FINAL PROJECT REPORT

::NUMERICAL METHODS::

MTD701 PROJECT

Department of Mathematics
Indian Institute of Technology,
New Delhi

Project Supervisor: Prof. Mani Mehra

Rakesh Prasad (2019MAS7093)
Manish K Kumawat(2019MAS7085)

Contents

1 Introduction

The fractional differential calculus has attracted a lot of attention by many researchers of different fields, such as Science and Engineering ,Physics, Chemistry, Biology, Economics, Control theory and Biophysics, etc. Since most fractional differential equations do not have exact analytic solutions, approximate and numerical techniques, are used extensively, such as the Adomian decomposition method and Power series method.
The Adomian decomposition method (ADM) is a powerful tool for solving both linear and nonlinear initial value problem of fractional differential equations .
The power series method is used to obtain solution of fractional differential equations with different types of initial conditions. Power series method is the most transparent method of solution of fractional differential equations.

2 Preliminaries

A real function $f(x)$ is said to be of class C, if $f(x)$ is piecewise continuous on $(0, \infty)$ and integrable on any finite subinterval of $(0, \infty)$

Definition: Let $f(x)$ be a function of class C, then the Riemann Liouville fractional integral of order $\alpha > 0$, is defined as
$I^\alpha f(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-s)^{\alpha-1} f(s) ds, \alpha > 0, t > 0$

Definition: Let α be a positive real number, such that $m - 1 < \alpha \leq m, m \in \mathbb{N}$ and let $f^{(m)}(x)$ exist and be a function of class C. Then the Caputo fractional derivative of f is defined as

$D^\alpha f(x) = I^{m-\alpha} \left(\frac{d^m}{dt^m} f(x) \right)$

Some properties of fractional integrals and derivatives:
(1) $\quad I^\alpha I^\beta f = I^{\alpha+\beta} f, \quad \alpha, \beta \geq 0$
(2) $\quad I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} x^{\gamma+\alpha}, \quad \alpha > 0, \gamma > -1$
(3) $I^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{x^k}{k!}, \quad m - 1 < \alpha \leq m$

3 The Modified Adomian decompositon method for solving different type of inhomogeneous nonlinear fractional differential equation

3.1 Type-I

Consider

$$D^{\alpha}u(x) + \alpha_0 u(x) + \beta f(u(x)) = g(x) \; u(0) = c_0, 0 < \alpha \leq 1$$

We suppose that $g(x)$ is analytic, so has Taylor expansion series:

$$g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$$

by properties of the fractional integral and derivatives we have

$$I^{\alpha}D^{\alpha}u(x) = u(x) - c_0,$$

$$I^{\alpha}g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds = \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds.$$

Now, applying the integral operator I^{α} to both sides of given differential equation we get

$$u(x) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds$$
$$- \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(u(s)) ds \ldots (1)$$

using

$$u(x)=\sum_{n=0}^{\infty}u_n\ldots(2)$$

and

$$\beta f(u(x))=\sum_{n=0}^{\infty}A_n\ldots(3)$$

where$A_n=A_n\left(u_0,u_1,\ldots,u_n\right)$ are the well-known Adomian polynomials

By equations $(2),(3)$ equation (1) becomes,

$$\sum_{n=0}^{\infty}u(x)=c_0+\frac{1}{\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{g_n}{n!}\int_0^x(x-s)^{\alpha-1}s^n ds$$

$$-\frac{\alpha_0}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sum_{n=0}^{\infty}u(s)ds-\frac{\beta}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sum_{n=0}^{\infty}A_n(s)ds$$

Now, we set the following recursion scheme:

$u(0)=c_0$,

$$u_{n+1}(x)=\frac{g_n}{\Gamma(\alpha)n!}\int_0^x(x-s)^{\alpha-1}s^n ds-\frac{\alpha_0}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}u_n(s)ds$$

$$-\frac{\beta}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}A_n(s)ds$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1}=\sum_{n=0}^{m}u_n(x)$$

$$\text{that gives: }\lim_{m\to\infty}\phi_{m+1}(x)=\sum_{n=0}^{\infty}u_n(x)=u^*(x)$$

Example(1): Consider the following type (I) inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha}u(x) + u^{2}(x) = 1, u(0) = c_0 = 0, 0 < \alpha \leq 1$$

Solution:-
The Exact solution when $\alpha = 1$ is $\frac{e^{2x}-1}{e^{2x}+1}$
Here

$$\alpha_0 = 0, \beta = 1$$

and the inhomogeneous term:

$$g(x) = 1\,(g_0 = 1, g_n = 0, n \geq 1)$$

and the first few Adomian polynomials for $f(u(x)) = u^{2}(x)$ are:

$$\begin{aligned} A_0(x) &= u_0^2(x) \\ A_1(x) &= 2u_0(x)u_1(x) \\ A_2(x) &= 2u_0(x)u_2(x) + u_1^2(x) \end{aligned}$$

by the similar way,
by the new modified recursion scheme for $\alpha = 0.98$,

$$\begin{aligned} &u_0(x) = c_0 = 0, \\ &u_1(x) = \tfrac{1}{\Gamma(\alpha)} \textstyle\int_0^x (x-s)^{\alpha-1} ds = 1.008x^{0.98}, \\ &u_2(x) = 0, \end{aligned}$$

$u_3(x) = \frac{-1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_1^2(s) ds = -0.350x^{2.94}$, ...
so

$$u(x) = 1.008x^{0.98} - 0.350x^{2.94} + \ldots$$

which is the required equation.

Example(2): Consider the following type (I) inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha}u(x) + u(x) = \sin(x), \quad u(0) = c_0 = 0, 0 < \alpha \leq 1$$

Solution:-The Exact Solution when $\alpha = 1$ is $\frac{-1}{2}\cos(x) + \frac{1}{2}\sin(x) + \frac{1}{2}e^{-x}$
Here

$$\alpha_0 = 1, \beta = 0$$

and the nonhomogeneous term:

$$g(x) = \sin(x)$$

where the first few coefficients $g_n = g^{(n)}(0)$ are given by:

$$g_0 = 0, g_1 = 1, g_2 = 0, g_3 = -1, \ldots$$

by the new modified recursion scheme for $\alpha = 0.98$,

$$u_0(x) = c_0 = 0$$
$$u_1(x) = 0$$
$$u_2(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} s ds = 0.509x^{1.98}$$

$$u_3(x) = \frac{-0.504}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} s^{1.98} ds = -0.175x^{2.96}$$

$\ldots$

so

$$u(x) = 0.509x^{1.98} - 0.175x^{2.96} + \ldots$$

which is the required equation.

3.2 Type II:

We now consider the following inhomogeneous nonlinear fractional differential equation

$$D^{\alpha}u(x) + \lambda_1 D^{\alpha-1}u(x) + \alpha_0 u(x) + \beta f\left(u(x), D^{\alpha-1}u(x)\right) = g(x) \ldots (4)$$
$$u(0) = c_0, u'(0) = c_1, 1 < \alpha \leq 2$$

Now, applying the integral operator I^{α} to both sides of equation(4),we obtain:

$$u(x) - (c_0 + c_1 x) = I^{\alpha}g(x) - I^{\alpha}\left(\lambda_1 D^{\alpha-1}u(x) + \alpha_0 u(x)\right) - I^{\alpha}(\beta f(u(x)))$$

By the properties of fractional integrals and derivatives, we have

$$I^{\alpha}D^{\alpha-1} = I^{1}I^{\alpha-1}D^{\alpha-1}, \quad \text{and so}$$
$$u(x) = c_0 + c_1 x + \tfrac{1}{\Gamma(\alpha)}\textstyle\sum_{n=0}^{\infty}\frac{g_n}{n!}\int_0^x (x-s)^{\alpha-1}s^n ds - \lambda_1 I\left(u(x) - c_0\right)$$

$$-\frac{\alpha_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u(s)ds - \frac{\beta}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(u(s))ds$$

Similarly to the type (I), we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_{n+2}(x) = \frac{g_n}{\Gamma(\alpha)n!}\int_0^x (x-s)^{\alpha-1}s^n ds - \lambda_1\int_0^x u_{n+1}(s)ds$$
$$-\frac{\alpha_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u_n(s)ds - \frac{\beta}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}A_n(s)ds$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x)$$

that gives:

$$\lim_{m\to\infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x)$$

Example: Consider the following type (II) inhomogeneous non linear fractional differential equation:

$$D^{\alpha}u(x)+e^{-2u(x)} = 0, u(0) = c_0 = 0, u'(0) = c_1 = 1, \quad 1 < \alpha \leq 2$$

solution- The Exact Dolution when $\alpha = 2$ is $\ln(1 + x)$,
Here

$$\lambda_1 = \alpha_0 = 0, \beta = 1$$

and the inhomogeneous term:

$$g(x) = 0$$

and the first few Adomian polynomials for $f\left(u(x), D^{\alpha-1}u(x)\right) = e^{-2u(x)}$ is:

$$\begin{aligned}
A_0(x) &= e^{-2u_0(x)}, \\
A_1(x) &= -2u_1(x)e^{-2u_0(x)}, \\
A_2(x) &= -2u_2(x)e^{-2u_0(x)} + 2u_1^2(x)e^{-2u_0(x)}
\end{aligned}$$

by the new modified recursion scheme for $\alpha = 1.98$,

$$\begin{aligned}
&u_0(x) = c_0 = 0,\\
&u_1(x) = c_1 x = x,\\
&u_2(x) = \tfrac{-1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds = -0.509x^{1.98},\\
&u_3(x) = \tfrac{2}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} s ds = 0.341x^{2.98},\\
&\dots\\
&u(x) = x - 0.509x^{1.98} + 0.341x^{2.98} + \dots
\end{aligned}$$

It is the required equation.

3.3 Type III:

We consider the following inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha}u(x) + \lambda_1 D^{\alpha-1}u(x) + \lambda_2 D^{\alpha-2}u(x) + \alpha_0 u(x)$$

$$+\beta f\left(u(x), D^{\alpha-2}u(x), D^{\alpha-1}u(x)\right) = g(x)..........(A)$$
$$u(0) = c_0, u'(0) = c_1, u''(0) = c_2, 2 < \alpha \leq 3$$

Now, applying the integral operator I^{α} to both sides of equation (A), we obtain:

$$u(x) - \left(c_0 + c_1 x + c_2 \frac{x^2}{2!}\right) = I^{\alpha} g(x) - I^{\alpha}\left(\lambda_1 D^{\alpha-1}u(x)\right.$$

$$\left.+\lambda_2 D^{\alpha-2}u(x) + \alpha_0 u(x)\right) - I^{\alpha}(N(u(x)))$$

Similarly to the type (II), we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_2(x) = c_2 \tfrac{x^2}{2!},$$
$$u_{n+3}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 \int_0^x u_{n+2}(s) ds$$
$$-\lambda_2 \textstyle\int_0^x (x-s) u_{n+1}(s) ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds$$
$$-\frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x)$$

that gives:

$$\lim_{m\to\infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x)$$

Example:- Consider the following type (III) inhomogeneous non linear fractional differential equation:

$$D^\alpha u(x) + D^{\alpha-2} u(x) + u^4(x) = (1+\sin(2x))^2$$
$$u(0) = c_0 = 1, u'(0) = c_1 = 1, u''(0) = c_2 = -1, 2 < \alpha \leq 3$$

Solution: Solution when $\alpha = 3$ is $\sin(x) + \cos(x)$,
Here

$$\lambda_2 = 1, \lambda_1 = 0, \alpha_0 = 0, \beta = 1$$

and the inhomogeneous term:

$$g(x) = (1+\sin(2x))^2$$

where the first few coefficients $g_n = g^{(n)}(0)$ are given by:

$$g_0 = 1, g_1 = 4, g_2 = 8, g_3 = -16, \ldots$$

and

the first few Adomian polynomials for

$f\left(u(x), D^{\alpha-1}u(x), D^{\alpha-2}u(x)\right) = u^4(x)$ is:

$$\begin{aligned}
&A_0(\text{x})\text{=}u_0^4,\\
&A_1(x) = 4u_0^3(x)u_1(x),\\
&A_2(x) = 4u_0^3(x)u_2(x) + 6u_0^2(x)u_1^2(x)
\end{aligned}$$

by the new modified recursion scheme for $\alpha = 2.98$,

$$\begin{aligned}
&u_0(x) = c_0 = 1,\\
&u_1(x) = c_1 x = x,\\
&u_2(x) = \tfrac{-x^2}{2},\\
&u_3(x) = -\int_0^x (x-s)s\,ds = \tfrac{-x^3}{6},\\
&u_4(x) = \tfrac{1}{2}\int_0^x (x-s)^{\alpha-1}s^2 ds = \tfrac{x^4}{24},\\
&\ldots
\end{aligned}$$

so

$$u(x) = 1 + x - \tfrac{x^2}{2} - \tfrac{x^3}{6} + \tfrac{x^4}{24} + \ldots.$$

3.4 Generalized form of inhomogeneous Nonlinear fractional differential Equation

We consider the follwing generalized form of inhomogeneous nonlinear fractional differential equation:

$$\begin{aligned}
&D^\alpha u(x) + \alpha_0 u(x) + \textstyle\sum_{i=1}^{m-1} \lambda_i D^{\alpha-i}u(x)\\
&+\beta f\left(u(x), D^{\alpha-1}u(x), \ldots, D^{\alpha-(m-1)}u(x)\right) = g(x)
\end{aligned}$$

$u(0) = c_0, u'(0) = c_1, \ldots, u^{(m-1)}(0) = c_{m-1}, \quad m-1 < \alpha \leq m, 2 \leq m$

Proceeding as before, we set the following recursion scheme:

$$\begin{aligned}
&u_0(x) = c_0,\\
&u_1(x) = c_1 x,\\
&u_2(x) = c_2 \tfrac{x^2}{2!},\\
&\vdots\\
&u_{m-1}(x) = c_{m-1} \tfrac{x^{m-1}}{(m-1)!},
\end{aligned}$$

$$\begin{aligned}
u_{n+m}(x) = &\frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds\\
&- \sum_{i=1}^{m-1} \frac{\lambda_i}{(i-1)!} \int_0^x (x-s)^{(i-1)} u_{n+m-i}(s) ds
\end{aligned}$$

$$-\frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x)$$

that gives:

$$\lim_{m \to \infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x).$$

Example: Consider the following inhomogeneous non-linear fractional differential equation:

$$
\begin{aligned}
&D^{\alpha}u(x) - 24e^{-5u(x)} = 0\\
&u(0) = c_0 = 1,\\
&u'(0) = c_1 = e^{-1},\\
&u''(0) = c_2 = -e^{-2},\\
&u'''(0) = c_3 = 2e^{-3},\\
&u^{(iv)}(0) = c_4 = -6e^{-4}, 4 < \alpha \leq 5
\end{aligned}
$$

Solution:- Solution when $\alpha = 5$ is $\ln(e + x)$, which we can consider here is

$$\alpha_0 = 0, \beta = -24, g(x) = 0, \lambda_i = 0 \text{ for } 1 \leq i \leq 4$$

here we have Approximate solution
for $\alpha = 2.98$
by the new modified recursion scheme for $\alpha = 4.98$,

$$
\begin{aligned}
&u_0(x) = 1\\
&u_1(x) = e^{-1}x\\
&u_2(x) = -e^{-2}\tfrac{x^2}{2}\\
&u_3(x) = e^{-3}\tfrac{x^3}{3}\\
&u_4(x) = -e^{-4}\tfrac{x^4}{4}\\
&u_5(x) = e^{-5}(0.206)x^{4.98}
\end{aligned}
$$

so

$$u(x) = 1+e^{-1}x-e^{-2}\frac{x^2}{2}+e^{-3}\frac{x^3}{3}-e^{-4}\frac{x^4}{4}+e^{-5}(0.206)x^{4.98}+\ldots$$

4 Power series method:

The idea is to obtain the solution in the power series form whose coefficients are to be determined.
The solution can be computed approximately as partial sum of series, that is why this method is frequently used for solving applied problems. One-term equation with zero, nonzero and $\left[{}_0D_t^{\alpha-1}y(t)\right]_{t=0} = B$ type initial conditions are considered.

4.1 One-term equation with zero initial condition:

Consider the one-term fractional initial value problem

$$ {}_0D_t^{\alpha}y(t) = f(t), \quad y(0) = 0, t > 0, \quad 0 < \alpha < 1.......(A)$$

Let

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+\alpha}..........(B)$$

be the power series solution of equation (A).Assume that the function $f(t)$ can be expanded in the Taylor series converging for $0 \leq t \leq R$, where R is the radius of convergence:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}t^n..........(C)$$

Using equation (B) and (C) in equation (A), we have

$${}_0D_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{n+\alpha}\right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}t^n$$

applying the formula ${}_aD_t^{\alpha}t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)}t^{\nu-\alpha}, \nu > -1$,
$\alpha > 0$
we obtain

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(1+n+\alpha)}{\Gamma(1+n)} t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$$

Equating coefficients of like powers of t, one can determine the coefficients by formula

$$a_n = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} \frac{f^{(n)}(0)}{n!} = \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)}, \quad n = 0, 1, 2, 3 \ldots \cdots (D)$$

Substituting these coefficients in equation (B), the series solution of given fractional initial value problem is

$$y(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)} t^{n+\alpha}$$

Clearly the series converges absolutely with region of convergence $(0, \infty)$.

Example: Consider the one-term fractional initial value problem

$${}_0D_t^{\frac{1}{2}} y(t) = \text{sint}, \quad y(0) = 0 \quad t > 0$$

Now, we determine the coefficients $a_n, n = 0, 1, 2\dots$ by above equation (D)

$$\begin{aligned}
&a_0 = \tfrac{f(0)}{\Gamma\left(\frac{3}{2}\right)},\\
&a_1 = \tfrac{f'(0)}{\Gamma\left(\frac{5}{2}\right)},\\
&a_2 = \tfrac{f''(0)}{\Gamma\left(\frac{7}{2}\right)},\\
&a_3 = \tfrac{f'''(0)}{\Gamma\left(\frac{9}{2}\right)},\\
&\dots\\
&a_n = \tfrac{f^{(n)}(0)}{\Gamma\left(n+\frac{3}{2}\right)},\\
&\dots
\end{aligned}$$

since

$$\begin{aligned}
&f(t) = \text{sint},\\
&f'(t) = \text{cost},\\
&f''(t) = -\text{sint},\\
&f^{(3)}(t) = -\text{cost},\\
&f^{(4)}(t) = \text{sint}
\end{aligned}$$

so that

$$\begin{aligned}
&f(0) = 0,\\
&f'(0) = 1,\\
&f''(0) = 0,\\
&f^{(3)}(0) = -1,\\
&f^{(4)}(0) = 0
\end{aligned}$$

Hence

$$\begin{aligned}
a_0 &= 0, \\
a_1 &= \tfrac{1}{\Gamma\left(\frac{5}{2}\right)}, \\
a_2 &= 0, \\
a_3 &= \tfrac{-1}{\Gamma\left(\frac{9}{2}\right)}, \\
a_4 &= 0, \\
a_5 &= \tfrac{1}{\Gamma\left(\frac{13}{2}\right)}, \ldots
\end{aligned}$$

Thus the series solution of given fractional initial value problem is

$$y(t) = \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{16}{105\sqrt{\pi}}t^{\frac{7}{2}} + \frac{64}{10395\sqrt{\pi}}t^{\frac{11}{2}} - \frac{256}{2027025\sqrt{\pi}}t^{\frac{15}{2}} + \ldots$$

Remark: If the function $f(t)$ on RHS of above (A) has the form $f(t) = t^{\beta}g(t), \quad \beta >$ -1 then also we can discuss the power series method. Assume that $g(t)$ can be expanded in the Taylor series converging for $0 \leq t \leq R$, where R is the radius of convergence, as

$$g(t) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} t^n$$

The coefficients $a_n, \quad n = 0, 1, 2 \ldots$ are given by

$$a_n = \frac{\Gamma(1+n+\beta)}{\Gamma(1+n+\alpha+\beta)} \frac{g^{(n)}(0)}{\Gamma(1+n)} \cdots (E)$$

Then series solution of fractional initial value problem(A) is

$$y(t) = \sum_{n=0}^{\infty} \frac{\Gamma(1+n+\beta)}{\Gamma(1+n+\alpha+\beta)} \frac{g^{(n)}(0)}{\Gamma(1+n)} t^{n+\alpha+\beta}$$

Example: Consider the one-term fractional initial value problem

$$ {}_0D_t^{\frac{1}{2}} y(t) = t^2 \text{ sint} \ , \quad y(0) = 0, \quad t > 0 $$

Determine the coefficients $a_n, n = 0, 1, 2\ldots$ from (E) as

$$
\begin{aligned}
a_0 &= 0, \\
a_1 &= \tfrac{\Gamma(4)}{\Gamma(\frac{9}{2})\Gamma(2)}, \\
a_2 &= 0, \\
a_3 &= \tfrac{\Gamma(6)}{\Gamma(\frac{13}{2})\Gamma(4)}, \\
a_4 &= 0, \\
a_5 &= \tfrac{\Gamma(8)}{\Gamma(\frac{17}{2})\Gamma(6)}, \ldots
\end{aligned}
$$

Thus the series solution of fractional initial value problem is

$$ y(t) = \frac{96}{105\sqrt{\pi}} t^{\frac{7}{2}} - \frac{1280}{10395\sqrt{\pi}} t^{\frac{11}{2}} + \frac{10572}{2027025\sqrt{\pi}} t^{\frac{15}{2}} - \ldots $$

4.2 One -term equation with nonzero initial condition:

Consider the one-term fractional initial value problem

$$ {}_0D_t^{\alpha} y(t) = f(t), \quad y(0) = A, \quad A \neq 0, \quad t > 0, \quad 0 < \alpha < 1....(A) $$

Assume that

$$ f(t) = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha} .....(B) $$

where the coefficients f_n are known.

Let

$$y(t) = \sum_{n=0}^{\infty} a_n t^n .....(C)$$

be the series solution of fractional initial value problem(A). Thus, the fractional initial value problem (A) reduces to

$${}_0D_t^{\alpha} \left[\sum_{n=0}^{\infty} a_n t^n \right] = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha}$$

Applying equation ${}_aD_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}, \nu > -1, \alpha > 0$ to yield

$$\sum_{n=0}^{\infty} a_n \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha} = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha}$$

Equating coefficients of like powers of t, we obtain

$$a_0 = A, \quad a_n = \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} f_n, \quad n = 1, 2, 3, \ldots$$

Hence, the series solution of fractional initial value problem (A) is

$$y(t) = A + \sum_{n=1}^{\infty} \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} f_n t^n$$

Example: Consider the one-term fractional initial value problem
${}_0D_t^{\frac{1}{2}}y(t) = f(t), \quad y(0) = \frac{3}{2}, t > 0 \quad$ and $f(n) = n$
determine the coefficients $a_n, n = 0, 1, 2\ldots$ as

$$\begin{aligned}
&a_0 = \tfrac{3}{2},\\
&a_1 = \tfrac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \times 1,\\
&a_2 = \tfrac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} \times 2,\\
&a_3 = \tfrac{\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)} \times 3,\\
&a_4 = \tfrac{\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)} \times 4\\
&\ldots
\end{aligned}$$

therefore, the series solution of the given fractional initial value problem is

$$y(t) = \frac{3}{2} + \frac{1}{2}\sqrt{\pi}t\left[1 + \frac{3}{2}t + \frac{15}{4}t^2 + \ldots\right]$$

which is the required solution.

5 Boundary value Problem

5.1 Some Results for Boundary Value Problem

(H1) The function $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is jointly continuous.
(H2) There exists a constant $l > 0$ such that
$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq l[|u - \bar{u}| + |v - \bar{v}|]$ for each $t \in J$, and each $u, \bar{u} \in \mathbb{X}, v, \bar{v} \in \mathbb{X}$
(H3) $k : \Delta \times X \to X$ is continuous and there exists a constant $l_1 > 0$, such that $|k(t, s, u) - k(t, s, v)| \leq l_1|u - v|, \forall u, v \in X, \quad t, s \in \Delta$.

Theorem: Assume that (H1) to (H3) holds.
If l (1+l_1)A <1 , where

$$A=\frac{1}{\Gamma(\alpha+1)}+\frac{|\beta|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+\alpha+1)\,|\Gamma(p+1)-\beta\eta^{p}|}$$

then

$$cD^{\alpha}y(t)=f\left(t,y(t),\int_0^t k(t,s,y(s))ds\right),\quad t\in J=[0,1],t\neq t_k\ldots(A)$$
$$y\left(t_k^{+}\right)=y\left(t_k^{-}\right)+y_k\ldots(B)$$
$$y(0)=\beta I^{p}y(\eta),0<\eta<1\ldots(C)$$

where $k=1,\ldots,m,0<\alpha\leq 1,{}^{c}D^{\alpha}$ is the Caputo fractional derivative
has a unique solution on J.
Theorem :(H4) There exists a constant $M>0$ such that $|f(t,u,v)|\leq M$ for each $t\in J$ and each $u,v\in\mathbb{X}$.
(H5) There exists a constant $M^{*}>0$ such that $\sum_{i=1}^{k}|y_i|\leq M^{*}$ for each

$$k=1,\ldots,m$$

are satisfied. The problem above problem (A) -(C) has at least one solution on J provided that

$$\frac{1}{\Gamma(\alpha+1)}+\frac{|\beta|\eta^{p+\alpha}\Gamma(p+1)}{|\Gamma(p+\alpha+1)|\Gamma(p+1)-\beta\eta^{p}\,|}<1$$

.

Example

Let us consider the impulsive fractional boundary-value problem,

$$
\begin{gathered}
{}^{c}D^{\frac{1}{2}}y(t)=\frac{|y|}{2\left(e^{t}+1\right)^{2}(1+|y|)}+\frac{1}{2}\int_{0}^{t} e^{-\frac{1}{4}y(s)}ds, \quad \alpha\in(0,1], \quad t\in J:[0,1]....(i)\\
y\left(t_1^{+}\right)=y\left(t_1^{-}\right)+\frac{1}{3}.....(ii)\\
y(0)=\sqrt{3}I^{1/2}y\left(\frac{1}{3}\right).....(iii)
\end{gathered}
$$

$$
\begin{gathered}
f(t,x(t),By(t))=\frac{|y|}{2\left(e^{t}+1\right)^{2}(1+|y|)}+By(t), \quad (t,x)\in J\times[0,\infty)\\
By(t)=\frac{1}{2}\int_{0}^{t} e^{-\frac{1}{4}y(s)}ds
\end{gathered}
$$

Let $x,y\in[0,\infty)$ and $t\in J$.

Then we have $\alpha=\frac{1}{2},\beta=\sqrt{3},p=\frac{1}{2},\eta=\frac{1}{3}$

$$
\begin{aligned}
|Bx(t)-By(t)|=&\frac{1}{2}\left|\int_{0}^{t} e^{-\frac{1}{4}x(s)}ds-\int_{0}^{t} e^{-\frac{1}{4}y(s)}ds.\right|\\
&\leq\frac{1}{8}|x(t)-y(t)|\leq\frac{1}{8}|x-y|
\end{aligned}
$$

and

$$
|f(t,x,Bx(t))-f(t,y,By(t))|=\frac{1}{2\left(e^{t}+1\right)^{2}}\left|\frac{x}{1+x}-\frac{y}{1+y}+\frac{1}{8}(Bx(t)-By(t))\right|
$$

$$
\begin{aligned}
&=\frac{1|x-y|}{2\left(e^{t}+1\right)^{2})(1+x)(1+y)}+\frac{1}{8}|(Bx(t)-By(t))|\\
&\leq\frac{1}{2\left(e^{t}+1\right)^{2}}|x-y|+\frac{1}{8}|(Bx(t)-By(t))|\\
&\leq\frac{1}{8}[|x-y|+|(Bx(t)-By(t))|]
\end{aligned}
$$

Hence the conditions (H2) $-$ (H3) holds with $l=\frac{1}{8}$ and $l_1=\frac{1}{2}$. We shall check that condition,indeed

$$
l\left(1+l_1\right)\left[\frac{1}{\Gamma(\alpha+1)}+\frac{|\beta|\eta^{p+\alpha}\Gamma(p+1)}{\Gamma(p+\alpha+1)\left|\Gamma(p+1)-\beta\eta^{p}\right|}\right]<1
$$

which is satisfied for some $\alpha \in (0,1]$. Then by above Theorem the problems $(i) - (iii)$ has unique solution.

5.2 Boundary Value problem

consider the following nonlinear Reimann-Liouville fractional differential equation boundary value problem:-

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t), x(t)) + g(t, x(t)), \quad 0 < t < 1, 1 < \alpha < 2, \\ x(0) = x(1) = 0 \text{............}(A) \end{cases}$$

and

$$\begin{cases} D_{0+}^{\alpha} x(t) = f(t, x(t), x(t)) + g(t, x(t)), \quad 0 < t < 1, 1 < \alpha < 2 \\ x(0) = 0, \quad x(1) = \beta x(\eta) \text{...........}(B) \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville fractional derivative and $\eta, \beta\eta^{\alpha-1} \in (0,1)$

For solving this equation,following results must be satisfied :-

($R_{1.}$) $f(t,x,y) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous and increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$ and $y \in [0,+\infty)$ decreasing in $y \in [0,+\infty)$ for fixed $t \in [0,1]$ and $x \in [0,+\infty)$.

($R_{2.}$) $g(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1], g(t,0) \neq 0$.

($R_{3.}$) there exists a constant $\delta_0 > 0$ such that
$f(t,x,y) \geq \delta_0 g(t,x), t \in [0,1], x, y \geq 0$.

($\mathrm{R}_{4.}$) $g(t, \lambda x) \geq \lambda g(t, x)$ for $\lambda \in (0,1), t \in [0,1]$,
u $\in$ $[0, +\infty)$ and there exists a constant ξ $\in$(0,1) such that $f\left(t, \lambda x, \lambda^{-1} y\right) \geq \lambda^{\xi} f(t, x, y), \forall t \in [0,1], x, y \in [0, +\infty)$.
Then there exist $x_0, y_0 \in P_h$ and $\gamma \in (0,1)$ such that $\gamma y_0 \leq x_0 < y_0$ and

$$\begin{array}{ll} x_0(t) \leq \int_0^1 G(t,s)\left[f\left(t, x_0(s), y_0(s)\right) + g\left(s, x_0(s)\right)\right] ds, & t \in [0,1] \\ y_0(t) \geq \int_0^1 G(t,s)\left[f\left(s, x_0(s), y_0(s)\right) + g\left(s, x_0(s)\right)\right] ds, & t \in [0,1] \end{array}$$

where $h(t) = t^{\alpha-1}(1-t), t \in [0,1]$.

Example: Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\frac{5}{3}} x(t) = x^{\frac{1}{3}} + \arctan x + y^{-\frac{1}{3}} + t^2 + t^3 + \frac{\pi}{2}, & 0 < t < 1 \\ x(0) = x(1) = 0 \end{cases}$$

In this case, $\alpha = \frac{5}{3}$.
This Problem can be regarded as a boundary value problem of the form (A) with
$f(t, x, y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$ and $g(t, x) = \arctan x + t^3$.
Now we verify that conditions(R_1) $-$ (R_4) are satisfied.
Firstly, it is easy to see (R_1) and (R_2) are satisfied and $g(t, 0) = t^3 \not\equiv 0$.
Secondly, take $\delta_0 \in (0,1]$, we obtain

$$f(t, x, y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$$

$$
\begin{aligned}
&\geq t^2 + \tfrac{\pi}{2} \\
&\geq t^3 + \arctan x \\
&\geq \delta_0 \left(t^3 + \arctan x\right) \\
&= \delta_0 g(t, x)
\end{aligned}
$$

Thus, (R_3) is satisfied. Moreover, for any $\lambda \in (0,1), t \in [0,1], x \in [0,\infty), y \in [0,\infty)$, we get $\arctan(\lambda x) \geq \lambda \arctan x$. Therefore,

$$
\begin{aligned}
g(t, \lambda x) &\geq \lambda g(t, x), \\
f\left(t, \lambda x, \lambda^{-1} y\right) &= \lambda^{\frac{1}{3}} x^{\frac{1}{3}} + \lambda^{\frac{1}{3}} y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2} \\
&\geq \lambda^{\frac{1}{3}} \left(x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \tfrac{\pi}{2}\right) \\
&= \lambda^{\gamma} f(t, x, y)
\end{aligned}
$$

where $\gamma = \frac{1}{3}$.

We conclude that condition (H_4) is satisfied. Therefore, above theorem ensures that the given BVP has a unique positive solution in P_h with $h(t) = t^{\frac{1}{3}}(1-t)$.

Example: Consider the following Boundary Value problem :

$$
\begin{cases}
-D_{0+}^{\frac{3}{2}} x(t) = 2x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + t^3, \quad 0 < t < 1 \\
x(0) = 0, \quad x(1) = \frac{1}{2} x\left(\frac{1}{2}\right)
\end{cases}
$$

In this case, $\alpha = \frac{3}{2}$.

This problem can be regard as a boundary value problem of form (B) with $f(t, x, y) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2$ and $g(t, x) = x^{\frac{1}{2}} + t^3$.

Now we verify that conditions (R_1) – (R_4) are satisfied.

Firstly, it is easy to see (R_1) and (R_2) are satisfied and $g(t,0)=t^3\neq 0$.
Secondly, take $\delta_0\in(0,1]$, we obtain

$$f(t,x,y)=x^{\frac{1}{2}}+y^{-\frac{1}{2}}+t^2\geq x^{\frac{1}{2}}+t^3\geq \delta_0\left(x^{\frac{1}{2}}+t^3\right)=\delta_0 g(t,x)$$

Thus, (R_3) is satisfied. Moreover, for any $\lambda\in(0,1), t\in[0,1], x\in[0,\infty), y\in[0,\infty)$, we have

$$g(t,\lambda x)=\lambda^{\frac{1}{2}}x^{\frac{1}{2}}+t^3\geq\lambda^{\frac{1}{2}}\left(x^{\frac{1}{2}}+t^3\right)\geq\lambda g(t,x)$$
$$f\left(t,\lambda x,\lambda^{-1}y\right)=\lambda^{\frac{1}{2}}x^{\frac{1}{2}}+\lambda^{\frac{1}{2}}y^{-\frac{1}{2}}+t^2\geq\lambda^{\frac{1}{2}}\left(x^{\frac{1}{2}}+y^{-\frac{1}{2}}+t^2\right)=\lambda^{\gamma}f(t,x,y)$$

where $\gamma=\frac{1}{2}$. We conclude that condition (R_4) is satisfied. Therefore, above results ensures that the above Boundary value problem has a unique Positive Solution in P_h with $h(t)=t^{\frac{1}{2}}$.

6 Conclusion

In this project we have introduced the modification of the Adomian decomposition method presented for solving initial value problem for fractional differential equation. The great advantage of this new modified method is that, we apply I^{α} on Taylor expansion series of function g and this technique allows us to admit various type of analytic functions as g, since sometimes it is difficult to compute $I^{\alpha}(g)$ when I^{α} is a fractional integral and g is a trigonometric or exponential function. The results for numerical examples demonstrate that the present method can give a more accurate approximation.
And the power series method best for One-term equation with zero, nonzero initial value fractional differential equations.
We also introduce the existance and uniqueness of fractional boundary value problem.

References:

1.Numerical Solutions Of The Initial Value Problem For Fractional Differential Equations By Modification Of the Adomian Decomposition Method by Neda Khodabakhshi,S. Mansour Vaezpour and Dumitru Baleanu(Nov 2014).

2.Methods of Solving Fractional Differential equations of order $0 < \alpha < 1$ by J.A.Nanware and Gunwant A.Birajdar Bulletin of the Marathawada Mathematical Society Vol.15(2014).

3.Fractional boundary value problems with Riemann-Liouville fractional derivatives by Jingjing Tan and Cao-zong Cheng (2015).

4.Riemann-Liouville integral boundary value problems for impulsive fractional integro-differential equations by P. Karthikeyan and R.Arul(Article -2013).