FRACTIONAL CALCULUS

Introduction to fractional calculus

It deals with the investigation and application of integrals of integrals and derivatives of arbitrary order. we know that

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots\cdots\cdots \frac{d^ny}{dx^n}$$

$$\int f(x), \int f(x,y) \ldots\ldots\ldots$$

what happens if this order is arbitrary number's i.e.the order is not an integer it can be negative,fraction or it can be complex number?

Fractional derivative

Fractional derivative of order α of a function f(t) will be denoted by

$$\mathrm{D}_{a,t}^{\alpha} f(t)$$

where α is an integer or fraction or complex and a and t are the bounds of the operation fractional differentiation.

Fractional integral

Fractional integral of order α of a function f(t) will be denoted by

$$\mathrm{D}_{a,t}^{-\alpha} f(t)$$

where α is an integer or fraction or complex and a and t are the bounds of the operation fractional differentiation.

Fractional derivative formula of arbitrary order for monomials

$$let \qquad f(x) = x^k$$

$$f'(x) = k.x^k$$

$$f''(x) = k.(k-1).x^k$$

$$......$$

$$(f^n)(x) = \frac{k!}{(k-n)!}.x^{k-n}$$

$$\Longrightarrow = \frac{\Gamma(k+1)}{\Gamma(k-n+1)}.x^{k-n}$$

$\Longrightarrow$ derivative of arbitrary order α

$$D^{\alpha}x^{k} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)}.x^{k-n} e.g. \alpha = \frac{1}{2}, k = 1$$

$$D^{\frac{1}{2}}x = \frac{\Gamma\left(1\right)}{\Gamma\left(\frac{3}{2}\right)}.x^{\frac{1}{2}}$$

Riemann-Liouville integral

with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as

$$\mathrm{D}_{a,t}^{-\alpha} f(t) = RL\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \mathrm{d}s$$

and

$$\mathrm{D}_{t,b}^{-\alpha} f(t) = RL\mathrm{D}_{t,b}^{-a} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{a-1} f(s) \mathrm{d}s$$

where $\Gamma(\cdot)$ is the Euler's gamma function.

Grünwald-Letnikov derivatives

with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as

$$GL\mathrm{D}^{\alpha}_{a,1} f(t) = \lim_{\substack{h \to 0 \\ Nh = l - a}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t - jh)$$

and

$$GL\mathrm{D}^{\alpha}_{t,b} f(t) = \lim_{\substack{h \to 0 \\ Nh = b - t}} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} f(t + jh)$$

Riemann-Liouville derivatives

with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as

$$RLD^{\alpha}_{a,t}f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\mathrm{D}^{-(m-\alpha)}_{a,d}f(t)\right]$$
$$= \frac{1}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}r^m}\int_a^t (t-s)^{m-\alpha-1}f(s)\mathrm{d}s$$

and

$$RLD^{\alpha}_{t,b}f(t) = (-1)^m\frac{d^m}{dt^m}[\mathrm{D}^{-(m-\alpha)}_{t,b}f(t)]$$

$$= \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\int_t^b (s-t)^{m-\alpha-1}f(s)\mathrm{d}s$$

where m is a positive integer satisfying

$$m-1 < \alpha < m$$

Caputo derivative

with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ are defined as

$$cD_{a,t}^{\alpha} f(t) = D_{a,t}^{-(m-\alpha)} \left[f^{(m)}(t) \right]$$

$$= \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s$$

and

$$cD_{t,b}^{\alpha} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_t^b (s-t)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s$$

where m is a positive integer satisfying $\mathrm{m} - 1 < \alpha \leq \mathrm{m}$

Riesz derivative

with order $\alpha > 0$ of the given function $f(t), t \in (a, b)$ is defined as

$$RZ\mathrm{D}_t^\alpha f(t) = c_\alpha \left(RL\mathrm{D}_{a,t}^\alpha f(t) + RL\mathrm{D}_{t,b}^\alpha f(t) \right)$$

where

$$c_\alpha = -\frac{1}{2\cos(\alpha\pi/2)}, \alpha \neq 2k+1, k = 0, 1, \cdots ,$$

$$RZ\mathrm{D}_t^\alpha f(t)$$

is sometimes expressed as

$$\frac{\partial^\alpha f(t)}{\partial |t|^\alpha}$$

Riemann-Liouville fractional integral operators satisfy the following semi-group properties

$$\mathrm{D}_{a,t}^{-\alpha}\mathrm{D}_{a,t}^{-\beta}f(t) = \mathrm{D}_{a,t}^{-\beta}\mathrm{D}_{a,t}^{-\alpha}f(t) = \mathrm{D}_{a,t}^{-\alpha-\beta}f(t)$$

$$\mathrm{D}_{t,b}^{-\alpha}\mathrm{D}_{t,b}^{-\beta}f(t) = \mathrm{D}_{tb}^{-\beta}\mathrm{D}_{t,b}^{-\alpha}f(t) = \mathrm{D}_{t,b}^{-a-\beta}f(t)$$

where $\alpha, \beta > 0$. If $f(t)$ is continuous on $[a, b]$, then

$$\lim_{t\to a}\mathrm{D}_{a,t}^{-\alpha}f(t) = \lim_{t\to b}\mathrm{D}_{t,b}^{-\alpha}f(t) = 0, \quad \forall\alpha > 0$$

Riemann-Liouville fractional derivative operators satisfy the following properties

$$RLD^{\alpha}_{a,t} D^{-a}_{a,t} f(t) = f(t)$$
$$RLD^{\alpha}_{t,b} D^{-a}_{tb} f(t) = f(t)$$

where $\alpha > 0$.

Proposition:- Let $\alpha > 0$, $n - 1 < \beta < n$, n is a positive integer, f $\in C^n[a, b]$ then

$$\mathrm{D}_{a,\lambda}^{-a}\left(c\mathrm{D}_{a,1}^{\beta} f(t)\right) = c\mathrm{D}_{a,t}^{\beta-a} f(t), \alpha \neq \beta$$
$$\mathrm{D}_{t,b}^{-a}\left(c\mathrm{D}_{tb}^{\beta} f(t)\right) = c\mathrm{D}_{t,b}^{\beta-a} f(t), \alpha \neq \beta$$

where

$$c\mathrm{D}_{a,x}^{\beta-\alpha} = \mathrm{D}_{a,d}^{\beta-\alpha} \text{ and } c\mathrm{D}_{tb}^{\beta-\alpha} = \mathrm{D}_{t,b}^{\beta-\alpha} \text{ if } \beta < \alpha$$

Especially

$$\mathrm{D}_{a,t}^{-\beta}\left(c^{-}\mathrm{D}_{a,t}^{\beta}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(k+1)}(t-a)^{k}$$

$$\mathrm{D}_{t,b}^{-\beta}\left(c\mathrm{D}_{tb}^{\beta}f(t)\right)=f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(b)}{\Gamma(k+1)}(b-t)^{k}$$

Some other properties of fractional derivatives

The fractional integrals and derivatives are linear operators, i.e.,

$$D^{\alpha}(\lambda f(t) + \mu g(t)) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t)$$

where D^{α} denotes any fractional integral or derivative hereafter.

Leibnitz Rule for fractional Derivative:-

if $f(t)$ and $g(t)$ together with their derivatives are continuous in $[a, t]$. Under these conditions the Leibniz rule for fractional differentiation takes the form:

$$RLD_{a,t}^{\alpha}(g(t)f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} g^{(k)}(t) f^{(\alpha-k)}(t)$$

Fractional Derivative of a Composite Function

The Leibniz rule for the fractional derivative can be used to obtain the fractional derivative for a composite function.

$$RLD^{\alpha}_{a,t}F(h(t)) = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}F(h(t))$$

$$+\sum_{k=1}^{\infty}\binom{\alpha}{k}\frac{k!(t-a)^{k-\alpha}}{\Gamma(1-\alpha+k)}\sum_{m=1}^{k}F^{(m)}(h(t))\sum\prod_{r=1}^{k}\frac{1}{a_r!}\left(\frac{h^{(r)}(t)}{r!}\right)^{a_r}$$

where the sum $\sum$ extends over all combinations of non-negative integer values of $a_1, a_2, \cdots, a_k$ such that

$$\sum_{r=1}^{k} r a_r = k \quad \text{and} \quad \sum_{r=1}^{k} a_r = m$$

Laplace Transforms of Fractional Derivatives

The Laplace transform of the Riemann-Liouville derivative of function f(t) of order $\alpha > 0$

$$L\left\{ {}_{RL}\mathrm{D}^{\alpha}_{0,t} f(t); s \right\} = s^{\alpha} L\{f(t); s\} - \sum_{k=0}^{m-1} s^{k} \left[RL\mathrm{D}^{\alpha-k-1}_{0,t} f(t) \right]_{t=0}$$

$$m - 1 \leq \alpha < m$$

Fourier Transforms of Fractional Derivatives

The Fourier transform of the Riemann-Liouville derivative of function f(t) of order $\alpha > 0$

$$F\left\{ {}_{RL}\mathrm{D}^{\alpha}_{-\infty,t} f(t);\omega \right\} = (i\omega)^{\alpha} F\{f(t);\omega\},$$

$$F\left\{ {}_{RL}\mathrm{D}^{\alpha}_{t,\infty} f(t);\omega \right\} = (-i\omega)^{\alpha} F\{f(t);\omega\}.$$

Directional Integrals and Derivatives in R^2

Let $\alpha > 0, \theta \in [0, 2\pi)$ be given. The α -th order fractional integral in the direction of θ is given by

$$\mathrm{D}_{\theta}^{-\alpha} u(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \xi^{\alpha-1} u(x - \xi \cos\theta, y - \xi \sin\theta) \mathrm{d}\xi$$

Remark : It is easy to see that for special directions as $\theta = 0, \pi/2, \pi$ and $3\pi/2$,
the directional operator is reduced to left and right Riemann-Liouville integral operators,
i.e.

$$\mathrm{D}_0^{-\alpha} u(x,y) = \mathrm{D}_{-\infty,x}^{-\alpha} u(x,y)$$

$$\mathrm{D}_\pi^{-a} u(x,y) = \mathrm{D}_{x,\infty}^{-\alpha} u(x,y)$$

$$\mathrm{D}_{\pi/2}^{-a} u(x,y) = \mathrm{D}_{-\infty,y}^{-\alpha} u(x,y)$$

$$\mathrm{D}_{3\pi/2}^{-a} u(x,y) = \mathrm{D}_{y,\infty}^{-\alpha} u(x,y)$$

the α -th order fractional derivative

Let n be a positive integer satisfying $n-1 \leq \alpha < n, \theta \in [0, 2\pi)$. Then the α -th order fractional derivative in the direction θ is defined by

$$\mathrm{D}_\theta^\alpha u(x,y) = \mathrm{D}_\theta^n \mathrm{D}_\theta^{-(n-\alpha)} u(x,y)$$

where D_θ^n is given by

$$\mathrm{D}_\theta^n u(x,y) = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)^n u(x,y)$$

Proposition-The fractional directional integral operator satisfies the following semi-group properties

$$\mathrm{D}_\theta^{-\alpha}\mathrm{D}_e^{-\beta}u(x,y)=\mathrm{D}_\theta^{-\alpha-\beta}u(x,y)$$

where $\alpha,\beta>0,\theta\in[0,2\pi),u\in L^2\left(\mathbb{R}^2\right)$
Proposition :- For $\alpha>0,\theta\in[0,2\pi),u\in L^2\left(\mathbb{R}^2\right)$, the following relation holds

$$\mathrm{D}_\theta^{\alpha}\mathrm{D}_e^{-\alpha}u(x,y)=u(x,y)$$

Proposition :- The fractional directional integral operator $D_\theta^{-\alpha}$ satisfies the following Fourier transform property

$$F\left\{\mathrm{D}_\theta^{-\alpha} u(x,y);\omega\right\} = \left(i\omega_1\cos\theta + i\omega_2\sin\theta\right)^{-\alpha} F\{u(x,y);\omega\}$$

where $\omega = (\omega_1, \omega_2)$ and

$$F\{u(x,y);\omega\} = \int_{\mathbb{R}^2} e^{-i(\omega_1 x+\omega_2 y)} u(x,y)\mathrm{d}x\ \mathrm{d}y$$

Proposition :- For $u \in C_0^\infty(\Omega), \Omega \in \mathbb{R}^2$ and $\alpha > 0$, we have

$$F\left\{\mathrm{D}_\theta^{\alpha} u(x,y);\omega\right\} = \left(i\omega_1\cos\theta + i\omega_2\sin\theta\right)^{\alpha} F\{u(x,y);\omega\}$$

Partial Fractional Derivatives

Similar to the classical partial derivatives, we can also define the partial fractional derivatives . For example, let $0 < \alpha_1, \alpha_2 < 1$, the partial fractional derivative ${}_{RL}\mathrm{D}^{\alpha_1+\alpha_2}_{x^{\alpha}1y^{\alpha_2}} u(x,y)$ is defined by
Obviously, if $u(x,y)$ is "good" enough, then one can easily obtain

$$RL^{\mathrm{D}}_{x^{\alpha}1\mathrm{y}^{\alpha_2}}{}^{\alpha_1+\alpha_2} u(x,y) = RL^{\mathrm{D}}_{0,y}{}^{\alpha_2}\left[RL^{\mathrm{D}}_{0,x}{}^{\alpha_1} u(x,y)\right]$$

$$={}_{RL}\ \mathrm{D}^{\alpha_2}_{0,y}\left[\frac{1}{\Gamma(1-\alpha_1)}\frac{\partial}{\partial x}\int_0^x (x-s)^{-\alpha_1} u(s,y)\mathrm{d}s\right]$$

$$=\frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)}\frac{\partial^2}{\partial x\partial y}\int_0^x\int_0^y (x-s)^{-\alpha_1}(y-\tau)^{-\alpha_2} u(s,\tau)\mathrm{d}\tau\mathrm{d}s$$

Obviously, if $u(x,y)$ is "good" enough, then one can easily obtain

$$RL\mathrm{D}^{\alpha_1+\alpha_2}_{x^{\alpha}1y^{\alpha_2}} u(x,y) = RL^{\mathrm{D}}_{y^{\alpha}2x^{\alpha_1}}{}^{\alpha_2+\alpha_1} u(x,y)$$

Hadamard Type fractional Integrals and Fractional derivatives

Let $(a, b)(0 \leqq a < b \leqq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$ and let $\Re(\alpha) > 0$ and $\mu \in \mathbb{C}$. We consider the left-sided and right-sided integrals of fractional order $\alpha \in \mathrm{C}(\mathrm{R}(\alpha) > 0)$ defined by

(A) $\left(\mathcal{J}_{a+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t} \quad (a < x < b)$

and

(B) $\left(\mathcal{J}_{b-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_z^b \left(\log \frac{t}{x}\right)^{a-1} \frac{f(t)dt}{t} (a < x < b)$

respectively.

When $a = 0$ and $b = \infty$, these relations are given by

$$\left(\mathcal{J}_{0+}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log \frac{x}{t}\right)^{\alpha-1} \frac{f(t)dt}{t} (x > 0)$$

and

$$\left(\mathcal{J}_{-}^{\alpha} f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_z^\infty \left(\log \frac{t}{x}\right)^{\alpha-1} \frac{f(t)dt}{t} (x > 0)$$

properties of Hadamard type fractional integral

If $\Re(\alpha) > 0, \Re(\beta$, and $0 < a < b < \infty$, then

$$\left(\mathcal{J}_{a+}^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}$$

$$\left(D_{a+}^{a}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}$$

and

$$\left(\mathcal{J}_{b-}^{\alpha}\left(\log\frac{b}{t}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{b}{x}\right)^{\beta+\alpha-1}$$

$$\left(D_{b-}^{\alpha}\left(\log_{i}^{b}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{b}{x}\right)^{\beta-\alpha-1}$$

The Hadamard and Hadamard type fractional integrals satisfy the following semigroup property

Let $\Re(\alpha) > 0, \Re(\beta) > 0$, and $1 \leqq p \leqq \infty$

(a) If $0 < a < b < \infty$, then, for $f \in L^p(a,b)$

$$\mathcal{J}^{\alpha}_{a+}\mathcal{J}^{\beta}_{a+}f = \mathcal{J}^{\alpha+\beta}_{a+}f(c \leqq 0) \text{ and } \mathcal{J}^{\alpha}_{b-}\mathcal{J}^{\beta}_{b-}f = \mathcal{J}^{\alpha+\beta}_{b-}f(c \geqq 0)$$

(b) If $\mu \in \mathbb{C}$, $c \in \mathbb{R}$, a=0 and b=∞, then, for $f \in X^p_c(\mathbb{R}^+)$

$$\mathcal{J}^{\alpha}_{0+,\mu}\mathcal{J}^{\beta}_{0+,\mu}f = \mathcal{J}^{\alpha+\beta}_{0+,\mu}f(\Re(\mu) > c) \text{ and } \mathcal{J}^{\alpha}_{-,\beta}\mathcal{J}^{\beta}_{-,i1}f = \mathcal{J}^{\alpha+\beta}_{-,\mu}f(\Re(\mu) > -c)$$

In particular, when $\mu = 0$,

$$\mathcal{J}_{0+}^{\alpha}\mathcal{J}_{0+}^{\beta}f = \mathcal{J}_{0+}^{\alpha+\beta}f(c<0), \quad \mathcal{J}_{-}^{\alpha}\mathcal{J}_{-}^{\beta}f = \mathcal{J}_{-}^{\alpha+\beta}f(c>0)$$

Examples of fractional derivative and integrals

(1):-

$$\frac{d^2}{dx^2}\left(x^5\right) = \frac{\Gamma(5+1)}{\Gamma(5+1-\frac{1}{2})}x^{5-1/2}$$
$$= \frac{\Gamma(6)}{\Gamma(1/2)}x^{\frac{9}{2}}$$

$$= \frac{(5)!}{\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}x^{\frac{9}{2}}$$
$$= \frac{120\times 16}{9\times 7\times 5\times 3\sqrt{\pi}}x^{\frac{9}{2}}$$

(2)

$$\frac{d^{1/2}(x)}{dx^{1/2}} = \frac{\Gamma(1+1)}{\Gamma(1+1-\frac{1}{2})}x^{(1-\frac{1}{2})} = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2}$$

$$= \frac{1}{\frac{1}{2}\times\sqrt{\pi}}x^{1/2} = \frac{2}{\sqrt{\pi}}\sqrt{x}$$

(3)

$$\frac{d^{1/2}}{dx^{1/2}}\left(\frac{d^{1/2}}{dx^{1/2}}x\right)$$

solution:- $\frac{d^{1/2}}{dx^{1/2}}\left(\frac{2}{\sqrt{\pi}}\sqrt{x}\right)$

$$= \frac{2}{\sqrt{\pi}} \cdot \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\sqrt{x}$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{1}{2}+1-\frac{1}{2})}x^{\frac{1}{2}-\frac{1}{2}} = \frac{2}{\sqrt{\pi}} \times \Gamma(\frac{3}{2})$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{1}{2} \times \sqrt{\pi}$$

$$= 1$$

$$\frac{d^{1/2}}{dx^{1/2}}\left(\frac{2}{\sqrt{\pi}}\sqrt{x}\right) = 1$$

Note:- If $P(x)$ is a polynomial then $P'(X)$ is also a polynomial. since polynomial and continuous and differentiable So $P^n(x)$ is also continuous and differentiable. But this doesn't happen in case of fractional derivative .
As $P(X) = 1$ is continuous and differentiable and $P'(X) = 0$ is also continuous and differentiable. But

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}(1) = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{x}}$$ is not continuous or not defined at zero.

$$\frac{d^\alpha}{dx^\alpha}(C)$$

is not continuous at the orgin for α being a fraction.

Grunwalad- Letnikov fractional derivative :- **we know forward difference**

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

backward difference

$$\frac{d}{dx}f(x) = \lim_{h \to 0} \frac{f(x) - f(x-h)}{h}$$

$$\frac{d^2}{dx^2}f(x) = \frac{d}{dx}f'(x)$$

$$= \lim_{h \to 0} \frac{f'(x) - f'(x-h)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x)-f(x-h)}{h} - \frac{f(x-h)-f(x-2h)}{h}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h^2}\{f(x) - 2f(x-h) + f(x-2h)\}$$

$$\frac{d^3}{dx^3} f(x) = \lim_{h\to 0} \frac{1}{h^3}[f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)]$$

....

$$\frac{d^k}{dx^k} f(x) = \lim_{h \to 0} \left(\frac{1}{h} \right)^k \sum_{\ell=0}^{k} (-1)^l \left({}^k c_l \right) f(x - lh)$$

our goal is to extend to N to R

$$\frac{d^K}{dx^k} f(x) = \lim_{h \to 0} \left(\frac{1}{h} \right)^k \sum_{l=0}^{k} (-1)^l \frac{k!}{\ell!(k-l)!} f(x - lh)$$

$$\frac{d^K}{dx^k} f(x) = \lim_{h \to 0} \left(\frac{1}{h} \right)^k \quad \sum_{\ell=0}^{k} (-1)^l \frac{(k)!}{\ell!(k-l)!} f(x - lh)$$

extend $k \to \alpha$ where $\alpha \in \mathbb{R}$.

$$\frac{d^\alpha}{dx^\alpha} f(x) = \lim_{k \to \infty} \left(\frac{k}{x-a} \right)^\alpha \sum_{l=0}^{k} (-1)^l \frac{(\alpha)!}{\ell!(\alpha - l)!} \times f\left(x - \frac{l}{k}(x-a) \right)$$

Connection between derivative and integral

consider $\alpha = 1$

$$
{}_a D^{-1} x = \lim_{n \to \infty} (h)^1 \sum_{\ell=0}^{h} \frac{\Gamma(1+l)}{\ell! \sqrt{1}} f(x - lh)
$$

$$
{}_a D^{-1} x(f) = \lim_{n \to \infty} \sum_{\ell=0}^{h} f(x - lh)
$$

$$
{}_a D^{-1} x(f) = \int_0^{x-a} f(x-t) dt
$$

$$
dt = -du
$$

$$
u = x - t
$$

$$
du = -dt
$$

$$
t = 0 \Rightarrow u = x
$$

$$
{}_a D^{-1} X(f) = \int^{x} f(t) dt
$$

consider $\alpha = 2$

$$a^{D^{-2}}xf = \lim_{n\to\infty} h^2 \sum_{\ell=0}^{n} \frac{\Gamma(1+2)}{\ell!\sqrt{2}} f(x-lh)$$

$$a^{D^{-2}}xf = \lim_{n\to\infty} h^2 \sum_{\ell=0}^{n} (l+1) f(x-lh)$$

$$a^{D^{-2}}xf = \lim_{n\to\infty} h^2 \sum_{\ell=1}^{n+1} lf(x-lh)$$

$$= \lim_{n\to\infty} \sum_{\ell=1}^{n+1} lhf(x-lh)h$$

$$= \int_0^{x-a} tf(x-t)dt$$

$$a^{D^{-2}}xf = \int^{x} tf(x-t)dt$$

Similarly

for $\alpha = 3$

$_aD^{-3}_x f = \frac{1}{2!}\int_a^x (x-t)^2 f(x-t)dt$

$_aD^{-4}_x f = \frac{1}{3!}\int_a^x (x-t)^3 f(x-t)dt$

In genral

$_aD^{-(k+1)}{}_x f = \frac{1}{k!}\int_a^x (x-t)^k f(x-t)dt$

$\cong$**cauchy integral representation**

$_aD^{(-\alpha)}{}_x f = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt \to$ **fractional integral formula**

Fractional Derivative of Constant From a physical point of view, it is reasonable to have the fractional derivative of a constant equal to zero. For the Riemann-Liouville fractional derivative, it holds that

$$ {}_{\mathrm{RL}}D^{a}_{0,t}c = \frac{c}{\Gamma(1-\alpha)}t^{-a} \neq 0 $$

where c is a constant.

The following property is one of the advantages of the Caputo fractional derivative over the RL fractional derivative.

Lemma:- For the Caputo fractional derivative, it holds that

$$D^{\alpha} c = 0$$

Proof- According the definition of the Caputo fractional derivative, we have

$$D^{\alpha} c = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{c^{(m)}}{(t-\tau)^{a-m+1}} \, \mathrm{d}\tau = 0$$

where $m - 1 \leq \alpha < m$ and $m \in Z^{+}$.

Fractional Derivative of the Power Function

To facilitate the analysis of the Caputo fractional

derivative for the power function, we first give the Taylor expansion, as follows

f(t)=f(0)+f′(0)$t + \frac{f''(0)}{2!}t^2 + \frac{f'''(0)}{3!}t^3 + \cdots$

If $D^a t^p$ is known, the Caputo fractional derivative for arbitrary function can be

described as

$$D' f(t) = D^a \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} D^a t^k$$

the *RL* fractional derivative of the power function satisfies

$$\mathrm{RL}D_{0,t}^{\alpha} t^{p} = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha}$$

where $m-1 < \alpha < m, p > -1,$ and $p \in \Re$

Furthermore, the Caputo fractional derivative of the power function satisfies

$D^a t^p$

$$= \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-a} = & m-1 < \alpha < m, \quad p > n-1, \quad p \in \Re \\ 0, & m-1 < \alpha < m, \quad p \leq n-1, \quad p \in Z^+ \end{cases}$$

For $p > m - 1$, the Caputo fractional derivative of the power function is a generalization of the integer-order derivative of the power function. We have

$$(t^p)^{(n)} = \left(pt^{p-1}\right)^{(n-1)} = \left(p(p-1)t^{p-2}\right)^{(n-2)}$$

$= \cdots = p(p-1)\cdots(p-n+1)t^{p-n}$

$= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} t^{p-n}, \quad n \in Z^+, p \in \Re$

On the basis of this analysis, the Caputo fractional derivative for an arbitrary function $f(t)$ can be computed by the following equation:

$$D^a \mathrm{e}^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+m} t^{k+m-a}}{\Gamma(k+1+m-\alpha)} = \lambda^m t^{m-a} E_{1,m-a+1}(\lambda t)$$

where λ is a positive constant.

Fractional Derivatives of sine and Cosine Functions

The behavior of the Caputo fractional derivative for sine and cosine functions is discussed in this section. For the sine function we have

$$
\begin{aligned}
D^a \sin \lambda t &= D^a \frac{\mathrm{e}^{\mathrm{i}\lambda t} - \mathrm{e}^{-\mathrm{i}\lambda t}}{2\mathrm{i}} \\
&= \frac{1}{2\mathrm{i}} \left(D^a \mathrm{e}^{\mathrm{i}\lambda t} - D^\alpha \mathrm{e}^{-\mathrm{i}t} \right)
\end{aligned}
$$

$$= \frac{1}{2\mathrm{i}}\left((\mathrm{i}\lambda)^m t^{m-a} E_{1,m-a+1}(\mathrm{i}\lambda t) - (-\mathrm{i}\lambda)^m t^{m-a} E_{1,\ \mathrm{m}-a+1}(-\mathrm{i}\lambda t)\right)$$

$$= \frac{1}{2\mathrm{i}}(\mathrm{i}\lambda)^m t^{m-\alpha}\left(E_{1,m-\alpha+1}(\mathrm{i}\lambda t) - (-1)^m t^{m-a} E_{1,m-a+1}(-\mathrm{i}\lambda t)\right)$$

For the cosine function, we obtain

$$D^a \cos \lambda t = \frac{1}{2}(i\lambda)^m t^{m-a}\left(E_{1,m-a+1}(i\lambda t) + (-1)^m t^{m-\alpha} E_{1,m-a+1}(-i\lambda t)\right)$$

Example:-Solve the following three-term initial-value problem.

$$\mathcal{D}_0^3 y(t) + 2\mathcal{D}_0^1 y(t) + y(t) = 0$$

$$\mathcal{D}_0^{\frac{1}{3}} y(t)\Big|_{t=0} = 1$$

$$\mathcal{D}_0^{-\frac{1}{2}} y(t)\Big|_{t=0} = 2$$

The Laplace transform gives the formula:

$$s^{\frac{4}{3}}Y(s) - 1 - 2s^{\frac{5}{6}} + 2s^{\frac{1}{2}}Y(s) - 4 + Y(s) = 0$$

and under the assumption $|_{s^{\frac{4}{3}}+2s^{\frac{1}{2}}}|< 1$ we can write: Then we use the term-by-term inversion and get the result:

$$y(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Big[2t^{13}k - \frac{4}{3} E^{(k)}_{\frac{5}{6},\frac{4}{3}k-\frac{1}{3}} \left(-2t^5\right) + 5t^{13}k$$

$$-\frac{1}{2} E^{(k)}_{\frac{5}{6},\frac{4}{3}k+\frac{1}{2}} \left(-2t^5\right)$$

Application of fractional calculus

(1)**The Tautochrone Problem:** This well-known example was for the first time studied by Abel in the early 19^{th} century. It was one of the basic problems where the framework of the fractional calculus was used although it is not essentially necessary .

The problem is the following:

To find a curve in the (x, y) -plane such that the time required for a particle to slide down the curve to its lowest point is independent of its initial placement on the curve; suppose the homogeneous gravity field and no friction. Let us fix the lowest point of a curve at the origin and the position of a curve in the positive quadrant of the plane, denoting by (x, y) the initial point and (x^*, y^*) any point intermediate between (0,0) and (x, y).

According to the energy conservation law we may write

$$\frac{m}{2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 = mg\,(y - y^*)$$

where σ- the length along the curve measured from the origin,

m- the mass of the particle,

g-the gravitational acceleration.

Considering $\frac{\mathrm{d}\sigma}{\mathrm{d}t} < 0$ and $\sigma = \sigma\left(y^*(t)\right)$,

we rewrite the formula in the form

$$\sigma' . \frac{dy^*}{dt} = -\sqrt{2g\left(y - y^*\right)}$$

which we integrate from $y^* = y$ to $y^* = 0$ and from $t = 0$ to $t = T$.

After some calculations we get the integral equation:

$$\int_0^y \frac{\sigma'\left(y^*\right)}{\sqrt{y - y^*}} \, \mathrm{d}y^* = \sqrt{2g}\,T$$

Here one can easily recognize the Caputo differintegral and write

$$
{}^{\mathrm{C}}\mathrm{D}_0^1\sigma(y) = \frac{\sqrt{2}g}{\Gamma\left(\frac{1}{2}\right)}T
$$

Let us note that T is the time of descent, so it is a constant.

By applying the $\frac{1}{2}$ -integral to both sides of the equation and by using the formulas for the composition of the Caputo differintegrals and for the fractional integral of the constant, we get the relation between the length along the curve and the initial position in y direction.

$$\sigma(y) = \frac{\Gamma(1)\sqrt{2g}\,T}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)} y^{\frac{1}{2}} = \frac{2\sqrt{2g}\,T}{\pi} y^{\frac{1}{2}}$$

The formula describing coordinates of points generating the curve can be written by the help of the relation:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}y} = \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}$$

which after the substitution of $\sigma(y)$ gives It can be shown that the solution of this equation is so-called tautochrone, i.e. one arch of the cycloid which arises by rolling of the circle along the green line in figure

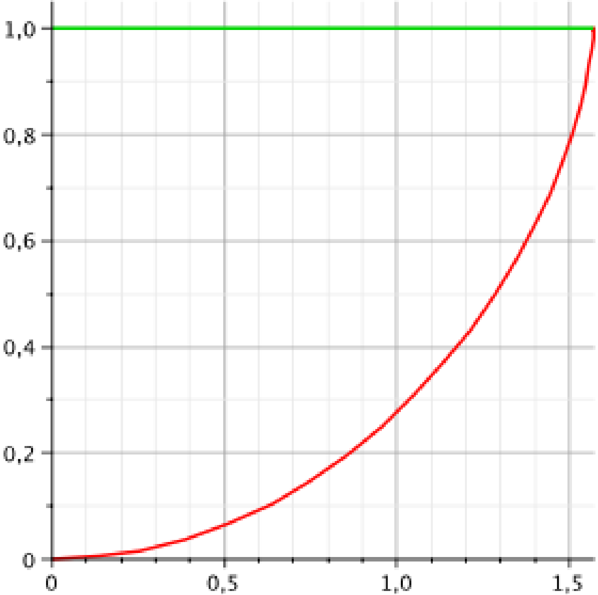
.

The parametric equations of the tautochrone are

$$x = \frac{A}{2}[u + \sin(u)]$$
$$y = \frac{A}{2}[1 - \cos(u)]$$

where $A = \frac{2gT^2}{\pi^2}$.

In particular for $T = \frac{\pi}{\sqrt{2g}}$, i.e. for $A = 1$, the tautochrone is drawn in figure by the red color.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0,6
0,4
0,2
0
0
0,5
1,0
1,5

(2) WORLD POPULATION GROWTH:-

(2)-(a)**Standard Approach :-**There exist several attempts to describe the World Population Growth . The simplest model is the following, known as the Malthusian law of population growth, which is used to predict populations under ideal conditions. Let $N(t)$ be the number of individuals in a population at time t, B and M the birth and mortality rates, respectively, so that the net growth rate is given by

$$N'(t) = (B - M)N(t) = PN(t)$$

where $P := B - M$ is the production rate. Here, we assume that B and M are constant, and thus P is also constant.

The solution of this differential equation is the function

$$\mathrm{N(t)}{=}\mathrm{N_0 e}^{Pt}, \quad t \geq 0$$

where N_0 is the population at $t = 0$. Because of the solution , this model is also known as the exponential growth model.

(A)(b)Fractional approach- Considered now that the World Population Growth model is ruled by the fractional differential equation

$$ {}_0^C D_t^{\alpha} N(t) = PN(t), \quad t \geq 0, \alpha \in (0, 1) $$

Observe that, taking the limit $\alpha \to 1^-$, Eq (3) converts into Eq. 1, but if we consider $\alpha \in (1, 2)$ and take the limit $\alpha \to 1^+$, we obtain $N'(t) - N'(0) = PN(t)$

The solution of this fractional differential equation is the function

$$N(t) = N_0 E_\alpha \left(P t^\alpha \right)$$

where E_α is the Mittag-Leffler function

$$E_\alpha(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + 1)}, \quad t \in \mathbb{R}$$

(A)(b)Numerical experiment :- For our numerical treatment of the problem, we find several databases with the world population through the centuries. Here, we use the one provided by the United Nations from year 1910 until 2010 , where the initial value is $N_0 = 1750$. For the classical approach, the production rate is

$$P \approx 1.3501 \times 10^{-2}$$

and the error from the data with respect to the analytic solution is given by

$$E_{\text{classical}} \approx 7.0795 \times 10^{5}$$

If we take into consideration the fractional model (4), for

$\alpha \in (0, 2)$ we obtain that the best values are

$$\alpha = 1.393298754843208 \text{ and } P \approx 3.4399 \times 10^{-3}$$

with error

$$E_{\text{fractional}} \approx 2.0506 \times 10^{5}$$

The gain of the efficiency in this procedures is

$$\frac{7.0795 \times 10^{5} - 2.0506 \times 10^{5}}{7.0795 \times 10^{5}} \approx 0.71$$

The graph of Figure 2 illustrates the World Population Growth model with the data, the classical model and the fractional model.

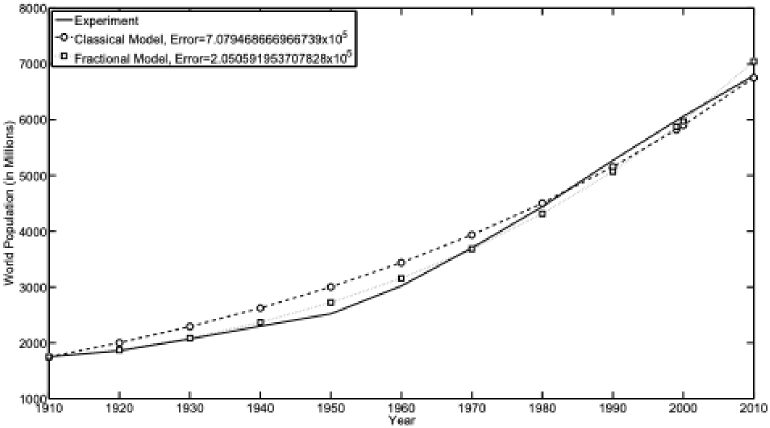

Figure 2: World Population Growth model.

References:-

1.Theory and Applications of Fractional Differential Equation (NORTH HOLLAND)
2.Numerical Methods for Fractional Calculus (CHAPMAN HALL/CRC)

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