Manish Sanwal
ms89322.

Final Exam.

Q.5.2 Given a symmetric tridiagonal matrix A derive an algorithm to compute $A = UDU^T$

We start by Partitioning.

$$A \rightarrow \left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline * & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline * & * & A_{LL} \end{array}\right) = \left(\begin{array}{ccc|c|ccc} \times & \times & & & & & \\ \times & \times & \times & & & & \\ & \times & \times & \alpha_{MF} & & & \\ \hline & & \alpha_{MF} & \alpha_{MM} & \alpha_{LM} & & \\ \hline & & & \alpha_{LM} & \times & \times & \\ & & & & \times & \times & \times \\ & & & & & \times & \times \end{array}\right)$$

Where e_F and e_L are standard basis vectors with a "1" in the first and the last element respectively.

$$\alpha_{MF} e_L = \alpha_{MF} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \alpha_{MF} \end{pmatrix}$$

and.

$$\alpha_{ML} e_F^T = \alpha_{ML} (1, 0 \ldots 0) = (\alpha_{ML}\ 0 \ldots 0)$$

We repartition

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline * & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline * & * & A_{LL} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} A_{00} & \alpha_{01} e_L & 0 & 0 \\ \hline * & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & * & * & A_{33} \end{array}\right)$$

and. $U \rightarrow \left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline 0 & 1 \end{array}\right)$ $\quad D \rightarrow \left(\begin{array}{c|c} D_{00} & 0 \\ \hline 0 & \delta_1 \end{array}\right)$

$$UDU^T = \left(\begin{array}{c|c} U_{00} & u_{01} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} D_{00} & 0 \\ \hline 0 & \delta_1 \end{array}\right) \left(\begin{array}{c|c} U_{00}^T & 0 \\ \hline u_{01}^T & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c} U_{00} D_{00} U_{00}^T + \delta_1 u_{01} u_{01}^T & u_{01}\delta_1 \\ \hline * & \delta_1 \end{array}\right)$$

Now equating $A = UDU^T$

$$\left(\begin{array}{cc|c|c} A_{00} & \alpha_{01} e_L & \boxed{\begin{matrix} 0 \\ \alpha_{12} \end{matrix}} & 0 \\ * & \alpha_{11} & & 0 \\ \hline * & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & * & * & A_{33} \end{array}\right) = \left(\begin{array}{c|c} U_{00} D_{00} U_{00}^T & u_{01} \delta_1 \\ \hline * & \delta_1 \end{array}\right)$$

The algorithm overwrites the strictly upper triangular part of A with strictly upper triangular part of U and the diagonal of A with D.

Now relative to algorithm in exercise 5.1 in referenced paper. we can see

- $\alpha_{22} := \delta_1 = \alpha_{22}$ (no-op)

- $\begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix} = u_{01} \delta_1$

$$\Rightarrow \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix} := \frac{1}{\alpha_{22}} \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha_{12}/\alpha_{22} \end{pmatrix}.$$

$$\Rightarrow \alpha_{12} := \frac{\alpha_{12}}{\alpha_{22}}.$$

- equivalent to $A_{00} := A_{00} - u_{01} a_{01}^T$ would be.

$$A_{00} := A_{00} - \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix} \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix}^T = A_{00} - \begin{pmatrix} 0 \\ \alpha_{12} \end{pmatrix} \begin{pmatrix} 0 & \alpha_{12} \end{pmatrix}.$$

$$:= \left(\begin{array}{c|c} A_{00} & \alpha_{01} e_L \\ \hline * & \alpha_{11} \end{array}\right) - \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \alpha_{12}^2 \end{array}\right)$$

$$A_{00} := \left(\begin{array}{c|c} A_{00} & \alpha_{01} e_L \\ \hline * & \alpha_{11} - \alpha_{12}^2 \end{array}\right)$$

* Continue with computing $A_{00} \rightarrow U_{00} D_{00} U_{00}^T$
* The Algorithm will complete as long as $\delta_1 \neq 0$.

Algorithm $A := \mathrm{UDUT\_TRI}(A)$

Partition $A \rightarrow \left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L^T & 0 \\ \hline * & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline * & * & A_{LL} \end{array}\right)$

where $A_{LL} = 0 \times 0$

while $m(A_{LL}) < m(A)$ do

Repartition

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline * & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline * & * & A_{LL} \end{array}\right) \rightarrow \left(\begin{array}{c|c|c|c} A_{00} & \alpha_{01} e_L & 0 & 0 \\ \hline * & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & * & * & A_{33} \end{array}\right)$$

where

$$\alpha_{11} := \alpha_{11} - \alpha_{12}^2$$

$$\alpha_{12} := \frac{\alpha_{12}}{\alpha_{22}}$$

Continue with.

$$\left(\begin{array}{c|c|c} A_{FF} & \alpha_{FM} e_L & 0 \\ \hline * & \alpha_{MM} & \alpha_{ML} e_F^T \\ \hline * & * & A_{LL} \end{array}\right) \leftarrow \left(\begin{array}{c|c|c|c} A_{00} & \alpha_{01} e_L & 0 & 0 \\ \hline * & \alpha_{11} & \alpha_{12} & 0 \\ \hline * & * & \alpha_{22} & \alpha_{23} e_F^T \\ \hline * & * & * & A_{33} \end{array}\right)$$

endwhile.