Q 1.9 Bordered Algorithm for computing cholosky factorization.

Given A is symmetric Positive definite Matrix.

Lets Consider $A = LL^T$

Start by Partitioning the matrix A and L.

$$A = \left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad \text{and} \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)$$

now substituting these partitions to $A = LL^T$

$$\left(\begin{array}{c|c} A_{00} & * \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right)^T$$

$$= \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array}\right) \left(\begin{array}{c|c} L_{00}^T & l_{10} \\ \hline 0 & \lambda_{11} \end{array}\right)$$

$$= \left(\begin{array}{c|c} L_{00} L_{00}^T & * \\ \hline l_{10}^T L_{00}^T & l_{10}^T l_{10} + \lambda_{11}^2 \end{array}\right)$$

As these matrices are equal, their individual elements must be equal too.

so we can write
$$A_{00} = L_{00} L_{00}^T$$
$$a_{10}^T = l_{10}^T L_{00}^T \Rightarrow l_{10}^T = a_{10}^T L_{00}^{-T}$$
$$\alpha_{11} = l_{10}^T l_{10} + \lambda_{11}^2 \Rightarrow \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$$

with these 3 equations we can write the algorithm as.

Step 1 → Partition matrix A as $\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$

Step 2 → $A_{00} := L_{00}$ this is the Cholosky factorization. computed in the previous iteration.

i.e. $A_{00} := L_{00} = \text{Chol}(A_{00})$

Step 3 → overwrite $a_{10}^T := l_{10}^T = a_{10}^T L_{00}^{-T}$

Step 4 → overwrite $\alpha_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}}$

Repeat step 1 to step 4, until we traverse full matrix.

Pseudo code for the algorithm

Algorithm $A := \text{Chol-Bordered}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right)$

where A_{TL} is 0×0.

while $n(A_{TL}) < n(A)$ do.

$$\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c} A_{00} & * & * \\ \hline a_{10}^T & \alpha_{11} & * \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

where α_{11} is a scalar.

$a_{10}^T := a_{10}^T \, \text{TRIL}(A_{00})^{-T}$

$\alpha_{11} := \sqrt{\alpha_{11} - a_{10}^T a_{10}}$

$$\left(\begin{array}{c|c} A_{TL} & * \\ \hline A_{BL} & A_{BR} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c} A_{00} & * & * \\ \hline a_{10}^T & \alpha_{11} & * \\ \hline A_{20} & a_{21} & A_{22} \end{array}\right)$$

endwhile.

Q1-(b)

<u>Theorem</u> Given SPD matrix A, there exists a lower triangular matrix L such that $A = LL^T$.

if the diagonal elements of L are restricted to be Positive, L is unique.

Proving theorem for Bordered Cholesky factorization.

Before going through the prove, lets first establish a few lemma.

<u>Lemma 1</u> → Let $A \in \mathbb{R}^{n \times n}$ is a Symmetric positive definite matrix, and partitioned as $\left(\begin{array}{c|c} A_{00} & a_{10} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right)$ then A_{00} is also symmetric positive definite.

<u>Proof</u> → Let $x_1 \in \mathbb{R}^{n-1}$ be an arbitary non-zero vector. Define $x = \begin{pmatrix} x_1 \\ \chi_2 \end{pmatrix}$

where $\chi_2 = \dfrac{-(x_1^T a_{10} + a_{10}^T x_1)}{\alpha_{11}}$

then since clearly $x \neq 0$. and A is SPD.

$$x^T A x > 0.$$

replacing the values.

$$\begin{pmatrix} x_1 \\ \chi_2 \end{pmatrix}^T \left(\begin{array}{c|c} A_{00} & a_{10} \\ a_{10}^T & \alpha_{11} \end{array}\right) \begin{pmatrix} x_1 \\ \chi_2 \end{pmatrix} > 0.$$

$$= \begin{pmatrix} x_1^T & \chi_2 \end{pmatrix} \begin{pmatrix} A_{00} x_1 + a_{10} \chi_2 \\ a_{10}^T x_1 + \alpha_{11} \chi_2 \end{pmatrix}$$

$$= x_1^T A_{00} x_1 + x_1^T a_{10} \chi_2 + \chi_2 a_{10}^T x_1 + \chi_2 \alpha_{11} \chi_2.$$

replacing value of χ_2.

$$= x_1^T A_{00} x_1 - x_1^T a_{10} \frac{(x_1^T a_{10} + a_{10}^T x_1)}{\alpha_{11}} - a_{10}^T x_1 \frac{(x_1^T a_{10} + a_{10}^T x_1)}{\alpha_{11}}$$

$$+ \frac{(x_1^T a_{10} + a_{10}^T x)}{\cancel{\alpha_{11}}} \cancel{\alpha_{11}} \frac{(x_1^T a_{10} + a_{10}^T x_1)}{\alpha_{11}}$$

$$\Rightarrow \quad x_1^T A_{00} x_1 - \cancel{\frac{x_1^T a_{10} x_1^T a_{10}}{\alpha_{11}}} - \cancel{\frac{x_1^T a_{10} a_{10}^T x_1}{\alpha_{11}}} - \cancel{\frac{a_{10}^T x_1 x_1^T a_{10}}{\alpha_{11}}} - \cancel{\frac{a_{10}^T x_1 a_{10}^T x_1}{\alpha_1}}$$

$$+ \cancel{\frac{x_1^T a_{10} x_1^T a_{10}}{\alpha_{11}}} + \cancel{\frac{x_1^T a_{10} a_{10}^T x_1}{\alpha_{11}}} + \cancel{\frac{a_{10}^T x_1 x_1^T a_{10}}{\alpha_{11}}} + \cancel{\frac{a_{10}^T x_1 a_{10}^T x_1}{\alpha_{11}}}$$

simplying.. we can see.

$$x_1^T A_{00} x_1 > 0.$$

and we have choose x_1 to be a non-zero arbitary vector. this proves that A_{00} is SPD.

Lemma 2 → if $A \in \mathbb{R}^{n \times n}$ be SPD. then α_{11} is real and positive.

Proof → Let e_1 be the 1st unit basis vector, corrosponding to the position of α_{11}

so from the definition of SPD.

$$e_1^T A e_1 > 0$$

this implies $\alpha_{11} > 0$.

Now with these 2 proves, we can continue with the bordered Cholosky factorization. We will be employing proof by induction.

Base Case: $n = 1$

the base case for a 1×1 matrix $A = \alpha_{11}$: In this case the fact that A is SPD means that α_{11} is real and positive and the cholosky factor is given by $\lambda_{11} = \sqrt{\alpha_{11}}$ with uniqueness if we insist λ_{11} is positive

Inductive step – Assume the result is true for all matrices with $n = k$, we will show that it holds true for $n = k+1$.

Let $A \in \mathbb{R}^{(k+1)(k+1)}$ be SPD, Partition.

$$A = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array} \right) \quad \text{and} \quad L = \left(\begin{array}{c|c} L_{00} & 0 \\ \hline l_{10}^T & \lambda_{11} \end{array} \right)$$

so
$$\begin{cases} A_{00} = L_{00} L_{00}^T & \text{— (I)} \\ l_{10}^T = a_{10}^T L_{00}^{-T} & \text{— (II)} \\ \lambda_{11} = \sqrt{\alpha_{11} - l_{10}^T l_{10}} & \text{— (III)} \end{cases}$$

will be the component of L which is cholesky factor of A.

* From Lemma-1 we see the A_{00} is also a SPD matrix and the similar analysis can be applied recursively to find cholesky factorization.

* eqn 2 can be written as. $l_{10}^T L_{00}^T = a_{10}^T$
As L_{00} is the cholesky factor derived in previous step of SPD matrix A, so by inductive assumption we know that it exists and is non-singular, similarly we know that l_{10} exists from previous step. So we can deduce $l_{10}^T L_{00}^T = a_{10}^T$ has a solution that is unique.

* From Lemma 2 we know α_{11} is real and positive, which implies as long as. $\alpha_{11} \geq l_{10}^T l_{10}$., we will have a. positive and well defined value of λ_{11}

Thus a $(k+1)(k+1)$ matrix has a unique cholesky factorization as long as at every step
$$\alpha_{11} \geq l_{10}^T l_{10}.$$

Substituting value of l_{10} from eqn (11).

$$\alpha_{11} \geq (a_{10}^T L_{00}^{-T})^T (a_{10}^T L_{00}^{-T}).$$

$$\alpha_{11} \geq L_{00}^{-1} a_{10} a_{10}^T L_{00}^{-T}.$$

With this condition in place, by the principle of mathematical Induction this result holds. and we establish the bordered Cholesky factorization is a well defind for a matrix A that is SPD.