Last class: $X \sim N[\mu, \sigma^2]$

known

$H_0: \mu = \mu_0$

Test statistic: $Z = \dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N[0,1]$ when H_0 is true.

- compute critical point and p-value using $N(0,1)$ as the null distribution

- Z-test

Still: $X \sim N[\mu, \sigma^2]$, σ^2 = unknown. ; s^2 = sample variance

Case 2: t-**test (unknown σ^2):** $H_0 : \mu = \mu_0$

Test statistic: $T = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \sim t_{n-1}$ when H_0 is true

Critical point for the level α test:

One-sided alternative: $t_{n-1,\alpha}$ ~ $-t_{n-1,\alpha} = t_{n-1,1-\alpha}$

Two-sided alternative: $t_{n-1,\alpha/2}$

p**-value:**

H_1	reject when	p-value	computing p-value
$\mu \neq \mu_0$	$\lvert t_{\text{obs}}\rvert \geq t_{n-1,\alpha/2}$	$P(\lvert t\rvert \geq \lvert t_{\text{obs}}\rvert \,\lvert H_0)$	$2(1 - F(\lvert t_{\text{obs}}\rvert))$
$\mu > \mu_0$	$t_{\text{obs}} \geq t_{n-1,\alpha}$	$P(t \geq t_{\text{obs}} \,\lvert H_0)$	$1 - F(t_{\text{obs}})$
$\mu < \mu_0$	$t_{\text{obs}} \leq -t_{n-1,\alpha}$	$P(t \leq t_{\text{obs}} \,\lvert H_0)$	$F(t_{\text{obs}}) = pt(t_{\text{obs}}, n-1)$

F = CDF a t_{n-1} df.

One-sample test for μ when X is nonnormal

Large-sample z-test: $H_0 : \mu = \mu_0$

- Need large n but works for mean of any (non-normal) population
- Use the z-test with test statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim \frac{\bar{X} - \mu_0}{s/\sqrt{n}} \overset{\cdot}{\sim} N[0,1] \text{ under } H_0.$$

- When n is large, the null distribution is approximately $N(0, 1)$ due to central limit theorem.
- This test has approximate level α.

A general large-sample test (Z-test) based on MLE

Recall: Often, $\hat{\theta} \overset{\cdot}{\sim} N[\theta, \hat{V}]$ when n is large.

$\uparrow$ MLE

$H_0: \theta = \theta_0$

Then: $Z = \dfrac{\hat{\theta} - \theta_0}{\sqrt{\hat{V}}} \overset{\cdot}{\sim} N(0,1)$ when H_0 is true and n is large.

$\Rightarrow$ Can do a Z-test if n is large.

One-sample test for population prop p

— Just a special case of the previous situation. $X \sim$ Bernoulli(p)

- The large-sample z-test works because in this case $X \sim$ Bernoulli(p) and $E(X) = \underline{p}$.
- Use the z-test with test statistic

$$z = \frac{\hat{p} - p_0}{SE(\hat{p})}, \qquad SE(\hat{p}) = \begin{cases} \sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}} \\ \text{or} \\ \sqrt{\dfrac{p_0(1-p_0)}{n}} \end{cases}$$

"book"

both are valid; Can use either one.

- This test has approximate level α.

Ex 1: A long-time authorized user of a computer account takes 0.2 seconds on average between keystrokes. One day, when a user typed in the username and password, 15 times between keystrokes were recorded. These data had mean of 0.3 seconds and standard deviation of 0.12 seconds. Do these data give evidence of an unauthorized login attempt? Assume normality for the times between keystrokes and 5% level of significance.

n $\bar{X}$ s

Recall: X = time b/w two keystrokes for the person trying to log in (sec)

$\mu = E(X)$

$H_0: \underbrace{\mu = 0.2}_{\text{authorized attempt}}$ vs $H_1: \underbrace{\mu \neq 0.2}_{\text{unauthorized attempt}}$

Assume: $X \sim N[\mu, \sigma^2]$. $\alpha = 0.05$

$T_{obs} = \dfrac{0.3 - 0.2}{0.12/\sqrt{15}}$

Since σ^2 is unknown, do a t-test.

see R code

Ex 2: The number of concurrent users for an ISP has historically averaged 5000. After a marketing campaign, the management would like to know if it has resulted in an increase in the number of concurrent users. To test this, data were collected by observing the number of concurrent users at 100 (n) randomly selected moments of time. Suppose that the average ($\bar{X}$) and the standard deviation (s) of the sample data are 5200 and 800, respectively. Is there evidence that the mean number of concurrent users has increased? Assume 5% level of significance (α).

Recall: X = # concurrent users

$\theta = E[X]$

$H_0: \underbrace{\theta = 5000}_{\text{campaign not successful}}$ vs. $H_1: \underbrace{\theta > 5000}_{\text{Campaign successful}}$

Do - large-sample z-test

$$z = \frac{5200 - 5000}{800/\sqrt{100}}$$

see R code

Ex 3: A recent poll of 1,000 American people estimated that the approval rating of the current congress is 31%. Do these data give evidence that less than 30% of the American people approve the performance of the congress? Assume 5% level of significance.

$\hat{p}$

Recall: $X = \begin{cases} 1, & \text{"approve"} \\ 0, & \text{"not approve"} \end{cases}$ $X \sim \text{Bernoulli}(p)$

$\theta = E[X] = P[X=1] = p$

$H_0: p = 0.3$ vs $H_1: p < 0.3$

Use large-sample z-test. Can use either $\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ or $\sqrt{\frac{p_0(1-p_0)}{n}}$ in the denom.

$$Z_{obs} = \frac{\hat{p} - p_0}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} = \frac{0.31 - 0.3}{\sqrt{\frac{(0.31)(1-0.31)}{1000}}}$$

Recap:

Test of hypothesis on p

$H_0 : p = p_0$

Is n large?

Yes → $Z = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$ or $\dfrac{\hat{p} - p_0}{\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}}$

No → Take advanced class

$H_1 : p \neq p_0$

- Reject H_0 if $|Z| > Z_{\alpha/2}$
- p-value: $2\{1 - F(|Z_{obs}|)\}$

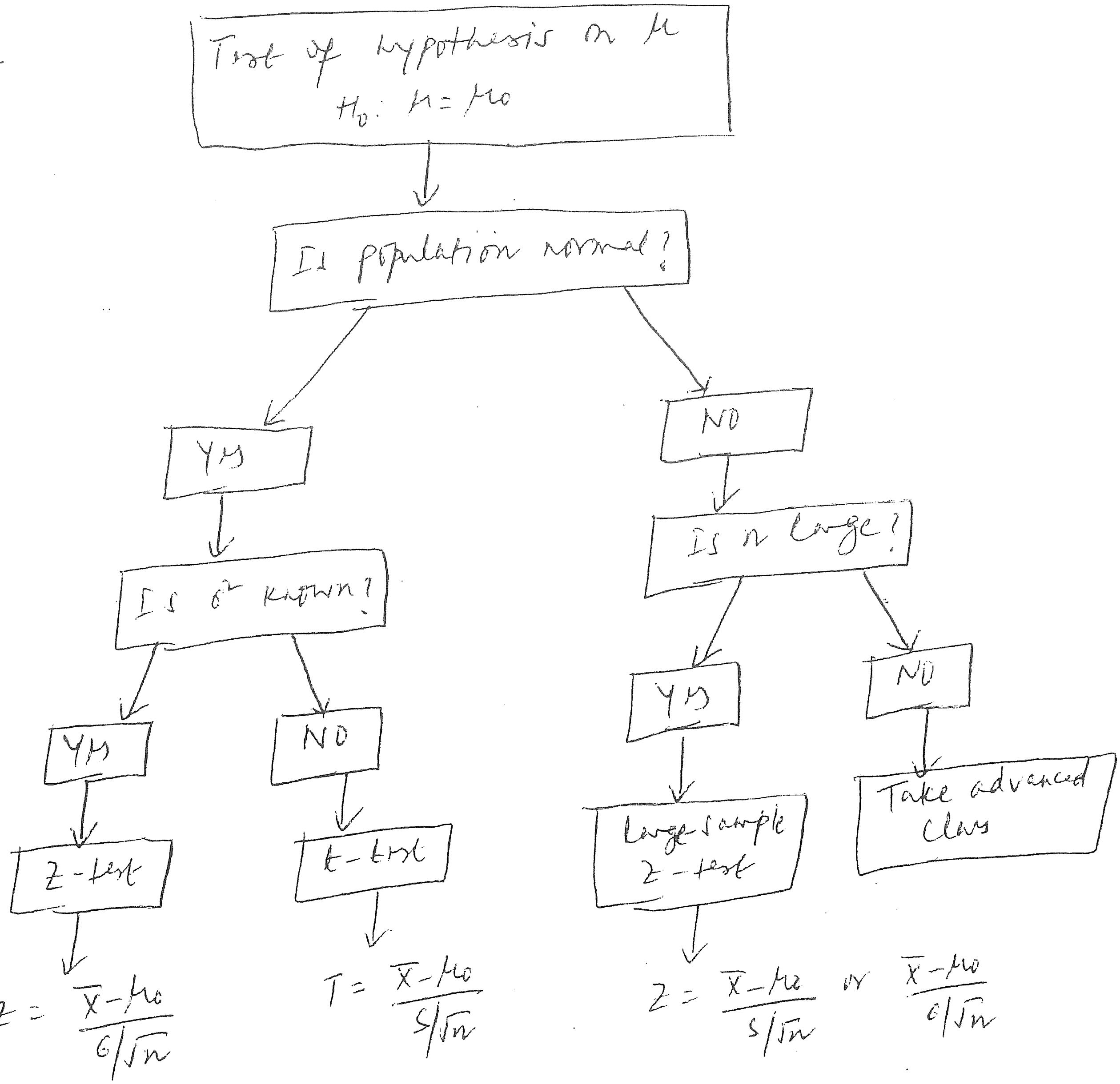
$H_1 : p > p_0$

- Reject H_0 if $Z > Z_\alpha$
- p-value: $1 - F(Z_{obs})$

$H_1 : p < p_0$

- Reject H_0 if $Z < -Z_\alpha$
- p-value $= F(Z_{obs})$

Recap:



$H_1: \mu \neq \mu_0$

- Reject H_0 if $|Z| > z_{\alpha/2}$ or $|T| > t_{n-1,\alpha/2}$
- p-value: $2\{1 - F(|Z_{obs}|)\}$ or $2\{1 - F(|T_{obs}|)\}$

$H_1: \mu > \mu_0$

- Reject H_0 if $Z > z_\alpha$ or $T > t_{n-1,\alpha}$
- p-value: $1 - F(Z_{obs})$ or $1 - F(T_{obs})$

$H_1: \mu < \mu_0$

- Reject H_0 if $Z < -z_\alpha$ or $T < -t_{n-1,\alpha}$
- p-value: $F(Z_{obs})$ or $F(T_{obs})$

Two-sample tests for $\mu_X - \mu_Y$ for normal populations

Set up: $X \sim N(\mu_X, \sigma_X^2)$, $Y \sim N(\mu_Y, \sigma_Y^2)$

- X sample: $X_1, \ldots, X_n$ — i.i.d. as X
- Y sample: $Y_1, \ldots, Y_m$ — i.i.d. as Y
- $H_0 : \underbrace{\mu_X - \mu_Y}_{\theta} = \underset{\theta_0}{\Delta}$, where Δ is given and may be zero

θ

Case 1: Paired samples, i.e., (X_i, Y_i) comes from subject $i = 1, \ldots, n$.

$= E[D] = \mu_X - \mu_Y$

$H_0 : \mu_D = 0$

- $D = X - Y \sim N(\mu_D, \sigma_D^2)$ where
- Define the differences $D_i = X_i - Y_i$ — i.i.d. as
- Apply one-sample procedures to the differences — **paired z-test** or **paired t-test**

Case 2: Independent samples with known variances σ_X^2 & σ_Y^2

Test statistic: $H_0: \mu_X - \mu_Y = \Delta$

$$Z = \frac{(\bar{X} - \bar{Y}) - \Delta}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N[0,1] \quad \text{if } H_0 \text{ is true.}$$

- Know how to get critical points and p-values for z-test
- **Two-sample z-test**

Case 3: Independent samples with unknown variances σ_X^2 & σ_Y^2

Test statistic: (w/o making any assumption about equality of variances).

$$T = \frac{(\bar{X} - \bar{Y}) - \Delta}{\sqrt{\frac{S_X^2}{n} + \frac{S_Y^2}{m}}} \dot{\sim} t_\nu \quad \text{if } H_0 \text{ is true,}$$

where ν is computed using Satterthwaite's formula (Welch's formula)

- Know how to get get critical points and p-values for a t-test
- Approximate **Two-sample t-test**
- No assumption regarding equality of variances

Case 4: Independent samples with unknown but equal variances $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ common value.

Estimation of common variance σ^2:

$$S_p^2 = \frac{(n-1)S_X^2 + (m-1)S_Y^2}{n+m-2}$$

$$\mathrm{Var}(\bar{X}-\bar{Y}) = \sigma^2\left(\frac{1}{n}+\frac{1}{m}\right)$$

Test statistic:

$$\frac{\bar{X}-\bar{Y}-\Delta}{\sqrt{S_p^2\left(\frac{1}{n}+\frac{1}{m}\right)}} \sim t_{n+m-2} \text{ if } H_0 \text{ is true.}$$

- Know how to get get critical points and p-values for a t-test
- **Two-sample t-test**