $\{(x_1, \ldots, x_n) \in V : f(x_1, \ldots, x_n) = 0\} = \{(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1})) : (x_1, \ldots, x_{n-1}) \in U\}$. In other words, the set $\{x \in V : f(x) = 0\}$ is a graph of a function over U. Moreover, g is differentiable at $(y_1, \ldots, y_{n-1})$, and we have $\frac{\partial g}{\partial x_j}(y_1, , y_{n-1}) = -\frac{\partial f}{\partial x_j}(y) / \frac{\partial f}{\partial x_n}(y)$ for all $1 \leq j \leq n-1$.*

Basically, the point is that if you know that $f(x_1, \ldots, x_n) = 0$ then as long as $\partial f \partial x_n \neq 0$, the variables x_n is implicitly defined in terms of the other $n-1$ variables, and one can differentiate the above identity in, say, the x_j direction using the chain rule to obtain $\frac{\partial f}{\partial x_j} + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial x_j} = 0$. Thus, the implicit function theorem allows one to define a dependence implicitly, by means of a constraint rather than by a direct formula.

Sets which look like graphs of continuous functions at every point are called manifolds.

18 Lebesgue measure

It is now natural to consider the question of integration in several variable calculus. Given some subset Ω of $\boldsymbol{R}^n$, and some real valued function $f : \Omega \to \boldsymbol{R}$, is it possible to integrate f on Ω to obtain some number $\int_\Omega f$? The notion of Riemann integral $\int\limits_{[a,b]} f$, answered this question in one dimension. However not all functions are Riemann integrable. It is possible to extend this notion of a Riemann integral to higher dimensions, but it will only be able to integrate 'Riemann integrable' functions. The notion of **Lebesgue integral** can handle a very large class of functions, even discontinuous functions.

The process of calculating length/area/volume of Ω is connected to the calculation of ${}_\Omega f$. One can see it by integrating 1 on Ω - in one dimension it leads to length, in two dimensions it leads to area, and in three dimensions it leads to volume. Generally, we call it the **measure** of Ω. Ideally, to every subset Ω of $\boldsymbol{R}^n$ we would like to associate a non-negative number $m(\Omega)$, which will be the measure of Ω, with the possibility of $m(\Omega)$ being 0 or ∞. This measure should obey certain reasonable properties - the measure of unit cube $(0,1)^n := \{(x_1,\ldots,x_n) : 0 < x_i < 1\}$ should be equal to 1, we should have $m(A \cup B) = m(A) + m(B)$ if A and B are disjoint, we should have $m(A) \leq m(B)$ whenever $A \subseteq B$, and we should have $m(x + A) = m(A)$ for any $x \in \boldsymbol{R}^n$.

Remarkably, it turns out that such a measure does not exist! Under axiom of choice **Banach-Tarski paradox** shows that a unit ball in $\boldsymbol{R}^3$ is decomposed into five pieces, and then the five pieces are reassembled via translations and rotations to form two complete and disjoint unit balls, thus violating any concept of conservation of volume. These paradoxes mean that it is impossible to find a reasonable way to assign a measure to every single subset of $\boldsymbol{R}^n$. However, if we only measure a certain class of sets in $\boldsymbol{R}^n$, called **measurable sets** things can work out. Once we restrict our attention to measurable sets, which are almost all the sets one encounter in real life (e.g. all open and closed sets), one recovers all the above properties again and we can do analysis on it.

18.1 The goal: Lebesgue measure

Let $\boldsymbol{R}^n$ be a Euclidean space. We intend to define measurable set, a special kind of subset of $\boldsymbol{R}^n$, and for every such measurable set $\Omega \subset \boldsymbol{R}^n$, we indent to define the **Lebesgue measure** $m(\Omega)$ to be a certain number in $[0,\infty]$. Measurable set will obey the following properties:

- Borel property: Every open set in $\boldsymbol{R}^n$ is measurable, as is every closed set.
- Complementarity: If Ω is measurable, then $\boldsymbol{R}^n \backslash \Omega$ is also measurable.
- Boolean algebra property: If $(\Omega_j)_{j\in J}$ is any finite collection of measurable sets (so J is finite), then the union $\bigcup\limits_{j\in J} \Omega_j$ and intersection $\bigcap\limits_{j\in J} \Omega_j$ are also measurable.
- σ-algebra property: If $(\Omega_j)_{j\in J}$ are any countable collection of measurable sets (so J is countable), then the union ${}_{j\in J}\Omega_j$ and intersection $\bigcap\limits_{j\in J} \Omega_j$ are also measurable.

These properties ensure that virtually every set one cares about is measurable; though there do exist non-measurable sets. To every measurable set Ω, we associate the **Lebesgue measure** $m(\Omega)$ of Ω, which will obey the following properties:

- Empty set: The empty set $\emptyset$ has measure $m(\emptyset) = 0$.
- Positivity: We have $0 \leq m(\Omega) \leq \infty$ for every measurable set Ω.
- Monotonicity: If $A \subseteq B$, and A and B are both measurable, then $m(A) \leq m(B)$.
- Finite sub-additivity: If $(A_j)_{j\in J}$ are a finite collection of measurable sets, then $m(\bigcup\limits_{j\in J} A_j) \leq \sum\limits_{j\in J} m(A_j)$.
- Finite additivity: If $(A_j)_{j\in J}$ are a finite collection of disjoint measurable sets, then $m(\bigcup\limits_{j\in J} A_j) = \sum\limits_{j\in J} m(A_j)$.

- Countable sub-additivity: If $(A_j)_{j\in J}$ are a countable collection of measurable sets, then $m(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m(A_j)$.
- Countable additivity: If $(A_j)_{j\in J}$ are a countable collection of disjoint measurable sets, then $m(\bigcup_{j\in J} A_j) = \sum_{j\in J} m(A_j)$.
- Normalization: The unit cube $[0,1]^n = \{(x_1,\dots,x_n) \in \boldsymbol{R}^n : 0 \leq x_j \leq 1 forall 1 \leq j \leq n\}$ has measure $m([0,1]^n) = 1$.
- Translation invariance: If Ω is a measurable set, and $x \in \boldsymbol{R}^n$, then $x + \Omega := \{x + y : y \in \Omega\}$ is also measurable, and $m(x+\Omega) = m(\Omega)$.

Theorem 18.1. *(Existence of Lebesgue measure) There exists a concept of a measurable set, and a way to assign a number $m(\Omega)$ to every measurable subset $\Omega \subseteq \boldsymbol{R}^n$, which obeys all of the above properties.*

Measure theory generalizes the concept of measures for other domains than Euclidean spaces $\boldsymbol{R}^n$. We first try to cover the set by boxes, and then add up the volume of each box. This approach will almost work, giving the concept of outer measure and obeys all the properties except for finite and countable additivity. We refine the outer measure to recover the additional property.

18.2 First attempt: Outer measure

Definition 18.2.1. *(Open box) An open box $B \in \boldsymbol{R}^n$ is any set of the form $B = \prod_{i=1}^{n}(a_i, b_i) := \{(x_1 \dots, x_n) \in \boldsymbol{R}^n : x_i \in (a_i, b_i)$ for all $1 \leq i \leq n\}$, where $b_i \geq a_i$ are real numbers. We define the volume $vol(B)$ of this box to be the number $vol(B) := \prod_{i=1}^{n}(b_i - a_i)$.*

In one dimension boxes are the same as open intervals. We expect the measure $m(B)$ of a box to be the same as the volume $vol(B)$ of that box.

Definition 18.2.2. *(Covering by boxes) Let $\Omega \subseteq \boldsymbol{R}^n$ be a subset of $\boldsymbol{R}^n$. We say that a collection $(B_j)_{j\in J}$ of boxes cover Ω iff $\Omega \subseteq \bigcup_{j\in J} B_j$.*

For $\Omega \subseteq \boldsymbol{R}^n$ covered by a finite or countable collection of boxes $(B_j)_{j\in J}$, Ω is measurable and obeys the monotonicity and sub-additivity properties and $m(B_j) = vol(B_j)$ for every box j only if $m(\Omega) \leq m(\bigcup_{j\in J} B_j) \leq \sum_{j\in J} m(B_j) = \sum_{j\in J} vol(B_j)$. We thus conclude $m(\Omega) \leq inf\Big\{\sum_{j\in J} vol(B_j) : (B_j)_{j\in J}$ covers Ω; J at most countable$\Big\}$.

Definition 18.2.3. *(Outer measure) If Ω is a set, we define the outer measure $m^*(\Omega)$ of Ω to be the quantity $m^*(\Omega) := inf\Big\{\sum_{j\in J} vol(B_j) : (B_j)_{j\in J}$ covers Ω; J at most countable$\Big\}$*

Lemma 18.2.1. *(Properties of outer measure) Outer measure has the following properties:*

- *Empty set: The empty set $\emptyset$ has outer measure $m^*(\emptyset) = 0$.*
- *Positivity: We have $0 \leq m^*(\Omega) \leq +\infty$ for every measurable set Ω.*
- *Monotonicity: If $A \subseteq B \subseteq \boldsymbol{R}^n$, then $m^*(A) \leq m^*(B)$.*
- *Finite sub-additivity: If $(A_j)_{j\in J}$ are a finite collection of subsets of $\boldsymbol{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$.*
- *Countable sub-additivity: If $(A_j)_{j\in J}$ are a countable collection of subsets of $\boldsymbol{R}^n$, then $m^*(\bigcup_{j\in J} A_j) \leq \sum_{j\in J} m^*(A_j)$.*
- *Translation invariance: If Ω is a subset of $\boldsymbol{R}^n$, and $x \in \boldsymbol{R}^n$, then $m^*(x+\Omega) = m^*(\Omega)$.*

Proposition 18.2.1. *(Outer measure of closed box) For any closed box* $B = \prod_{i=1}^{n} [a_i, b_i] := \Big\{(x_1, \ldots, x_n) \in \boldsymbol{R}^n : x_i \in [a_i, b_i] \text{ for all } 1 \leq i \leq n\Big\}$, *we have* $m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$.

Corollary 18.2.1. *For any open box* $B = \prod_{i=1}^{n} (a_i, b_i) := \Big\{(x_1, \ldots, x_n) \in \boldsymbol{R}^n : x_i \in (a_i, b_i) \text{ for all } 1 \leq i \leq n\Big\}$, *we have* $m^*(B) = \prod_{i=1}^{n} (b_i - a_i)$.

To fine the one-dimensional measure of $\boldsymbol{Q}$ we see that for each rational number $\boldsymbol{Q}$, the point $\{q\}$ has outer measure $m^*(q) = 0$. Since $\boldsymbol{Q}$ is clearly the union $\boldsymbol{Q} = \bigcup_{q \in \boldsymbol{Q}} \{q\}$ of all these rational points q, and $\boldsymbol{Q}$ is countable, we have $m^*\boldsymbol{Q} \leq \sum_{q \in \boldsymbol{Q}} m^*(\{q\}) = \sum_{q \in \boldsymbol{Q}} 0 = 0$, and so $m^*(\boldsymbol{Q})$ must be equal to zero. To compute the one-dimensional measure of the irrationals $\boldsymbol{R} \backslash \boldsymbol{Q}$ we note from sub-additivity that $m^*(\boldsymbol{R}) \leq m^*(\boldsymbol{R} \backslash \boldsymbol{Q}) + m^*(\boldsymbol{Q})$. Since $\boldsymbol{Q}$ has outer measure 0, and $m^*(\boldsymbol{R})$ has outer measure $+\infty$, we thus see that the irrationals $\boldsymbol{R} \backslash \boldsymbol{Q}$ have an outer measure $+\infty$. Similarly $[0,1] \backslash \boldsymbol{Q}$ have outer measure 1.

The unit interval $[0, 1]$ in $\boldsymbol{R}$ has one-dimensional outer measure 1, but the unit interval $\{(x, 0) : 0 \leq x \leq 1\}$ in $\boldsymbol{R}^2$ has two-dimensional outer measure 0.

18.3 Outer measure is not additive

Proposition 18.3.1. *(Failure of countable additivity) There exists a countable collection* $(A_j)_{j \in J}$ *of disjoint subsets of* $\boldsymbol{R}$, *such that* $m^*(\bigcup_{j \in J} A_j) \neq \sum_{j \in J} m^*(A_j)$.

Proof: Let $\boldsymbol{Q}$ be the rationals, and $\boldsymbol{R}$ be the reals. We say a set $A \subset \boldsymbol{R}$ is a coset of $\boldsymbol{Q}$ if it is of the form $A = x + \boldsymbol{Q}$ for some real number x. For instance $\sqrt{2} + \boldsymbol{Q}$ and $\boldsymbol{Q} = 0 + \boldsymbol{Q}$, both, are coset of $\boldsymbol{Q}$. A given coset A could correspond to several values of x; for instance $2 + \boldsymbol{Q}$ is exactly the same coset as $0 + \boldsymbol{Q}$. Also, two cosets can not partially overlap; if $x + \boldsymbol{Q}$ and $y + \boldsymbol{Q}$ intersect even at just a single point, then $x - y$ must be rational, and thus $x + \boldsymbol{Q}$ and $y + \boldsymbol{Q}$ must be equal. So any two cosets are either identical or distinct.

We observe that every coset A of the rationals $\boldsymbol{Q}$ has a non-empty intersection with $[0, 1]$. To see this, if A is a coset, then $A = x + \boldsymbol{Q}$ for some real number x. If we then pick a rational number q in $[-x, 1 - x]$ then we see that $x + q \in [0, 1]$, and thus $A \cap [0, 1]$ contains $x + q$.

Let $\boldsymbol{R} \backslash \boldsymbol{Q}$ denote the set of all cosets of $\boldsymbol{Q}$; note that this is a set whole elements are themselves sets of real numbers. For each coset A in $\boldsymbol{R} \backslash \boldsymbol{Q}$, let us pick an element x_A of $A \cap [0, 1]$. This requires us to make an infinite number of choices, and thus requires the axiom of choice. Let E be the set of all such x_A, i.e., $E := \{x_A : A \in \boldsymbol{R} \backslash \boldsymbol{Q}\}$. Note that $E \subseteq [0, 1]$ by construction.

Now consider the set $X = \bigcup_{q \in \boldsymbol{Q} \cap [-1,1]} (q + E)$. Clearly this set is contained in $[-1, 2]$. We claim that this set contains the interval $[0, 1]$. To see is, notice that for any $y \in [0, 1]$, we know that y must belong to some coset A, e.g, it belongs to the coset $y + \boldsymbol{Q}$. But we also have x_A belonging to the same coset, and thus $y - x_A$ is equal to some rational q. Since y and x_A both live in $[0, 1]$, then q lives in $[-1, 1]$. Since $y = q + x_A$, we have $y \in q + E$, and hence $y \in X$ as desired.

We claim that $m^*(X) \neq \sum_{q \in \boldsymbol{Q} \cap [-1,1]} m^*(q + E)$, which would prove the claim. Observe that since $[0, 1] \subseteq X \subseteq [-1, 2]$, that we have $1 \leq m^*(X) \leq 3$ by monotonicity and the outer measure of closed box. For the right hand side, observe from translation invariance that $\sum_{q \in \boldsymbol{Q} \cap [-1,1]} m^*(q + E) = \sum_{q \in \boldsymbol{Q} \cap [-1,1]} m^*(E)$. The set $\boldsymbol{Q} \cap [-1, 1]$ is countably infinite. Thus the right-hand side is either 0, if $m^*(E) = 0$ or $+\infty$, if $m^*(E) > 0$. Either way, it can not be between 1 and 3, and the claim follows. $\square$

One can refine the above argument and show that m^* is not finitely additive either.

Proposition 18.3.2. *(Failure of finite additivity) There exists a finite collection $(A_j)_{j\in J}$ of disjoint subsets of $\boldsymbol{R}$, such that* $m * \left(\bigcup\limits_{j\in J} A_j\right) \neq \sum\limits_{j\in J} m^*(A_j)$.

Proof: Suppose for the sake of contradiction that m^* was finitely additive. Let E and X be the sets introduced in the previous proposition. From countable sub-additivity and translation invariance we have $m^*(X) \leq \sum\limits_{q\in \boldsymbol{Q}\cap[-1,1]} m^*(q+E) = \sum\limits_{q\in \boldsymbol{Q}\cap[-1,1]} m^*(E)$. Since we know that $1 \leq m^*(X) \leq 3$, we thus have $m^*(E) \neq 0$, since otherwise we would have $m^*(E) \leq 0$, a contradiction.

Since $m^*(E) \neq 0$, there exists a finite integer $n > 0$ such that $m^*(E) > 1/n$. Now let J be a finite subset of $\boldsymbol{Q} \cap [-1,1]$ of cardinality $3n$. If m^* were finitely additive, then we would have $m * \left(\sum\limits_{q\in J} q+E\right) = \sum\limits_{q\in J} m^*(q+E) = \sum_{q\in J} m^*(E) > 3n\frac{1}{n} = 3$. But we know that $\sum\limits_{q\in J} q+E$ is a subset of X, which has outer measure at most 3. This contradicts monotonicity. Hence m^* cannot be finitely additive. □

The examples here are related to the Banach-Tarski paradox, which using axiom of choice demonstrates that one can partition the unit ball in $\boldsymbol{R}^3$ into a finite number of pieces which, when rotated and translated, can be reassembled to form two complete unit balls! Of course, this partition involves non-measurable sets.

18.4 Measurable sets

We saw that some pathological sets were badly behaved with respect to outer measure, contradicting the finite or countable additivity property. Using Constantin Caratheodory clever definition we can exclude these sets and recover finite and countable additivity.

Definition 18.4.1. *(Lebesgue measurability) Let E be a subset of $\boldsymbol{R}^n$. We say that E is Lebesgue measurable, or measurable for short, iff we have the identity $m^*(A) = m^*(A\cap E) + m^*(A\backslash E)$ for every subset A of $\boldsymbol{R}^n$. If E is measurable, we define the Lebesgue measure of E to be $m(E) = m^*(E)$; if E is not measurable, we leave $m(E)$ undefined.*

In other words, E being measurable means that if we use the set E to divide up an arbitrary set A into two parts, we keep the additivity property. We know that not every set is finitely additive. One can think of the measurable sets as sets for which finite additivity works. We use this definition to prove various useful properties of measurable sets, and there after rely on those properties to identify measurable sets, leaving behind this hard to work with definition.

It can be shown that a large number of sets are indeed measurable, like empty set $E = \emptyset$ and the whole space $E = \boldsymbol{R}^n$. Half-spaces $\{(x_1,\dots,x_n) \in \boldsymbol{R}^n : x_n > 0\}$ are also measurable. In fact, any half-space of the form $\{(x_1,\dots,x_n) \in \boldsymbol{R}^n : x_j > 0\}$ or $\{(x_1,\dots,x_n) \in \boldsymbol{R}^n : x_j < 0\}$ for some $1 \leq j \leq n$ is measurable.

Lemma 18.4.1. *(Properties of measurable sets)*

- *If E is measurable, then $\boldsymbol{R}^n$ is also measurable.*
- *Translation invariance: If E is measurable, and $x \in \boldsymbol{R}^n$, then $x+E$ is also measurable, and $m(x+e) = m(E)$.*
- *If E_1 and E_2 are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.*
- *Boolean algebra property: If $E_1, E_2, \dots, E_N$ are measurable, then $\bigcup\limits_{j=1}^{N} E_j$ and $\bigcap\limits_{j=1}^{N} E_j$ are measurable.*
- *Every open box, and every closed box, is measurable.*
- *Any set E of outer measure zero, i.e., $m^*(E) = 0$ is measurable.*

Lemma 18.4.2. *(Finite additivity) If $(E_j)_{j\in J}$ are a finite collection of disjoint measurable sets and any set A, no necessarily measurable, we have $m^*\left(A \cap \bigcup\limits_{j\in J} E_j\right) = \sum\limits_{j\in J} m^*(A\cap E_j)$. Furthermore, we have $m(\bigcup\limits_{j\in J} E_j) = \sum\limits_{j\in J} m(E_j)$.*

This lemma implies that there exists non-measurable sets.

Corollary 18.4.1. *If $A \subseteq B$ are two measurable sets, then $B \backslash A$ is also measurable and $m(B \backslash A) = m(B) - m(A)$.*

Lemma 18.4.3. *(Countable additivity) If $(E_j)_{j \in J}$ are a countable collection of disjoint measurable sets, then $\bigcup_{j \in J} E_j$ is measurable, and $m(\bigcup_{j \in J} E_j) = \sum_{j \in J} m(E_j)$.*

Lemma 18.4.4. *(σ-algebra property) If $(\Omega_j)_{j \in J}$ are any countable collection of measurable sets, so J is countable, then union $\bigcup_{j \in J} \Omega_j$ and the intersection $\bigcap_{j \in J} \Omega_j$ are also measurable.*

Lemma 18.4.5. *Every open set can be written as a countable or finite union of open boxes.*

Lemma 18.4.6. *(Borel Property) Every open set, and every closed set, is Lebesgue measurable.*

18.5 Measurable functions

Definition 18.5.1. *(Measurable functions) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f :\to \boldsymbol{R}^m$ be a function. A function f is measurable iff $f^{-1}(V)$ is measurable for every open set $V \subseteq \boldsymbol{R}^m$.*

Lemma 18.5.1. *(Continuous functions are measurable) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}^m$ be continuous. Then f is also measurable.*

Lemma 18.5.2. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}^m$ be a function. Then f is measurable iff $f^{-1}(B)$ is measurable for every open box B.*

Corollary 18.5.1. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}^m$ be a function. Suppose that $f = (f_1, \ldots, f_m)$, where $f_i : \Omega \to \boldsymbol{R}$ is the jth coordinate of f. Then f is measurable iff all of the f_j are individually measurable.*

Unfortunately, it is not true that the composition of two measurable functions is automatically measurable; however we can do the next best thing: a continuous function applied to a measurable function is measurable.

Lemma 18.5.3. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let W be an open subset of $\boldsymbol{R}^m$. If $f : \Omega \to W$ is measurable, and $g : W \to \boldsymbol{R}^p$ is continuous, then $g \circ f : \Omega \to \boldsymbol{R}^p$ is measurable.*

Corollary 18.5.2. *Let Ω be a measurable subset of $\boldsymbol{R}^n$. If $f : \Omega \to \boldsymbol{R}$ is a measurable function, then so is $|f|$, $max(f, 0)$, and $min(f, 0)$.*

Corollary 18.5.3. *Let Ω be a measurable subset of $\boldsymbol{R}^n$. If $f : \Omega \to \boldsymbol{R}$ and $g : \Omega \to \boldsymbol{R}$ are measurable functions, then so is $f + g, f - g, fg, max(f, g)$, and $min(f, g)$. If $g(x) \neq 0$ for all $x \in \Omega$, then f/g is also measurable.*

Lemma 18.5.4. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}$ be a function. Then f is measurable iff $f^{-1}((a, \infty))$ is measurable for every real number a.*

We can extend the notion of a measurable function to the extended real number system $\boldsymbol{R}^* := \boldsymbol{R} \cup \{+\infty\} \cup \{-\infty\}$.

Definition 18.5.2. *(Measurable functions in the extended reals) Let Ω be a measurable subset of $\boldsymbol{R}^n$. A function $f : \Omega \to \boldsymbol{R}^*$ is said to be measurable iff $f^{-1}([a, +\infty])$ is measurable for every real number a.*

Measurability behaves well with respect to limits.

Lemma 18.5.5. *(Limits of measurable functions are measurable) Let Ω be a measurable subset of $\boldsymbol{R}^n$. For each positive integer n, let $f_n : \Omega \to \boldsymbol{R}^*$ be a measurable function. Then the functions $\sup_{n \geq 1} f_n$, $\inf_{n \geq 1} f_n$, $\limsup_{n \to \infty} f_n$, and $\liminf_{n \to \infty} f_n$ are also measurable. In particular, if the f_n converge pointwise to another function $f : \Omega \to \boldsymbol{R}^*$, then f is also measurable.*

Almost always, the only way to construct non-measurable functions is via artificial means such as invoking the axiom of choice.

19 Lebesgue integration

Like for Riemann integral where we start with piecewise constant functions and then move to a general function, in Lebesgue integral we begin by first considering a special subclass of measurable functions - the simple functions. We fist show to integrate the simple functions, then from there move on to the absolutely integrable measurable functions.

19.1 Simple functions

Definition 19.1.1. *(simple functions) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}$ be a measurable function. We say that f is a simple function if the image $f(\Omega)$ is finite. In other words, there exists a finite number of real numbers $c_1, c_2, , c_N$ such that for every $x \in \Omega$, we have $f(x) = c_j$ for some $1 \leq j \leq N$.*

Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let E be a measurable subset of Ω. We define the characteristic function or indicator function $\chi_E : \Omega \to \boldsymbol{R}$ by setting $\chi_E(x) := 1$ if $x \in E$ and $\chi_E(x) := 0$ if $x \notin E$. Then χ_E is a measurable function and is a simple function, because the image $\chi_E(\Omega)$ is in $\{0, 1\}$. We have the following three lemmas describing three basic properties of simple functions:

Lemma 19.1.1. *Let Ω be a measurable subset of $\boldsymbol{R}^n$.*

1. *Simple functions form a vector space: Let $f : \Omega \to \boldsymbol{R}$ and $g : \Omega \to \boldsymbol{R}$ be simple functions. Then $f + g$ is also a simple function. Also, for any scalar $c \in \boldsymbol{R}$, the function cf is also a simple function.*

2. *Simple functions are linear combinations of characteristic functions: Let $f : \Omega \to \boldsymbol{R}$ be a simple function. Then there exists a finite number of real numbers $c_1, \ldots, c_N$, and a finite number of disjoint measurable sets $E_1, E_2, \ldots, E_N$ in Ω, such that $f = \sum\limits_{i=1}^{N} c_i \chi_{E_i}$.*

3. *Simple functions approximate measurable functions: Let $f : \Omega \to \boldsymbol{R}$ be a measurable function. Suppose that f is always non-negative, i.e., $f(x) \geq 0$ for all $x \in \Omega$. Then there exists a sequence $f_1, f_2, \ldots$ of simple functions, $f_n : \Omega \to \boldsymbol{R}$, such that the f_n are non-negative and increasing, $0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$ for all $x \in \Omega$ and converge pointwise to f: $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$.*

Definition 19.1.2. *(Lebesgue integral of simple functions) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}$ be a simple function which is non-negative; thus f is measurable and the image $f(\Omega)$ is finite and contained in $[0, \infty)$. We then define the Lebesgue integral $\int\limits_\Omega f$ of f on Ω by $\int\limits_\Omega f := \sum\limits_{\lambda \in f(\Omega); \lambda > 0} \lambda m(x \in \Omega : f(x) = \lambda)$.*

Sometimes we write $\int\limits_\Omega f$ as $\int\limits_\Omega f dm$ to emphasize the role of Lebesgue measure m, or use a dummy variable such as x, e.g., $\int\limits_\Omega f(x)dx$. Simple integral of a simple function can be $+\infty$. The reason why we restrict this integral to non-negative functions is to avoid every encountering the indefinite form $+\infty + (-\infty)$. This notion also corresponds to the area under the graph intuition.

Lemma 19.1.2. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $E_1, \ldots, E_N$ are a finite number of disjoint measurable subsets in Ω. Let $c_1, \ldots, c_N$ be non-negative numbers, not necessarily distinct. Then we have $\int\limits_\Omega \sum\limits_{j=1}^{N} c_j \chi_{E_j} = \sum\limits_{j=1}^{n} c_j m(E_j)$.*

Proposition 19.1.1. *Let Ω be a measurable set, and let $f : \Omega \to \boldsymbol{R}$ and $g : \Omega \to \boldsymbol{R}$ be a non-negative simple functions.*

1. *We have $0 \leq \int\limits_\Omega f \leq \infty$. Furthermore, we have $\int\limits_\Omega f = 0$ iff $m(\{x \in \Omega : f(x) \neq 0\}) = 0$.*

2. *We have $\int\limits_\Omega (f + g) = \int_\Omega f + \int\limits_\Omega g$.*

3. *For any positive number c, we have $\int\limits_\Omega cf = c \int\limits_\Omega f$.*

4. *If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.*

For the first part we say that $\int_\Omega f = 0$ iff f is zero for almost every point in Ω - if a property holds for all point in Ω, except for a set of measure zero, then we say that property holds for almost every point in Ω.

19.2 Integration of non-negative measurable functions

Definition 19.2.1. *(Majorization) Let $f : \Omega \to \boldsymbol{R}$ and $g : \Omega \to \boldsymbol{R}$ be functions. We say that f majorizes g, or g minorizes f, iff we have $f(x) \geq g(x)$ for all $x \in \Omega$.*

Definition 19.2.2. *(Lebesgue integral for non-negative functions) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to [0, \infty]$ be measurable and non-negative. Then we define the Lebesgue integral $\displaystyle\int_\Omega f$ of f on Ω to be*

$$\int_\Omega f := \left\{ \int_\Omega s : s \text{ is simple and non-negative, and minorizes } f \right\}.$$

Note that $\int_\Omega f$ is always at least 0, since 0 is simple, non-negative, and minorizes f. Unlike Riemann integral, we will not need to match this lower integral with an upper integral here. Further if Ω' is any measurable subset of Ω, then we can define $\int_{\Omega'} f$ as well by restricting f to Ω', thus $\int_{\Omega'} f := \int_{\Omega'} f|_{\Omega'}$.

Proposition 19.2.1. *Let Ω be a measurable set, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be non-negative measurable functions.*

1. *We have $0 \leq \int_\Omega f \leq \infty$. Furthermore, we have $\int_\Omega f = 0$ iff $f(x) = 0$ for almost every $x \in \Omega$.*

2. *For any positive number c, we have $\int_\Omega cf = c \int_\Omega f$.*

3. *If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.*

4. *IF $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.*

5. *If $\Omega' \subseteq \Omega$ is measurable, then $\int_{\Omega'} f = \int_\Omega f\chi_{\Omega'} \leq \int_\Omega f$.*

Point 4 implies that as if no individual point, or even a measure zero collection of points, has any "vote" in what the integral of a function should be; only the collective set of points has an influence on an integral.

We cannot always interchange an integral with a limit, or with limit-like concepts such as supremum. However, with the Lebesgue integral it is possible to do so if the functions are increasing.

Theorem 19.1. *(Lebesgue monotone convergence theorem) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $(f_n)_{n=1}^\infty$ be a sequence of non-negative measurable functions from Ω to $\boldsymbol{R}$ which are increasing in the sense that $0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots$ for all $x \in \Omega$. Then we have $0 \leq \int_\Omega f_1 \leq \int_\Omega f_2 \leq \int_\Omega f_3 \leq \ldots$ and $\int_\Omega \sup_n f_n = \sup_n \int_\Omega f_n$.*

Lemma 19.2.1. *(Interchange of addition and integration) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be measurable functions. Then $\int_\Omega (f + g) = \int_\Omega f + \int_\Omega g$.*

Once one can interchange an integral with a sum of two functions, one can handle an integral and any finite number of functions by induction. More surprisingly, one can handle infinite sums as well of non-negative functions.

Corollary 19.2.1. *If Ω is a measurable subset of $\boldsymbol{R}^n$, and $g_1, g_2, \ldots$ are a sequence of non-negative measurable functions from Ω to $[0, \infty]$, then $\displaystyle\int_\Omega \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty \int_\Omega g_n$.*

Note that we need not assume anything about the convergence of the above sums, however we do need to assume non-negativity. However, interchanging the limit and integral is not always admissible. The following "*moving bump*" examples shows it. For each $n = 1, 2, 3, \ldots$, let $f_n : \boldsymbol{R} \to \boldsymbol{R}$ be the function $f_n = \chi_{[n,n+1)}$. Then $\lim\limits_{n\to\infty} f_n(x) = 0$ for every x, but $\int\limits_{\boldsymbol{R}} f_n = 1$ for every n, and hence $\lim\limits_{n\to\infty} f_n = 1 \neq 0$. In other words, the limiting function $\lim\limits_{n\to\infty} f_n$ can end up having significantly smaller integral than any of the original integrals. However Fatou's lemma shows that the reverse can not happen - there is no way the limiting function has larger integral than the limit of the original integrals.

Lemma 19.2.2. *(Fatou's lemma) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of non-negative functions from Ω to $[0, \infty]$. Then* $\int\limits_{\Omega} \liminf\limits_{n\to\infty} f_n \leq \liminf\limits_{n\to\infty} \int\limits_{\Omega} f_n$.

Lemma 19.2.3. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to [0, \infty]$ be a non-negative measurable function such that $\int\limits_{\Omega} f$ is finite. Then f is finite almost everywhere, i.e., the set $\{x \in \Omega : f(x) = +\infty\}$ has measure zero.*

Lemma 19.2.4. *(Borel-Cantelli lemma) Let $\Omega_1, \Omega_2, \ldots$ be measurable subsets of $\boldsymbol{R}^n$ such that $\sum\limits_{n=1}^{\infty} m(\Omega_n)$ is finite. Then the set $\{x \in \boldsymbol{R}^n : x \in \Omega_n$ for infinitely many $n\}$ is a set of measure zero. In other words, almost every point belongs to only finitely many Ω_n.*

19.3 Integration of absolutely integrable functions

Now we consider function which can be both positive and negative. We wish to avoid the indefinite expression $+\infty + (-\infty)$, so we restrict our attention to a subclass of measurable functions - *absolutely integrable functions.*

Definition 19.3.1. *(Absolutely integrable functions) Let Ω be a measurable subset of $\boldsymbol{R}^n$. A measurable function $f : \Omega \to \boldsymbol{R}^*$ is said to be absolutely integrable if the integral $\int\limits_{\Omega} |f|$ is finite.*

Absolutely integrable functions are also known as $L^1(\Omega)$ functions. If $f : \Omega \to \boldsymbol{R}^*$ is a function, we define the positive part $f^+ : \Omega \to [0, \infty]$ and negative part $f^- : \Omega \to [0, \infty]$ by the formulae $f^+ := max(f, 0)$, $f^- := -min(f, 0)$. Bother f^+ and f^- are measurable, are non-negative and $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Definition 19.3.2. *(Lebesgue integral) Let $f : \Omega \to \boldsymbol{R}^*$ be an absolutely integrable function. We define the Lebesgue integral $\int\limits_{\Omega} f$ of f to be the quantity $\int\limits_{\Omega} f := \int\limits_{\Omega} f^+ - \int\limits_{\Omega} f^-$.*

Since f is absolutely integrable, $\int\limits_{\Omega} f^+$ and $\int\limits_{\Omega} f^-$ are less than or equal to $\int\limits_{\Omega} |f|$ and hence are finite. Thus $\int\limits_{\Omega} f$ is always finite. We also have a useful triangle inequality

$$|\int_{\Omega} f| \leq \int_{\Omega} f^+ + \int_{\Omega} f^- = \int_{\Omega} |f|.$$

Proposition 19.3.1. *Let Ω be a measurable set, and let $f : \Omega \to \boldsymbol{R}$ and $g : \Omega \to \boldsymbol{R}$ be absolutely integrable functions.*

1. *For any real number c, we have that cf is absolutely integrable and $\int\limits_{\Omega} cf = f \int\limits_{\Omega} f$.*

2. *The function $f + g$ is absolutely integrable, and $\int\limits_{\Omega} (f + g) = \int\limits_{\Omega} f + \int\limits_{\Omega} g$.*

3. *If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int\limits_{\Omega} f \leq \int\limits_{\Omega} g$.*

4. *If $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int\limits_{\Omega} f = \int\limits_{\Omega} g$.*

One cannot necessarily interchange limits and integrals, as the moving bump example showed. However, it is possible to exclude the moving bump example, and successfully interchange limits and integrals, if we know that the functions f_n are all majorized by a single absolutely integrable function.

Theorem 19.2. *(Lebesgue dominated convergence theorem) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of measurable functions from Ω to $\boldsymbol{R}^*$ which converge pointwise. Suppose also that there is an absolutely integrable function $F : \Omega \to [0, \infty]$ such that $|f(x)| \leq F(x)$ for all $x \in \Omega$ and all $n = 1, 2, 3, \ldots$. Then*

$$\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n.$$

Definition 19.3.3. *(Upper and lower Lebesgue integral) Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}$ be a function, not necessarily measurable. We define the upper Lebesgue integral $\overline{\int}_\Omega f$ to be*

$$\overline{\int}_\Omega f := \inf\left\{ \int_\Omega g : g \text{ is an absolutely integrable function from } \Omega \text{ to } \boldsymbol{R} \text{ that majorizes } f \right\}$$

and the lower Lebesgue integral $\underline{\int_\Omega} f$ to be

$$\underline{\int_\Omega} f := \sup\left\{ \int_\Omega g : g \text{ is an absolutely integrable function from } \Omega \text{ to } \boldsymbol{R} \text{ that minorizes } f \right\}$$

Lemma 19.3.1. *Let Ω be a measurable subset of $\boldsymbol{R}^n$, and let $f : \Omega \to \boldsymbol{R}$ be a function, not necessarily measurable. Let A be a real number, and suppose $\overline{\int}_\Omega f = \underline{\int_\Omega} f = A$. Then f is absolutely integrable and $\int_\Omega f = A$.*

19.4 Comparison with the Riemann integral

Now we show that Lebesgue integral is a generalization of the Riemann integral.

Proposition 19.4.1. *Let $I \subseteq \boldsymbol{R}$ be an interval, and let $f : I \to \boldsymbol{R}$ be a Riemann integrable function. Then f is also absolutely integrable, and the two integrals are equivalent.*

The converse is not true. Take for instance the function $f : [0, 1] \to \boldsymbol{R}$ defined by $f(x)" = 1$ where x is rational, and $f9x) := 0$ when x is irrational. f is not Riemann integrable, while the Lebesgue integral is 0. Thus the Lebesgue integral can handle more functions than the Riemann integral. The Lebesgue integral also interacts well with limits, as the Lebesgue monotone convergence theorem, Fatou's lemma, and Lebesgue dominated convergence theorem already attest. There are no comparable theorems for the Riemann integral.

19.5 Fubini's theorem

We saw that in one dimension the Lebesgue integral is connected to the Riemann integral. Now we try to understand that connection in higher dimensions. We look at the integrals of the form $\int_{\boldsymbol{R}^2} f$. Once we know how to integrate on $\boldsymbol{R}^2$, we can integrate on measurable subset Ω of $\boldsymbol{R}^2$, since $\int_\Omega f$ can be rewritten as $\int_{\boldsymbol{R}^2} f\chi_\Omega$. Let $f(x, y)$ be a function of two variables. We have three different ways to integrate f on $\boldsymbol{R}^2$. First, we can use the two-dimensional Lebesgue integral, to obtain $\int_{\boldsymbol{R}^2}$. Second, we can fix x and compute a one-dimensional integral of y, and then take that quantity and integrate in x, thus obtaining $\int_{\boldsymbol{R}} \left(\int_{\boldsymbol{R}} f(x, y)dy \right) dx$. Third, we could fix y and integrate x, and then integrate y, thus obtaining $\int_{\boldsymbol{R}} \left(\int_{\boldsymbol{R}} f(x, y)dx \right) dy$. Fortunately, if the function f is absolutely integrable on f, then all three integrals are equal.

Theorem 19.3. *(Fubini's theorem) Let $f : \boldsymbol{R}^2 \to \boldsymbol{R}$ be an absolutely integrable function. Then there exists absolutely integrable functions $f : \boldsymbol{R} \to \boldsymbol{R}$ and $G : \boldsymbol{R} \to \boldsymbol{R}$ such that for almost every x, $f(x, y)$ is absolutely integrable in y with $F(x) = \int_{\boldsymbol{R}} f(x, y)dy$, and for almost every y, $f(x, y)$ is absolutely integrable in x with $G(y) = \int_{\boldsymbol{R}} f(x, y)dx$. Finally, we have $\int_{\boldsymbol{R}} F(x)dx = \int_{\boldsymbol{R}^2} f = \int_{\boldsymbol{R}} G(y)dy$.*

Thus, Fubini's theorem allows us to compute two-dimensional integrals by splitting them into one-dimensional integrals. It is possible that the integral $\int_{\boldsymbol{R}} f(x,y)dy$ does not actually exist for every x, and similarly the integral $\int_{\boldsymbol{R}} f(x,y)dx$ does not exist for every y; Fubini's theorem only asserts that these integrals only exist for almost every x and y. For instance, if $f(x,y)$ is the function which equals 1 when $y > 0$ and $x = 0$, equals -1 when $y < 0$ and $x = 0$, and is zero otherwise, then f is absolutely integrable on $\boldsymbol{R}^2$ and $\int_{\boldsymbol{R}^2} f = 0$, since f equals zero almost everywhere in $\boldsymbol{R}^2$, but $\int_{\boldsymbol{R}} f(x,y)dy$ is not absolutely integrable when $x = 0$, though it is absolutely integrable for every other x.