SteepestDescent-Example

Question 1

The following questions analyze the behavior of the steepest-descent for ellipsoids. Let

$$f(x, y) = \frac{1}{2}(cx^2 + y^2)$$

where c is a constant. Notice that the minimum of $f(x, y)$ is at the point $x = 0, y = 0$, regardless of the value of c. For $c = 1$ cross sections of f are circles, and we expect the steepest-descent to perform very well.

1. Show that the exact steepest-descent is given by the equations:

$$x_{k+1} = (1 - a_k c)x_k, \qquad y_{k+1} = (1 - a_k)y_k,$$

where

$$a_k = \frac{c^2 + m_k^2}{c^3 + m_k^2}, \qquad m_k = \frac{y_k}{x_k}$$

2. Show that if we start with $x = 1, y = c$, then at the k'th iteration $a_k = 2/(1 + c)$ and

$$x_k = \left(\frac{1-c}{1+c}\right)^k, \qquad y_k = (-1)^k c \left(\frac{1-c}{1+c}\right)^k$$

3. Use the results of the previous question to estimate the number of iterations required to obtain $|x| \leq 10^{-6}, |y| \leq 10^{-6}$, for $c = 0.001$. (Answer : $k = 6908$)

4. Write a program that implements a variant of the exact steepest-descent algorithm in which x is updated by the rule:

$$x = x + \beta a r$$

where a is as given by the exact steepest-descent algorithm and $\beta > 0$.

Write the program for minimizing $f(x, y)$ with $c = 0.001$, and output the number of iterations needed in order to obtain $|x| \leq 10^{-6}, |y| \leq 10^{-6}$. Assume that the program always starts with $x = 1, y = c$. Choose $\beta = 1$ and verify the results of the previous problem. How many iterations are needed for the following values of β: $0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1$?

Notice that here we you cannot use $a = 2/(1 + c)$. Instead you should be using $a = (c^2 + m^2)/(c^3 + m^2)$, with $m = y/x$.

5. Update your program for the previous question to compute the number of iterations when in each iteration β is chosen at random from the interval $[0, 1]$.

6. Implement the $\epsilon - step$ steepest-descent algorithm for the same function. What value of ϵ is needed in order to get accuracy as above? How many iterations are needed?