Constraint Aggregation

In the constraint aggregation method we replace several constraints by a single one. Let us show the technique trough the example of the knapsack problem. The original knapsack problem is formulated as follows.

$$\max Z = \sum_{i=1}^{N} p_i x_i$$

Subject to

$$\begin{aligned} \sum_{i=1}^{N} c_i x_i &\leq C \\ x_i &\in \{0,1\}, \qquad i = 1, \ldots, N \end{aligned}$$

Let us assume now that instead of a single inequality constraint we have two. Assume further that all constants are nonnegative integers. Thus, the task becomes this:

$$\max Z = \sum_{i=1}^{N} p_i x_i$$

Subject to

$$\begin{aligned} \sum_{i=1}^{N} a_i x_i &\leq A \\ \sum_{i=1}^{N} b_i x_i &\leq B \\ x_i &\in \{0,1\}, \qquad i = 1, \ldots, N \end{aligned}$$

A possible motivation is, refering to the Capital Budgeting problem, that there are two kinds of costs associated with each item, and the budgets for the different types come from different sources, so they may have to be obeyed separately. Or, in the knapsack terminology, each item has a size and a weight parameter, and the knapsack can carry only a certain total size and a certain total weight, so we have to satisfy both constraints.

Let us first convert the inequality constraints into equations, by introducing slack variables. Let x_{N+1} be the slack variable in the first inquality, and x_{N+2} in the second. Further, let $a_{N+1} = 1,\ b_{N+1} = 0,\ b_{N+2} = 1$. In this way we can simply include the slack variables in the summation, without extra terms. Moreover, we can observe that the size of the "gap" the slack variable has to fill can be at most A in the first constraint, and at most B in the second.

$$\max\ Z = \sum_{i=1}^{N} p_i x_i$$

Subject to

$$\begin{aligned}
\sum_{i=1}^{N+2} a_i x_i &= A \\
\sum_{i=1}^{N+2} b_i x_i &= B \\
x_i &\in \{0,1\}, \qquad i = 1, \ldots, N \\
x_{N+1} &\in \{0, 1, \ldots, A\} \\
x_{N+2} &\in \{0, 1, \ldots, B\}
\end{aligned}$$

Now the question is this: can we replace the two equality constraints above by a single one, so that this single constraint is equivalent to the joint effect of the two? By equivalence we mean that they generate the same set of feasible solutions.

The answer to the above question is yes, even though it may seem counter-intuitive at first. Let us define the parameters of the new constraint by

$$c_i = a_i + Mb_i$$

for every i, and

$$C = A + MB$$

where M is a constant that we are going to choose later.

It is clear that the two constraints

$$\sum_{i=1}^{N+1} a_i x_i = A \quad (1)$$

$$\sum_{i=1}^{N+2} b_i x_i = B \quad (2)$$

imply the new one

$$\sum_{i=1}^{N+2} (a_i + Mb_i) x_i = A + MB \quad (3)$$

(taking $a_{N+2} = 0$), since (3) is just a linear combination of (1) and (2). The question, however, is this: can we choose M such that the implication also holds in the opposite direction, that is, such that the aggregated constraint (3) implies (1) and (2)?

The answer is yes. To see it, let us rearrange (3) in the following form:

$$\underbrace{\sum_{i=1}^{N+1} a_i x_i - A}_{Y} + M \underbrace{\left(\sum_{i=1}^{N+2} b_i x_i - B \right)}_{Z} = 0 \quad (4)$$

Now observe that the value of

$$Y = \sum_{i=1}^{N+1} a_i x_i - A$$

must be at least $-A$, and cannot be more than $\sum_{i=1}^{N} a_i + A - A = \sum_{i=1}^{N} a_i$. So we have the bound

$$|Y| \leq A' = \max\{A, \sum_{i=1}^{N} a_i\}.$$

Using $Z = \sum_{i=1}^{N+2} b_i x_i - B$, we can rewrite (4) as

$$Y + MZ = 0. \tag{5}$$

How can this equality hold? One possibility is that $Y = Z = 0$. In that case, (1) and (2) are both satisfied, since

$$Y = \sum_{i=1}^{N+1} a_i x_i - A \quad \text{and} \quad Z = \sum_{i=1}^{N+2} b_i x_i - B.$$

On the other hand, if $Z \neq 0$, then $|Z| \geq 1$, being an integer. In this case Y must make the left hand side 0 in (5). If, however, we choose $M = A' + 1$, then it becomes impossible, since with this choice we have $M|Z| \geq A' + 1$, and we already know $|Y| \leq A'$.

Thus, with the choice of $M = A' + 1$, we can achieve that whenever the aggregated constraint (3) is satisfiable, both original constraints (1) and (2) will also be satisfiable. Of course, this is only guaranteed when the variables take their values from the allowed ranges.

Remark: We can assume that $A < \sum_{i=1}^{N} a_i$, since otherwise the original constraint $\sum_{i=1}^{N} a_i x_i \leq A$ would always be satisfied, so we could simply leave it out. With this assumption the choice for the constant M becomes

$$M = 1 + \sum_{i=1}^{N} a_i.$$