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Solving fractional two-point boundary value problems using continuous analytic method

Ahmad El-Ajou [a], Omar Abu Arqub [a,*], Shaher Momani [b]

[a] *Department of Mathematics, Faculty of Science, Al Balqa Applied University, Salt 19117, Jordan*
[b] *Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan*

Abstract   In this article, the homotopy analysis method is applied to provide approximate solutions for linear and nonlinear two-point boundary value problems of fractional order. The solution was calculated in the form of a convergent power series with easily computable components. In this method, one has great freedom to select auxiliary functions, operators, and parameters in order to ensure the convergence of the approximate solution and to increase both the rate and region of convergence. Numerical examples are provided to demonstrate the accuracy and efficiency of the present method. Meanwhile, further iterations can produce more accurate results and decrease the error.

1. Introduction

Fractional differential equations (DEs) have received considerable attention in the recent years due to their wide applications in the areas of applied mathematics, physics, engineering, economy, and other fields. Many important phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are well described by fractional DE [1–7]. It is well known that the fractional order differential and integral operators are non-local operators. This is one reason why fractional differential

operators provide an excellent instrument for description of memory and hereditary properties of various physical processes. For example, half-order derivatives and integrals proved to be more useful for the formulation of certain electrochemical problems than the classical models [5–9]. An excellent account in the study of fractional DEs can be found in [10,11]. Motivated by increasing number of applications of fractional DEs, considerable attention has been given to provide efficient methods for exact and numerical solutions of fractional DEs.

In general, most of fractional DEs do not have exact solutions. Particularly, there is no known method for solving fractional boundary value problems (BVPs) exactly. As a result, numerical and analytical techniques have been used to study such problems. It should be noted that much of the work published to date concerning exact and numerical solutions is devoted to the initial value problems for fractional order ordinary DEs. The theory of BVPs for fractional DEs has received attention quiet recently. The attention drawn to

* Corresponding author. Address: P.O. Box, Al-Salt 19117, Jordan.
E-mail address: o.abuarqub@bau.edu.jo (O. Abu Arqub).

the theory of existence and uniqueness of solutions to BVPs for fractional order DEs is evident from the increased number of recent publications. The reader is asked to refer [12–18] in order to know more details about the fractional BVPs, including its history and kinds, existence and uniqueness of solution, applications and methods of solutions, etc.

The purpose of this article is to extend the application of the homotopy analysis method (HAM) to provide symbolic approximate solution for two-point BVP of fractional order which is as follows:

$$D_*^\alpha u(x) = f(x, u(x), u'(x), u''(x), \ldots, u^{(n)}(x)), \quad a \leqslant x \leqslant b, \quad (1)$$

subject to the boundary conditions

$$u^{(i)}(a) = a_i, \quad u^{(i)}(b) = b_i, \quad i \in \{0, 1, 2, \ldots, k-1\}, \quad (2)$$

and subject to the constraints conditions

$$k - 1 < \alpha \leqslant k, \quad k \geqslant n,$$
$$\#\{a_i, b_i\} = k, \quad i \in \{0, 1, 2, \ldots, k-1\},$$

where $D_*^\alpha u(x)$ is the fractional derivative of order α of $u(x)$ in the sense of Caputo, $n, k \in \mathbb{N}$, $\#$ denote the number of elements, $u(x)$ is an unknown function of independent variable x to be determined, $f : [a, b] \times \mathbb{R}^{n+1} \to \mathbb{R}$ is nonlinear continuous function of x, $u(x)$, $u'(x)$, $u''(x)$, $\ldots$, $u^{(n)}(x)$, and a, b, a_i, b_i are real finite constants.

The numerical solvability of fractional BVPs and other related equations have been pursued by several authors. To mention a few, in [12], the authors have discussed the Chebyshev spectral method for solving equation $D_*^{1.5} u(x) + u''(x) + u(x) = g(x)$. Furthermore, the collocation-shooting method is carried out in [13] for the fractional equation $D_*^{1.5} u(x) + Au''(x) + Bu(x) = g(x)$. The monotone iterative sequences method has been applied to solve the fractional equation $D_*^\alpha u(x) + g(x, u(x)) = 0$, where $1 < \alpha \leqslant 2$ as described in [14]. Recently, the piecewise polynomial collocation approach for solving linear fractional BVP of the form $D_*^{1.5} u(x) + u''(x) + u(x) = g(x)$ is proposed in [15]. However, none of previous studies propose a methodical way to solve fractional BVP (1) and (2). Moreover, previous studies require more effort to achieve the results, they are not accurate, and usually, they are suited for linear form of fractional BVP (1) and (2).

But on the other aspects as well, the applications of other versions of series solutions to linear and nonlinear problems can be found in [19–21], and for numerical solvability of different categories of two-point BVPs, one can consult Refs. [22,23].

The HAM, which proposed by Liao [24–29], is effectively and easily used to solve some classes of linear and nonlinear problems without linearization, perturbation, or discretization. The HAM is based on the homotopy, a basic concept in topology. The auxiliary parameter $\hbar$ and the auxiliary function $H(x)$ are introduced to construct the so-called zero-order deformation equation. Thus, unlike all previous analytic techniques, the HAM provides us with a family of solution expressions in auxiliary parameter $\hbar$. As a result, the convergence region and rate of solution series are dependent upon the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$, and thus can be greatly enlarged by means of choosing a proper value of $\hbar$ and (or) $H(x)$. This provides us with a convenient way to adjust and control convergence region and rate of solution series given by the HAM.

In the last years, extensive work has been done using HAM, which provides analytical approximations for nonlinear equations. This method has been implemented in several differential and integral equations, such as nonlinear water waves [27], unsteady boundary-layer flows [28], solitary waves with discontinuity [29], Klein-Gordon equation [30], fractional initial value problems [31], fractional SIR model [32], BVPs for integro-DEs [33], and others.

The organization of this paper is as follows: in the next section, we present some necessary definitions and preliminary results that will be used in our work. In Section 3, basic idea of the HAM is introduced. In Section 4, we utilize the statement of the HAM for solving fractional BVPs. In Section 5, numerical examples are given to illustrate the capability of HAM. The discussion and conclusion are given in the final part, Section 6.

2. Preliminaries

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

For the concept of fractional derivative, there are several definitions. The two most commonly used are the Riemann–Liouville and Caputo definitions. Each definition uses Riemann–Liouville fractional integration and derivatives of whole order. The difference between the two definitions is in the order of evaluation. The Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. The Riemann–Liouville fractional derivative is computed in the reverse order. Therefore, the Caputo fractional derivative allows traditional initial and boundary conditions to be included in the formulation of the problem, but the Riemann–Liouville fractional derivative allows initial conditions in terms of fractional integrals and their derivatives. For homogeneous initial condition assumption, these two operators coincide.

Definition 2.1. A real function $f(x)$, $x > 0$ is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_μ^n iff $f^{(n)}(x) \in C_\mu$, $n \in \mathbb{N}$.

Definition 2.2. The Riemann–Liouville fractional integral operator of order $\alpha \geqslant 0$, of a function $f(x) \in C_\mu$, $\mu \geqslant -1$ is defined as $J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t) dt$, $\alpha > 0$, $x > 0$ and $J^0 f(x) = f(x)$, where Γ is the well-known Gamma function.

Properties of the operator J^α can be found in [4–7], we mention only the following: for $f \in C_\mu$, $\mu \geqslant -1$, α, $\beta \geqslant 0$, and $\gamma \geqslant -1$, we have $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x) = J^\beta J^\alpha f(x)$ and $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$.

Next, we shall introduce a modified fractional differential operator D_*^α proposed by Caputo in his work on the theory of viscoelasticity [4].

Definition 2.3. The fractional derivative of $f \in C_{-1}^n$ in the Caputo sense is defined as $D_*^\alpha f(x) = J^{n-\alpha} D^n f(x)$, $n - 1 < \alpha < n, x > 0$ and $D_*^\alpha f(x) = f^{(n)}(x), \alpha = n$, where $n \in \mathbb{N}$ and α is the order of the derivative.

Lemma 2.1. *If $n - 1 < \alpha \leqslant n$, $n \in \mathbb{N}$, and $f \in C_\mu^n$, $\mu \geqslant -1$, then $D_*^\alpha J^\alpha f(x) = f(x)$ and $J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x}{k!}$, $x > 0$.*

For mathematical properties of fractional derivatives and integrals, one can consult the mentioned references.

3. Homotopy analysis method

The principles of the HAM and its applicability for various kinds of DEs are given in [24–33]. For convenience of the reader, we will present a review of the HAM [24–29] then we will extend the HAM to construct a symbolic approximate solution to fractional BVPs.

To achieve our goal, we consider the following nonlinear equation:

$$N[u(x)] = 0, \quad x \geqslant a, \tag{3}$$

where N is a nonlinear operator and $u(x)$ is an unknown function of independent variable x.

Let $u_0(x)$ denote an initial guess approximation of the exact solution of Eq. (3), $\hbar \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, and $\mathcal{L}$ an auxiliary linear operator with the property $\mathcal{L}[f(x)] = 0$ when $f(x) = 0$. The auxiliary parameter $\hbar$, the auxiliary function $H(x)$, and the auxiliary linear operator $\mathcal{L}$ play important roles within the HAM to adjust and control the convergence region of series solution [24].

Liao [24–29] constructs, using $q \in [0, 1]$ as an embedding parameter, the so-called zero-order deformation equation

$$(1 - q)\mathcal{L}[\phi(x; q) - u_0(x)] = q\hbar H(x)N[\phi(x; q)], \tag{4}$$

where $\phi(x; q)$ is the solution of Eq. (3) which depends on $\hbar$, $H(x)$, $\mathcal{L}$, $u_0(x)$, and q. When $q = 0$, the zero-order deformation Eq. (4) becomes

$$\phi(x; 0) = u_0(x), \tag{5}$$

and when $q = 1$, since $\hbar \neq 0$ and $H(x) \neq 0$, the zero-order deformation Eq. (4) reduces to

$$N[\phi(x; 1)] = 0. \tag{6}$$

So, $\phi(x; 1)$ is exactly the solution of the nonlinear Eq. (3). Thus, according to Eqs. (5) and (6), as q increasing from 0 to 1, the solution $\phi(x; q)$ various continuously from the initial approximation $u_0(x)$ to the exact solution $u(x)$.

Define the so-called mth-order deformation derivatives

$$u_m(x) = \frac{1}{m!} \frac{\partial^m \phi(x; q)}{\partial q^m} \bigg|_{q=0}, \tag{7}$$

expanding $\phi(x; q)$ in Taylor series with respect to the embedding parameter q, using Eqs. (5) and (7), we obtain

$$\phi(x; q) = u_0(x) + \sum_{m=1}^{\infty} u_m(x) q^m. \tag{8}$$

Assume that the auxiliary parameter, the auxiliary function, the initial approximation, and the auxiliary linear operator are properly chosen so that the power series (8) of $\phi(x; q)$ converges at $q = 1$. Then, we have under these assumptions the series solution $u(x) = u_0(x) + \sum_{m=1}^{\infty} u_m(x)$.

Define the vector $\vec{u}_n(x) = \{u_0(x), u_1(x), u_2(x), \ldots, u_n(x)\}$. Differentiating Eq. (4) m-times with respect to embedding parameter q, then setting $q = 0$ and dividing them by $m!$, using Eq. (7), we have the so-called mth-order deformation equation

$$\mathcal{L}[u_m(x) - \chi_m u_{m-1}(x)] = \hbar H(x) R_m(\vec{u}_{m-1}(x)),$$
$$m = 1, 2, \ldots, n, \tag{9}$$

where $\chi_m = 1$, $m > 1$ and $\chi_1 = 0$ and

$$R_m(\vec{u}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N(\phi(x; q))}{\partial q^{m-1}} \bigg|_{q=0}. \tag{10}$$

For any given nonlinear operator $\mathcal{L}$, the term $R_m(\vec{u}_{m-1})$ can be easily expressed by Eq. (10). Thus, we can gain $u_0(x)$, $u_1(x)$, $u_2(x), \ldots, u_n(x)$ by means of solving the linear high-order deformation Eq. (9) one after the other in order. The mth-order approximation of $u(x)$ is given by $u^m(x) = \sum_{k=0}^{m} u_k(x)$.

It should be emphasized that the so-called mth-order deformation Eq. (9) is linear, which can be easily solved by symbolic computation software's such as Maple or Mathematica.

4. Solution of fractional two-point BVP by HAM

In this section, we employ our algorithm of the HAM to find out series solution for the fractional two-point BVP subject to a given boundary conditions.

Accordingly, we extend the application of the HAM to solve the following fractional problem

$$N_\alpha[u(x)] := D_*^\alpha u(x) - f(x, u(x), u'(x), u''(x), \ldots, u^{(n)}(x)),$$
$$a \leqslant x \leqslant b, \tag{11}$$

subject to the boundary conditions (2).

First of all, we assume that the nonlinear fractional Eq. (11) satisfies the initial conditions $u^{(i)}(a) = c_i$, $i = 0, 1, 2, \ldots, k - 1$, where the unknown constants c_i can be determined later by substituting the boundary conditions $u^{(i)}(b) = b_i$, $i \in \{0, 1, 2, \ldots, k - 1\}$ into the obtained solution. It is worth noting that some of c_i are known from the given initial conditions.

According to the last description, the initial guess u_0 will be of the form $u_0 = u_0(x; c_0, c_1, c_2, \ldots, c_{k-1})$. If we take the auxiliary linear operator $\mathcal{L} = D_*^\alpha$; the Caputo fractional derivative of order $\alpha > 0$, then the so-called zero-order deformation equation can be defined as

$$(1 - q)D_*^\alpha[\phi(x; q, c_0, c_1, \ldots, c_{k-1}) - u_0(x; c_0, c_1, \ldots, c_{k-1})]$$
$$= q\hbar H(x)N_\alpha[\phi(x; q, c_0, c_1, \ldots, c_{k-1})]. \tag{12}$$

Obviously, when $q = 0$, Eq. (12) reduces to $\phi(x; q, c_0, c_1, \ldots, c_{k-1}) = u_0(x; c_0, c_1, \ldots, c_{k-1})$. In this case, the so-called mth-order deformation equation can be constructed as

$$D_*^\alpha[u_m(x; c_0, c_1, \ldots, c_{k-1}) - \chi_m u_{m-1}(x; c_0, c_1, \ldots, c_{k-1})]$$
$$= \hbar H(x) R_m(\vec{u}_{m-1}(x; c_0, c_1, \ldots, c_{k-1})), \tag{13}$$

where

$$R_m(\vec{u}_{m-1}(x; c_0, c_1, \ldots, c_{k-1}))$$
$$= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial q^{m-1}} N_\alpha[\phi(x; q, c_0, c_1, \ldots, c_{k-1})] \bigg|_{q=0}. \tag{14}$$

Operating the operator J^α; the inverse operator of D_*^α to both sides of Eq. (13), then the mth-order deformation equation will have the form

$$u_m(x; c_0, c_1, \ldots, c_{k-1}) = \chi_m u_{m-1}(x; c_0, c_1, \ldots, c_{k-1})$$
$$+ \hbar J^\alpha [H(x) R_m(\vec{u}_{m-1}(x; c_0, c_1, \ldots, c_{k-1}))]. \quad (15)$$

So, the mth-order approximation of $u(x)$ can be given as

$$u^m(x) = \sum_{k=0}^{m} u_k(x; c_0, c_1, \ldots, c_{k-1}). \quad (16)$$

Finally, if we substitute the boundary conditions $u^{(i)}(b) = b_i$, $i \in \{0, 1, 2, \ldots, k-1\}$ into Eq. (16), then we obtain a system of nonlinear algebraic equations in the variables c_0, c_1, c_2, ..., c_{k-1} which can be easy solved using symbolic computation software.

5. Numerical results and discussion

The HAM provides an analytical approximate solution in terms of an infinite power series. However, there is a practical need to evaluate this solution and to obtain numerical values from the infinite power series. The consequent series truncation and the practical procedure are conducted to accomplish this task, transforms the otherwise analytical results into an exact solution, which is evaluated to a finite degree of accuracy. In this section, we consider four fractional BVPs to demonstrate the performance and efficiency of the present technique.

Throughout this paper, we will try to give the results of the all examples; however, in some cases, we will switch between the results obtained for the examples in order not to increase the length of the paper without the loss of generality for the remaining results.

Example 5.1. Consider the following Bagley–Torvik BVP:

$$D_*^{3/2} u(x) = -u(x) - u''(x) + x^2 + x + \frac{4}{\sqrt{\pi}} \sqrt{x} + 3,$$
$$1 < \alpha \leqslant 2, \quad 0 \leqslant x \leqslant 1, \quad (17)$$

subject to the boundary conditions
$$u(0) = 1, \quad u(1) = 3. \quad (18)$$

According to our extension of the HAM, if we take the auxiliary function $H(x) = 1$, then the mth-order deformation Eq. (15) becomes

$$u_m(x; c) = \chi_m u_{m-1}(x; c) + \hbar J^{3/2} [R_m(\vec{u}_{m-1}(x; c))],$$

where

$$R_m(\vec{u}_{m-1}(x; c)) = D_*^{3/2} u_{m-1}(x; c) + u_{m-1}(x; c) + u''_{m-1}(x; c)$$
$$- \left(x^2 + x + \frac{4}{\sqrt{\pi}} \sqrt{x} + 3 \right)(1 - \chi_m).$$

Choose the initial guess approximation as $u_0(x; c) = x^2 + cx + 1$, which satisfies the initial conditions $u_0(0; c) = 1$ and $u'_0(0; c) = c$. Then, the first few terms of the HAM solution for Eq. (17) according to these initial conditions are as follows:

$$u_1(x; c) = -\frac{\hbar}{6}(1 - c)x^3,$$

$$u_2(x; c) = u_1(x; c) - \hbar^2(1 - c)\left(\frac{1}{6}x^3 + \frac{16}{105\sqrt{\pi}}x^{7/2} + \frac{1}{120}x^5 \right),$$

$$u_3(x; c) = -u_1(x; c) + 2u_2(x; c)$$
$$- (1 - c)\hbar^3 \left(\frac{1}{6}x^3 + \frac{32}{105\sqrt{\pi}}x^{7/2} + \frac{1}{24}x^4 + \frac{1}{60}x^5 \right.$$
$$\left. + \frac{128}{10395\sqrt{\pi}}x^{11/2} + \frac{1}{5040}x^7 \right),$$

$$\vdots$$

and so on. To determine the value of c, we must chose a value to the auxiliary parameter $\hbar$. It was proved that if we set $\hbar = -1$, then we have the Adomian decomposition method (ADM) solution which is a special case of the HAM solution [34,35]. Therefore, -1 is available value for $\hbar$. Now, substitute $\hbar = -1$ into the 3rd-order approximation of $u(x)$ to get

$$u^3(x; c) = 1 + cx + x^2 + (1 - c)$$
$$\times \left[\frac{1}{6}x^3 - \frac{16}{105\sqrt{\pi}}x^{7/2} + \frac{1}{24}x^4 - \frac{1}{120}x^5 \right.$$
$$\left. + \frac{128}{10395\sqrt{\pi}}x^{11/2} + \frac{1}{5040}x^7 \right]. \quad (19)$$

Now, we determine an introductory value of a constant c by substituting the boundary condition $u(1) = 3$ into Eq. (19) to obtain $c = 1$. Hence, the exact solution for Eqs. (17) and (18) will be $u(x) = x^2 + x + 1$.

Example 5.2. Consider the following nonlinear fractional BVP:

$$D_*^\alpha u(x) = -2(u'(x))^2 - 8u(x), \quad 1 < \alpha \leqslant 2, \quad 0 \leqslant x \leqslant 1, \quad (20)$$

subject to the boundary conditions

$$u(0) = 0, \quad u'(1) = -1. \quad (21)$$

If we select the auxiliary function $H(x) = 1$, then according to Eq. (15), the mth-order deformation equation can be given as

$$u_m(x; c) = \chi_m u_{m-1}(x; c) + \hbar J^\alpha [R_m(\vec{u}_{m-1}(x; c))],$$

where

$$R_m(\vec{u}_{m-1}(x; c)) = D_*^\alpha u_{m-1}(x; c) + 2\sum_{i=0}^{m-1} (u'_i(x; c)u'_{m-1-i}(x; c))$$
$$+ 8u_{m-1}(x; c).$$

Assuming the initial approximation of Eq. (20) has the form $u_0(x; c) = cx$ which satisfies the initial conditions $u_0(0; c) = 0$ and $u'_0(0; c) = c$. Consequently, the first few terms of the HAM solution for Eq. (20) according to these initial conditions are as follows:

$$u_1(x; c) = \frac{2\hbar}{\Gamma(2 + \alpha)}[4cx^{1+\alpha} + c^2(1 + \alpha)x^\alpha],$$

$$u_2(x; c) = u_1(x; c) + \frac{2c\hbar^2}{\Gamma(2 - \alpha)\Gamma(1 + \alpha)\Gamma(2 + \alpha)}$$
$$\times [-4\pi\alpha(\alpha - 1)\csc(\pi\alpha)x^{1+\alpha} + \ldots],$$

$$\vdots$$

Table 1 Numerical values of c at $\hbar = -1$ for different values of α generated from the 7th-order approximation HAM solution of $u(x)$ for Eqs. (20) and (21).

Order of derivative	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
c	0.014214	0.046539	0.115495	0.238885

Table 2 The valid region of the auxiliary parameter $\hbar$ for different values of α derived from Fig. 1.

Order of derivative	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$
Valid region	$(-0.85, -0.15)$	$(-0.9, -0.2)$	$(-0.95, -0.3)$	$(-1, -0.25)$

and so on. As in the previous example, we determine an introductory value of a constant c by substituting $\hbar = -1$ and the boundary condition $u'(1) = -1$ into the obtained HAM solution. However, various values of c have been listed in Table 1 when $\alpha = 1.25$, $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2$ which are generated from the 7th-order approximation HAM solution of $u(x)$.

The HAM yields rapidly convergent series solution by using a few iterations. For the convergence of the HAM, the reader is referred to [24]. According to [36], it is to be noted that the series solution contains the auxiliary parameter $\hbar$ which provides a simple way to adjust and control the convergence of the series solution. To this end, we have plotted the so-called $\hbar$-curves of $u''(0.5)$ when $\alpha = 1.25$, $\alpha = 1.5$, $\alpha = 1.75$, and $\alpha = 2$ in Fig. 1. In fact, the numerical values of c have been identified previously in Table 1. So, the valid region for the values of $\hbar$ can be obtained directly by $\hbar$-curve which corresponds to the line segment nearly parallel to the horizontal axis as shown in that figure.

These valid regions have been listed in Table 2 for the various α in (1,2]. Furthermore, these valid regions ensure us the convergence of the obtained series.

Our next goal is to show how the auxiliary parameter $\hbar$ affects the value of the constant c. Table 3 shows some numerical values of the constant c for different values of α and $\hbar$ which generated from substituting the right boundary condition $u'(1) = -1$ into 7th-order approximation HAM solution of $u(x)$.

As we mentioned earlier, the ADM is a special case of the HAM when $H(x) = 1$ and $\hbar = -1$. Author in [37] constructed the ADM solution of Eqs. (20) and (21) when $\alpha = 2$, which was the same solution as the HAM solution when $H(x) = 1$

and $\hbar = -1$. For this special case, the exact solution is $u(x) = x - x^2$. Fig. 2 shows the HAM solution of Eqs. (20) and (21) for different values of $\hbar$ when $\alpha = 2$. It is clear from the figure that the HAM solution when $\hbar = -0.43$ is more closed to the exact solution than the ADM solution. Therefore, we can obtain an approximate solution with more accuracy by choosing suitable values of the auxiliary parameter $\hbar$. It is evident that the overall error can be made smaller by computing more terms of the HAM series solution.

Representation the 7th-order approximate HAM solution of Eqs. (20) and (21) at different values of $\hbar$ and α mentioned in Table 3 is depicted in Fig. 3. However, this figure shows the correspondence between these solutions. It is worth mentioning that for various values of α, the values of $\hbar$ have been selected such that the HAM solutions satisfies the given initial-boundary conditions.

Example 5.3. Consider the following nonlinear fractional BVP:

Table 3 Numerical values of c for different values of α and $\hbar$ generated from the 7th-order approximation HAM solution of $u(x)$ for Eqs. (20) and (21).

α	$\hbar$	c
2	-0.5	0.873737
2	-0.43	1.002362
1.75	-0.32	1.113691
1.5	-0.245	1.087526
1.25	-0.175	1.173733
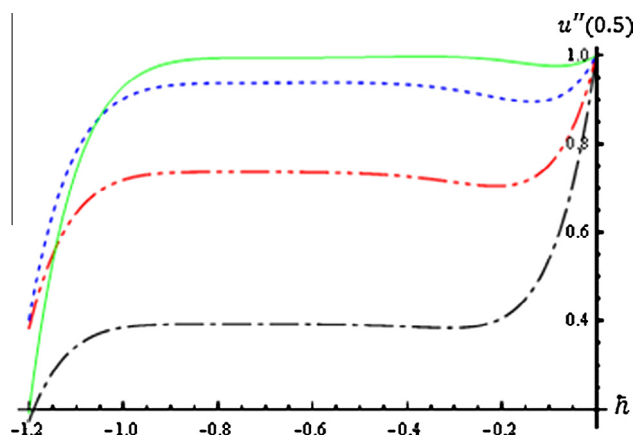
Figure 1 The $\hbar$-curves of $u''(0.5)$ which are corresponding to the 7th-order approximation HAM solution of Eqs. (20) and (21) at different values of α: solid line: $\alpha = 1.25$; dotted line: $\alpha = 1.5$; dash-dot-dotted line: $\alpha = 1.75$; dash-dotted line: $\alpha = 2$.
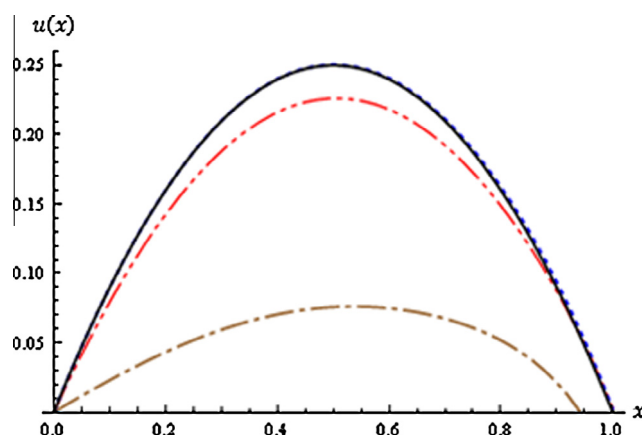
Figure 2 Solutions of Eqs. (20) and (21) when $\alpha = 2$: dash-dotted line: ADM (HAM) at $\hbar = -1$ and $c = 0.238885$; dash-dot-dotted line: HAM at $\hbar = -0.5$ and $c = 0.873737$; dotted line: HAM at $\hbar = -0.43$ and $c = 1.002362$; solid line: exact solution.
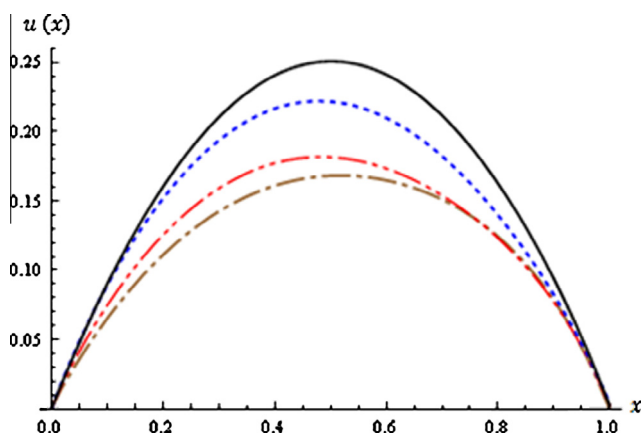
Figure 3 HAM solutions of Eqs. (20) and (21) at different values of α and $\hbar$: dash-dotted line: $\alpha = 1.25$ and $\hbar = -0.175$; dash-dot-dotted line: $\alpha = 1.5$ and $\hbar = -0.245$; dotted line: $\alpha = 1.75$ and $\hbar = -0.32$; solid line: $\alpha = 2$ and $\hbar = -0.43$.

$$D_*^\alpha u(x) = -(1+x)u(x) + 6(u''(x))^2 + x,$$
$$2 < \alpha \leqslant 3, \quad 0 \leqslant x \leqslant 1, \tag{22}$$

subject to the boundary conditions

$$u(0) = 0, \quad u'(1) = 0.25, \quad u''(1) = -0.25. \tag{23}$$

Select the auxiliary function $H(x) = x^\gamma$, $\gamma > -1$. Then according to Eq. (15) the mth-order deformation equation is

$$u_m(x; c_1, c_2) = \chi_m u_{m-1}(x; c_1, c_2) + \hbar J^\alpha [x^\gamma R_m(\vec{u}_{m-1}(x; c_1, c_2))],$$

where

$$R_m(\vec{u}_{m-1}(x; c_1, c_2)) = D_*^\alpha u_{m-1}(x; c_1, c_2) + (1+x)u_{m-1}(x; c_1, c_2)$$
$$- 6 \sum_{i=0}^{m-1} \left(u_i''(x; c_1, c_2) u_{m-1-i}''(x; c_1, c_2) \right)$$
$$- x(1 - \chi_m).$$

Choose the initial guess approximation as $u_0(x; c_1, c_2) = c_1 x + \frac{1}{2} c_2 x^2$, which satisfies the initial conditions $u_0'(0; c_1, c_2) = c_1$ and $u_0''(0; c_1, c_2) = c_2$. Using the previous iteration formula, we can directly obtain the following approximation terms of the HAM solution for Eqs. (22) and (23):

$$u_1(x; c_1, c_2) = -\hbar x^{\gamma+\alpha} \left[\frac{6c_2^2 \Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} + \frac{(1-c_1)\Gamma(2+\gamma)}{\Gamma(2+\gamma+\alpha)} x \right.$$
$$\left. - \frac{3c_1 \Gamma(3+\gamma)}{2\Gamma(3+\gamma+\alpha)} x^2 - \frac{c_1 \Gamma(4+\gamma)}{2\Gamma(4+\gamma+\alpha)} x^3 \right],$$

$$u_2(x; c_1, c_2) = u_1(x; c_1, c_2)$$
$$+ \hbar^2 \left[\frac{12c_2 \Gamma(1+\alpha)\Gamma(2+\gamma)\Gamma(2\gamma+\alpha)}{\Gamma(\alpha)\Gamma(2(\gamma+\alpha))\Gamma(2+\gamma+\alpha)} x^{2\gamma+2\alpha-1} + \ldots \right],$$

$$\vdots$$

and so on. To find the numerical values of constants c_1 and c_2, substitute $\hbar = -1$ and the boundary conditions $u'(1) = 0.25$ and $u''(1) = -0.25$ into the 6th-order approximation HAM solution of $u(x)$. However, various values of these constants have been listed in Table 4 for different values of α and γ.

Next, we show how the parameter γ in the auxiliary function $H(x)$ affects the solutions. As a special case, if $\alpha = 3$, then Eqs. (22) and (23) have the exact solution $u(x) = \frac{x}{1+x}$. Fig. 4

shows the 6th-order approximation HAM solution of Eqs. (22) and (23) at $\hbar = -1$ when $\alpha = 3$ and for different values of γ. It is clear from the figure that the best solution is obtained when $\gamma = 4$. Also, from that figure, we can conclude that if we select a suitable value of the auxiliary function $H(x)$, then we may obtain a series solution which is closed to the exact solution and may accelerate the rapid convergence of the series solution.

Fig. 5 shows a comparison between the HAM solution of Eqs. (22) and (23) at $\hbar = -1$ and $\gamma = 4$ for different values of α obtained from the 6th-order approximation HAM solution of $u(x)$. From this figure, we can confirm that the solution is continuously depends on the fractional derivative α, where $2 < \alpha \leqslant 3$.

Example 5.4. Consider the following nonlinear fractional BVP:

$$D_*^\alpha u(x) = u^2(x) - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4$$
$$+ 120x - 48, \quad 3 < \alpha \leqslant 4, \quad 0 \leqslant x \leqslant 1, \tag{24}$$

subject to the boundary conditions

$$u(0) = 0, \quad u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 1. \tag{25}$$

If we select the auxiliary function as $H(x) = 1$, then according to Eq. (15) the mth-order deformation equation will be

Table 4 Numerical values of c_1 and c_2 at $\hbar = -1$ for different values of α and γ generated from the 6th-order approximation HAM solution of $u(x)$ for Eqs. (22) and (23).

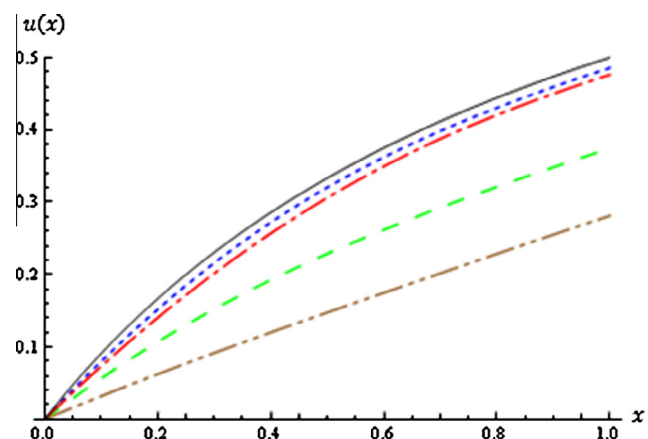| α	γ	c_1	c_2
3	0	0.326056	−0.154811
3	2	0.593535	−0.584094
3	4	0.841203	−0.806600
3	6	0.762626	−0.601025
2.75	4	0.604612	−0.481414
2.50	4	0.404940	−0.241865
2.25	4	0.184292	−0.069527

Figure 4 HAM solutions of Eqs. (22) and (23) at $\hbar = -1$ when $\alpha = 3$ and for different values of γ: solid line: exact solution; dotted line: HAM at $\gamma = 4$; dash-dotted line: HAM at $\gamma = 6$; dash line: HAM at $\gamma = 2$; dash-dot-dotted line: ADM (HAM) at $\gamma = 0$.
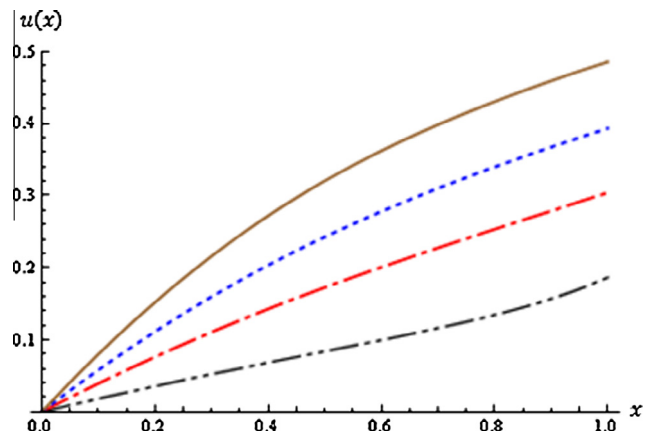
Figure 5 HAM solutions of Eqs. (22) and (23) at $\hbar = -1$ when $\gamma = 4$ and for different values of α: solid line: $\alpha = 3$; dotted line: $\alpha = 2.75$; dash-dotted line: $\alpha = 2.5$; dash-dot-dotted line: $\alpha = 2.25$.

$$u_m(x; c_1, c_2) = \chi_m u_{m-1}(x; c_1, c_2) + \hbar J^\alpha[R_m(\vec{u}_{m-1}(x; c_1, c_2))],$$

where

$$R_m(\vec{u}_{m-1}(x; c_1, c_2)) = D_*^\alpha u_{m-1}(x; c_1, c_2)$$

$$- \sum_{i=0}^{m-1}(u_i(x; c_1, c_2)u_{m-1-i}(x; c_1, c_2))$$

$$- (-x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4$$

$$+ 120x - 48)(1 - \chi_m).$$

Choose the initial guess approximation to be $u_0(x; c_1, c_2) = \frac{1}{2}c_1 x^2 + \frac{1}{6}c_2 x^3$ which satisfies the initial conditions $u''(0; c_1, c_2) = c_1$ and $u'''(0; c_1, c_2) = c_2$. Using the previous iteration formula, we can directly obtain the following approximation terms of the HAM series solution for Eq. (24) subject to these initial conditions:

$$u_1(x; c_1, c_2) = 2\hbar x^\alpha \left[\frac{\Gamma(5)}{\Gamma(1+\alpha)} - \frac{\Gamma(6)}{2\Gamma(2+\alpha)}x - \frac{3(16 - c_1^2)}{2\Gamma(5+\alpha)}x^4 \right.$$

$$- \frac{10c_1 c_2}{\Gamma(6+\alpha)}x^5 - \frac{10(96 + c_2^2)}{\Gamma(7+\alpha)}x^6 - \frac{2\Gamma(8)}{\Gamma(8+\alpha)}x^7$$

$$\left. + \frac{2\Gamma(9)}{\Gamma(9+\alpha)}x^8 - \frac{2\Gamma(10)}{\Gamma(10+\alpha)}x^9 + \frac{\Gamma(11)}{\Gamma(11+\alpha)}x^{10} \right],$$

$$u_2(x; c_1, c_2) = u_1(x; c_1, c_2)$$

$$+ \hbar^2 x^{\alpha-1}\left[-\frac{1728\pi \csc(\pi\alpha)}{\Gamma(4-\alpha)\Gamma(\alpha-3)\Gamma(1+\alpha)} + \cdots \right],$$

$$\vdots$$

and so on. Similar to the previous discussion, one can substitute $\hbar = -1$ and the boundary conditions $u(1) = 1$ and $u'(1) = 1$ into the 3rd-order approximation HAM solution of $u(x)$ to obtain various values to the constants c_1 and c_2. Anyway, various numerical values of c_1 and c_2 have been listed in Table 5 for different values of α.

To show the accuracy of the present method for Eqs. (24) and (25), we report the exact error, Ex, which is defined as $\text{Ex}(x) := |u_{\text{Exact}}(x) - u_{\text{HAM}}^m(x)|$, where $a \leqslant x \leqslant b$, u_{Exact} is the exact solution, and u_{HAM}^m is the mth-order approximation of $u(x)$ obtained by the HAM.

As a special case when $\alpha = 4$, Eqs. (24) and (25) have the exact solution $u(x) = x^5 - 2x^4 + 2x^2$. Fig. 6 shows the exact error for 3rd-order approximation HAM solution of Eqs. (24) and (25) at $\hbar = -1$ when $\alpha = 4$. It is clear from the figure that the HAM solution of $u(x)$ admits the exact solution of the problem.

Fig. 7 shows the 3rd-order approximation HAM of Eqs. (24) and (25) at $\hbar = -1$ and for different values of α. Also, as shown in Example 5.3, the solutions continuously depend on the fractional derivative α, where $3 < \alpha \leqslant 4$.

Table 5 Numerical values of c_1 and c_2 at $\hbar = -1$ for different values of α generated from the 3rd-order approximation HAM solution of $u(x)$ for Eqs. (24) and (25).

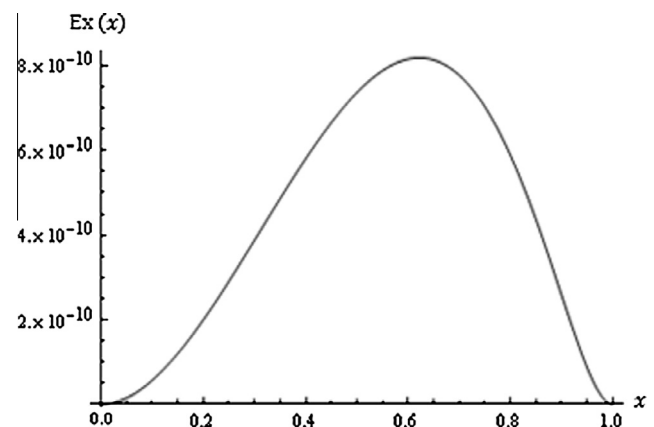| α	c_1	c_2
4	3.999990	4.0869×10^{-6}
3.5	6.759610	-3.287180
3.75	4.992550	-0.755718
3.25	9.641795	-8.650150

Figure 6 Exact error of $u_{\text{HAM}}^3(x)$ for Eqs. (24) and (25) at $\hbar = -1$ when $\alpha = 4$.
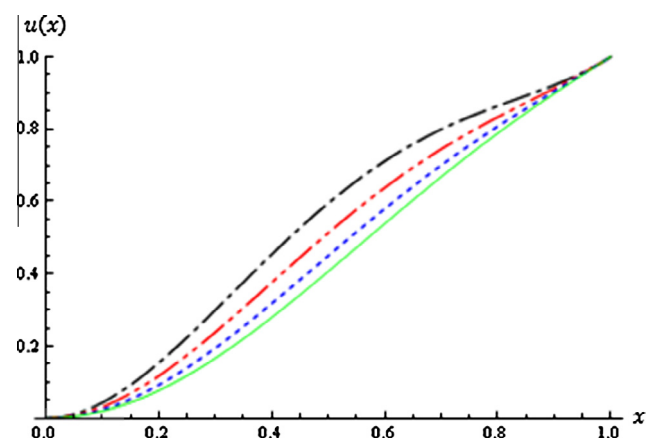

Figure 7 HAM solutions of Eqs. (24) and (25) at $\hbar = -1$ and for different values of α: solid line: $\alpha = 4$; dotted line: $\alpha = 3.75$; dash-dot-dotted line: $\alpha = 3.5$; dash-dotted line $\alpha = 3.25$.

6. Discussion and conclusion

The main concern of this work has been to propose an efficient algorithm for the solution of linear and nonlinear fractional two-point BVPs. The goal has been achieved by extending the HAM to solve this class of BVPs. We can conclude that the HAM is powerful and efficient technique in finding exact solutions as well as approximate solutions for linear and nonlinear fractional two-point BVPs. The proposed algorithm produced a rapidly convergent series by choosing suitable values of the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$. After computing several approximants and using the boundary conditions at the boundary points, we can easily determine the solution.

There are three important points to make here. First, the HAM provides us with a simple way to adjust and control the convergence region of the series solution by introducing the auxiliary parameter $\hbar$ and the auxiliary function $H(x)$. Second, the comparison of the result obtained by the HAM with that obtained by the ADM reveals that HAM is very effective and convenient in nonlinear cases. Finally, the HAM requires less computational work. This confirms our belief that the efficiency of our technique gives it much wider applicability for general classes of linear and nonlinear fractional DEs.

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Dr. Ahmad El-Ajou earned his Ph.D. degree in mathematics from the university of Jordan (Jordan) in 2009. He then began work at Al-Balqa Applied university in 2012 as assistant professor of applied mathematics until now. His research interests focus on numerical analysis, fractional differential and integral equations, and fuzzy differential and integral equations.

Dr. Omar Abu Arqub received his Ph.D. from the university of Jordan (Jordan) in 2008. He then began work at Al-Balqa Applied university in 2008 as assistant professor of applied mathematics until now. His research interests focus on numerical analysis, optimization techniques, fuzzy differential equations, and fractional differential equations.

Prof. Shaher Momani received his Ph.D. from the university of Wales (UK) in 1991. He then began work at Mutah university in 1991 as assistant professor of applied mathematics and promoted to full professor in 2006. He left Mutah university to the university of Jordan in 2009 until now. His research interests focus on the numerical solution of fractional differential equations in fluid mechanics, non-Newtonian fluid mechanics, and numerical analysis. Prof. Momani has written over 200 research papers and warded several national and international prizes. Also, he was classified as one of the top ten scientists in the world in fractional differential equations according to ISI web of knowledge.