Regression with Time-Series Data: Nonstationary Variables
Part 1

BUAN / MECO 6312

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Stationary and Nonstationary Variables

- Formally, a time series y_t is stationary if:

1. Its mean is constant over time and,
2. Its variance is constant over time and,
3. If the covariance between two values from the series depends only on the length of time separating the two values, and not on the actual times at which the variables are observed.

$$E(y_t) = \mu \qquad \text{(constant mean)}$$

$$\operatorname{var}(y_t) = \sigma^2 \qquad \text{(constant variance)}$$

$$\operatorname{cov}(y_t, y_{t+s}) = \operatorname{cov}(y_t, y_{t-s}) = \gamma_s \qquad \text{(covariance depends on } s\text{, not } t\text{)}$$

- The AR(1) is useful for explaining the difference between stationary and non-stationary series:

$$y_t = \rho y_{t-1} + v_t, \quad |\rho| < 1$$

  - The errors v_t are not correlated, with zero mean and constant variance σ_v^2

- The mean of the AR(1) process:

$$y_1 = \rho y_0 + v_1$$

$$y_2 = \rho y_1 + v_2 = \rho(\rho y_0 + v_1) + v_2 = \rho^2 y_0 + \rho v_1 + v_2$$

$$\vdots$$

$$y_t = v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \dots\dots + \rho^t y_0$$

- If $\rho < 1$ and t is sufficiently large, then $\rho^t y_0$ is negligible:

$$E(y_t) = E\left(v_t + \rho v_{t-1} + \rho^2 v_{t-2} + \dots\right) = 0$$

- The variance of the AR(1) process

$$\text{var}(y_t) = E(y_t^2) - (E(y_t))^2 = E(y_t^2) = \frac{\sigma_v^2}{(1-\rho^2)}$$

- And the covariance between two errors *s* period apart is:

$$\text{cov}(y_t, y_{t-s}) = E(y_t \cdot y_{t-s}) = \frac{\sigma_v^2 \rho^s}{1-\rho^2}$$

- **AR(1) process is stationary if** $\rho < 1$
- **The AR(1) in this example is stationary around 0 mean *(figure a)***

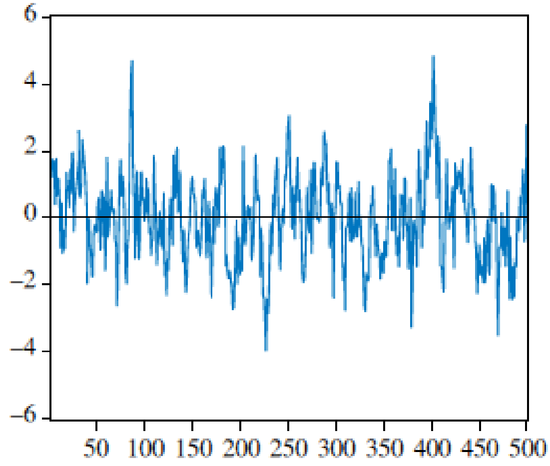
Stationarity around a Non – 0 Mean

- Real-world data often don't have a zero mean
- We say that the "**de-meaned**" variable is $y_t - \mu$ **stationary around 0.**
- Or that the variable y_t is stationary around μ *(figure b).*

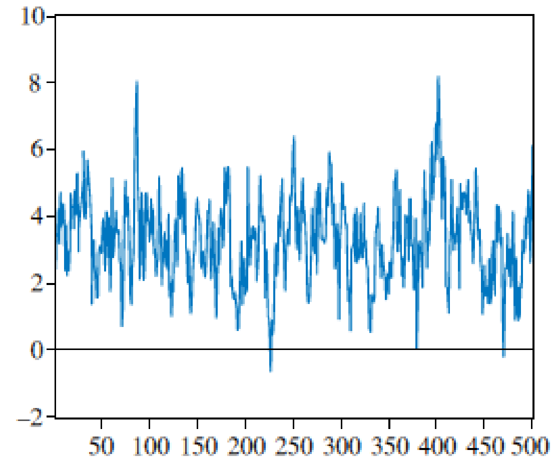
Stationarity around a Linear Trend

- AR(1) model can also fluctuate around a linear trend: ($\mu + \delta t$)
- y_t is **not stationary**, however….
- The "**de-trended**" series ($y_t - \mu - \delta t$) will be **stationary.**
- y_t is often described as stationary around a deterministic trend $\mu + \delta \cdot t$ *(figure c)*

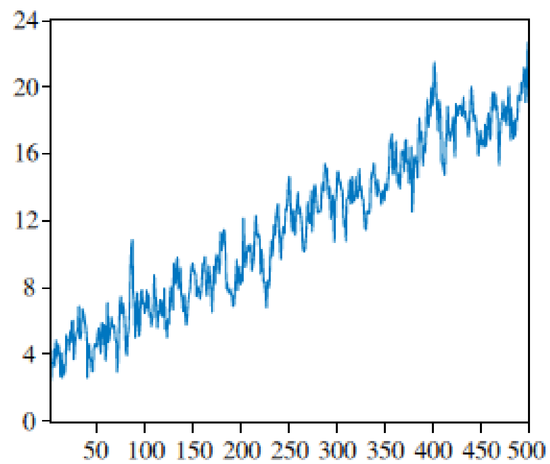
Time-Series Models



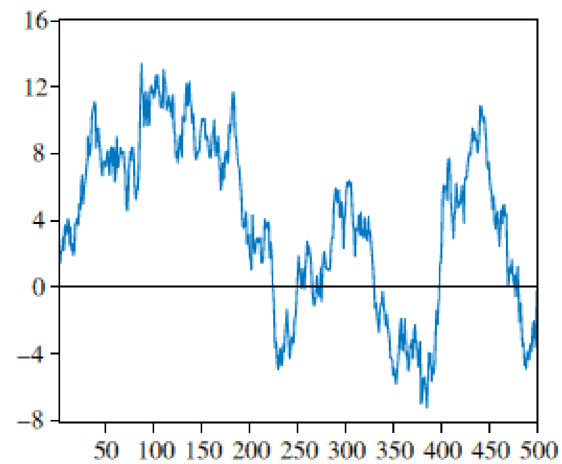
(*a*) $y_t = 0.7y_{t-1} + v_t$



(*b*) $y_t = 1 + 0.7y_{t-1} + v_t$



(*c*) $y_t = 1 + 0.01t + 0.7y_{t-1} + v_t$



(*d*) $y_t = y_{t-1} + v_t$

Random Walk Models

- Consider the special case of $\rho = 1$:

$$y_t = y_{t-1} + v_t$$

- These time series are called **random walks** because they appear to wander slowly upward or downward with no real pattern *(figure d)*.

- $$y_1 = y_0 + v_1$$
  $$y_2 = y_1 + v_2 = (y_0 + v_1) + v_2 = y_0 + \sum_{s=1}^{2} v_s$$
  $$\vdots$$
  $$y_t = y_{t-1} + v_t = y_0 + \sum_{s=1}^{t} v_s$$

- $\sum_{s=1}^{t} v_s$ is called **stochastic trend**
  – a stochastic component v_t is added for each time t, and causes the time series to trend in unpredictable directions

- Random walk is **NOT** stationary.
  - Mean is constant:

$$E(y_t) = y_0 + E(v_1 + v_2 + \ldots + v_t) = y_0$$

  - However, the variance increases over time, eventually becoming infinite

$$\operatorname{var}(y_t) = \operatorname{var}(v_1 + v_2 + \ldots + v_t) = t\sigma_v^2$$

  – This implies that the series may not return to its mean, so sample mean taken for different periods are not the same.

Random Walk with a Drift

- A variation of model is obtained by adding a constant term:

$$y_t = \alpha + y_{t-1} + v_t$$

  - This model is known as the **random walk with drift**

- A better understanding is obtained by applying recursive substitution:

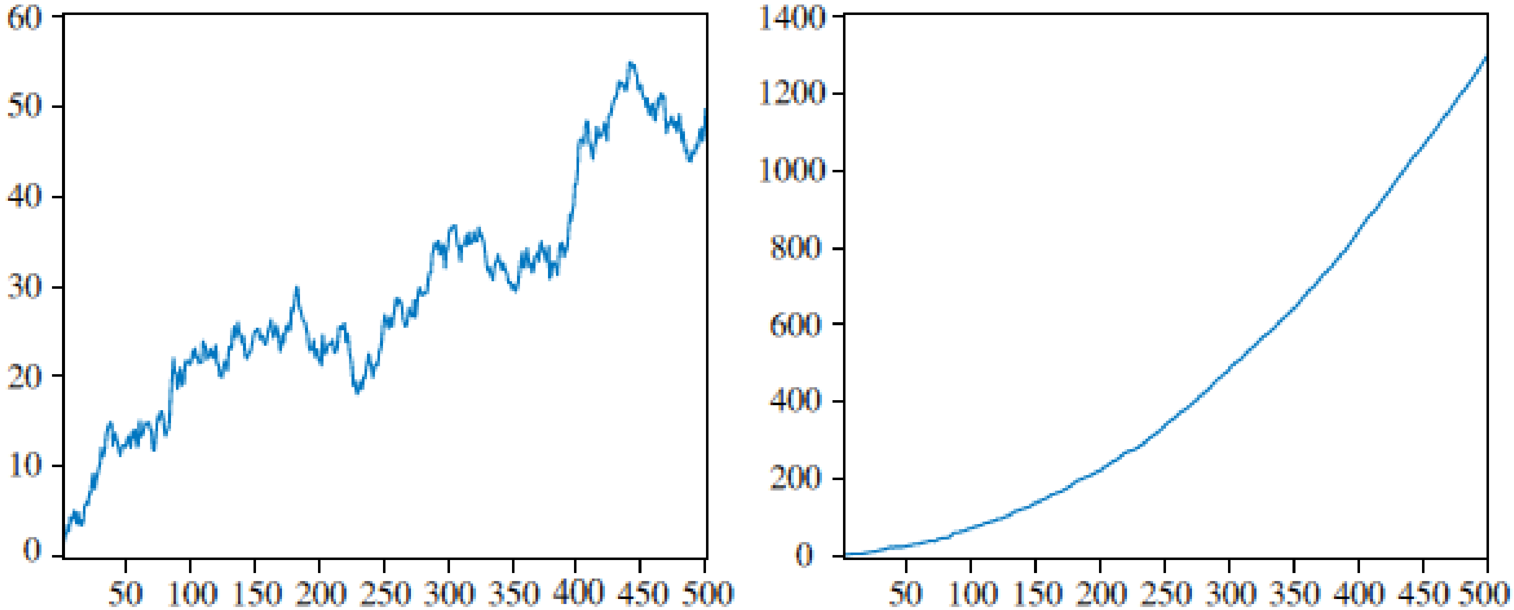
$$y_1 = \alpha + y_0 + v_1$$

$$y_2 = \alpha + y_1 + v_2 = \alpha + (\alpha + y_0 + v_1) + v_2 = 2\alpha + y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = \alpha + y_{t-1} + v_t = t\alpha + y_0 + \sum_{s=1}^{t} v_s$$

- The term $t\alpha$ is a **deterministic trend** component because a fixed value α is added for each time t
- The variable y wanders up and down as well as increases by a fixed amount at each time *t (figure e)*



$(e)\ y_t = 0.1 + y_{t-1} + v_t$ $\quad (f)\ y_t = 0.1 + 0.01t + y_{t-1} + v_t$

- Now both mean and variance of y_t are not constant:

$$E(y_t) = t\alpha + y_0 + E(v_1 + v_2 + v_3 + ... + v_t) = t\alpha + y_0$$

$$\operatorname{var}(y_t) = \operatorname{var}(v_1 + v_2 + v_3 + ... + v_t) = t\sigma_v^2$$

- We can extend the random walk model even further by adding a time trend

$$y_t = \alpha + \delta t + y_{t-1} + v_t$$

- The addition of a time-trend variable *t* strengthens the trend behavior *(figure f):*

$$y_1 = \alpha + \delta + y_0 + v_1$$

$$y_2 = \alpha + \delta 2 + y_1 + v_2 = \alpha + 2\delta + (\alpha + \delta + y_0 + v_1) + v_2 = 2\alpha + 3\delta + y_0 + \sum_{s=1}^{2} v_s$$

$$\vdots$$

$$y_t = \alpha + \delta t + y_{t-1} + v_t = t\alpha + \left(\frac{t(t+1)}{2}\right)\delta + y_0 + \sum_{s=1}^{t} v_s$$

where we used: $1 + 2 + 3 + \cdots + t = t(t+1)/2$

Regression with Time-Series Data: Nonstationary Variables
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Spurious Regressions

- It is important to know whether a time series is stationary or non-stationary.

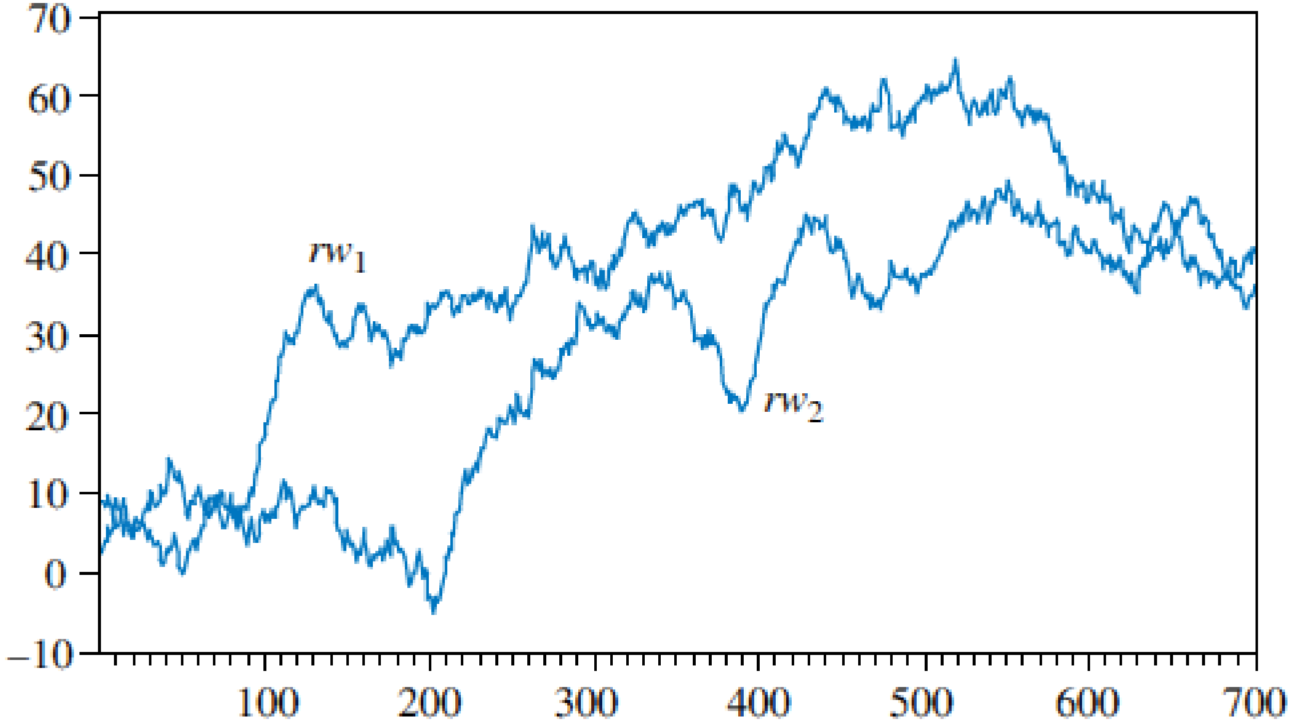
  - When nonstationary series are used in regression analysis there is a danger of obtaining apparently significant regression results from unrelated data.
  - Such regressions are said to be **spurious**

- Consider two independent random walks:

$$
\begin{aligned}
rw_1 &: \quad y_t = y_{t-1} + v_{1t} \\
rw_2 &: \quad x_t = x_{t-1} + v_{2t}
\end{aligned}
$$

  - These series were generated independently and, in truth, have no relation to one another.

Time series and scatter plot of two random walk variables



(*a*) Time series

Yet when plotted, we see a positive relationship between them!

- A simple regression of series one (rw_1) on series two (rw_2) yields:

$$\widehat{rw}_{1t} = 17.818 + 0.842\, rw_{2t} \qquad R^2 = 0.70$$

(t) (40.837)

  - These results are completely meaningless even though t statistic is huge and R^2 is very high.
  - Results will be even stronger for random walk with a drift

- Since many macroeconomic time series are nonstationary, it is particularly important to take care when estimating regressions with macroeconomic variables

Unit Root Tests for Stationarity

Dickey – Fuller Unit Root Tests for Stationarity

- Consider the AR(1) model:

$$y_t = \rho y_{t-1} + v_t$$

  – We can test for nonstationarity by testing the null hypothesis that $\rho = 1$ against the alternative that $\rho < 1$

- A more convenient form is:

$$y_t - y_{t-1} = \rho y_{t-1} - y_{t-1} + v_t$$

$$\Delta y_t = (\rho - 1) y_{t-1} + v_t$$

$$= \gamma y_{t-1} + v_t$$

  – The hypotheses are:

$$H_0 : \rho = 1 \quad \Leftrightarrow \quad H_0 : \gamma = 0$$

$$H_1 : \rho < 1 \quad \Leftrightarrow \quad H_1 : \gamma < 0$$

- The second Dickey–Fuller test includes a constant term in the test equation:

$$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$$

  - The null and alternative hypotheses are the same as before

- The third Dickey–Fuller test includes a constant and a trend in the test equation:

$$\Delta y_t = \alpha + \gamma y_{t-1} + \lambda t + v_t$$

  – The null and alternative hypotheses are as before:

$$H_0: \gamma = 0 \text{ and } H_1: \gamma < 0$$

- If we **reject** the null hypothesis we conclude that the series is **stationary**.

- If we **do not reject** the null hypothesis, we conclude that the series is **not stationary** or that the series has **a unit root**

- The Dickey-Fuller testing procedure:
  - First plot the time series of the variable and select a suitable Dickey-Fuller test based on a visual inspection of the plot
    - If the series appears to be wandering or fluctuating around a sample average of zero, use test 1
    - If the series appears to be wandering or fluctuating around a sample average which is nonzero, use test 2.
    - If the series appears to be wandering or fluctuating around a linear trend, use test 3.

Critical Values for the Dickey–Fuller Test

This t-statistic no longer has the t-distribution under the null hypothesis.

Instead, we use the statistic often called a **τ (*tau*) statistic**

Critical values for one tail (left) are given in the following table:

Model	1%	5%	10%
$\Delta y_t = \gamma y_{t-1} + v_t$	−2.56	−1.94	−1.62
$\Delta y_t = \alpha + \gamma y_{t-1} + v_t$	−3.43	−2.86	−2.57
$\Delta y_t = \alpha + \lambda t + \gamma y_{t-1} + v_t$	−3.96	−3.41	−3.13
Standard critical values	−2.33	−1.65	−1.28

Note: These critical values are taken from R. Davidson and J. G. MacKinnon (1993), *Estimation and Inference in Econometrics*, New York: Oxford University Press, p. 708.

Augmented Dickey–Fuller Test

- An important extension of the Dickey-Fuller test allows for the possibility that the error term is auto-correlated.
- Such auto-correlation is likely to occur if our earlier models did not have sufficient lags to capture all full dynamic nature in the process.
- The solution: **we will add lags of the dependent variable until we eliminate auto-correlation in the residuals, and only then we will conduct the Dickey-Fuller test.**
- This is the augmented Dickey-Fuller test.

- As an example, consider the two interest rate series:
  - The federal funds rate (F_t)
  - The three-year bond rate (B_t)

The Dickey-Fuller Tests: An Example

```
.
. dfuller f , regress lags(1)

Augmented Dickey-Fuller test for unit root         Number of obs   =       102

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -2.505            -3.509            -2.890            -2.580
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.1143

------------------------------------------------------------------------------
         D.f |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           f |
         L1. |  -.0446213   .0178142    -2.50   0.014    -.0799685   -.0092741
         LD. |   .5610582   .0809827     6.93   0.000     .4003708    .7217455
             |
       _cons |   .1725221   .1002333     1.72   0.088    -.0263625    .3714067
------------------------------------------------------------------------------

. dfuller b , regress lags(1)

Augmented Dickey-Fuller test for unit root         Number of obs   =       102

                               ---------- Interpolated Dickey-Fuller ---------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -2.703            -3.509            -2.890            -2.580
------------------------------------------------------------------------------
MacKinnon approximate p-value for Z(t) = 0.0735

------------------------------------------------------------------------------
         D.b |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
           b |
         L1. |  -.0562412   .0208081    -2.70   0.008     -.097529   -.0149534
         LD. |   .2903078   .0896069     3.24   0.002     .1125084    .4681072
             |
       _cons |    .236873   .1291731     1.83   0.070    -.0194345    .4931804
------------------------------------------------------------------------------
```

- Consider the random walk model:

$$y_t = y_{t-1} + v_t$$

  - This can be rendered stationary by taking the first difference:

$$\Delta y_t = y_t - y_{t-1} = v_t$$

  - The variable y_t is said to be a **first difference stationary** series

- Series like y_t, which can be made stationary by taking the first difference, are also said to be **integrated of order one**, and denoted as **I**(**1**)
  - Stationary series are said to be integrated of order zero, **I**(**0**)

- In general, the order of integration of a series is the minimum number of times it must be differenced to make it stationary

- The results of the Dickey–Fuller test for a random walk applied to the first differences are:

```
.
. dfuller D.f, noconstant lags(0)

Dickey-Fuller test for unit root                     Number of obs   =       102

                               ---------- Interpolated Dickey-Fuller ----------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -5.487            -2.600            -1.950            -1.610

.
. dfuller D.b, noconstant lags(0)

Dickey-Fuller test for unit root                     Number of obs   =       102

                               ---------- Interpolated Dickey-Fuller ----------
                  Test         1% Critical       5% Critical      10% Critical
               Statistic           Value             Value             Value
------------------------------------------------------------------------------
 Z(t)             -7.662            -2.600            -1.950            -1.610
```

- Based on the large negative value of the *t* statistic (-5.487 < -1.95), we **reject** the null hypothesis that ΔF_t is nonstationary and **accept the alternative that it is stationary**.

- We similarly conclude that ΔB_t is **stationary** (-7.662 < -1.95)

Conclusion: F and B are both integrated of order 1.

First Difference Stationary

- If the series are **difference stationary** – by taking **first differences**

- Suppose that *y* and *x* are I(1), an example of an ARDL(1,1) regression equation is:

$$\Delta y_t = \alpha + \theta \Delta y_{t-1} + \beta_0 \Delta x_t + \beta_1 \Delta x_{t-1} + e_t$$