Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 3

1. Let $\{X_n\}$ be a sequence of random variables defined on the probability space $([0,\infty),\beta_1,P),\ P(\{0\})=0$.

   (a) Define $X_n(s)=\frac{1}{s}\left(1-\frac{1}{n}\right),\ s\in[0,\infty)$. Then show that $X_n\overset{a.s}{\to}X$, where $X(s)=\frac{1}{s},\ s\in[0,\infty)$.

   (b) Define $X_n(s)=\frac{1}{ns},\ s\in[0,\infty)$. Then show that $X_n\overset{a.s}{\to}0$.

   (c) Let $P(I)=\displaystyle\int_I e^{-x}dx$. Define

   $$X_n(s)=\begin{cases}0, & \text{if } s \text{ is rational}\\ (-1)^n, & \text{if } s \text{ is irrational}\end{cases}$$

   Then show that $\{X_n\}$ diverges almost surely.

2. Consider a sequence of random variables $\{X_n\}$ with $P(X_n=0)=1-\frac{1}{n^\alpha},\ P(X_n=\pm n)=\frac{1}{2n^\alpha}$. Determine the value of α for which the sequence obeys WLLN.

3. Consider a sequence of random variables $\{X_n\}$ with $E(X_n)=m$ and

   $$Cov(X_i,X_j)=\begin{cases}\sigma^2, & i=j\\ a\sigma^2, & i=j\pm1,\ \text{where } |a|<1,\ \sigma^2>0 \text{ are given contants}\\ 0, \text{otherwise}\end{cases}$$

   Show that WLLN holds for $\{X_n\}$.

4. Use CLT to show that $\displaystyle\lim_{n\to\infty}e^{-n}\sum_{i=0}^{n}\frac{n^i}{i!}=0.5$

5. Consider a sequence of independent random variables $\{X_n\}$ such that

   $$P(X_n=0)=1-\frac{2}{n^3},\ P(X_n=\pm n)=\frac{1}{n^3},\ n>1$$

   Does the sequence $\{X_n\}$ obey CLT ?

6. If $X_n\overset{p}{\to}0$, then find the median of $X_n\ \to\ 0$ as $n\to\infty$.

7. Consider a sequence $\{X_n\}$ of identically distributed random variables with the property that $nP(|X_i|>n)\to 0$ as $n\to\infty$. Show that $\frac{1}{n}\max_{1\le i\le n}X_i\overset{p}{\to}0$

8. Suppose $|X_n-X|\le Y_n$, almost surely for some random variable X, then show that of $E(Y_n)\to 0$, then $E(X_n)\to E(X)$ and $X_n\overset{p}{\to}X$.

9. Show that $X_n\overset{2}{\to}X\ \Rightarrow\ E(X_n)\to E(X),\ E(X_n^2)\to E(X^2)$ as $n\to\infty$.

10. Let $\{X_i\}$ be a sequence of independent random variables, such that each X_i has mean 0 and variance 1. Show that

    $$\sqrt{n}\frac{X_1+X_2+\cdots+X_n}{X_1^2+X_2^2+\cdots+X_n^2}\overset{l}{\to}Z\sim N(0,1)$$

11. Does WLLN hold for the following sequences

    (a) $P(X_k=\pm 2^k)=\frac{1}{2}$

    (b) $P(X_k=\pm\frac{1}{k})=\frac{1}{2}$

12. For what values of α, does the strong law of large numbers hold for the sequence $\{X_n\}$, where $P(X_k=\pm k^\alpha)=\frac{1}{2},\ k=1,2,\ldots$.

13. Let $\{X_n\}$ be a sequence of independent random variables with the following probability distribution. In each case, does the Lindeberg Condition for CLT hold ?

(a) $P(X_k = \pm\frac{1}{2^n}) = \frac{1}{2}$

(b) $P(X_k = \pm\frac{1}{2^{n+1}}) = \frac{1}{2^{n+3}},\ P(X_n = 0) = 1 - \frac{1}{2^{n+2}}$

14. From an urn containing 10 identical balls numbered 0 through 9, n balls are drawn with replacement,

(a) What does the law of large number tell you about the appearance of 0's in n drawings.

(b) How many drawings must be made in order that with probability atleast 0.95, the relative frequency of occurrence of 0's will be between 0.09 and 0.11 ?