NUMERICAL METHODS

The Modified Adomian decompositon method

Type-I

Consider

$$D^{\alpha}u(x) + \alpha_0 u(x) + \beta f(u(x)) = g(x)$$
$$u(0) = c_0, 0 < \alpha \leq 1$$

We suppose that $g(x)$ is analytic, so has Taylor expansion series:

$$g(x) = \sum_{n=0}^{\infty} g_n \frac{x^n}{n!}$$

by properties of the fractional integral and derivatives we have

$$I^{\alpha} D^{\alpha} u(x) = u(x) - c_0,$$

$$I^{\alpha} g(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s) ds$$

$$= \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds.$$

Now, applying the integral operator I^{α} to both sides of given differential equation we get

$$u(x) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds$$

$$-\frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds$$

$$-\frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(u(s)) ds \ldots (1)$$

using

$$u(x) = \sum_{n=0}^{\infty} u_n \ldots (2)$$

and

$$\beta f(u(x)) = \sum_{n=0}^{\infty} A_n \ldots (3)$$

where

$$A_n = A_n\left(u_0, u_1, \ldots, u_n\right)$$

are the well-known Adomian polynomials.

By equations (2), (3) equation (1) becomes,

$$\sum_{n=0}^{\infty} u(x) = c_0 + \frac{1}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds$$

$$-\frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \sum_{n=0}^{\infty} u(s) ds$$

$$-\frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \sum_{n=0}^{\infty} A_n(s) ds$$

Now, we set the following recursion scheme:

$$u(0) = c_0,$$

$$u_{n+1}(x) = \frac{g_n}{\Gamma(\alpha)n!}\int_0^x (x-s)^{\alpha-1}s^n ds$$

$$-\frac{\alpha_0}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}u_n(s)ds$$

$$-\frac{\beta}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}A_n(s)ds$$

Hence, we can approximate the solution $u^*(x)$ by:

$$\phi_{m+1} = \sum_{n=0}^{m} u_n(x)$$

that gives:

$$\lim_{m\to\infty} \phi_{m+1}(x) = \sum_{n=0}^{\infty} u_n(x) = u^*(x)$$

Example: Consider the following type (I) inhomogeneous nonlinear fractional differential equation:

$$D^{\alpha}u(x) + u^2(x) = 1, u(0) = c_0 = 0, 0 < \alpha \leq 1$$

Solution:-

The Exact solution when $\alpha = 1$ is $\frac{e^{2x}-1}{e^{2x}+1}$

Here

$$\alpha_0 = 0, \beta = 1$$

and the inhomogeneous term:

$$g(x) = 1\,(g_0 = 1, g_n = 0, n \geq 1)$$

and the first few Adomian polynomials for $f(u(x)) = u^2(x)$ are:

$$\begin{aligned} A_0(x) &= u_0^2(x), \\ A_1(x) &= 2u_0(x)u_1(x), \\ A_2(x) &= 2u_0(x)u_2(x) + u_1^2(x), \\ &\vdots \end{aligned}$$

by the new modified recursion scheme for $\alpha = 0.98$,

$$u_0(x) = c_0 = 0,$$
$$u_1(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds = 1.008x^{0.98},$$
$$u_2(x) = 0,$$

$u_3(x) = \frac{-1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_1^2(s) ds = -0.350x^{2.94}$, ...
so

$$u(x) = 1.008x^{0.98} - 0.350x^{2.94} + \ldots$$

which is the required equation.

Type II:

$$\begin{aligned}&D^{\alpha}u(x)+\lambda_1 D^{\alpha-1}u(x)+\alpha_0 u(x)\\&+\beta f\left(u(x),D^{\alpha-1}u(x)\right)=g(x)\ldots(4)\\&u(0)=c_0,u'(0)=c_1,1<\alpha\leq 2\end{aligned}$$

Now, applying the integral operator I^{α} to both sides of equation(4),we obtain:

$$u(x)-(c_0+c_1x)=I^{\alpha}g(x)-I^{\alpha}\left(\lambda_1 D^{\alpha-1}u(x)\right.$$

$$\left.+\alpha_0 u(x)-I^{\alpha}(\beta f(u(x))\right)$$

By the properties of fractional integrals and derivatives, we have

$$I^{\alpha} D^{\alpha-1} = I^{1} I^{\alpha-1} D^{\alpha-1},$$

and so

$$\begin{aligned} u(x) = c_0 + c_1 x + \tfrac{1}{\Gamma(\alpha)} \textstyle\sum_{n=0}^{\infty} \tfrac{g_n}{n!} \int_0^x (x-s)^{\alpha-1} s^n ds \\ -\lambda_1 I\left(u(x) - c_0\right) - \tfrac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u(s) ds \\ -\tfrac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(u(s)) ds \end{aligned}$$

Similarly to the type (I), we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$

$$u_{n+2}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 \int_0^x u_{n+1}(s) ds$$
$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s)$$

Type III:

$$D^{\alpha}u(x)+\lambda_1 D^{\alpha-1}u(x)+\lambda_2 D^{\alpha-2}u(x)+\alpha_0 u(x)$$

$$+\beta f\left(u(x), D^{\alpha-2}u(x), D^{\alpha-1}u(x)\right)=g(x)..........(A)$$
$$u(0)=c_0, u'(0)=c_1, u''(0)=c_2, 2<\alpha\leq 3$$

Now, applying the integral operator I^{α} to both sides of equation (A), we obtain:

$$u(x)-\left(c_0+c_1x+c_2\frac{x^2}{2!}\right)=I^{\alpha}g(x)-I^{\alpha}\left(\lambda_1 D^{\alpha-1}u(x)\right.$$
$$\left.+\lambda_2 D^{\alpha-2}u(x)+\alpha_0 u(x)\right)-I^{\alpha}(\beta f\left(u(x), D^{\alpha-2}u(x), D^{\alpha-}\right.$$

Similarly to the type (II), we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_2(x) = c_2 \frac{x^2}{2!},$$

$$u_{n+3}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds - \lambda_1 \int_0^x u_{n+2}(s)$$

$$-\lambda_2 \int_0^x (x-s) u_{n+1}(s) ds - \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds$$

$$-\frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds$$

Generalized form of inhomogeneous Nonlinear fractional differential Equation:

$$D^{\alpha}u(x)+\alpha_0 u(x)+\sum_{i=1}^{m-1}\lambda_i D^{\alpha-i}u(x)$$
$$+\beta f\left(u(x),D^{\alpha-1}u(x),\ldots,D^{\alpha-(m-1)}u(x)\right)=g(x)$$

$$\begin{gathered}u(0)=c_0,\\ u'(0)=c_1,\\ \vdots\\ u^{(m-1)}(0)=c_{m-1},m-1<\alpha\leq m,2\leq m\end{gathered}$$

Proceeding as before, we set the following recursion scheme:

$$u_0(x) = c_0,$$
$$u_1(x) = c_1 x,$$
$$u_2(x) = c_2 \frac{x^2}{2!},$$
$$\vdots$$
$$u_{m-1}(x) = c_{m-1} \frac{x^{m-1}}{(m-1)!},$$

$$u_{n+m}(x) = \frac{g_n}{\Gamma(\alpha) n!} \int_0^x (x-s)^{\alpha-1} s^n ds$$

$$- \sum_{i=1}^{m-1} \frac{\lambda_i}{(i-1)!} \int_0^x (x-s)^{(i-1)} u_{n+m-i}(s) ds$$

$$- \frac{\alpha_0}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} u_n(s) ds - \frac{\beta}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} A_n(s) ds.$$

Power series method

One-term equation with zero initial condition:

$$ {}_0D_t^{\alpha} y(t) = f(t), \quad y(0) = 0, t > 0, \quad 0 < \alpha < 1 \ldots\ldots.(A) $$

Let

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n+\alpha} \text{..........} (B)$$

be the power series solution of equation (A).Assume that the function $f(t)$ can be expanded in the Taylor series converging for $0 \leq t \leq R$, where R is the radius of convergence:

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n \text{..........} (C)$$

Using equation (B) and (C) in equation (A), we have

$$
{}_0D_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^{n+\alpha}\right] = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n
$$

applying the formula ${}_aD_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}, \nu > -1, \alpha > 0$

we obtain

$$
\sum_{n=0}^{\infty} a_n \frac{\Gamma(1+n+\alpha)}{\Gamma(1+n)} t^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n
$$

Equating coefficients of like powers of t, one can determine the coefficients by formula

$$a_n = \frac{\Gamma(1+n)}{\Gamma(1+n+\alpha)} \frac{f^{(n)}(0)}{n!} = \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)}, \quad n = 0, 1,$$

Substituting these coefficients in equation (B), the series solution of given fractional initial value problem is

$$y(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{\Gamma(1+n+\alpha)} t^{n+\alpha}$$

Clearly the series converges absolutely with region of convergence $(0, \infty)$.

Example: Consider the one-term fractional initial value problem

$${}_0D_t^{\frac{1}{2}} y(t) = \text{sint}, \quad y(0) = 0 \quad , t > 0$$

Now, we determine the coefficients $a_n, n = 0, 1, 2 \ldots$ by above equation (D)

$$a_0 = \frac{f(0)}{\Gamma\left(\frac{3}{2}\right)}, a_1 = \frac{f'(0)}{\Gamma\left(\frac{5}{2}\right)}, a_2 = \frac{f''(0)}{\Gamma\left(\frac{7}{2}\right)}, a_3 = \frac{f'''(0)}{\Gamma\left(\frac{9}{2}\right)}, \ldots$$

$$a_n = \frac{f^{(n)}(0)}{\Gamma\left(n+\frac{3}{2}\right)}, \ldots$$

since

$$\begin{aligned} f(t) &= \mathrm{sint}, \\ f'(t) &= \mathrm{cost}, \\ f''(t) &= -\mathrm{sint}, \\ f^{(3)}(t) &= -\mathrm{cost}, \\ f^{(4)}(t) &= \mathrm{sint} \end{aligned}$$

so that

$$\begin{aligned} f(0) &= 0, \\ f'(0) &= 1, \\ f''(0) &= 0, \\ f^{(3)}(0) &= -1, \\ f^{(4)}(0) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
a_0 &= 0, \\
a_1 &= \frac{1}{\Gamma\left(\frac{5}{2}\right)}, \\
a_2 &= 0, \\
a_3 &= \frac{-1}{\Gamma\left(\frac{9}{2}\right)}, \\
a_4 &= 0, \\
a_5 &= \frac{1}{\Gamma\left(\frac{13}{2}\right)}, \ldots
\end{aligned}
$$

Thus the series solution of given fractional initial value

$$y(t) = \frac{4}{3\sqrt{\pi}}t^{\frac{3}{2}} - \frac{16}{105\sqrt{\pi}}t^{\frac{7}{2}} + \frac{64}{10395\sqrt{\pi}}t^{\frac{11}{2}} - \ldots$$

One -term equation with nonzero initial condition:

$$ {}_0D_t^{\alpha} y(t) = f(t), y(0) = A, A \neq 0, t > 0, 0 < \alpha < 1....($$

Assume that

$$ f(t) = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha} .....(B) $$

where the coefficients f_n are known.
Let

$$ y(t) = \sum_{n=0}^{\infty} a_n t^n .....(C) $$

be the series solution of fractional initial value problem(A).

Thus, the fractional initial value problem (A) reduces to

$$ {}_0D_t^{\alpha}\left[\sum_{n=0}^{\infty} a_n t^n\right] = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha} $$

Applying equation ${}_aD_t^{\alpha} t^{\nu} = \frac{\Gamma(1+\nu)}{\Gamma(1+\nu-\alpha)} t^{\nu-\alpha}, \nu > -1, \alpha > 0$ to yield

$$ \sum_{n=0}^{\infty} a_n \frac{\Gamma(1+n)}{\Gamma(1+n-\alpha)} t^{n-\alpha} = \frac{At^{-\alpha}}{\Gamma(1-\alpha)} + \sum_{n=1}^{\infty} f_n t^{n-\alpha} $$

Equating coefficients of like powers of t, we obtain

$$a_0 = A, \quad a_n = \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} f_n, \quad n = 1, 2, 3, \ldots$$

Hence, the series solution of fractional initial value problem (A) is

$$y(t) = A + \sum_{n=1}^{\infty} \frac{\Gamma(1+n-\alpha)}{\Gamma(1+n)} f_n t^n$$

Example: Consider the one-term fractional initial value problem

$${}_0D_t^{\frac{1}{2}}y(t) = f(t), \quad y(0) = \frac{3}{2}, t > 0 \quad \text{and } f(n) = n$$

determine the coefficients $a_n, n = 0, 1, 2 \ldots$ as

$$\begin{aligned}
a_0 &= \frac{3}{2}, \\
a_1 &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)} \times 1, \\
a_2 &= \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3)} \times 2, \\
a_3 &= \frac{\Gamma\left(\frac{7}{2}\right)}{\Gamma(4)} \times 3, \\
a_4 &= \frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma(5)} \times 4
\end{aligned}$$

$\ldots$

therefore, the series solution of the given fractional initial value problem is

$$y(t) = \frac{3}{2} + \frac{1}{2}\sqrt{\pi}t\left[1 + \frac{3}{2}t + \frac{15}{4}t^2 + \dots\right]$$

which is the required solution.

Boundary value Problem

consider the following nonlinear Reimann-Liouville fractional differential equation boundary value problem:

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x(t), y(t)) + g(t, x(t)), & 0 < t < 1 \\ x(0) = x(1) = 0 \ldots\ldots\ldots\ldots (A) \end{cases}$$

and

$$\begin{cases} D_{0+}^{\alpha}x(t) = f(t, x(t), y(t)) + g(t, x(t)), & 0 < t < 1 \\ x(0) = 0, \quad x(1) = \beta x(\eta) \ldots\ldots\ldots\ldots (B) \end{cases}$$

where D_{0+}^{α} is the standard Riemann-Liouville frac-

For solving this equation,following results must be satisfied :-

$(\mathrm{R}_{1.})$ $f(t, x, y) : [0, 1] \times [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and increasing in $x \in [0, +\infty)$ for fixed $t \in [0, 1]$ and $y \in [0, +\infty)$ decreasing in $y \in [0, +\infty)$ for fixed $t \in [0, 1]$ and $x \in [0, +\infty)$.

$(\mathrm{R}_{2.})$ $g(t, x) : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is continuous and increasing in $x \in [0, +\infty)$ for fixed $t \in [0, 1], g(t, 0) \neq 0$.

$(\mathrm{R}_{3.})$ there exists *a* constant $\delta_0 > 0$ such that $f(t, x, y) \geq \delta_0 g(t, x), t \in [0, 1], x, y \geq 0$.

$(\mathrm{R}_{4.})$ $g(t,\lambda x) \geq \lambda g(t,x)$ for $\lambda \in (0,1)$, $t \in [0,1]$, u $\in [0,+\infty)$ and there exists a constant $\xi \in(0,1)$ such that $f\left(t,\lambda x,\lambda^{-1}y\right) \geq \lambda^{\xi} f(t,x,y), \forall t \in [0,1], x,y \in [0,+\infty)$.

Then there exist $x_0, y_0 \in P_h$ and $\gamma \in (0,1)$ such that $\gamma y_0 \leq x_0 < y_0$ and

$$x_0(t) \leq \int_0^1 G(t,s)\left[f\left(t,x_0(s),y_0(s)\right)+g\left(s,x_0(s)\right)\right]ds,$$
$$y_0(t) \geq \int_0^1 G(t,s)\left[f\left(s,x_0(s),y_0(s)\right)+g\left(s,x_0(s)\right)\right]ds,$$

where $h(t)=t^{\alpha-1}(1-t), t \in [0,1]$.

Example: Consider the following boundary value problem:

$$\begin{cases} -D_{0+}^{\frac{5}{3}} x(t) = x^{\frac{1}{3}} + \arctan x + y^{-\frac{1}{3}} + t^2 + t^3 + \frac{\pi}{2}, \quad 0 \\ x(0) = x(1) = 0 \end{cases}$$

In this case, $\alpha = \frac{5}{3}$.
This Problem can be regarded as a boundary value problem of the form (A) with $f(t, x, y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$ and $g(t, x) = \arctan x + t^3$.

Now we verify that conditions$(\mathrm{R}_1) - (\mathrm{R}_4)$ are satisfied. Firstly, it is easy to see (R_1) and (R_2) are satisfied and $g(t, 0) = t^3 \neq 0$.
Secondly, take $\delta_0 \in (0, 1]$, we obtain

$$f(t, x, y) = x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$$

$$\begin{aligned} &\geq t^2 + \tfrac{\pi}{2} \\ &\geq t^3 + \arctan x \\ &\geq \delta_0 \left(t^3 + \arctan x\right) \\ &= \delta_0 g(t, x) \end{aligned}$$

Thus, (R_3) is satisfied.

Moreover, for any $\lambda \in (0,1), t \in [0,1], x \in [0,\infty), y \in [0,\infty)$, we get $\arctan(\lambda x) \geq \lambda \arctan x$. Therefore,

$$g(t,\lambda x) \geq \lambda g(t,x),$$

$$f\left(t,\lambda x,\lambda^{-1}y\right) = \lambda^{\frac{1}{3}}x^{\frac{1}{3}} + \lambda^{\frac{1}{3}}y^{-\frac{1}{3}} + t^2 + \frac{\pi}{2}$$

$$\geq \lambda^{\frac{1}{3}}\left(x^{\frac{1}{3}} + y^{-\frac{1}{3}} + t^2 + \tfrac{\pi}{2}\right)$$
$$= \lambda^{\gamma} f(t,x,y)$$

where $\gamma = \frac{1}{3}$.

We conclude that condition (H_4) is satisfied. Therefore,the given BVP has a unique positive solution.

Example: Consider the following Boundary Value problem :

$$\begin{cases} -D_{0+}^{\frac{3}{2}}x(t) = 2x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2 + t^3, \quad 0 < t < 1 \\ x(0) = 0, \quad x(1) = \frac{1}{2}x\left(\frac{1}{2}\right) \end{cases}$$

In this case, $\alpha = \frac{3}{2}$.
This problem can be regard as a boundary value problem of form (B) with $f(t, x, y) = x^{\frac{1}{2}} + y^{-\frac{1}{2}} + t^2$ and $g(t, x) = x^{\frac{1}{2}} + t^3$.

Now we verify that conditions $(\mathrm{R}_1)-(\mathrm{R}_4)$ are satisfied.

Firstly, it is easy to see (R_1) and (R_2) are satisfied and $g(t,0)=t^3\neq 0$.

Secondly, take $\delta_0\in(0,1]$, we obtain

$$f(t,x,y)=x^{\frac{1}{2}}+y^{-\frac{1}{2}}+t^2\geq x^{\frac{1}{2}}+t^3\geq\delta_0\left(x^{\frac{1}{2}}+t^3\right)=\delta_0$$

Thus, (R_3) is satisfied.

Moreover, for any $\lambda \in (0,1), t \in [0,1], x \in [0,\infty), y \in [0,\infty)$, we have

$$g(t,\lambda x) = \lambda^{\frac{1}{2}} x^{\frac{1}{2}} + t^3 \geq \lambda^{\frac{1}{2}}\left(x^{\frac{1}{2}} + t^3\right) \geq \lambda g(t,x)$$

$$f\left(t,\lambda x,\lambda^{-1}y\right) = \lambda^{\frac{1}{2}} x^{\frac{1}{2}} + \lambda^{\frac{1}{2}} y^{-\frac{1}{2}} + t^2 \geq \lambda^{\frac{1}{2}}\left(x^{\frac{1}{2}} + y^{-\frac{1}{2}}\right.$$

where $\gamma = \frac{1}{2}$. We conclude that condition (R_4) is satisfied.

Therefore, above results ensures that the above Boundary value problem has a unique Positive Solution.

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