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SECOND EDITION

Discrete Mathematical Structures

G. Shanker Rao

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(SECOND EDITION)

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To

my wife,

Usha Rani

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Preface to the Second Edition

This edition is a revision of 2002 edition of the book. Considerable attention has been given to improve the first edition. As far as possible efforts were made to keep the book free from typographic and other errors. Most of the changes were made at the suggestions of the individuals who have used the first edition of the book and who were kind enough to send their comments. Enhancements to the material devoted to mathematical logic methods of proof, combinations and graph theory are designed to help the readers master the subject.

I am thankful to the chief editor and the editors of New Age International (P) Limited, Publishers for the interest and cooperation during the production of the second edition of the book.

The author would like to express his appreciation to Sri Saumya Gupta, Managing Director, New Age International (P) Limited, for his encouragement.

Any suggestions for future improvements of this book will be gratefully received

G. SHANKER RAO

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Preface to the First Edition

This book explains some of the fundamental concepts in discrete structures. It can be used by the students in mathematics and computer science as an introduction to the fundamental ideas of discrete mathematics. The topics mathematical logic, sets, relations, function, Boolean algebra, logic gates, combinations, algebraic structures, graph theory and finite state machines have been discussed in this book. Throughout I have made an extensive use of worked examples to develop the general ideas.

Chapter 1 deals with mathematical logic. Propositions, logical equivalence, tautologies, fallacies, quantifiers, and methods of proof were briefly discussed in this chapter.

Chapter 2 is devoted to set theory.

Chapter 3 deals with relations. Reflexive, symmetric and transitive relations, have been discussed.

Chapter 4 deals with functions and recurrence relations.

Chapter 5 covers Boolean algebra. Lattices, Boolean functions, karnaugh maps, canonical forms have been discussed in this chapter.

Chapter 6 covers logic gates.

Chapter 7 deals with Elementary combinatorics. Permutation combinations and Binomial theorem have been discussed in this chapter.

Chapter 8 deals with graph theory. Isomorphism, colouring of graphs, trees, spanning trees have been explored in this chapter.

Chapter 9 covers Algebraic Structures. Groups, rings and fields, their properties have been briefly discussed in this chapter.

Chapter 10 explains finite state machines.

I am much indebted to Sri Siva Kumar, Manager, New Age International (P) Limited, Publishers Hyderabad Branch, whose suggestions and criticism helped me in writing the book. I am thankful to Sri Arvind Mishra of New Age International (P) Limited, Publishers.

G. Shanker Rao

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Contents

1

Mathematical Logic

1.1 INTRODUCTION

In this chapter we shall study mathematical logic, which is concerned with all kinds of reasoning. Mathematical logic has two aspects. On one hand it is analytical theory of art of reasoning whose goal is to systematize and codify principles of valid reasoning. It may be used to judge the correctness of statements which make up the chain. In this aspect logic may be called 'classical' mathematical logic. The other aspect of Mathematical logic is inter-related with problems relating the foundation of Mathematics. G. Frege (1884–1925) developed the idea, regarding a mathematical theory as applied system of logic.

Principles of logic are valuable to problem analysis, programming and logic design.

1.2 STATEMENTS

A statement is a declarative sentence which is either true or false but not both. The truth or falsity of a statement is called its truth value. The truth values 'True' and 'False' of a statement are denoted by T and F respectively. They are also denoted by 1 and 0.

Example 1: Bangalore is in India.

Example 2: 3 + 7 = 9.

Example 3: Roses are red.

Statements are usually denoted by the letters p, q, r, ... The capital letters A, B, C, ..., P, Q, ... with the exception of T and F are also used.

1.3 LAWS OF FORMAL LOGIC

Now we state two famous laws of Formal Logic.

1.3.1 Law of Contradiction

According to the law of Contradiction the same predicate cannot be both affirmed and denied precisely of the same subject; i.e., for every proposition p it is not the same that p is both true and false.

1.3.2 Law of Excluded Middle

If p is a statement (proposition), then either p is true or p is false, and there cannot be middle ground.

1.4 CONNECTIVES AND COMPOUND STATEMENTS

Statements can be connected by words like ‘not’, ‘and’, etc.

These words are known as logical connectives. The statements which do not contain any of the connectives are called atomic statements or simple statements.

The common connectives used are: negation (~) [or ($\neg$)], and ($\wedge$) or ($\vee$), if ... then ($\rightarrow$), if and only if ($\leftrightarrow$), equivalence ($\equiv$) or ($\Leftrightarrow$). We will use these connectives along with symbols to combine various simple statements.

1.4.1 Compound Statement

A statement that is formed from atomic (Primary) statements through the use of sentential connectives is called a compound statement.

1.4.2 Truth Table

The table showing the Truth values of a statement formula is called ‘Truth Table’.

1.4.3 Conjunction

A compound statement obtained by combining two simple statements say p and q, by using the connective “and” is called conjunction, i.e., the conjunction of two statements p and q is the statement $p \wedge q$. It is read as “p and q”.

The statement $p \wedge q$ has the truth value T, whenever both p and q have the truth value T, otherwise $p \wedge q$ has the truth value F. The above property can also be written in the form of the table below, which we regard as defining $p \wedge q$:

Table 1.1 *Truth table for conjunction*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 1: Form the conjunction of

p: Delhi is in India.

q: 5 + 7 = 12.

Solution: $p \wedge q$ is the statement:

“Delhi is in India and 5 + 7 = 12”

Example 2: From the conjunction of

p: It is raining.

q: The sun is shining.

Solution: $p \wedge q$. It is raining and the sun is shining.

Example 3: Construct a Truth Table for the conjunction of "$n > 3$" and "$n < 10$" when $n \in N$.

Solution: When $n > 3$ and $n < 10$ are true, the conjunctive statement "$n > 3$ and $n < 10$" is true. The Truth Table is given below:

Table 1.2

$n > 3$	$n < 10$	$n > 3$ and $n < 10$
T	T	T
T	F	F
F	T	F
F	F	F

1.4.4 Disjunction

Any two simple statements can be combined by the connective "or" to form a statement called the disjunction of the statements; i.e., if p and q are simple statements, the sentence "p or q" is the disjunction of p and q.

The disjunction of p and q is denoted symbolically by $p \vee q$

$p \vee q$ is read as "p or q"

If p is 'True' or q is 'True' or both p and q are 'True', then $p \vee q$ is true, otherwise $p \vee q$ is false. The truth table of $p \vee q$ is given below:

Table 1.3 *Truth table of* $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 1: Let p: $5 + 2 = 7$, q: $9 + 2 = 10$ then

$$p \vee q: 5 + 2 = 7 \text{ or } 9 + 2 = 10$$

Example 2: Let p: Roses are red

q: Violets are blue, then,

$p \vee q$: Roses are red or violets are blue.

1.4.5 Negation

Let p be any simple statement, then the negation of p is formed by writing "it is false that" before p. The negation of p is also obtained by writing "p is false".

The negation is p is denoted by $\sim p$.

If the statement p is true, then "$\sim p$ is false" and if p is false then $\sim p$ is true. The Truth Table for negation is given below:

Table 1.4

p	$\sim p$
T	F
F	T

Example 1: Let p: Tajmahal is in New York.

Then the negation of p is

$\sim p$: it is false that Tajmahal is in New York.

Example 2: Form the negation of the statement

p: It is cold

Solution: $\sim p$: It is not cold.

Example 3: Form the negation of the statement

p: $n > 12$

Solution: $\sim p$: $n > 12$ is false.

1.5 PROPOSITION

If $p, q, r, s, ...$ are Simple Statements then the Compound Statement $P(p, q, r, s, ...)$ is called a Proposition. The statement $p, q, r, ...$ are called the Sub-statements or Variables of P.

The truth value of proposition P depends on the truth values of the variables, $p, q, r, ...$ If the truth values of the variables are known to us, then we can find the truth value of the proposition P. A truth table is a simple way to show this relationship.

Example: Find the truth table of the Proposition $\sim p \wedge q$

Solution: The truth table of $\sim p \wedge q$ is:

Table 1.5 *Truth table* $\sim p \wedge q$

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

1.6 SOLVED EXAMPLES

Example 1: Let p be "it is cold" and q be "it is raining". Give a simple verbal sentence which describes each of the following:

(*i*) $\sim p$ (*ii*) $\sim p \wedge \sim q$

Solution:

(*i*) $\sim p$: It is not cold

(*ii*) $\sim p \wedge \sim q$: It is not cold and it is not raining.

Example 2: Let p be "He is tall" and let q be "He is Handsome". Write each of the following statements in symbolic form using p and q.

(i) He is tall and handsome.

(ii) He is neither tall nor handsome.

Solution: (i) $p \wedge q$ (ii) $\sim p \wedge \sim q$

Example 3: Write the disjunction of:

Roses are red. Violets are blue.

Solution: Let p: Roses are red

q: Violets are blue then the disjunction of p and q is $p \vee q$: Roses are red or violets are blue.

Example 4: Determine the truth value of each of the following statements (Propositions):

(i) $3 + 5 = 8$ or $2 + 1 = 9$

(ii) $4 + 3 = 7$ and $5 + 2 = 7$

(iii) Agra is in England or $1 + 9 = 8$

Solution: (i) Let p: $3 + 5 = 8$, q: $2 + 1 = 9$

p is true, q is false

hence $p \vee q$ is true

i.e., Truth Value of $p \vee q$ is T

(ii) Let p: $4 + 3 = 7$, q: $5 + 2 = 7$

p is true and q is true $\Rightarrow$ $p \wedge q$ is true (T)

(iii) Let p: Agra is in England

q: $1 + 9 = 8$

p is false; q is false $\Rightarrow$ $p \vee q$ is false.

Example 5: Construct a truth table for $p \wedge \sim p$.

Solution: The truth table for $p \wedge \sim p$ is given below:

Table 1.6

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Example 6: Construct the truth table for $p \vee \sim p$

Solution:

Table 1.7

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

Example 7: Find the truth table for $p \wedge (q \vee r)$

Solution:

Table 1.8

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Example 8: Find the truth table for $\sim(\sim p)$ (Double negation)

Solution:

Table 1.9

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

EXERCISE 1.1

1. Determine the truth value of each of the following:

(*a*) 4 + 2 = 6 and 2 + 2 = 4

(*b*) 5 + 4 = 9 and 3 + 3 = 5

(*c*) 6 + 4 = 10 and 1 + 1 = 3

(*d*) Charminar is in Hyderabad or 7 + 1 = 6

(*e*) It is not true that Delhi is in Russia

(*f*) It is false that 3 + 3 = 6 and 2 + 2 = 8

2. Construct truth tables for the following:

(*a*) $\sim(p \vee q)$

(*b*) $\sim(p \vee \sim q)$

(*c*) $(p \wedge q) \vee (p \wedge q)$

(*d*) $(p \vee q) \vee \sim p$

(*e*) $\sim(\sim p \vee \sim q)$

(*f*) $p \wedge (q \wedge p)$

(*g*) $p \vee \sim(p \wedge q)$

3. Write the negation of each statement

(*a*) Violets are blue (*b*) Delhi is in America

(*c*) $3 + 3 = 7$

4. Let p be "Mark is rich" and q be "Mark is happy"

Write each of the following in symbolic form.

(*a*) Mark is poor but happy

(*b*) Mark is neither rich nor happy

(*c*) Mark is either rich or happy

(*d*) Mark is either poor or else; he is both rich and happy

5. Let p be "It is cold" and let q be "It is raining"

Give a simple verbal sentence which describes each of the following statements:

(*a*) $\sim p$ (*b*) $p \wedge q$

(*c*) $p \vee q$ (*d*) $\sim p \wedge \sim q$

6. Write the symbols for connectives in the following sentences:

(*a*) Either p or not p (*b*) p and not q

(*c*) not p or not q (*d*) not p and q

7. Write the conjunction of:

(*a*) It is raining; It is snowing (*b*) $4 + 7 = 11$; $2 \times 4 = 7$

8. Let p be "He is tall" and q be "He is handsome"

Write each of the following statements in symbolic form using p and q

(*a*) He is tall and handsome

(*b*) He is tall but not handsome

(*c*) He is neither tall nor handsome

Answers:

2. (*a*)

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

(*b*)

p	q	$\sim q$	$p \vee \sim q$	$\sim(p \vee \sim q)$
T	T	F	T	F
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

(*c*)

p	q	$p \wedge q$	$(p \wedge q) \vee (p \wedge q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(*d*)

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

(*e*)

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$\sim(\sim p \vee \sim q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(*f*)

p	q	$q \wedge p$	$p \wedge (p \wedge q)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(*g*)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$p \wedge \sim(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	F
F	F	F	T	T

3. (*a*) Violets are not blue
(*b*) Delhi is not in America
(or It is not the case that Delhi is in America)

(*c*) $3 + 3 = 7$
(or It is not the case that $3 + 3 = 7$)

4. (*a*) $\sim p \wedge q$ (*b*) $\sim p \wedge \sim q$ (*c*) $p \vee \sim q$ (*d*) $\sim p \vee (p \wedge \sim q)$

5. (*a*) It is not cold (*b*) It is cold and raining
(*c*) It is cold or it is raining (*d*) It is not cold and it is not raining.

6. (*a*) $p \vee \sim p$ (*b*) $p \wedge \sim q$ (*c*) $\sim p \vee \sim q$ (*d*) $\sim p \wedge q$

7. (*a*) It is raining and it is snowing
(*b*) $4 + 7 = 11$ and $2 \times 4 = 7$

8. (*a*) $p \wedge q$ (*b*) $\sim p \wedge \sim q$ (*c*) $\sim p \wedge \sim q$

1.7 CONDITIONAL STATEMENTS

1.7.1 Conditional $p \rightarrow q$

If p and q are any two statements then the statement $p \rightarrow q$ which is read as "if p then q" is called a Conditional statement.

The symbol $\rightarrow$ is used to denote connective "If ... then"

The conditional $p \rightarrow q$ can also be read:

(*a*) p only if q (*b*) p implies q (*c*) p is sufficient for q (*d*) q if p

The conditional $p \rightarrow q$ has two simple statements p and q connected by "if ... then"

The statement p is called the antecedent and the statement q is called the consequent (or conclusion). If p is true and q is false, then conditional $p \rightarrow q$ is false. In other cases $p \rightarrow q$ is true.

The truth values of $p \rightarrow q$ are given in Table 1.10.

Table 1.10 *Truth table for* $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example 1: If Delhi is in India, then $3 + 3 = 6$

Example 2: Let p: He is a graduate
q: He is a lawyer then,
$p \rightarrow q$: If he is a graduate, then he is a lawyer.

1.7.2 Biconditional

A statement of the form "p if and only if q" is called a Biconditional statement. It is denoted by $p \rightleftarrows q$ (or by $p \leftrightarrow q$).

A Biconditional statement contains the connective "if and only if " and has two conditions. If p and q have the same truth value, then $p \leftrightarrow q$ is true. The truth values $p \leftrightarrow q$ are given in Table 1.11.

Table 1.11 *Truth table for $p \leftrightarrow q$*

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example 1: Bangalore is in India, if and only if 4 + 4 = 8.

Example 2: 3 + 3 = 6 if and only if 4 + 3 =7.

1.7.3 Converse, Inverse and Contrapositive Propositions

If $p \rightarrow q$, is a conditional statement, then

(*a*) $q \rightarrow p$ is called its converse

(*b*) $\sim p \rightarrow \sim q$ is called its inverse

(*c*) $\sim q \rightarrow \sim p$ is called its contrapositive.

The truth values of these propositions are given in Tables 1.12, 1.13 and 1.14, respectively.

Table 1.12 *Truth table for the converse of $p \rightarrow q$*

p	q	$p \rightarrow q$	$q \rightarrow p$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Table 1.13 *Truth table for the inverse of $p \rightarrow q$*

p	q	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Table 1.14 *Truth table for contraposition*

p	q	$\sim q$	$\sim p$	$\sim q \to \sim p$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Example: Write the contrapositive of the implication

"if it is raining, then I get wet"

Solution: let p: It is raining

q: I get wet

then the contrapositive is

$\sim q \to \sim p$: If I do not get wet, then it is not raining.

1.8 WELL FORMED FORMULAS

Statement formulas contain one or more simple statements and some connectives. If p and q are any two statements, then

$$p \vee q, (p \wedge q) \vee (\sim p), (\sim p) \wedge q$$

are some statement formulas derived from the statement variables p and q where p and q called the components of the statement formulas. A statement formula has no truth value. It is only when the statement variables in a statement formula are replaced by definite statements that we get a statement, which has a truth value that depends upon the truth values of the statements used in replacing the variables. A statement formula is a string consisting of variables, parentheses and connective symbols. A statement formula is called a well formed (w f f) if it can be generated by the following rules:

1. A statement variable p standing alone is a well formed formula.
2. If p is a wellformed formula, then $\sim p$ is a well formed formula.
3. If p and q are wellformed formulas, then $(p \wedge q)$, $(p \vee q)$, $(p \to q)$ and $(p \leftrightarrow q)$ are well formed formulas.
4. A string of symbols is a well formed formula if and only if it is obtained by finitely many applications of the rules 1, 2 and 3.

According to the above recursive definition of a well formed formula $\sim(p \vee q)$, $(\sim p \wedge q)$, $(p \to (p \vee q))$ are well formed formulas.

A statement formula is not a statement and has no truth values. But if we substitute definite statements in place of variables in given formula we get a statement. The truth value of this resulting statement depends upon the truth values of the statements substituted for the variables, which appears as one of the entries in the final column of the truth table constructed. Therefore the truth table of a well formed formula is the summary of truth values of the resulting statements for all possible assignments of values to the variables appearing in the formula. The final column entries of the truth table of a well formed formula gives the truth values of the formula.

1.9 TAUTOLOGY

A statement formula that is true for all possible values of its propositional variables is called a Tautology.

Example 1: $(p \vee q) \leftrightarrow (q \vee p)$ is a tautology.

Example 2: $p \vee \sim p$ is a tautology.

1.10 CONTRADICTION

A statement formula that is always false is called a contradiction (or absurdity).

Example: $p \wedge \sim p$ is an absurdity.

1.11 CONTINGENCY

A statement formula that can either be true or false depending upon the truth values of its propositional variables is called a contingency.

Example: $(p \rightarrow q) \wedge (p \wedge q)$ is a contingency.

1.12 LOGICAL EQUIVALENCE

Two propositions P and Q are said to be logically equivalent or simply equivalent if $P \rightarrow Q$ is a tautology.

Example: $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are logically equivalent.

Two formulas may be equivalent, even if they do not contain the same variables. Two statement formulas P and Q are equivalent if $P \rightleftarrows Q$ is a tautology and conversely, if $P \rightleftarrows Q$ is a tautology then P and Q are equivalent. If "P is equivalent Q" then we can represent the equivalence by writing "$P \Leftrightarrow Q$" which can also be written as $P \Leftrightarrow Q$. The symbol "$\Leftrightarrow$" is not a connective. We usually drop the quotation marks.

1.13 SOLVED EXAMPLES

Example 1: The converse of a statement is given. Write the inverse and the contrapositive statements "if I come early, then I can get the car".

Solution: Inverse: "If I cannot get the car, then I shall not come early"

Contrapositive: If I do not come early, then I cannot get the car.

Example 2: The inverse of a statement is given. Write the converse and contrapositive of the statement. "If a man is not a fisherman, then he is not a swimmer".

Solution: Converse: "If he is a swimmer, then the man is a fisherman".

Contrapositive: "If he is not a swimmer, then the man is not a fisherman".

Example 3: Determine a truth table of $\sim p \rightarrow (q \rightarrow p)$

Solution:

Table 1.15

p	q	$\sim p$	$q \to p$	$\sim p \to (q \to p)$
T	T	F	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Example 4: Show that $p \wedge \sim p$ is a contradiction.
Solution: The truth table for $p \wedge \sim p$ is given below:

Table 1.16

p	$\sim p$	$p \wedge \sim p$
T	F	F
T	F	F

$p \wedge \sim p$ is always false, hence $p \wedge \sim p$ is a contradiction.

Example 5: Show that $p \vee \sim p$ is a tautology.
Solution: We construct the truth table for $(p \vee \sim p)$

Table 1.17

p	$\sim p$	$(p \vee \sim p)$
T	F	T
T	T	T

$p \vee \sim p$ is always true.
Hence $p \vee \sim p$ is a tautology.

Example 6: Show that $(p \wedge q) \to p$ is tautology.
Solution: Let us construct the truth table for the statement $(p \wedge q) \to p$

Table 1.18

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

In Table 1.18, we notice that the column (4) has all its entries as T. Hence $(p \wedge q) \rightarrow p$ is a tautology.

Example 7: Show that $\sim(p \rightarrow q) \equiv (p \wedge \sim q)$

Solution: Let us construct the truth table for the given propositions:

Table 1.19

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the truth table it is clear that the truth values of $\sim(p \rightarrow q)$ and $p \wedge \sim q$ are identical.

Hence $\sim(p \rightarrow q) \equiv p \wedge \sim q$.

1.14 LAWS OF LOGIC

1. Idempotent Laws:

(a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$

2. Commutative Laws:

(a) $p \vee p \equiv q \vee p$ (b) $p \wedge q \equiv q \wedge p$

3. Associative Laws:

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

4. Distributive Laws:

(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \vee q) \vee (p \wedge r)$

5. Identity Laws:

(a) (i) $p \vee f \equiv p$ (ii) $p \vee t \equiv t$

(b) (i) $p \wedge f \equiv f$ (ii) $p \wedge t \equiv p$

6. Complement Laws:

(a) (i) $p \wedge \sim p \equiv t$ (ii) $p \wedge \sim p \equiv f$

(b) (i) $\sim\sim p \equiv p$ (ii) $\sim t \equiv f, \sim f \equiv t$

7. De Morgan's Laws:

(a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ (b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

where t and f are used to denote the variables which are restricted to the truth values true and false respectively.

1.15 THE DUALITY PRINCIPLE

The Principle of duality states that any established result involving statement formulas and connectives $\vee$ and $\wedge$ gives a corresponding dual result by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. If the formula contains special variables t and f, the corresponding dual is obtained by replacing t by f and f by t. The connectives $\wedge$ and $\vee$ are called duals of each other.

Definition 1.1: Two statement formulas P and P^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ and $\vee$ and $\vee$ by $\wedge$.

Example: Write the duals of

(*a*) $(p \wedge q) \vee r$ (*b*) $(p \wedge q) \vee r$ (*c*) $\sim(p \wedge q)$

Solution: The duals are

(*a*) $(p \vee q) \wedge r$ (*b*) $(p \vee q) \wedge r$ (*c*) $\sim(p \vee q)$

1.16 SOLVED EXAMPLES

Example 1: Simplify the following statements:

(*a*) $\sim(p \vee \sim q)$ (*b*) $\sim(\sim p \wedge q)$ (*c*) $\sim(\sim p \vee \sim q)$ (*d*) $(p \vee q) \wedge \sim p$

Solution:

(*a*) $\sim(p \vee \sim q) = \sim p \wedge \sim\sim q$ (De Morgan's law)
$= \sim p \wedge q$

(*b*) $\sim(\sim p \wedge q) = \sim\sim p \vee \sim q$ (De Morgan's law)
$= p \vee \sim q$

(*c*) $\sim(\sim p \wedge \sim q) = \sim\sim p \vee \sim q$
$= p \vee \sim q$

(*d*) $(p \vee q) \wedge \sim p = \sim p \wedge (p \vee q)$
$= (\sim p \wedge p) \vee (\sim p \wedge q)$
$= f \vee (\sim p \wedge q)$
$= \sim p \wedge q$

Example 2: Show that

$$(\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r) \Leftrightarrow r$$

where $\Leftrightarrow$ is the symbol for equivalence

Solution: $\sim p \wedge (\sim q \wedge r) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow (\sim p \wedge (\sim q \wedge r) \vee [(q \vee p) \wedge r]$

$\Leftrightarrow [(\sim p \wedge \sim q) \wedge r] \vee [(q \vee p) \wedge r]$

$\Leftrightarrow [(\sim p \wedge \sim q) \vee [(q \vee p)] \wedge r$

$\Leftrightarrow [(\sim p \vee \sim q) \vee (p \vee q)] \wedge r$

$\Leftrightarrow t \wedge r$ (t denotes tautology)

$\Leftrightarrow r$

Example 3: Simplify

(*i*) $p \vee (p \wedge q)$ (*ii*) $(p \vee q) \wedge (\sim p \wedge q)$

Solution: (*i*) $p \vee (p \wedge q) = (p \vee t) \wedge (p \vee q)$

$$= p \wedge (t \vee q) \qquad (t\text{: tautology})$$
$$= p \wedge t$$
$$= p$$

(*ii*) $(p \vee q) \wedge (\sim p \wedge q)$

$$= (\sim p \wedge \sim q) \vee (\sim p \wedge q)$$
$$= \sim p \wedge (\sim q \vee q)$$
$$= \sim p \wedge t \qquad (t\text{: tautology})$$
$$= \sim p$$

Example 4: Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution: Let us construct the truth table:

Table 1.20

p	q	$p \wedge q$	$p \vee q$	$p \wedge q \rightarrow p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

All the entries in the last column of the truth table are True (T). Hence given proposition is a tautology.

Example 5: Show that

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

Solution: We construct the truth table for given propositions:

Table 1.21

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

From the truth table it is clear that the truth values of $\sim(p \rightarrow q)$ and $p \wedge \sim q$ are identical. Hence

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

Example 6: Show that

$$\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q$$

Solution: We prove the equivalence by means of a truth table.

Table 1.22

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim p$	$\sim p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	T	F	T
F	F	T	F	T	F	T	F

The truth values of columns (4), (6) and (8) are alike; which proves the equivalence of the formulas $\sim(p \leftrightarrow q)$, $\sim p \leftrightarrow q$, and $p \leftrightarrow \sim q$.

Example 7: There are two restaurants next to each other. One has a sign that says "Good food is not cheap", and the other has the sign that says "cheap food is not good".

Are the signs saying the same thing?

If so verify.

Solution: Let p: food is good

q: food is cheap

Then we have, $\sim p$: food is not good

$\sim q$: food is not cheap

Therefore, the given statements are

$p \to \sim q$: Good food is not cheap

$q \to \sim p$: Cheap food is not good

The truth table for the statements is given below:

Table 1.23

p	q	$\sim p$	$\sim q$	$p \to \sim q$	$q \to \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

From the table, it is clear that both the signs say the same thing.

1.17 LOGICAL IMPLICATION

We state the following theorem:

Theorem 1.1: Let $P(p_1, p_2, ...)$ and $Q(p_1, p_2, ...)$ be two propositions. Then the following conditions are equivalent:

1. $\sim P(p_1, p_2, ...) \vee Q(p_1, p_2, ...)$ is a Tautology.
2. $P(p_1, p_2, ...) \wedge Q(p_1, p_2, ...)$ is a Contradiction.
3. $P(p_1, p_2, ...) \to Q(p_1, p_2, ...)$ is a Tautology.

Definition 1.2: A proposition $P\,(p_1, p_2, ...)$ is said to logically imply a proposition $Q\,(p_1, p_2, ...)$ if one of the conditions in Theorem 1.1 holds.

If $P\,(p_1, p_2, ...)$ logically implies $Q\,(p_1, p_2, ...)$ then we symbolically denote it by writing $P\,(p_1, p_2, ...) \Rightarrow Q\,(p_1, p_2, ...)$

Example 1: $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction.

Hence $p \wedge q \Rightarrow p \vee q$

Example 2: $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

Hence $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow (p \rightarrow r)$

Theorem 1.2: The relation in propositions defined by

$$P\,(p_1, p_2, ...) \Rightarrow Q\,(p_1, p_2, ...)$$

is reflexive, anti-symmetric and transitive.

Note: The symbols $\rightarrow, \Rightarrow$ are not the same $\Rightarrow$ is not a connective nor $P \Rightarrow Q$ is a statement formula (proposition). $P \Rightarrow Q$ defines a relation in composite propositions $P \rightarrow Q$. The symbol $\rightarrow$ is a connective and note that $P \rightarrow Q$ is just a proposition.

1.18 OTHER CONNECTIVES

We now introduce the connectives NAND, NOR which have useful applications in the design of Computers.

The word NAND is a combination of "NOT" and "AND" where "NOT" stands for negation and "AND" for the conjunction. It is denoted by the symbol $\uparrow$.

If P and Q are two formulas then $P \uparrow Q \leftrightarrow \sim(P \wedge Q)$

The connective $\uparrow$ has the following equivalence:

$$P \uparrow P \leftrightarrow \sim(P \wedge P) \leftrightarrow \sim P \vee \sim P \Leftrightarrow \sim P$$

$$(P \uparrow Q) \uparrow (P \uparrow Q) \leftrightarrow \sim (P \uparrow Q) \leftrightarrow P \wedge Q$$

$$(P \uparrow P) \uparrow (Q \uparrow Q) \leftrightarrow \sim P \uparrow \sim Q \Leftrightarrow \sim (\sim P \wedge \sim Q) \leftrightarrow P \vee Q$$

The connective NAND is commutative but not associative:

i.e., $P \uparrow Q \leftrightarrow Q \uparrow P$ but $P \uparrow (Q \uparrow R) \leftrightarrow \sim P \vee (Q \wedge R)$ and

$(P \uparrow Q) \uparrow R \leftrightarrow \sim (P \wedge Q) \sim R$. Therefore the connective $\uparrow$ is not associative.

The connective NOR is a combination of "NOT" and "OR", where NOT stands for negation and "OR" stands for the disjunction.

The connective NOR is denoted by the symbol $\downarrow$.

The connective $\downarrow$ has the following equivalence:

$$P \downarrow P \leftrightarrow \sim (P \vee P) \leftrightarrow \sim P \wedge \sim P \Leftrightarrow \sim P$$

$$(P \downarrow P) \downarrow (P \downarrow Q) \leftrightarrow \sim (P \downarrow Q) \leftrightarrow P \vee Q$$

$$(P \downarrow P) \downarrow (Q \downarrow Q) \leftrightarrow \sim P \uparrow \sim Q \leftrightarrow P \wedge Q$$

The connective $\downarrow$ is commutative, but not associative, i.e.

$$P \downarrow Q \Leftrightarrow Q \downarrow P \text{ but } (P \downarrow Q) \downarrow Q \leftrightarrow (P \vee Q) \wedge \sim R$$

$$P \downarrow (Q \downarrow R) \leftrightarrow \sim P \wedge (Q \vee R)$$

Therefore the connective $\downarrow$ is not associative.

The connectives $\wedge, \vee, \sim$ can be expressed in terms of the connective $\downarrow$ as follows:

(*i*) $\sim p \equiv p \downarrow p$ (*ii*) $\sim q \equiv q \downarrow q$

(*iii*) $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$ (*iv*) $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$

Let us verify the above by means of the following truth tables:

Table 1.24

p	q	$\sim p$	$P \downarrow p$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	T

From the above truth table it is clear that $\sim p \equiv p \downarrow p$

Similarly, $\sim q \equiv q \downarrow q$

Now consider the table

Table 1.25

p (1)	q (2)	$p \wedge q$ (3)	$p \downarrow p$ (4)	$q \downarrow q$ (5)	$(p \downarrow p) \downarrow (q \downarrow q)$ (6)
T	T	T	F	F	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	F	T	T	F

The identical truth values of columns (3) and (6) reveal that

$$p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$$

In order to verify (*iv*) we construct the truth table

Table 1.26

p (1)	q (2)	$p \vee q$ (3)	$p \downarrow q$ (4)	$(p \downarrow q) \downarrow (p \downarrow q)$ (5)
T	T	T	F	T
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

The truth values of columns (3) and (5) are alike, which proves the equivalence

$$p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$$

1.19 NORMAL FORMS

Definition 1.3: If a given statement formula A ($p_1, p_2, \ldots p_n$) involves n atomic variables, we have 2^n possible combinations of truth values of statements replacing the variables.

The formula A is a tautology if A has the truth value T for all possible assignments of the truth values to the variables p_1, p_2, ... p_n and A is called a contradiction if A has the truth value F for all possible assignments of the truth values of the n variables. A is said to be satisfiable if A has the truth value T for atleast one combination of truth values assigned to $p_1, p_2, \ldots p_n$.

The problem of determining whether a given statement formula is a Tautology, or a Contradiction is called a decision problem.

The construction of truth table involves a finite number of steps, but the construction may not be practical. We therefore reduce the given statement formula to normal form and find whether a given statement formula is a Tautology or Contradiction or atleast satisfiable.

A formula, which is a product (conjunction) of the variables and their negations is called an Elementary product.

If p and q are atomic values then p, $\sim p$, $\sim p \wedge q$, $p \wedge \sim p$ are some examples of Elementary products.

The sum of (disjunction) of variables and their negations in a formula is called Elementary sum.

If p and q are any two atomic variables p, $\sim p \vee q$, $p \vee \sim p$ and $\sim q \vee p \vee \sim p$ are some examples of Elementary sums.

1.19.1 Disjunctive Normal Form

Definition 1.4: Let A denote a given formula. Another formula B which is equivalent to A is called a Disjunctive normal form of A if B is a sum of elementary products.

A disjunctive normal form of a given formula is constructed as follows:

(*i*) Replace '$\rightarrow$', '$\leftrightarrow$' by using the logical connectives $\wedge$, $\vee$ and ~.

(*ii*) Use De Morgan's laws to eliminate ~ before sums or products.

(*iii*) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

Example 1: Obtain disjunctive normal form of $p \wedge (p \rightarrow q)$

Solution: $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q) \equiv (p \wedge \sim p) \vee (p \wedge q)$

Example 2: Obtain disjunctive normal form of

$$p \vee (\sim p \to (q \vee (q \to \sim r)))$$

Solution:

$$p \vee (\sim p \to (q \vee (q \to \sim r)))$$
$$\equiv p \vee (\sim p \to q \vee (\sim q \vee \sim r)))$$
$$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$$
$$\equiv p \vee p \vee q \vee \sim q \vee \sim r$$
$$\equiv p \vee q \vee \sim q \vee \sim r$$

Therefore, the disjunctive normal form of

$$p \vee (\sim p \to (q \vee (\sim q \to \sim r))) \text{ is } p \vee q \vee \sim q \sim r$$

1.19.2 Conjunctive Normal Form

Let A denote a given formula, another formula B which is equivalent to A is called conjunctive normal formula if B is a product of an elementary sum.

Example: Obtain conjunctive normal of

$$p \wedge (p \to q)$$

Solution: $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$

Hence $p \wedge (\sim p \vee q)$ is the conjunctive normal form of $p \wedge (p \to q)$

1.19.3 Principal Disjunctive Normal Form

Let p and q be the two statement variables. Then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, and $\sim p \wedge \sim q$ are called minterms of p and q. They are called Boolean Conjunctives of p and q. Each minterm has the truth value T for exactly one combination of truth values of the variables p and q. There are 2^2 possible minterms for the two variables p and q. Note that none of the minterms should contain both a variable and its negation. The number of minterms in n variables is 2^n.

We now introduce one more normal form called the principal normal form in the next definition.

Definition 1.5: If A is a given formula, then an equivalent formula B, consisting of disjunctives of minterms only is called the Principal disjunctive normal form of the formula A.

The principle disjunctive normal formula of A is also called the sum-of-products canonical form of A.

Example: Obtain the principal disjunctive normal form of $(\sim p \vee \sim q) \to (\sim p \wedge r)$

Solution: $(\sim p \vee \sim q) \to (\sim p \wedge r)$

$$\Leftrightarrow \sim(\sim p \vee \sim q) \vee (\sim p \wedge r)$$
$$\Leftrightarrow \sim (\sim(p \wedge q)) \vee (\sim p \wedge r)$$
$$\Leftrightarrow (p \wedge q) \vee (\sim p \wedge r)$$
$$\Leftrightarrow (p \wedge q \wedge (r \vee \sim r)) \vee (\sim p \wedge r \wedge (q \vee \sim q))$$
$$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge r \wedge q) \vee (\sim p \wedge r \wedge \sim q)$$

The principal disjunctive normal form of the given formula is

$$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$$

1.19.4 Principal Conjunctive Normal Form

The dual of a minterm is called a Maxterm. For a given number of variables the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once. Each of the maxterm has the truth value F for exactly one combination of the truth values of the variables. Now we define the principal conjunctive normal form.

Definition 1.6: If A is a given formula, then an equivalent formula B is called principle conjunctive normal form of A if B is a product of maxterms.

The principal conjunctive normal form of A is also called the Product-of-sums canonical form.

Example: Obtain the principal conjunctive normal form of

$$(p \wedge q) \vee (\sim p \wedge r)$$

Solution: $(p \wedge q) \vee (\sim p \wedge r)$

$$\Leftrightarrow ((p \wedge q) \vee \sim p) \vee ((p \wedge q) \vee r)$$

$$\Leftrightarrow (p \vee \sim p) \wedge (q \vee \sim p) \wedge (p \vee r) \wedge (q \vee r)$$

$$\Leftrightarrow (q \vee \sim p \vee (r \wedge \sim r)) \wedge (p \vee r \vee (q \wedge \sim q)) \wedge (q \vee r \vee (p \wedge \sim p))$$

$$\Leftrightarrow (q \vee \sim p \vee r) \wedge (q \vee \sim p \vee \sim r) \wedge (p \vee r \vee q)$$

$$\wedge (p \vee r \vee \sim q) \wedge (q \vee r \vee p) \wedge (q \vee r \vee \sim p)$$

$$\Leftrightarrow (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee \sim q \vee r)$$

EXERCISE 1.2

1. The following statement is of the form $p \vee q$. Write out the contradictory statement in the form $\sim p \wedge \sim q$:

   "Either he is a fool or he has some evil design."
2. Let p: A triangle is equilateral

   q: It is equiangular

   then write $p \to q$ the conditional $p \to q$
3. The converse of a statement is given. Write the inverse and contrapositive statements.

   "If he is considerate of others, then a man is a gentleman."
4. The converse of a statement is: If a steel rod or stretcher, then it has been heated:

   Write inverse and contrapositive statements.
5. The contrapositive of a statement is given as

   "If $x < 2$, then $x + 4 < 6$"

   Write the converse and inverse.

6. Let p: It is cold and q: It is raining. Give a simple verbal sentence which describes each of the following statements:

(a) $p \wedge \sim(q)$ (b) $q \rightarrow p$ (c) $p \leftrightarrow \sim q$

7. Write an equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which does not contain biconditional.

8. Show that

(a) $\sim (p \wedge q) \rightarrow (\sim p \vee)(\sim p \vee q) \Leftrightarrow \sim p \vee q$

(b) $(p \vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow (\sim p \wedge q)$

(c) $p \rightarrow q \Leftrightarrow \sim p \vee q$

9. By means of a truth table prove that

$$p \wedge q \equiv (p \downarrow q) \downarrow (q \downarrow p) \equiv \sim p \downarrow \sim q$$

10. Show that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

11. Show that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

12. Show that $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ is a tautology.

13. By means of a truth table prove that

(a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (q \wedge r)$

(b) $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

14. Let p be "He is rich" and let q be "He is honest". Write each of the following statements in symbolic forms using p and q:

(a) To be poor is to be honest.

(b) It is necessary to be poor in order to be honest.

(c) He is poor only if he is dishonest.

(d) If he is poor if he is dishonest.

15. Prove that

(a) $p \rightarrow q \equiv \sim p \rightarrow \sim q$ (b) $p \rightarrow q \equiv \sim p \vee q$

16. Write the contradiction of each of the following disjunction statements:

(1) $x = 2$ or $x = 4$ (2) $x > 3$ or $x < 3$

17. Show that $p \leftrightarrow \sim q$ does not logically imply that $p \rightarrow q$

18. Prove the following:

(a) $p \vee \sim(p \wedge q)$ is a Tautology.

(b) $(p \wedge q) \wedge \sim (p \vee q)$ is a Contradiction.

(c) $(p \wedge q) \rightarrow (p \vee q)$ is a Tautology.

19. Show that $p \wedge q$ logically implies $p \leftrightarrow q$.

20. Decide whether each of the following is true or false:

(a) $p \Rightarrow p \wedge q$ (b) $p \Rightarrow p \vee q$ (c) $p \wedge q \Rightarrow p$

(d) $p \vee q \Rightarrow p$ (e) $q \Rightarrow p \rightarrow q$

21. Write the disjunction, the conjunction, and two implications involving the two statements. I like cats. I like dogs.

22. If

A: The Eiffel Tower is in Australia

B: Australia is below the Equator

C: The Eiffel Tower is in Paris

D: Paris is in France

E: France is in Australia

Prove the following:

(1) The argument (A $\wedge$ B) $\rightarrow$ is valid

(2) (C $\wedge$ D) $\rightarrow$ is invalid

(3) (A $\vee$ E) $\rightarrow$ is invalid

23. Simplify the following compound propositions

(a) $(p \vee q) \wedge \sim[(\sim p \vee q]$

(b) $\sim[\sim\{(p \vee q) \wedge r\} \vee \sim q]$

24. Show that $[(r \rightarrow s) \wedge \{(r \rightarrow s) \rightarrow (t \rightarrow u)\}] \rightarrow [\sim t \vee u]$ is a tautology

Answers:

1. Either the man is born free or he is nowhere in chains.

2. If a triangle is equilateral, then it is equiangular.

3. Inverse: If a man is not a gentleman, then he is not considerate of others.

Contrapositive: If he is not considerate of others, then the man is not a gentleman.

4. Inverse: If a steel rod is not heated, then it does not stretch.

Contrapositive: If a steel rod does not stretch, then it has not been heated.

5. Converse: If $x > 2$, then $x + 4 > 6$

Inverse: If $x + 4 > 6$, then $x > 2$

6. $p \rightarrow \sim q$: It is cold, then it is not raining.

$q \leftrightarrow p$: It is raining if and only if it is raining.

$p \leftrightarrow \sim q$: It is cold if and only is it is not raining.

7. $p \wedge (q \rightarrow r) \wedge (r \rightarrow q) \vee (r \rightarrow p) \wedge (p \rightarrow r)$

14. (a) $\sim p \leftrightarrow \sim q$ (b) $q \rightarrow \sim p$

(c) $\sim p \rightarrow \sim q$ (d) $\sim p \wedge q$

16. (1) $x \neq 2$ and $x > \neq 4$ (2) $x > 3$ and $x = 3$

20. (a) False (b) True (c) True (d) False (e) True

21. (1) I like cats or I like dogs.

(2) I like cats and I like dogs.

(3) If I like cats then I like dogs.

(4) I like cats if I like dogs.

23. (a) $p \wedge (\sim q)$ (b) $q \wedge r$

1.20 SOLVED EXAMPLES

Example 1: Show that

$$(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q) \equiv q$$

Solution: Consider L.H.S.

$$\begin{aligned}
&(p \wedge (\sim p \vee q)) \vee (q \wedge \sim (p \wedge q) \\
&\equiv ((p \wedge \sim p) \vee (p \wedge q)) \vee (q \wedge (\sim p \vee \sim q)) \\
&\equiv f \vee (p \wedge q) \vee (q \wedge \sim p) \vee (q \wedge \sim p) \vee (q \wedge \sim q) && (\because p \wedge \sim p = f) \\
&\equiv (p \wedge q) \vee (q \wedge \sim p) \vee f \\
&\equiv (p \wedge q) \vee (q \vee \sim p) \\
&\equiv (q \wedge p) \vee (q \vee \sim p) \\
&\equiv q \wedge (p \vee \sim p) \\
&\equiv q \wedge (p \vee \sim p) \\
&\equiv q \wedge t && (\because p \vee \sim p = t) \\
&\equiv q \\
&\equiv \text{R.H.S.}
\end{aligned}$$

Hence $(p \wedge (\sim p \vee q)) \vee (q \wedge \sim(p \wedge q) \equiv q$

Example 2: Obtain the disjunctive normal form of

(*a*) $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

Solution:

(*a*) $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$

$$\begin{aligned}
&\equiv p \vee (\sim p \rightarrow q \vee (\sim q \vee \sim r)) \\
&\equiv p \vee (p \vee q \vee (\sim q \vee \sim r)) \\
&\equiv p \vee p \vee q \vee \sim q \vee \sim r \\
&\equiv p \vee q \vee \sim q \vee \sim r
\end{aligned}$$

Example 3: Show that

$((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology.

Solution: Consider

$$\begin{aligned}
&((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q \\
&\equiv ((p \vee \sim q) \wedge \sim p \vee (p \vee \sim q) \wedge \sim q) \vee q \\
&\equiv ((p \wedge \sim p) \vee (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee (\sim q \wedge \sim q)) \vee q \\
&\equiv (f \vee (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee \sim q) \vee q \\
&\equiv (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee \sim q \vee q \\
&\equiv (\sim q \wedge \sim p) \vee (p \wedge \sim q) \vee t && (\text{Since } \sim q \vee q \equiv t) \\
&\equiv t
\end{aligned}$$

Hence $((p \vee \sim q) \wedge (\sim p \vee \sim q)) \vee q$ is a tautology.

Example 4: Obtain the principal disjunctive normal form of $\sim p \vee q$

Solution: $\sim p \vee q \equiv (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$$

Hence $(\sim p \wedge q) \wedge (\sim p \wedge \sim q) \wedge (p \wedge q)$ is the required principal disjunctive normal form.

Example 5: Prove the following logical equivalencies:

(*i*) $[(p \vee q) \wedge (p \vee \sim q)] \vee q \Leftrightarrow p \vee q$

(*ii*) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

(*iii*) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow p \vee q \vee r$

(*iv*) $[(\sim p \vee q) \wedge (p \wedge (p \wedge q))] \Leftrightarrow p \wedge q$

Solution:

(*i*) $(p \vee q) \wedge (p \vee \sim q) \Leftrightarrow p \vee (q \wedge \sim q)$ (by distributive law)

$\Leftrightarrow p \vee f$ (f: fallacy)

$\Leftrightarrow p$ (by using identity law)

(*ii*) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p \vee q$

$\Leftrightarrow p$ (by an idempotent law)

(*iii*) $[p \vee q \vee (\sim p \wedge \sim q \wedge r)] \Leftrightarrow (p \vee q) \vee [(\sim (p \vee q) \wedge r)]$

$\Leftrightarrow [(p \vee q) \vee \sim (p \vee q)] \wedge [(p \vee q) \vee r]$

$\Leftrightarrow t \wedge (p \vee q \vee r)$ (t: tautology)

$\Leftrightarrow p \vee q \vee r$

(*iv*) $(\sim p \vee q) \wedge [p \wedge (p \wedge q)] \Leftrightarrow (\sim p \vee q) \wedge (p \wedge q)$

$\Leftrightarrow [\sim p \wedge (p \wedge q)] \vee [q \wedge (p \wedge q)]$

$\Leftrightarrow [(\sim p \wedge p) \wedge q] \vee [q \wedge (p \wedge q)]$

$\Leftrightarrow [(f \wedge q) \vee (q \wedge (p \wedge q))]$ (f: fallacy)

$\Leftrightarrow f \vee (p \wedge q)$

$\Leftrightarrow p \wedge q$

EXERCISE 1.3

1. Construct truth tables for the following:
   (*a*) $\sim(\sim p \wedge \sim q)$
   (*b*) $p \wedge (p \vee q)$
   (*c*) $(q \wedge (p \rightarrow q)) \rightarrow p$
2. Prove $(p \rightarrow q) \Leftrightarrow (\sim p \vee q)$
3. Show that $p \rightarrow (q \rightarrow r) \Leftrightarrow (\sim q \vee r) \Leftrightarrow (p \wedge q) \rightarrow r$
4. Show that $((p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))) \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ is a tautology.

5. Write the duals of

(a) $(p \vee q) \wedge r$

(b) $(p \wedge q) \vee t$

(c) $\sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim s))$

6. Show that $(p \vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow \sim p \wedge q$

7. Prove the following implication:

(a) $(p \wedge q) \Rightarrow (p \to q)$

(b) $(p \to (q \to r) \Rightarrow (p \to q) \to (p \to r))$

8. Write an equivalence formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which does not contain biconditional.

9. Obtain disjunctive normal forms of

(a) $p \wedge (p \to q)$

(b) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

10. Obtain the principal disjunctive normal forms of

(i) $\sim p \vee q$

(ii) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

(iii) $p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

11. Obtain the principal conjunctive normal forms of

(i) $(\sim p \to r) \wedge (q \leftrightarrow p)$

(ii) $(q \to p) \wedge (\sim p \wedge q)$

(iii) $q \wedge (p \vee \sim q)$.

12. Show that $(P \to Q) \wedge (R \to Q)$ and $(P \vee R) \to Q$ are equivalent. ***(MCA, Oct., 2001, MKU)***

13. Define Tautology and contradiction. Find which of the following is a tautology and which is a contradiction:

$(P \wedge Q) \wedge \rceil (P \vee Q), P \vee \rceil (P \wedge Q)$ ***(MCA, Oct., 2001, MKU)***

14. Show that if $p \to q, q \to r, \rceil (p \wedge r)$ and $(p \vee r)$ then r. ***(MCA, Oct., 2001, MKU)***

15. Construct a truth table for the formula

$(P \wedge Q) \vee (\rceil P \wedge Q) \vee (P \wedge \rceil Q) \vee (\rceil P \wedge \rceil Q)$ ***(MCA, May 2001, MKU)***

16. (a) Construct the truth table for the following compound statements and which of them are tautologies:

(i) $(q \wedge r) \to (p \wedge \rceil r)$

(ii) $p \to q \rightleftarrows (\rceil p \vee q)$

write an equivalent formula for

$P \wedge (Q \rightleftarrows R) \vee (R \rightleftarrows P)$

which contains neither the biconditional nor the conditional

(b) Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises

$P \vee Q, Q \to R, P \to M$ and $\rceil M$. ***(MCA, May 2001, MKU)***

17. For any propositions p, q prove the following:

(*a*) $\sim (p \downarrow q) \Leftrightarrow (\sim p \uparrow \sim q)$

(*b*) $\sim (p \uparrow q) \rightleftarrows (\sim p \downarrow \sim q)$

18. For any proposition p, q, r prove the following:

(*a*) $p \uparrow (q \uparrow r) \rightleftarrows \sim p \vee (q \wedge r)$

(*b*) $p \downarrow (q \downarrow r) \rightleftarrows \sim p \wedge (q \vee r)$.

1.21 QUANTIFIERS

In this section we introduce, two logical notions called quantifiers. So far we have discussed the propositions in which each statement has been about a particular object. In this section we shall see how to write propositions that are about whole classes of objects.

In grammar a predicate is the word in a sentence which expresses what is said of the object. It is a part of a declarative sentence describing the properties of an object or relation among objects (The word 'Predicate' and property will be used to mean the same thing) for example 'is a cricket player', 'is a teacher' 'is short' are predicates. In logic the word predicate has a broader role than in grammar. The basis for this is the observation that a predicate is supplemented by, including a variable x as a place holder, for the intended subject, the result behaves as 'a statement function', in the sense that for each value of x a statement results. Consider the statement

$$p : x \text{ is an even number}$$

The truth value of p depends on the value of x. p is true when $x = 4$, and false when $x = 11$. The statement p is not a proposition. In this section we extend the system of logic to include such statements.

In grammar 'Rajan loves' is not a predicate. If 'x' is introduced as a place holder for the object, then we get the result as

'Rajan loves x'.

which is a statement function. Thus we can define, a predicate $p(x)$ as an expression having the quality that on an assignment of values to the variable x, from an appropriate domain, a statement results.

Definition 1.7: Let $P(x)$ be a statement involving variable x and a set D. We call P a propositional function if for each x in D, $P(x)$ is a proposition. The set D is called the domain of discourse (or universe of discourse) of P. It is the set of all possible values which can be assigned to variables in statements involving predicates.

For example the domain of discourse for $P(x)$: "x is a cricket player" can be taken as the set of all human beings and the statement.

$$x^2 - 3x - 7 = 0$$

is a propositional function. The domain of discourse is the set of real numbers.

1.21.1 Universal Quantifier

Consider the proposition

'All odd prime numbers are greater than 2'. The word 'all' in this proposition is a logical quantifier. The proposition can be translated as follows:

"For every x, if x is an odd prime then x is greater than 2"

Similarly, the proposition:

'Every rational number is a real number' may be translated as.

For every x, if x is a rational and number, then x is a real number.

The phrase 'for every x' is called a universal quantifier. In symbols it is denoted by $\forall x$.

The phrases 'for every x', 'for all x' and 'for each x', have the same meaning and we can symbolize each by $\forall x$.

If $P(x)$ denotes a predicate (propositional function), then the universal quantification for $P(x)$, is the statement.

"For all values of x, $P(x)$ is true"

Example 1: Let $A = \{x : x$ is a natural number less than $9\}$

Here $P(x)$ is the sentence "x is a natural number less than 9"

The common property is "is a natural number less than 9"

$P(1)$ is true, therefore, $1 \in A$

$P(12)$ is not true, therefore $12 \notin A$

Example 2: Let $P(x)$: $x + 5 < 9$, then for all $x \geq 0$, $P(x)$ is a false statement because $P(5)$ is not true.

1.21.2 Existential Quantifier

In some situations we only require that there be at least one value for each the predicate is true. This can be done by prefixing $P(x)$ with the phrase "there exists an x'". The phrase "there exists an x'" is called an existential quantifier. The existential quantification for a predicate is the statement "There exists a value of x" for which $P(x)$.

The symbol $\exists$, is used to denote the logical quantifier 'there exists' the phrases 'There exists an x', 'There is a x', for some x' and 'for at least one x' have the same meaning.

The existential quantifier for $P(x)$ is denoted by $\exists\ x\ P(x)$.

Example 1: The proposition:

There is a dog without a tail can be written as

($\exists$ a dog) (the dog without tail)

Example 2: The proposition:

There is an integer between 2 and 8 inclusive may be written as

($\exists$ an integer) (the integer is between 2 and 8)

The propositions which include quantifiers may be negated as follows:

Example 3: Negate the proposition

All integers are greater than 8.

Solution: We can write the given proposition as

($\forall$ integers x) ($x > 8$)

The negation is

($\exists$ an integer x) ($x \leq 8$)

i.e., the negated proposition is: There is an integer less than or equal to 8.

In the negation a proposition 'for all' becomes 'there is' and 'there is' becomes 'for all' i.e., the symbol $\forall$ becomes $\exists$ and $\exists$ becomes $\forall$.

Example 4: The negated proposition of

($\exists$ an integer x) ($0 \leq x \leq 8$) is

($\forall$ integers x) ($x < 0$ or $x > 8$)

The following table gives us the equivalences involving quantifiers.

Table 1.27 *Equivalences involving quantifiers*

I_1^1	Distributivity of $\exists$ over $\vee$
	$\exists x\,(P(x) \vee Q(x)) \equiv \exists x\,P(x) \vee \exists x\,Q(x)$
	$\exists x\,(P \vee Q(x)) \equiv P \vee (\exists x\,Q(x))$
I_2^1	Distributivity of $\forall$ over $\wedge$
	$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x\,P(x) \wedge \forall x\,Q(x)$
	$\forall x\,(P \wedge Q(x)) \equiv P \wedge (\forall x\,Q(x))$
I_3^1	$\rceil(\exists x\,P(x)) \equiv \forall x \rceil(P(x))$
I_4^1	$\rceil(\forall x\,P(x)) \equiv \exists x \rceil(P(x))$
I_5^1	$\exists x\,(P \wedge Q(x)) \equiv P \wedge (\exists x\,Q(x))$
I_6^1	$\forall x\,(P \vee Q(x)) \equiv P \vee (\forall x\,Q(x))$
I_7^1	$\forall x\,P(x) \Rightarrow \exists x\,P(x)$
I_8^1	$\forall x\,P(x) \vee \forall x\,Q(x) \Rightarrow \vee\,(P(x) \vee Q(x))$
I_9^1	$\exists x\,(P(x) \wedge Q(x)) \Rightarrow \exists x\,P(x) \wedge \exists x\,Q(x)$

Rules of inference for addition and deletion of quantifiers are given by the following table:

Table 1.28 *Rules of inference for addition and deletion of quantifiers*

R_1	Universal instantiation.
	$\dfrac{\forall x\,P(x)}{\therefore P(k)}$
	k is some element of the universe.
R_2	Existential instantiation
	$\dfrac{\exists x\,P(x)}{\therefore P(k)}$
	k is some element for which $P(k)$ is true.

Contd.

R_3 Universal generalization

$$\frac{P(x)}{\forall x\, P(x)}$$

R_4 Existential generalization

$$\frac{P(k)}{\therefore \exists\, x\, P(x)}$$

k is some element on the universe.

1.22 METHODS OF PROOF

In this section, we discuss different types of Proof: Direct Proof, Indirect Proof, Proof by counter example and proof by cases.

1.22.1 Direct Proof

We assume that P is true, and from the available information the conclusion q is shown to be true by valid reference. In this methods of proof we construct a chain of statements P, P_1, P_2, P_3, ..., P_n, ... , q where P is either a hypothesis of the theorem or an axiom and each of the implications $p \Rightarrow p_1,\ p_1 \Rightarrow p_2, ...,\ p_n \Rightarrow q$ is either an axiom or is implied by the implication preceding it.

Example 1: If x is an even integer then x^2 is an even integer.

Solution: Direct Proof

Let p: x is an even integer

q: x^2 is an even integer.

Consider, the hypothesis p. If x is an even integer they by the definition of an even integer.

$x = 2m$ for some integer m.

Hence $$x^2 = (2m)^2 \Rightarrow x^2 = 4m^2$$

$x^2 = 4m^2$ is clearly divisible by 2. Therefore x^2 is an even integer. Thus $p \rightarrow q$.

Example 2: If a and b are odd integer, then $a + b$ is an even integer.

Solution: (Direct Proof) An odd integer is of the form $2k + 1$, where k is some integer given that a and b are even integers, therefore $a = 2m_1 + 1$, $b = 2m_2 + 1$ for some integers m_1 and m_2.

Then
$$\begin{aligned} a + b &= (2m_1 + 1) + (2m_2 + 1) \\ &= 2m_1 + 1 + 2m_2 + 1 \\ &= 2m_1 + 2m_2 + 2 \\ &= 2\,(m_1 + m_2 + 1) \end{aligned}$$

But $m_1 + m_2 + 1$ is an integer, therefore $a + b$ is an even integer.

Example 3: If a is number such that $a^2 - 7a + 12 = 0$, then show that $a = 3$, $a = 4$ by direct proof.

Solution: $a^2 - 7a + 12 = 0$ using the rules of algebra, we can write

$$a^2 - 7a + 12 = (a - 3)\,(a - 4) = 0$$

i.e., product of the two numbers $(a - 3)$ and $(a - 4)$ is zero. Therefore, $a - 3 = 0$ or $a - 4 = 0$

$$a - 3 = 0 \Rightarrow a = 3,\ a - 4 = 0 \Rightarrow a = 4$$

Hence $$a = 3 \text{ or } a = 4$$

Example 4: Prove that if $|x| > |y|$ then $x^2 > y^2$, by direct method.

Solution: Since $|x| > |y|$ then $|x|^2 > |y|^2$

Now $|x|^2 = x^2$ and $|y|^2 = y^2$, hence $x^2 > y^2$

1.22.2 Method of Contraposition

Indirect Proof: This method of proof is very useful and is powerful at all levels of the subject mathematics. Indirect method follows from the Tautology $(p \to q) \leftrightarrow ((\sim q) \to (\sim p))$. This states that the implication $p \Rightarrow q$ is equivalent to $\sim q \Rightarrow \sim p$. To prove $p \Rightarrow q$ indirectly, we assume that q is false and then show that p is false.

Example 1: For any integer $n > 2$, prove that n Prime $\Rightarrow$ n odd.

Solution: Let p: n Prime

q: n odd

then $\sim q$: n even

$\sim p$: n not prime

If n is an even number greater than 2, then $n = 2m$ for some integer $m > 1$. Thus n is divisible by 2 and $n \neq 2$, therefore n cannot be prime thus we have $\sim q \Rightarrow \sim p$ i.e., if n is any number bigger than 2, then n cannot be prime.

Example 2: Prove: If α^2 is an even integer, then α is an even integer.

Solution: Let p: α^2 is an even integer

q: α is an even integer

let $\sim q$ be true then, α is not an even integers therefore α must be odd. α is of the form $\alpha = 2m + 1$ for some integer m.

$$\alpha = 2m + 1$$

$$\Rightarrow \quad \alpha^2 = (2n + 1)^2$$

$$= 4n^2 + 4n + 1$$

$$= 2\,(2m^2 + 2m) + 1$$

α^2 is of the form $\alpha^2 = 2n + 1$ where $n = (2m^2 + 2m)$

i.e., α^2 is odd

Thus, we have $\sim q \Rightarrow \sim p$

Hence by contraposition α is even.

1.22.3 Proof by Contradiction

In this method of proof, we assume the opposite of what we are trying to prove and get a logical contradiction. Hence our assumption must have been false. Therefore what we were originally required to prove must be true. To prove $p \to q$ is true, in this the proof can be constructed as follows:

(i) Assume $p \wedge (\sim q)$ is true.

(ii) On the basis of the assumption find some conclusion that is false.

(iii) Then the contradiction discovered in step (ii) leads us to the conclusion that $p \wedge (\sim q)$ is false which powers that $p \rightarrow q$ is true.

Example 1: Suppose that the integers 1, 2, 3, ..., 10 are randomly positioned around a circular wheel. Show that the sum of some set of 3 consecutively positioned numbers is at least 15.

Solution: (Proof by Contradiction)

Let a_r respect the integer at position r on the wheel. Then we are to prove

$$\left.\begin{array}{ll} & a_1 + a_2 + a_3 \geq 15 \\ \text{or} & a_2 + a_3 + a_4 \geq 15 \\ & \vdots \\ \text{or} & a_{10} + a_1 + a_2 \geq 15 \end{array}\right\} \quad \text{... (1)}$$

where $a_1 + a_2 + ... + a_{10} = 1 + 2 + 3 + ... + 10$

Let us assume that, the above conclusion is false. Then we must have

$$\begin{array}{l} a_1 + a_2 + a_3 < 15 \\ a_2 + a_3 + a_4 < 15 \\ \vdots \\ a_{10} + a_1 + a_2 < 15 \end{array}$$

We can write the above the inequalities as

$$\begin{array}{l} a_1 + a_2 + a_3 \leq 14 \\ a_2 + a_3 + a_4 \leq 14 \\ \vdots \\ a_{10} + a_1 + a_2 \leq 14 \end{array}$$

Taking the sum: we get

$$3(a_1 + a_2 + ... + a_{10}) \leq 10 \times 14$$

i.e., $$3(1 + 2 + ... + 10) \leq 140$$

or $$3 \cdot \frac{10 \cdot (10+1)}{2} \leq 140$$

or $$3 \times 5 \times 11 \leq 140$$

or $$165 \leq 140$$

a contradiction

Hence the given proposition i.e. (1) is true

Example 2: Show that $\sqrt{2}$ is not a rational number.

Solution: Let us assume that $\sqrt{2}$ is rational. Then we can find integers such that

$$\sqrt{2} = \frac{p}{q}$$

where p and q have no common factor. After canceling the common factors squaring on both sides, we get

$$2 = \frac{p^2}{q^2}$$

$$\Rightarrow p^2 = 2q^2$$

$$\Rightarrow p^2 \text{ is even}$$

$$\Rightarrow p \text{ is even}$$

$$\Rightarrow p = 2m \text{ for some integer } m.$$

$$\Rightarrow (2m)^2 = 2q^2$$

$$\Rightarrow 4m^2 = 2q^2$$

$$\Rightarrow q^2 = 2m^2$$

$$\Rightarrow q \text{ is even}$$

Hence p and q have common factor of 2, which is a contradiction to the statement that a and b have no common factors.

Hence our assumption that $\sqrt{2}$ is rational leads to a contradiction. Thus $\sqrt{2}$ is irrational.

1.22.4 Proof by Counter Example

To show that $\forall x, P(x)$ it is sufficient to give specific example k, in the universe such that $P(k)$ is false, where the object k is called a counter example to the assertion $\forall x, P(x)$.

Example: Prove or disprove the statement:

If x and y are real number

$$(x^2 = y^2) \Leftrightarrow (x = y)$$

-3, 3 are real number and $(-3)^2 = 3^2$ but $-3 \neq 3$

Hence the result is false and implication is false.

1.22.5 Proof by Cases

To prove $p \to q$ by cases, we take p to be in the form $p_1 \vee p_2 \vee ... \vee p_n$ by proving separately, each of the following $p_1 \to q$, $p_2 \to q$, ..., $p_n \to q$ we can establish $(p_1 \vee p_2 \vee ... \vee p_n) \to q$.

In this section, we discuss rules of inference. Which are criteria for determining the validity of an argument. The rules of inference will be given in terms of statement formulas. Before discussing the rules of inference, we define consistency, which is an extremely important notion in mathematical logic.

Definition 1.8: A collection of statements is consistent if the statements can all be true simultaneously.

A set of formulas $H_1, H_2, H_3, ..., H_n$ is said to be consistent if their conjunction $H_1 \wedge H_2 \wedge ... \wedge H_n$ has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, ..., H_n$. And a set of formulae $H_1, H_2, ..., H_n$ is inconsistent if their conjunction $H_1 \wedge H_2 \wedge ... \wedge H_n$ implies a contradiction, that is $H_1 \wedge H_2 \wedge ... \wedge H_n \Rightarrow S \wedge \urcorner S$ (a contradiction) where is S is any formula.

We use the notion of inconsistency in a method of proof called proof by contradiction (or indirect proof) we now begin our discussion by stating the following two rules of inference.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by any one or more of true preceding formulas in the derivation.

1.22.6 Rules of Inference

The following tables give us the rules of inference:

Table 1.29

Implications:

I_1 $P \wedge Q \Rightarrow P$ (Simplification)

I_2 $P \wedge Q \Rightarrow Q$ (Simplification)

I_3 $P \Rightarrow P \vee Q$ (Addition)

I_4 $Q \Rightarrow P \vee Q$ (Addition)

I_5 $\urcorner P \Rightarrow P \rightarrow Q$

I_6 $Q \Rightarrow P \rightarrow Q$

I_7 $\urcorner(P \rightarrow Q) \Rightarrow P$

I_8 $\urcorner(P \rightarrow Q) \Rightarrow \urcorner Q$

I_9 $P, Q \Rightarrow P \wedge Q$

I_{10} $\urcorner P, P \vee Q \Rightarrow Q$ (Disjunctive syllogism)

I_{11} $P, P \rightarrow Q \Rightarrow Q$ (Modus Ponens)

I_{12} $\urcorner Q, P \rightarrow Q \Rightarrow \urcorner P$ (Modus Tollens)

I_{13} $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (Hypothetical syllogism)

I_{14} $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (Dilemma)

Equivalences:

E_1 $\urcorner\urcorner P \Leftrightarrow P$ (Double negation)

E_2 $P \wedge Q \Leftrightarrow Q \wedge P$ (Commutative law)

E_3 $P \vee Q \Leftrightarrow Q \vee P$ (Commutative law)

E_4 $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ (Associative law)

Contd.

E_5 $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (Associative law)

E_6 $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ (Distributive law)

E_7 $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee Q)$ (Distributive law)

E_8 $\rceil (P \wedge Q) \Leftrightarrow \rceil P \vee \rceil Q$ (De Morgan's law)

E_9 $\rceil (P \vee Q) \Leftrightarrow \rceil P \wedge \rceil Q$ (De Morgan's law)

E_{10} $P \vee P \Leftrightarrow P$

E_{11} $P \wedge P \Leftrightarrow P$

E_{12} $R \vee (P \wedge \rceil P) \Leftrightarrow R$

E_{13} $R \wedge (P \vee \rceil P) \Leftrightarrow R$

E_{14} $R \vee (P \vee \rceil P) \Leftrightarrow T$

E_{15} $R \wedge (P \wedge \rceil P) \Leftrightarrow F$

E_{16} $P \rightarrow Q \Leftrightarrow \rceil P \vee Q$

E_{17} $\rceil (P \rightarrow Q) \Leftrightarrow P \wedge \rceil Q$

E_{18} $P \rightarrow Q \Leftrightarrow \rceil Q \rightarrow \rceil P$

E_{19} $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E_{20} $\rceil (P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \rceil Q$

E_{21} $P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E_{22} $(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\rceil P \wedge \rceil Q)$

The rules 'Modus Ponens' and 'Hypothetical Syllogism' are known as the fundamental rules of inference.

De Morgan's laws and the law of contraposition are the other fundamental rules, from which other rules follow. Modus Ponens is also called the rule of detachment. It can be stated as follows:

Whenever the statements p and $(p \rightarrow q)$ are accepted as true, then we must accept the statement q as true.

The tabular form of the rule is given below

$$\begin{array}{l} p \\ \underline{p \rightarrow q} \\ \therefore q \end{array}$$

In the above tabular presentation p and $(p \rightarrow q)$, which are above the horizontal line are the Premises (Hypotheses). The assertion q which below the Horizontal line is the conclusion.

The rule of Hypothetical Syllogism is also known as the transitive rule. It can be stated as follows:

If two implications $(p \to q)$ and $(q \to r)$ are true, then the implication $(p \to r)$ is true.

The tabular form of the rule is given below:

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

The transitive rule can be extended to a larger number of implications as follows:

$$\begin{array}{r} p \to q \\ q \to r \\ r \to s \\ \hline \therefore\ p \to s \end{array}$$

$$\begin{array}{r} p_1 \to p_2 \\ p_2 \to p_3 \\ p_3 \to p_4 \\ \vdots \\ p_{n-1} \to p_n \\ \hline \therefore\ p_1 \to p_n \end{array}$$

Most of the arguments are based on the two fundamental rules of inference. In an argument premises are always taken to be true. Whereas the conclusion may or may not be true. The conclusion is true only when the argument is true. To list the validity of an argument we can also employ the laws of logic, logical equivalences and tautologies

Modus Tollens is a rule of denying. It can be stated as follows:

If $p \to q$ is true and q is false, then p is false.

The tabular form of the rule is gives below:

$$\begin{array}{r} p \to q \\ \sim q \\ \hline \therefore \sim\ p \end{array}$$

Rule of Disjunctive Syllogism, States that if $p \vee q$ is true and p is false, then q is true.

In tabular form, the rule can be written as follows:

$$\begin{array}{r} p \vee q \\ \sim q \\ \hline \therefore \sim\ q \end{array}$$

1.22.7 Fallacies

Faulty inferences are know as fallacies. There are three forms of fallacies:

1. The fallacy of affirming the consequent
2. The fallacy of denying the antecedent
3. The non Sequitur fallacy

The fallacy of affirming the consequent has the following tabular form:

$$\begin{array}{l} p \to q \\ \underline{q \qquad\quad} \\ \therefore\ p \end{array} \qquad \text{(fallacy)}$$

The fallacy of denying antecedent is presented in the following form:

$$\begin{array}{l} p \to q \\ \underline{\quad\ \sim p} \\ \therefore \sim\ q \end{array} \qquad \text{(fallacy)}$$

Fallacies of assuming converse and all logical errors are Special Cases of the non Sequitur fallacy. It can be presented as follows.

$$\begin{array}{l} \underline{\ p\ } \\ \therefore q \end{array}$$

Example 1: Prove that the following arguments are valid

$$(a) \quad \begin{array}{l} p \to r \\ \underline{q \to r \qquad\quad} \\ \therefore (p \vee q) \to r \end{array}$$

$$(b) \quad \begin{array}{l} p \to r \\ \sim p \to q \\ \underline{q \to s \quad\ } \\ \therefore \sim r \to s \end{array}$$

Solution (a) Consider

$(p \to r) \wedge (q \to r)$

We have

$$\begin{aligned} (p \to r) \wedge (q \to r) &\Leftrightarrow (\sim p \vee r) \wedge (\sim q \vee r) \\ &\Leftrightarrow (r\ u \sim p) \wedge (r \vee q) && \text{(commutative law)} \\ &\Leftrightarrow r \vee (\sim p \wedge \sim q) && \text{(distributive law)} \end{aligned}$$

$\Leftrightarrow r \vee \sim (p \vee q)$ (De Morgan's law)

$\Leftrightarrow \sim (p \vee q) \vee r$ (commutative law)

$\Leftrightarrow (p \vee q) \rightarrow r$

$(p \rightarrow r) \wedge (q \rightarrow r)$ and $(p \vee q) \rightarrow r$ are logically equivalence. Hence the given argument is valid

(b) we have

$(p \rightarrow r) \wedge (\sim p \rightarrow q) \wedge (q \rightarrow s) \Rightarrow (p \rightarrow r) \wedge (\sim p \rightarrow s)$ (by rule of Syllogism)

$\Leftrightarrow (\sim r \rightarrow \sim p) \wedge (\sim p \rightarrow s)$ (by the rule of contrapositive)

$\Rightarrow \sim r \rightarrow s$ (by rule of Syllogism)

$(p \rightarrow r) \wedge (\sim p \rightarrow q) \wedge (q \rightarrow s)$ and $\sim r \rightarrow s$ are logically equivalent. Hence, the given argument is valid

***Example 2*:** Prove that the argument given below is a valid argument

$$\begin{array}{l} p \rightarrow (q \rightarrow r) \\ \sim q \rightarrow \sim p \\ \underline{\quad p \qquad\qquad} \\ \therefore r \end{array}$$

Solution: Consider $[p \rightarrow (q \rightarrow r)] \wedge (\sim q \rightarrow \sim p) \wedge p$

We have

$[p \rightarrow (q \rightarrow r)] \wedge (\sim q \rightarrow \sim p) \wedge p$

$\Leftrightarrow [(p \rightarrow (q \rightarrow r)) \wedge p] \wedge (\sim q \rightarrow \sim p)$

$\Rightarrow (q \rightarrow r) \wedge (\sim q \rightarrow \sim p)$ (by the rule of Modus Ponens)

$\Leftrightarrow (q \rightarrow r) \wedge (\sim \sim p \rightarrow \sim \sim q)$ (by contraposition)

$\Leftrightarrow (q \rightarrow r) \wedge (p \rightarrow q)$

$\Leftrightarrow p \rightarrow r$ (by the rule of Syllogism)

$\Leftrightarrow r$ (since p is true)

$[p \rightarrow (q \rightarrow r) \wedge [\sim q \rightarrow \sim p) \wedge p$ and r are logically equivalent. Therefore the given argument is valid

Example 3: Text the validity of the argument:

If I drive to work, then I will arrive tired

$$\begin{array}{l} \underline{\quad \text{I am not tired} \quad} \\ \therefore \text{I do not drive to work} \end{array}$$

Solution:

Let p: I drive to work

q: I arrive tired

The argument has the following Symolic form

$$\begin{array}{l} p \to q \\ \underline{\sim q} \\ \therefore \sim p \end{array}$$

By the rule of Modus Tollens, the argument is valid

Example 4: Text the validity of the argument

If a person is poor, he is unhappy

$$\begin{array}{c} \underline{\text{If a person is unhappy, he dies young}} \\ \therefore \text{Poor person die young} \end{array}$$

Solution:

Let p: a person is poor

q: a person is unhappy

r: a person dies young

Then the argument takes the following form:

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

by the law of hypothetical; Syllogism, (i.e., transitive rule) the argument is valid

Example 5: Prove $\urcorner Q, P \to Q \Rightarrow \urcorner P$

Solution: A formal proof is as follows:

1.	$P \to Q$	P
2.	$\urcorner Q \to \urcorner P$	T, (1) and E_{18}
3.	$\urcorner Q$	P
4.	$\urcorner P$	T, (2), (3) and I_{11}

Example 6: Show that $\urcorner P$ follows, logically from $\urcorner(P \wedge \urcorner Q)$, $\urcorner Q \vee P$, $\urcorner R$.

Solution: A formal proof is as follows:

1.	$\urcorner(P \wedge \urcorner Q)$	P
2.	$\urcorner P \vee Q$	$\because \urcorner(P \wedge Q) \Leftrightarrow \urcorner P \vee \urcorner Q$
3.	$P \to Q$	$\because P \to Q \Leftrightarrow \urcorner P \vee Q$
4.	$\urcorner Q \vee R$	P

5.	$Q \to R$	
6.	$P \to R$	(3), (5)
7.	$\neg R$	P
8.	$\neg P$	$\neg Q, P \to Q \Rightarrow \neg P$

We now introduce another rule of inference called rule CP or rule of conditional proof.

Rule *CP:* If we can derive S from R and a set of premises, then we can derive $R \to S$ from the set of premises alone.

The above rule is also called deduction theorem.

Example 7: Show that $R \to S$ can be derived from the premises $P \to (P \to S)$, $\neg R \vee P$, and Q.

Solution: We include R as an additional premise and show S, so that $R \to S$ can be derived.

1.	$\neg R \vee S$	P
2.	R	P (assumed premises)
3.	P	T, (1), (2) and I_{10}
4.	$P \to (Q \to S)$	P
5.	$Q \to S$	T, (3), (4) and I_{11}
6.	Q	P
7.	S	T, (5), (6) and I_{11}
8.	$R \to S$	$C\ P$

We shall now give an example to prove inconsistency in the given set of formula.

Example 8: Show that $(R \to \neg Q)$, $R \vee S$, $S \to \neg Q$, $P \to Q \Leftrightarrow \neg P$ are inconsistent.

Solution: A formal proof is:

1.	P	(assumed)
2.	$P \to Q$	Rule P
3.	Q	(1) and (2)
4.	$S \to \neg Q$	P
5.	$Q \to \neg S$	$P \to Q \rightleftarrows \neg Q \to \neg P$
6.	$7S$	(3), (5)
7.	$R \vee S$	P
8.	$\neg R \to S$	$P \to Q \rightleftarrows \neg P \vee \neg Q$
9.	$\neg S \to R$	$P \to Q \rightleftarrows \neg Q \to \neg P$
10.	R	(6), (9)
11.	$R \to \neg Q$	P
12.	$\neg Q$	(10), (11)
13.	$Q \wedge \neg Q$	(3), (12)

Inconsistent

EXERCISE 1.4

1. If a real number x is such that $|x| > 5$, then $x^2 > 25$ (give the proof by cases).

2. Prove the statement by contradiction: "In a room of 15 people, 2 or more people have their birthday in the same month.

3. Prove deductively that all right angles are equal.

4. Prove by using direct method: If an integer a is such that $a - 2$, is divisible by 3, then $a^2 - 1$ is divisible by 3.

5. Prove by using contrapositive method: If x^2 is an odd integer, then x is an odd integer.

6. Prove by using direct method:

(*i*) Sum of two even integers is an even integer.

(*ii*) The sum of an even integer and an odd integer is an odd integer.

(*iii*) The product of an even integer and an odd integer is an even integer.

7. Disprove the proposition (by counter example) for every integer x there is an integer y where $y^2 = x$.

8. Prove that $\sqrt{5}$ is not a rational number (prove by contradiction).

9. Prove that if $x^2 - 4 = 0$, then $n \neq 0$ by the method of contradiction.

10. Prove by direct method:

If x and y are rational numbers then $n + y$ is rational.

11. Find a counter example:

If $a > b$ then $a^2 > b^2$

12. Give a direct proof that if a and b are odd integers then $a + b$ is even.

13. Prove using contrapositive that if $x^2 - 4 < 0$, then $-2 < x < 2$.

14. Rewrite the following propositions using the symbols $\forall$ and $\exists$.

(*a*) There is a cat without a tail.

(*b*) There is an integer between 2 and 15 inclusive.

(*c*) All odd prime numbers are bigger than 2.

(*d*) All elephants have trunks.

(*e*) All cats like cream.

(*f*) All students are clever.

(*g*) Every clever student is successful.

(*h*) There are some successful students who are not clever.

15. Negate the following propositions:

(*a*) $(\forall\, a > 0)\ (\exists\, b > 0)\ a + b$ is prime.

(*b*) ($\forall$ integers m) ($\exists$ an integer n) $(m^2 = n)$.

(*c*) All good students study hard.

(*d*) All fish swim.

(*e*) ($\exists$ an integer x) $(5 \leq x < 25)$.

(*f*) There is a triangle whose sum of angles $\neq 180^\circ$.

(*g*) If the teacher is absent, then some students do not complete their home work.

(*h*) All the students completed their home work and the teacher is present.

16. Prove that the following argument is valid. If a baby is hungry, then the baby cries. If the baby is not mad, then he does not cry. If a baby is mad, then he has a red face. Therefore if a baby is hungry, then he has a red face.

17. Suppose that the 10 integers 1, 2, …, 10 are randomly positioned around a circular wheel. Show that the sum of some set of 3 consecutively positioned numbers is at least 17.

18. Compute the truth value of the statement

$(p \to q) \leftrightarrow (\sim q \to \sim p)$ (***MCA, 2000, VTU***)

Answer: T

19. Determine whether each of the following statements are a tautology, a contingency or an absurdity

(*a*) $p \to (q \to p)$

(*b*) $(p \wedge q) \to p$

(*c*) $(p \wedge (p \to q)) \to q$

(*d*) $(q \wedge (\sim p)) \leftrightarrow r$ (***B.E, Mar. 2001, VTU***)

20. Show that $(P \vee Q) \rceil (\rceil P \wedge (\rceil Q \vee \rceil R))) \vee (\rceil P \wedge \rceil Q) \vee (\rceil P \wedge \rceil R)$ is a tautology (***B.E., Feb. 2002, VTU***)

21. Let k be an integer. If k^2 is odd then show that k is an odd integer

(***B.E Mar. 2001, VTU***)

22. Explain the differences between tautology and contingency

Answer:

19. (a) Tautology (b) Tautology (c) Tautology (d) Contingency

2

Set Theory

2.1 INTRODUCTION

The notion of a set is elementary to all of Mathematics and every branch of mathematics can be considered as a study of sets of objects of one kind or another. Cantor was the founder of the theory of sets. The word set is a primitive term and is regarded as one of the basic undefined ideas of mathematics. But we must have an Intuitive idea of what we mean by a set. Let us now consider the idea of a set.

2.2 SETS AND OPERATIONS ON SETS

2.2.1 Set

Definition 2.1:

A set is collection of well defined objects.

In the above definition the words set and collection for all practical purposes are Synonymous. We have really used the word set to define itself.

2.2.2 Notation

Each of the objects in the set is called a member of an element of the set. The objects themselves can be almost anything. Books, cities, numbers, animals, flowers, etc.

Elements of a set are usually denoted by lower-case letters. While sets are denoted by capital letters of English larguage.

The symbol $\in$ indicates the membership in a set.

If "a is an element of the set A", then we write $a \in A$.

The symbol $\in$ is read "is a member of " or "is an element of ".

The symbol $\notin$ is used to indicate that an object is not in the given set.

The symbol $\notin$ is read "is not a member of " or "is not an element of ".

If x is not an element of the set A then we write $x \notin A$.

2.2.3 Specifying Sets

There are five different ways of specifying sets:

(*i*) One method of specifying a set is to list all the members of the set between a pair of braces. Thus $\{1, 2, 3\}$ represents a set. This method is called "The listing method".

Example 1:

(*i*) {3, 6, 9, 12, 15}

(*ii*) $\{a, b, c, d\}$

This method of listing the elements of the set is also known as 'Tabulation'. In this method the order in which the elements are listed is immaterial, and is used for small sets.

(*ii*) Another method of defining particular sets is by a description of some attribute or characteristic of the elements of the set. This method is more general and involves a description of the set property.

$$A = \{x \mid x \text{ has the property } P\}$$

Designates "the set A of all objects 'x' such that x has the property P". This notation is called Set-Builder notation. The vertical bar | is read as "such that".

Example 2:

(*i*) $A = \{x \mid x$ is a positive Integer greater then 100}.
This is read as "the set of all x is a positive Integer less than 25".

(*ii*) $B = \{x \mid x$ is a complex number}.

Note: Repetition of objects is not allowed in a set, and a set is collection of objects without ordering.

(*iii*) We can describe a set by its characteristic function.

$$\mu A(x) = \begin{cases} 1 & \text{if } \ x \in A \\ 0 & \text{if } \ x \notin A \end{cases}$$

(*iv*) In another method we describe the set by a recursive formula:

Example 3: Let $x_0 = 2,\ x_1 = 1$ and $x_{i+1} = x_i + x_{i-1};\ i \geq 1$ and $A = \{x_i : i \geq 0\}$.

(*v*) We can also describe a set by an operation on some other sets.

2.3 SUBSETS

Definition 2.2: A set A is a subset of the set B if and only if every element of A is also an element of B. We also say that A is contained in B, and use the notation $A \subseteq B$.

Symbolically: If $x \in A \Rightarrow x \in B$, then $A \subseteq B$.

If $A \subseteq B$, it is possible that $A = B$, to emphasize this fact we write $A \subseteq B$.

If A is contained in B, then we may also state that B contains A and write $B \supseteq A$.

2.3.1 Proper Subset

Definition 2.3: A set A is called proper subset of the set B. If (*i*) A is subset of B and (*ii*) B is not a subset A

i.e., A is said to be a proper subset of B if every element of A belongs to the set B, but there is atleast one element of B, which is not in A. If A is a proper subset of B, then we denote it by $A \subset B$.

Note: Every set is a subset to itself.

2.3.2 Equal Sets

If A and B are sets such that every element of A is an element of B and every element of B is an element of, A then A and B are equal (Identical). We write "$A = B$", and it is read as A and B are identical.

2.3.3 Super Set

If A is subset of B, then B is called a superset of A.

Example:

(i) If $A = A\{0, 2, 9\}$, $B = \{0, 2, 7, 9, 11\}$ then $A \subset B$ (A is a proper subset of B).

(ii) If $A = \{a, a, b\}$, $B = \{a, b\}$, then A and B denoted the same set, i.e., $A = B$.

(iii) If $A = \{1, 2, 4\}$, $B = \{2, 4, 6, 8\}$ A is proper subset of B and B is a superset of A.

2.4 NULL SET

Definition 2.4: The set with no elements is called an empty set or null set. A Null set is designated by the symbol ϕ.

The null set is a subset of every set, i.e., If A is any set then $\phi \subset A$.

Example:

(i) The set of real roots of the polynomial $x^2 + 9 = 0$.

(ii) $\{x \mid 5x = 5x + 2\}$.

2.5 SINGLETON

Definition 2.5: A set having only one element is called a singleton.

Example: (i) $A = \{8\}$, (ii) $\{\phi\}$

Theorem 2.1: Two sets A and B are equal if and only if $A \subseteq B$ and $B \subseteq A$.

Proof: If $A = B$, every member of A is a member of B and every member of B is a member of A.

Hence $A \subseteq B$ and $B \subseteq A$

Conversely let us suppose that $A \neq B$, then there is either an element of A that is not in B or there is an element of B that is not in A. But $A \subseteq B$, therefore every element of A is in B and $B \subseteq A$, therefore every element of B is in A. Therefore, our assumption that $A \neq B$ leads to a contradiction, hence $A = B$.

Theorem 2.2: If ϕ and ϕ' are empty sets, then $\phi = \phi'$.

Proof: Suppose $\phi \neq \phi'$. Then one of the following statements must be true:

1. There is an element $x \in \phi$ such that $x \notin \phi'$
2. There is an element $x \in \phi'$ such that $x \notin \phi$.

But both these statements are false, since neither ϕ nor ϕ' has any elements. If follows that $\phi = \phi'$.

2.6 FINITE SET

Definition 2.6: A set is said to be finite, if it has finite number of elements.

Example:

(*i*) {1, 2, 3, 5}

(*ii*) The letters of the English alphabet.

2.7 INFINITE SET

Definition 2.7: A set is infinite, if it is not finite.

Example:

(*i*) The set of all real numbers.

(*ii*) The points on a line.

2.8 UNIVERSAL SET

Definition 2.8: In many discussions all the sets are considered to be subsets of one particular set. This set is called the universal set for that discussion.

The Universal set is often designated by the script letter U (or by X).

Universal set in not unique, and it may change from one discussion to another.

Example: If $A = \{0, 2, 7\}$, $B = \{3, 5, 6\}$, $C = \{1, 8, 9, 10\}$ then the universal set can be taken as the set.

$$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

2.9 THE POWER SET

Definition 2.9: The set of all subsets of a set A is called the power set of A.

The power set of A is denoted by $P(A)$.

Hence $$P(A) = \{x \mid x \subseteq A\}$$

The power set of A is also denoted sometimes by $2A$

If A has n elements in it, then $P(A)$ has 2^n elements:

Example 1: If $A = \{a, b\}$ then

$$P(A) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$$

***Example 2*:** The empty set ϕ, has only subset, therefore $P(\phi) = \{\phi\}$.

Note: A set is never equal to its power set. In the programming language Pascal, the notion power set is used to define data type in the language.

2.10 DISJOINTS SETS

Definition 2.10: Two sets are said to be disjoint if they have no element in common.

Example: The sets, $A = \{0, 4, 7, 9\}$ and $B = \{3, 6, 10\}$ are disjoint.

2.11 PROPERTIES OF SET CONTAINMENT

2.11.1

If A is any set then $A \subseteq A$

Proof: If $x \in A$, then $x \in A$ (by the repetition of the statement)

Hence $A \subseteq A$

2.11.2

If $A \subseteq B$, and $B \subseteq C$ then $A \subseteq C$ where A, B and C are sets

Proof: Let $x \in A$. Then

$$\begin{aligned} x \in A &\Rightarrow x \in B\ (A \subseteq B) \\ &\Rightarrow x \in C\ (A \subseteq C) \end{aligned}$$

if $x \in A$ then $x \in C$, therefore $A \subseteq C$.

2.12 OPERATIONS ON SETS: UNION OF SETS

Definition 2.11: The union of two sets A and B is the set whose elements are all of the elements in A or in B or in both.

The union of sets A and B denoted by $A \cup B$ is read as "A union B".

Symbolically: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Example:

(i) If $A = \{5, 7, 8\}$, $B = \{2, 7, 9, 10, 11\}$ then, $A \cup B = \{2, 5, 7, 8, 9, 10, 11\}$

(ii) If $A = \{x | x \in Z, \text{ and } x \geq 3\}$ and $B = \{x | x \in Z, \text{ and } x \geq 8\}$

then $A \cup B = \{x | x \in Z,\ x \geq 3\}$

Where Z denoted the set of integers.

Union of More than Two Sets

Definition 2.12: If $A_1, A_2, A_3, \ldots, A_n$ denote, sets then the union of these sets denoted by $\bigcup_{i=1}^{n} A_i$ is defined as $\bigcup_{i=1}^{n} A_i = \{x | x \in A_i$ for at least one set $A_i\}$.

2.13 PROPERTIES OF UNION OPERATION

2.13.1

If A and B are two sets then:

(i) $A \subseteq (A \cup B)$

(ii) $B \subseteq (A \cup B)$

(iii) $A \cup U = U$ where U is the universal set.

Proof:

(i) Let $x \in A$. Then

$x \in A \;\Rightarrow\; x \in A$ or $x \in B$

$\Rightarrow\; x \in (A \cup B)$

Thus $A \subseteq A \cup B$

(ii) Let $x \in B$

Then $x \in B \;\Rightarrow\; x \in A$ or $x \in B$

$\Rightarrow\; x \in (A \cup B)$

Hence $B \subseteq A \cup B$

2.13.2

If A is any set then (i) $A \cup \phi = A$ (ii) $A \cup A = A$

Proof:

(i) Clearly $A \subseteq (A \cup \phi)$ … (1)

Conversely, let $x \in A \cup \phi$

$$x \in A \cup \phi \;\Rightarrow\; x \in A \text{ or } x \in \phi$$

$$\Rightarrow\; x \in A$$

Thus $A \cup \phi \subset A$ … (2)

from (1) and (2), we have

$$A \cup \phi = A$$

(ii) Clearly $A \subseteq A \cup A$ … (1)

Now let $x \in A \cup A$

$$x \in A \cup A \;\Rightarrow\; x \in A \text{ or } x \in A$$

$$\Rightarrow\; x \in A$$

Thus $A \cup A \subseteq A$ … (2)

Combining (1) and (2), we get

$$A \cup A = A$$

(iii) In order to prove that $A \cup U = U$ we have to prove that $(A \cup U) \subseteq U$ and $U \subseteq (A \cup U)$

Every set is a subset of the universal set.

i.e., $(A \cup U) \subseteq U$ … (1)

also $U \subseteq (A \cup U)$ … (2)

Combining (1) and (2), we get

$$A \cup U = U.$$

2.13.3

Union of sets is commutative, i.e., If A and B are any sets, then $A \cup B = B \cup A$

Proof: Let $x \in A \cup B$, then

$$x \in A \cup B \Rightarrow x \in A \text{ or } x \in B \qquad \ldots (1)$$
$$\Rightarrow x \in B \text{ or } x \in A$$
$$\Rightarrow x \in (B \cup A)$$

Hence $$(A \cup B) \subseteq (B \cup A) \qquad \ldots (1)$$

Conversely, let $x \in (B \cup A)$, then

$$x \in (B \cup A) \Rightarrow x \in B \text{ or } x \in A$$
$$\Rightarrow x \in A \text{ or } x \in B$$
$$\Rightarrow x \in (A \cup B)$$

Thus $$(B \cup A) \subseteq (A \cup B) \qquad \ldots (2)$$

From (1) and (2) we have

$$A \cup B = B \cup A.$$

2.13.4 Associative Law for Addition

Union of sets is Associative, i.e., If A, B and C are any three sets, then $(A \cup B) \cup C = A \cup (B \cup C)$

Proof: Let $x \in (A \cup B) \cup C$ then

$$x \in (A \cup B) \cup C \Rightarrow x \in (A \cup B) \text{ or } x \in C \qquad \ldots (1)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$$
$$\Rightarrow x \in A \text{ or } x \in (B \cup C)$$
$$\Rightarrow x \in A \cup (B \cup C)$$

Hence $$(A \cup B) \cup C \subseteq A \cup (B \cup C)$$

conversely, let $x \in A \cup (B \cup C)$

$$x \in (A \cup B) \cup C \Rightarrow x \in A \text{ or } x \in (B \cup C) \qquad \ldots (1)$$
$$\Rightarrow x \in A \text{ or } (x \in B \text{ or } x \in C)$$
$$\Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C$$
$$\Rightarrow x \in A \cup B \text{ or } x \in C$$
$$\Rightarrow x \in (A \cup B) \cup C$$

Thus $$A \cup (B \cup C) \subseteq (A \cup B) \cup C \qquad \ldots (2)$$

Hence from (1) and (2), we have

$$(A \cup B) \cup C = A \cup (B \cup C)$$

2.14 INTERSECTION OF SETS

Definition 2.13: The intersection of two sets A and B is the set whose elements are all of the elements common to both A and B.

The intersection of the sets of "A" and "B" is denoted by $A \cap B$ and is read as "A intersection B" ***symbolically***: $A \cap B = \{x | x \in A \text{ and } x \in B\}$

Intersection of More than Two Sets

Definition 2.14: If $A_1, A_2, A_3, \ldots A_n$ denote sets, then the intersection of these sets denoted by

$\bigcap_{i=1}^{n} A_i$ is defined as follows

$$\bigcap_{i=1}^{n} A_i = \{x \mid x \in A_i \text{ for every } i \ (i = 1, 2, \ldots, n)\}$$
$$= \{x \mid x \text{ belongs to all sets } A_i\}$$

Example: (*i*) $A = \{1, 2, 3, 8\}$, $B = \{5, 8, 9\}$ than $A \cap B = \{8\}$.

(*ii*) If $A = \{a, b, c, d\}$, $B = \{b, d, e, f, g\}$ then $A \cap B = \{b, d\}$.

2.15 PROPERTIES OF INTERSECTION OPERATION

2.15.1

If A and B are any two sets then

(*i*) $A \cap B \subseteq A$ (*ii*) $A \cap B \subseteq B$

Proof:

(*i*) Let x be an element of the set $A \cap B$, then

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in A$$

Hence $A \cap B \subseteq A$

(*ii*) Let x be a number of the set $A \cap B$, then

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$

Hence $\Rightarrow A \cap B \subseteq B$

2.15.2

If A is any set then

(*i*) $A \cap \phi = \phi$, (*ii*) $A \cap A = A$, (*iii*) $A \cap U = A$, where U is the universal set.

Proof:

(*i*) ϕ is a subset of every set, therefore

$$\phi \subseteq A \cap \phi \qquad \text{... (1)}$$

and $$A \cap \phi \subseteq \phi$$

from (1) and (2) we have $$A \cap \phi = \phi \qquad \text{... (2)}$$

(*ii*) Clearly $A \cap A \subseteq A$ … (1)

Let x be a member of A, then

$$x \in A \Rightarrow x \in A \text{ and } x \in A \quad \text{(by the repetition of the statement)}$$
$$\Rightarrow x \in A \cap A$$

Thus $$A \subseteq A \cap A \qquad \text{… (2)}$$

Combining (1) and (2), we get

$$A \cap A = A$$

(*iii*) Clearly $A \cap U \subseteq A$

Let x be any member of A, then

$$x \in A \Rightarrow x \in U \qquad (\because A \subseteq U)$$
$$\Rightarrow x \in A \text{ and } x \in U$$
$$\Rightarrow x \in A \cap U$$

Hence $$A \subseteq A \cap U \qquad \text{… (2)}$$

Combining (1) and (2), we get

$$A \cap U = A$$

2.15.3 Commutative Law

Intersection of sets is commutative, i.e., if A and B are any two sets, then $A \cap B = B \cap A$

Proof: Let $x \in A \cap B$, then

$$x \in A \cap B \Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in B \text{ and } x \in A$$
$$\Rightarrow x \in B \cap A$$

Thus $$A \cap B \subseteq B \cap A \qquad \text{… (1)}$$

Conversely, Let $x \in B \cap A$, then

$$x \in B \cap A \Rightarrow x \in B \text{ and } x \in A$$
$$\Rightarrow x \in A \text{ and } x \in B$$
$$\Rightarrow x \in A \cap B$$

Hence $$B \cap A \subseteq A \subseteq B \qquad \text{… (2)}$$

Combining (1) and (2) $A \cap B = B \cap A$

2.15.4 Associative Law for Intersection

Intersection of sets is associative, i.e., if A, B and C are any three sets, then $(A \cap B) \cap C = A \cap (B \cap C)$

Proof: Let $x \in (A \cap B) \cap C$, then

$$\begin{aligned} x \in (A \cap B) \cap C &\Rightarrow x \in (A \cap B) \text{ and } x \in C \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C) \\ &\Rightarrow x \in A \text{ and } x \in B \cap C \\ &\Rightarrow x \in A \cap (B \cap C) \end{aligned}$$

Hence $\quad (A \cap B) \cap \subseteq A \cap (B \cap C)$

Conversely, Let $x \in A \cap (B \cap C)$, then

$$\begin{aligned} x \in A \cap (B \cap C) &\Rightarrow x \in A \text{ and } (x \in B \cap C) \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ and } x \in C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ &\Rightarrow x \in (A \cap B) \text{ and } x \in C \\ &\Rightarrow x \in (A \cap B) \in C \end{aligned}$$

Thus $\quad A \cap (B \cap C) \subseteq (A \cap B) \cap C \quad \ldots (2)$

Combining (1) and (2), we get

$$(A \cap B) \cap C = A \cap (B \cap C)$$

2.16 DISTRIBUTIVE LAWS

2.16.1

Intersection of sets is distributive over the union of sets, i.e., if A, B and C are any three sets, then

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Proof: Let $x \in A \cap (B \cup C)$, Then

$$\begin{aligned} x \in A \cap (B \cup C) &\Rightarrow x \in A \text{ and } x \in (B \cup C), \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\ &\Rightarrow x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Thus $\quad A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C) \quad \ldots (1)$

conversely let $x \in (A \cap B) \cup (A \cap C)$, then

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \end{aligned}$$

$$\Rightarrow\ x \in A \text{ and } (x \in B \text{ or } x \in C)$$
$$\Rightarrow\ x \in A \text{ and } x \in (B \cup C)$$
$$\Rightarrow\ x \in A \cap (B \cup C)$$

Hence $\qquad (A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ … (2)

From (1) and (2), we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

2.16.2

Union of sets is distributive over the intersection, i.e., if A, B and C are any three sets, then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

2.17 COMPLEMENT OF A SET

2.17.1 Relative Complement (or Difference of Sets)

Definition 2.15: If A and B are subsets of the universal set U, then the relative complement of B in A is the set of all elements in A which are not in A. It is denoted by $A - B$ thus:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Example: Let $A = \{a, b, c\}$ and $B = \{b, c, d, e, f, g\}$, then $A - B = \{a\}$.

2.17.2 Complement of Set

Definition 2.16: If U is a universal set containing the set A, then $U - A$ is called the complement of A. It is denoted by A' or by $\overline{A}$.

Thus $\qquad \overline{A} = A' = \{x : x \notin A\}$

2.18 PROPERTIES OF COMPLEMENTATION

If A and B are two subsets of universal set U, then

(1) $\overline{U} = \phi$

(2) $\overline{(\phi)} = U$

(3) $A \cup \overline{A} = \cup$

(4) $A \cap \overline{A} = \phi$

(5) $\overline{(\overline{A})} = A$

(6) $A \subseteq B \Rightarrow \overline{B} \subseteq \overline{A}$

(7) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$ (De Morgan's laws)

(8) $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ (De Morgan's laws)

Proof: (1) Clearly $\phi \subseteq \overline{U}$

Conversely $\qquad x \in \overline{U} \Rightarrow x \notin U$

$$\Rightarrow x \in \phi$$

Hence $\qquad \overline{U} \subseteq \phi$

Now $$\phi \subseteq \bar{U} \text{ and } \bar{U} \subseteq \phi$$
$$\Rightarrow \bar{U} = \phi$$

Proof: (2) Let $x \in \bar{\phi}$, then

$$x \in \bar{\phi} \Rightarrow x \notin \phi$$
$$\Rightarrow x \in U$$

Thus $$\bar{\phi} \subseteq U$$

Conversely let $x \in U$, then

$$x \in U \Rightarrow x \notin \phi$$
$$\Rightarrow x \in \bar{\phi}$$

Hence $$U \subseteq \bar{\phi}$$

Therefore $$\bar{\phi} \subseteq U \text{ and } U \subseteq \bar{\phi} \Rightarrow \bar{\phi} = U$$

The properties (3) and (4) are very simple and follow immediately from the definition of complement, we prove the remainig.

Proof: (5) If $x \in \overline{(\bar{A})}$ then $x \notin A$

Hence $$x \in \overline{(\bar{A})} \Rightarrow x \notin \bar{A}$$
$$\Rightarrow x \in A$$

Thus $$\overline{(\bar{A})} \subseteq A \qquad \ldots (1)$$

Conversely, let $x \in A$, then

$$x \in A \Rightarrow x \notin \bar{A}$$
$$\Rightarrow x \in \overline{(\bar{A})} \qquad \ldots (2)$$

Hence $A \subseteq \overline{(\bar{A})}$

Therefore from (1) and (2)

$$\overline{(\bar{A})} = A$$

Proof: (6) We have $A \subseteq B$

Let $x \in \bar{B}$, then

$$x \in \bar{B} \Rightarrow x \notin B$$
$$\Rightarrow x \notin A \quad (\therefore A \subseteq B)$$
$$\Rightarrow x \in \bar{A}$$

Hence $\bar{B} \subseteq \bar{A}$

Proof: (7) Let $x \in \overline{(A \cap B)}$; then

$$x \in \overline{(A \cap B)} \Rightarrow x \notin (A \cap B)$$

$$\Rightarrow x \notin A \text{ or } x \notin B$$
$$\Rightarrow x \in \overline{A} \text{ or } x \in \overline{B}$$
$$\Rightarrow x \in \overline{A} \cup \overline{B}$$

Hence $\overline{(A \cap B)} \subseteq \overline{A} \cup \overline{B}$ ... (1)

Conversely, let $x \in \overline{A} \cup \overline{B}$, then

$$x \in (\overline{A} \cup \overline{B}) \Rightarrow x \in \overline{A} \text{ or } x \in \overline{B}$$
$$\Rightarrow x \notin A \text{ or } x \notin B$$
$$\Rightarrow x \notin (A \cap B)$$
$$\Rightarrow x \in \overline{(A \cap B)}$$

Thus $\overline{A} \cup \overline{B} \subseteq \overline{(A \cap B)}$ ... (2)

From (1) and (2), we have

$$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

Proof: (8) Let $x \in \overline{(A \cup B)}$, then

$$x \in \overline{(A \cup B)} \Rightarrow x \notin A \cup B$$
$$\Rightarrow x \notin A \text{ and } x \notin B$$
$$\Rightarrow x \in \overline{A} \text{ and } x \in \overline{B}$$
$$\Rightarrow x \in \overline{A} \cap \overline{B}$$

Hence $\overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$ ... (1)

Conversely, let $x \in A \cap B$, then

$$x \in \overline{A} \cap \overline{B} \Rightarrow x \in \overline{A} \text{ and } x \in \overline{B}$$
$$\Rightarrow x \notin A \text{ and } x \notin B$$
$$\Rightarrow x \notin (A \cup B)$$
$$\Rightarrow x \in \overline{(A \cup B)}$$

Thus $\overline{A} \cap \overline{B} \subseteq \overline{(A \cup B)}$ ... (2)

Combining (1) and (2), we have $\left(\overline{A \cup B}\right) = \overline{A} \cap \overline{B}$

The above properties of union and intersection of sets are called set Identities.

2.19 PROPERTIES OF DIFFERENCE

If A and B are two subsets of a universal set, then

1. $A - B = A \cap \overline{B}$
2. $\overline{A} = U - A$
3. $A - A = \phi$
4. $A - \phi = A$

5. $A - B = B - A$, if and only if $A = B$

6. $A - B = A$ if and only if $A \cap B = \phi$

7. $A - B = \phi$ if and only if $A \subseteq B$

Proof: (1) Let $x \in A - B$, then

$$\begin{aligned} x \in A - B &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A \text{ and } x \in \overline{B} \\ &\Rightarrow x \in A \cap \overline{B} \end{aligned}$$

Hence $$A - B \subseteq A \cap \overline{B} \qquad \ldots (1)$$

Conversely let $x \in A \cap B$, then

$$\begin{aligned} x \in A \cap B &\Rightarrow x \in A \text{ and } x \in \overline{B} \\ &\Rightarrow x \in A \text{ and } x \notin B \\ &\Rightarrow x \in A - B \end{aligned}$$

Therefore $$A \cap \overline{B} \subseteq A - B \qquad \ldots (2)$$

From (1) and (2), we have $A - B = A \cap \overline{B}$

2.20 SYMMETRIC DIFFERENCE

Definition 2.17: The symmetric difference of two sets A and B is the relative complement of $A \cap B$ with respect to $A \cup B$. It is denoted by $A \Delta B$ (or by $A \oplus B$)

Symbolically: $A \Delta B = \{x | x \in A \cup B \text{ and } x \notin A \cap B\}$

Example: Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ $B = \{3, 4, p, q, r, s\}$

Then, we have $A \cup B = \{1, 2, 3, 4, 5, 6, 7, p, q, r, s\}$ and $A \cap B = \{3, 4\}$

We get $A \Delta B = \{1, 2, 5, 6, 7, p, q, r, s\}$

Note: We can also find the symmetric difference by using the identity.

$$A \Delta B = (A - B) \cup (B - A)$$

In the above example, we have $A - B = \{1, 2, 5, 6, 7\}$;
and $B - A = \{p, q, r, s\}$

Hence $A \Delta B = (A - B) \cup (B - A) = \{1, 2, 5, 6, 7, p, q, r, s\}$

2.21 PROPERTIES OF SYMMETRIC DIFFERENCE

If A and B are any two sets, then

1. $A \Delta A = \phi$
2. $A \Delta B = B \Delta A$
3. $A \Delta \phi = A$
4. $(A \Delta B) \Delta C = A \Delta (B \Delta C)$
5. $A \Delta B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

2.22 VENN DIAGRAMS

Set operation can be illustrated by Venn diagrams. A venn diagram is a useful device to our consideration of relationships that may exist between the subsets of a given universe. The universal set U is represented by the set of points inside and on the boundary of a single closed curve (usually a rectangle). If A is a subset of U is then represented by the points inside and on the boundary of another simple closed curve (usually a circle) inside the rectangle. Several venn diagrams, together with their interpretations are shown below (usually the label U is ommitted).

Using venn diagram, we can produce a geometrical interpretation for any expression involving sets and set operations. However, it should be noted that, venn diagrams do not prove the truth of a relationship between sets, they only illustrate plausibility.

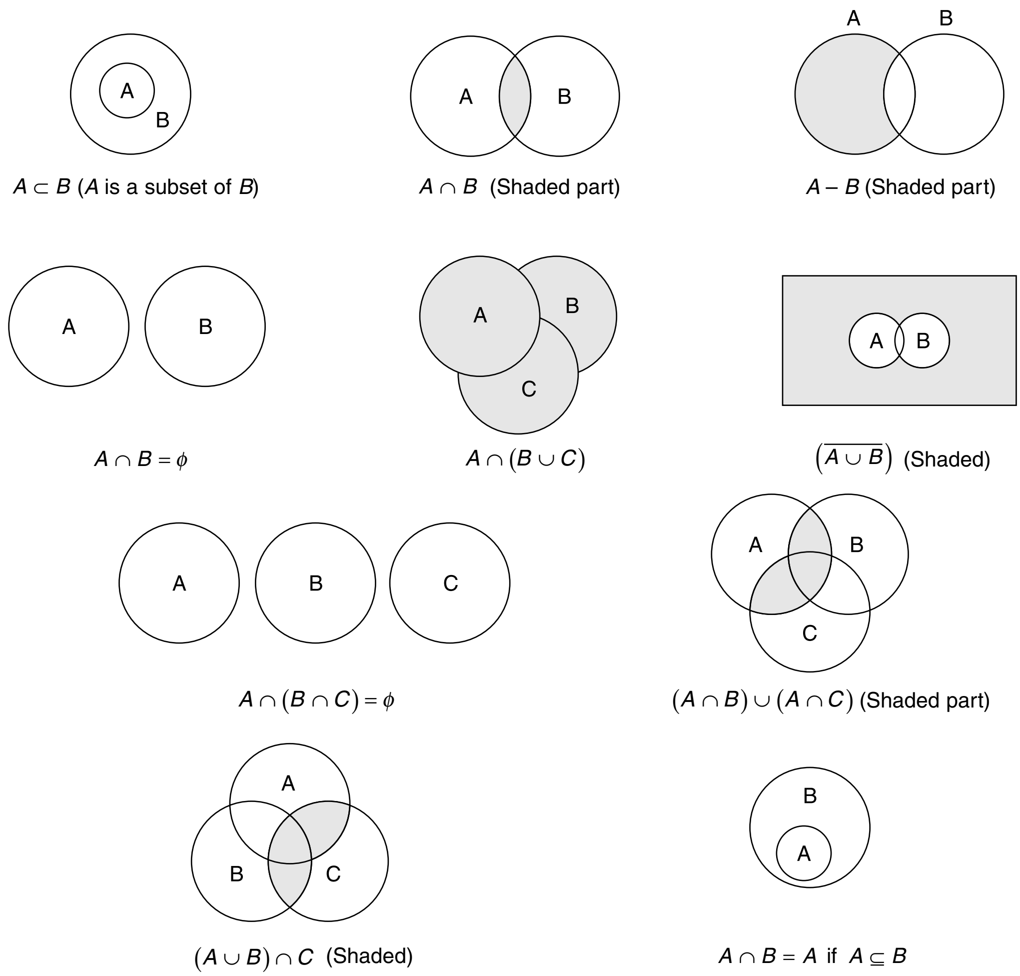


Fig. 2.1 Venn Diagrams

2.23 PRINCIPLE OF DUALITY

The principle of duality states that any established result involving sets and complements and operations of union and intersection gives a corresponding dual result by replacing U by ϕ and $\cup$ by $\cap$, and vice versa.

Example: Consider $A \cup \overline{A} = U$

Applying the principle of dualilty, we get $A \cap \overline{A} = \phi$

2.24 SOLVED EXAMPLES

Example 1: Prove that $A \cap (B - C) \subset A - (B \cap C)$

Solution: Let $x \in A \cap (B - C)$, then

$$\begin{aligned} x \in A \cap (B - C) &\Rightarrow x \in A \text{ and } x \in (B - C) \\ &\Rightarrow x \in A \text{ and } (x \in B, \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin (B \cap C) \\ &\Rightarrow x \in A - (B \cap C) \end{aligned}$$

Hence $$A \cap (B - C) \subset A - (B \cap C)$$

Example 2: Show that $\left[A \cap \left(B \cup \overline{A}\right)\right] \cup B = B$

Solution:
$$\begin{aligned} \left[A \cap \left(B \cup \overline{A}\right)\right] \cup B &= \left((A \cap B) \cup \left(A \cap \overline{A}\right)\right) \cup B \\ &= ((A \cap B) \cup \phi) \cup B \\ &= (A \cap B) \cup B = B \end{aligned}$$

Example 3: Give that ϕ is an empty set, find $P(\phi)$, $P(P(\phi))$, $P(P(P(\phi)))$

Solution:
$$\begin{aligned} P(\phi) &= \{\phi\} \\ P(P(\phi)) &= \{\phi, (\phi)\} \\ P(P(P(\phi))) &= \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\} \end{aligned}$$

EXERCISE 2.1

I(*a*) Illustrate the following identities by means of venn diagrams:

1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2. $A \cap U = A$
3. $A \cup U = U$
4. $A \cup B = \overline{(A \cup B)} \cup (A \cap B)$

5. $\overline{(A \cup B)} = A \cap B$
6. $(A \cup B) \cap B = A \cap \overline{B}$
7. $\overline{A} \cup (A \cap B) = \overline{A} \cup B$
8. $A \cap (A \cup B) = A$

(*b*) 1. $A \subseteq B \Leftrightarrow A \cup B = B$ ***(MCA, OSM, 1999)***

2. If A and B and subsets of universal set, then $A \subset B$; if and only if $\overline{B} \subseteq \overline{A}$ ***(MCA, OSM, 1996)***
3. Use Venn diagram to show that
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ***(MCA, OSM, 1998)***
4. Prove that $A - (A \cap B) = (A - B)$ ***(MCA, OSM, 1998)***
5. If A_0 and A_k are sets of real numbers defined as
   $A_0 = \{a / a \le 1\}$
   $A_k = \{a / a \le 1 + 1/k;\ k = 1, 2, \overline{...}\}$
   $\prod_{k=1}^{\infty} = A_0$ ***(MCA, OSM, 1999)***
   $k = 1$
6. Show that $P(A) \cup P(B) \subseteq P(A \cup B)$ where $P(X)$ is the power set of set X ***(MCA, OSM, 1998)***
7. Let A be a set with k elements and $P(A)$ its power set show that the cardinclity of $P(A)$ is 2^k.

II

1. $A = \{1, 2\}$, $B = \{1, 2, 4, 5\}$ and $C = \{5, 7, 9, 10\}$ find
   (*a*) $A \cup B$ (*b*) $A \cap B$ (*c*) $(A \cup B) \cup C$
   (*d*) $(A \cap B) \cap C$ (*e*) $(A \cup B) \cap C$ (*f*) $(A \cap B) \cup C$
2. If $A = \{a, b, c, d\}$, $B = \{c, d, e\}$ and $C = \{e, f, g, h\}$ state the elements of the sets.
   (*a*) $A \cup C$ (*b*) $B \cap A$
   (*c*) $B \cap (A \cup C)$ (*d*) $(B \cap A) \cup (B \cap C)$
3. If $U = \{x \in Z \mid -5 < x < 5\}$ and $A = (x \in Z \mid -2 < x < 3)$ state the elements of the sets.
   $\overline{A}$, $\overline{A} \cap \overline{A}$, $A \cap U$, $A \cup U$ $\{Z$ is the set of integers$\}$
4. If $U = \{1, 2, 3, 4, \ldots 10\}$, $A = \{x \in U \mid x \text{ is a prime}\}$, $B = \{x \in U \mid x$ is odd$\}$ Show that $A \cap B = (A \cup B)$
5. $U = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c, d\}$, $B = \{a, b, c, d, e, f\}$ and $C = \{a, b, g\}$ find $\overline{A}$, $\overline{B}$, $\overline{C}$,
   $A - B$, $B - C$, $A \cap B(A \cup B)$ and $B \cap C$.
6. If $A = \{a, b, c, e, f,\}$, $B = \{b, e, f, r, s\}$ and $C = \{a, t, u, v\}$ find $A \cap B$, $A \cap C$ and $B \cap C$.

7. If $A = \{x \mid x$ is an integer and $x \leq 4\}$ and $U = Z$, then write A.
8. If $U = \{a, b, c, d, e, f, g, h\}$, $A = \{a, c, f, g\}$, $B = \{a, e\}$, $C = \{b, h\}$ compute
   (*a*) $A - B$ (*b*) A (*c*) B (*d*) $A \cup B$
9. List all the subsets of (*a*) $A = \{a, b, c\}$, (*b*) $\{a, b, c, d\}$.
10. If $A_1 = \{1, 5\}$, $A_2 = \{1, 2, 4, 6\}$, $A_3 = \{3, 4, 7\}$, $B = \{2, 4\}$ and $I = \{1, 2, 3\}$
    Verify the Identies.
    (*a*) $B \cup \left(\bigcap_{i=1}^{3} A_i\right) = \bigcap_{i=1}^{3} (B \cup A_i)$ (*b*) $B \cup \left(\bigcup_{i=1}^{3} A_i\right) = \bigcup_{i=1}^{3} (B \cap A_i)$
11. If $A_1 = \{\{1, 2\}, \{3\}\}$, $A_2 = \{\{1\}, \{2, 3\}\}$, and $A_3 = \{1, 2, 3\}$ these show that A_1, A_2, A_3 are mutually disjoint.
12. If A and B are two given sets then, prove that $A \cap (B - A) = \phi$
13. If $A = \{1, 3, 5, 7, 8, 9\}$ and $B = \{3, 5, 8\}$ then verify:
    $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
14. If $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$and $C = \{1, 3, 4, 5, 6, 7\}$ verify
    $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
15. If $A \oplus$ denotes the symmetric difference of two sets A and B, then find $A \oplus B$ for the following:
    (*a*) $A = \{a, b\}, B = \{a, c\}$ (*b*) $A = \{a, b\}, B = \{b, c\}$
16. If $A = (1, 3, 5, 7, 9\}$ and $B = \{3, 5, 8\}$ then find $A \Delta B$ (symmetric difference of the sets A and B).
17. Prove (or disprove) by Venn diagram or otherwise that
    $(A \cup B) \cap (B \cup \overline{C}) \subset (A \cap \overline{B})$ (***MCA, OSM, 1997***)
18. A and B are two independent events. The Probability that both A and B occur is 1/12. The Probability that neither A nor B occures is 1/2. Find the values of $P(A)$ and $P(B)$.
19. A and B are two events such that $P(A) = 0.3$, $P(B) = 0.4$ and $P\ \overline{(AB)} = 0.5$ find $P\left(B \mid \left(A \cup \overline{B}\right)\right)$.
20. A die of 6 faces is thrown 4 times. What is the probability that the minimul value is not less than 2 and the maximum value is not grater than 5
21. Define a partition of a set.
22. Define a power set. Illustrate with an example.

Answers:

II 1. (*a*) $\{1, 2, 4, 5\}$; (*b*) $\{1, 2\}$; (*c*) $\{1, 2, 4, 5, 7, 9, 10\}$; (*d*) ϕ; (*e*) $\{5\}$; (*f*) $\{1, 2, 5, 7, 9, 10\}$

2. (*a*) $\{a, b, c, d, e, f, g, h\}$, (*b*) $\{c, d\}$, (*c*) $\{c, d, e\}$, (*d*) $\{c, d, e\}$

3. $\overline{A} = \{x \in Z \mid -5 < x < -2 \cup 3 < x < 5\}$

   $\overline{A} \cap \overline{A} = U$

   $A \cap U = A,\ A \cup U = U$

5. $\overline{A} = \{e, f, g\}, \overline{B} = \{b, g\}, \overline{C} = \{c, d, e, f\},$
   $A - B = \{b\}, B - C = \{c, d, e, f\}$
   $A \cap B = \{a, c, d\}$
   $A \cup B = \{a, b, c, d, e, f\}$
   $B \cap C = \{a, b\}$
6. $A \cap B = \{b, e, f\}, A \cap C = \{a\}, B \cap C = \phi$
7. $\overline{A} = \{x | x \in Z, x > 4\}$
8. (*a*) $A - B = \{c, f, g\}$
   (*b*) $\overline{A} = \{b, d, e\}$
   (*c*) $\overline{B} = \{b, c, d, f, g\}$
   (*d*) $\{a, c, e, f, g\}$
9. (*a*) $P(A) = \{\{f\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, A\}$
   (*b*) $\{f, \{a\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}$ $\{b, c, d\}\{a, b, c, d\}\}$
15. (*a*) $\{b, c\}$, (*b*) $\{a, c\}$
16. $A \Delta B = \{1, 7, 8, 9\}$
18. $\frac{1}{4}, \frac{1}{3}$
19. $\frac{1}{4}$
20. $\frac{16}{81}$

2.25 SETS OF NUMBERS

We now introduce here several sets and their notations that will be used throughout this book.

(*a*) $Z^+ = \{x \mid x \text{ is a positive integer}\} = \{1, 2, 3, 4\ldots\}$

(*b*) $N = \{x \mid x \text{ is a positive integer and zero}\} = \{0, 1, 2, 3, 4\ldots\}$

Note: 1, 2, 3, 4 … are natural numbers.

(*c*) The rational integers are the members of the set Z, where

$$Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$$

(*d*) The rational numbers are the members of the set.

$$Q = \{p/q \mid p \in Z, q \in Z, \text{ and } q \neq 0\}$$

(*e*) The irrational numbers are the members of the set of all real numbers that cannot be expressed as the quotient p/q of two integers.

Example: $\sqrt{2}, \sqrt{3}, \pi$, …, etc., are all irrational.

(f) The real numbers are the members of the set formed by the union of the sets of rational and irrational numbers it is denoted by R.

$$\therefore\ R = \{x \mid x \text{ is a real number}\}$$

2.26 CARDINALITY

2.26.1 Finite Set

Definition 2.18: A set A is called a finite set if it has n distinct members (elements) where $n \in N$ (refer 1.5).

2.26.2 Cardinality of a Set

Definition 2.19: If A is finite set with n distinct elements, then n is called the cardinality of A. The cardinality of A is denoted by $|A|$ [or by $n(A)$].

Example: Let $A = \{a, b, c, d\}$, then A is a finite set and $|A| = 4$

2.26.3 Cardinality of Union of Two Sets

Number of elements in $A \cup B$: (Cardinality of union) If A and B are any two finite sets then, the numbers of elements in $A \cup B$, denoted by $|A \cup B|$ is given by $|A \cup B| = |A| + |B| - |A \cap B|$

Note: The number of elements in $A \cap B$, is also denoted by $n(A \cup B)$.

2.26.4 Cardinality of Union of Three Sets

Number of elements in $A \cup B \cup C$: If A, B and C are any three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C| - |C \cap A| - |A \cap B| + |A \cap B \cap C|$$

Example 1: If $n(A) = 2$, $n(B) = 3$, $n(A \cap B) = 1$, find $n(A \cup B)$

Solution: Given $|A| = 2$, $|B| = 3$, $|A \cap B| = 1$

Using the formula $|A \cup B| = |A| + |B| - |A \cap B|$

We get $|A \cup B| = 2 + 3 - 1 = 4$

Example 2: Verify:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$

Where $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 3, 4, 6\}$, $C = \{3, 4, 6, 8\}$

Solution: We have $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 8\}$

$A \cap B \cap C = \{3, 4\}$

$A \cap B = \{2, 3, 4\}$, $B \cap C = \{3, 4, 6\}$

$C \cap A = \{3, 4\}$

$$|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$= 5 + 4 + 4 - 3 - 3 - 2 + 2$$
$$= 7 = |A \cap B \cap C|$$

Example 3: Out of 30 students in a dormitory, 15 take an art course, 8 take a biology course and 6 take a chemistry course. It is known that 3 students take all the three courses. Show that 7 or more students take none of the courses.

Solution: Let A be the set of students taking an art course
B be the set of students taking a biology course
C be the set of students taking a chemistry course

Then we have

$$|A| = 15, |B| = 8, |C| = 6, |A \cap B \cap C| = 3$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$= 15 + 8 + 6 - |A \cap B| - |B \cap C| - |C \cap A| + 3$$
$$= 32 - |A \cap B| - |B \cap C| - |C \cap A| \qquad \text{... (1)}$$

But $|A \cap B| \geq |A \cap B \cap C|, |B \cap C| \geq |A \cap B \cap C|, |C \cap A| \geq |A \cap B \cap C|$

Therefore, $|A \cap B| + |B \cap C| + |C \cap A| \geq 3\,|A \cap B \cap C|$

From (1), we have

$$|A \cup B \cup C| \geq 32 - 3\,|A \cap B \cap C| = 32 - 3 \times 3$$

Hence $|A \cup B \cup C| \geq 23$

The number of students taking atleast one course ≥ 23. The students taking none of the courses $\geq 30 - 23 = 7$.

Hence, seven or more students take none of the courses.

2.26.5 Comparable Sets

Definition 2.20: Two sets A and B are said to be comparable if $A \subset B$ or $B \subset A$

2.26.6 Sets Not Comparable

Definition 2.21: Two sets A and B are said to be not comparable if $A \not\subset B$ and $B \not\subset A$.

Example 1: Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 6\}$ then A is comparable to B, since A is a subset of B.

Example 2: If $A = \{a, c\}, B = \{b, c, d, e, f\}$ then $A \not\subset B$ and $B \not\subset A$. Therefore the sets A and B are not comparable.

2.26.7 Multiset

Definition 2.22: A collection of objects that are not necessarily distinct is called a multiset.

Example: $\{a, a, b, b\ c, c\}$

2.26.8 Multiplicity

Definition 2.23: Let S be a multiset and $x \in S$. The multiplicity of x is defined to be the numbers of times the element x appears in the multiset S.

Example 1: Let $S = \{a, a, b, b, b, d, d, d, e\}$

Then Multiplicity of a is 2

Multiplicity of b is 3

Multiplicity of d is 3

Multiplicity of e is 1

If A and B are multisets then $A \cup B$ and $A \cap B$ are also multisets. The multiplicity of an element $x \in A \cup B$ is equal to the maximum of the multiplicity of x in A and B.

The multiplicity of $x \in A \cap B$ is equal to the minimum of the multiplicities of x in A and in B.

Example 2: Let $A = \{a, a, a, b, b, c, c, d, d\}$

and $B = \{a, a, b, b, c, d\}$

Then $A \cup B = \{a, a, a, b, b, c, c, d, d\}$

$A \cap B = \{a, a, b, c, d\}$

2.27 CARTESIAN PRODUCT OF SETS

2.27.1 Ordered Pair

Definition 2.24: If $a \in B$, and $b \in B$ then the ordered pair is the set $\{\{a\}, \{a, b,\}\}$ consisting of the pair $\{a, b\}$ and the singleton $\{a\}$. It is represented by (a, b).

In the ordered pair (a, b), the element is a called the first element and the element b is called the second element.

Example: If $P(x, y)$ is a point in the plane, then the coordinates of P from an pair. The first member x, is called the x-coordinate of P and y is called y-coordinate of P.

Note: If (a, b) and (a', b') are two ordered pairs such that $(a, b) = (a', b')$ then $a = a'$ and $b = b'$.

2.27.2 Cartesian Product

Definition 2.25: If A and B are two non-empty sets, then the cartesian product of A and B is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

The cartesian product of the sets A and B is denoted by $A \times B$.

Using set notation we can write $A \times B$ as

$$A \times B = \{(a, b) \mid a \in A, \text{ and } b \in B\}$$

Example: If $A = (0, 1, 2)$, $B = \{3, 5\}$, then

$$A \times B = \{(0, 3), (0, 5), (1, 3), (1, 5), (2, 3), (2, 5)\}$$

Note: If A and B are both the set of real numbers then $A \times B$ is the cartesian plane. The cartesian product of sets can also be represented by tree diagrams.

2.27.3

If A, B, C are sets then

(*i*) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(*ii*) $A \times (B \cap C) = (A \times B) \cap (A \times C)$ ***(V.T.U., B.E., 2000)***

Proof: (*i*) $A \times (B \cup C)$

$$= \{(x, y) : x \in A,\ y \in B \cup C\}$$
$$= \{(x, y) : x \in A \text{ and } y \in B \text{ or } y \in C\}$$
$$= \{(x, y) : x \in A,\ y \in B \text{ or } x \in A,\ y \in C\}$$
$$= \{(x, y) : (x, y) \in A \times B \text{ or } (x, y) \in A \times C\}$$
$$= (A \times B) \cup (A \times C)$$

(*ii*) $A \times (B \cap C)$

$$= \{(x, y) : x \in A,\ y \in B \cap C\}$$
$$= \{(x, y) : x \in A \text{ and } y \in B \text{ and } y \in C\}$$
$$= \{(x, y) : x \in A,\ y \in B \text{ and } x \in A,\ y \in C\}$$
$$= \{(x, y) : (x, y) \in A \times B \text{ and } (x, y) \in A \times C\}$$
$$= (A \times B) \cap (A \times C)$$

2.27.4

If A, B, and C are non-empty sets then

$$A \subseteq B \Rightarrow A \times C \subseteq B \times C$$

Proof: Let (a, b) be any element $A \times C$, then

$$\Rightarrow a \in B \text{ and } b \in C \qquad (\because A \subseteq B)$$
$$\Rightarrow (a, b) \in B \times C$$

Hence $A \times C \subseteq B \times C$

2.27.5 Cartesian Product of *n* Sets

Let $A_1, A_2, A_3, ..., A_n$ denote n sets where $n \geq 2$, then the Cartesian product $A_1 \times A_2 \times ... A_n$ is the set of all n-tuples of the form $(a_1, a_2, a_3, ..., a_n)$ where $a_1 \in A_1$, $a_2 \in A_2$, $a_3 \in A_3$, $a_n \in A_n$.

From the definition we have

$$A_1 \times A_2 \times A_3 \times ... A_n = \{(a_1, a_2, a_3, ..., a_n) : a_i \in A_i,\ 1 \leq i \leq n\}$$

EXERCISE 2.2

1. Let $A = \{1, 2, 4\}$, $B = \{0, 2\}$ find $A \times B$.
2. Prove that $A \times B = \phi$ if $A = \phi$, $B = \phi$.
3. Prove that $A \times B = B \times A$ if and only of $A = B$.
4. If $A = \{a, b\}$, $B = \{2, 3\}$ and $C = \{3, 4\}$
   Find
   (1) $A \times (B \cup C)$, $(A \times B) \cup (A \times C)$ and show that
   $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   (2) Find $A \times (B \cap C)$ and show that
   $A \times (B \cap C) = (A \times B) \cap (A \times C)$
5. If A, B and C are sets then
6. Prove that
   (*i*) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
   (*ii*) Show that $\overline{A} - \overline{B} = B - A$
7. If A, B and C are any sets
   Show that $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$
8. If A is a proper subset of B, then show that $A \cup (B - A) = B$
9. Prove (or disprove) by Venn diagram or otherwise that
   $(A \cup B) \cap (B \cup \overline{C}) \subset (A \cap \overline{B})$ ***(OU, MCA, 1997)***
10. Define the following:
    (*i*) Power set
    (*ii*) Partition of a set give examples
11. Prove the following and represent by Venn diagram:
    (*i*) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
    (*ii*) $A - (B \cup C) = (A - B) \cap (A - C)$ ***(OU, MCA, 1991)***
12. Show by an example that
    (*i*) $A \times B \neq B \times A$
    (*ii*) $(A \times B) \times C = A \times (B \times C)$ ***(OU, MCA, 1994)***
13. Write the sets
    (*i*) $\phi \cup \{\phi\}$
    (*ii*) $\{\phi\} \cup \{\phi\}$ and
    (*iii*) $\{\phi, \{\phi\}\} - \phi$ ***(OU, MCA, 1995)***
14. Show that $A \subseteq B$ implies that $A \cup (B - A) = 0$ ***(OU, MCA, 1995)***
15. Show that $A - B \subseteq C$ if and only if $A - C \subseteq B$ ***(OU, MCA, 1995)***

16. If A and B subsets of the Universal set U, then show that

$\overline{A} \oplus \overline{B} = A \oplus B$ ***(OU, MCA, 1995)***

17. For the sets S, T and V prove that

$(S \cap T) \times V = (S \times V) \cap (T \times V)$ ***(OU, MCA, 1995, 96)***

18. Use Venn diagrams to show that

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ***(OU, MCA, 1998)***

19. Show that $P(A) \cup P(B) \subseteq P(A \cup B)$ where $P(X)$ is the powerset of X. ***(OU, MCA, 1998)***

20. Let A, B, C be subsets of U prove or disprove

$(A \cup B) \cap \left(B \cup \overline{C}\right) \subset A \cap B$ ***(OU, MCA, 1998)***

21. (*a*) Show that $A \subseteq B \Leftrightarrow A \cup B = B$

(*b*) $A \cap \overline{A} = \phi$; $A + \phi = A$ ***(OU, MCA, 1999)***

22. 35 children of a class draw a map. 26 children use red colour and some children use yellow colour. If 19 use both the colours. Find the number of children who used the yellow colour.

(Ans: 28)

23. In a class of 42 students each play atleast one of three games—hockey, cricket, and football. It is found that the play cricket, 20 play hockey and 24 play football, 3 play both cricket and football, 2 play both hockey and football and none play all the three games. Find the number of students who play cricket but not hockey. ***(Ans: 31)***

24. Use Venn diagram to show that the following argument is valid:

P_1: All dictionaries are useful

P_2: Many owns only romance novels

P_3: No violance novel is useful

P_4: Many does not own a dictionary.

25. In a survey of 500 people 285 are interested in football game, 195 are interested in hockey game, 115 are interested in basketball game, 45 in football and basketball, 70 in football and hockey and 50 in hockey and basketball games; and 50 are not interested in any of these three games.

(*i*) How many people are interested in all the three of the games?

(*ii*) How many people are interested in exactly one of the games? ***(VTU, BE, Aug. 2000)***

26. If there are 200 faculty members that speak French, 50 that speak Russian, 100 that speak Spanish, 20 that speak French and Russian, 60 that speak French and Spanish, 35 that speak Russian and Spanish, while only 10 that speak French, Russian and Spanish. Determine how many speak either French or Russian or Spanish? ***(VTU, MCD, Sep. 1999)***

27. If A_k are sets such that $A_0 = \{a/a \le 1\}$ and $A_k = \{a/a \le 1 + {}^1\!/\!_k\}$, prove that $\underset{k=1}{\overset{\infty}{\pi}} A_k = A_0$.

3

Relations

3.1 CONCEPT OF RELATION

This chapter deals primarily with the concept of a Relation.

A relation may involve equality or inequality. The mathematical concept of a relation deals with the way the variables are related or paired. A relation may signify a family tie between such as "is the son of" "is the brother of", " is the sister of". In mathematics the expressions like, "is less than", "is greater than", "is perpendicular to", "is parallel to" are relations. In this chapter, we shall only consider relations called binary relations. The 'equivalence relation' in sets is also discussed.

Definition 3.1: Let A and B be non-empty sets, then any subset of R of the cartesian product $A \times B$ is called a relation from A to B.

Example 1: Let $A = \{3, 6, 9\}$, $B = \{4, 8, 12\}$

Then $R = \{(3, 4), (3, 8), (4, 12)\}$ is a relation from A to B.

Example 2: Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

Then $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

If $R = \{(1, a), (3, b)\}$, then $R \subseteq A \times B$ and R is a relation from A to B.

Example 3: Let A denote the set of real numbers define

$R = \{(a, b): 4a^2 + 25b^2 \leq 100\}$ clearly R is a relation on A.

If $(a, b) \in R$, we often write aRb and state "a is related to b".

If $R \subseteq A \times A$, then R is a relation from A to A, and R is called a relation in A.

If R is a relation from A to B, then the set of all first elements of the ordered pairs (a, b), which belong to R is called the domain of R. The range of R is the set of all second coordinates of the ordered pairs (a, b) which belong to R. From the definition it is clear that relation R is also a set and many operations can be applied to relation R to obtain a new relation.

If R_1 and R_2 are two relations with the same domain D and same range then we can define the relation $R_1 \cup R_2$ and $R_1 \cap R_2$ with the same domain D and same range R.

Null set $\varnothing$ is a subset of every set. Therefore for any specified non-empty domain, and range, $\varnothing$ is a relation, called null relation or empty relation.

3.2 PROPERTIES OF RELATIONS

3.2.1 Reflexive Relation

Definition 3.2: Let R be a relation defined in a set A; then R is reflexive if aRa holds for all $a \in A$, i.e., if $(a, a) \in R$ for all $a \in A$.

Example 1: Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c)\}$ then R is a reflexive relation in A.

Example 2: 'Equality' is a reflexive relation, since an element equals itself.

3.2.2 Symmetric Relation

Definition 3.3: A relation R defined in set A is said to be 'symmetric' if bRa holds whenever aRb holds for $b \in A$, i.e., R is symmetric in A if

$$(a, b) \in R \Rightarrow (b, a) \in R.$$

Example: Let R be relation 'is perpendicular to' in the set of all straight lines, then R is a symmetric relation.

3.2.3 Transitive Relation

Definition 3.4: A relation R in set A is said to be transitive if

$$(a, b) \in R \;\; (b, c) \in R \Rightarrow (a, c) \in R$$

i.e., if aRb and $bRc \Rightarrow aRc$, $a, b, c \in A$.

Example 1: Let A denote the set of straight lines in a plane and R be a relation in A defined by 'is parallel to' then R is a transitive relation in A.

Example 2: Let $A = \{1, 2, 3\}$ and $R = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$ then R is transitive.

3.2.4 Equivalence Relation

Definition 3.5: A relation R in a set A is said to be an equivalence relation in A, if R is reflexive, symmetric and transitive.

Example: (*i*) Let A be the set of all triangle in a plane and let R be a relation in A defined by 'is congruent to', then R is reflexive, symmetric and transitive.

$\therefore$ R is an Equivalence relation in A.

(*ii*) Let $A = \{a, b, c\}$, and $R = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$ then R is an equivalence relation in A.

3.2.5 Anti-symmetric Relation

Definition 3.6: Let R be a relation in a set A, then R is called anti-symmetric.

$$(a, b) \in R, \; (b, a) \in R \Rightarrow a = b \;\forall\; a, b \in R$$

i.e., aRb and $bRb \Rightarrow a = b$

Example: Let N denote the set of Natural Numbers R be a relation in N, defined by 'a is a divisor' of b, i.e., aRb if a divides b then R is anti-symmetric since a divides b and b divides $a \Rightarrow a = b$.

3.2.6 The Inverse of a Relation

Let R be a relation from A to B. Then the relation $R^{-1} = \{(b, a): (a, b) \in R\}$ from B to A is called the inverse of R.

Example: Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$ and $R = \{(1, 4), (2, 5), (3, 5)\}$ be a relation from A to B. then $R^{-1} = \{(4, 1), (5, 2), (5, 3)\}$

3.3 MISCELLANEOUS EXAMPLES

Example 1: If a relation R is transitive then prove that its inverse relation R^{-1} is also transitive.

Solution: Let (a, b) and $(b, c) \in R^{-1}$ then $(b, a) \in R$ and $(c, b) \in R$ is transitive, therefore R

$$(c, b) \in R \text{ and } (b, a) \in R \Rightarrow (c, a) \in R$$

$$\Rightarrow (a, c) \in R^{-1}$$

i.e., $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1} \Rightarrow (a, c) \in R^{-1}$

Hence R^{-1} is transitive.

Example 2: $A = \{2, 3\}$, $B = \{3, 4, 5, 6\}$ and R is a relation from A to B defined as follows:

$(a, b) \in R$ if "a divides b" write the solution set of R.

Solution: 2 divides 4 and 2 divides 6

$$\Rightarrow (2, 4) \in R \text{ and } (2, 6) \in R$$

3 divides 3, 3 divides 6

$$\Rightarrow (3, 3) \in R \text{ and } (3, 6) \in R$$

4 divides 4 $\Rightarrow (4, 4) \in R$

Thus $R = \{(2, 4), (2, 6), (3, 3), (3, 6), (4, 4)\}$

Example 3: Let A be the set of triangles in the Euclidean plane, and R is the relation in A defined by "a is similar to b" then show that R is an equivalence relation in A.

Solution: Every triangle is similar to itself: the relation R is reflexive. If "a is similar to b" then "b is similar to a", i.e. $(a, b) \Rightarrow (b, a) \in R$

Hence R is symmetirc.

Clearly, if "a is similar to b", "b is similar to c" then "a is similar to c".

Therefore, the relation R is transitive. R is reflexive, symmetric and transitive. Thus R is an equivalence relation.

Example 4: X is a family of sets and R is relation in X defined by "x is subset of y" show that R is anti-symmetric and transitive.

Solution: Let $(A, B) \in R$ and $(B, A) \in R$ then $A \subset B$ and $B \subset A \Rightarrow A = B$.

Thus R is anti-symmetric also $A \subset B$, $B \subset C \Rightarrow A \subset C$. Therefore R is transitive.

Example 5: Show that the relation "Equality" defined in any set A, is an Equivalence relation.

Solution: (*i*) $a = a$ for every $a \in A$

Thus R is reflexive

$a = b$ implies $b = a$ for all $a, b \in A$

$\therefore$ R is symmetric

and (*ii*) $a = b$, $b = c$ implies $a = c$

for all $a, b, c \in A$

$\therefore$ R is transitive.

Thus R is an equivalence relation in A.

Example 6: Let Z denote the set of integers and the relation R in Z be defined by "aRb" iff $a - b$ is an even integer". Then show that R is an equivalence relation.

1. R is reflexive; since
   $\varnothing = a - a$ is even, hence aRa for every $a \in Z$.
2. R is symmetric:
   If $a - b$ is even then $b - a = -(a - b)$ is also even hence $aRb \Rightarrow bRa$
3. R is transitive: for if aRb and bRc then both $a - b$ and $b - c$ are even.
   Consequently, $a - c = (a - b) + (b - c)$ is also even.
   $\therefore$ aRb and $bRc \Rightarrow aRc$

Thus, R is an equivalence relation.

3.4 IRREFLEXIVE RELATION

Definition 3.7: A relation R on a set A is irreflexive if $a \not R a$ for every $a \in A$

Example: Let $A = \{1, 2, 3\}$ and

$$R = \{(1, 2), (2, 3), (3, 1), (2, 1)\}$$

Then the relation R is irreflexive on A.

3.5 ASYMMETRIC RELATION

Definition 3.8: A relation R defined on a set A is asymmetric if whenever aRb, then $b \not R a$.

Example: Let $A = \{a, b, c\}$ and $R = \{(a, b), (b, c)\}$ be a relation on A. Clearly R is a symmetric.

3.6 COMPATIBLE RELATION

Definition 3.9: A relation R in A is said to be a compatible relation if it is reflexive and symmetric.

If R is an equivalence relation on A, then R is compatible relation on A.

3.7 UNIVERSAL RELATION

Definition 3.10: A relation R in a set A is said to be universal relation if

$$R = A \times A.$$

Example 1: Let $A = \{a, b\}$, then

$R = \{(a, a), (a, b), (b, a), (b, b)\}$ is a universal relation on set A.

Example 2: Let $A = \{1, 2, 3\}$, then

$$R = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$$

is a universal relation on A.

3.8 COMPLEMENTARY RELATION

Definition 3.11: Let R be a relation from A to B, then the complement of R denoted by R' and is expressed in terms of R as follows;

$$aRb \text{ if } a\not{R}b$$

Example: Let $A = \{1, 2, 3\}$ and

$$R = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 3)\}$$

Then $$R' = \{(2, 1), (2, 3), (3, 1), (3, 2)\}$$

3.9 RELATION-RELATED SETS

Definition 3.12: Let A and B denote two non-empty sets and R be a relation from A to B. We can define various sets related to the relation R:

3.9.1 *R*-relative Set of an Element

Let $x \in A$, then the R-relative set of x is defined to be the set of all elements y in B with the property that x is related to y, where R is relation from A to B. It is denoted by $R(x)$ or by $[x]R$. Thus in symbols we can write:

$$R(x) = [x]R = \{y \in B/xRy\}$$

Example: Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $R = \{(1, a), (2, b), (3, c), (3, d)\}$

Then $R(3) = \{c, d\}$.

3.9.2 *R*-relative Set of a Subset

Definition 3.13: Let $A_1 \subseteq A$, then the R-relative set of A_1 is said to be the set of all elements y in such that x is R-related to y for some $x \in A_1$. It is denoted by

$R(A_1)$. Thus in symbols:

$$R(A_1) = \{ y \in B \mid xRy \text{ for some } x \text{ in } A_1\}$$

Example: Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d\}$ and $R = \{(1, a), (2, b), (2, d), (3, c), (4, d), (5, c), (5, d)\}$

If $A_1 = \{2, 5\}$ then $R(A_1) = \{b, c, d\}$

3.10 EQUIVALENCE CLASSES

Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$ be a relation defined on A.

Clearly R is reflexive, symmetric and transitive. Therefore R is an Equivalence relation on A.

Consider $[1]_R$, $[2]_R$, and $[4]_R$

We have $[1]_R = \{1, 2\}$, $[2]_R = \{1, 2\}$,
$[3]_R = \{3\}$ and $[4]_R = \{4\}$

Observe that

$$[1]_R = [2]_R$$

$$[1]_R \cap [3]_R = [1]_R \cap [4]_R = [2]_R \cap [3]_R = [2]_R \cap [4]_R = \varnothing$$

and $$[1]_R \cup [2]_R \cup [3]_R \cup [4]_R = A$$

Thus, the relation R is such that it gives rise to subset $[x]_R$ of A for which either $[x]_R = [y]_R$ or $[x]_R \cap [y]_R = \varnothing \ \forall\ x, y \in A$

Thus, the relation R on A, induces a partition in A.

Theorem 3.1: Every equivalence relation on a set generates a unique partition of the set. The blocks of this partition correspond to the R-equivalence classes.

Proof: Let R be an equivalence defined on a set A. For any $x \in A$ the $[x]_R \subseteq A$ be given by

$$[x]_R = \{y \mid y \in A \text{ and } x R y\}$$

The relation R is an equivalence relation

$\Rightarrow$ R is reflexive

$\Rightarrow$ $x R x$ is true

$\Rightarrow$ $x \in [y]_R$

Let $y \in A$ such that xRy, then $y \in [y]_R$

R is symmetric, therefore $xRy \Rightarrow yR x$

$$x \in [y]_R$$

Let $z \in [y]_R$ then

$$xR\, y,\ xR\, z \Rightarrow yR\, z$$

$$\Rightarrow [y]_R \subseteq [x]_R$$

by symmetry, $$[x]_R \subseteq [y]_R$$

$\therefore$ $$[x]_R \subseteq [y]_R \text{ and } [y]_R \subseteq [x]_R \Rightarrow [x]_R \subseteq [y]_R$$

If xRy, then it is shown that $[x]_R = [y]_R$. If $x \not R y$, then we must show that $[x]_R$ and $[y]_R$ are disjoint. To prove that, let us assume that, there is atleast one element

say $$z \in [x]_R \cap [y]_R$$

now, $xR\, z$ and $yR\, y$

$\Rightarrow$ $xR\, z$ and $yR\, y$

$\Rightarrow$ $xR\, y$ [by Transitivity]

which is a contradiction

Hence $$xR\, y \Rightarrow [x]_R \cap [y]_R = \varnothing$$

From the above it is clear that each element of A generates an R-equivalence class which is non-empty. The R-equivalence classes generated by any two elements are either equal or disjoint, and the union of R-equivalence classes generated by the element of A is the set A. Hence the R-equivalence classes generated by the elements of A defines a partition of A.

Note: The family of equivalence classes generated by the elements of A is denoted by A/R and is called quotient set of A and R. Each element of A/R is a set.

Example: Let Z be the set of integers and let R be the relation called "congruent modulo 5" defined by

$$R = \{(x, y) \mid x \in Z, y \in Z \text{ and } (x - y) \text{ is divisible by } 5\}$$

Determine the equivalence classes generated by the element of Z.

Solution: The equivalence classes are

$[0]_R = \{ \ldots \ldots -10, -5, 0, 5, 10, 15, \ldots\}$

$[1]_R = \{ \ldots \ldots -9, -4, 1, 6, 11, 16, \ldots\}$

$[2]_R = \{ \ldots \ldots -8, -3, 2, 7, 12, 17, \ldots\}$

$[3]_R = \{ \ldots \ldots -7, -2, 3, 8, 13, 18, \ldots\}$

$[4]_R = \{ \ldots \ldots -6, -1, 4, 9, 14, 19, \ldots\}$

$\therefore \quad Z/R = \{[0]_R, [1]_R, [2]_R, [3]_R, [4]_R\}$

3.11 RELATIONS ON COORDINATE DIAGRAMS

If A and B are two subsets of the set of real numbers and R is relation from A to B; then the relation R can be displayed on a coordinate diagram of $A \times B$ in which the ordered pairs of R are represented by points in a cartesian plane.

Example: $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5\}$ and

$R = \{(1, 3), (1, 5), (2, 3) (3, 3), (3, 5), (4, 5)\}$

Sketch R on the coordinate diagram of $A \times B$

Solution: The sketch of R on the coordinate diagram of $A \times B$ is a follows.

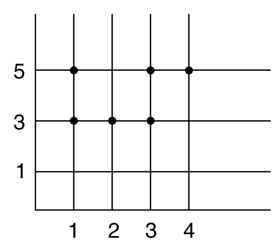


Fig. 3.1

3.12 TABULAR FORM OF A RELATION

A relation (Binary relation) on a set A can be represented in the tabular form. The tabular form of a relation is useful in determining whether the binary relation is a reflexive relation.

For example: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 2), (2, 2), (1, 4), (2, 4), (3, 2), (4, 3)\}$ be a relation on A

The tabular form of the relation R is given below:

	1	2	3	4
1	✓	✓		✓
2		✓		✓
3		✓		
4			✓	

Fig. 3.2

The check marks in the cells represent the elements (the ordered pairs) of R. If the cells in the main diagonal of table contain check marks then R is reflexive. For example: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 1), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

The tabular form of R is:

	1	2	3	4
1	✓			✓
2		✓		✓
3			✓	
4				✓

Fig. 3.3

The cells in the main diagonal of the table contain check marks. Thus R is a reflexive relation in A.

If the relation R is a symmetrical relation in a set A, then the check marks will be symmetrical with respect to the main diagonal in the table. For example, consider the relation

$R = \{(1, 2), (1, 4), (2, 1), (2, 3), (3, 2), (3, 4), (4, 1), (4, 3), (4, 4)\}$

Defined on the set $A = \{1, 2, 3, 4\}$

The table given below represents R:

	1	2	3	4
1		✓		✓
2	✓		✓	
3		✓		✓
4	✓		✓	✓

Fig. 3.4

The check marks are in cells that are symmetric with respect to the main diagonal. Therefore R is symmetric.

3.13 TRANSITIVE EXTENSION

Definition 3.14: Let R be a relation on the set A. Another relation R_1 defined on A is called the transitive extension of R if R_1 contains R and

$(a, b) \in R,\ (b, c) \in R \Rightarrow (a, c) \in R_1$

Example: Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (2, 3), (3, 2), (2, 4)\}$ and

$R_1 = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

Clearly R_1 contains R and $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R_1$

	1	2	3	4
1		✓		
2			✓	✓
3		✓		
4				

(*a*) Relation R

	1	2	3	4
1		✓	✓	
2		✓	✓	✓
3		✓	✓	✓
4				

(*b*) Relation R_1

Fig. 3.5

R is shown in Fig. 3.5 (*a*) and R_1 is shown in Fig. 3.5 (*b*). Note that the ordered pairs in R_1 which are not in R are marked with heavy check marks.

3.14 TRANSITIVE CLOSURE

Definition 3.15: Let R be a relation on the set A. R_1 denote the transitive extension of R, R_2 denote the transitive extension of R_1 and in general R_{i+1} denote the transitive extension of R_i, then the transitive closure of R is defined as the set union of R, R_1, R_2, … R_i, R_{i+1}…. It is denoted by R^+. Thus

$$R^+ = R \cup R_1 \cup R_2 \cup ... \cup R_i \cup R_{i+1} \cup ....$$

R^+ is the smallest transitive relation containing R.

Example: Let $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$

Then $R^+ = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$ is the transitive closure of R.

Theorem 3.2: Let R be a relation from A to B and let A_1 and A_2 be two subsets of A, then

(*i*) $A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$

(*ii*) $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

(*iii*) $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Proof: (*i*) Let $y \in R(A_1)$

$$y \in R(A_1) \Rightarrow xR\,y \text{ for some } x \in A_1$$

$$\Rightarrow x \in A_2 \qquad [\text{Since } A_1 \subseteq A_2]$$

$$\therefore \quad R(A_1) \subseteq R(A_2)$$

(*ii*) Let $y \in R\left(A_1 \cup A_2\right)$ then xRy for some x in $A_1 \cup A_2$ now $x \in A_1 \cup A_2 \Rightarrow x \in A_1$ or $x \in A_2$

If $x \in A_1$, then $xR\,y \Rightarrow y \in R\left(A_1\right)$ by the same argument; if $x \in A_2$ then $y \in R\left(A_2\right)$ in either case $y \in R\left(A_1\right) \cup R\left(A_2\right)$

$$\therefore R\left(A_1 \cup A_2\right) \subseteq R\left(A_1\right) \cup R\left(A_2\right)$$

Conversely,

$$A_1 \subseteq A_1 \cup A_2 \Rightarrow R\left(A_1\right) \subseteq R\left(A_1 \cup A_2\right) \text{ [by } (i)]$$

similarly $\quad A_2 \subseteq A_1 \cup A_2 \Rightarrow R\left(A_2\right) \subseteq R\left(A_1 \cup A_2\right)$

therefore $\quad R\left(A_1\right) \cup R\left(A_2\right) \subseteq R\left(A_1 \cup A_2\right)$

Thus (*ii*) is true.

(*iii*) Let $y \in R\left(A_1 \cap A_2\right)$ then xRy for some x in $A_1 \cap A_2$ now $x \in A_1 \cap A_2 \Rightarrow x \in A_2$ and $x \in A_2$

$$\Rightarrow y \in R\left(A_1\right) \text{ and } y \in R\left(A_2\right)$$

$$\Rightarrow y \in R\left(A_1\right) \cap \left(A_2\right)$$

Thus $\quad R\left(A_1 \cap A_2\right) \subseteq R\left(A_1\right) \cap R\left(A_2\right)$

Example 1: Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d, e, f\}$ consider the relation

$$R = \{(1, a), (1, c), (2, d), (2, e), (2, f), (3, b)\}$$

Let $A_1 = \{1, 3\}$ and $A_2 = \{2, 3\}$ then we have

$$R\left(A_1\right) = \{a, b, c\}$$

and $\quad R\left(A_2\right) = \{b, d, e, f\}$

Hence $\quad R\left(A_1\right) \cup R\left(A_2\right) = \{a, b, c, d, e, f\}$

and $\quad R\left(A_1\right) \cap R\left(A_2\right) = \{b\}$

now $\quad R\left(A_1 \cup A_2\right) = R(A) = \{a, b, c, d, e, f\}$

$$R\left(A_1\right) \cup R\left(A_2\right)$$

also $\quad R\left(A_1 \cap A_2\right) = R\{3\} = \{b\}$

$\Rightarrow R\left(A_1 \cap A_2\right) \subseteq R\left(A_1\right) \cap R\left(A_2\right)$ holds.

Example 2: Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e, f\}$ consider the relation

$$R = \{(1, a), (1, c), (1, e), (2, b), (2, d), (2, f), (3, c), (3, d), (4, a), (4, f)\}$$

Let $A_1 = \{1, 4\}$ $A_2 = \{1, 3, 4\}$ then $A_1 \subset A_2$

and $\quad R\left(A_1\right) = \{a, c, e, f\}$

$$R\left(A_2\right) = \{a, c, d, e, f\}$$

Clearly $R(A_1) \subseteq R(A_2)$

Thus $A_1 \subseteq A_2 \Rightarrow R(A_1) \subseteq R(A_2)$

3.15 MATRIX REPRESENTATION OF RELATIONS

Suppose A and B are both finite sets and R is a relation from A to B, then R may be represented as a matrix called the relation matrix of R. It is denoted by M_R.

If $A = \{a_1, a_2, \ldots a_m\}$ and $B = \{b_1, b_2, \ldots b_n\}$ are two finite sets containing m and n elements respectively and R is relation from A to B, then the Relation Matrix of R is the $m \times n$, matrix,

$M_R = [m_{ij}]_{m\times n}$ is defined by

$$m_{ij} = \begin{cases} 0 \text{ if } (a_i, b_j) \notin R \\ 1 \text{ if } (a_i, b_j) \in R \end{cases}$$

Where m_{ij} is the element in the ith row and jth column. M_R can be first obtained by first constituting a table, whose columns are preceded by a column consisting of successive elements of A and where rows are headed by a row consisting of successive elements of B. If $(a_i, b_i) \in R$, then we enter 1 in the ith row and jth column and if $(a_k, a_l) \notin R$, then we enter zero in the kth row and ith column.

Example 1: Let $A = \{1, 2, 3\}$ and $R = \{(x, y) \mid x < y\}$, find M_R.

Solution: We have, $R = \{(1, 2), (1, 3), (2, 3)\}$

The table and corresponding relation Matrix for the R are given below

	1	2	3
1	0	1	1
2	0	0	1
3	0	0	0

$$M_R = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Fig. 3.6

Example 2: Let $A = \{1, 4, 5\}$ and $\{(1, 4), (1, 5), (4, 1), (4, 4), (5, 5)\}$, find M_R.

Solution: Given that $R = \{(1, 4), (1, 5), (4, 1), (4, 4), (5, 5)\}$

The relation Matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: If A and B are two finite sets with $|A| = m$; and $|B| = n$, then a $m \times n$ matrix, whose entries are zeros and ones determine a relation from A to B.

If R is symmetric relation on a set A, and M_R denotes the Matrix of relation R, then

$$m_{ij} = 1 \Rightarrow m_{ji} = 1$$

and $$m_{ji} = 0 \Rightarrow m_{ji} = 0 \text{ in } M_R = [m_{ij}]$$

i.e., $M_R = M_R^T$, where M_R^T denotes the transpose of M_R.

If R is an anti-symmetric relation on A, then $m_{ij} = 0$ or $m_{ji} = 0$ for all $i \neq j$ in M_R and if R is a transitive relation on A then

$m_{ij} = 1$ and $m_{jk} = 1 \Rightarrow m_{ik} = 1$ is satisfied by M_R. Moreover, if R is a relation from A to B and S is a relation from B to C, where A, B and C are finite sets m, n and p elements respectively, then $M_R \,.\, M_s$ can be computed. Provided M_R is $m \times n$ matrix and M_s is a $n \times p$ matrix. The Matrices $M_R \,.\, M_S$ and M_{soR} are equal.

Example 3: Let A = {1, 2, 3}

R and S be two relations defined A as follows:

$$R = \{(1,1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2)\}$$

and
$$S = \{(1, 1), (2, 2), (2, 3), (3, 1), (3, 3)\}$$

then
$$SoR = \{(1, 1), (1, 3), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$$

we get

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}; \quad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

and $M_{SOR} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = M_R\, M_S$ can easily be verified

If

$$M_R = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and the relational Matrices of the relation R and S defined on a set $A = \{1, 2, 3,4\}$ for which

We know that

$$M_{SoR} = M_R \,.\, M_s$$

Therefore
$$M_{SoR} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence
$$M_{SoR} = \{(1, 1), (1, 3), (1, 4), (3, 3)\}$$

3.16 RELATIONS AND DIGRAPHS

A relation can be represented pictorially by drawing its graph. Let R be a relation on the set $A = \{a_1, a_2, \ldots a_n\}$. The element a_i of A are represented by points (or circles) called nodes (or vertices). If $(a_i\,, a_j) \in R_j$

then we connect the vertices a_i and a_j by means of an arc and put an arrow in the direction from a_i to a_j. If $(a_i, a_j) \in R$ and $(a_j, a_i) \in R$ then we draw two arcs between a_i and a_j (sometimes by one arc which starts from node a_i and relatives to node x_i (such an arc is called a loop). When all the nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows, we get a graph of the relation R. If R is reflexive, then there must be a loop at each node in the graph of R. If R is symmetric, then $(a_i, a_j) \in R$ implies $(a_j, a_i) \in R$ and the nodes a_i and a_j will be connected by two arcs (edges) one from a_i to a_j and the other from a_j to a_i.

Example 1: Let $A = \{a, b, d\}$ and R be a relation on A given by

$$R = \{(a, b), (a, d), (b, d), (d, a), (d, d)\}$$

Construct the digraph of R.

Solution: The digraph of R is as shown in Fig. 3.7.

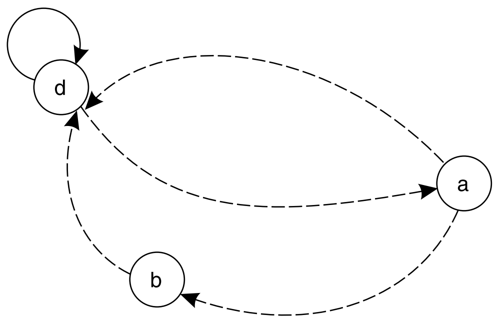


Fig. 3.7

Example 2: Let $A = \{1, 2, 3, 4\}$

$$R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1), (4, 4)\}$$

Construct the digraph of R.

Solution: The digraph of R is shown in Fig. 3.8.

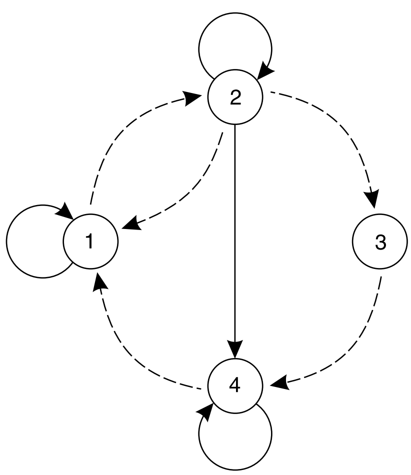


Fig. 3.8

Example 3: Find the relation determined by Fig. 3.9.

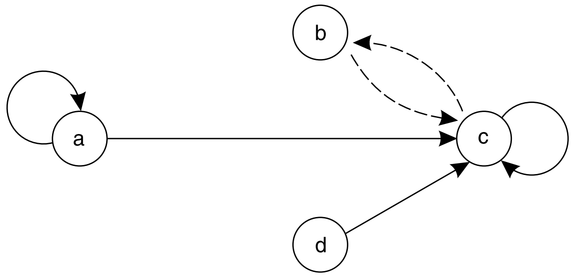


Fig. 3.9

Solution: The relation R of the digraph is

$$R = \{(a, a), (a, c), (b, c), (c, b), (c, c), (d, c)\}$$

Paths in Relation and Digraph

If R is a relation on a set A, a path of length n in R from a_i to a_j is a finite sequence P: a_i, a_1, a_2, … a_{n-1}, a_j, beginning with a_i and ending with a_j such that:

$$a_i \, Ra_1, \; a_1 \, Ra_2, \; \ldots, \; a_{n-1}Ra_j$$

A path in a digraph of the relation R is succession of edges, where the indicated directions of the edges are followed. The length of a path in a digraph is the number of edges in the path. If n is a positive integer then the relation R^n on the set A can be defined as follows:

$(a_i , a_j) \in R^n$ means there is a path of length n from a_i to a_j in R. The relation R^∞ can be defined on A, by letting $(a_i , a_j) \in R^\infty$ means, that there is some path in R from a_i to a_j.

Definition 3.16: A cycle in a digraph is a path of length $n \geq 1$ from a vertex to itself.

Example 1: Let $A = \{1, 2, 3, 4, 5\}$ and

$$R = \{(1, 1), (1, 2), (2, 3), (3, 5), (3, 4), (4, 5)\}$$

Compute (a) R^2 (b) R^∞.

Solution: The digraph of R is shown in Fig. 3.10.

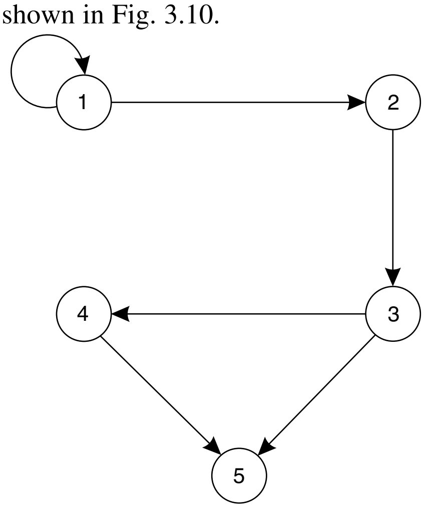


Fig. 3.10

$$(1, 1) \in R \text{ and } (1, 1) \in R \Rightarrow (1, 1) \in R^2$$
$$(1, 1) \in R \text{ and } (1, 2) \in R \Rightarrow (1, 2) \in R^2$$
$$(1, 2) \in R \text{ and } (2, 3) \in R \Rightarrow (1, 3) \in R^2$$
$$(2, 3) \in R \text{ and } (3, 5) \in R \Rightarrow (2, 5) \in R^2$$
$$(2, 3) \in R \text{ and } (3, 4) \in R \Rightarrow (2, 4) \in R^2$$
$$(3, 4) \in R \text{ and } (4, 5) \in R \Rightarrow (3, 5) \in R^2$$

Hence $R^2 = \{(1, 1), (1, 2), (1, 3), (2, 5), (2, 4), (3, 5)\}$

(*b*) There is a path from 1 to $4 \Rightarrow (1, 4) \in R^\infty$, whose length is 3.

There is a path from 1 to 5 $\Rightarrow \in R^\infty$, whose length is 3 and

There is a path from 1 to 5 whose length is 5

Hence $R^\infty = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Example 2: Give an example of a non-empty set and a relation on the set that satisfies each of the following combinations of properties: draw a digraph of the relation:

(1) Symmetric and reflexive but not transitive
(2) Transitive and reflexive; but not anti-symmetric
(3) Anti-symmetric and reflexive, but not transitive.

Solution: (1) Let $A = \{a, b, c\}$ and $R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, a), (c, b)\}$ clearly R is symmetric and reflexive but not transitive. The digraph of R is given below:

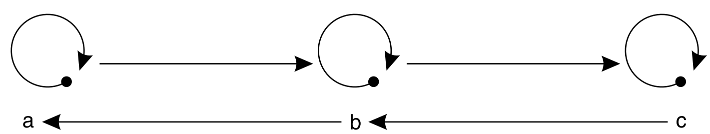


Fig. 3.11

(2) Let $A = \{a, b, c\}$, and

$$R = \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c), (b, a), (c, b), (c, a)\}$$

The relation R is reflexive and transitive but not anti-symmetric.

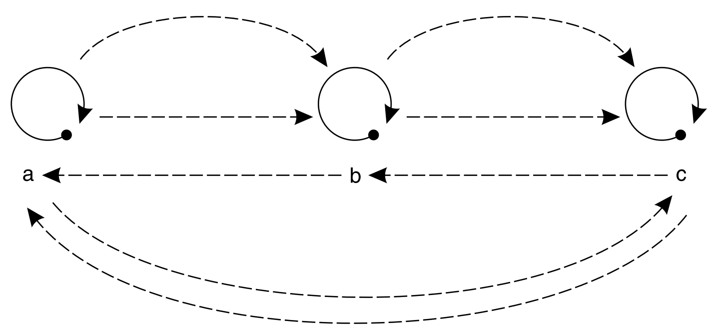


Fig. 3.12

Example 3: Let $A = \{a, b, c\}$ and $R = \{(a, b), (b, b), (c, c), (a, b), (b, c)\}$

The relation on A is symmetric and reflexive but not transitive. Figure 3.13 illustrates the relation.

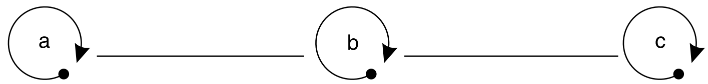


Fig. 3.13

Example 4: For the following digraph which of the special properties are satisfied by digraph's relation?

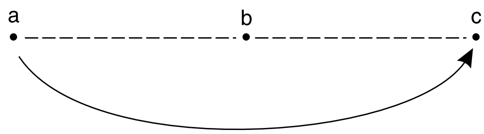


Fig. 3.14

Solution: $R = \{(a, b), (b, c), (b, c), (a, c)\}$

R is transitive and anti-symmetric on $A = \{a, b, c\}$.

3.17 COMPOSITION OF RELATIONS

Definition 3.17: Let R be a relation from A to B and S be a relation from B to C. Then we can define a relation, the composition of R and S written as SoR. The relations SoR is a relation from the set A to the set C and is defined as follows:

If $a \in A$, and $c \in A$, then $(a, c) \in S o R$ if and only if for some $b \in B$, we have $(a, b) \in R$ and $(b, c) \in S$.

Example: Let $A = \{1, 2, 3, 4\}$ and R, S be two relations on A defined by

$$R = \{(1, 2), (1, 3), (2, 4), (3, 2)\};$$

$$S = \{(1, 4), (4, 3), (2, 3), (3, 1)\} \text{ find } SoR.$$

Solution:

$$(1, 2) \in R \text{ and } (2, 3) \in S \Rightarrow (1, 3) \in S o R$$

$$(1, 3) \in R \text{ and } (3, 1) \in S \Rightarrow (1, 3) \in S o R$$

$$(3, 2) \in R \text{ and } (2, 3) \in S \Rightarrow (3, 3) \in S o R$$

$$(2, 4) \in R \text{ and } (4, 3) \in S \Rightarrow (2, 3) \in S o R$$

Thus $$S oR = \{(1, 3), (1, 1), (3, 3), (2, 3)\}$$

Theorem 3.3: If R is relation from A to B, S is a relation from B to C and T is a relation from C to D. Then

$$To\,(SoR) = (ToS)\,oR$$

Proof: Let M_R, M_S, M_T denote the Matrices related to relations R, S and T respectively, then

$$M_{To\,(SoR)} = M_{SoR} \cdot M_T$$
$$= (M_R \cdot M_S) \cdot M_T = (M_R \cdot M_S) \cdot M_T$$
(Q Multiplication of matrices is associative)
$$= M_R \cdot (M_{ToS}) = M_{(ToS)\,oR}$$
$$\Rightarrow To\,(SoR) = (ToS)\,oR$$

Theorem 3.4: Let R be a relation from the set A to the set B and S be a relation from the set B to set C, then

$$(SoR)^{-1} = R^{-1}\,oS^{-1}$$

Proof: Let $(c, a) \in (SoR)^{-1}$ for some $c \in C$ and $a \in A$. Then

$$(c, a) \in (SoR)^{-1} \text{ if } (a, c) \in SoR$$

$\therefore$ There is an element $b \in B$ with $(a, b) \in R$ and $(b, c) \in S$ now $(a, b) \in R$ and $(b, c) \in S$

$$\Rightarrow (b, a) \in R^{-1} \text{ and } (c, b) \in S^{-1}$$
$$\Rightarrow (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1}$$
$$\Rightarrow (c, a) \in R^{-1}\ S^{-1}$$

thus $(SoR)^{-1} = R^{-1}\ S^{-1}$

Example: Let $A = \{a, b\}$

$$R = \{(a, a), (b, a), (b, b)\} \text{ and } S = \{(a, b), (b, a), (b, b)\}$$

Then, verify $(SoR)^{-1} = R^{-1}\,oS^{-1}$

Solution:

$$S\,oR = \{(a, b), (b, a), (b, b)\}$$

$\Rightarrow$ $$(S\,oR)^{-1} = \{(b, a), (a, b), (b, b)\}$$

and $$R^{-1} = \{(a, a), (a, b), (b, b)\},\ S^{-1} = \{(a, a), (a, b), (b, b)\}$$

$\Rightarrow$ $$R^{-1}\,oS^{-1} = \{(b, a), (a, b), (b, b)\} = (SoR)^{-1}$$

EXERCISE 3.1

1. Let $A = \{1, 2, 3, 4\}$, determine whether the relations are reflexive, symmetric, anti-symmetric or transitive.

1. $R = \varnothing$
2. $R = \{(1, 1), (2, 2), (3, 3)\}$
3. $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$
4. $R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
5. $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4)\}$
6. $R = \{(1, 3), (4, 2), (2, 4), (3, 1), (2, 2)\}$
7. $R = A \times A$
8. $R = \{(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)\}$

2. Write down the relations in the square of the set $\{1,2, 4, 8, 16, 32, 64\}$.

3. The following relations in N, the set of natural numbers. Give their domains and ranges.
   (*i*) {(1, 1), (16, 2), (81, 3), (256,}
   (*ii*) {(2, 1), (4, 2), (10, 5), (18, 9), (20, 10)}

4. Determine the domain and range of relation R, on set of Integers
$R = \{(x, y) \mid x$ is a multiple of 3 and y is a multiple of $5\}$.

5. Tabulate the element of the following relations from A to B:
   (*a*) $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and
   $R = \{(x, y) \mid y = x^2 + 3x + 3\}$
   (*b*) $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and
   $R = \{(x, y) \mid 5x + 2\,y$ is a prime number$\}$

6. Let f on a mapping of a set X onto a set R. Then if we define $(a, b) \in R$, for $a, b \in X$ provided $f(a) = f\ (b)$. Prove that R is an equivalence relation.

7. Determine whether the relation
$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (3, 1), (1, 4), (4, 1)\}$ is an equivalence relation in $\{1, 2, 3, 4\}$

8. Let R be the relation in the natural numbers $N = \{1, 2, 3, \ldots\}$
Define by "$x + 2y = 10$" i.e., let $R = \{(x,\ y) \mid x \in N,\ y \in N,\ x + 2y = 10\}$.
Find (*a*) The domain and range of R (*b*) R^{-1}

9. Let $A = \{1, 2, 3, 4, 5, 6\}$, construct pictorial descriptions of the relation R on A for the following as:
   (*a*) $R = \{(j, k) \mid j$ is a multiple of $k\}$
   (*b*) $R = \{(j, k) \mid (j - k)^2 \in A\}$
   (*c*) $R = \{(j, k) \mid (j$ divides $k\}$
   (*d*) $R = \{(j, k) \mid j\,k$ is a prime

10. Let R be the relation from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 3, 5\}$ which is defined by "x is less than y", write R as a set of ordered pairs:

11. Let L be the set of lines in the Euclidean plane and let R be the relation in L defined by "x is parallel to y". Is R a symmetric relation? Why? Is R a transitive relation?

12. Prove that if R is a symmetric relation, then $R \cap R^{-1} = R$.

13. Let $A = \{1, 2, 3\}$. Give an example of a relation R in A. Such that R is neither symmetric nor anti-symmetric.

14. If A is a set with the element and B is a set with a elements. Then find the number of relations possible firm A to B.

15. In $N \times N$ show that the relation defined by $(a, b)\ R\ (c, d)$ if and only if $ad = bc$ is an equivalence relation.

16. On the set of Natural numbers N, the relation R is defined "aRb" iff "a divides b". Show that R is anti-symmetric.

17. On the set of Integers, the relation R is defined by "aRb" iff "$(a - b)$ is even integer". Show that R is an equivalence relation.

18. Give an example of a non-empty set and a relation on the set that satisfies each of the following properties; draw a digraph of the relation.
   (*a*) Reflexive (*b*) irreflexive (*c*) an anti-symmetric relation

19. Let $A = \{1, 2, 3\}$ determine whether the relation R whose matrix M_R is given is an equivalence relation:

$$(a)\quad M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad (b)\quad M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

20. Determine whether the relation whose digraph is given below (Fig. 3.15) is an equivalence relation.

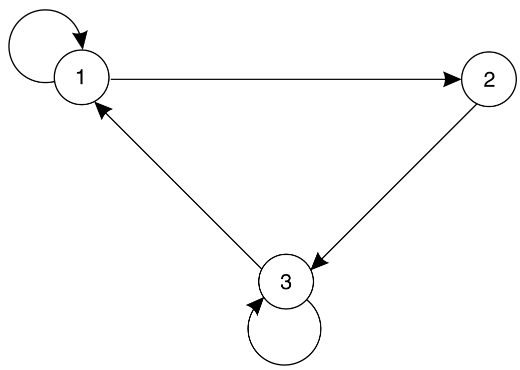


Fig. 3.15

21. Let $A = \{1, 2, 3\}$ and
$R = \{(1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3)\}$
Write the Matrix of R and sketch its graph.

22. Let $R = \{(1, 2), (3, 4), (2, 2)\}$ and $S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$
Find $R\ oS$, $S\ oR$, $(S\ oR)$, $(R\ oS)\ oR$, $R\ oR$, $S\ oS$, and $(R\ oR)\ oR$.

23. Let R and S be two relations on a set of positive integers.
$= \{(x, 2x) \mid x \in I\}$, $S = \{(x, 7x) \mid x \in 1\}$
Find $R\ oS$, $R\ oR$, $R\ oR\ oR$ and $R\ oS\ oR$.

24. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and
$R = \{(x, y) \mid x - y \text{ is divisible by } 3\}$
Show that R is an equivalence relation. Draw the graph of R.

25. If $A = \{1, 2, 3, 4\}$ and $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$
Show that the transitive closure $R\infty$ is
$\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4)\}$

26. $A = \{a, b, c\}$, and R, S are relations on A whose matrices are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad M_S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Show that $M_{SoR} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

27. Find all the partitions of $A = \{1, 2, 3\}$

28. Find the numbers of partitions on $A = \{a, b, c, d\}$

29. Let $N = \{1, 2, 3, \ldots\}$ and a relation R is defined in $N \times N$ as follows (a, b) is related to (c, d) if and only if

$$a + b = b + c$$

then show that R is an equivalence relation.

30. If a finite set A has n elements. Prove the following:

(*a*) There are $2^{n^2 - n}$ reflexive relations on A

(*b*) There are $2^{n^2 - n}$ irreflexive relations on A

(*c*) There are $2^{(n^2 + n)/2}$ symmetric relations on A

(*d*) There are $2^{(n^2 - n)/2}$ compatibility relations on A

(*e*) There are $2^n \cdot 3^{(n^2 - n)/2}$ anti-symmetric relations on A

Answers:

1. (1) Symmetric (2) Symmetric and transitive (3) Transitive (4) Transitive (5) Equivalence relation (6) Symmetric (7) Equivalence relation (8) Transitive.

2. $R = \{(1, 1), (4, 2), (16, 4), (64, 8)\}$

3. (*i*) Domain = {1, 16, 81, 256}

Range = {1, 2, 3, 4}

(*ii*) Domain = {2, 4, 10, 18, 20}

Range = {1, 2, 5, 9, 10}

4. Domain = $\{ x \in Z / x$ is multiple of 3}

= {… –12, –9, –6, –3, 0, 3, 6, 9, …}

Range = $y \in Z / y$ is a multiple of 5}

= {…, –15, –10, –5, –0, –5, 10, 15, …}

5. (*a*) {(1, 1), (2, 1), (3, 3), (4, 7)}

(*b*) {(1, 1), (1, 3), (1, 4), (3, 1), (3, 2), (3, 4)}

6. Yes, equivalence relation.

7. $R = \{(8, 1), (6, 2), (4, 3), (2, 9)\}$

$R^{-1} = \{(1, 8), (2, 6), (3, 4), (4, 2)\}$

9. $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

10. Symmetric and transitive; since:

(*i*) x is parallel to y P y is parallel to y

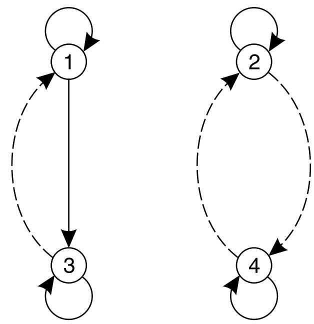
(*ii*) if x is parallel to y and y is parallel to z then x is parallel to z.

17. (*a*) $R = \{(x, x), (y, y), (z, z)\ (z, y)\}$

19. (*a*) Yes (*b*) No

20. No.

21. $$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$



22. $R\,oS = \{(4, 2), (3, 2), (1, 4)\}$
$S\,oR = \{(1, 5), (3, 2), (2, 5)\}$
$(R\,oS)\,oR = \{(3, 2)\}$
$R\,oR = \{(1, 2)\}$;
$S\,oS = \{(4, 5), (3, 3)\}$
$(R\,oR)\,o\,R = \varnothing$

23. $R\,oS = \{(x, 14x) \mid x \in I\} = S\,oR$

$R\,oR = \{(x, 4x) \mid x \in I\}$

$R\,oR\,oR = \{(x, 8x) \mid x \in I\}$

$R\,oS\,oR = \{(x, 2x) \mid x \in I\}$

27. The different partitions of A are
{{1, 2, 3}}, {{1}, {2, 33}}, {{2}, {1, 33}}, {{3}, {1, 3}}, {{13, 52}, {3}}

28. Number of different partitions on A is 15.

4

Functions and Recurrence Relations

4.1 INTRODUCTION

The concept of relation was defined very generally in the preceding chapter. We shall now discuss a particular class of relations called Functions. They are widely used in Mathematics and the concept is basic to the idea of computation.

4.2 FUNCTION

Definition 4.1: Let A and B be any two sets. A relation f from A to B is called function if for every $a \in A$ there is a unique element $b \in B$, such that $(a, b) \in f$.

If f is a function from A to B, then f is a function from A to B such that

(*i*) Domain $f = A$

(*ii*) Whenever $(a, b) \in f$ and $(a, c) \in f$, then $b = c$

The notation $f: A \to B$, means f is a function from A to B.

Functions are also called Mappings or Transformations. The terms such as "correspondence" and "operation" are used as synonyms for "function".

Given any function $f: A \to B$, the notation $f(a) = b$ means $(a, b) \in f$. It is customary to write $b = f(a)$. The element $a \in A$ is called an argument of the function f, and $f(a)$ is called the value of the function for the argument a or the image of a under f.

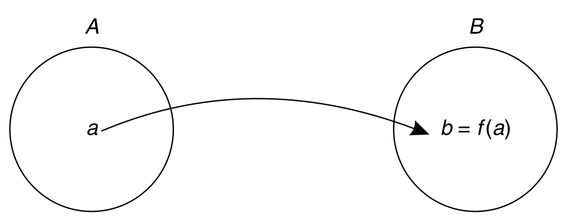


Fig. 4.1 Representation of a function

Example 1: Let $A = \{1, 2, 3\}$, $B = \{p, q, r\}$ and

$f = \{(1, p), (2, q), (3, r)\}$. Then $f(1) = p$, $f(2) = q$, $f(3) = r$, clearly f is a function from A to B.

Example 2: Consider the sets $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$.

Let $f = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ every element of A is related to exactly one element of B. Hence f is function (*see* Fig. 4.2).

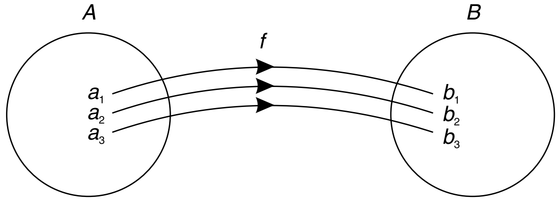


Fig. 4.2

If f: $A \to B$ is a function, then A is called the Domain of f and the set B is called the codomain of f. The range of f is defined as the set of all images under f.

It is denoted by $f(A) = \{b \mid \text{for some } a \text{ in } A, f(a) = b\}$ and is called the image of A in B. The Range f is also denoted by R_f.

If D_f denotes the domain of f: $A \to B$, and R_f denotes the Range of f, then $D_f = A$ and $R_f \subseteq B$.

A function need not be defined by a formula. While defining the property, it is customary to identify the function by a formula for example $f(x) = x^3$ for $x \in R$ represents the function $f = \{(x, x^3)\text{: } x \in R\}$. Where R is the set of real numbers.

4.2.1 Restriction and Extension

Definition 4.2: If f: $A \to B$ and $P \subseteq A$, then $f \cap (P \times B)$ is a function from $P \to B$, called the Restriction of f to P. Restriction of f to P is written as $f \mid P$: $P \to B$ is such that $(f \mid P) = f(a)\ \forall\ a \in P$. If g is a restriction of f, then f is called the extension of g.

The domain of $f \mid P$ is P.

If g is a restriction of f, then $D_g\ \hat{I}\ D_f$ and $g(a) = f(a)\ a\ \hat{I}\ D_g$ and $g\ \hat{I}\ f$.

Example: Let $f : R \to R$, be defined by $f(x) = x^3$.

If N is the set of Natural numbers = $\{0, 1, 2,\ldots\}$ then $N \subseteq R$ and $f \mid N = \{(0, 0), (1, 1), (2, 2)\ldots\}$

4.3 ONE-TO-ONE MAPPING (INJECTION ONE-TO-ONE FUNCTION)

Definition 4.3: A mapping f: $A \to B$ is called one-to-one mapping if distinct elements of A are mapped into distinct elements of B,

i.e., f is one-to-one if

$$a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$$

or equivalently $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Example: f: $R \to R$ defined by $f(x) = 3x\ \forall\ x \in R$ is one–one since

$$f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in R.$$

4.4 ONTO-MAPPING (SURJECTION)

Definition 4.4: A mapping $f: A \to B$ is called onto-mapping if the range set $R_f = B$.

If $f: A \to B$ is onto, then each element of B is f-image of atleast one element of A.

i.e., $\{f(a) : a \in A\} = B$

If f is not onto, then it is said to be into mapping.

Example: $f: R \to R$, given by $f(x) = 2x \ \forall\ x \in R$ is onto.

4.5 BIJECTION (ONE-TO-ONE, ONTO)

Definition 4.5: A mapping $f: A \to B$ is called one-to-one, onto if it is both one-to-one and onto.

Example: $f: R \to R$, defined by $f(x) = 3x + 2$ is a bijection.

4.6 IDENTITY MAPPING

Definition 4.6: If $f: A \to A$ is a function such that every element of A is mapped onto itself then f is called an Identity mapping it is denoted by I_A.

i.e., $f(a) = a \ \forall\ a \in A$ then $f: A \to A$ is an Identity mapping.

We have $$I_A = \{(a, a) : a \in A\}$$

4.7 COMPOSITION OF FUNCTIONS

Definition 4.7: Let $f: A \to B$, and $g: B \to C$ be two mappings. Then the composition of two mappings f and g denoted by gof is the mapping from A into C defined by $gof = \{(a, c) \mid$ for some b, $(a, b) \in f$ some $(b, c) \in g\}$.

i.e., $gof: A \to C$ is a mapping defined by

$$(gof)(a) = g(f(a)) \text{ where } a \in A$$

Note: In the above definition it is assumed that the range of the function f is a subset of B (the Domain of g), i.e., $R_f \subseteq D_g$ If $R_f \subseteq D_g$, then gof is empty.

(*i*) The composition of functions is not commutative, i.e., $fog \neq gof$ where f and g are two functions.

(*ii*) gof is called the left composition g with f.

Example: Let $f: R \to R$; $g : R \to R$ be defined by $f(x) = x + 1$, $g(x) = 2x^2 + 3$, then

$$(gof)(x) = g[f(x)] = g[(x + 1)] = 2(x + 1)^2 + 3$$
$$(fog)(x) = f[g(x)] = f(2x^2 + 3) = 2x^2 + 3 + 1 = 2x^2 + 4$$

gof and fog are both defined but $gof \neq fog$

Theorem 4.1: Let $f: A \to B$, then $g: B \to C$ be both one-one and onto functions, then $gof: A \to C$ is also one-one and onto.

Proof:

Let $a_1,\ a_2 \in A$, then

$$(gof)\,(a_1) = (gof)\,(a_2) \Rightarrow g\,[f\,(a_1)] = g\,[f\,(a_2)]$$

$$\Rightarrow f\,(a_1) = f\,(a_2) \qquad (\because\ g \text{ is one-one})$$

$$\Rightarrow a_1 = a_2 \qquad (\because\ f \text{ is one-one})$$

Hence *gof* is one-to-one

Now, from the definition, *gof*: $A \to C$ is a function g: $B \to C$ is onto, then $c \in C \Rightarrow$ There is some element $b \in B$ such that $\Rightarrow$ $c = g\,(b)$ and f: $A \to B$ is onto, then by definition there exists an element $a \in A$, such that $f\,(a) = b$

We have $$c = g\,(b) = g\,[f\,(a)] = (gof)\,(a)$$

$$\Rightarrow (gof): A \to C \text{ is onto}$$

Hence *gof* is both one-one and onto.

4.8 ASSOCIATIVITY OF MAPPINGS

Definition 4.8: If f: $A \to B$, g: $B \to C$ and h: $C \to D$ are three functions, then *gof*: $A \to C$, *hog*: $C \to D$ and (*hog*) *of*: $A \to D$, can also be formed assuming that $a \in A$, we have

$$(hog)\ of\,(a) = (hog)\,[f\,(a)]$$

$$= h\,[gf\,(a)]$$

$$= h[gof\,(a)] = ho(gof)\,(a)$$

Thus the composition of functions is associative.

4.9 CONSTANT FUNCTION

Definition 4.9: Let f: $A \to B$, f said to be a constant function if every element of A is mapped on to the same element of B.

i.e., If the Range of f has only one element then f is called a constant mapping.

Example: f: $R \to R$, defined by

$f(x) = 5\ \forall\ x \in R$ is a constant mapping we have $R_f = \{5\}$

4.10 INVERSE MAPPING

Definition 4.10: Let f: $A \to B$, be one-one, onto mapping (bijection), then f^{-1}: $B \to A$ is called the inverse mapping of f.

f^{-1} is the set defined as

$$f^{-1} = \{(b,\ a) \mid (a,\ b) \in f\}$$

Note:

(i) In general the inverse f^{-1} of a function $f: A \to B$, need not be a function. It may be a relation.

(ii) If $f: A \to B$ is a bijection and $f(a) = b$, then $a = f^{-1}(b)$ where $a \in A$, and $b \in B$

Example:

(i) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $f = \{(a, 1), (b, 3), (c, 2)\}$ clearly f is both one-to-one and onto

$\therefore f^{-1} = \{(1, a), (2, c), (3, b)\}$ is a function from B to A.

(ii) Let R be a set of real numbers and $f: R \to R$ be given by

$$f(x) = x + 5 \ \forall \ x \in R \text{ , i.e., } f = \{(x, x + 5) \mid x \in R\}$$

then $f^{-1} = \{(x + 5, x) \mid x \in R\}$ is a function from R to R.

Theorem 4.2: If $f: A \to B$ be both one-one and onto, then $f^{-1}: B \to A$ is both one-one and onto.

Proof: Let $f: A \to B$ be both one-one and onto. Then there exist elements $a_1, a_2 \in A$, and elements $b_1, b_2 \in B$ such that

$$f(a_1) = b, \text{ and } f(a_2) = b_2$$

or

$$a_1 = f^{-1}(b_1) = \text{ and } a_2 = f^{-1}(b_2)$$

Now, let $f^{-1}(b_2) = f^{-1}(b_2)$ then

$$f^{-1}(b_1) = f^{-1}(b_2)$$
$$\Rightarrow a_1 = a_2$$
$$\Rightarrow f(a_1) = f(a_2)$$
$$\Rightarrow b_1 = b_2$$

Thus f^{-1} is one-one

Again since f is onto, for $b \in B$, there is some element $a \in A$, such that $f(a) = b$.

Now
$$f(a) = b$$
$$\Rightarrow a = f^{-1}(b)$$

$\Rightarrow f^{-1}$ is onto

Hence f^{-1} is both one-one and onto

Theorem 4.3: The inverse of an invertible mapping is unique.

Proof: Let $f: A \to B$

By any invertible mapping. If possible let

$$g: B \to A$$

and

$$h: B \to A$$

be two different inverse mappings of f.

Let $b \in B$ and

$$g(b) = a_1, a_1 \in A$$

$$h(b) = a_2, a_2 \in A$$

Now $g(b) = a_1, \Rightarrow b = f(a_1)$

and $h(b) = a_2 \Rightarrow b = f(a_2)$

Furthermore $b = f(a_1)$ and $b = f(a_2)$

$$\Rightarrow f(a_1) = f(a_2)$$

$$\Rightarrow a_1 = a_2 \qquad (\because f \text{ is one-one})$$

This proves that $g(b) = h(b) \ \forall \ b \in B$

Thus, the inverse of f is unique

This completes the proof of the theorem.

Theoem 4.4: If $f: A \rightarrow B$ is an invertible mapping, then $f \, o f^{-1} = I_B$ and $f^{-1} \, of = I_A$.

Proof: f is invertible, then f^{-1} is defined by $f(a) = b \Leftrightarrow f^{-1}(b) = a$ where $a \in A$ and $b \in B$

To prove that $f \, of^{-1} = I_B$

Let $b \in B$ and $f^{-1}(b) = a, a \in A$ then

$$f \, of^{-1}(b) = f[f^{-1}(b)]$$
$$= f(a) = b$$

Therefore $fof^{-1}(b) = b \ \forall \ b \in B$

$$\Rightarrow f \, of^{-1} = I_B$$

and $f^{-1} \, of(a) = f^{-1}[f(a)] = f^{-1}(b) = a$

Therefore $f^{-1} \, of(a) = a \ \forall \ a \in A$

$$\Rightarrow f^{-1} \, of = I_A$$

Theorem 4.5: If $f: A \rightarrow B$ is invertible then $f^{-1} \, of = I_A$, and $f \, of^{-1} = I_B$

Proof: Left as an exercise.

Theorem 4.6: Let $f: A \rightarrow B$, and $g: B \rightarrow C$. The function $g = f^{-1}$, only if $gof = I_A$, and $fog = I_B$.

Proof: Left as an exercise.

Theorem 4.7: If $f: A \rightarrow B$, and $g: B \rightarrow C$, are both one-one and onto, then $(gof)^{-1} = f^{-1} \, og^{-1}$.

Proof:

$f: A \rightarrow B$ is one-one and onto

$g: B \rightarrow C$ is one-one and onto,

Hence $gof: A \rightarrow C$ is one-one and onto

$\Rightarrow (gof)^{-1} : C \rightarrow A$ is one-one and onto

Let $a \in A$, then there exists and element $b \in B$ such that $f(a) = b \Rightarrow a = f^{-1}(b)$.

Now, $b \in B \Rightarrow$ there exists an element $c \in C$ such that $g(b) = c \Rightarrow b = g^{-1}(c)$

Then $(gof)(a) = g[f(a)] = g(b) = c \Rightarrow a = (gof)^{-1}(c)$ … (1)

$(f^{-1} \, og^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a \Rightarrow a = (f^{-1} \, og^{-1})(c)$ … (2)

Combining (1) and (2), we have

$$(gof)^{-1} = f^{-1} \, og^{-1}$$

4.11 CHARACTERISTIC FUNCTION OF A SET

Definition 4.11: Let U be a universal set and A be a subset of U. Then the function

$$\psi_A : U \to [0, 1] \text{ defined by}$$

$$\psi_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

is called a characteristic function of the set A.

4.11.1 Properties of Characteristic Functions

Let A and B be any two subsets of a universal set U. Then the following properties hold for all $x \in U$:

(1) $\psi_A(x) = 0 \Leftrightarrow A = \varnothing$

(2) $\psi_A(x) = 1 \Leftrightarrow A = U$

(3) $\psi_A(x) \subseteq \psi_B(x) = \Leftrightarrow A \subseteq B$

(4) $\psi_A(x) = \psi_B(x) \Leftrightarrow A = B$

(5) $\psi_{A \cap B}(x) = \psi_A(x) \, . \, \psi_B(x)$

(6) $\psi_{A \cup B}(x) = \psi_A(x) + \psi_B(x) - \psi_{A \cap B}(x)$

(7) $\psi_A(x) = 1 - \psi_A(x)$

(8) $\psi_{A-B}(x) = \psi_A(x) - \psi_{A \cap B}(x)$

The operations I, =, +, . and –, used above are the usual arithmetical operations.

The values of characteristic functions are always either 1 or 0. The properties (1) to (8) can easily be proved using the definition of characteristic functions.

Set identities can also be proved by using the properties characteristic functions.

Example: Show that $\overline{\overline{(A)}} = A$

Solution:

$$\begin{aligned} \psi\left(\overline{\overline{A}}\right)(x) &= 1 - \psi\left(\overline{A}\right)(x) \\ &= 1 - \left(1 - \psi_A(x)\right) \\ &= \psi_A(x) \Rightarrow \left(\overline{\overline{A}}\right) = A \end{aligned}$$

4.12 SOLVED EXAMPLES

Example 1: On which sets A will identity function $I_A : A \to A$ be (*i*) one-one (*ii*) an onto function.

Solution: A can be any set

(*i*) The identity function is always one-one and

(*ii*) The identity function is always onto.

Example 2: Can a constant function be (*i*) one-one (*ii*) onto.

Solution: If f is a constant function, the co-domain of f consists of single element. Therefore

(i) a constant function is one-one if the domain of f contains a single element.

(ii) a constant function is always onto.

Example 3: $f: R \to R$ is defined by $f(x) = a\,x + b$, where $a, b, x \in R$ and $a \neq 0$.

Show that f is invertible and find the inverse of f.

Solution: Firstly, we shall show that f is one-to-one.

Let $x_1, x_2 \in R$ such that $f(x_1) = f(x_2)$

$$\Rightarrow a\,x_1 + b = a\,x_2 + b$$

$$\Rightarrow ax_1 = ax_2$$

$$\Rightarrow x_1 = x_2$$

Thus f is one-one

To show that f is onto

Let $y \in R$ such that $y = f(x)$

$$\Rightarrow y = a\,x + b$$

$$\Rightarrow a\,x = y - b$$

i.e., given $y \in R$, there exists an element

$$x = \frac{1}{a}(y - b) \in R, \text{ such that } f(x) = y$$

this proves that f is onto

Hence f is one-one and onto

Hence f is invertible and

$$f^{-1}_{(y)} = \frac{1}{a}(y - b)$$

Example 4: Let $U = \{a, b, c, d, e, f\}$ and $A = \{a, d, e\}$ then find X_A, where X_A denotes the characteristic function of A.

Solution:

$$X_A = \{(a, 1), (b, 0), (c, 0), (d, 1), (e, 1), (f, 0)\}$$

Since $a \in A \Rightarrow X_A(a) = 1$, $d \in A \Rightarrow X_A(d) = 1$, and $e \in A \Rightarrow X_A(e) = 1$, b, c, f are not the members of A,

Thus $X_A(b) = X_A(c) = X_A(f) = 0$

Example 5: Let A and B be subsets of a universal set U then prove $X_{A \cap B} = X_A\, X_B$.

Solution:

$$\Rightarrow X_A(a) = X_B(a) = 1$$

$$\Rightarrow X_A(a)\, X_B(a) = 1$$

Let $b \in (A \cap B)'$, then $b \notin A \cap B \Rightarrow X_A \cap X_B (b) = 0$

Now $$b \in (A \cap B)' \Rightarrow X_A \cap X_B (b) = 0$$

also $$b \in (A \in B)' \Rightarrow b \in (A' \cup B')$$
$$\Rightarrow b \in A' \text{ or } b \in B'$$
$$\Rightarrow X_A (b) = 0 \text{ or } X_B (b) = 0$$
$$\Rightarrow (X_A X_B)(b) = X_A (b) X_B (b) = 0$$

Hence by definition

$$X_{A \cap B} = X_A X_B.$$

EXERCISE 4.1

1. Define the terms:
   (*i*) Function
   (*ii*) One-one function
   (*iii*) Onto function
   (*iv*) Identify function
2. Let $f: N \to N, f(x) = 2x + 3$, N being the set of natural numbers. Prove that f is injection but not surjection.
3. Show that the function f from the reals into the reals defined by $f(x) = x^2 + 1$ is one-to-one, onto function and find f^{-1}.
4. A binary operation b on a set A is said to be associative if $b(x, b(y, z)) = b(b(x, y), z)$ for all $x;\ y,\ z \in A$, which of the operations are associative?
   (*a*) $b(x, y) = x - y$
   (*b*) $b(x, y) = x^2 + y^2$
   (*c*) $b(x, y) = max\ \{x, y\}$, where $max\ \{x, y\}$ denotes the larger of the two numbers x and y. (For example $max\ \{5, 7\} = 7$, $max\ \{-6, 2\} = 2$)
5. If $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$, determine if the following functions are one-to-one or onto.
   (*a*) $f = \{(1, a), (2, a), (3, b), (4, d)\}$
   (*b*) $g = \{(1, e), (2, b), (3, a), (4, a)\}$
   (*c*) $h = \{(1, d), (2, b), (3, a), (4, c)\}$
6. $A = \{-1, 0, 2, 5, 6, 11\ f : A \to B$ is a function given by $f(x) = x^2 - x - 2$ for all $x \in A$. Find the range of f. If f an onto function.
7. Let R_0 be the set of all non-zero real numbers, show that f: $R_0 \to R_0$ defined by $f(x) = \dfrac{1}{x}$ for all $x \in R_0$ is one-to-one and onto for all $x \in R_0$ is one-to-one and onto.
8. Let f: $S \to T$ be a function; Let A and B be subsets of S, and D and E be subsets of T. Prove the following theorems:

(*i*) $A \subset B$, then $f(A) \subset f(B)$

(*ii*) $D \subset E$, then $f^{-1}(D) \subset f^{-1}(E)$

(*iii*) $f(A \cap B) \subset f(A) \cap f(B)$

9. $A = \{1, 2, 3\}$, $B = \{p, q\}$, $C = \{a, b$, $f: A \to B$ and $g: B \to C$ are given by $f = \{(1, p), (2, q), (3, q)\}$, $g = \{(p, b), (q, b)\}$ show that: $gof = \{(1, b), (2, b), (3, b)\}$.

10. $X = \{1, 2, 3\}$ and f, g and h are function from X to X given by $f = \{(1, 2), (2, 3), (3, 1)\}$, $g = \{(1, 2), (2, 1), (3, 3)\}$, $h = \{(1, 1), (2, 2), (3, 1)\}$ find $f\,og$, $g\,of$, $f\,og\,of$, $h\,of$, $h\,oh$, and $g\,oh$.

11. R is the set of real numbers, given that $f(x) = x + 2$, $g(x) = x - 2$, and $h(x) = 3x \ \forall \ x \in R$.

Find $g\,of$, $f\,og$, $f\,of$, $g\,og$, $f\,oh$, $h\,og$, $h\,of$, and $f\,ogoh$

12. Let $A = \{1, 2, 3 \ldots, n\}$, $\{n \geq 2)$, and $B = \{a, b\}$ find the number of subjections from A onto B.

13. Let $f: R \to R$ be defined by $f(x) = 3x + 4$, show that f is one-one and onto. Give a formula that defines f^{-1}.

14. Let $A = R - \{3\}$, and let $f: A \to B$ be defined by $f(x) = \dfrac{x-2}{x-3}$, show that f is one-to-one and onto. Find f^1.

15. Let the functions $f: R \to R$ and $g: R \to R$ be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$. Find formulas which define the functions $g\,of$ and $f\,og$.

16. Let $U = \{a, b, c, d, e\}$, $A = \{c, d\}$ and $B = \{a, d, e\}$, find (1) X_A (2) X_B.

17. Show that the functions $f: N \times N \to N$ and $g: N \times N \to N$ given by $f(x, y) = x + y$ and $g(x, y) = xy$ are onto but not one-to-one. ***(OU Mar. 2002)***

18. If x and y are finite sets, find a necessary condition for the existence of one-to-one mapping from x to y. ***(OU Mar. 2001)***

19. Let $A = B = R$, the set of real number. Let $f: A \to B$ be given by the formula $f(x) = 2x^2 - 1$ and let $g: B \to A$ be given by $g(y) = 3\sqrt{\frac{1}{2}y + \frac{1}{2}}$. Show that f is a bijection between A and B and g is a bijection between B and A. ***(OU Dec. 2000)***

Answers:

3. $f^{-1}(Y) = (Y - 1)^{1/2}$.

4. (*c*)

5. (*c*)

6. $f(A) = B = \{-2, 0, 18, 28, 108\}$

10. $f\,og = \{(1, 3), (2, 2), (3, 1)\}$

$g\,of = \{(1, 1), (2, 3), (3, 2)\}$

$f\,og\,of = \{(1, 2), (2, 1), (3, 3)\}$

$h\,of = \{(1, 2), (2, 1), (3, 1)\}$

$h\,oh = \{(1, 1), (2, 2), (3, 1)\}$

$g\,oh = \{(1, 2), (2, 3), (3, 2)\}$

11. $g\ of\ (x) = x$

$f\ og\ (x) = kx$

$f\ of\ (x) = x + 4$

$go\ g\ (x) = x - 4$

$f\ oh\ (x) = 3x + 2$

$h\ og\ (x) = 3x - 6$

$h\ of\ (x) = 3x + 6$

$f\ ogoh\ (x) = 3x$

12. $2^n - 2.$

13. $f^1\ (y) = \dfrac{y-4}{3}$

14. $f^1\ (y) = \dfrac{2-3y}{1-4}$

15. $g\ of\ (x) = 4x^2 + 4x - 1, f\ og = 2x^2 - 2$

16. $X_A = \{(a, 0), (b, 0), (c, 1), (d, 1), (e, 0)\}$

$X_B = \{(a, 1), (b, 0), (c, 0), (d, 1), (e, 1)\}$

4.13 RECURSION AND RECURRENCE RELATIONS

Recursion is a technique of defining a function, a set or an algorithm in terms of itself. For example: consider the set of natural numbers, we introduce the method of generating the set of natural numbers by recursion.

The natural numbers (including zero) are those objects which can be generated by starting with an initial object 0 (zero) and from any object "n" already generated passing to another uniquely determined object n^+, the successor of n. The objects differently generated are always distinct. Thus the natural numbers appear as a set of objects $0, 0^+, 0^{++}, 0^{+++}, \ldots$

The transition to the usual notation is made upon introducing 1, 2, 3, 4, … to stand for $0^+, 0^{++}, 0^{+++,}$ $0^{++++}, \ldots$ and then employing the notation. The set of Natural numbers is denoted by N.

The set of natural numbers can also be generated by starting with usual null set $\varnothing$ and notion of a successor set.

If A is a given set, then the successor of A is the set $A \cup \{A\}$. It is denoted by A^+

Thus $$A^+ = A \cup \{A\}.$$

Let $\varnothing$ be the null set, then find the successor sets of $\varnothing$ these sets are:

$$\varnothing, \varnothing^+ = \varnothing \cup \{\varnothing\}, \varnothing^{++} = \varnothing \cup \{\varnothing\} \cup \{\varnothing, \{\varnothing\}\}$$

They can also be written as $\varnothing, \varnothing^+ = \{\varnothing\},$

$$\varnothing^{++} = \{\varnothing, \cup \{\varnothing\}\}$$

renaming the $\varnothing$ as 0 (zero),

$$\varnothing^+ = 0^+ = \{\varnothing\} = 1$$

$$\varnothing^{++} = 1^+ = \{\varnothing, \{\varnothing\}\} = \{0, 1\} = 2, \ldots$$

We get the set {0, 1, 2, 3, …}. Each element in the above set is a successor set of the previous element, except for the element 0. (0 is not the successor of any element).

Now we consider recursion in terms of successor:

Let S denote the successor. We define

(*i*) $x + 0 = x$ (*ii*) $x + S(y) = S(x + y)$

In this definition (*i*) is the basis, and it defines addition of 0. The recursive part defines addition of the successor of y.

Example 1:

$$\begin{aligned} 3 + 2 &= 3 + S(1) \\ &= S(3 + 1) \\ &= S(S(3 + 0)) = S(S(3)) = S(4) = 5 \end{aligned}$$

Now we start with a set of three functions called basis functions or initial functions of:

1. Zero function: Z: $Z(x) = 0$
2. Successor function S: $S(x) = x + 1$
3. Projection function : $\cup_1^n : \cup_1^n \left(x_1, x_2, x_3, ...x_n\right) = x_1$ (or generalized identify function).

Example 2:

$S(3) = 3 + 1 = 4$, $S(4) = 4 + 1 = 5$,

$\cup_1^2 (x, y) = x$, $\cup_2^3 (\alpha, \beta, r) = \beta$, $\cup_3^5 (1, 7, 6, 2, 4) = 6$, $\cup_3^6 (9, 3, 2, 5, 6, 8) = 2$, etc.

The initial functions are used in defining other functions by induction.

4.13.1 Sequences

The idea of a sequences is important to computer science. *A* sequence is defined as the list of objects in order. There are several ways of representing a sequence. One way is to list first terms of the sequence till the rule for writing down other terms is obtained.

For example: The sequence {2, 4, 6, 8, …} is a sequence whose nth terms in $2n$. Another way is to give a rule of writing the nth terms of the sequence.

The sequence {1, 4, 7, 10, …} can be written as

$$\{S_n\} \text{ where } S_n = 3n + 1, n = 0, 1, 2, ...$$

We can represent a sequence by using a recursive relation:

The recurrence relation

$a_n = a_{n-1} + 5$, with $a_1 = 6$, recursively defines the sequence 6, 11, 16, …

$a_1 = 6$ is called the initial condition.

4.13.2 Strings

Definition 4.12: Let A be a non-empty set, which we refer to as the alphabet. *A* string on the set A is a finite sequence of elements of A.

The set of all strings on A is denoted by A^* (or by A^+).

Example 1:

(*i*) $a_3\, a_2\, a_1\, a_4$ is a string on $A = \{a_1, a_2, a_3, a_4\}$

(*ii*) 246107 is a string on the alphabet consisting of the ten digits {0, 1, 2, 3, … 9}

Usually we write a string on A without using commas between the elements of the string.

If $a_1, a_2, a_3, \ldots, a_n$ is a string then the length of the string is n.

Example 2: The length of the string 621708 is 6

If $a_1, a_2, \ldots, a_n$ is a segment string, then $a_{r+1}\, a_{r+2} \ldots a_s$ where $1 \le r \le s \le n$ is called a (proper) segment of the string. If $r = 0$, then $a_{r+1}\, a_{r+2} \ldots a_s$ is called an initial segment of the string.

Example 3: 537 is an initial segment of the string 537108 and 710 is a segment of the string 537108.

Let $a_1\, a_2 \ldots, a_m$ and $b_1\, b_2 \ldots b_n$ be strings. There concatenation is $a_1\, a_2$ and $b_1\, b_2 \ldots b_n$. It is also called the product or Join and is a length $m + n$. For example: 537 and 108 are two strings and their catenation is the string 537108.

4.13.3 Floor and Ceiling Functions

Definition 4.13: Let x be any real number, then the greatest integer that does not exceed x is called the Floor of x.

The FLOOR of x is denoted by $\lfloor x \rfloor$

Examples:

$$\lfloor 5.14 \rfloor = 5$$

$$\lfloor \sqrt{5} \rfloor = 2$$

$$\lfloor -7.6 \rfloor = -8$$

$$\lfloor 6 \rfloor = 6$$

$$\lfloor -3 \rfloor = -3$$

Definition 4.14: Let x be a real number, then the least integer that is not less than x is called the CEILING of x.

The CEILING of x is denoted by $\lceil x \rceil$

Examples:

$$\lceil 2.15 \rceil = 3$$

$$\lceil \sqrt{5} \rceil = 3$$

$$\lceil -7.4 \rceil = -7$$

$$\lceil -2 \rceil = -2$$

Note: If x is an integer, then $\lceil x \rceil = \lfloor x \rfloor$

4.13.4 Integer Value of Function

Definition 4.15: Let x be any real number. The Integer value of x is the value of x converted into an integer by deleting the fractional part of x. It is denoted by $INT\,(x)$.

Example:

$$INT\ (2.33) = 2, INT\ (-6.4) = -6$$
$$INT\ (9) = 9$$

Note: If x is a positive Integer then $INT\ (x) = \lfloor x \rfloor$ and if x is a negative integer then $INT\ (x) = \lceil x \rceil$.

4.13.5 Absolute Value Function

Definition 4.16: Let x be a real number, then the absolute value of x denoted by $ABS\ (x)$ or $|\ x\ |$ is defined as follows:

$$|\ x\ | = -x \text{ if } x < 0$$
$$|\ x\ | = x \text{ if } x \geq 0$$

Example:

$$|\ 4.14\ | = 4.12, |\ 7\ | = 7$$
$$|\ -3.5\ | = 3.5, |\ -0.12\ | = 0.12$$

Note: $|\ x\ | = |\ -x\ |$

4.13.6 Logarithmic Function

Definition 4.17: Let M be positive number, then the Logarithm of M to the base a, written $log_a M$ represents the exponent to which a must be raised to obtain M. $a^x = M$ and $log_a M = x$ are equivalent statements.

Example:

If $3^4 = 81$ then $log_3 81 = 4$

$$log_{10} 0.01 = -2 \text{ since } 10^{-2} = 0.01$$

Note: Logarithms to base 10 are called common logarithms and logarithms to the base e are called natural logarithms where $e = 2.718281$.

Usually log x mean $log_{10} x$.

4.13.7 Partial and Total Functions

Definition 4.18: Let X and Y be sets and A be a subset of X. A function f from A to Y is called a partial function from X to Y. The set A is a called the domain of f. If the domain of f is the set X, then f is called a total function from X to Y.

Let f be a function from N (the set of Natural Numbers) to N given by $f\ (a) = b$ if $a = b^2$. Then the domain of the function f is the set of squares and f is a partial function. We define N^n to be the set of all n-tuples of elements of N. Any function $f : N^n \to N$ is a total function. If A is a subset of N^n, then the function $f : A \to N$ is a partial function. The function $f\ (x, y) = x + y$, for all elements $x, y \in N$ is a total function. But the function defined by $S\ (x, y) = x - y$, for all $x, y \in N$ is a partial functions.

4.13.8 Primitive Recursive Functions

Definition 4.19: Let g: $N^n \to N$ and h: $N^{n+2} \to N$ be functions. We say that f: $N^{n+1} \to N$ is defined from g and h by Primitive recursion if f satisfies the conditions:

$$f(x_1, x_2, \ldots, x_n, 0) = g(x_1, x_2, \ldots, x_n)$$

and $f(x_1, x_2, \ldots, x_n, y + 1) = h(x_1, x_2, \ldots, x_n, y, f(x_1, x_2, \ldots, x_n, y))$ (where y is the inductive variable).

From the above definition it is clear that Primitive recursion (or recursion) is the operation in the operation in which a function f of $(n + 1)$ variables is defined by using two other functions g and h of n and $(n + 2)$ variables.

Definition 4.20: A function f is said to be Primitive recursive if it can be obtained from the initial functions by a finite number of operations of composition and recursion (i.e., Primitive recursion).

Example 1: Show that addition is primitive recursive.

Solution: Addition is defined by $x + 0 = x$, $x + (y + 1) = (x + y) + 1$ for all natural numbers x, y

We define $f(x, y) = x + y$ such that

$$\begin{aligned} f(x, y + 1) &= x + y + 1 = (x + y) + 1 \\ &= f(x, y) + 1 \\ &= S(f(x, y)) \end{aligned}$$

also $f(x, 0) = x$

More formally we define $f(x, y)$ as

$$f(x, 0) = x = \cup_1^1(x)$$

$$f(x, y + 1) = S\left(\cup_3^3\left(x, y, f(x, y)\right)\right)$$

If follows that f comes from Primitive recursion from $\cup_1^1$ and $\cup_3^3$ and so is f Primitive recursive

Example 2: Show that multiplication * defined by $x * 0 = 0$, $x_* (y + 1) = x_* y + x$ is Primitive recursive.

Solution: We define $\mu(x, y)$ to be $x_*\ y$. so that $\mu(x, 0) = 0 = Z(x)$,

$$\begin{aligned} \mu(x, y + 1) &= \mu(x, y) + x \\ &= f(x, \mu(x, y)) \end{aligned}$$

$$\mu(x, y + 1) = f\left(\cup_3^3\left(x, y, \mu(x, y), \left(\cup_1^3\left(x, y, \mu(x, y)\right)\right)\right)\right)$$

Where f is the addition function.

Example 3: Let x, y be positive integers and suppose Q is defined recursively as follows:

$$Q(x, y) = \begin{cases} 0 & \text{if } x < y \\ Q(x - y, y) + 1 & \text{if } y \le x \end{cases}$$

find (*i*) $Q(4, 7)$ (*ii*) $Q(14, 6)$.

Solution:

(*i*) $Q(4, 7) = 0$ since $4 < 7$

(*ii*) $$\begin{aligned} Q(14, 6) &= Q(14 - 6, 6) \\ &= Q(8, 6) + 1 \\ &= Q(8 - 6, 6) + 1 + 1 \\ &= Q(2, 6) + 2 \end{aligned}$$

$= 0 + 2$
$= 2$

Example 4: Compute (*i*) A (1, 1), (*ii*) A (1, 2), (*iii*) A (2, 1) where $A: N^2 \to N$, (called Ackerman's function) is defined by

$$A(0, y) = y + 1$$
$$A(x + 1, 0) = A(x, 1)$$
$$A(x + 1, y + 1) = A(x + 1, y)$$

Solution:

(*i*) $A(1, 1) = A(0 + 1, 0 + 1)$
$= A(0, A(1, 0))$
$= A(0, a(0, 1))$
$= A(0, 1 + 1)$
$= A(0, 2)$
$= 2 + 1 = 3$

(*ii*) $A(1, 2) = A(0 + 1, 1 + 1)$
$= A(0, A(1, 1))$
$= A(0, 3) = 3 + 1 = 4$

(*iii*) $A(2, 1) = A(1 + 1, 0 + 1)$
$= A(1, A(2, 0))$
$= A(1, A(1, 1))$
$= A(1, 3)$
$= A(0 + 1, 2 + 1)$
$= A(0, 4)$
$= 4 + 1 = 5$

The functions given below are Primitive recursive:

1. Sign function

   The sign of x, denoted by $S_g(x)$ is defined by

$$S_g(0) = 0$$
$$S_g(x) = 1 \text{ for } x \neq 0$$

or
$$S_g(0) = Z(0)$$
$$S_g(y + 1) = S\left(Z\left(\cup_2^2\left(y, S_g(y)\right)\right)\right)$$

   Sign function is also called non-zero test function.

2. Zero test function:

   It is denoted by $\overline{S_g}$ and is defined as $\overline{S}_g(0) = 1,\ \overline{S}_g(y + 1) = 0$

3. Predecessor function:

   It is denoted by P, and is defined as

$$P(0) = 0,\ P(y + 1) = y = \cup_1^2\left(y, P(y)\right)$$

4. Proper subtraction function:
   It is denoted by $\dot{-}$ and is given by
   $$x \dot{-} 0 = x, x - (y + 1) = P (x - y)$$
   that is $x - y = x - y$ for $x > y$
   and $x \dot{-} y = 0$ for $x < y$
   ($x \dot{-} y$ does not map into N, so we do not consider it here)
5. Absolute function: $| \quad |$
   It is defined as
   $$| x - y | = (x \dot{-} y) + (y \dot{-} x)$$

4.13.9 Hashing

Hashing Function

In this section, we discuss Hashing. A Hashing function is used to store and retrieve data. We know that 'files; are used to store information on a computer. Each file contains many records and each record contains a field which is designated as a key to that record. The key has a value that identifies a record. Any transformation which maps the internal bit representation of the set of keys to a set of addresses is called a Hashing function. Ideally, each key would map to a unique hash address. A good hash function maps a random selection of keys uniformly across the hash table. A poor hash function results in frequent collisions. Various hash functions are available. Extraction and compression are two well-known methods of hashing which are practical for relatively small hash tables. Probabilistic hashing and virtual hashing are advanced methods for hashing. We now explain a method known as the division method, with the help of an example; in which the hashing function h defined by the division is

$$h(k) = k \text{ (mod } n)$$

The set $\{0, 1, 2, \ldots, n - 1\}$ is the address set, and $h(k)$ is the remainder of dividing k by the integer n. The remainder is a member of the address set.

Example 1: Assume that there are 10,000 customer account records to be stored and processed. The company's computer is capable of searching a list of 100 items in an acceptable amount of time and 101 lists are available for storage. If hashing function h is defined from the set of 7-digit account number to the set $\{0, 1, 2, \ldots, 100\}$ as

$$h(k) = k \text{ (mod 101)}$$

The customer with account number 3563821 will be assigned to the list 36.

Example 2: Suppose that 7,500 customer account records must be stored and processed the company's computer is capable of searching a list of 58 items in an acceptable amount of time. There are 59 linked lists of storage. A hashing function h is defined from the set of 7-digit account numbers to the set $\{0, 1, 3, \ldots, 58\}$ is defined as follows:

h takes the first three digits of the account number as one number and the last four digits as another number, adds them and then applies the mod 59 function. Determine which list the customer with customer account number (key) 2614902 should be attached.

Solution: Consider 2614902

We split up the number as 261 and 4902 adding we get 261 + 4902 = 5163.

Dividing by 59, we get 30 as the remainder the record with account number 2614902 will be assigned to the list 30

i.e., $h(2614902) = 30$

Folding: In this method of hashing, the key (customer record number) is divided in several parts. The parts are then added to for another number in the required range.

For example: Consider the customer record number 37124865 (having 8-digit key). If 3-digit address is to be obtained then $h(37124865) = 371 + 248 + 65 = 684$.

EXERCISE 4.2

1. Show that exponentiation defined by $f(x, y) = x^y$ is Primitive recursive.
2. Show that the function $[x / 2]$ which is equal to the greatest integer which is $< x / 2$ is Primitive recursive.
3. Show that the cosine of x, $C\,ox$ defined by $C\,00 = 1$, $C\,ox = 0$ for $x > 0$ is Primitive recursive.
4. Prove that the square function given by $f(y) = y^2$ is primitive recursive.
5. Show that the Ackerman's function A: $N^2 \to N$ which is defined by:

$$A(0, y) = y + 1$$
$$A(x + 1, 0) = A(x, 1)$$
and
$$A(x + 1, y + 1) = A(x + 1, y)$$

is not primitive recursive.

6. Show that the function

$$f(x) = \begin{cases} x/2 & \text{when } x \text{ is even} \\ (x-1)/2 & \text{when } x \text{ is odd} \end{cases}$$

is primitive recursive.

4.14 RECURRENCE RELATIONS AND SOLUTIONS OF RECURRENCE RELATIONS

In the previous section, we defined the set of recursive function. In this section we shall first explain what is meant by a numeric function and then study recurrence relations.

Functions whose domain is the set of natural numbers and whose range is the set of real numbers are called numeric function.

For example

$$a_r = \begin{bmatrix} 0, 0 \le r \le 2 \\ 2^{-r} + 5,\ r \ge 3 \end{bmatrix}$$

is a numeric function. Bold face lowercase letters are used to denote Numeric functions for example

$$a = \{a_0, a_1, a_2, a_3, \ldots a_r, \ldots\}$$

is a numeric function where a_0, a_1, a_2, a_3, $\ldots a_r, \ldots$ denotes the values a at 0, 1, 2, 3, … r, …. If a and b denote two numeric functions then their sum and product are also numeric functions. If k is a scalar and a is a numeric function. The k is also numeric function.

If $a = \{ a_0, a_1, a_2, a_3\}$ is a

If $a = \{ a_0, a_1, a_2, \ldots a_r, \ldots\}$ is a numeric function, then $a_{r+1} - a_r$ is called its forward difference at r. It is denoted by Δa_r. The backward difference at r 1 is $a_r - a_{r-1}$. The backward difference of a at r is denoted by ∇a_r.

The convolution of two numeric function a and b is a numeric function c such that

$$C_r = a_0\, b_r + a_1\, b_{r-1} + a_2\, b_{r-2} + .. + a_{r-1}\, b_1 + a_r\, b_0$$

$$= \sum_{i=0}^{r} a_i\, b_{r-i}$$

The convolution of a and b is denoted by $a * b$

For example, let a and b denote two numeric functions such that

$$a_r = 5^r\, r^3,\ r \geq 0$$

and

$$b_r = 3^r\, r^3\, 0,\ r \geq 0$$

Then the convolution $a * b$ is given by

$$C_r = \sum_{i=0}^{r} 5^i\, 3^{i-1}$$

4.15 GENERATING FUNCTIONS

We introduce now an alternative way to represent numeric functions. For the numeric function

$$a = \{ a_0, a_1, a_2, \ldots a_r, \ldots\}$$

we define an infinite series

$a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + \ldots$ which is called the generating function of a. It is denoted by $A\,(Z)$. The coefficient of z^r is $A\,(z)$ is the value of the numeric function a. For example, the generating function of $a = (2^0, 2^1, 2^2, \ldots 2^r, \ldots)$ is $z^0 + 2z + 2z^2 + 2^3 z^3 + \ldots + 2^r z^r + \ldots$

The above infinite series can be written in the closed form as

$$A(z) = \frac{1}{1 - 2z}$$

In $\quad A\,(z) = a_0\, z^0 + a_1\, z^1 + a_2\, z^2 + \ldots + a_r\, z^r + \ldots$

The term $a_0\, z^0 = a_0$ is called the constant term, the term $a_r\, z^r$ is the term of degree r. Note that $A(z)$ generates its coefficients. If all the coefficients are zero from some point on, $A(z)$ is just a polynomial. If $a_r \neq 0$ and $a_s = 0$ for $s \geq r + 1$ then $A\,(z)$ is a polynomial degree r.

Let $\quad A\,(z) = a_0\, z^0 + a_1\, z^1 + a_2\, z^2 + \ldots + a_r\, z^r + \ldots$

and $\quad B\,(z) = b_0\, z^0 + b_1\, z^1 + \ldots + b_r\, z^r + \ldots$

denote the generating functions. $A\,(z)$ and $B\,(z)$ are equal if $a_r = b_r$ for each

$$r \geq 0 \text{ and } A(z) + B(z) = \sum_{r=0}^{\alpha} (a_r + b_r)\, z^r$$

If k is a scalar then

$$k\,A(z) = k\,(a_0 z^0 + a_1 z^1 + \ldots + a_r z^r + \ldots)$$

$$= k\sum_{r=0}^{\alpha} a_r\, z^r$$

The product $A(z)\,B(z)$ is defined as

$$A(z)\,B(z) = a_0 b_0 z^0 + (a_0 b_1 + a_1 b_0)\,z + (a_0 b_2 + a_1 b_1 + a_2 b_0)\,z^2 + \ldots + (a_0 b_r + a_1 b_{r-1} + \ldots + a_r b_0)\,z^r + \ldots$$

for example, the generating function of the Numeric function

$$a_r = 5.2^r,\; r \geq 0$$

is
$$A(z) = \frac{5}{1-2z}$$

the generating function of the Numeric function

$$a_r = 5.2^r,\; r \geq 0$$

is
$$A(z) = \frac{25}{1-5z}$$

also the generating function of the numeric function

$$a_r = 5^{r+2},\; r \geq 0$$

is
$$A(z) = \frac{1}{1-2z} + \frac{1}{1-5z}$$

4.15.1 Asymptotic Behaviour of Numeric Function

Let a be a numeric function. By the asymptotic behaviour of a, we mean how the volume of the function a varies for large r.

For example: for

$$a_r = 3r^2,\; r \geq 0$$

The value of the Numeric function increases for increasing r, and for

$$b_r = \frac{2}{r},\; r > 0$$

The value of the function decreases for increasing r. Finally for

$$C_r = 7,\; r > 0$$

The value of the numeric function remains constant for increasing r.

Let a and b be two numeric functions.

If there exist two positive constants r and k such that

$|b_r| \geq |a_r|$ for $r \geq k$

then we say that the numeric function a asymptotically dominates b, or the Number function b is dominated by a.

For example, let a and b be two numeric functions such that

$$a_r = r + 5,\ r \geq 0$$

$$b_r = \frac{1}{r} + 9,\ r > 0$$

Then the numeric functions a dominates b.

If a is a numeric function then $|a|$ asymptotically dominates a and if the numeric function b is asymptotically dominated by a, then for any constant λ, $\lambda\, b$ is also dominated by a.

If a is a given numeric function, then the set of all numeric functions that are dominated by a is called the order "a" or "big – Oh a" and is denoted by $O(a)$.

4.15.2 Recurrence Relations

In Section 4.1, we have discussed recursive definition of a sequence. The recursive formula for defining a sequence (or numeric function) is called a recurrence relation. If $a = (a_0, a_1, \ldots, a_r, \ldots)$ a numeric function, then the recurrence relation for a is an equation relating a_r for any r, to one or more a_i 's $(i < r)$. Every recursive formula includes a starting point. The information accompanying a recursive formula about the beginning of the sequence (or numeric function) is called initial condition. A recurrence relation is also called a difference equation.

Example 1: The recurrence relation $a_n = a_{n-1} + 5$, with $a_1 = 2$ recursively defines the sequence 7, 12, 17, … where $a_1 = 2$ is the initial condition.

Example 2: The recurrence relation $a_n = a_{n-1} + a_{n-2}$, with $a_1 = a_2 = 1$ defines the Fibonacci sequence 1, 1, 1, 2, 3, 5, 8, 13, … $a_0 = 1$, $a_1 = 1$, are called the initial conditions.

4.15.3 Linear Recurrence Relations

Suppose r and k are non-negative integers. A recurrence relation of the form

$$c_0(r)\, a_r + c_1(r)\, a_{r-1} + \ldots + c_k(r)\, a_{r-k} = f(r) \text{ for } r \geq k$$

where $c_0(r), c_1(r), \ldots c_k(r)$ and $f(r)$ are functions of r is said to be a recurrence relation. If $c_0(r) > 0$ and $c_k(r) > 0$, then it said to be a linear recurrence relation of degree k. If $c_0(r), c_1(r), \ldots c_k(r)$ are constants then the recurrence relation is called a linear recurrence relation with constant coefficients. if $f(r) = 0$, then the recurrence relation is called a linear homogenous relation. A recurrence relation which is not homogeneous is said to be inhomogeneous relation.

Examples: the recurrence relations

$$a_n - 6a_{n-1} + 11a_{n-1} + 6a_{n-3} = 2n$$

$$a_n - 9a_{n-1} + 26a_{n-1} - 24a_{n-3} = 5n$$

are linear recurrence relations with constant coefficients

The relation

$$a_n - 6a_{n-1} + 11a_{n-2} + 6a_{n-3} = 0$$

is homogeneous and $a_r + 9a_{r-2} = 0$ is a second order recurrence relation with constant coefficients.

4.15.4 Solutions of Recurrence Relations

We shall now describe some methods of solving recurrence relations in this section. We know that every recurrence relation is accompanied by boundary condition. Any numeric function that can be described by a recurrence relation together with an appropriate set of boundary conditions is called a solution of the recurrence relation. If $a = (a_0, a_1, \ldots, a_r, \ldots)$ is a solution to a recurrence relation then it is said to satisfy the relation. A given recurrence relation may or may not have a solution. We shall now consider the methods of solving of Homogeneous recurrence relations (*i*) the intervention method (substitution method) of solving a recurrence relation. In this method the recurrence relation for a_r is used repeatedly to solve the recurrence for a general expression for a_r in terms of r. We illustrate this method in the examples given below:

Example 1: Solve the recurrence relation $a_r = 2a_{r-1} + 1$ with $a_1 = 7$ for $r > 1$ by substitution.

Solution:

$a_1 = 7$ the initial conditions

$$
\begin{aligned}
a_2 &= 2a_1 + 1 = 2\,.7 + 1\\
a_3 &= 2a_2 + 1 = 2\,.(2.7 + 1) + 1\\
&= 2^2\,.\,7 + 2 + 1\\
a_4 &= 2a_3 + 1 = 2\,(2^2\,.\,7 + 2 + 1) + 1\\
&= 2^3\,.\,7 + 2^2 + 2 + 1\\
&\ldots\\
a_r &= 2^{r-1}\,.\,7 + 2^{r-2} + 2^{r-3} + \ldots + 2 + 1
\end{aligned}
$$

We have

$$a_r = 7.2^{r-1} + 2^{r-1} - 1,\ \ r \geq 1 \qquad (\because\ 1 + 2 + \ldots + 2^{r-2} = 2^{r-1} - 1)$$

Example 2: Solve the recurrence relation $a_r = a_{r-1} + f(r)$ for $r \geq 1$ by substitution.

Solution: We have

$$
\begin{aligned}
a_1 &= a_0 + f(1)\\
a_2 &= a_1 + f(2) = a_0 + f(1) + f(2)\\
a_3 &= a_2 + f(3) = a_0 + f(1) + f(2) + f(3)\\
a_r &= a_0 + f(1) + f(2) + \ldots + f(r)\\
&= a_0 + \sum_{n=1}^{r} f(n)
\end{aligned}
$$

(*ii*) Method of generating functions

Recurrence relations can also be solved by using generating functions. Some equivalent expressions used are given below:

$$A(z) = \sum_{r=0}^{\infty} a_r\, z^r \text{ then}$$

$$A(z) = \sum_{r=k}^{\infty} a_r\, z^r = A(z) - a_0 - a_1\, z - \ldots - a_{k-1}\, z^{k-1}$$

$$A(z)=\sum_{r=k}^{\infty} a_{r-1}\, z^r = z\left(A(z)-a_0-a_1 z-\ldots-a_{k-2}\, z^{k-2}\right)$$

$$A(z)=\sum_{r=k}^{\infty} a_{r-2}\, z^r = z^2\left(A(z)-a_0-a_1\, z-\ldots-a_{k-3}\, z^{k-3}\right)$$

$$A(z)=\sum_{r=k}^{\infty} a_{r-k}\, z^r = z^k\, A(z)$$

Example 3: Solve the recurrence relation

$$a_r - 7a_{r-1} + 10a_{r-2} = 0 \text{ for } r \geq 2$$

Solution: Multiplying each term of the recurrence relation by z^r, we get

$$a_r z^r - 7a_{r-1}\, z^r + 10a_{r-2}\, z^r = 0$$

Taking the sum 2 to ∞, we get

$$\sum_{r=2}^{\infty} a_r\, z^r - 7\sum_{r=2}^{\infty} a_{r-1}\, z^r + 10\sum_{r=2}^{\infty} a_{r-2}\, z^r = 0$$

replacing each infinite sum by the corresponding equivalent expression given above, we get

$$[A(z) - a_0 - a_1\, z] - 7\, z\, \{A(z) - a_0\} - 10z^2 A(z) = 0$$

Simplifying, we get

$$A(z)\,(1 - 7z + 10\, z^2] = a_0 + a_1\, z - 7a_0\, z$$

or
$$A(z)=\frac{a_0+(a_1-7a_0)z}{1-7z+10z^2}=\frac{a_0+(a_1-7a_0)z}{(1-2z)(1-5z)}$$

Decomposing $A(z)$ as a sum of partial fractions, we can write

$$A(z)=\frac{c_1}{1-2z}+\frac{c_2}{1-5z}=c,\sum_{r=0}^{\infty} 2^r z^r + c_2\sum_{r=0}^{\infty} 5^r z^r$$

The solution is $a_r = c_1\, 2^r + c_2\, 5^r$

(*iii*) Method of characteristic roots

Now we explain the method of solving linear recurrence relation by the methods of characteristic roots:

Consider

$$c_0\, a_r + c_1\, a_{r-1} + c_2\, a_{r-2} + \ldots + c_k\, a_{r-k} = 0 \qquad \ldots (1)$$

Substituting $A\alpha^r$ for a_r in (1), we get

$$c_0\, A\alpha^r + c_1\, A\alpha^{r-1} + c_2\, A\alpha^{r-2} + \ldots + c_k\, A\alpha^{r-k} = 0 \qquad \ldots (2)$$

(the constant A is to be determined by the boundary conditions)

(2) Can be simplified as

$$c_0\, \alpha^k + c_1\, \alpha^{k-1} + \ldots + c_k = 0 \qquad \ldots (3)$$

The equation (3) is called the characteristic equation of the difference equation.

We get $$P = \frac{1}{6}$$

The particular solution is $a_r^{(p)} = \frac{1}{6}$

(*ii*) $f(r)$ may be polynomial in r and is of degree m. say. In this case the corresponding particular solution is of the form

$$P_1 r^m + P_2 r^{m-1} + \ldots + P_{m+1}$$

Example 2: Find the particular solution of

$$a_r + 5a_{r-1} + 6a_{r-2} = 3r^2$$

Solution: Let the form of the particular solution be $P_1 r^2 + P_2 r + P_3$ where P_1, P_2 and P_3 are constants to be determined. Substituting the expression into the left hand side of given difference equation, we get

$$P_1 r^2 + P_2 r + P_3 + 5P_1 (r-1)^2 + 5 P_2 (r-1) + 5P_3 + 6P_1 (r-2)^2 + 6P_2 (r-1) + 6P_3 = 3r^2$$

Simplifying, we get

$$12P_1 r^2 - (34p_1 - 12p_2) r + (29P_1 - 17P_2 + 12P_3) = 3r^2$$

$$12P_1 = 3 \qquad \ldots \text{(i)}$$

$$34P_1 - 12P_2 = 0 \qquad \ldots \text{(ii)}$$

$$29P_1 - 17P_2 + 12P_3 = 0 \qquad \ldots \text{(iii)}$$

solving we get

$$P_1 = \frac{1}{4},\ P_2 = \frac{17}{24},\ P_3 = \frac{115}{288}$$

The particular solutions is

$$a_r^{(p)} = \frac{1}{4}r^2 + \frac{17}{24}r + \frac{115}{288}$$

(*iii*) When $f(r)$ is of the form λ^r, and λ is not a characteristic root of the recurrence relation, the particular solution is of the form $P\lambda^2$ further more when $f(r)$ is of the form

$\left(b_1 r^m + b_2 r^{m-1} + \ldots + b_{m+1}\right)\lambda^r$, the corresponding particular solution is of the form

$\left(P_1 r^m + P_2 r^{m-1} + \ldots + P_{m+1}\right)\lambda^r$,

Where λ is not a characteristic root of the recurrence relation. When λ is a characteristic root with multiplicity and $f(r)$ is of the form

$$\left(b_1 r^m + b_2 r^{m-1} + \ldots + b_{m+1}\right)\lambda^r,$$

the corresponding particular solution is of the form

$$r^{p-1}\left(P_1 r^m + P_2 r^{m-1} + \ldots + P_{m+1}\right)\lambda^r.$$

Example 3: Find a general expression for a solution to the recurrence relation

$$a_n - 5a_{n-1} + 6a_{n-2} = 4^n,\ n \geq 2.$$ ***(OU Dec. 2000)***

Solution: The characteristic equation of the given relation is

$$\alpha^2 - 5\alpha + 6 = 0$$

or

$$(\alpha - 2)(\alpha - 3) = 0$$

$\alpha = 2, \alpha = 3$ are the characteristic roots

The homogeneous solution is

$$a_n^{(h)} = A_1 2^n + A_2 3^n$$

The particular solution $a_n^{(p)}$ will be of the form

$$a_n^{(p)} = \lambda \cdot 4^n$$

Substituting $a_n = \lambda \cdot 4^n$ in

$$a_n - 5a_{n-1} + 6a_{n-2} = 4^n$$

we get

$$\lambda 4^n - 5\lambda 4^{n-1} + 6\lambda 4^{n-2} = 4^n$$

or

$$4^{n-2}[\lambda 4^2 - 5\lambda \cdot 4 + 6\lambda] = 4^{n-2} \cdot 4^2$$

or

$$16\lambda - 20\lambda + 6\lambda = 16$$

or

$$2\lambda = 16$$

or

$$\lambda = 8$$

Therefore, the general solution of the given recurrence relation is

$$a_n = A_1 \cdot 2^n + A_2 \cdot 3^n + 8 \cdot 4^n$$

Example 4: Find the particular integral of $a_r + a_{r-1} = 3r\,2^r$.

Solution: The general form of the particular solution is $(p_1 r + P_2)\,2^r$ substituting into $a_r + a_{r-1} = 3\,r\,2^r$, we get $(P_1 r + P_2)\,2^r + (P_1(r-1) + P_2)2^{r-1} = 3r\,2^r$ simplifying, we get

$$\frac{3}{2} P_1 r + \left(-\frac{1}{2} P_1 + \frac{3}{2} P_2\right)2^r = 3r2^r$$

Comparing, we get

$$\frac{3}{2} P_1 = 3$$

$$-\frac{1}{2} P_1 + \frac{3}{2} P_2 = 0$$

Solving the above equation

$$P_1 = 2,\ P_2 = \frac{2}{3}$$

The particular solution is

$$a_r^{(p)} = \left(2r + \frac{2}{3}\right)2^r$$

Example 5: Find the general solution of

$$a_r - 7a_{r-1} + 10a_{r-2} = 7.3^r, r \geq 2$$

Solution: The characteristic equation is $\left(\alpha^2 - 7\alpha + 10\right) = 0$

or $$\left(\alpha - 2\right)\left(\alpha - 5\right) = 0$$

The characteristic roots are 2, 5 the homogeneous solution is

$$a_r^{(h)} = A_1\, 2^r + A_2\, 5^r$$

Let $P \cdot 3^r$ be the particular solution of the given recurrence relation substitution $P, 3^r$ for a_r in the recurrence relation given

$$P \cdot 3^r - 7\, P\, 3^{r-1} + 10\, P\, 3^{r-2} = 7.3^r$$

$$\Rightarrow (-2)\, P = 7.3^2$$

$$\Rightarrow P = -63/2$$

The particular solutions is $a_r^{(p)} = (-63/2)3^r$

The general solutions is

$$a_r = a_r^{(p)} + a_r^{(h)} = A_1\, 2^r + A_2\, 5^r \left(-63/2\right)3^r$$

Example 6: Solve $a_r - 6a_{r-1} + 8a_{r-2} = r.4^r$ where $a_0 = 8$ and $a_1 = 22$

Solution: The characteristic equation of the given relation is

$$\alpha^2 - 6\alpha + 8 = 0$$

$$\Rightarrow \left(\alpha - 2\right)\left(\alpha - 4\right) = 0$$

The characteristic roots are 2, 4 the homogeneous solution is

$$a_r^{(h)} = A_1\, 2^r + A_2\, 4^r$$

Hence, 4 is a characteristic root with multiplicity 1. The particular solution takes the form $r\,(P_1 + P_2\, r)\, 4^r$

Substituting this expression into recurrence relation, we get

$$16r\,(P_1 + P_2\, r) - 24\,(r)\,(r-1)\,[P_1 + P_2\,(r - 1)] + 8\,(r - 2)\,[P_1 + P_2\,(r - 2)] = 16r$$

The above expression holds for all values of and in particular for $r = 0$.

We obtained the simplified equation $P_1 + P_2 = 0$ for $r = 1$, we get $P_1 + 3P_2 = 2$

Which give $P_1 = -1$, $P_2 = 1$

The particular solution is $a_r^{(p)} = r\,(-1 + r)\, 4^r = r\,(r - 1)4^r$

The general solution is $a_r = a_r^{(h)} + a_r^{(p)}$

$$= A_1\, 2^r + A_2\, 4^r + r\,(r - 1)\, 4^r$$

the initial conditions $a_0 = 8$, $a_1 = 22$ give $A_1 = 3$.

The general solution is $a_r = r\,(-1 + r)\, 4^r + 5.2^r + 3.4^r$.

EXERCISE 4.3

I. Solve the recurrence relations

(*a*) $a_r = 7a_{r-1} - 10a_{r-2}$, $a_0 = 4$, $a_1 = 17$

(*b*) $a_r - 8a_{r-1} + 16a_{r-2} = 0$, $a_2 = 16$, $a_3 = 80$

(*c*) $a_r - 4a_{r-1} - 11a_{r-2} + 30a_{r-3} = 0$
given the initial condition
$a_0 = 0$, $a_1 = -35$, and $a_2 = -85$

(*d*) $a_r - a_{r-1} - 6a_{r-2} = -30$ with $a_0 = 20$, $a_1 = -5$

II. Solve:

(*a*) $S_k - 2S_{k-1} + S_{k-2} = 2$, with $S_0 = 25$, $S_1 = 16$

(*b*) $G_k - 7G_{k-1} + 10G_{k-2} = 6 + S_k$ with $G_0 = 1$, $G_1 = 2$

(*c*) $a_r - 3a_{r-1} - 4a_{r-2} = 4^r$

(*d*) $a_r - 4a_{r-1} + 4a_{r-2} = 3r + 2^r$

Answers:

I. (*a*) $a_r = 1.2^r + 3.5^r$

(*b*) $a_r = (2 + r)\, 4^{r-1}$

(*c*) $a_r = 4\,(-3)^r + 1.2^r + (-5)\,.\,5^r$

(*d*) $a_r = 11\,.\,(-2)^r + 4\,.\,3^r + 5$

II. (*a*) $S_k = 25 - 10k + k^2$

(*b*) $G_k = -9.2^k + 2.5\text{k} + (8 + 2k)$

(*c*) $a_r = A_1\,.\,(-1)^{\text{r}} + A_2\,.\,4^r + \dfrac{4r \cdot 4^r}{5}$

(*d*) $a_r = (r^2 + 7r - 22)\,.\,2^{r-1} + (12 + 3k)$.

5

Boolean Algebra

5.1 INTRODUCTION

In this chapter, we study partially ordered sets, Lattices and Boolean algebras. George Boole in 1854 has introduced a new kind of algebraic system known as Boolean algebra. It is relatively very simple and can be used to analyse and design completes circuits. Before we study Boolean algebra in this chapter we consider ordering relations and Lattices.

5.2 PARTIAL ORDERING

Definition 5.1: A relation R on a set A is called a partial order relation in A if R is, Reflexive anti-symmetric and transitive.

If R is a partial order on a set A, then A is said to be partially ordered by R. The partial order R on A is simply called an order relation on A. The set A with partial order R on it is called a partially ordered set or n ordered set or a Poset. We write (A, R) when we want to specify the partial order relation R usually we denote a partial order relation by the symbol $\leq$. This symbol does not necessarily mean "less than or equal to".

Example 1: If A is a non-empty set and $P(A)$ denotes the power set of A, then the relation set inclusion denoted by $\leq$ in $P(A)$ is a partial ordering.

Example 2: Let $A = \{2, 3, 6, 12, 24, 36\}$ and R be a relation in A which is defined by "a divides b". Then R is a partial order in A.

5.2.1 Comparability

Definition 5.2: Let R be a partial order on A and $a,\ b \in A$ whenever $aR\,b$ or $bR\,a$, we say that a and b are comparable otherwise a and b are non-comparable.

5.3 TOTALLY ORDERED SET

Definition 5.3: Let $(A, \leq)$ be a partially order set. If for every $a,\ b \in A,$ we have $a \leq b$ or $b \leq a$, then $\leq$ is called a simple ordering (or linear ordering) on A, and the set $(A, \leq)$ is called a totally ordered set or a chain.

Note: If A is a partially ordered set, then some of the elements of A are non-comparable. On the other land, if A is totally ordered then every pair of elements of A are comparable.

Example 1: Let N be the set of positive integers ordered by divisibility. The elements 5 and 15 are comparable. Since 5/15 on similarly the elements 7 and 21 are comparable since 7/21. The positive integers 3 and 5 are non-comparable since neither 3/5 nor 5/3. Similarly the integers 5 and 7 are non- comparable.

Example 2: Let A be a non-empty set with two or more elements and $P(A)$ denote the power set of A. Then $P(A)$ is not linearly ordered.

Example 3: The set N of positive integers with the usual order $\leq$ (less than equal) is a linear order on N. The set $(N, \leq)$ is a totally ordered set.

5.4 DUAL ORDER

Definition 5.4: If $\leq$ is a partial order on a set A, then the converse of R is also a partial order on A. i.e., if $\leq$ is a partial ordering on A, then $\geq$ is also a partial ordering on A. $(A, \geq)$ is called the dual of $(A, \leq)$.

Corresponding to every partial ordering on $\leq$ on A, we can define another relation on A which is denoted by < and is defined as follows:

$$a < b \Leftrightarrow a \leq b\text{: for all } a,\ b \in A, \text{ where } a \neq b$$

Similarly corresponding to the partial ordering $\geq$, we can define the a relation >, such that $a > b \Leftrightarrow a \geq b$ for $b \in A$ where $a \neq b$. The relations < and > are irreflexive, but both the relations < and > are transitive.

5.5 HASSE DIAGRAM

Definition 5.5: A Hasse diagram is a pictorial representation of a finite partial order on a set. In this representation, the objects i.e., the elements are shown as vertices (or dots).

Two related vertices in the Hasse diagram of a partial order are connected by a line if and only if they are related.

Example 1: Let $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\leq$ be such that $a \leq b$ if a divides b. The Hasse diagram of $(A, \leq)$ is shown in Fig. 5.1.

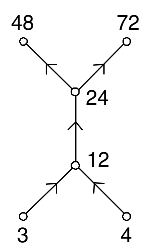


Fig. 5.1

We avoid arrows in a Hasse diagram and draw lines to show that the elements are related. Hasse diagrams can be drawn for any relation which is anti-symmetric and transitive but not necessarily reflexive.

Example 2: Let $A = \{1, 2, 3\}$, and $\leq$ be the relation "less than of equal to" on A. Then the Hasse diagram of $(A, \leq)$ is as shown in Fig. 5.2.

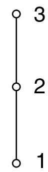


Fig. 5.2

Example 3: Let $A = \{a\}$, and $\leq$ be the inclusion relation on the elements of $P(A)$. The Hasse diagram of $(P(A), \leq)$, can drawn as shown in Fig. 5.3.

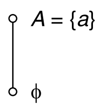


Fig. 5.3

Example 4: Draw the Hasse diagram representing the positive divisions of 36 (i.e., D_{36})
Solution: We have $D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

Let R denote the partial order relation on D_{36}, i.e., aRb if and only a divides b. The Hasse diagram for R is shown in Fig. 5.3 (a).

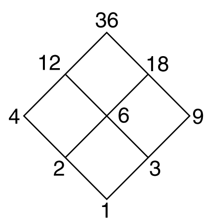


Fig. 5.3 (a)

Note:

(i) Two unequal relations R_1 and R_2 may have the same Hasse diagram.
(ii) The Hasse diagram of Poset A need not be connected.

If $(A, \leq)$ is partially ordered set, the Hasse diagram of $(A, \leq)$ is not unique. For example, consider the set $A = \{a, b\}$. The relation of inclusion $(\leq)$ on $P(A)$ is a partial ordering. The Hasse diagrams of $(P(A), \leq)$ are given in Fig. 5.4.

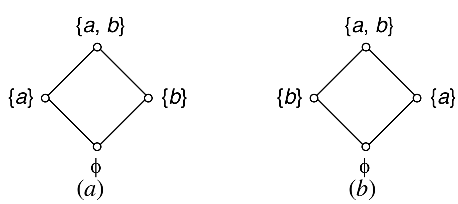


Fig. 5.4 Hasse diagram

The Hasse diagram which represent the partial ordered set $(A, \leq)$ show that the Hasse diagram of a poset is not unique.

If $(A, \leq)$ is a poset, the Hasse diagram of $(A, \geq)$ can be obtained by rotating the Hasse diagram of $(A, \leq)$ through 180^o. So that the points at the top become the points at bottom. Some Hasse diagrams may have a unique point which is above all the other points in the diagram and in some cases, the Hasse diagrams have a unique point which is below all the other points.

Definition 5.6: A relation R on a set A is said to be connected if for every pair of distinct elements $a, b \in A$, either $aR\,b$ or $bR\,a$.

Definition 5.7: A partial ordering on a set A is said to be linear ordering if it is connected.

5.6 LEXICOGRAPHIC ORDERING

Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. We define another partial order on $A \times B$, denoted by $\prec$, and is defined as follows:

$(a, b) \prec (a', b')$ if $a < a'$ or if $a = a'$ and $b \leq b'$. The order $\prec$ is called Lexicographic ordering (or dictionary ordering). In the above ordering of elements in the first coordinate dominates except in case of ties. In this case of equality. We consider the second coordinate. The Lexicographic ordering defined above can be extended as follows:

Let $(A_1, \leq)$, $(A_2, \leq)$, ... $(A_n, \leq)$ denote partially ordered sets. We define a partial order $\propto$ on $A_1 \times A_2 \times ... A_n$ as follows:

$(a_1, a_{2,} ..., a_n) \prec (a_1', a_2', ..., a_n')$ if and only if

$a_1 < a_1'$ or

$a_1 = a_1'$ and $a_2' < a_2'$ or

$a_1 = a_1', a_2' = a_2'$, and $a_3 < a_3'$ or

$a_1 = a_1', a_2 = a_2' ... a_{n-1} = a_{n-1}'$and $a_n \leq a_n'$

in the above ordering:

The first coordinate dominates, in the case of equality, we consider the second coordinate, if the equality holds again, we pass to the next coordinate and so on.

The order in which the words in an English dictionary appear is an example of lexicographic ordering.

Example 1: Let $A = (a, b, c, ... z)$ and let A be linearly ordered in the usual way $(a \leq b), b \leq c, ... y \leq z)$. The set $A^n = A \times A \times ... \times A$ (n factors) can be identified with the set of all words having length n. The Lexicographic ordering on A^n has the S^n has the property that if $w_1 < w_2$ (where w_1 and w_2 are two words in S^n), then w_1 would precede w_2 in the dictionary listing.

Thus

Card $\prec$ Cart

Loss $\prec$ Lost

Park $\prec$ Part

Salt $\prec$ Seat
Mark $\prec$ Mast

We can extend Lexicographic order to posets. If A is a partially ordered set and A^* denotes the set of all finite sequences of elements of A we can extend Lexicographic order to A^* as follows:

Let $\alpha = a_1 a_2 \ldots a_m$ and $\beta = b_1 b_2 \ldots b_n$ belong to A^* with $m \leq n$., we say that $\alpha \propto \beta$ if $(a_1, a_2, \ldots a_m) \propto (b_1, b_2, \ldots b_n)$ in $A^n = A \times A \times \ldots \times A$ (n factors) under the Lexicographic ordering of A^n.

Example 2: Let $A = \{a, b, c, \ldots z)$ and let R be a simple ordering on A denoted by $\leq$, where $(a \leq b \leq c \ldots \leq z)$

Let $S = A \cup A^2 \cup A^3$

Then S consists of all words (strings) of three or fewer than 3 letters from A. Let $\prec$ Denote the lexicographic ordering on S. We have

Be $\prec$ Bet
Leg $\prec$ Let
Peg $\prec$ Pet
Sea $\prec$ See
...

Lexicographic ordering is used in sorting character data on a computer. The Lexicographic ordering is also called dictionary ordering. We can use the names "Lexically less than" or "Lexically equal to" or "Lexically greater then" to denote a lexicographic ordering.

Example 3: Let $A = \{a, b, c, \ldots z)$ with simple alphabetical order and let $A^2 = A \times A$, then $bq > ae$, $df > ab$ and $dy < ez$.

5.7 COVER OF AN ELEMENT

Definition 5.8: Let $(A, \leq)$ be a partially ordered set. An element $b \in A$ is said to cover an element $a \in A$, if a $< b$ and if there does not exist any element $c \in A$ such that $a \leq c$ and $a \leq b$.

If b covers a, then a line is drawn between the elements a and b in the Hasse diagram of $(A, \leq)$.

5.8 LEAST AND GREATEST ELEMENTS

Definition 5.9: Let $(A, \leq)$ denote a partially ordered set. If there exists an element $a \in A$ such that $a \leq x \ \forall \ x \in A$, then a is called the least member in A, relative to the partial ordering $\leq$. Similarly, if there exists an element $b \in A$, such that $x \leq b \ \forall \ x \in A$, then b is called greatest member in A relative to $\leq$.

Note:

(*i*) The least member of is usually denoted by 0, and the greatest member in a poset is usually denoted by 1.
(*ii*) For a given poset, the greatest or least member may or may not exist.
(*iii*) The least member in poset, if it is unique, and the greatest member if it exists is unique.
(*iv*) In every chain, the least and greatest members always exist.

Example 1: Let $A = \{1, 2, 3, 4, 5\}$ and $\leq$ be the relation "less than or equal to" then the Hasse diagram of $(A, \leq)$ is as shown in Fig. 5.5.

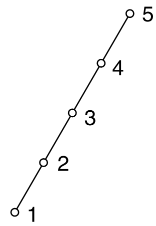


Fig. 5.5 Hasse diagram

From Fig. 5.5, it is clear that 1 is the least member and 5 is the greatest element in $(A, \leq)$.

Example 2: Let $A = \{a, b\}$ and $P(A)$ denote the power set of A. Then $P(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$. Let $\leq$ be the inclusion relation on the elements of $P(A)$. Clearly $\varnothing$ is the least member and $A = \{a, b\}$ is the greatest member in $(P(A), \leq)$.

We now discuss certain elements which are of special importance.

5.9 MINIMAL AND MAXIMAL ELEMENTS (MEMBERS)

Definition 5.10: Let $(A, \leq)$ denote a partially ordered set. An element $a \in A$ is called a minimal member of a relative to $\leq$ if for no $x \in A$, is $x < a$.

Similarly an element $b \in A$ is called a maximal member of A relative to the partial ordering $\leq$ if for no $x \in A$, is $b < x$.

The minimal and maximal members of a partially ordered set need not unique.

Example 1: Consider the poset shown in Fig. 5.6

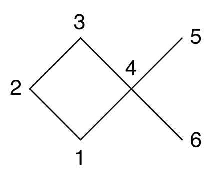


Fig. 5.6

There are two maximal elements and two minimal elements.

The elements 3, 5 are maximal and the elements 1 and 6 are minimal.

Example 2: Let $A = \{a, b, c, d, e\}$ and let Fig. 5.7 represent the partial order on A in the natural way. The element a is maximal. The elements d and e are minimal.

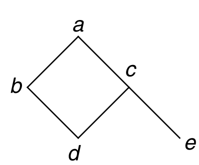


Fig. 5.7

Distinct minimal members of a partially ordered set are incomparable and distinct maximal members of a poset are also incomparable.

5.10 UPPER AND LOWER BOUNDS

Definition 5.11: Let $(A, \leq)$ be a partially ordered set and let $B \leq A$. Any element $m \in A$ is called an upper bound for B if for all $x \in A$, $x \leq m$. Similarly an element $l \in A$ is called a lower bound for B if for all $x \in A$, $l \leq x$.

Example 1: $A = \{1, 2, 3, ..., 6\}$ be ordered as pictured in Fig. 5.8.

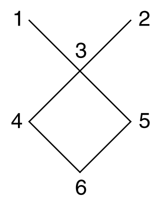


Fig. 5.8

If $B = \{4, 5\}$ then

The upper bounds of B are 1, 2, 3

The lower bound of B is 6.

Example 2: Let $A = \{a, b, c\}$ and $(P(A) \leq)$ be the partially ordered set. The Hasse diagram of the Poset be as pictured in Fig. 5.9.

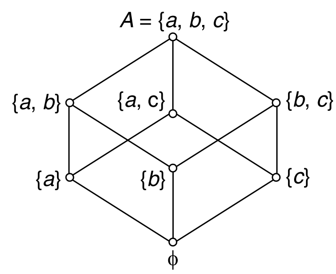


Fig. 5.9

If B is the subset $\{a, c\}$, $\{c\}$. Then the upper bounds of B are $\{a, c\}$ and A, while the lower bounds of B are $\{c\}$ and $\varnothing$.

From the above, it is clear that the upper and lower bounds of a subset are not unique.

5.10.1 Least Upper Bound (Supremum)

Definition 5.12: Set A be a partially ordered set and B a subset of A. An element $m \in A$ is called the least upper bound of B if M is an upper bound of B and $M \leq M'$ whenever M' is an upper bound of B.

A least upper bound of a partially ordered set if it exist is unique.

Example: Let $A = \{a, b, c, d, e, f, g, h\}$ denote a partially ordered set. Whose Hasse diagram is shown in Fig. 5.10:

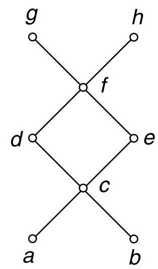


Fig. 5.10

If $B = \{c, d, e\}$ then f, g, h are upper bounds of B. The elements f is least upper bound.

5.10.2 The Greatest Lower Bound (Infimum)

Definition 5.13: Let A be a partially ordered set and B denote a subset of A. An element L is called a greatest lower bound of B if l is a lower of B and $L' \leq L$ whenever L' in a lower bound of B.

The greatest lower bound of a poset if it exists is unique.

Example: Consider the poset $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ whose Hasse diagram is shown in Fig. 5.11 and let $B = \{3, 4, 5\}$

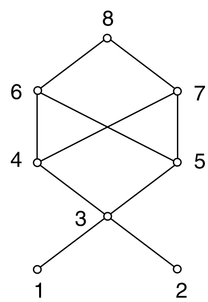


Fig. 5.11

The elements 1, 2, 3 are lower bounds of B. 3 is greatest lower bound.

The least upper bound (LUB) and the greatest lower bound (GLB) of subset B are also called the supremum and infimum of the subset B.

Note:

(i) The least upper bound of a set is abbreviated as $l \cdot u \cdot b$ or *sup* and the greatest lower bound is abbreviated as "$g \cdot l \cdot b$" or "*inf*".
(ii) If A is a chain, then every subset S: has a supremum and an infimum.
(iii) Let N be the set of positive integers and let N by ordered by divisibility. If a and b are two elements of N, then

$inf\,(a, b) = gcd\,(a, b)$
and Sup (a, b) = Lcm (a, b)

(*iv*) If $(A, \leq)$ is a poset, then its dual $(A, \geq)$ is also a poset. The least member of $(A, \leq)$ is the greatest member in $(A, \geq)$ relative to $\geq$ and vice versa. Similarly the $g \cdot l \cdot b$ of A with respect to the relation $\leq$ is the as $g \cdot l \cdot b$ of A with respect to the relation $\geq$ and vice versa.

(*v*) $g \cdot l \cdot b$ of a and b is called the meet or product of a and b and the l.u.b of a and b is called the join or sum of a and b where $a, b \in N$. The symbols such as * and $\oplus$ are also used to denote meet and join respectively.

Theorem 5.1: Let $(A, \leq)$ be a partially ordered set and S be a subset of A. Then

(*i*) The least upper bound of set, if it exists is unique.

(*ii*) The greatest lower bound of S, if it exists is unique.

i.e., S can have at most, one least upper bound and at most one greatest lower bound.

Proof: (*i*) If possible let there be two least upper bounds for S, say b_1 and b_2. Now b_2 is supremum and b_1 is an upper bound of $S \Rightarrow b_2 \leq b_1$. Similarly b_1 is supremum and b_2 is an upper bound of $S \Rightarrow b_1 \leq b_2$. $S \subseteq A$, therefore by symmetric property $b_2 \leq b_2, b_1 \leq b_2 \Rightarrow b_1 = b_2$. Hence, least upper bound of S is unique.

(*ii*) Left as an exercise.

Theorem 5.2: Let A be finite non-empty poset with partial order $\leq$. Then A has atleast, one maximal element.

Proof: Let $a \in A$. If a is not the maximal element. Then we can find an element $a_1 \in A$ such that $a < a_1$. It a_1 is not a maximal element of A, then we can find an element $a_2 \in A$ such that $a_1 < a_2$. Continuing this argument we get a chain

$$a < a_1 < a_2 < a_3 < \dots a_{r-1} < a_r$$

Since A is finite this chain cannot be extended and for any $b \in A$, we cannot have $a_r < b$. Hence a_r is a maximal element of $(A, \leq)$.

By the same argument, the dual poset $(A, \geq)$ has a maximal element such that $(A, \leq)$ has a minimal element.

Theorem 5.3: If $(A, \leq)$ and $(B, \leq)$ are partially ordered sets, then $(A \times B, \leq)$ is a partially ordered set with the partial order $\leq$, defined by $(a, b) \leq (a', b')$ if $a \leq a'$ in A and $b \leq b'$ in B.

Proof: $a \leq a'$ in A and $b \leq b'$ in B

$\therefore$ $$(a, b) \in A \times B \text{ implies } (a, b) \leq (a, b)$$

Hence $\leq$ satisfies reflexive property in $A \times B$.

Let $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$ where a, a' are the members of A and b, b' are the members of B.

Then $a \leq a'$ and $a' \leq a$ in A and $b \leq b'$ and $b' \leq b$ in B

Now $$a \leq a' \text{ and } a' \leq a \Rightarrow a = a'$$

(since A is a partially ordered set)

and $$b \le b' \text{ and } b' \le b \Rightarrow b = b'$$

(since B is a partially ordered set)

$\therefore \le$ is anti-symmetric in $A \times B$

Also $(a, b) \le a', b'$ and $(a', b') \le (a'', b'')$ in $A \times B$ where $a, a', a'' \in A$ and $b, b', b'' \in A$ implies that

$$a \le a' \text{ and } a' \le a''$$

and $$b \le b' \text{ and } b' \le b''$$

by the transitive property of the partial orders in A and B, we have

$$a \le a', a' \le a'' \Rightarrow a \le a''$$

and $$b \le b', b' \le b'' \Rightarrow b \le b''$$

Hence $(a, b) \le (a'', b'')$

Therefore transitive property holds for partial order in $A \times B$. Hence $A \times B$ is a partially ordered set.

5.11 WELL-ORDER SET

Definition 5.14: A set with an ordering relation is well-ordered if every non-empty subset of the set has a least element.

Example: The set of natural numbers is well-ordered.

5.12 BINARY AND N-ARY OPERATIONS

Definition 5.15: Let A be a non-empty set and f be a mapping f: $A \times A \to A$. Then f is called a binary operation on the set A and the mapping

$$f: A^n \to A$$

is called an n-ary operation on A. If, f is an n-ary operation; then n is called the order of the operation f: $A \to A$ (i.e., $n = 1$), is called a unary operation.

Examples: (i) Addition is a binary operation on the set of natural numbers.

(ii) Addition, multiplication and subtraction are binary operations on the set of integers.

It is customary to denote a binary operation by symbols such as +, –, 0, *, $\cup, \cap$, etc. A binary operation on the elements of a set produces images which are again the members of the same set. A given set with the given binary operation is said to be closed with respect to the binary operation.

5.13 CHOICE FUNCTIONS

Definition 5.16: Let $\{A_i : i \in I\}$ be a collection of non-empty disjoint sets and let $A_i \le X \ \forall \ i$. A function

$f: \{A_i\} \to \times$ defined by

$f(A_i) = a_i \in A_i$ is called a choice function.

5.14 AXIOM OF CHOICE

There exists a choice functions for any non-empty collection of non-empty sets.

We now state a theorem called the well-ordering theorem without proof.

5.15 WELL-ORDERING THEOREM

Theorem 5.4: Every set A can be well-ordered.

5.16 LATTICES

In this section, we introduce lattices which have important applications in the theory and design of computers.

Definition 5.17: A lattice is a partially ordered set $(L, \leq)$ in which every pair of elements $a, b \in L$ has a greatest lower bound and a least upper bound.

Example 1: Let Z^+ denote the set of all positive integers and let R denote the relation 'division' in Z^+. Such that for any two elements $a, b \in Z^+$, $aR\, b$, if a divides b. Then (Z^+, R) is a lattice in which the join of a and b is the least common multiple of a and b, i.e. $a \vee b = a \oplus b$ = LCM of a and b, and the meet of a and b, i.e. $a * b$ is the greatest common divisor (GCD) a and b i.e.,

$$a \wedge b = a * b = \text{GCD of } a \text{ and } b$$

We can also write $a + b = a \vee b = a \oplus b =$ LCM of a and b and $a \,.\, b = a \wedge b = a * b =$ GCD of a and b.

Example 2: Let S be a non-empty set and $L = P(S)$; $(P(S); \leq)$ i.e., $(L, \leq)$ is a partially ordered set. If A and B are two elements of L, then we have $A \cup B = A \vee B$ and $A \cap B = A \wedge B$

Hence the $(L, \leq)$ is a Lattice.

Example 3: Let $\cap$ be a positive integer and S_n be the set of all divisors of n ... S_n. If $n = 30$, $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$. Let R denote the relation division as defined in example 1. Then (S_{30}, R) is a Lattice see Fig. 5.12.

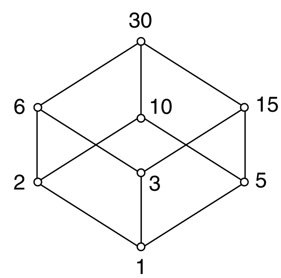


Fig. 5.12

Different lattices can be represented by the same, Hasse diagram. If $(L, \leq)$ is a lattice, then $(L, \geq)$ is also a lattice. The operations of meet and join on $(L, \leq)$ become the operations of join and meet on

$(L, \geq)$. The statement involving the operations * and $\oplus$ and $\leq$ hold if we replace * by $\oplus$, $\oplus$ by * and $\leq$ by $\geq$. The lattices $(L, \leq)$ and $(L, \geq)$ are duals of each other.

Example 4: Let A be a non-empty set and $L = P(A)$. Then $(L, \leq)$ is a lattice. Its dual $(L, \geq)$ is also a lattice.

5.17 SOME PROPERTIES OF LATTICES

Let $(L, \leq)$ be a lattice and '·' and '+' denote the two binary operation meet and join on $(L, \leq)$. Then for any $a, b, c \in L$ we have

$(L-1)$ $a \cdot a = a$, $(L-1)'$ $a + a = a$ (Idempolint laws)

$(L-2)$ $a \cdot a = b \cdot a$, $(L-2)'$ $a + b = b + a$ (Commutative laws)

$(L-3)$ $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $(L-3)'$ $(a + b) + c = a + (b + c)$ (Associative laws)

$(L-4)$ $a \cdot (a + b) = a$, $(L-4)'$ $a + (a \cdot b) = a$ (Absorption laws).

The above properties (L, – 1) to (L, – 4) can be proved easily by using definitions of meet and join. We can apply the principle of duality and obtain $(L-1)'$ to $(L-4)'$.

Theorem 5.5: Let $(L, \leq)$ be a lattice in which '·' And '+' denote the operations of meet and join respectively. Then

$$a \leq b \Leftrightarrow a \cdot b = a \Leftrightarrow a + b = b \quad \forall\ a, b, c \in L$$

Proof: Let $a \leq b$

We know that $a \leq a$, therefore $a \leq a \cdot b$ but from the definition we have $a \cdot b \leq a$

$$\therefore \qquad a \leq b \Rightarrow a \cdot b = a$$

let us assume that $a \cdot b = a$

but this is possible only if $a \leq b$

i.e., $a \cdot b = a \Rightarrow a \leq b$

$$\therefore \qquad a \leq b \Rightarrow a \cdot b = a \text{ and } a \cdot b = a \Rightarrow a \leq b$$

combining these two, we get

$$a \leq b \Leftrightarrow a \cdot b = a$$

now let $a \cdot b = a$, then we have

$$b + (a \cdot b) = b + a + a + b$$

but $b + (a \cdot b) = b$

Hence $a + b = b$

Similarly by assuming $a + b = b$ we can show that $a \cdot b = a$

Hence $a \leq b \Leftrightarrow a \cdot b = a \Leftrightarrow a + b = b = a$

Theorem 5.6: Let $(L, \leq)$ be a lattice. Then

$$b \leq c \Rightarrow \begin{cases} a \cdot b \leq a \cdot c \\ a + b \leq a + c \end{cases}$$

For all $a, b, c \in L$

Proof: From Theorem 5.4

$$b \leq c \Leftrightarrow b \cdot c = b$$

now $$(a \cdot b),\ (a \cdot c) = (a \cdot a)\ (b \cdot c) = a\ (b \cdot c)\ \ a \cdot b$$

$$\Rightarrow a \cdot b \leq a \cdot c$$

Similarly we can prove $a + b \leq a + c\ \forall\ a, b, c \in L$

Note: The above properties of a Lattice are called properties of Isotonicity

We now state the following theorem without proof:

Theorem 5.7: Let $(L, \leq)$ be a Lattice. Then

$$a + (b \cdot c) \leq (a + b) \cdot (a + c)$$

$$a \cdot (b + c) \leq (a \cdot b) + (a \cdot c)$$

for all $a, b, c \in L$

Proof: The proof is left as an exercise.

5.18 LATTICE AS AN ALGEBRAIC SYSTEM

We now define lattice as an algebraic system, so that we can apply many concepts associated with algebraic systems to lattices.

Definition 5.18: A lattice is an algebraic system $(L, \cdot, +)$ with two binary operation '·' and '+' on L which are both commutative and associative and satisfy absorption laws.

5.19 BOUNDED LATTICES

If L is a lattice, then every pair of elements of L has a least upper bound and a greatest lower bound. If A is a finite subset of A, then A has both least upper bound and greatest lower bound. This property may not hold if A is not a finite subset of L, we find greatest lower bound and least upper bound of a subset of a lattice as follows.

Let $(L, \cdot, +)$ be a lattice and $A \leq L$ be a finite subset of L. The greatest and least upper bound of A are defined as

$$g\,l\,b\ \ A = \underset{i=1}{\overset{n}{\cdot}}\ a_i \text{ and } l \cup b\ \ A = \underset{i=1}{\overset{n}{+}}\ a_i$$

Where $A = \{a_1, a_2, \ldots a_n\}$

Example: Show that for a bounded, distributive lattice, complement of an element is unique.

(VTU Aug. 2000)

Solution: Let L be a bounded distributive lattice.

Let $e \in L$. If possible let e' and e'' be the complements of e in L.

Then $e + e' = 1$ and $e + e'' = 1$

$$e \cdot e' = 0 \text{ and } e \cdot e'' = 0$$

now
$$\begin{aligned} e' &= e' + 0 \\ &= e' + (e \cdot e'') \\ &= (e' + e) \cdot (e' + e'') \\ &= (e + e') \cdot (e' + e'') \\ &= 1 \cdot (e' + e'') \end{aligned}$$

Thus
$$e' = e' + e'' \qquad \text{... (1)}$$

Also
$$\begin{aligned} e'' &= e'' + 0 \\ &= e'' + (e \cdot e') \\ &= (e'' + e) \cdot (e'' + e') \end{aligned}$$

Definition 5.19: A lattice is called complete if each of its non-empty subsets has *a* least upper bound and a greatest lower bound.

The least and greatest elements of a lattice L, if they exist are called the bounds of the lattice L, they are denoted by 0 and 1 respectively. A lattice which has both 0 and 1 is called a bounded lattice. Every finite lattice must be complete, and every complete lattice must have a least element and a greatest elements. The bounds of L satisfy the following:

$$a + 0 = a, \; a \cdot 1 = a$$
$$a + 1 = 1, \; a \cdot 0 = 0 \text{ for any } a \in L$$

If L is a bounded lattice, then the elements 0 and 1 duals of each other. If L is a bounds lattice, then we denote it by $(L, \cdot, +, 0, 1)$

Definition 5.20: Let $(L, \cdot, +, 0, 1)$ be a bounded lattice and $a \in L$. If there exists an element $b \in L$ such that

$$a \cdot b = 0 \text{ and } a + b = 1$$

then b is called the complement of a

Example 1: Let A be a non-empty set and $L = P(A)$. Then the every element of L has a complement.

Example 2: In the lattice shown in Fig. 5.13 the elements a and d are complements of each other

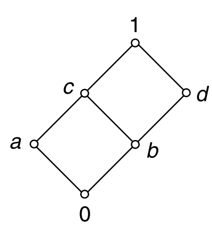


Fig. 5.13

Definition 5.21: A lattice L is complemented if it is bounded and if every element in L has a complement.

Example: If S is non-empty set and $L = P(S)$. Then each element of L has a unique complement in L. Therefore $L = P(S)$ is a complemented lattice.

5.20 SUB LATTICES, DIRECT PRODUCTS

Definition 5.22: Let $(L, \leq)$ be a lattice. A non-empty subset A of L is called a sub lattice of L if $a + b \in A$ and $a \cdot b \in A$ whenever $a \in A$ and $b \in A$.

If A is a sub lattice of L, then A is closed under the operations of '·' and '+'.

Example 1: Let Z^+ be the set of all positive integers and let D denote the relation "division" in Z^+ such that for any $a, b \in Z^+$, $a\ D\ b$ if a divides 6. Then (Z^+, D) is a lattice in which $a + b$ = LCM of a and b and $a \cdot b$ = GCD of a and b.

Example 2: Let n be a positive integer and S_n be the set of all divisiors of n. If D denote the relation as defined above (in example 1). Then (S_n, D) is a sub lattice of (Z^+, D).

Example 3: Consider the lattice L shown in Fig. 5.14. The subset $A = \{a, c, d, y\}$ is a sub lattice of L.

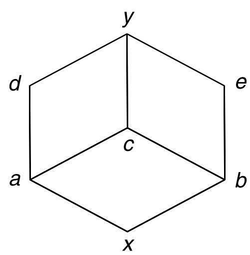


Fig. 5.14

Definition 5.23: Let $(L_1, *, +)$ and $(L_2, \wedge, \vee)$ be two lattices. The algebraic system $(L_1 \times L_2, ..., +)$ in which the binary operation + and '·' are on $L_1 \times L_2$ defined as

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$$

$$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$$

for all (a_1, b_1) and (a_2, b_2) $(a_2, b_2) \in L_1 \times L_2$

is called the direct product of the lattices L_1 and L_2.

Example 4: Let $L_1 = \{1, 2, 4\}$ and $L_2 = \{1, 3, 9\}$, clearly L_1 and L_2 are chains and division is a partial ordering on L_1 and L_2 and L_1 consists of divisors of 4 and L_2 consists of divisor of 9. $L_1 \times L_2$ consists of 36 where each node in the diagram of $L_1 \times L_2$ is shown as (a, b) (instead of $a\ b$).

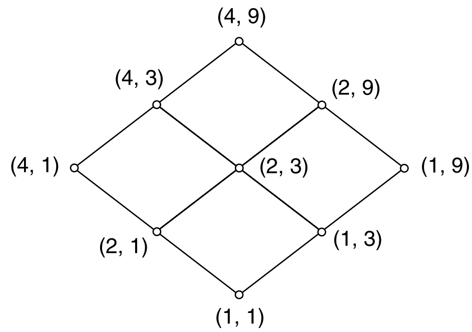
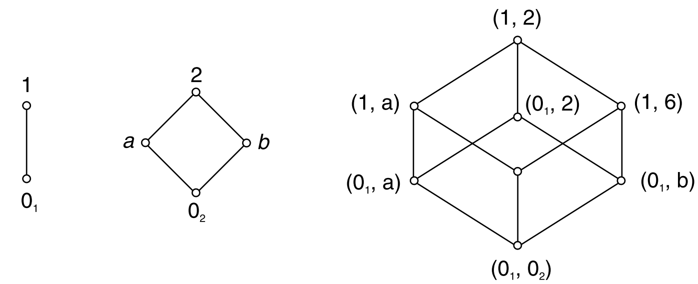


Fig. 5.15 Direct products of two lattices

Example 5: Let L_1 and L_2 be two lattices shown Fig. 5.16 (*a*) and (*b*) respectively. Then $L_1 \times L_2$ is the lattice shown Fig. 5.16. (*c*).



(*a*) (*b*) (*c*)

Fig. 5.16

We can use the direct product of lattices to construct larger lattices from the smaller ones. If L is a lattice we can form lattices $L \times L$, $L \times L \times L$, $L \times L \times L \times L$, ... which are denoted L^2, L^3, L^4, ... respectively.

Example 6: If $L = (0, 1)$ and $(L, \leq)$ is a lattice, then $(L^n, \leq_n)$ is a lattice of n-tuples of 0 and 1. Any element in the lattice $(L^n, \leq_n)$ can be written as $(a_1, a_2, \ldots a_n)$ in which a_1 is either 0 and 1 for $i = 1, 2, 3, \ldots, 4$.

The partial ordering relation on L^n can be defined for any a, b in L^n as

$$a \leq_n b \Leftrightarrow a_i, \leq_n b_i \text{ for } i = 1, 2, \ldots, n$$

where $a = (a_1, a_2, \ldots a_n)$ and

$$b = (b_1, b_2, \ldots, b_n)$$

In general the diagram of $(L^n, \leq_n)$ is an n-cube.

Definition 5.24: Let $(L, \cdot, +, 0, 1)$ be a lattice L is said to complemented lattice if every element has atleast one complement.

Example 7: Let $(L_3, \leq_3)$ be a lattice of 3-tuples of 0 and 1. The complement of an element of L_3 can be obtained by changing 1 by 0 and 0 by 1 in the 3-tuples representing the element.

The complement of (0, 1, 1) is (1, 0, 0), the complement of (1, 0, 1) is (0, 1, 0) and so an. The bounds of (0, 0, 0) and (1, 1, 1).

Definition 5.25: A lattice $(L, \cdot, +)$ is called a distributive lattice if for any $a, b, c \in L$

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c) \text{ and}$$
$$a + (b \cdot c) = (a + b) \cdot (a + c)$$

Example 8: $(L_3, \leq_3)$ is distributive.

Example 9: The power set of a non-empty set A is a lattice under the operation $\cap$ and $\cup$ is a distributive lattice.

Definition 5.26: Let $(L, \leq)$ be a lattice, with a lower bound 0. An element $a \in L$ is said to be join irreducible if $a = x + y \Rightarrow a = x$ or $a = y$.

Example 10: $0 \in L$ is join irreducible.

Let $a \neq 0$ be an element of L. The element a is join irreducible if and only if a has unique immediate predecessor.

Definition 5.27: Let $(L, \leq)$ be a lattice, with an upper bound 1. An element $a \in L$ is said to be meet irreducible if $a = x \cdot y$ implies $a = x$ or $a = y$.

If $a \neq 0$ then a is meet irreducible if and only if a has unique immediate sucessor.

Example 11: Find the join irreducible and meet irreducible elements of the lattice shown in Fig. 5.17.

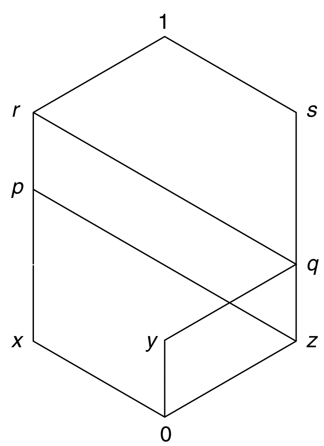


Fig. 5.17

Solution: The elements x, y, z and s are join irreducible. The elements x, y, p, r and s are meet irreducible.

Definition 5.28: The join irreducible elements of a lattice L, which immediately succeed 0 are called atoms.

Example 12: In the lattice shown in Fig. 5.18, 2 is the 0 element (lower bound) 3 succeeds 2, hence atom of L is the element 3.

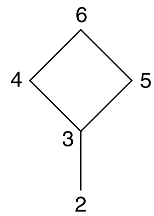
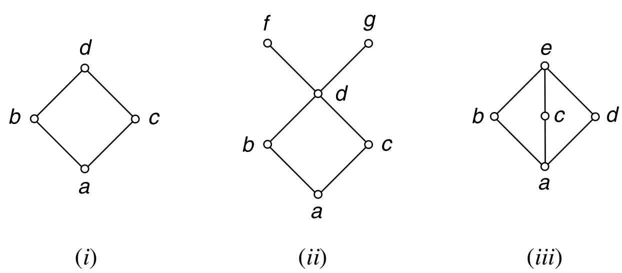


Fig. 5.18

EXERCISE 5.1

1. Define the terms
   (*a*) Partially ordered set
   (*b*) Linearly ordered set.
   Give examples.
2. Let $\leq$ be a Partial ordering of a set S. Define the dual order on S. How is the dual order related to the iverse of the relation $\leq$.
3. Define Lenilographical order on $A \times B$ where A and B are two linearly ordered sets.
4. Define the terms 'immediate predecessor' and 'immediate successor' and show that each element of a linearly ordered set can have at most one immediate predecessor.
5. What is meant by a 'Hasse diagram'? Draw the Hasse diagram of the relation R on A where $A = \{1, 2, 3, 4\}$ and
   $R = \{(1, 1), (1, 2), (2, 2), (2, 4), (1, 3), (3, 3), (3, 4), (1, 4), (4, 4)\}$.
6. Let n be a positive integer and S_n be the set of all divisors of n. Let D denote the relation of 'division' '*ns*, such that $a\ D\ b$' iff a divides b. Draw the Hasse diagram for (S_n, D).
7. Let $A = \{1, 2, 3, 4, 6, 8, 9, 12, 24\}$ be ordered by divisibility.
8. Determine the greatest and least elements, if they exist of the poset
   $A = \{2, 4, 6, 8, 12, 18, 24, 36, 72\}$ with the partial order of divisibility.
9. Which of the Hasse diagram in the figure given below represents lattices:



10. If $(L_1, \leq)$ and $(L_2, \leq)$ are two lattices, then show that $(L_1, \times L_2, \leq)$ is also a lattice.

11. Define that term ‘sub lattice’. Give an example.

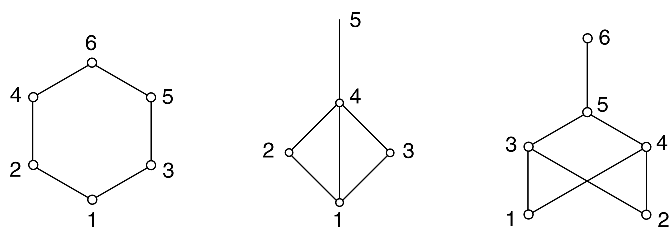
12. What is meant by a bounded lattice? If L is a finite lattice show that L is bounded.

13. Show that a subset of a linearly ordered poset is a sub lattice.

14. Show that a linearly ordered poset is a distributive lattice.

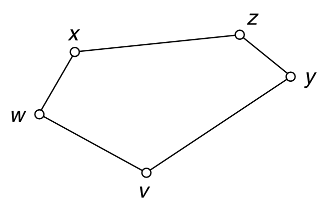
15. Let L be a bounded lattice with atleast two elements. Show that no two elements of L is its own complement.

16. Which of the partially ordered sets shown in the figure below are lattices.



(a) (b) (c)

17. Consider the lattices $D = \{v, w, x, y, z\}$ shown in the figure given below. Find all the sub lattices with three or more elements.



18. Suppose the following collection of sets is ordered by set inclusion:

$$A = \{\{a\}, \{a, b\}, \{a, b, c, d\}, \{a, b, c, d, e, f\}\}.$$

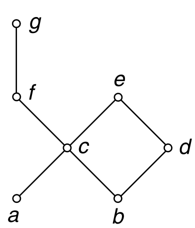
Is A well-ordered?

19. Define the dual of a statement in a lattice L. Why does principle of duality apply to L?

20. Suppose L is a linearly ordered set. Show that S has almost one maximal element.

21. $S = \{2, 4, 6, 12, 20\}$ is ordered by divisibility. Find the maximal and minimal elements of S.

22. Find all the maximal and minimal elements of the poset B diagrammed in the figure below:



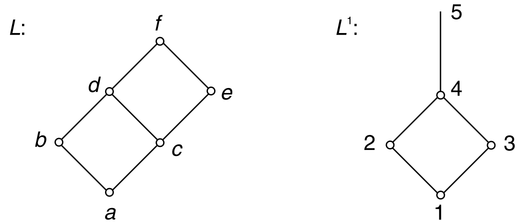
23. Show that every chain is a distributive lattice.

24. Show that the operations of meet and join in a lattice are, commutative, associative and idempotent.

25. If $(L, \leq)$ is a lattice in which * and $\oplus$ denote the operations of meet and join respectively then show that

$$a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$$
$$\forall \; a, b, c \in L$$

26. Define Isomorphic lattices. Show that the lattices L and L^1 given below are not isomorphic:



27. Define the terms

(*a*) Distributive Lattice.

(*b*) Join irreducible elements of a Lattice.

28. Show that if a bounded lattice has two or more elements then $0 \neq 1$.

29. L is a bounded lattice. If L is distributive and the complement of an element $a \in L$ exists, then show that it is unique.

5.21 BOOLEAN ALGEBRA

Definition 5.29: A Boolean algebra is a distributive complemented lattice having atleast two elements as well as 0 and 1.

A Boolean algebra is generally denoted by a 6-tuple, $(B, +, \cdot, {}^1, 0, 1)$ where $(B, +, \cdot)$ is a lattice with two binary operations + and $\cdot$, called the join and meet respectively is a unary operation in B. The elements 0 and 1 are the least and greatest elements of the lattice $(B, +, \cdot)$. The following axioms are satisfied:

1. There exist at least two elements a, b in B and that $a \neq b$.

2. $\forall \; a, b \in B$

   (*i*) $a + b \in B$

   (*ii*) $a \cdot b \in B$

3. for all $a, b \in B$

   $$\left.\begin{aligned} &(i)\ a + b = b + a \\ &(ii)\ a \cdot b = b \cdot a \end{aligned}\right\} \text{ commutative laws}$$

4. Associative laws: for all $a, b, c \in B$

   (*i*) $a + (b + c) = (a + b) + c$

   (*ii*) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

5. Distributive laws: for all $a, b, c \in B$

 (*i*) $a + (b \cdot c) = (a + b) \cdot (a + c)$

 (*ii*) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

6. (*i*) Existence of zero: There exists of B such that

$$a + 0 = a \ \forall \ a \in B$$

 The element 0 is called the zero element

 (*ii*) Existence of unit: There exists $1 \in B$ sum that

$$a \cdot 1 = a \ \forall \ a \in B$$

 The element 1 is called the unit element.

7. Existence of complement: $\forall \ a \in B$ there exists an element $a' \in B$ such that

 (*i*) $a + a' = 1$ and (*ii*) $a \cdot a' = 0$

Example 1: Let $A_1, A_2, ..., A_n$ be subsets of a universal set X. The set of all subsets of $\{A_1, A_2, ..., A_n\}$. Which can be formed from A_i by union intersection and complement together with the binary operation $\cup$ and $\cap$, and the unary operation is a Boolean algebra.

Example 2: Let $B = \{0, 1\}$ and let $+, \cdot$ be two operations in B defined by the following operation tables (*a*) and (*b*):

+	1	0
1	1	1
0	1	0

(*a*)

·	1	0
1	1	0
0	0	0

(*b*)

Suppose that the complements are defined by $1^1 = 0$ and $0^1 = 1^1$, then B is a Boolean algebra.

Example 3: Let B_n denote the set of n-bit sequences. Let the operations of sum. Product and complement in B_n defined as follows:

For all $a, b \in B_n$ ... $a + b$ contains 1 if a, b contains 1, $a \cdot b$ contains 1 if a and b contain: a and b contains 1 if a contains 0, then B_n is a Boolean algebra.

Example 4: Let $\pi 1$ be the set of all propositions $T1$ is a Boolean algebra under the operations $\vee$ and A with ~ being the complement. The contradiction f is the zero element and the taulogy T is unit element of in $\pi 1$.

When $\cdot$ operations are performed before + operation the parentheses are not used, we use the letter B to represent a Boolean algebra $(B, +, \cdot, ^1, 0, 1)$ and we often use the symbols $\vee$ and $\wedge$ in the place of + and operation. The dual of any statement S in B is the statement obtained by interchanging the operations + and $\cdot$ and interchanging the identity elements 0 and 1, in the original statement S. Also the dual of any theorem in B is also a theorem. We shall now prove some theorems.

Theorem 5.8: $\forall \ a \in B, a + a = 0$

Proof: $a + a = (a + a) \cdot 1$ (axiom 6 (ii))

$= (a + a) \cdot (a + a')$ (axiom 7(i))

$= a + a \cdot a^1$ (axiom 5(i))

$= a + 0$ (axiom 7(ii))

$= a$ (axiom 6(i))

Theorem 5.9: $\forall\ a \in B,\ a \cdot a = a$

Proof: $a \cdot a = a \cdot a + 0$ (axiom 6 (i))

$= a \cdot a + a \cdot a'$ (axiom 7(ii))

$= a \cdot (a + a')$ (axiom 5(ii))

$= a \cdot 1$ (axiom 7(i))

$= a$

Theorem 5.10: The elements 0 and 1 in B are unique.

Proof: Assume that 0_1 and 0_2 are two zero elements in B. Such that $a_1 + 0_1 = a_1$ and $a_2 + 0_2 = a_2$ $\forall\ a_1 \in B$ and $a_2 \in B$

Consider $a_1 + 0_1 = a_1$

taking $a_1 = 0_2$, we get

$0_2 + 0_1 = 0_2$

Similarly by taking $a_2 = 0_1$ in $a_2 + 0_2 = a_2$, we get $0_1 + 0_2 = 0_1$

Hence $0_2 + 0_1 = 0_1 + 0_2 \Rightarrow 0_1 = 0_2$

by the principle of duality we can easily show that the unit element 1 in unique in B.

Theorem 5.11: In each Boolean algebra

(*i*) $0' = 1$

(*ii*) $1' = 0$

Proof: (*i*) we have $0' = 0 + 0' = 1$

and (*ii*) $1' = 1 \cdot 1 = 0$

Theorem 5.12: For any $a \in B$, (*i*) $a + 1 = 1$ (*ii*) $a \cdot 0 = 0$

Proof: (*i*) $a + 1 = (a + 1) \cdot 1$

$= (a + 1) \cdot (a + a')$

$= a + 1 \cdot a'$

$= a + a'$

$= 0$

(*ii*) applying principle of duality, we get

$$a \cdot 0 = 0$$

Theorem 5.13: For any $a, b \in B$

(i) $a + a \cdot b = a$ (ii) $a \cdot (a + b) = a$ (absorption laws)

Proof: (i) We have

$$\begin{aligned} & a + a \cdot b \\ &= a \cdot 1 + a \cdot b \\ &= a \cdot (1 + b) \\ &= a \cdot 1 \\ &= a \end{aligned}$$

(ii) We have

$$\begin{aligned} a \cdot (a + b) &= (a + 0)\,(a + b) \\ &= a + 0 \cdot b \\ &= a + 0 \\ &= a \end{aligned}$$

Theorem 5.14: For each $a \in B$, there exists a unique complement.

Proof: Let a_1' and a_2' be two complements of a in B

then $$a + a_1' = 1,\ a + a_2' = 1$$

and $$a \cdot a_1' = 0,\ a \cdot a_2' = 0$$

now
$$\begin{aligned} a_1' &= 1 \cdot a_1' \\ &= (a + a_2') \cdot a_1' \\ &= a \cdot a_1' + a_2' \cdot a_1' \\ &= 0 + a_2' \cdot a_1' \\ &= a\, a_2' + a_1' \cdot a_2' \\ &= (a + a_1') \cdot a_2' \\ &= 1 \cdot a_2' \\ &= a_2' \end{aligned}$$

Hence the complement a^1 for each $a \in B$ is unique.

Theorem 5.15: For each $a \in B$, $(a')' = a$ (Involution law)

Proof: By definition of complement

$$a + a' = 1 \text{ and } a' \cdot a = 0$$

now $$a + a' = a \Rightarrow a' + 1 \Rightarrow a' + a$$

and $$a \cdot a' = 0 \Rightarrow a' \cdot a = 0$$

by uniqueness a is the complement of a^1

hence $$(a')' = a \ \forall\ a \in B$$

Theorem 5.16: In a Boolean algebra B,

$$a + (b + c) = (a + b) + c$$

and $a \cdot (b \cdot c) = (a \cdot b) \cdot c \ \ \forall\ a \in B$, (Associative laws)

Proof: Left as an exercise

Theorem 5.17: For any $a; b \in B$

$$(a + b)' = a' \cdot b'$$

and $(a \cdot b) = a' + b'$ (De Morgan's laws)

Proof: (*i*) The theorem is proved if we show that

$$(a + b) = (a' \cdot b') = 1 \text{ and } (a + b), (a' \cdot b') = 0$$

Consider $(a + b)\,(a' \cdot b') = b + a + (a' \cdot b')$

$$= b + (a + a') \cdot (a + b')$$

$$= b + 1 \cdot (a + b')$$

$$= b + a + b'$$

$$= b + b' + a$$

$$= 1 + a$$

$$= 1$$

also

$$(a + b) \cdot (a' \cdot b')$$

$$= ((a + b) \cdot a') \cdot b'$$

$$= ((a \cdot a') + (b \cdot a')) \cdot b'$$

$$= (0 + (b \cdot a')) \cdot b'$$

$$= (b \cdot a') \cdot b'$$

$$= (b \cdot b') \cdot a'$$

$$= 0 \cdot a'$$

$$= 0$$

Hence $(a + b') = a' \cdot b'$

(*ii*) Follows from 5.15 (*i*) by the principle of duality.

5.22 SUB-BOOLEAN ALGEBRA

Definition 5.30: Let $(B, +, \cdot, ', 0, 1)$ be a Boolean algebra and $S \leq B$. If S contains the elements 0 and 1 and is closed under the operations +, and $\cdot$, then $(S, +, \cdot, ', 0, 1)$ is called a sub-Boolean algebra)

For any Boolean algebra, the set {0, 1} and B are both sub-Boolean algebra of B.

Example: Consider the Boolean algebra B shown in Fig. 5.19.

The subset $S_1 = \{a, a', 0, 1\}$ is a sub-Boolean algebra of B. The subset $S_2 = \{a \cdot b', b', 0, 1\}$ is not a sub-Boolean algebra.

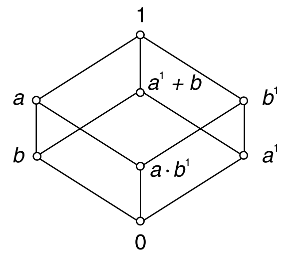


Fig. 5.19

5.23 DIRECT PRODUCTS

Definition 5.31: Let $(B_1, +_1, \cdot, ', 0_1, 1_1)$ and $(B_2, +_2, \cdot, '', 0_2, 1_2)$ be Boolean algebra. The direct product of the Boolean algebras denoted by $(B_1 \times B_2, +_3, \cdot_3, ''', 0_3, 1_3)$ is a Boolean algebra in which the operation are defined as follows:

$$(a_1, b_1) +_3 (a_2, b_2) = (a_1 +_1 a_2, b_1 +_2 b_2)$$

$$(a_1, b_1) \cdot_3 (a_2, b_2) = (a_1 \cdot_1 a_2, b_1 \cdot_2 b_2)$$

$$(a_1, b_1)''' = (a_1', b_1'') \ \forall \ (a_1, b_1), \text{ and } (a_2, a_2) \in B$$

also $\qquad 0_3 = (0_1, 0_2)$ and $1_3 = (1_1, 1_2)$

using the definition given above we can generate new Boolean algebras. If B is a Boolean algebra we can generate the Boolean algebras $B \times B = B^2$, $B \times B \times B = B^3$, $B \times B \times B \times B = B^4$, ...

5.24 HOMOMORPHISM

Definition 5.32: Let $(B, +, \cdot, ', 0, 1)$ and $(B_1, +_1, \cdot_1, ^-, 0_1, 1_1)$ be two Boolean algebras. A function $f: B \to B,$ is called a Boolean algebra homomorphism, if f preserve the two binary operation and the unary operation i.e., for all $a, b \in B$

$$f(a + b) = f(a) +_1 f(b)$$

$$f(a \cdot b) = f(a) \cdot, f(b)$$

$$f(a') = f(\overline{a})$$

$$f(0) = 0_1$$

and $\qquad f(1) = 1_1$

5.25 ATOMS OF BOOLEAN ALGEBRA

Let $(L, +, \cdot)$ be a lattice. An element $a \in L$ is called join-irreducible if it cannot expressed as the join of two distinct elements of L.

i.e., $a \in L$ is join irreducible, if for any $a_1, a_2, \in L$

$$a = a_1 + a_2 \Rightarrow (a = a_1) \text{ or } (a = a_2)$$

In the case of a Boolean algebra, the elements which cover the least element 0, are the only elements which are join irreducible. They are called atoms of the Boolean algebra. An element $a \in B$ is called an atom if $a \neq 0,$ and either $a \cdot b = 0$ or $a \cdot b = 1 \ \forall \ b \in B.$ The atoms a Boolean algebra are also called minterms. We can also represent the elements of a Boolean algebra in terms of the meet of their anti-atoms. The anti-atoms in Boolean algebra are those elements of the Boolean algebra which are covered by the greatest element 1. Anti-atoms are also called Maxterms. They are the complements of the atoms.

5.26 BOOLEAN EXPRESSIONS AND MINIMIZATION OF BOOLEAN FUNCTIONS

Boolean expressions are formed by application of the basic operations +, ·, and ^{1}, to one or more constants of variables. The simplest expression consists of a single constant or a variable such as 0 or a.

Definition 5.33: A Boolean expression or form, in n variables $x_1, x_2, ..., x_n$ is any finite string of symbols formed as given below:

1. 0 and 1 are Boolean expression.
2. $x_1, x_2, ..., x_n$ are Boolean expressions.
3. If α and β are Boolean expression, then $(\alpha) \cdot (\beta)$ and $(\alpha) + (\beta)$ are also Boolean expressions.
4. If α is a Boolean expression then $(\alpha)^1$ is also a Boolean expression.
5. No strings symbols except those formed in accordance with the above rules are Boolean expressions.

If α is a Boolean expression in n variables, say $x_1, x_2, ..., x_n$ then α can be written as α $(x_1, x_2, ..., x_n)$. A Boolean expression in n variables may or may not contain all the n variables.

Some examples of Boolean expressions are

$$x_1 \ x_2', \ x_1(x_2 + x_3)', \ x_1 + x_1' \ x_2 + x_3 \ x_1'$$

Parentheses are added to specify the order in which the operations are performed and some of them can be dropped whenever possible.

0, 1, x_1, x_2, ... x_3 are Boolean expressions. If α and β are two Boolean expression, then $(\alpha), \alpha^1, \alpha + \beta$ and $\alpha \ldots \beta$ are also Boolean expression.

If $\alpha = (x_1, x_2, \ldots x_n)$ is a Boolean expression then we can assign values $a_1, a_2, \ldots a_n$ respectively to the variables where each a_i is either 0 or 1.

For example

Consider the Boolean expression

$$\alpha = (x_1, x_2, \ldots x_n) = [(x_1 \cdot x_2) + x_3]'$$

If $x_1 = 1, x_2 = 0,$ and $x_3 = 0,$ then

$$\begin{aligned} \alpha \ (x_1, x_2, x_3) &= \alpha \ (1, 0, 0) \\ &= [(1.0) + 0] \end{aligned}$$

$$= (0 + 0)'$$
$$= 0'$$
$$= 1$$

Definition 5.34: Two Boolean expression α $(x_1, x_2, ..., x_n)$ and β $(x_1, x_2, ..., x_n)$ are said to be equal (or equivalent) if one can be obtained from the other by a finite number of application of the identities of Boolean algebra.

Definition 5.35: A literal is defined to be a Boolean variable or its complement.

Example 1: x, and x' are literals.

Definition 5.36: A literal or a product of two or more literals in which no two literals involve the same variable is called a fundamental product.

Example 2: $x_1\ x_2'\ x_3$, $x_1\ x_2\ x_3$, xy' are fundamental products.

Definition 5.37: A Boolean expression generated by $x_1, x_2, ..., x_n$ over B, which has the form of conjunction (product) of n literals is called a minterm.

The number of minterm generated by n variables in B_2 is 2^n.

The two variables x_1 and x_2 generate, the minterms x_1, x_2, $x_1' x_2$, $x_1 x_2'$, and $x_1' x_2$ in B_2. A minterm form of a Boolean expression is also called sum-of-products form or complete product of n variables.

We shall denote a particular *min* term by min_j or m_j where j is the decimal representation of $a_1\ a_2$ ... a_n and each a_i is either 0 or 1 for i = 1, 2, ..., n. The minterms satisfies the following properties.

$$m_i \cdot m_j = 0 \text{ for } i \neq j$$

and
$$\mathop{+}_{i=0}^{2^n-1} m_i = 1$$

For $i \neq j$ the minterm m_i and m_j are not equal every Boolean expression except 0, can be expressed in an equivalent form consisting of the sums of minterms. When an expression is written as a sum of minterms, the equivalent form obtained is called a sum of products canonical form or a minterm expansion. In a minterm expansion any particular minterm may or may not be present. The number of different sum of products canonical forms is 2^{2^n}. These include minterms expansion of 0, in which no minterm is present in the sum, and also the minterm expansion of 1, where all the minterms are present in the sum. Therefore, the set of Boolean expression can be partitioned into 2^{2^n} equivalence classes. The set of Boolean expression under the operation +, $\cdot$ and ^{1}, form a Boolean algebra called a free Boolean algebra.

A Boolean expression can also be written as a product of sums of Maxterm, the equivalent expression obtained is called a product sums canonical form or maxterm expansion. Each maxterm is the complement of the corresponding minterm. In general, a maxterm of n variables is a sum of n literals in which each variable appears exactly once in either true or complement form, but not both. The maxterm expansion for an expression is unique.

The minterm expansion i.e., sum of products canonical form is called the disjunctive normal form and the maxterm expansion or products of sums canonical form is called the conjunction normal form.

Definition 5.38: Let $(B, +, \cdot, ', 0, 1)$ be a Boolean algebra. A function $f: B_n \to B$ which is associated with a Boolean expression in n variables is called a Boolean function.

Every function from B^n to B, may n of be Boolean, and there are functions from B^n to B which are not Boolean. Different expressions may determine the same Boolean functions. Absorption laws, De Morgan's laws, distributive laws and the other identities for Boolean algebras bring out the redundancy of Boolean expression. To transform Boolean expression E into a sums-of-products form; we first use De Morgan's laws and involution law, to convert E into a form which contains only sum and products of literals. We next use distributive law to transform E into a sums-of-products form. The next step is to transform each product in E into 0 or a fundamental product. This done by using the commutative, idempotent and complement laws. Finally, we use absorption law to get the sums-of-product form of E. To obtain complete sum-of-product form of E. We involve all the variables in each product of the sum-of-product form of E.

Example 3: Transform $((x_1 x_2)' x_3)' ((x_1' + x_3)(x_2' + x_3'))'$ into a sums-of-products form.

Solution: $((x_1 x_2)' x_3)' ((x_1' + x_3)(x_2' + x_3'))'$

$$= [((x_1 x_2)')' + x_3'] [(x_1' + x_3)' + (x_2' + x_3')']$$

$$= [x_1 x_2 + x_3'] [((x_1 x_3')')' + ((x_2 x_3)')']$$

$$= [x_1 x_2 + x_3'] [x_1 x_3' + x_2 x_3]$$

$$= x_1 x_2 x_1 x_3' + x_1 x_2 x_2 x_3 + x_3' x_1 x_3' + x_3' x_2 x_3$$

$$= x_1 x_2 x_3' + x_1 x_2 x_3 + x_1 x_3' + 0$$

$$= x_1 x_3' + x_1 x_2 x_3 + x_1 x_2 x_3'$$

Example 4: Express (*i*) $x_1 \cdot x_2$ (*ii*) $x_1(x_2' x_3)'$

In an equivalent sum-of-products canonical form in three variables x_1, x_2 and x_3.

Solution: (*i*) $x_1 \cdot x_2 = x_1 \cdot x_2 \cdot (x_3 + x_3')$

$$= x_1 \cdot x_2 \cdot x_3 + x_1 \cdot x_2 \cdot x_3' = m_7 + m_6$$

$$= m_6 + m_7$$

$$\therefore \quad x_1 \cdot x_2 = m_6 + m_7$$

(*ii*) $x_1(x_2' x_3)'$

$$= x_1[(x_2')' + x_3']$$

$$= x_1[x_2 + x_3']$$

$$= x_1 x_2 + x_1 x_3'$$

$$= x_1 x_2(x_3 + x_3') + x_1(x_2 + x_2')x_3'$$

$$= x_1 x_2 x_3 + x_1 x_2 x_3' + x_1 x_2 x_3' + x_1 x_2' x_3'$$

$$= x_1 x_2 x_3 + x_1 x_2 x_3' + x_1 x_2' x_3'$$

$$= m_7 + m_6 + m_4$$

$$= m_4 + m_6 + m_7$$

$$= +4, 6, 7 = \sum m\ (4, 6, 7)$$

Example 5: Write $f(x_1, x_2, x_3) = x_1' x_2 x_3 + x_1 x_2' x_3 + x_1 x_2 x_3' + x_1 x_2 x_3 + x_1 x_2' x_3'$ in term of m-notation.

Solution: $f = x_1' x_2 x_3 + x_1 x_2' x_3 + x_1 x_2 x_3' + x_1 x_2 x_3 + x_1 x_2' x_3'$

$$= min_3 + min_5 + min_6 + min_7 + min_4$$
$$= min_3 + min_4 + min_5 + min_6 + min_7$$
$$= m_3 + m_4 + m_5 + m_6 + m_7$$
$$= \sum m\ (3, 4, 5, 6, 7)$$

Example 6: $f = x_1'(x_2' + x_4) + x_1 x_3 x_4'$

Solution: $f = x_1'(x_2' + x_4) + x_1 x_3 x_4'$

$$= x_1' x_2' + x_1' x_1' x_4 + x_1 x_3 x_4'$$

$$= x_1' x_2'(x_3 + x_3')\ (x_4 + x_4') + x_1(x_2 + x_2')\ (x_3 + x_3')x_4 + x_1(x_2 + x_2')x_3 x_4'$$

$$= x_1' x_2' x_3 x_4 + x_1' x_2' x_3' x_4 + x_1' x_2' x_3 x_4' + x_1' x_2' x_3' x_4' + x_1 x_2 x_3 x_4 +$$
$$x_1 x_2 x_3' x_4 + x_1 x_2' x_3 x_4 + x_1 x_2' x_3' x_4 + x_1 x_2 x_3 x_4' + x_1 x_2' x_3 x_4'$$

$$= m_0 + m_1 + m_3 + m_5 + m_7 + m_{10} + m_{14}$$

$$= \sum m\ (0, 1, 3, 5, 7, 10, 14)$$

Example 7: Write $Z = (x_1 + x_2 + x_3)\ (x_1 + x_2 + x_3')\ (x_1 + x_2' + x_3)$ in M-notation.

Solution: $Z = (x_1 + x_2 + x_3)\ (x_1 + x_2 + x_3')\ (x_1 + x_2' + x_3)$

$$= \text{Max}_0\ \text{Max}_1\ \text{Max}_3$$

$$= \prod M\ (0, 1, 2)$$

Example 8: $f = \sum m\ (0, 1, 2, 5)$ is a three input function. Transform f into its canonical sum-of-products form.

Solution: Let x_1, x_2 and x_3 denote the three inputs then

$$m_0 = 000 = x_1' x_2' x_3'$$

$$m_1 = 000 = x_1' x_2' x_3$$

$$m_2 = 010 = x_1' x_2 x_3'$$

$$m_3 = 101 = x_1 x_2' x_3$$

$\therefore$ Canonical sum-of-products form of the expression is

$$f = x_1' x_2' x_3' + x_1' x_2' x_3 + x_1' x_2 x_3' + x_1 x_2' x_3'$$

Example 9: Covert $f(x_1 x_2 x_3) = \prod (0, 2, 4, 5)$ into its canonical products-of-sums form.

Solution: $f(x_1 x_2 x_3) = \prod (0, 2, 4, 5)$

$= M_0 \ M_1 \ M_2 \ M_5$

we have

$$M_0 = 000 = x_1 x_2 x_3$$

$$M_1 = 000 = x_1 x_2' x_3$$

$$M_2 = 010 = x_1' x_2 x_3$$

$$M_3 = 101 = x_1' x_2 x_3'$$

$\therefore$ The required canonical product-of-sums form is

$$f = (x_1 x_2 x_3)\ (x_1 x_2' x_3)\ (x_1' \ x_2 \ x_3)\ (x_1' x_2 x_3')$$

Example 10: Obtain the three variable product-of-sums canonical form of the Boolean expression $x_1 \cdot x_2$.

Solution: Let x_3 denote the variable then

$$x_1 \cdot x_2 = [x_1 + (x_2 \cdot x_2')]\ [x_2 + (x_1 \cdot x_1')]$$

$$= (x_1 + x_2) \cdot (x_1 + x_2') \cdot (x_1 + x_2) \cdot (x_1' + x_2)$$

$$= (x_1 + x_2) \cdot (x_1 + x_2')\ (x_1' + x_2)$$

$$= [(x_1 + x_2) \cdot (x_3 + x_2')]\ [(x_1 + x_2') \cdot (x_3 x_3')]\ [(x_1' + x_2) \cdot (x_3 x_3')]$$

$$= (x_1 + x_2 + x_3) \cdot (x_1 + x_2 + x_3') \cdot (x_1 + x_2' + x_3)$$

$$(x_1 + x_2' + x_3')\ (x_1' + x_2 + x_3)\ (x_1' + x_2 + x_3')$$

$$= \text{Max}_0 \cdot \text{Max}_1 \cdot \text{Max}_2 \cdot \text{Max}_3 \cdot \text{Max}_4 \cdot \text{Max}_5$$

$$= M_0 \cdot M_1 \cdot M_2 \cdot M_3 \cdot M_4 \cdot M_5$$

$$= \prod M\ (0, 1, 2, 3, 4, 5)$$

Let $(B, +, \cdot, ', 0, 1)$ be any Boolean algebra and $(a_1, a_2, \ldots, a_n) \in B^n$ where $a_i \in B$. If α $(x_1, x_2, \ldots x_n)$ is a Boolean expression, we can find the value of α $(x_1, x_2, \ldots x_n)$ for $(a_1, a_2, \ldots a_n)$ by replacing x_1 by a_1, x_2 by a_2, ..., x_n by a_n.

Example 11: Find the value of $x_1 + (x_1 x_2)$ over the ordered pairs of the two-element Boolean algebra.

Solution: Let $B = \{0, 1)$ then the (0, 0), (0, 1), (1, 0) and (1, 1) are the elements of $B^2 = B \times B$.

The values of $x_1 + (x_1 \cdot x_2)$ are listed in Table 5.1 given below.

Table 5.1

$(x_1 \cdot x_2)$	$x_1 + (x_1 \cdot x_2)$
(0, 0)	0
(0, 1)	0
(1, 0)	1
(1, 1)	1

Definition 5.39: Let P_1 and P_2 be fundamental products such that exactly one variable say x_i appears in complemented form in one of the products P_1 and P_2 and uncomplemented in the other. The consensus of P_1 and P_2 is the product of the literals P and the literals of P_2 after deleting x_i and x_i^1.

Example 12: (*i*) The consensus of $A\ B$ and $A'C$ is BC.

(*ii*) The consensus of $A\ B'C$ and $A'B'C'$ is 0.

(*iii*) The consensus of $x_1\ x_2\ x_3'\ x_4$ and $x_1\ x_2'\ x_5$ is $x_1\ x_3'\ x_4\ x_5$.

Consensus method is very useful in simplifying Boolean expressions. It is used to eliminate redundant terms in a Boolean expression. The redundant terms which are eliminated are called consensus terms.

Example 13: In the expression in $E = x_1\ x_2 + x_1'\ x_3 + x_2\ x_3$ the terms $x_2\ x_3$ is redundant. It is referred to as the consensus term. Eliminating $x_2\ x_3$ be can write $E = x_1\ x_2 + x_1'\ x_3$ as the simplified expression for E.

Definition 5.40: Let E be a Boolean expression. A fundamental product P is called a prime implicant of E if $P + E = E$ but no other fundamental product included in P has this property.

Example 14: $x_1\ x_3'$ is a prime implicant of the Boolean expression

$$E = x_1\ x_2' + x_1\ x_2\ x_3' + x_1'\ x_2\ x_3'.$$

5.27 MINIMIZATION OF BOOLEAN EXPRESSIONS

Boolean expressions are practically implemented in the form of gates. The cost of a circuit depends upon the number of gates in the circuit. Hence we reduce the member of gates in the circuit to a minimum so that the cost of the circuit is decreased to a miximum extent.

In this section, we explain to methods for simplification of Boolean expressions, namely (*i*) Algebraic method and (*ii*) Karnaugh map method.

5.27.1 Algebra Method

In this method, we make use of Boolean positates rules and theorem to simplify given Boolean expressions.

Example 1: Simplify

$$F = \overline{A}\ \overline{B}\ \overline{C} + \overline{A}\ B\ \overline{C} + A\ \overline{B}\ \overline{C} + A\ B\ \overline{C}$$

Solution:

$$F = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,B\,\overline{C} + A\,\overline{B}\,\overline{C} + A\,B\,\overline{C}$$
$$= (\overline{A}\,\overline{B} + \overline{A}\,B + A\,\overline{B} + A\,B)\,\overline{C}$$
$$= (\overline{A}(\overline{B} + B) + A(\overline{B} + B))\,\overline{C}$$
$$= (\overline{A}\cdot(1) + A\,(1))\cdot\overline{C}$$
$$= (\overline{A} + A)\,\overline{C}$$
$$= 1\cdot\overline{C}$$
$$= \overline{C}$$

Example 2: Simplify $z\,(y + z)\,(x + y + z)$

Solution:

$$z\,(y + z)\,(x + y + z)$$
$$= (z\,y + z\,z)\,(x + y + z)$$
$$= (z\,y + z)\,(x + y + z)$$
$$= z\,(y + 1)\,(x + y + z)$$
$$= z\,(x + y + z)$$
$$= z\,x + z\,y + z\,z$$
$$= z\,x + z\,y + z$$
$$= z\,(x + y + 1)$$
$$= z\,(x + 1)$$
$$= z$$

Example 3: Simplify $Y = (P + Q)\,(P + Q')\,(P' + Q)$

Solution:

$$Y = (P + Q)\,(P + Q')\,(P' + Q)$$
$$= (P\,P + P\,Q' + P\,Q + Q\,Q')\,(P' + Q)$$
$$= (P + P\,Q' + P\,Q + 0)\,(P' + Q)$$
$$= (P + P\,Q' + P\,Q)\,(P' + Q)$$
$$= P\,P^1 + P\,Q + P\,Q'\,P' + P\,Q'\,Q + P\,Q\,P' + P\,Q\,Q$$
$$= 0 + P\,Q + 0 + 0 + 0 + P\,Q$$
$$= P\,Q + P\,Q$$
$$= P\,Q$$

Example 4: Show that $Y = P\,Q\,R + P\,Q'\,R + P\,Q\,R'$ can be simplifies as $Y = P\,(Q + R)$

Solution:

$$Y = P\,Q\,R + P\,Q'\,R + P\,Q\,R'$$
$$= P\,R\cdot(Q + Q') + P\,Q\,R'$$
$$= P\,R\cdot 1 + P\,Q\,R'$$
$$= P\,(R + Q\,R')$$
$$= P\,(R + Q)$$
$$= P\,(Q + R)$$

Example 5: Minimize the expression $\overline{A\,B} + \overline{A} + A\,B$

Solution:

$$
\begin{aligned}
& \overline{A\,B} + \overline{A} + A\,B \\
&= \overline{A} + \overline{B} + \overline{A} + A\,B \\
&= \overline{A} + \overline{A} + \overline{B} + A\,B \\
&= \overline{A} + \overline{B} + A\,B \\
&= \overline{A} + A\,B + \overline{B} \\
&= (\overline{A} + \overline{\overline{A}}\,B)\,\overline{B} \\
&= \overline{A} + B + \overline{B} \\
&= \overline{A} + 1 = 1
\end{aligned}
$$

Hence $\overline{A\,B} + \overline{A} + A\,B = 1$

5.27.2 Karnaugh Maps

A Boolean expression generally denoted the structure of a logical circuit, while the Boolean function describes the behaviour of the circuit. Many different circuits or programs can be used to compute the same Boolean function: It is often desirable to select the one that is simplest. The algebraic techniques used to simplify Boolean functions are difficult to apply in a systematic way. The Karnaugh method (named after Maurice Karnaugh) is a systematic method for simplifying switching (Boolean) functions.

The Karnaugh map is a graphical representation of the truth table with a square representing each minterm. If f is a function of n variables, then the Karnaugh map will have 2^n squares. 1-variable Karnaugh map is shown in Fig. 5.20. Note that the map has $2^1 = 2$ cells.

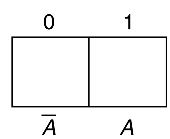


Fig. 5.20 Karnaugh map for 1-variable

Consider the Truth table shown in Table 5.2 for a function Z of two variables. To convert the table into its Karnaugh map, we begin by drawing 5.21 (*a*) (i.e., a blank map):

Table 5.2

A	B	Z
0	0	0
0	1	0
1	0	1
1	1	1

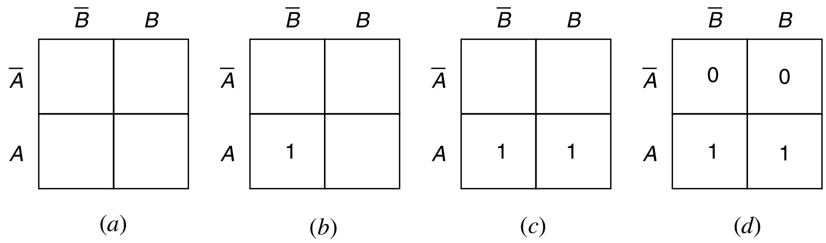


Fig. 5.21

The first out put 1 appears for $A = 1$ and $B = 0$.

The input condition for this fundamental product is $A\overline{B}$. Enter this input condition in the Karnaugh map as shown in Fig. 5.21 (*b*). Table 5.2 has an output 1 appearing for $A = 1$ and $B = 1$. This fundamental product is $A\,B$. Enter this fundamental product (i.e., $A\,B$) as shown in Fig. 5.21 (*c*). Finally enter 0s in the remaining spaces (See Fig. 5.21 (*d*). A two variable Karnaugh map, can be represented as shown in Fig. 5.22:

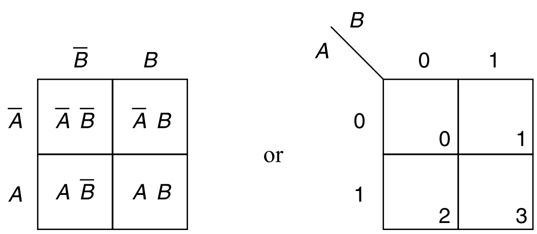


Fig. 5.22 Two variable Karnaugh map representing minterms

If the top horizontal line represents $\overline{A}$ and A and the vertical line represents $\overline{B}$ and B, then the Karnaugh map, can be drawn as shown in Fig. 5.23:

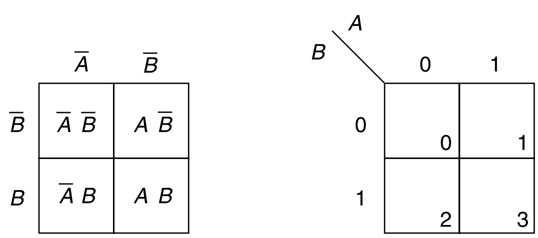


Fig. 5.23 2-Variable Karnaugh map

In the above Karnaugh map we observe that in every square a number is written. Each number is a minterm. If the number 0 is given to a square means it represents the minterm m_0. Similarly, a square (cell) with the number 1 represents the minterm m_1 a square with the number 2 represents m_2 and so on.

The binary number in the Karnaugh map differ by only one place, when moving from left to right. That is two adjust squares in Karnaugh map differ only by one variable.

The successive numbers are 00, 01, 11 and 10, where

00 represents $\overline{A}\,\overline{B}$

01 represents $\overline{A}B$

11 represents AB

and 10 represents $A\overline{B}$

In these products only one variable changes from complemented to uncomplemented form. The above code is called gray code. The Karnaugh map for the truth Table 5.3 is shown in Fig. 5.24.

Table 5.3

A	B	Z
0	0	1
0	1	1
1	0	0
1	1	0

	$\overline{B}$	B
$\overline{A}$	1	1
A	0	0

Fig. 5.24

Figure 5.25 shows a Karnaugh map for 3 variables.

	$\overline{A}\,\overline{B}$	$\overline{A}B$	AB	$A\overline{B}$
$\overline{C}$	$\overline{A}\,\overline{B}\,\overline{C}$	$\overline{A}B\overline{C}$	$AB\overline{C}$	$A\overline{B}\,\overline{C}$
C	$\overline{A}\,\overline{B}C$	$\overline{A}BC$	ABC	$A\overline{B}C$

$C \backslash AB$	00	01	11	10
0	0	1	3	2
1	4	5	7	6

Fig. 5.25 Karnaugh map for 3 variables

Note that the numbering scheme here is 0, 1, 3, 2 then 4, 5, 7, 6.

Figure 5.26 shows a four variables Karnaugh map.

	$\bar{A}\bar{B}$	$\bar{A}B$	AB	$A\bar{B}$
$\bar{C}\bar{D}$	$\bar{A}\bar{B}\bar{C}\bar{D}$	$\bar{A}B\bar{C}\bar{D}$	$AB\bar{C}\bar{D}$	$A\bar{B}\bar{C}\bar{D}$
$\bar{C}D$	$\bar{A}\bar{B}\bar{C}D$	$\bar{A}\bar{B}\bar{C}D$	$AB\bar{C}D$	$A\bar{B}\bar{C}D$
CD	$\bar{A}\bar{B}CD$	$\bar{A}BCD$	$ABCD$	$A\bar{B}CD$
$C\bar{D}$	$\bar{A}\bar{B}C\bar{D}$	$\bar{A}BC\bar{D}$	$ABC\bar{D}$	$A\bar{B}C\bar{D}$

$CD \backslash AB$	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	9	11	10

Fig. 5.26 Karnaugh map for four variables

Example 1: Draw a Karnaugh map for Table 5.4:

Table 5.4

A	*B*	*C*	*Z*
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

Solution: We first draw the blank map of Fig. 5.27 (*a*) output 1 appears *A B C* inputs of 000, 001, 010 and 110. The fundamental products for these conditions are $\bar{A}\bar{B}\bar{C}$, $\bar{A}\bar{B}C$, $\bar{A}B\bar{C}$ and $AB\bar{C}$. Enter *Is* for these products on the Karnaugh map (Fig. 5.27 (*b*)).

Finally enter 0*S* in the remaining spaces as shown in Fig. 5.27 (*c*).

$C \backslash AB$	00	01	11	10
0				
1				

(*a*)

$C \backslash AB$	00	01	11	10
0	1	1	1	
1	1			

(*b*)

$C \backslash AB$	00	01	11	10
0	1	1	1	0
1	1	0	0	0

(*c*)

Fig. 5.27

Figure 5.27 (*c*) represents the Karnaugh map for Table 5.4.

Pair in a Karnaugh Map

Consider the Karnaugh map shown in Fig. 5.28. The map contains a pair is which are adjust to each other. The map represents the sum-of-products equation $Z = A\ B\ C\ D + A\ B\ C\ \overline{D}$. Where the first minterm represents the product $A\ B\ C\ D$ and the second minterm stands for the product $A\ B\ C\ \overline{D}$. The variable D in the uncomplemented form goes to the complemented form and the variables A, B and C remain uncomplemented. The variable D can be eliminated.

	$\overline{A}\overline{B}$	$\overline{A}B$	AB	$A\overline{B}$
$\overline{C}\overline{D}$	0	0	0	0
$\overline{C}D$	0	0	0	0
CD	0	0	1	0
$C\overline{D}$	0	0	1	0

	$\overline{A}\overline{B}$	$\overline{A}B$	AB	$A\overline{B}$
$\overline{C}\overline{D}$	0	0	0	0
$\overline{C}D$	0	0	0	0
CD	0	0	1	0
$C\overline{D}$	0	0	1	0

Fig. 5.28 Example of pair

In a pair, the variable which changes its state from complemented to uncomplemented (or vise versa) is removed. This rule is called pair reduction rule.

To draw a Karnaugh map for a truth table with don't care conditions; we first treat the don't conditions as 1s and encircle actual 1s in the largest groups. The remaining don't cares (which are not included in the groups) are regarded by visualizing them as 0*s*.

Thus, *A* pair in a Karnaugh map eliminates are variable and its complement.

In the above expression $Z = A\ B\ C\ D + A\ B\ C\ \overline{D}$ can be factored as $Z = A\ B\ C\ (D + \overline{D})$.

Quads

In a Karnaugh map, a quad is a group of four 1s that are horizontally or vertically adjacent. The 1s in a quad may be end-to-end or in the form of a square as shown in Fig. 5.29. A quad eliminates two variables and their complements. The rule is known as quad reduction rule. The quads in a Karnaugh map are always encircled as shown in Fig. 5.29.

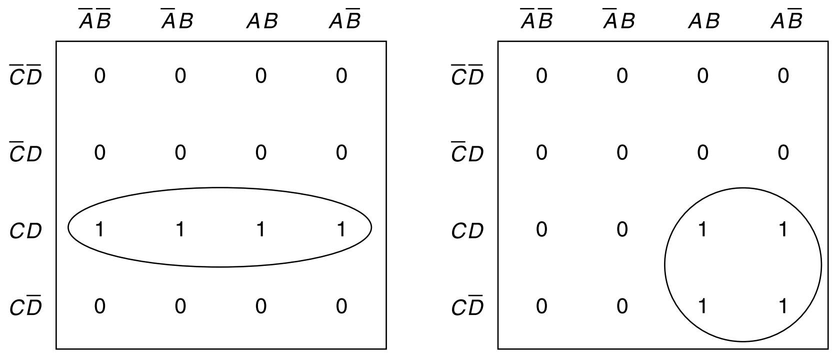


Fig. 5.29 Examples of quads

Octet

The octet in a Karnaugh map is a group of eight *Is*. An octet eliminates three variables and their complements.

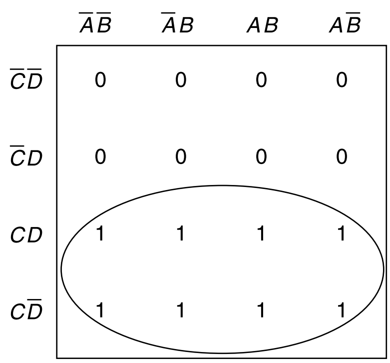


Fig. 5.30 Example of octet

Don't Care Conditions

In some problems certain input combinations may never occur in the circuit therefore the corresponding output never appears. But they appear in the truth table. In such case an *X* is entered in the truth table as functional value. *X* is called a don't-care condition. The logic designer can later assign a functional value 0 or 1 to the corresponding entries in the truth table (Table 5.5).

Table 5.5 *Don't care conditions*

A	B	C	F
0	0	0	1
0	0	1	X
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	X
1	1	1	1

Rolling a Karnaugh Map

Pairs; quads and octets in a Karnaugh map are marked after rolling the map; in which we consider the map as if its left edge are touching the right edges and the top edges are touching the bottom edges.

Redundant Group

In a Karnaugh map, a group whose is are already used by others is called a redundant group (*See* Fig. 5.31). The removal of a redundant group leads to much simpler expression.

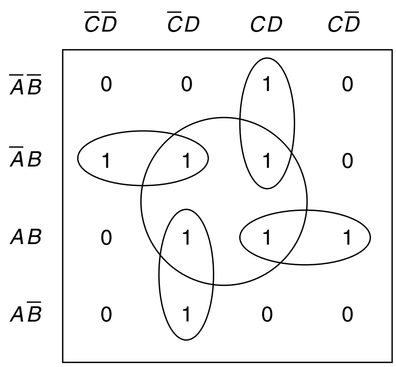


Fig. 5.31 Karnaugh map with redundant group

In the Fig. 5.31 all the 1s of the quad are used by pairs of the map, therefore quad is redundant on Fig. 5.31 and it can be eliminated as shown in Fig. 5.32.

Summary of the rules for simplifying Boolean expression:

1. Construct the truth table for the given expression.
2. Begin with empty Karnaugh map and enter *a* 1 in the Karnaugh map for each fundamental product that produces *a* 1 output in the truth table. Enter 0s 1*s* elsewhere.
3. Encircle the octets, quads and pairs roll the Karnaugh map.
4. Eliminate redundant groups if any.

5. Write the simplified Boolean expression by ORing the products corresponding to the encircled groups.

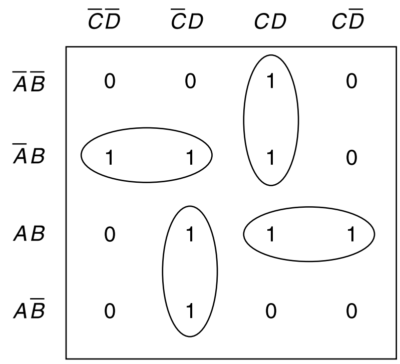


Fig. 5.32 Karnaugh map without redundant group

Example 2: Simplify the Boolean expression

$$Y = A\ B\ \overline{C} + A\ \overline{B}\ \overline{C} + A\ B\ C + A\ \overline{B}\ C$$

Solution: The truth table for Y is shown in Table 5.6:

Table 5.6

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Figure 5.33 showns the Karnaugh map for Y.

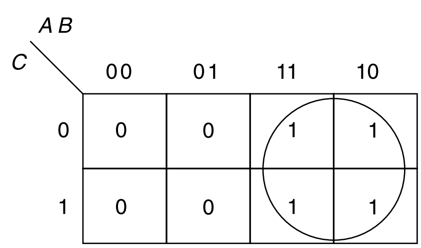


Fig. 5.33

There is quad in the map.

There are no redundant groups in the map A is the only variable which remained unchanged in the map.

$\therefore$ The simplified expression $Y = A$.

This can be proves as follows:

$$
\begin{aligned}
Y &= A\,B\,\overline{C} + A\,\overline{B}\,\overline{C} + A\,B\,C + A\,\overline{B}\,C \\
&= A(B\,\overline{C} + \overline{B}\,\overline{C} + B\,C + \overline{B}\,C) \\
&= A((B + \overline{B})\,\overline{C} + (B + \overline{B})\,C) \\
&= A(1 \cdot \overline{C} + 1 \cdot C) \\
&= A(\overline{C} + C) \\
&= A \cdot 1 \\
&= A
\end{aligned}
$$

Example 3: Obtain a simplified expression for a Boolean expression F (x. y. z) the Karnaugh map for which is given below:

X \ YZ	00	01	11	10
0	 0	1 1	1 3	 2
1	 4	1 5	1 7	 6

Fig. 5.34

Solution: Completing the given Karnaugh map by entering 0s in the empty square, we get the following Karnaugh map (See Fig. 5.35):

X \ YZ	00	01	11	10
0	0 0	1 1	1 3	0 2
1	0 4	1 5	1 7	0 6

Fig. 5.35

There are no pairs, no octets. There is a quad in the map. The quad consists of the minterms m_1, m_3, m_5 and m_7. Moving horizontally from m_1 to m_3 i.e., $\overline{X}\;\overline{Y}\;Z$ to $\overline{X}\;Y\;Z$ we observe that Y changes from complemented form to uncomplemented for,... Therefore Y is eliminated. Moving vertically from m_1 to m_5 or m_3 to m_7, we find that the variable $\overline{X}$ changes to X. Hence we eliminate X.

$\therefore$ The simplified expression for $F(x, y, z)$ is

$$F(x, y, z) = Z$$

Example 4: Simplify $F(A, B, C, D) = \sum (0, 2, 7, 8, 10, 15)$ using Karnaugh map.

Solution: The minterms the function F are

$$m_0 = 0000 = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$$

$$m_2 = 0010 = \overline{A}\,\overline{B}\,C\,\overline{D}$$

$$m_7 = 0111 = \overline{A}\,B\,C\,D$$

$$m_8 = 1000 = A\,\overline{B}\,\overline{C}\,\overline{D}$$

$$m_{10} = A\,\overline{B}\,C\,\overline{D}$$

$$m_{15} = 1111 = A\,B\,C\,D$$

Karnaugh map of the given function is shown in Fig. 5.36:

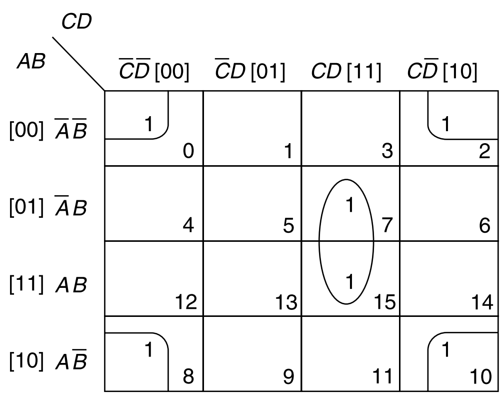


Fig. 5.36

The Karnaugh map has one pair, and one quad.

There are no over lappings.

Consider the pair $m_7 + m_{15}$.

A is the only variable which changes its form, have A is removed.

The reduced expression for the pair $m_7 + m_{15}$ is $B\,C\,D$ quad is $m_0 + m_3 + m_8 + m_{10}$ in the map moving horizontally we observe that the variable C changes its form and then moving vertically. We find that A changes its form. Therefore A and C are removed. The reduced expression for the quad $m_0 + m_3 + m_8 + m_{10}$ is $\overline{B}\,\overline{D}$.

Hence simplified expression for F is

$$B\,C\,D + \overline{B}\,\overline{D}$$

Example 5: Simplify

$$Y = \sum m\,(0, 1, 4, 5, 6, 8, 9, 12, 13, 14.)$$

Solution: The Karnaugh map can be constructed as shown in Fig. 5.37.

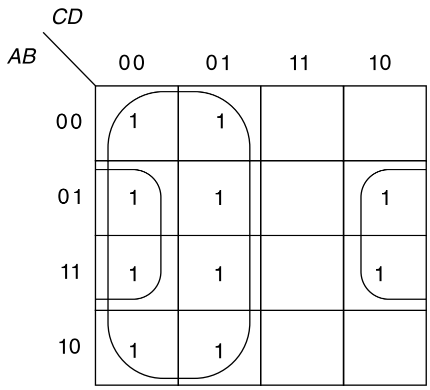


Fig. 5.37

There is one octet and a quad in the *K*-map. The quad is obtained by rolling vertically, such that the left and right edges are joined and over lappings. *A*, *B* and *D*, are the variables eliminated by the Octet in the map. Quad eliminated the variables *A* and *C*.

Octet gives $\overline{C}$ and quad gives $B\,\overline{D}$.

Hence, the reduced expression is $f = \overline{C} + B\,\overline{D}$.

Example 6: Simplify the Boolean function $f(A, B, C, D) = \sum m\,(1, 3, 7, 11, 15) + d\,(A, B, C, D)$ where the don't care conditions are given by $d\,(A, B, C, D) = \sum m\,(0, 2, 5)$.

Solution: The Karnaugh map for *f* can be constructed as shown in Fig. 5.38 (*b*).

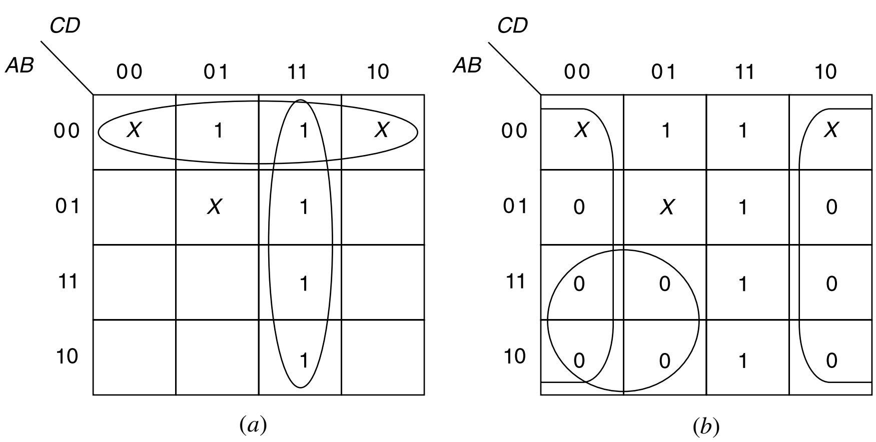


Fig. 5.38

The minterm for d may produce either 0 or 1 for f. The $I\ S$ and $X\ S$ are combined so as to endure maximum number of adjustment squares. The remaining cells are marked 0 as shown in Fig. 5.38 (b). There are two quads in Fig. 5.38 (a). The don't care condition in cell 5 is left free; as it does not contribute to any group in the map. The simplified expression in sum-of-products form is $f = \overline{A}\ \overline{B} + C\ D$. Combining $O\ S$ and $X\ S$ in Fig. 5.38 (b). We get the simplified products-of-sums equation as

$$f = \overline{(\overline{D} + A\ \overline{C})} = D(\overline{A} + C).$$

In this case, a variable can be taken for 0 and its complement is taken for 1.

EXERCISE 5.2

1. Define a Boolean algebra.

2. Define a sub algebra.

3. Define the dual of a statement S in a Boolean algebra.

4. Define an atom in a Boolean algebra B.

5. Define a Boolean expression and give examples.

6. Define a literal and a fundamental products and give examples.

7. What are idempotent laws for Boolean algebras?

8. What is involution law for Boolean algebras?

9. Write the dual of each statement:

(a) $(x + y)\,(x + 1) = x + x\,y + y$

(b) $\overline{(\overline{x} + \overline{y})} = x\,y$

(c) $x\,\overline{y} = 0$ if and only $x\,y = x$.

10. What is a minterm?

11. What is a maxterm?

12. Define a sum-of-products form.

13. Define a product-of-sums form.

14. Prove the following Boolean identities:

(i) $a + (\overline{a} \cdot b) = a + b$

(ii) $a \cdot (\overline{a} + b) = a \cdot b$

(iii) $(a \cdot b \cdot c) + (a \cdot b) = a \cdot b$

15. Simplify the following Boolean expressions:

(a) $(a \cdot b)' + (a \cdot b)'$

(b) $(a' \cdot b' \cdot c) + (a \cdot b' \cdot c) + (a \cdot b' \cdot c')$

16. Write the following Boolean expressions in equivalent sum-of-products form in three variables:

(a) $x_1\,x_2$ (b) $x_1 + x_2$

(*c*) $(x_1 \cdot x_2)' \cdot x_3$ (*d*) $x_1 + (x_2 \cdot x_3')$

(*e*) $(x_1 + x_2) + (x_1' \cdot x_3)$

17. Obtain the sum-of-products and product-of-sums of canonical forms of the following expressions:

(*a*) $x_1 \cdot x_2$ (*b*) $x_1\, x_2' + x_3$

18. Obtain simplified Boolean expressions which are equivalent to these expressions:

(*i*) $m_0 + m_1 + m_2 + m_3$

(*ii*) $m_0 + m_1 + m_2 + m_5 + m_6 + m_7$

(*iii*) $m_2 + m_3 + m_5 + m_6$

19. Express $F = x\,(y^1\ \ z)^1$ in complete sum-of-products form.

20. Write each of the following Boolean expressions in complete sum-of-products form:

(*i*) $E = x(x\,y' + x\,y' + y_2')$

(*ii*) $E = (x + y)'\,(x\,y')'$

(*iii*) $E = x_3\,(x_1' + x_2) + x_2'$

21. Simplify

(*i*) $X = \overline{A}\,\overline{B} + A\,\overline{B}$

(*ii*) $X = A\,B\,\overline{C} + A\,B\,C$

(*iii*) $X = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,B\,\overline{C} + A\,B\,\overline{C} + A\,\overline{B}\,C$

(*iv*) $X = \overline{A}\,\overline{B}\,\overline{C}\,D + \overline{A}\,B\,\overline{C}\,D + A\,B\,\overline{C}\,D + A\,\overline{B}\,\overline{C}\,D$

(*v*) $X = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + A\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}\,\overline{C}\,D + A\,\overline{B}\,C\,\overline{D}$

(*vi*) $X = A\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,B\,\overline{C}\,D + \overline{A}\,B\,C\,D + A\,B\,\overline{C}\,D + A\,B\,C\,D$

22. Simplify the following Boolean function in product of sums form:

$$F\,(A, B, C, D) = \sum\ (0, 1, 2, 5, 8, 9, 10)$$

6

Logic Gates

6.1 INTRODUCTION

Boolean algebra can be applied to the solution of any electronic circuit involving two possible states. We begin our study by examining switching circuits. The most elementary circuit is shown in the figure given below (Fig. 6.1):

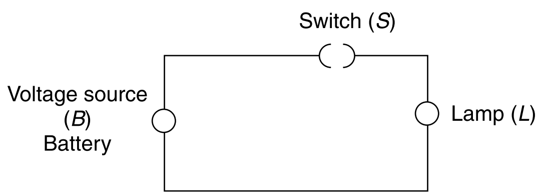
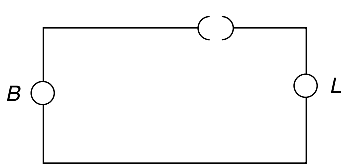


Fig. 6.1

The battery B, a single pole throw switch and an indicating lamp are connected in a simple switching circuit. The electronic circuit is a two-state device. It is for turning ‘on’ and ‘off’ an electronic light in the circuit. We can also construct a device which permit not only electric current but any quantity that can go through such as water, information etc. For general discussion we replace the word ‘switch’ by the word ‘gate’.

Consider the switching circuit displayed in Fig. 6.2 (*a*). When the switch S is open there is no current flowing in the circuit and the lamp L is off. This condition is indicated by the numeric value ‘0’. When the switch is closed the lamp L is on. This condition is indicated by the numerical value ‘1’. Therefore, the value 0 (or dark lamp) will show an open switch and the value 1 will show the closed switch in the circuit (the value 1 indicated a glowing lamp). The condition tables is the truth table which lists all the possible combinations of input binary variables is given in Fig. 6.2 (*b*):



(*a*) Simple switching circuit

S	L
0	0
1	1

(*b*) Condition table

Fig. 6.2

We can express the above condition table in different form as shown in Table 6.1:

Table 6.1

State of the switch	*State of the lamp*
Open	Off
Closed	On

Let P and Q denote the following statements:

P: The switch S is closed.

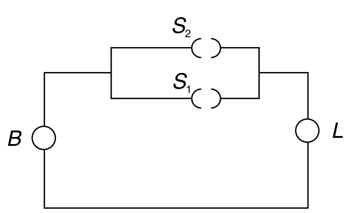
Q: The lamp L is on.

We can rewrite the condition table as follows:

Table 6.2

$P(S)$	$Q(L)$
0	0
1	1

Now consider the circuit shown in Fig. 6.3. The circuit has the switches S_1 and S_2 which are connected in parallel. When the switch S_1 is closed, the current flows exclusively through the upper branch, and the light (L) is on and when the switch S_2 is closed the current flows exclusively through the lower branch and the light is on. With both the switches closed the current divides equally between both branches still permitting the lamp to glow. If both the switches S_1 and S_2 are open the circuit becomes an open circuit and the light is off. This shows that an *OR* function can be obtained connected the switches in parallel. The light is or when either S_1 or S_2 is closed.



(*a*)

S_1	S_2	L
0	0	0
0	1	1
1	0	1
1	1	1

(*b*)

Fig. 6.3

Let P: The switch S_1 is closed.

Q: The switch S_2 is closed.

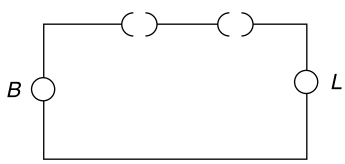
Y: The lamp L is on.

Then the truth Table 6.3 (*b*) can be written as:

Table 6.3

P	Q	Y
0	0	0
0	1	1
1	0	1
1	1	1

Fig. 6.4 (a) Contains the circuit in which two switches S_1 and S_2 are connected in 'series'. If both the switches S_1 and S_2 are closed then the circuit permits the flow of current in the circuit and the light is on. In all other combination the circuit is open and the light is off, showing that '*AND*' function is obtained (See Fig. 6.4):



(a)

S_1	S_2	L
0	0	0
0	1	0
1	0	0
1	1	1

(b)

Fig. 6.4

Let P: The switch S_1 is closed.

Q: The switch S_2 is closed.

Y: The light L is on.

The table of Fig. 6.4 (b) can be written as:

Table 6.4

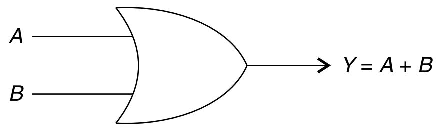
P	Q	Y
0	0	1
0	1	1
1	0	1
1	1	1

6.2 GATES AND BOOLEAN ALGEBRA

Boolean algebra is different from ordinary algebra. It is a system of mathematical logic. A switching network (governs) the flow of current through a circuit. Now we will discuss logic gates which are basically electronic circuits that can be used to actually implement the most elementary logical expressions, known as logic gates. There are three basic logic gates, the OR-gate, the AND-gate and the NOT-gate. Other logic gates that are derived from these three basic gates are NAND-gate, the NOR gate, the EX-OR gate and the EX-NOR gate.

6.2.1 OR-Gate

An OR-gate is a logic circuit with two or more than two inputs and outputs. The output of an OR-gate is '0' only when all of its inputs are at logic '0'. For all other input combinations the output is '1'. The symbol for the OR-gate is shown in Fig. 6.5. along with the associated truth table.



(*a*)

P	*Q*	*Y*
0	0	1
0	1	1
1	0	1
1	1	1

(*b*)

Fig. 6.5 OR-gate

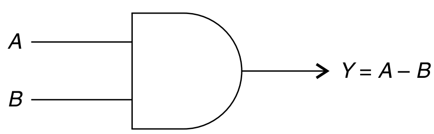
The operation of an OR-gate which is explained by the expression $Y = A + B$ is read as Y equal to A OR B.

Note: Performing OR operation is the same as taking the maximum of two bits.

6.2.2 AND-Gate

An AND-gate is a logic circuit having two or more than two inputs and one output. The output of an AND-gate is logic '1', only when all of its inputs are in '1' state. In all other possible combination the output is '0'. Fig. 6.6 (*a*) shows the symbol of an AND-gate. The truth table is given in Fig. 6.6 (*b*). The operation of an AND-gate is expressed by:

$$Y = A + B$$



(*a*)

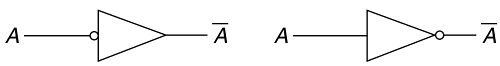
A	*B*	*Y*
0	0	0
0	1	0
1	0	0
1	1	1

(*b*)

Fig. 6.6

6.2.3 NOT-Gate (Inverter)

A NOT-gate is one input and one out put logic gate. The out put of a NOT-gate is always the complement of the input. A NOT-gate is also known as an inverter or a complementing circuit. Fig. 6.7 shows the NOT-gate with the associated truth table.



(*a*) (*b*)

A	$\overline{A}$
0	1
1	0

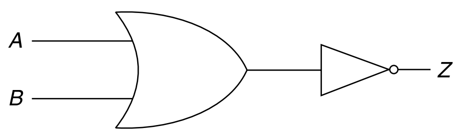
(*c*)

Fig. 6.7 A NOT-gate, equivalent symbol

6.2.4 NOR-Gate

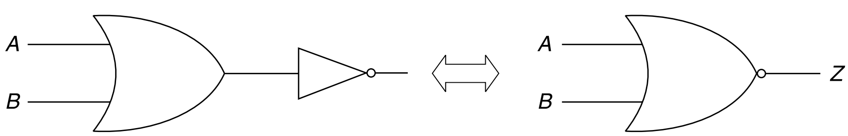
The NOR-gate has two or more input, but produces only are output. The NOR operations is symbolised as $Z = A \downarrow B$.

NOR action is illustrated in Fig. 6.8. The equivalent symbol is shown in Fig. 6.9.



A	B	Z
0	0	1
0	1	0
1	0	0
1	1	0

Fig. 6.8 NOR-gate symbol

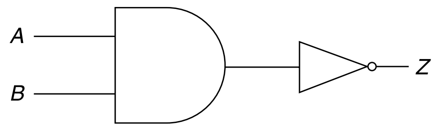


(*a*) (*b*)

Fig. 6.9 NOR-gate equivalent symbol

6.2.5 NAND-Gate

NAND-gate function is a composite function. In this case an AND function is complemented to produce a NAND function placing an *N* in front of AND. The NAND-gate has two or more input signals but only one output signal. The NAND-gate is a contraction of NOT AND. It can have many inputs as desired. The symbol for NAND-gate along with the truth table is shown in Fig. 6.10.



(*a*)

A	B	Z
0	0	1
0	1	1
1	0	1
1	1	0

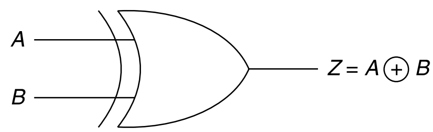
(*b*)

Fig. 6.10 NAND-gate

NAND operation is symbolised as $\uparrow$ i.e., A NAND B is written as $A \uparrow B$.

6.2.6 Exclusive OR-Gate (EX-OR Gate or XOR Gate)

EX-OR stands for exclusive OR. It differs from an OR gate in only are of the entries in the truth table. The EX-OR gate can also L are have two or more inputs but produces one output signal. The symbol for Exclusive OR is shown in Fig. 6.11. Where the operation of the gate is expressed by $Z = A \oplus B$.



(*a*)

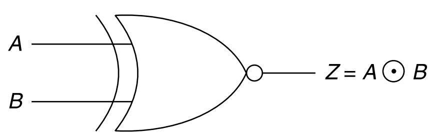
A	B	Z
0	0	0
0	1	1
1	0	1
1	1	0

(*b*)

Fig. 6.11 Two input—EX-OR gate

6.2.7 Exclusive NOR Gate (EX-NOR Gate or XNOR Gate)

The EX-NOR gate is logically equivalent to an inverted EX-OR gate i.e., EX-OR gate followed by a NOT-gate. The symbol for EX-NOR is shown in Fig. 6.12. The operation of the gate is expressed by $Z = A \odot B$.



(*a*) 2-input EX-NOR gate

A	B	Z
0	0	1
0	1	0
1	0	0
1	1	1

(*b*) Truth table for EX-NOR

Fig. 6.12

Sometimes it is convenient to use more than one representation for a given type of gate, Fig. 6.13 shows alternative symbols used for the logic gates.

We can use the above symbols to analyse and design complex digital systems. We can also apply the techniques of Boolean algebra to electronic logic. Generally a logic gate network is drawn so that the flow of information is from the left to right.

The inputs to a logic gate network will be formed on the left of schematic diagram and the outputs will be found on the right, which makes it easier for us to find the algebraic equation of the total network, Boolean algebra is useless unless it can be converted into hardware in the form of logic gates. Science of Boolean algebra can be a useful technique for analysing circuits only if the hardware can be translated into Boolean expression.

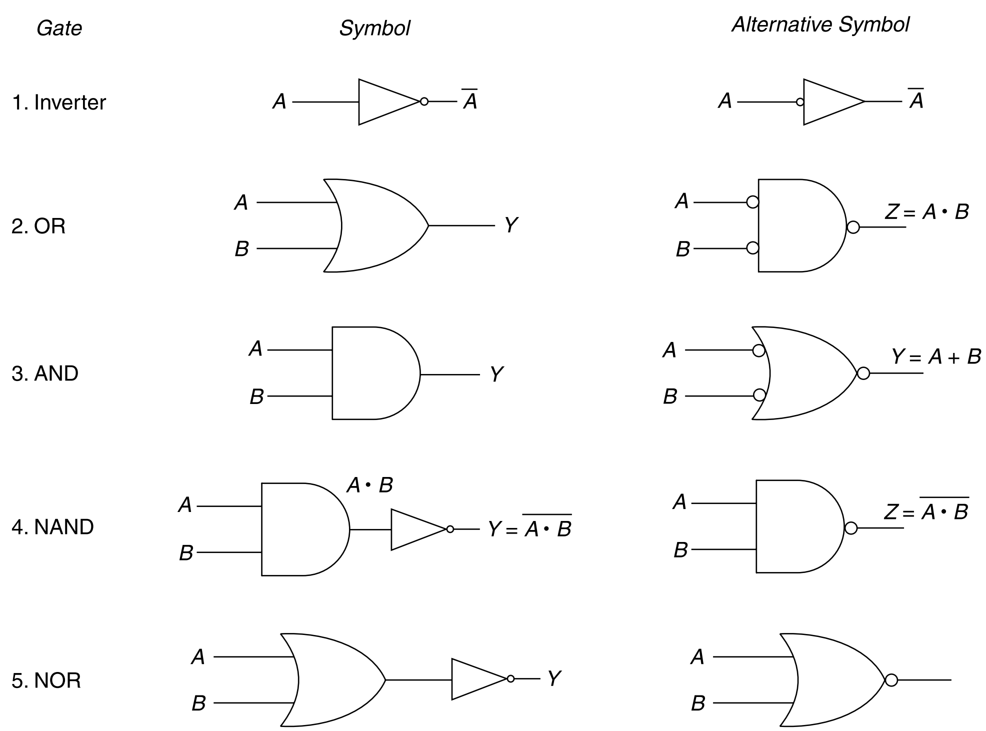


Fig. 6.13 Logic gates, equivalent symbols

Example 1: For the logic circuit shown in Fig. 6.14. write the input-output Boolean expression.

Solution: The output Boolean expression is

$$Z = (A + B) \cdot \left(\overline{A} + \overline{C}\right) \cdot (B + C).$$

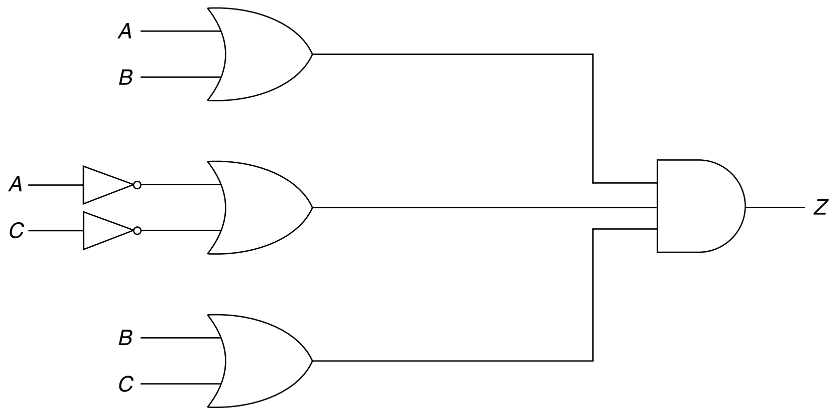


Fig. 6.14

Example 2: Represent $(A + B)(B + C)(C + A)$ in NOR-to-NOR form.

Solution: $(A + B)(B + C)(C + A) = (A \text{ NOR } B) \text{ NOR } (B \text{ NOR } C) \text{ NOR } (C \text{ NOR } A)$

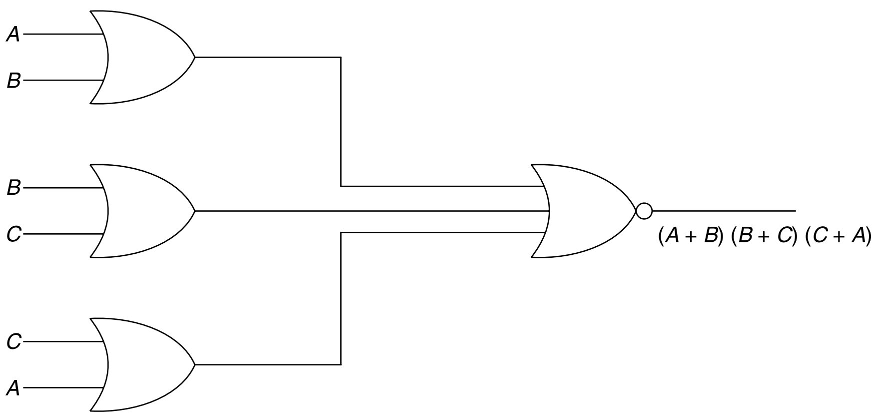


Fig. 6.15

Example 3: Implement the logic expression

$$F = \overline{A}\,\overline{B}C + \overline{A}BC + A\overline{B}$$

with logic gates.

Solution: The expression $F = \overline{A}\,\overline{B}C + \overline{A}BC + A\overline{B}$ requires three and gates and one OR-gate. It can be implemented as shown in Fig. 6.16:

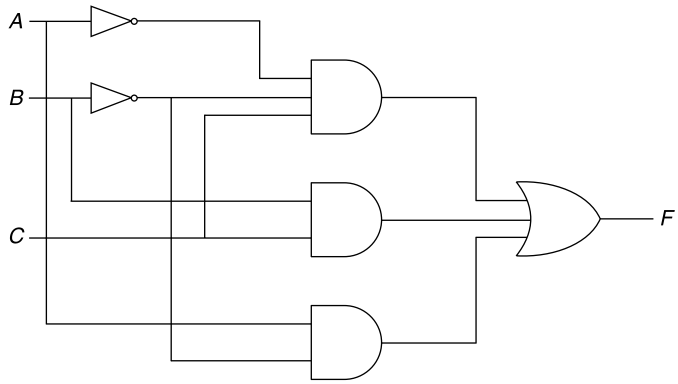


Fig. 6.16

Example 4: For the equation $Z = X\,Y + \overline{W}\,Y$ construct a gate structure and minimize it.

Solution: The gate structure for $Z = X\,Y + \overline{W}\,Y$ is shown in Fig. 6.17:

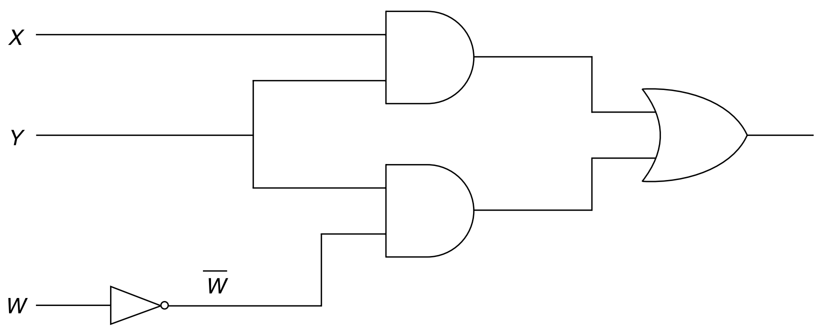


Fig. 6.17

Now consider $Z = X\,Y + \overline{W}\,Y$

The equation can be factored as $Z = Y\left(X + \overline{W}\right)$. The gate structure of Z has only two input gates.

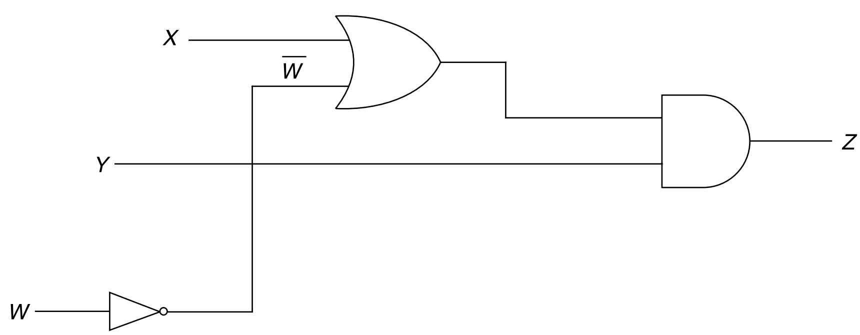


Fig. 6.18

Example 5: Construct a NAND-gate structure for the expression

$$Z = \left(\overline{A} + B\right) C + \overline{F} + D\ E$$

Solution: Figure 6.19 shows the NAND-gate network of Z:

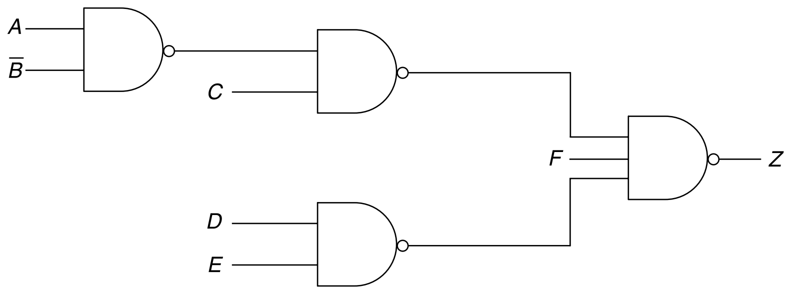


Fig. 6.19 NAND-gate network for *Z*

The equivalent AND-OR network is given in Fig. 6.20:

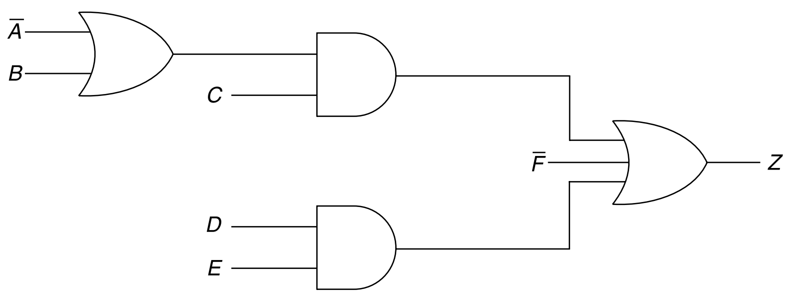


Fig. 6.20 Equivalent AND-OR network

Example 6: Use NAND-gates and draw a circuit diagram for $F = X\,\overline{Y}\,Z + \overline{Z}\,Y$.

Solution: $F = X\,\overline{Y}\,Z + \overline{Z}\,Y$

= (X NAND (NOT Y) (AND Z) NAND (NOT C NAND B)

The circuit diagram for F can be drawn as shown in Fig. 6.21:

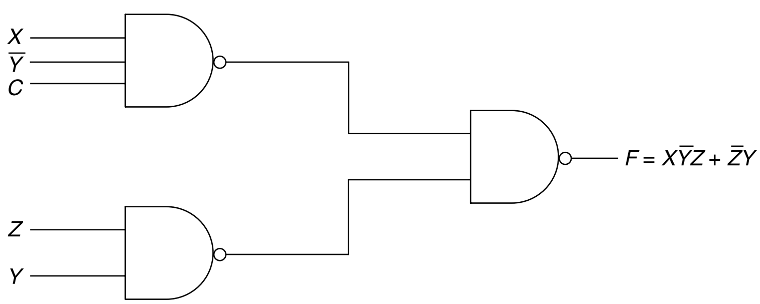


Fig. 6.21

Example 7: Show that the combination circuits of Fig. 6.22 are equivalent:

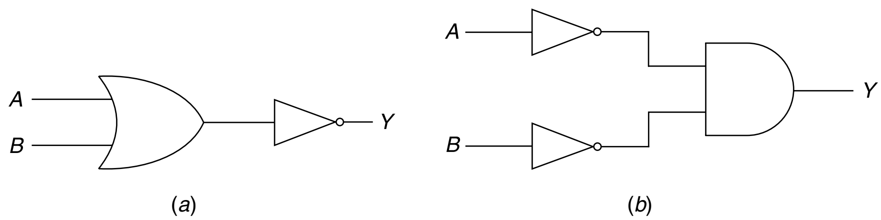


Fig. 6.22

Solution: The logic table for the circuit is shown in Fig. 6.22 (*a*) is:

Table 6.5

A	*B*	*Y*
1	1	0
1	0	0
0	1	0
0	0	1

(*a*)

A	*B*	*Y*
1	1	0
1	0	0
0	1	0
0	0	1

(*b*)

Table 6.5 (*a*) is logic table for the circuit Fig. 6.22 (*a*). The logic table for the circuit shown in Fig. 6.22 (*b*) is given in Table 6.5(*b*).

The logic table for the circuits is shown in Fig. 6.22 (*a*) and Fig. 6.22 (*b*) are identical. Hence the circuits are equivalent.

6.3 APPLICATIONS

Logic gates have several applications to the computers. They are used in the following:

1. Adders,
2. Encoder, and
3. Decoder

6.3.1 Adders

The application of binary bits consists the following elementary operations, namely

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 10.$$

We observe that the left bit gives the carry. When the augend and addend numbers contain more significant digits, the carry obtained from the addition of two bits is added to the next higher order pair of significant digits.

6.3.1.1 *Half adder*

A logic circuit that performs the addition of two bits is called a half adder. The half adder circuit needs two binary inputs and two binary outputs. The inputs variables designate augend and addend bits, the output variables produce the sum and carry. It is necessary to specify two output variables because the sum of 1 + 1 is binary 10. We assign symbols A and B to the input variables, S for the sum function and C for carry function. Both S and C are out put symbols. The truth table for half adder is given below:

Table 6.6 *2-input Half Adder*

A	*B*	*Carry (C)*	*Sum (S)*
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

From the table, it is clear that half adder performs binary operation (electronically): at a faster rate.

Logic circuit for half adder is given in Fig. 6.23:

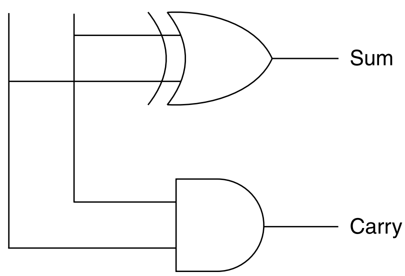


Fig. 6.23 (2-input) Half adder

6.3.1.2 *Full adder*

A logic circuit that performs the addition of three bits is a full adder. It consists of three inputs and two outputs. The two outputs are SUM and CARRY. Let us denote two of the input variables by A_1 and B_1, (to represent two significant bits to be adder) and the represents the carry from the previous lower significant position. The sum of three binary digits from 0 to 3. The binary numbers 2 and 3 need two binary digits. Hence we need two outputs designated by the symbols S (SUM) and C (CARRY). The truth table for the full adder is given below:

Table 6.7 *Full Adder*

A_1	B_1	C_1	C	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Logic circuit for Full adder is shown in Fig. 6.24.

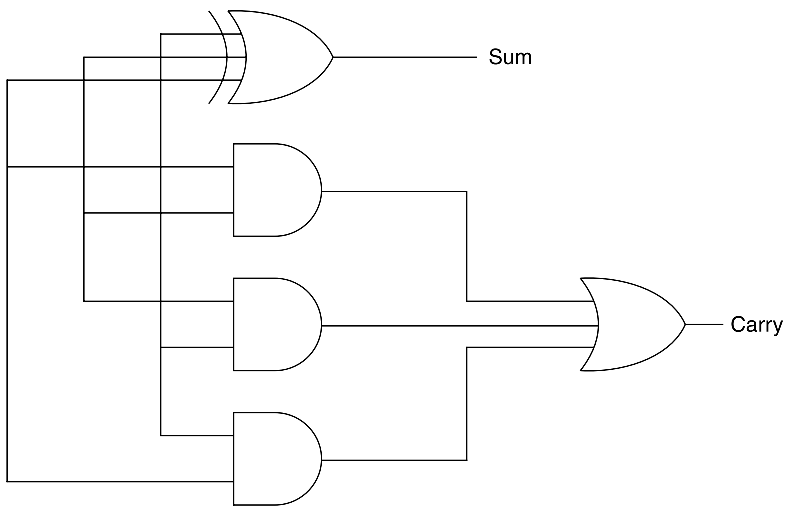


Fig. 6.24 (3-input) Full adder

6.3.2 Decoder

A decoder is a combinational logic circuit that converts n input lines to a maximum of 2^n unique output lines. In a 3 to 8 lines decoder three inputs are decoded into 8 outputs where each output represents one of the minterms of the 3 input variables. If the input variables represent a binary number then the outputs will be digits of octal system (contain 8 digits). *A* 3 to 8 lines decoder can also be used for decoding any 3 bit code to provide 8 outputs. The following is the diagram of 3 to 8 decoder:

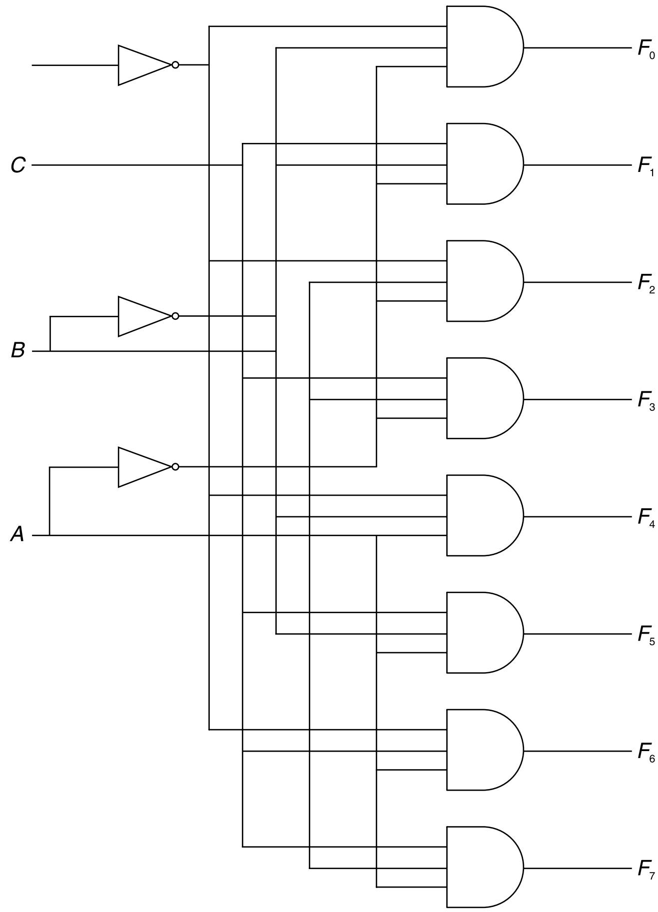


Fig. 6.25 A 3 to 8 decoder

The truth table of 3 to 8 lines decoder is given below:

Table 6.8

Inputs			*Outputs*							
A	*B*	*C*	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

We observe that the outputs variables are mutually exclusive. A decoder with n input variables can generate 2^n minterms.

6.3.3 Encoder

A logical circuit which performs the inverse operation of a decoder is called an Encoder. Decimal to binary encoder converts decimal numbers and a Hexadecimal to binary encoder converts Hexadecimal number to its binary equivalent. An encoder with 2^n input lines will have n output lines. The block diagram for 2^4 to 4 encoder is given in Fig. 6.26.

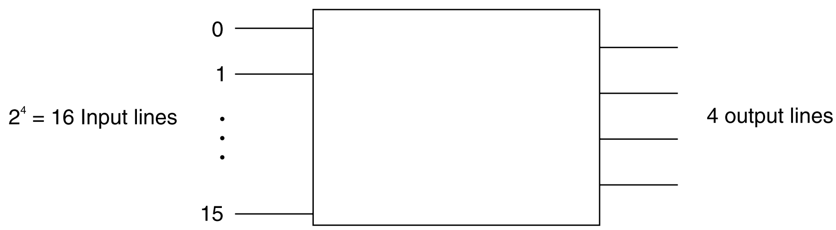


Fig. 6.26

6.3.4 Multiplexer (MUX)

Multiplexers are used to transmit large number of data over a small number of lines. Multiplex means many to one. A digital multiplexer is a combinational circuit that selects binary information from many input lines and directs it to a single output line. The selection of a particular input line is controlled by a set of selection lines. A multiplexer receives binary information from the 2^n lines and transmit information on a single output line (selected from the bit combination of n selection lines). The block diagram of 4 to 1 line multiplexer is shown in Fig. 6.27.

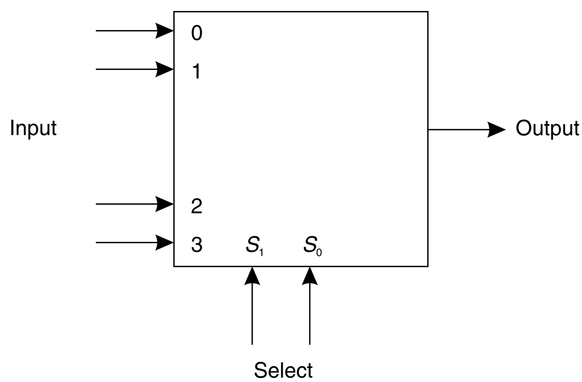


Fig. 6.27 Block diagram for 4 × 1 multiplexers

If a Boolean function F has n variables, we take $n-1$ of these variables and connect them to the selection lines of a multiplexer. The remaining variable say X of the function F is used for the inputs of the multiplexer. The inputs of the MUX are chosen to be either X or X' or 1 or 0. We can implement F with a MUX by choosing the four values X, X', 1 and 0 for the inputs and by connecting the other variables to the selection lines. We now explain the method of implementing a Boolean function F of n variables with 2^{n-1} to 1 multiplexer, with the help of an example.

Example: Implement

$$F = (a, b, c) = \Sigma\ (0, 3, 6, 7)$$

With a multiplexer

Solution: We first express F in its sum of minterms form. The ordered sequence of variables chosen for the minterms is $a\ b\ c$. Where a is the left most variable in the ordered sequences. Thus, we have

$$n = 3 \text{ (no of variables)}$$
$$n - 1 = 3 - 1 = 2$$

we use 2^{n-1} to 1, i.e. 2^{3-1} to 1 i.e., 8 to 1 multiplexes for the implementation of F. The selected variable a is in the highest order position in the sequences of variables m_0, m_1, m_2, m_3, m_4, m_5, m_6, m_7 are the minterms. We list all the minterm in two rows as shown in Table 6.9. The singled out variable a will be in the complemented form in the first row and will be in the uncomplemented form in the second row.

Table 6.9

	I_0	I_1	I_2	I_3
a^1	0	1	2	3
a	4	5	6	7

We circle the minterms of the function F and inspect each column separately: we apply the following rules:

(*i*) If the two minterms in a column are not circle we apply 0 to the corresponding multiplexer input.

(*ii*) If the two minterms are circled, then we apply 1 to the corresponding input of MUX.

(*iii*) If the bottom row minterm in a column is circled and the top minterm is not circled, apply a to the corresponding multiplexer input.

(*iv*) If the minterm of top row is encircled and the bottom row minterm in a column is not encircled, then we apply a^1 to the corresponding multiplexer input applying the above rules we obtain the values as shown in Table 6.10.

Table 6.10 *Implementation table*

	I_0	I_1	I_2	I_3
a^1	0	1	2	3
a	4	5	6	7
	a^1	0	a	1

The inputs I_0, I_1, I_2, I_3, are applied the values as shown below:

Input	*Value applied to the input*
I_0	a'
I_1	0
I_2	a
I_3	1

The function F can be implemented by using MUX as shown in Fig. 6.28.

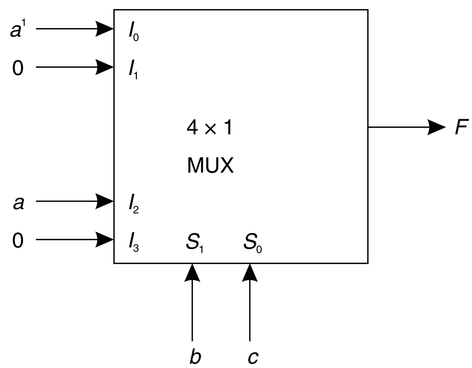


Fig. 6.28

6.4 SPECIAL SEQUENCES

If L is a logic circuit with n inputs devices and $A_1, A_2, \ldots A_n$ denote n-input sequences. Then each A_i must contain 2^n bits. There are many ways to form $A_1, A_2, \ldots A_n$, so that each A_i contain 2^n different possible combinations of input bits. One assignment scheme is as follows:

A_1: Assign 2^{n-1} bits which are 0's and 2^{n-1} bits which are 1*s*.

A_2: Repeatedly assign 2^{n-2} bits which are 0's followed by 2^{n-2} bits which are 1*s*.

A_3: Repeatedly assign 2^{n-3} bits which are 0's followed by 2^{n-3} bits which are 1*s*.

...

$A_1, A_2, A_s, \ldots$ are called special sequences. The complements of these special sequences can be obtained by replacing 0 by 1 and 1 by 0, in the sequences.

Example 1: Suppose a logic circuit has $n = 3$, input devices A, B and C, write down the special sequences for A, B and C and write their complements.

Solution: We have $n = 3$

There are $2^3 = 8$ bit special sequences for A, B and C. They can be written as follows:

$$A = 00001111$$
$$B = 00110011$$
$$C = 01010101$$

The special sequences for their complements are

$$\overline{A} = 11110000$$
$$\overline{B} = 11001100$$
$$\overline{C} = 10101010$$

Example 2: Determine how the pair of sequences 110001, 101101, is processed by an AND-gate.

Solution: 1's can occur as an outputs of an AND-gate. Only when both inputs are 1, therefore the pair 1100001, 101101, has the output 100001 by an AND-gate, (note that 1's occur in the first and last positions).

Example 3: How would a NOT-gate process the sequence 100011.

Solution: A NOT-gates changes 0 to 1 and 1 to 0. Hence the output of the given sequence is 01110000.

Example 4: If $A = 1100110110$, $B = 1110000111$ and $C = 1010010110$ find $A + B + C$.

Solution: 0s occur in the 4th, 7th positions. The remaining positions will have 1s.

Hence $A + B + C = 1110110111$.

Example 5: If $A = 00001111$, $B = 00110011$ and $C = 01010101$ find $A\ B\ C$.

Solution: A, B and C have 1s in the 8th position.

Hence $A\ B\ C = 00000001$

EXERCISE 6.1

I

1. What are logic gates? Name three basic logic gates.

2. What is an OR-gate? Explain in brief the function of an OR-gate.

3. What is an AND-gate.

4. What is an exclusive OR-gate? How does it differ from an OR-gate?

5. What is a NOT-gate? Explain its operations and draw its truth table.

6. Draw the circuit symbol of a NAND-gate.

7. Simplify:

1. $P'\ Q + P'\ Q\ R'\ S' + P\ Q\ R\ S'$
2. $(P' + Q' + R')\ (P' + Q' + R)\ (Q' + R)\ (P + R)\ (R + Q + R)\ (P' + Q)$
3. $P\ Q\ R\ S + P'\ R\ S + P\ Q\ S + P\ Q\ R\ S' + Q\ R'\ S$

II

1. Simplify the following expressions:

(a) $Z = \overline{(A \cdot B + B \cdot C)(B \cdot C + C \cdot D)(C \cdot D + A \cdot B)}$

(b) $Z = (A + B) \cdot (A + \overline{B}) \cdot (\overline{A} + B)$

2. What is the significant of Principle of duality. Write the duals of

(a) $(X + Y) * (Y + Z) * (Z + X)$

(b) $A + (B + E) + B * (C + A) + C * (A + B)$

3. $D_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$ (the division of 70} we define

$a + b = \text{lcm}\ (a, b)$

$a * b = \text{gcd}\ (a, b)$

$a' = {}^{70}\!/_{a}$

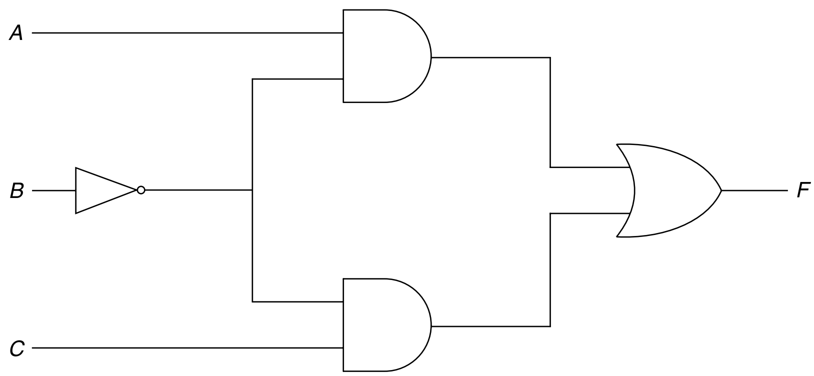
show that D_{70} is a Boolean algebra with 1 as the zero element and 70 as the unit element.

4. Find the sum of products form (disjunctive normal form) of the Boolean expression $E = ((xy)'\ z)$ $((n' + z)\ (y' + z'))$

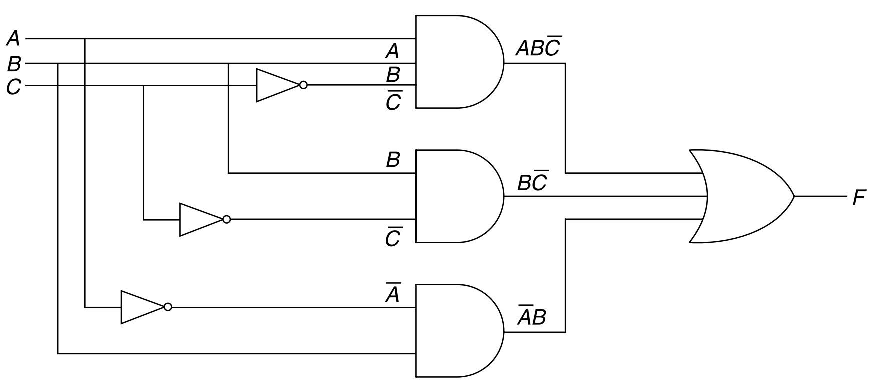
5. A logic circuit L has $n = 4$ inputs A, B, C, D write the 16 bit special sequence for A, B, C, D.

6. Given five inputs A, B, C, D and E find the special sequences which give all the different possible combinations of inputs bits.

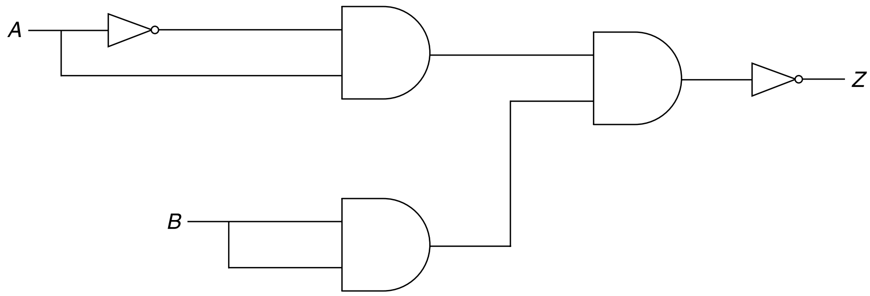
7. For each of the following networks find the output:



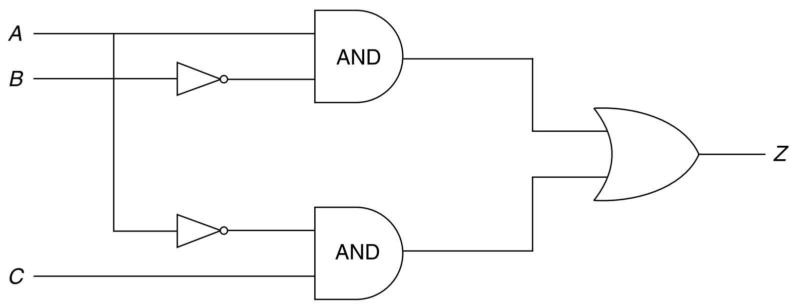
(*a*)



(*b*)



(*c*)

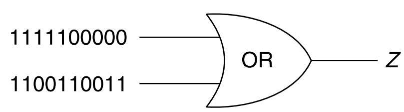


(*d*)

8. Simplify the Boolean expression and construct a network for the expression.

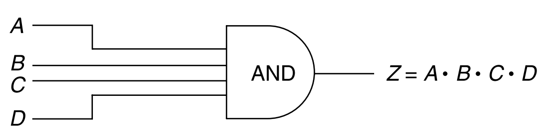
$$Z = \bar{A}BC + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$$

9. Determine the output of the gate



10. If $A = 11100111$
$B = 01111011$
$C = 01110011$
and $D = 11101110$

Determine the output of the gate



11. Implement $F(a, b, c) = \sum (1, 3, 5, 6)$ with a multiplexer.