Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 1

1. Show that $\Im \cap A$ is a $\sigma-$ field on $A \subset \Omega$ if $\Im$ is a $\sigma-$field on Ω.

2. Let $A_1, A_2, \ldots, A_n$ be events on a probability space $(\Omega, \Im, P)$.

   (i) For $n = 3$, if $P(A_1 \cap A_2 \cap A_3) = P(A_1^c \cap A_2^c \cap A_3^c)$, then each equals $\frac{1}{2}\left[1 - \sum_{i=1}^{3} P(A_i) + \sum_{i>j=1}^{3}\sum^{3} P(A_i \cap A_j)\right]$.

   (ii) $P(\cap_{i=1}^{n} A_i) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$.

3. The base and altitude of a right triangle are obtained by picking points randomly from $[0, a]$ and $[0, b]$, respectively. Show that the probability that the area of the triangle so formed will be less than $ab/4$ is $(1 + ln2)/2$.

4. Consider a randomly chosen group of $n(\leq 365)$ persons. What is the probability that at least two of them have the same birthdays?

5. In a movie theater that can accommodate $n + k$ persons, n persons are to be seated. What is the probability that $r \leq n$ given seats are occupied.

6. Let a sample of size 4 be drawn with(without) replacement from an urn containing 12 balls of which 5 are white. If the sample contains 3 white balls then find the probability the ball drawn on the third draw was white.

7. A certain test is 95% accurate on those who have cancer and 95% accurate on those who do not have this type. If $\frac{1}{4}$% of the people under test have this type of cancer. What is the probability that a person has this type of cancer when his test says he does ?

8. If A and B are independent and $A \subseteq B$ show that either $P(A) = 0$ or $P(B) = 1$.

9. A lot of five identical batteries is life tested. The probability assignment is assumed to be $P(A) = \int_A \frac{1}{\lambda} e^{-x/\lambda} dx$ for any event $A \subseteq [0, \infty)$, where $\lambda > 0$ is a known constant. Thus the probability that a battery fails after time t is given by $P(t, \infty) = \int_t^\infty \frac{1}{\lambda} e^{-x/\lambda} dx, \quad t \geq 0$. If the times to failure of the batteries are independent, what is the probability that at least one battery will operating after t_0 hours?

10. Consider measurable space (Ω, F), where $\Omega = \{a, b, c, d\}$ and $F = \{\phi, \Omega, \{a, b\}, \{c, d\}\}$. Examine if the function $X : \Omega \to R$ defined by $X(a) = X(b) = 1,\ X(c) = 1,\ X(d) = 2$ is $F-$measurable. Find the minimal $\sigma-$field w.r.t which X is measurable.

11. A die is tossed two times. Define X =sum of scores and Y =absolute difference of scores on the two tosses. Identify the probability space and verify that X, Y are random variables on this probability space. Write down the events: $\{X = 3\},\ \{Y \leq 1\},\ \{X > 5\},\ -1 < X \leq 4.5$ and determine their probabilities. Also find the p.m.f. of X and the distribution of Y. Using these determine $P(X \leq 7|X > 4)$ and $P(Y > 2.5|Y \leq 4)$.

12. If X is a random variable on a probability space (Ω, F, P), then show that $a + bX,\ |X|,\ X^+ = \begin{cases} X, & X \geq 0 \\ 0, & X < 0 \end{cases}$ are random variables on this probability space.

13. Let $\Omega = [0, 1]$. Define $X :\to R$ by $X(w) = \begin{cases} w, & 0 \leq w \leq \frac{1}{2} \\ w - \frac{1}{2}, & \frac{1}{2} < w \leq 1 \end{cases}$. For any interval $I \subseteq [0, 1]$, define $P(I) = \int 2x dx$. Identify the probability space and verify that X is a random variable on this space. Determine the distribution function of X and use this to find $P(X > \frac{1}{2}),\ P(\frac{1}{4} < X < \frac{1}{2}),\ P(X < \frac{1}{2}|X > \frac{1}{4})$.

14. An urn contains n cards numbered $1, 2, \ldots, n$. Let X be the least number on the m cards drawn randomly without replacement from the urn. Find probability distribution of random variable X. Compute $P(X \geq \frac{3}{2})$.

15. Let X be a continuous random variable taking values in the interval $[0,1]$. If $P(x < X \leq y)$, for all x, y, $0 \leq x < y \leq 1$ depends only on $(y-x)$, then show that X has uniform probability distribution on the interval $[0,1]$.

16. Let X be a random variable with distribution function

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{6} + \frac{x}{3}, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Show that $F_X(x) = \alpha F_1(x) + (1-\alpha) F_2(x)$ for some α where F_1 and F_2 are respectively distribution functions of a discrete and a continuous random variable. Find α. Evaluate conditional probability $P\left(\frac{1}{2} \leq X \leq 1 | X > \frac{1}{4}\right)$.

17. Let X be a random variable such that $P(X > \frac{1}{2}) = \frac{7}{8}$ and its pdf is:

$$f_X(x) = \begin{cases} ax, & 0 \leq x < 1 \\ b - x, & 1 \leq x < 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine a, b and find the distribution function of X.

18. Suppose X, the random number of eggs laid by a bird is a Poisson distributed random variable with parameter λ. The probability of an egg developing is p. Find the pmf of random variable Y, the number of eggs developing. Also find the pmf of $|Y - 4|$.

19. Let X be a standard normal distributed random variable. Find pdf of (i) $Y = a + bX$ (ii) $Y = \ln X$ (iii) $Z = X^2$

20. A discrete random variable X has a uniform probability distribution on the set $\{-k, -(k-1), \ldots, -1, 0, 1, 2, \ldots, r\}$. Find the probability distribution of (a) $|X|$ and (b) $(X+1)^2$.

21. Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{x^3}{64}, & 0 \leq x \leq 4 \\ 0, & \text{elsewhere} \end{cases}$$

Find pdf of the random variable $\min(\sqrt{X}, 2 - \sqrt{X})$.

22. Let X be a random variable with Poisson distribution with parameter λ. Show that the characteristic function of X is $\varphi_X(t) = exp\left[\lambda(e^{it} - 1)\right]$. Hence, compute $E(X^2)$, $Var(X)$ and $E(X^3)$.

23. Let X be a random variable with $N(0, \sigma^2)$. Find the moment generating function for the random variable X. Deduce the moments of order n about zero for the random variable X from the above result.

24. The moment generating function of a discrete random variable X is given by $M_X(t) = \frac{1}{6} + \frac{1}{2}e^{-t} + \frac{1}{3}e^{t}$. If μ is the mean and σ^2 is the variance of this random variable, find $P(\mu - \sigma < X < \mu + \sigma)$.