# An LP Formulation Example:
# Minimum Cost Flow

The Minimum Cost Flow (MCF) problem is a frequently used model. It can be described in our context as follows.

Model

Given a network with $N$ nodes and links among them. We would like to transport some entity among nodes (for example, data), so that it flows along the links. The goal is to determine the optimal flow, that is, how much flow is put on each link, according to the conditions discussed below.

Let us review the input for the problem, the objective and constraints and then let us formulate it as a linear programming task.

*Input data*:

- The link from node $i$ to $j$ has a given capacity $C_{ij} \geq 0$.
- Each node $i$ is associated with a given number $b_i$, the *source rate* of the node. If $b_i > 0$, the node is called a *source*, if $b_i < 0$, the node is a *sink*. If $b_i = 0$, the node is a "transshipment" node that only forwards the flow with no loss and no gain.
- Each link is associated with a *cost factor* $a_{ij} \geq 0$. The value of $a_{ij}$ is the cost of sending unit amount of flow on link $(i, j)$. Thus, sending $x_{ij}$ amount of flow on the link costs $a_{ij}x_{ij}$.

*Remarks:*

- The links are directed in this model, $C_{ij}$ and $C_{ji}$ may be different.
- If the link from $i$ to $j$ is missing from the network, then $C_{ij} = 0$. Thus, the capacities automatically describe the network topology, too.

*Constraints*:

- *Capacity constraint:* The flow on each link cannot exceed the capacity of the link.
- *Flow conservation:* The total outgoing flow of a node minus the total incoming flow of the node must be equal to the source rate of the node. That is, the difference between the flow out and into the node is exactly what the node produces or sinks. For transshipment nodes ($b_i = 0$) the outgoing and incoming flow amounts are equal.

*Objective*:

Find the amount of flow sent on each link, so that the constraints are satisfied and the total cost of the flow is minimized.

## LP Formulation

Let $x_{ij}$ denote the flow on link $(i, j)$. The $x_{ij}$ are the variables we want to determine.

Let us express the constraints:

- The flow is nonnegative (by definition):

$$x_{ij} \geq 0 \qquad (\forall i, j)$$

- Capacity constraints:

$$x_{ij} \leq C_{ij} \qquad (\forall i, j)$$

- Flow conservation:

$$\sum_{j=1}^{N} x_{ij} - \sum_{k=1}^{N} x_{ki} = b_i \qquad (\forall i)$$

  Here the first sum is the total flow out of node $i$, the second sum is the total flow into node $i$.

The objective function is the total cost, summed for all choices of $i, j$ :

$$Z = \sum_{i,j} a_{ij} x_{ij}$$

Thus, the LP formulation is:

$$\min\ Z = \sum_{i,j} a_{ij} x_{ij}$$

subject to

$$\begin{aligned}
\sum_{j=1}^{N} x_{ij} - \sum_{k=1}^{N} x_{ki} &= b_i && (\forall i) \\
x_{ij} &\leq C_{ij} && (\forall i, j) \\
x_{ij} &\geq 0 && (\forall i, j)
\end{aligned}$$

Is this in standard form? No, but can be easily transformed into standard form. Only the $x_{ij} \leq C_{ij}$ inequalities have to be transformed into equations. This can be done by introducing slack variables $y_{ij} \geq 0$ for each, and replacing each original inequality $x_{ij} \leq C_{ij}$ by $x_{ij} + y_{ij} = C_{ij}$.