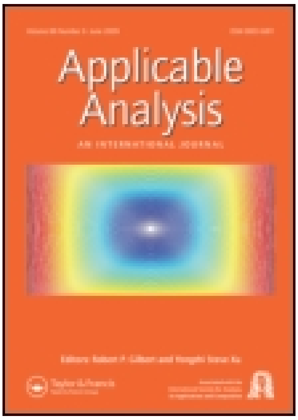

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New uniqueness results for fractional differential equations

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New uniqueness results for fractional differential equations

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We develop the Krasnoselskii–Krein type of uniqueness theorem for an initial value problem of the Riemann–Liouville type fractional differential equation which involves a function of the form $f(t, x(t), D^{q-1}x(t))$, for $1 < q < 2$ and establish the convergence of successive approximations. We prove a few other uniqueness theorems.

Keywords: fractional differential equations; uniqueness theorems; successive approximations

AMS Subject Classifications: 34A08; 34A12; 26A33

1. Introduction

The importance of uniqueness theorems in the study of initial value problems is well-known due to their relevance in establishing the well-posedness of the real-world problems arising in physical, engineering systems. Uniqueness results play a significant role in continuation of solutions and in the theory of autonomous systems. While the uniqueness results almost always come with the cost of stringent conditions, they are valuable, for without such uniqueness results it is impossible to make predictions about the behaviour of physical systems. For a monograph on different types of uniqueness theorems for differential equations, we refer to [1].

Fractional differential equations arise in a variety of real-world problems. There has been significant effort to develop the theory of FDE, along the lines of classical (integer order) differential equations [2–5]. Among the several different notions of fractional derivatives, we are concerned here with the Riemann–Liouville fractional derivative.

Recently, the Krasnoselskii–Krein, Nagumo's type uniqueness results and successive approximations have been extended to differential equations of fractional order $0 < q < 1$ of the form $D^q x = f(t, x)$, $x(t_0) = x_0$.

Most problems studied in these works, as is also the case with [1] involve only terms of the type $f(t, x)$. There is no other option when one tries to study uniqueness

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results for first-order ordinary differential equations and equations involving fractional derivatives $D^q x$ for $0<q<1$. However, in the context of fractional differential equations, with $1<q<2$ it is possible that terms of the type $f(t, x, D^{q-1}x)$ can arise. There are no known uniqueness results in such a context. We, therefore, in this article study a few uniqueness results for initial value problems of the form: for $1<q<2$,

$$D^q x = f(t, x(t), D^{q-1}x(t)), \tag{1}$$

$$x(0) = 0, \quad D^{q-1}x(0) = 0. \tag{2}$$

The organization of this article is as follows. In Section 2, we state some known uniqueness results for ordinary differential equations that we wish to generalize to the problem (1)–(2). Also, we provide some basic definitions and concerning fractional differential equations. The main Section 3 contains the proof of a Krasnoselskii–Krein type of uniqueness result, along with the convergence of successive approximations for (1)–(2). Sufficient conditions for Kooi and Roger's type of uniqueness results are given in Section 4. A few illustrative examples, which do not involve the derivative term are given in Section 5.

2. Preliminaries

Consider the following initial value problem:

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0. \tag{3}$$

It is well-known that if the function $f(t, x)$ is continuous and if it satisfies a Lipschitz condition in x, then the problem (3) has a unique solution that is the limit of Picard's iterates. Over the past several decades, uniqueness conditions less restrictive than requiring the Lipschitz continuity are obtained. A different type of uniqueness result involving two conditions instead of one Lipschitz type of condition was first given by Krasnoselskii and Krein [6], which we state below:

Let $S = \{(t, x) : |t - t_0| \le a,\ |x - x_0| \le b,\ a,\ b \in \mathbb{R}^+\}$.

THEOREM 2.1 *Let $f(t, x)$ be a continuous function on S, such that for all $(t, x), (t, \overline{x}) \in S$, it satisfies*

(i) $|f(t, x) - f(t, \overline{x})| \le k|t - t_0|^{-1}|x - \overline{x}|,\ t \ne t_0$,
(ii) $|f(t, x) - f(t, \overline{x})| \le c|x - \overline{x}|^\alpha$, *where c and k are positive constants, the real number α is such that* $0<\alpha<1$, *and* $k(1-\alpha)<1$.

Then the initial value problem (3) *has at most one solution in* $|t - t_0| \le a$.

Kooi's uniqueness theorem [7], replaces the condition (ii) by involving a singularity at $t = t_0$, and is given below:

THEOREM 2.2 *Let $f(t, x)$ be continuous in S and for all $(t, x), (t, \overline{x}) \in S$, it satisfies*

(i) $|t - t_0||f(t, x) - f(t, \overline{x})| \le k|x - \overline{x}|,\ t \ne t_0$, *where* $k>0$,
(ii) $|t - t_0|^\beta |f(t, x) - f(t, \overline{x})| \le c|x - \overline{x}|^\alpha$, *where c is a positive constant*, $0<\alpha<1$, $\beta<\alpha$ *and* $k(1-\alpha)<1-\beta$.

Then the initial value problem (3) *has at most one solution in* $|t - t_0| \le a$.

Rogers [8] proved a uniqueness theorem, generalizing the well-known Nagumo's result, by imposing a growth condition on f. However, it is known that Rogers theorem does not include Nagumo's result as a special case. We state Rogers theorem below:

THEOREM 2.3 *Let $f(t,x)$ be continuous in $T := \{(t,x): 0 \leq t \leq a, |x| < \infty\}$ and satisfy the condition*

$$f(t,x) = o(e^{1/t} t^{-2})$$

uniformly for $0 \leq x \leq \delta$, $\delta > 0$ arbitrary. Further, let

$$|f(t,x) - f(t,\overline{x})| \leq \frac{1}{t^2}|x - \overline{x}|, \quad t \neq 0$$

for all (t,x), $(t,\overline{x}) \in T$. Then the initial value problem

$$x'(t) = f(t, x(t)), \quad x(0) = x_0$$

has at most one solution in $[0,a]$.

It is our goal in this article to demonstrate how the above three important, different type of uniqueness results can be developed for the initial value problem (1)–(2) associated with a fractional differential equation. Below, we recall some basic definitions about fractional derivatives. For a detailed discussion on fractional derivatives and fractional differential equations, we refer to [9].

Definition 2.1 The fractional order integral of the function $h: (0,\infty) \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$D^{-\alpha}h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\mathrm{d}s,$$

where Γ is the Gamma function, provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2 For a function $h \in \mathcal{C}((0,\infty),\mathbb{R})$, Riemann–Liouville fractional derivative of h is defined by

$$D^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} h(s)\mathrm{d}s,$$

where $n = [|\alpha|] + 1$ and $[|\alpha|]$ denotes the integer part of α, provided that the right side is pointwise defined on $(0,\infty)$.

Let $R_0 = \{(t,x,y): 0 \leq t \leq 1, |x| \leq b, |y| \leq d\}$, for b, $d > 0$ are real numbers. By $\mathcal{C}(R_0,\mathbb{R})$, we mean the Banach space of all continuous functions $f: R_0 \to \mathbb{R}$, endowed with the supremum norm $\|f\| = \sup_{t \in [0,1]} |f(t)|$.

We assume that the function $f \in \mathcal{C}(R_0,\mathbb{R})$ and $f(0,0,0) \neq 0$.

A function $x(t)$ is called a solution of the IVP (1)–(2) on an interval $[0,1]$, if $x \in \mathcal{C}_p([0,1], \mathbb{R}) = \{x: x \in \mathcal{C}([0,1],\mathbb{R}), t^p x(t) \in \mathcal{C}([0,1],\mathbb{R})\}$, $p = 1 - q$, $D^q x(t)$ exists and is continuous on $[0,1]$ and $x(t)$ satisfies (1)–(2).

Any solution of IVP (1)–(2) is equivalent to the following Volterra fractional integral equation,

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s), D^{q-1}x(s))\mathrm{d}s. \tag{4}$$

3. Main result

Now, we state the Krasnoselskii–Krein type conditions for the initial value problem (1)–(2) of the Riemann–Liouville type fractional differential equation which involves derivative term in the function f.

THEOREM 3.1 *Let $f \in \mathcal{C}(R_0, \mathbb{R})$ satisfy the following Krein-type conditions*:

(a) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \Gamma(q)\dfrac{k+\alpha(q-1)}{2t^{1-\alpha(q-1)}}[|x-\overline{x}| + |y-\overline{y}|]$, $t \neq 0$, *where* $k > 0$ *and* $0 < \alpha < 1$,

(b) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq c[|x-\overline{x}|^{\alpha} + t^{\alpha(q-1)}|y-\overline{y}|^{\alpha}]$,

where c is a positive constant and $k(1-\alpha) < 1 + \alpha(q-1)$, for $(t,x,y), (t,\overline{x},\overline{y}) \in R_0$. Then the successive approximations given by

$$x_{n+1}(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x_n(s), D^{q-1}x_n(s))\mathrm{d}s, \quad x_0(t) = 0, n = 0, 1, \ldots, \tag{5}$$

converge uniformly to the unique solution $x(t)$ of (1)–(2) *on* $[0,\eta]$, *where* $\eta = \min\{1, (\frac{b\Gamma(1+q)}{M})^{1/q}, \frac{\mathrm{d}}{M}\}$, *$M$ is the bound for f on R_0.*

Proof We first establish the uniqueness. Suppose $x(t)$, $y(t)$ are any two solutions of (1)–(2) on $[0,\eta]$ and let $\phi(t) = |x(t) - y(t)|$ and $\theta(t) = |D^{q-1}x(t) - D^{q-1}y(t)|$. Note that $\phi(0) = 0$ and $\theta(0) = 0$.

Define $R(t) = \displaystyle\int_0^t [\phi^{\alpha}(s) + s^{\alpha(q-1)}\theta^{\alpha}(s)]\mathrm{d}s$, clearly $R(0) = 0$.

Further, we assert that $\phi(t) \leq \frac{ct^{q-1}}{\Gamma(q)} R(t)$ and $\theta(t) \leq cR(t)$. For, using (4) and condition (B), we have for $t \in [0,\eta]$

$$\begin{aligned}
\phi(t) &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x(s), D^{q-1}x(s)) - f(s, y(s), D^{q-1}y(s))|\,\mathrm{d}s \\
&\leq \frac{c}{\Gamma(q)} \int_0^t (t-s)^{q-1} [\phi^{\alpha}(s) + s^{\alpha(q-1)}\theta^{\alpha}(s)]\,\mathrm{d}s \\
&\leq \frac{c}{\Gamma(q)} t^{q-1} \int_0^t [\phi^{\alpha}(s) + s^{\alpha(q-1)}\theta^{\alpha}(s)]\,\mathrm{d}s \\
&= \frac{c}{\Gamma(q)} t^{q-1} R(t).
\end{aligned}$$

Further, we have for $t \in [0,\eta]$

$$x(t) = D^{-q} f(t, x(t), D^{q-1}x(t)),$$

$$D^{q-1}x(t) = D^{q-1}[D^{-q} f(t, x(t), D^{q-1}x(t))] = \int_0^t f(s, x(s), D^{q-1}x(s))\mathrm{d}s.$$

Using this and the condition (B), we get

$$\theta(t) \le \int_0^t |f(s, x(s), D^{q-1}x(s)) - f(s, y(s), D^{q-1}y(s))| \mathrm{d}s \le cR(t).$$

We will use the same symbol, C, to denote the different constants that arise throughout the rest of the proof of this theorem. Then,

$$\begin{aligned} R'(t) &= \phi^{\alpha}(t) + t^{\alpha(q-1)}\theta^{\alpha}(t) \\ &\le Ct^{\alpha(q-1)}R^{\alpha}(t). \end{aligned} \tag{6}$$

Since $R(t) > 0$ for $t > 0$, on multiplying (6) by $(1-\alpha)R^{-\alpha}(t)$ on both sides and integrating, we get for $t > 0$,

$$R(t) \le Ct^{\frac{\alpha}{1-\alpha}q+1}.$$

This leads to the following estimate on $\phi(t)$, for $t \in [0, \eta]$,

$$\phi(t) \le Ct^{\frac{q}{1-\alpha}}. \tag{7}$$

Define the function $\psi(t) = t^{-k}\max\{\phi(t), \theta(t)\}$ for $t \in (0, 1]$. If $\max\{\phi(t), \theta(t)\} = \phi(t)$ then we have

$$0 \le \psi(t) \le Ct^{\frac{q}{1-\alpha}-k}.$$

If $\max\{\phi(t), \theta(t)\} = \theta(t)$ then,

$$0 \le \psi(t) \le Ct^{\left[\frac{\alpha q - (\alpha-1)(1-k)}{1-\alpha}\right]}.$$

Since $k(1-\alpha) < 1 + \alpha(q-1)$, (by assumption) we have that the exponents of t in the above inequalities are positive. In either case, we have $\lim_{t\to 0^+} \psi(t) = 0$. Therefore, if we define $\psi(0) = 0$, the function $\psi(t)$ is continuous in $[0, \eta]$.

If $\psi(t) > 0$ at any point in $[0, \eta]$, there exists $0 < t_1 \le \eta \le 1$ and $m > 0$ such that $\psi(s) < m = \psi(t_1)$, for $s \in [0, t_1)$. If $\max\{\phi(t_1), \theta(t_1)\} = \phi(t_1)$ then from (A) we get

$$\begin{aligned} m &= \psi(t_1) = {t_1}^{-k}\phi(t_1) \\ &\le \frac{1}{\Gamma(q)}{t_1}^{-k}\int_0^{t_1}(t_1 - s)^{q-1}|f(s, x(s), D^{q-1}x(s)) - f(s, y(s), D^{q-1}y(s))|\,\mathrm{d}s \\ &\le \frac{k+\alpha(q-1)}{2}{t_1}^{-k}\int_0^{t_1}(t_1 - s)^{q-1}s^{-1+\alpha(q-1)}[\phi(s) + \theta(s)]\,\mathrm{d}s \\ &\le (k + \alpha(q-1)){t_1}^{-k+q-1}\int_0^{t_1} s^{k-1+\alpha(q-1)}\psi(s)\,\mathrm{d}s \\ &\quad < (k + \alpha(q-1)){t_1}^{-k+q-1}m\int_0^{t_1} s^{k-1+\alpha(q-1)}\,\mathrm{d}s \\ &= m{t_1}^{(q-1)(1+\alpha)} \le m, \end{aligned}$$

which is a contradiction.

On the other hand, if $\max\{\phi(t_1), \theta(t_1)\} = \theta(t_1)$, then from the condition (A), again using an argument similar to the above, we can show that $m = \psi(t_1) = \{t_1\}^{-k}\theta(t_1) < m$, which is a contradiction. This implies $\psi(t) \equiv 0$ and hence the uniqueness of the solution is established.

It can be easily shown that the successive approximations $\{x_{n+1}(t)\}$, $n = 0, 1, \ldots$ given by (5) are well-defined and continuous on $[0, \eta]$.

In fact, by (5), we get for $t \in [0, \eta]$,

$$|x_{n+1}(t)| \le \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} |f(s, x_n(s), D^{q-1}x_n(s))| \, \mathrm{d}s,$$

and

$$|D^{q-1}x_{n+1}(t)| \le \int_0^t |f(s, x_n(s), D^{q-1}x_n(s))| \, \mathrm{d}s.$$

For $n = 0$, and $t \in [0, \eta]$, we have

$$|x_1(t)| \le \frac{Mt^q}{q\Gamma(q)} \le b \quad \text{and} \quad |D^{q-1}x_1(t)| \le Mt \le d.$$

This implies, by induction that, the sequences $\{x_{n+1}(t)\}$ and $\{D^{q-1}x_{n+1}(t)\}$ are all well-defined and uniformly bounded on $[0, \eta]$. Further, it may be verified that $\{x_{n+1}(t)\}$, $\{D^{q-1}x_{n+1}(t)\}$ are equicontinuous families of functions. Hence by Arzela–Ascoli theorem, there exist subsequences $\{x_{n_k}\}$ and $\{D^{q-1}x_{n_k}\}$ such that $\{x_{n_k}\}$ and $\{D^{q-1}x_{n_k}\}$ converge uniformly on $[0, \eta]$. Further, if $\{x_n - x_{n-1}\} \to 0$ and $\{D^{q-1}x_n - D^{q-1}x_{n-1}\} \to 0$ as $n \to \infty$, then (5) implies that the limit of any such subsequence is the unique solution $x(t)$ of (1). It then follows that a selection of a subsequence is unnecessary and that the entire sequence $\{x_{n+1}(t)\}$ converges uniformly to $x(t)$, [10, Theorem 9.1].

Thus to prove Theorem 3.1, it is sufficient to show that

$$m(t) = \limsup_{n\to\infty} |x_n(t) - x_{n-1}(t)| \equiv 0$$

and

$$z(t) = \limsup_{n\to\infty} |D^{q-1}x_n(t) - D^{q-1}x_{n-1}(t)| \equiv 0.$$

Along the similar lines as in [5], we can show for $t_1, t_2 \in [0, \eta]$,

$$|x_{n+1}(t_1) - x_n(t_1)| \le |x_{n+1}(t_2) - x_n(t_2)| + \tfrac{4M}{\Gamma(1+q)}(t_2 - t_1)^q.$$

The right-hand side is at most $m(t_2) + \epsilon + \frac{4M}{\Gamma(1+q)}(t_2 - t_1)^q$ for large n if $\epsilon > 0$. Hence

$$m(t_1) \le m(t_2) + \epsilon + \tfrac{4M}{\Gamma(1+q)}(t_2 - t_1)^q \quad \text{for large } n \text{ if } \epsilon > 0.$$

Since $\epsilon > 0$ is arbitrary and t_1, t_2 can be interchangeable

$$|m(t_1) - m(t_2)| \le \tfrac{4M}{\Gamma(1+q)}(t_2 - t_1)^q.$$

This implies that $m(t)$ is continuous on $[0, \eta]$. Also, it can be verified that $z(t)$ is continuous on $[0, \eta]$.

Using condition (B) and the definition of successive approximations given by (5), we obtain

$$
\begin{aligned}
|x_{n+1}(t)-x_n(t)| &\le c\int_0^t (t-s)^{q-1}[|x_n(s)-x_{n-1}(s)|^\alpha \\
&\quad + s^{\alpha(q-1)}|D^{q-1}x_n(s)-D^{q-1}x_{n-1}(s)|^\alpha]\,\mathrm{d}s \\
&\le ct^{q-1}\int_0^t [|x_n(s)-x_{n-1}(s)|^\alpha \\
&\quad + s^{\alpha(q-1)}|D^{q-1}x_n(s)-D^{q-1}x_{n-1}(s)|^\alpha]\,\mathrm{d}s.
\end{aligned}
$$

For a fixed t in $[0,\eta]$, there are sequences of integers $n_1<n_2<\cdots$ and $p_1<p_2<\cdots$ such that

$$
|x_{n_{k+1}}(t)-x_{n_k}(t)|\to m(t) \quad \text{and} \quad |D^{q-1}x_{p_{k+1}}(t)-D^{q-1}x_{p_k}(t)|\to z(t) \text{ as } k\to\infty.
$$

Further,

$$
m^*(s)=\lim_{k\to\infty}|x_{n_k}(s)-x_{n_{k-1}}(s)| \quad \text{and} \quad z^*(s)=\lim_{k\to\infty}|D^{q-1}x_{p_k}(s)-D^{q-1}x_{p_{k-1}}(s)|
$$

exist uniformly on $0\le s\le\eta$ as $k\to\infty$.

Since $m^*(s)\le\lim\sup|x_n(s)-x_{n-1}(s)|=m(s)$ and $z^*(s)\le\lim\sup|D^{q-1}x_n(s)-D^{q-1}x_{n-1}(s)|=z(s)$, we get

$$
\begin{aligned}
m(t) &\le ct^{q-1}\int_0^t [(m^*(s))^\alpha + s^{\alpha(q-1)}(z^*(s))^\alpha]\,\mathrm{d}s \\
&\le ct^{q-1}\int_0^t [(m(s))^\alpha + s^{\alpha(q-1)}(z(s))^\alpha]\,\mathrm{d}s.
\end{aligned}
$$

Setting

$$
R(t)=\int_0^t [(m(s))^\alpha + s^{\alpha(q-1)}(z(s))^\alpha]\,\mathrm{d}s,
$$

we can define, as before,

$$
\psi^*(t)=t^{-k}\max\{m(t),z(t)\}, \qquad \text{and show that } \lim_{t\to 0^+}\psi^*(t)=0.
$$

We shall now show that $\psi^*(t)\equiv 0$. If $\psi^*(t)>0$ at any point in $[0,\eta]$, then there exists a point $t_1>0$ such that $0<\overline{m}=\psi^*(t_1)=\max_{0\le t\le\eta}\psi^*(t)$. Then, from the condition (A), we have

$$
\begin{aligned}
{t_1}^{-k}|x_{n+1}(t_1)-x_n(t_1)| &\le \frac{1}{\Gamma(q)}{t_1}^{-k}\int_0^{t_1}(t_1-s)^{q-1}|f(s,x_n(s),D^{q-1}x_n(s)) \\
&\quad - f(s,x_{n-1}(s),D^{q-1}x_{n-1}(s))|\,\mathrm{d}s \\
&\le \frac{k+\alpha(q-1)}{2}{t_1}^{-k}\int_0^{t_1}(t_1-s)^{q-1}s^{-1+\alpha(q-1)}[|x_n(s)-x_{n-1}(s)| \\
&\quad + |D^{q-1}x_n(s)-D^{q-1}x_{n-1}(s)|]\,\mathrm{d}s.
\end{aligned}
$$

If $\max\{m(t), z(t)\} = m(t)$ then, by suitably passing to the limit, we have

$$\begin{aligned}\overline{m} &= \psi^*(t_1) = {t_1}^{-k} m(t_1)\\ &\leq \frac{k+\alpha(q-1)}{2} {t_1}^{-k} \int_0^{t_1} (t_1 - s)^{q-1} s^{-1+\alpha(q-1)} [m(s) + z(s)]\, \mathrm{d}s\\ &\leq (k+\alpha(q-1)) {t_1}^{-k} \int_0^{t_1} (t_1 - s)^{q-1} s^{k-1+\alpha(q-1)} \psi^*(s)\, \mathrm{d}s,\end{aligned}$$

proceeding as before, we can show that $\overline{m} < \overline{m}$, which is a contradiction.

On the other hand, if $\max\{m(t), z(t)\} = z(t)$ then from the condition (A), again using an argument similar to the above, we can show that $\overline{m} = \psi^*(t_1) = {t_1}^{-k} z(t_1) < \overline{m}$, which is again a contradiction. So $\psi^*(t) \equiv 0$. Therefore, the successive approximations given by $x_{n+1}(t)$ converge uniformly to the unique solution $x(t)$ of (1)–(2) on $[0, \eta]$.

4. Other uniqueness results

We state below, without proof, an appropriate version of Kooi's uniqueness result for the problem (1)–(2). The proof runs along similar lines as that of Theorem 3.1, and we omit it here.

THEOREM 4.1 (Kooi's type uniqueness theorem) *Let the function f be in* (1) *holds the following conditions*:

(a) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \Gamma(q) \frac{k+\alpha(q-1)}{2t^{1-\alpha(q-1)}} [|x-\overline{x}| + |y-\overline{y}|]$, $t \neq 0$, *where* $k > 0$.

(b) $|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \frac{c}{t^\beta} [|x-\overline{x}|^\alpha + t^{\alpha(q-1)} |y-\overline{y}|^\alpha]$, *where c is a constant*, $0 < \alpha < 1$, $\beta < \alpha$, *and* $k(1-\alpha) < 1 + \alpha(q-1) - \beta$,

for $(t,x,y), (t,\overline{x},\overline{y}) \in R_0$. *Then the successive approximations given by* (5) *converge uniformly to the unique solution* $x(t)$ *of* $(1)-(2)$ *on* $[0, \eta]$, *where* $\eta = \min\{1, (\frac{b\Gamma(1+q)}{M})^{1/q}, \frac{\mathrm{d}}{M}\}$, *M being the bound for f on* R_0.

As mentioned in Section 1, the Rogers theorem is a generalization of Nagumo's result and to establish Roger's type of uniqueness theorem for (1)–(2), we need the following lemma.

LEMMA 4.1 *Let* $\phi(t)$ *and* $\theta(t)$ *be continuous, nonnegative functions in* $[0, a]$ *for a real number* $a > 0$. *Let* $\psi(t) = \int_0^t \frac{\phi(s) + s^{q-1}\theta(s)}{2s^{q+1}} \mathrm{d}s$. *Assume the following*:

(i) $\theta(t) \leq \psi(t)$,
(ii) $\phi(t) \leq t^{q-1} \psi(t)$,
(iii) $\phi(t) = o(t^{q-1} e^{-1/t})$,
(iv) $\theta(t) = o(e^{-1/t})$,

then $\phi(t) \equiv 0$.

Proof Differentiating $\psi(t)$ with respect to t and using (i) and (ii), we obtain for $t > 0$, $\psi'(t) = \frac{\phi(t) + t^{q-1}\theta(t)}{2t^{q+1}} \leq \frac{t^{q-1}\psi(t) + t^{q-1}\psi(t)}{2t^{q+1}} = \frac{\psi(t)}{t^2}$. Hence $[e^{1/t}\psi(t)]' \leq 0$ and $e^{1/t}\psi(t)$ is nonincreasing. Now, if $\epsilon > 0$, from (iii)and (iv), we obtain, for small t, that

$$\begin{aligned}e^{1/t}\psi(t) &= e^{1/t} \int_0^t \frac{\phi(s) + s^{q-1}\theta(s)}{2s^{q+1}} \mathrm{d}s\\ &\leq e^{1/t} \epsilon \int_0^t \frac{s^{q-1} e^{-1/s}}{s^{q+1}} \mathrm{d}s = \epsilon.\end{aligned}$$

Hence, $\lim_{t\to 0^+} e^{1/t}\psi(t) = 0$, and this implies that $e^{1/t}\psi(t) \leq 0$ for $t > 0$. This gives $\psi(t) \leq 0$. However, $\psi(t)$ is nonnegative, and thus $\psi(t) \equiv 0$. Since ϕ and θ are nonnegative, we have $\phi(t) \equiv 0$.

THEOREM 4.2 (Rogers type uniqueness theorem) *Let $f(t,x,y)$ be continuous in R_0 and satisfy the conditions*

(C)$f(t,x,y) = o(\Gamma(q)\frac{e^{-1/t}}{t^2})$ *uniformly for* $0 \leq x \leq \delta_1$, $0 \leq y \leq \delta_2$, δ_1, $\delta_2 > 0$ *arbitrary.*
(D)$|f(t,x,y) - f(t,\overline{x},\overline{y})| \leq \Gamma(q)\frac{1}{2t^{q+1}}[|x-\overline{x}| + t^{q-1}|y-\overline{y}|]$,

then the initial value problem (1)–(2) *has at most one solution in* $[0,a]$.

Proof Suppose $x(t)$, $y(t)$ are any two solutions of (1), we get for $t \in [0,a]$

$$\begin{aligned}
|x(t)-y(t)| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|\mathrm{d}s \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{2s^{q+1}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]\mathrm{d}s \\
&\leq t^{q-1}\int_0^t \frac{1}{2s^{q+1}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]\mathrm{d}s, \\
|D^{q-1}x(t) - D^{q-1}y(t)| &\leq \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|\mathrm{d}s \\
&\leq \Gamma(q)\int_0^t \frac{1}{2s^{q+1}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]\mathrm{d}s, \\
&\leq \int_0^t \frac{1}{2s^{q+1}}[|x(s)-y(s)| + s^{q-1}|D^{q-1}x(s) - D^{q-1}y(s)|]\mathrm{d}s,
\end{aligned}$$

which shows that the condition of (i) and (ii) of Lemma 4.1 is satisfied. Also, if $\epsilon > 0$, then from the condition (C) for small t, we have

$$\begin{aligned}
|x(t)-y(t)| &\leq \frac{t^{q-1}}{\Gamma(q)}\int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|\mathrm{d}s \\
&< t^{q-1}2\epsilon\int_0^t \frac{e^{-1/s}}{s^2}\mathrm{d}s = t^{q-1}e^{-1/t}2\epsilon \\
|D^{q-1}x(t) - D^{q-1}y(t)| &\leq \int_0^t |f(s,x(s),D^{q-1}x(s)) - f(s,y(s),D^{q-1}y(s))|\mathrm{d}s \\
&< 2\epsilon\Gamma(q)\int_0^t \frac{e^{-1/s}}{s^2}\mathrm{d}s. \\
&< 2\epsilon\int_0^t \frac{e^{-1/s}}{s^2}\mathrm{d}s = e^{-1/t}2\epsilon.
\end{aligned}$$

Now as an application of Lemma 4.1, we find that $|x(t) - y(t)| \equiv 0$, and this proves uniqueness of solutions of (1)–(2).

5. Example

The appropriate versions of the results obtained in Sections 3 and 4, for the case when the function f does not involve the fractional derivative and is of the form $f(t,x)$, are new and there are some differences in the conditions required to prove them. However, the proofs are analogous and so we merely state the results for this special case.

Remark 5.1 *If we consider the IVP with Riemann–Liouville derivative*

$$D^q x = f(t,x), \tag{8}$$

$$x(0) = 0, \quad D^{q-1}x(0) = 0, \tag{9}$$

where $1<q<2$, $f\in C(R_0,\mathbb{R})$ *with* $R_0=\{(t,x): 0\le t\le a,\ |x|\le b\}$, $f(0,0)\neq 0$, *then any solution of IVP* (8)–(9) *is equivalent to the following Volterra fractional integral equation,*

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x(s))\mathrm{d}s. \tag{10}$$

Let the function f in (8) *satisfy the following Krein-type conditions*:

(A) $|f(t,x)-f(t,y)| \le \Gamma(q)\frac{q(k-1)+1}{t^q}|x-y|$, $t\neq 0$, *where* $k>1$.
(B) $|f(t,x)-f(t,y)|\le c\Gamma(q)|x-y|^\alpha$, *where c is a constant*, $0<\alpha<1$, *and* $k(1-\alpha)<1$, *for* $(t,\ x)$, $(t,\ y)\in R_0$. *Then the successive approximations given by*

$$x_{n+1}(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s,x_n(s))\mathrm{d}s$$

converge uniformly to the unique solution $x(t)$ *of* (8)–(9) on $[0,\eta]$, *where* $\eta = \min\{a, (\frac{b\Gamma(1+q)}{M})^{1/q}\}$, *M is the bound for f on* R_0.

Example 5.1 Consider the initial value problem:

$$D^{3/2}x = f(t,x) = \begin{cases} At^{\frac{3}{2}\frac{\alpha}{1-\alpha}}, & 0\le t\le 1,\ -\infty<x<0,\\ At^{\frac{3}{2}\frac{\alpha}{1-\alpha}} - A\dfrac{x}{t^{3/2}}, & 0\le t\le 1,\quad 0\le x\le t^{\frac{3}{2}(1-\alpha)^{-1}},\\ 0, & 0\le t\le 1,\quad t^{\frac{3}{2}(1-\alpha)^{-1}}<x<\infty,\end{cases}$$

$x(0)=0$, where $0<\alpha<1$, $A=\Gamma(\frac{3}{2})(\frac{3}{2}k-\frac{1}{2})$, $q=3/2$, $c=\frac{A2^{1-\alpha}}{\Gamma(\frac{3}{2})}$, $k>1$ and $k(1-\alpha)<1$.

This function $f(t,\ x)$ is continuous in the strip. It can be verified in each of the cases, $0\le t\le 1$, $|x|<\infty$ (i) $0\le x,\overline{x}\le t^{\frac{3}{2}(1-\alpha)^{-1}}$, (ii) $t^{\frac{3}{2}(1-\alpha)^{-1}}<x<\infty$, $-\infty<\overline{x}<0$, (iii) $t^{\frac{3}{2}(1-\alpha)^{-1}}<x<\infty, 0\le\overline{x}\le t^{\frac{3}{2}(1-\alpha)^{-1}}$, (iv) $0\le x\le t^{\frac{3}{2}(1-\alpha)^{-1}}$, $-\infty<\overline{x}<0$, that the following estimates hold:

$$|f(t,x)-f(t,\overline{x})| \le \frac{A}{t^{3/2}}|x-\overline{x}|,$$
$$|f(t,x)-f(t,\overline{x})| \le A2^{1-\alpha}|x-\overline{x}|^\alpha.$$

Therefore, the initial value problem (8)–(9) has a unique solution in $[0,1]$.

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