Linear Programming

The *objective function* is of the form

$$Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

where

$c_1, \ldots, c_n$ are given constants

$x_1, \ldots, x_n$ are the variables to be optimized.

In the *standard form* of LP the *constraints* are formulated as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \\
x_1 &\geq 0 \\
&\vdots \\
x_n &\geq 0
\end{aligned}$$

We either want to minimize or maximize the objective function, subject to the constraints. That is, given the parameters a_{ij}, b_i, c_i as input, we want to find values for the x_i variables, such that all constraints are satisfied and the objective function takes minimum or maximum value, depending on whether we want to minimize or maximize.

Form the mathematical point of view, minimization and maximization are equivalent, since minimizing an objective function Z means the same as maximizing $-Z$, and vice versa.

Thus, a linear programming problem (or linear program, for short; abbreviated LP) looks like this for minimization:

$$\min Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$

subject to

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \\ x_1 & \geq & 0 \\ & \vdots & \\ x_n & \geq & 0 \end{array}$$

For maximization it would look the same, just the “min” would be replaced by “max”.

An important restriction is that we only allow *linear* expressions, both in the objective function and in the constraints. A linear expression (in terms of the x_i variables) can only be one of the following:

- a constant
- a variable x_i multiplied by a constant (such as c_ix_i)
- sum of expressions that belong to either the above types (Examples: $c_1x_1 + c_2x_2 + \ldots + c_nx_n$, or $3x_4 - 2$).

No other types of expressions are allowed. In particular, no product of variables, no logical case separation, etc.

The standard form LP in vector-matrix notation:

$$\min Z = \mathbf{cx}$$

subject to

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq 0 \end{aligned}$$

Other formulations are also possible. An often used version is shown below (some textbooks call this one the standard form, rather than the one above):

$$\min Z = \mathbf{cx}$$

subject to

$$\mathbf{Ax} \geq \mathbf{b}$$

$$\mathbf{x} \geq 0$$

Note: $\mathbf{x}$ is meant a column vector, which implies that $\mathbf{c}$ must be row a vector, so that the $\mathbf{cx}$ product makes sense. If $\mathbf{c}$ is a column vector, then we write $\mathbf{c}^T$. Usually it is clear from the context, which vector is column and which is row, so often the distinction is not shown explicitely.

When we write an inequality between vectors, then it means it holds componentwise. For example, $\mathbf{Ax} \leq \mathbf{b}$ means that when we take matrix-vector product $\mathbf{Ax}$, then each component of the resulting vector is $\leq$ than the corresponding component of the vector $\mathbf{b}$. Similarly, $\mathbf{x} \geq 0$ means that each component of $\mathbf{x}$ is ≥ 0.

The different formulations can be easily converted into each other, usually at the price of increasing the number of variables.

Why does it make sense to increase the number of variables? Because the standard form may be advantageous (For example, an LP solver program may require the input in standard form).

Exercises

1. Convert the following LP into standard form:

$$\min Z = 5x - 6y$$

subject to

$$\begin{aligned} 2x - 3y &\geq 6 \\ x - y &\leq 4 \\ x &\geq 3 \end{aligned}$$

Solution: Set $y = x_2 - x_3$ for the free (unbounded) variable y, using the fact that any number can be expressed as the difference of two *nonnegative* numbers. Furthermore, introduce a *slack variable* in each inequality. These slack variables, as discussed in class, "fill the gap" between the two sides of the inequality, to convert it into an equation.

For example, the first constraint $2x - 3y \geq 6$ will be transformed as follows. Let us use x_1 instead of x, for uniform notation. The variable y will be replaced by $x_2 - x_3$, where $x_2, x_3 \geq 0$. Then we get

$$2\underbrace{x_1}_{=x} - 3\underbrace{(x_2 - x_3)}_{\text{this was } y} - \underbrace{x_4}_{\text{slack variable}} = 6.$$

Notice that this will indeed force the original left-hand side (the part before x_4) greater than or equal to 6, as the original inequality required. The reason is that $x_4 \geq 0$ must hold, since all variables are forced to be nonnegative in the standard from. Therefore, the expression before x_4 must be ≥ 6, since *subtracting* a positive (or 0) quantity makes it equal to 6. If the original inequality had the $\leq$ direction, then the only difference would be that we would *add* the slack variable, rather than *subtracting* it.

After transforming the other inequalities, too, and also substituting the old variables in the objective function with the new ones, we get the entire standard form:

$$\min Z = 5x_1 - 6x_2 + 6x_3$$

subject to

$$\begin{aligned}
2x_1 - 3x_2 + 3x_3 - x_4 &= 6 \\
x_1 - x_2 + x_3 + x_5 &= 4 \\
x_1 - x_6 &= 3 \\
x_1,\, x_2,\, x_3,\, x_4,\, x_5,\, x_6 &\geq 0
\end{aligned}$$

2.

Convert the following problems to standard form:

$$
\begin{aligned}
\text{a)}\quad & \text{minimize} && x+2y+3z \\
& \text{subject to} && 2 \leqslant x+y \leqslant 3 \\
& && 4 \leqslant x+z \leqslant 5 \\
& && x \geqslant 0, \quad y \geqslant 0, \quad z \geqslant 0. \\
\text{b)}\quad & \text{minimize} && x+y+z \\
& \text{subject to} && x+2y+3z = 10 \\
& && x \geqslant 1, \quad y \geqslant 2, \quad z \geqslant 1.
\end{aligned}
$$

3.

A class of piecewise linear functions can be represented as $f(\mathbf{x}) = \text{Maximum}\ (\mathbf{c}_1^T\mathbf{x} + d_1, \mathbf{c}_2^T\mathbf{x} + d_2, \ldots, \mathbf{c}_p^T\mathbf{x} + d_p)$. For such a function f, consider the problem

$$
\begin{aligned}
\text{minimize} \quad & f(\mathbf{x}) \\
\text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\
& \mathbf{x} \geqslant \mathbf{0}.
\end{aligned}
$$

Show how to convert this problem to a linear programming problem.

4.

Convert the following problem to a linear program in standard form:

$$
\begin{aligned}
\text{minimize} \quad & |x| + |y| + |z| \\
\text{subject to} \quad & x + y \leqslant 1 \\
& 2x + z = 3.
\end{aligned}
$$

5.

A small computer manufacturing company forecasts the demand over the next n months to be d_i, $i = 1, 2, \ldots, n$. In any month it can produce r units, using *regular* production, at a cost of b dollars per unit. By using *overtime*, it can produce additional units at c dollars per unit, where $c > b$. The firm can store units from month to month at a cost of s dollars per unit per month. Formulate the problem of determining the production schedule that minimizes the cost, such that in each month the demand is satisfied.

6. * A telecommunications company sets up routes through its network to serve certain source-destination (S-D) pairs of traffic. We want to assign bandwidth to each route, under the following conditions:

- The routes are fixed and known in advance, each route goes through a known set of links. (These sets can possibly overlap, as the routes may share links.)
- Each link has a known available capacity, which cannot be exceeded by the routes that use the link, in the sense that the sum of the route bandwidths on the link cannot be more than the link capacity.
- Assigning bandwidth to a route has a cost. This cost is proportional to the bandwidth assigned. The cost of unit bandwidth is known for each route (may be different for different routes).
- Each route generates a profit, due to the traffic it carries. The profit of each route is proportional to the bandwidth assigned to the route. The profit generated by unit bandwidth is known for each route (may be different for different routes).

Under the above conditions, the company wants to decide how much bandwidth to assign to each route. The goal is that the *ratio of the total profit vs. the total cost is maximized.* In other words, they want to maximize the yield of the bandwidth investment in the sense that it brings the highest profit percentage. Formulate this optimization problem as a linear program.

Hint: Formulate it first as optimizing the ratio of two linear functions under linear constraints. This is still not an LP, since the objective function is a fraction. Then convert it into an LP in two steps, using the following idea.

Step 1. If you multiply both the numerator and the denominator of the fractional objective function by the same (nonzero) number, then the value of the fraction remains the same. Therefore, after introducing such as scaling factor, which will be an extra variable, we can fix the denominator at an arbitrary fixed nonzero value, say 1. This gives a new constraint. After adding the new constraint, we can aim at maximizing only the numerator in the new system, so we got rid of the fraction in the objective function.

Step2. While this new system is still nonlinear, containing the product of the original variables and the scaling factor, we can make it linear it by introducing another new variable for the product of variables.