Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 5

1. Prove that $\overline{X}$, the mean of a random sample of size n from a distribution that is $N(\theta, \sigma^2)$, $-\infty < \theta < \infty$ is an efficient estimator of θ for every known $\sigma^2 > 0$.

2. Assuming population to be $\mathcal{N}\ (\mu, \sigma^2)$, show that sample variance is a consistent estimator for population variance σ^2.

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from uniform distribution on an interval $(0, \theta)$. Show that $(\prod_{i=1}^{n} X_i)^{1/n}$ is consistent estimator of θe^{-1}.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the geometric distribution with pmf $P(x; p) = (1-p)^{x-1}p$, $x = 1, 2, \ldots$. Prove that maximum likelihood estimator of p is $\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{x}}$.

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from Poisson distribution $P(\lambda)$. Show that $\alpha\overline{X} + (1-\alpha)s^2, 0 \leq \alpha \leq 1$, is a class of unbiased estimators for λ. Also find an unbiased estimator for $e^{-\lambda}$.

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from binomial distribution $B(1, p)$. Find an unbiased estimators for p^2 if it exists.

7. Suppose that 200 independent observations $X_1, X_2, \ldots, X_{200}$ are obtained from random variable X. We are told that $\sum_{i=1}^{200} X_i = 300$ and that $\sum_{i=1}^{200} X_i^2 = 3754$. Using these values obtain unbiased estimates for $E(X)$ and $Var(X)$.

8. Find the maximum likelihood estimator based on a sample of size n from the two sided exponential family with pdf given as follows.
$$f(x) = \frac{1}{2} e^{-|x-\theta|}, \qquad -\infty < x < \infty$$
Is the estimator unbiased?

9. Using method of moments, find the estimators of the parameters for the following population distributions (a) $\mathcal{N}(\mu, \sigma^2)$ (b) $B(n, p)$.

10. Let X_1, X_2 and X_3 be three independent random variables having the Poisson distribution with the parameter λ. Show that
$$\hat{\lambda_1} = \frac{X_1 + 2X_2 + 3X_3}{6}$$
is an unbiased estimator of λ. Also compare the efficiency of $\hat{\lambda_1}$ with that of the alternate estimator.
$$\hat{\lambda_2} = \frac{X_1 + X_2 + X_3}{3}\ .$$

11. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with both mean and variance equal to an unknown parameter θ.
    (a) Is there a sufficient statistics?
    (b) What is the MLE?
    (c) What is the Cramer-Rao lower bound?

12. Prove that for the family of uniform distribution on $[0, \theta]$, $max(x_1, x_2, \ldots, x_n)$ is the MLE for θ.

13. Consider the normal distribution $N(0, \theta)$. With a random sample $X_1, X_2, \ldots, X_n$ we want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum_{i=1}^{n}|X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

14. Suppose that the random sample arises from a distribution with pdf

$$f(x;\theta) = \begin{cases} \theta x^{\theta-1}, & 0 < x < 1, \quad \theta \in \Omega = \{\theta;\ 0 < \theta < \infty\} \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\hat{\theta} = -\frac{n}{\ln \prod_{i=1}^{n} X_i}$ is the maximum likelihood estimator of θ. Further prove that in a limiting sense, $\hat{\theta}$ is the unbiased minimum variance estimator of θ and thus θ is asymptotically efficient.

15. Let $X_1, X_2, \ldots, X_n$ be random sample from a distribution with pdf

$$f(x;\theta) = \begin{cases} \theta^x(1-\theta), & x = 0, 1, 2, \ldots; \quad 0 \le \theta \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the MLE $\hat{\theta}$ of θ.

(b) Show that $\sum_{i=1}^{n} X_i$ is a complete sufficient statistics for θ.

(c) Determine the unbiased maximum variance estimator of θ.