Q-2.

To Prove - Backward Error analysis Results for bordered. LU factorization algorithm.

Prepration : Before starting the proof lets define a few results.

From Corollary 6.3.3.2 of the textbook.

For the dot Product Backward and Forward Error.

R-1B $\quad \check{\kappa} = (x+\delta x)^T y \quad$ where $|\delta x| \leqslant \gamma_n |x|$ — ①

R-2B $\quad \check{\kappa} = x^T(y+\delta y) \quad$ where $|\delta y| \leqslant \gamma_n |y|$ — ②

R-1F $\quad \check{\kappa} = x^T y + \delta \kappa \quad$ where $|\delta \kappa| \leqslant \gamma_n |x^T||y|$ — ③

From Theorem 6.6.2.14 of the textbook.

The Error analysis for the result of triangular system.

$(L+\Delta L)\check{x} = y \quad$ where $|\Delta L| \leqslant \max(\gamma_2, \gamma_{n-1})|L|$ — ④

From Corollary 6.6.2.15 of the textbook, the error analysis of the triangular system can also be written as.

R-1B. $\quad (L+\Delta L)\check{x} = y \quad$ where $|\Delta L| \leqslant \max(\gamma_2, \gamma_{n-1})|L|$ — ⑤

R-1F $\quad L\hat{x} = b+\delta b \quad$ where $|\delta b| \leqslant \max(\gamma_2, \gamma_{n-1})|L||x|$ — ⑥

Proof - we will imploy the proof by induction.

* Base Case $n=1$

the base case of $n=1$, where $A = (a)$ and $L = (1)$, $U = (a)$

clearly we can see

$$A + \Delta A = LU \quad \text{holds for any } \Delta A = (\delta_{11})$$

$$\text{where } |\Delta A| \leqslant \gamma_1 |\delta_{11}|.$$

* For the induction Step, we assume that the statement holds for matrix $A \in \mathbb{C}^{m \times n}$, with the nonsingular principle matrix. That is there exists ΔA such that

$$A + \Delta A = \check{L}\check{U} \qquad \text{with } |\Delta A| \leqslant \gamma_n |\check{L}||\check{U}|.$$

where $\check{L}$ is unit lower triangular and $\check{U}$ is upper triangular.

* Now we consider the case where $A \in \mathbb{C}^{m \times n+1}$ with a nonsingular leading principal submatrixes. we partition A as.

for step n.

$$A \rightarrow \left(\begin{array}{c|c} A_{TL} & A_{TR} \\ \hline A_{BL} & A_{BR} \end{array}\right), \quad L \rightarrow \left(\begin{array}{c|c} L_{TL} & 0 \\ \hline L_{BL} & L_{BR} \end{array}\right), \quad U \rightarrow \left(\begin{array}{c|c} U_{TL} & U_{TR} \\ \hline & U_{BR} \end{array}\right).$$

$$\text{with } \Delta A = \left(\begin{array}{c|c} \Delta A_{TL} & \Delta A_{TR} \\ \hline \Delta A_{BL} & \Delta A_{BR} \end{array}\right).$$

for step $n+1$, we partition as.

$$A \rightarrow \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) \quad L \rightarrow \left(\begin{array}{c|c} \check{L}_{00} & 0 \\ \hline \check{l}_{10}^T & 1 \end{array}\right) \quad U \rightarrow \left(\begin{array}{c|c} \check{U}_{00} & \check{u}_{01} \\ \hline 0 & \check{u}_{11} \end{array}\right)$$

where

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11}\end{array}\right) + \left(\begin{array}{c|c}\Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & \delta\alpha_{11}\end{array}\right) = \left(\begin{array}{c|c}\hat{L}_{00} & 0 \\ \hline \hat{l}_{10}^T & 1\end{array}\right)\left(\begin{array}{c|c}\hat{U}_{00} & \hat{u}_{01}\\ \hline 0 & \hat{u}_{11}\end{array}\right).$$

Now for the $(n+1)^{th}$ step of Bordered algorithm we know,

$L_{00} u_{01} = a_{01}$ ← triangular solve lower triangnalar mat

$l_{10}^T U_{00} = a_{10}^T$ ← triangular solve upper trianglar mat

$\alpha_{11} := u_{11} = \alpha_{11} - a_{10}^T a_{01}$ ← dot Product.

with eq^n ⑤ defined aboree.

$$\check{L}_{00} \check{u}_{01} = a_{01} + \delta a_{01} \quad \text{where} \quad \underbrace{|\delta a_{01}| \leq \max(\gamma_2, \gamma_{n-1}) |L_{00}||\hat{u}_{01}|}_{⑦}$$

similarly with the eq^n ⑥ we can extrapolate the result for upper triangular solve.

$\check{l}_{10}^T \check{U}_{00} = a_{10}^T$

← taking transpose on both sides.

$\check{U}_{00}^T \check{l}_{10} = a_{10}$.

this becomes a lower triangular solve. so.

$$\check{U}_{00}^T \check{l}_{10} = a_{10} + \delta a_{10} \quad \text{where} \quad \underbrace{|\delta a_{10}| \leq \gamma_n |U_{00}^T||\hat{l}_{10}|}_{⑧}$$

$$\alpha_{11} := \check{u}_{11} = \alpha_{11} - (a_{10} + \delta a_{10})^T (a_{01} + \delta a_{01})$$

$$= \alpha_{11} - a_{10}^T a_{01} - \delta\alpha_{11} \qquad ⑨$$

where $\delta\alpha_{11} = a_{10}^T \delta a_{01} + \delta a_{10}^T a_{01} + \delta a_{10}^T \delta a_{01}$

Putting eq^n ⑦, ⑧ and ⑨ togeather we get

$$\left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) = \left(\begin{array}{c|c} \hat{L}_{00}\hat{U}_{00} = A_{00} + \Delta A_{00} & \check{L}_{00}\check{u}_{01} = a_{01} + \delta a_{01} \\ \hline (\check{U}_{00}^T \hat{l}_{10})^T = (a_{10} + \delta a_{10})^T & \check{u}_{11} = \alpha_{11} - a_{10}^T a_{01} - \delta\alpha_{11} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{c|c} \hat{L}_{00}\hat{U}_{00} & \check{L}_{00}\check{u}_{01} \\ \hline \hat{l}_{10}^T \hat{U}_{00} & \alpha_{11} - a_{10}^T a_{01} \end{array}\right) = \left(\begin{array}{c|c} A_{00} & a_{01} \\ \hline a_{10}^T & \alpha_{11} \end{array}\right) + \left(\begin{array}{c|c} \Delta A_{00} & \delta a_{01} \\ \hline \delta a_{10}^T & -\delta\alpha_{11} \end{array}\right)$$

So with this result we can see that after step $(n+1)$ the resultant matrix A can be written as.

$$A + \Delta A = \check{L}\check{U}$$

$$\text{for } m(A) = n+1 \wedge \check{L}\check{U} = (A+\Delta A) \wedge |\Delta A| \leq \gamma_{n+1} |L||U|$$

Therefore we can see the backward error incurred with bordered LU factorization is

$$A + \Delta A = \check{L}\check{U} \quad \text{with} \quad |\Delta A| \leq \gamma_n |\check{L}||\check{U}|.$$

By The Priciple of mathematical Induction this result holds.