Department of Mathematics
MTL 601 (Probability and Statistics)
Tutorial Sheet No. 2

1. Let

$$F(x,y) = \begin{cases} 0, & x<0, y<0 \text{ or } x+y<1 \\ 1, & \text{otherwise} \end{cases}$$

Show that F is not a distribution function in R^2.

2. A point (u, v) is chosen as follows. First a u is chosen at random in the interval $(0, 1)$, then a point v is chosen at random on the interval $(0, u)$. Find the joint p.d.f. of (X, Y), defined by $X(u, v) = u,\ Y(u, v) = v$. Also find the conditional p.d.f. of X, given $Y = y$ and that of Y, given $X = x$.

3. Suppose 2 cards are drawn from a deck of 52 cards. Let X = number of aces obtained and Y = number of queens obtained. Discuss weather or not random variable X, Y are independent.

4. Suppose a two dimensional random variable has a joint p.d.f.:

$$f_{X,Y}(x,y) = \begin{cases} kx(x-y), & 0<x<2, -x<y<x \\ 0, & \text{otherwise} \end{cases}$$

(a) Evaluate the constant k

(b) Compute $P(X+Y<1),\ P(XY<1)$

(c) Compute $P(Y>-1/2|X=1)$

5. Let X_1, X_2 be i.i.d. random variables each having p.m.f. $P(X=\pm 1)=\frac{1}{2}$. Define $X_3 = X_1X_2$. Show that random variables X_1, X_2, X_3 are pairwise independent but not mutually independent.

6. Suppose X, Y are independent random variables each having binomial distribution with parameters n and $p,\ (0<p<1)$. FInd the joint pmf of $(X+Y, X-Y)$.

7. Suppose X, Y are independent Poisson random variables, show that the conditional distribution of X given $Z = X+Y$, is binomial.

8. Let X, Y be i.i.d each having standard normal distribution. Show that $U=\sqrt{X^2+Y^2}$ and $V=\frac{X}{Y}$ are independent.

9. Let (X, Y) be uniformly distributed on the region $\{(x,y) : 0<x<y<1\}$.

(a) Find $V(2X+3Y-4),\ Cov(X+Y, X-Y),\ E(X^3+XY^2-X^2Y)$.

(b) Find regression of Y on X and of X on Y.

(c) Find MGF of (X, Y) and use this to compute correlation coefficient between X, Y.

10. Let $X_1, X_2 \ldots, X_{m+n}$ be i.i.d. random variables each having a finite second order moment. Find the correlation coefficient between S_n and $S_{m+n}-S_n,\ n>m$, where $S_k=\sum_{i=1}^{k} X_i,\ k=1,2,\ldots,m+n$.

11. The joint pdf of (X, Y) is given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(1+xy(x^2-y^2)), & |x|\le 1,\ |y|\le 1 \\ 0, & \textit{otherwise} \end{cases}$$

Compute

(a) $E(Y|X),\ E(XY^2+Y|X),\ Cov(X^2, Y^2)$

(b) $MGF(X, Y),\ MGF(X+Y)$.

12. Let X, Y be discreet random variables with respective pmfs given by

$$p_X(x_1) = p_1,\ p_X(x_2) = 1 - p_1,\ p_Y(y_1) = p_2,\ p_Y(y_2) = 1 - p_2$$

Show that X, Y are independent iff X, Y are uncorrelated.

13. Using the concept of conditional expectation, prove that:

$$Var(X) = Var(E(X|Y)) + E(Var(X|Y))$$

14. Let $X_1, X_2, \ldots, X_n$ be the sequence of i.i.d. random variables and N be discrete random variable taking positive integer values. Suppose $X's$ and N are independent. Define random sum: $S_N = X_1 + X_2 + \ldots + X_N$.

(a) Show that $E(S_N) = E(N)E(X_1)$ and $Var(S_N) = (E(X_1))^2 Var(N) + E(N)Var(X_1)$.

(b) Find MGF of S_N in terms of mgf's of X_i and N.

15. From a point on the circumference of the circle of radius r, a chord is drawn in a random direction. Show that the expected value of length of the chord is $\frac{4r}{\pi}$ and its variance is $2r^2\left(1 - \frac{8}{\pi^2}\right)$.

16. Show that for a random variable with p.d.f. $f_X(x) = \frac{k\alpha^k}{(x+\alpha)^{k+1}},\ x > 0$, the α^{th} absolute moment exists for X, for $\alpha < k$.

17. For the random variable with p.m.f. $p_X(x) = \binom{r+x-1}{x} p^r q^x,\ x = 0, 1, 2, \ldots$, where $q = 1 - p$, find the m.g.f. and hence the mean and variance of X.

18. (Jensens's Inequality) If g is a convex function and $E(X)$ exists, then show that $g(E(X)) < E(g(X))$. Hence show that $E(X) \leq (E(|X|))^{1/r}$.

19. Let $g(X) \geq 0, \forall x \in [0, \infty)$ be a non-decreasing even function. Show that for any random variable X such that $E(g(X))$ exists $P(|X| \geq \epsilon) \leq \frac{E(g(X))}{g(\epsilon)}$.

20. Show that absolute moment of no order exists for the random variable having p.d.f $f_X(x) = \frac{1}{2|x|(\ln|x|)^2}$ for $|x| > e$.