Exercise 7.1. Compute x_0, X_1 and x_2 such that.

$$\begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \nu_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix} \begin{pmatrix} D_{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & & E_{22} \end{pmatrix} \begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \nu_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix}^T \begin{pmatrix} x_0 \\ X_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Lets multiply. and Solve, a X_1 is a free variable.

$$\begin{pmatrix} L_{00} & 0 & 0 \\ \lambda_{10} e_L^T & 1 & \nu_{12} e_F^T \\ 0 & 0 & U_{22} \end{pmatrix}^T \begin{pmatrix} x_0 \\ X_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} L_{00}^T & \lambda_{10} e_L & 0 \\ 0 & 1 & 0 \\ 0 & \nu_{12} e_F & U_{22}^T \end{pmatrix} \begin{pmatrix} x_0 \\ X_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$L_{00}^T x_0 + \lambda_{10} e_L X_1 = 0.$$
$$X_1 = 1.$$
$$\nu_{12} e_F X_1 + U_{22}^T x_2 = 0.$$

Solving the system we get

$$L_{00}^T x_0 = -\lambda_{10} e_L. \quad — (1)$$
$$U_{22}^T x_2 = -\nu_{12} e_F \quad — (2)$$
$$\text{and} \quad X_1 = 1$$

Now in both equation (1) and (2) the right hand side is a scalar multiplied by a unit basis vector.

$$\text{So} \quad \lambda_{10} e_L = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \lambda_{10} \end{pmatrix}.$$

$$\text{and.} \quad \nu_{12} e_F = \begin{pmatrix} \nu_{12} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

if we observe closely the left hand side both L_{00} and U_{22} have special bidiagonal structure. i.e. all the entries on the diagonal are 1.

Now, we can use system of linear equations to solve both eq^n(1) and equation 2

if A is a $n \times n$ matrix, then

$$m(L_{00}) + m(U_{22}) = n-1$$

we can see this just by observing twisted factorization.

So that means. there will be n linear equation each with the form

$$l_{ii} + 1 = 0$$

or $l_{11} + 1 = c$ where c is a constant.

Solving each of these equation would take 1 floating point computation., and hence all the entries for the vector can be solved in $\underline{\underline{O(n)}}$ time.