Exercise 6.1 Given

(a)
$$\begin{pmatrix} A_{00} & \alpha_{10} e_L & 0 \\ \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ 0 & \alpha_{21} e_F & A_{22} \end{pmatrix} = \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ 0 & 0 & \end{pmatrix} \begin{pmatrix} D_{00} & 0 & 0 \\ 0 & \phi_1 & 0 \\ 0 & 0 & E_{22} \end{pmatrix} \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \upsilon_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix}$$

show $\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$

Solution $\rightarrow$ As we know $A = LDL^T$

$$\Rightarrow \begin{pmatrix} A_{00} & \alpha_{10} e_L & 0 \\ \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ 0 & \alpha_{21} e_F & A_{22} \end{pmatrix} = \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & 0 \\ 0 & \lambda_{21} e_F & L_{22} \end{pmatrix} \begin{pmatrix} D_{00} & 0 & 0 \\ 0 & \delta_1 & 0 \\ 0 & 0 & D_{22} \end{pmatrix} \begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & \lambda_{21} e_F^T \\ 0 & 0 & L_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} L_{00} D_{00} & 0 & 0 \\ \lambda_{10} e_L^T D_{00} & \delta_1 & 0 \\ 0 & \lambda_{21} e_F \delta_1 & L_{22} D_{22} \end{pmatrix} \cdot \begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & \lambda_{21} e_F^T \\ 0 & 0 & L_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} L_{00} D_{00} L_{00}^T & L_{00} D_{00} \lambda_{10} e_L & 0 \\ \lambda_{10} e_L^T D_{00} L_{00}^T & \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \delta_1 & \delta_1 \lambda_{21} e_F^T \\ 0 & \lambda_{21} e_F \delta_1 & \lambda_{21} e_F \delta_1 \lambda_{21} e_F^T + L_{22} D_{22} L_{22}^T \end{pmatrix}$$

Comparing both sides we can see

$$\alpha_{11} = \lambda_{10} e_L^T D_{00} \lambda_{10} e_L + \delta_1 \qquad \text{--- } (1)$$

Similarly. $A = UEU^T$

$$\begin{pmatrix} A_{00} & \alpha_{10} e_L & 0 \\ \alpha_{10} e_L^T & \alpha_{11} & \alpha_{21} e_F^T \\ 0 & \alpha_{21} e_F & A_{22} \end{pmatrix} = \begin{pmatrix} U_{00} & \upsilon_{01} e_L & 0 \\ 0 & 1 & \upsilon_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix} \begin{pmatrix} E_{00} & 0 & 0 \\ 0 & \varepsilon_1 & 0 \\ 0 & 0 & E_{22} \end{pmatrix} \begin{pmatrix} U_{00} & \upsilon_{01} e_L & 0 \\ 0 & 1 & \upsilon_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix}^T$$

$$= \begin{pmatrix} U_{00}E_{00} & \nu_{01}e_L\mathcal{E}_1 & 0 \\ 0 & \mathcal{E}_1 & \nu_{12}e_F^T E_{22} \\ 0 & 0 & U_{22}E_{22} \end{pmatrix} \begin{pmatrix} U_{00}^T & 0 & 0 \\ \nu_{01}e_L^T & 1 & 0 \\ 0 & \nu_{12}e_F & U_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} U_{00}E_{00}U_{00}^T + \nu_{01}e_L\mathcal{E}_1\nu_{01}e_L^T & \nu_{01}e_L\mathcal{E}_1 & 0 \\ \mathcal{E}_1\nu_{01}e_L^T & \mathcal{E}_1 + \nu_{12}e_F^T E_{22}\nu_{12}e_F & \nu_{12}e_F^T E_{22}U_{22}^T \\ 0 & U_{22}E_{22}\nu_{12}e_F & U_{22}E_{22}U_{22}^T \end{pmatrix}$$

comparing both sides we get

$$\alpha_{11} = \mathcal{E}_1 + \nu_{12}e_F^T E_{22}\nu_{12}e_F \qquad \text{—— (2)}$$

Now lets multiply the twisted factorization.

$$\begin{pmatrix} A_{00} & \alpha_{10}e_L & 0 \\ \alpha_{10}e_L^T & \alpha_{11} & \alpha_{21}e_F^T \\ 0 & \alpha_{21}e_F & A_{22} \end{pmatrix} = \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10}e_L^T & 1 & \nu_{12}e_F^T \\ 0 & 0 & U_{22} \end{pmatrix} \begin{pmatrix} D_{00} & 0 & 0 \\ 0 & \phi_1 & 0 \\ 0 & 0 & E_{22} \end{pmatrix} \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10}e_L^T & 1 & \nu_{12}e_F^T \\ 0 & 0 & U_{22} \end{pmatrix}^T$$

$$= \begin{pmatrix} L_{00}D_{00} & 0 & 0 \\ \lambda_{10}e_L^T D_{00} & \phi_1 & \nu_{12}e_F^T E_{22} \\ 0 & 0 & U_{22}E_{22} \end{pmatrix} \begin{pmatrix} L_{00}^T & \lambda_{10}e_L & 0 \\ 0 & 1 & 0 \\ 0 & \nu_{12}e_F & U_{22}^T \end{pmatrix}$$

$$= \begin{pmatrix} L_{00}D_{00}L_{00}^T & L_{00}D_{00}\lambda_{10}e_L & 0 \\ \lambda_{10}e_L^T D_{00}L_{00}^T & \lambda_{10}e_L^T D_{00}\lambda_{10}e_L + \phi_1 + \nu_{12}e_F^T E_{22}\nu_{12}e_F & \nu_{12}e_F^T E_{22}U_{22}^T \\ 0 & U_{22}E_{22}\nu_{12}e_F & U_{22}E_{22}U_{22}^T \end{pmatrix}$$

again comparing both sides we get

$$\alpha_{11} = \lambda_{10}e_L^T D_{00}\lambda_{10}e_L + \phi_1 + \nu_{12}e_F^T E_2\nu_{12}e_F$$

Now replacing the values from eq^n (1) and (2)

$$\cancel{\alpha_{11}} = \cancel{\alpha_{11}} - \delta_1 + \phi_1 + \alpha_{11} - \mathcal{E}_1$$

$$\Rightarrow \quad \boxed{\phi_1 = \delta_1 + \mathcal{E}_1 - \alpha_{11}}$$

Proved

(b) Compute the cost of one twisted factorization.

Solution → For one twisted factorization we compute

$$\phi_1 = \delta_1 + \varepsilon_1 - \alpha_{11}$$

which equals one "addition" and one "subtraction" operation. So the complexity is $\underline{\underline{O(1)}}$

(c) What is the cost of all twisted factorizations.

Solution → We can calculate twisted factorization against each diagonal element. As an $n \times n$ matrix will have n diagonal elements. the cost of computing all twisted factorizations would be

$$\underline{\underline{O(n)}}.$$