Application of Maximum Flows to Solve Other Optimization Problems

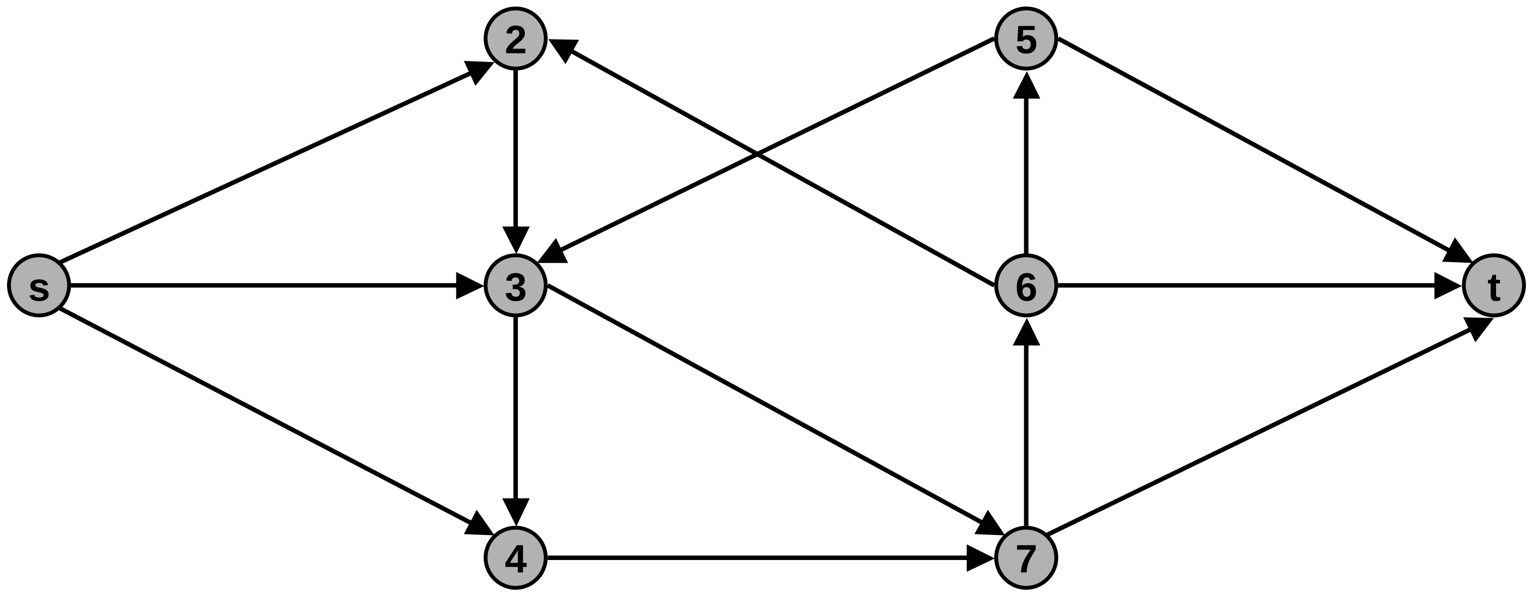
Disjoint Paths

Disjoint path network: G = (V, E, s, t).

- Directed graph (V, E), source s, sink t.
- Two paths are edge-disjoint if they have no arc in common.

Disjoint path problem: find max number of edge-disjoint s-t paths.

- Application: communication networks.



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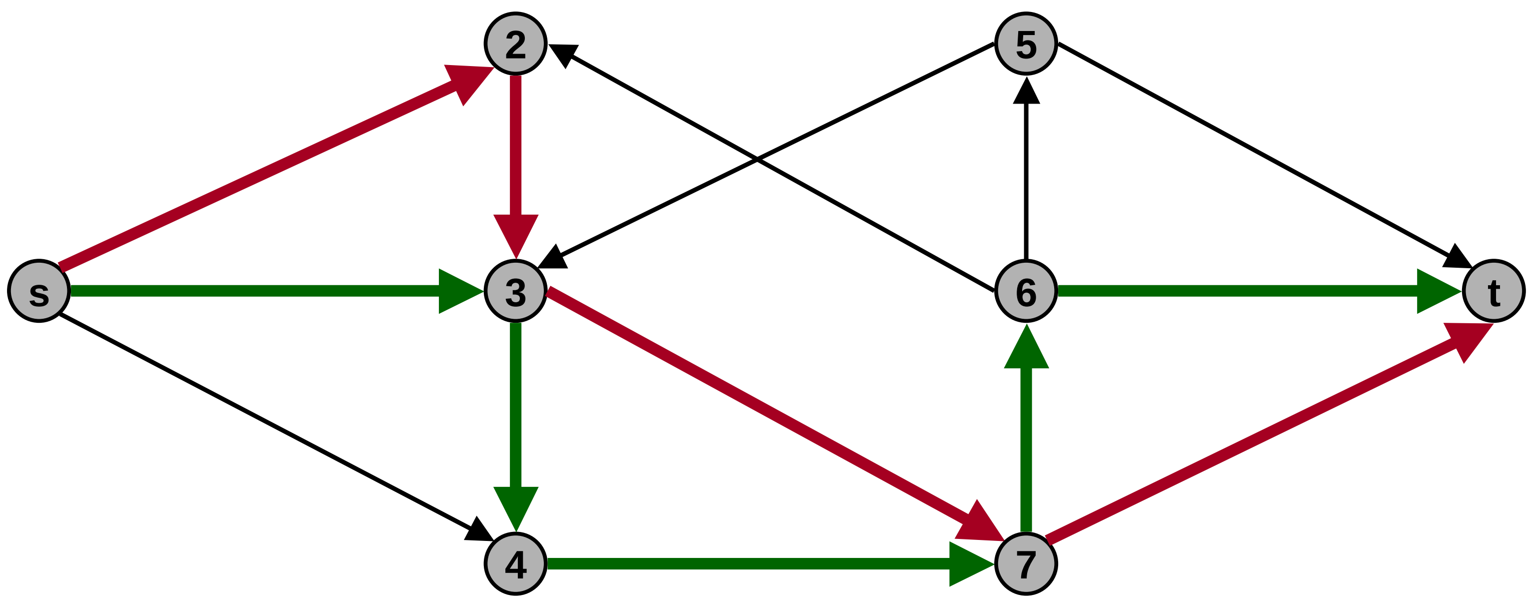
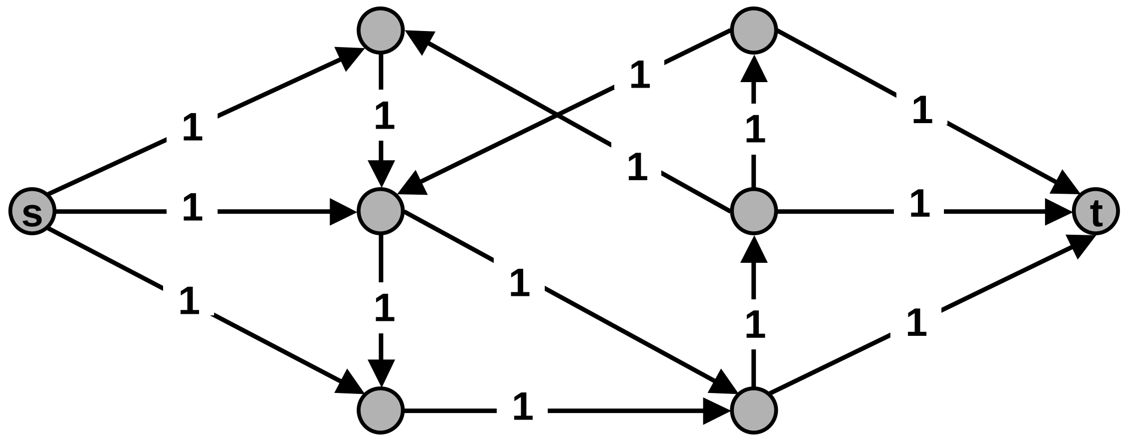


Disjoint Paths

Max flow formulation: assign unit capacity to every edge.



Theorem. There are k edge-disjoint paths from s to t if and only if the max flow value is k.

Proof. $\Rightarrow$

- **Suppose there are k edge-disjoint paths $P_1, \ldots, P_k$.**
- **Set f(e) = 1 if e participates in some path P_i; otherwise, set f(e) = 0.**
- **Since paths are edge-disjoint, f is a flow of value k.**

Disjoint Paths

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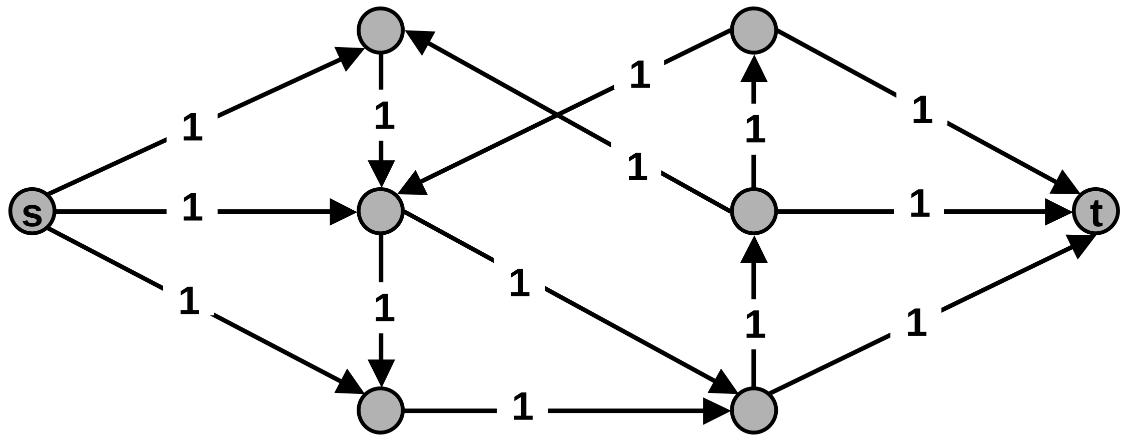


Theorem. **There are k edge-disjoint paths from s to t if and only if the max flow value is k.**

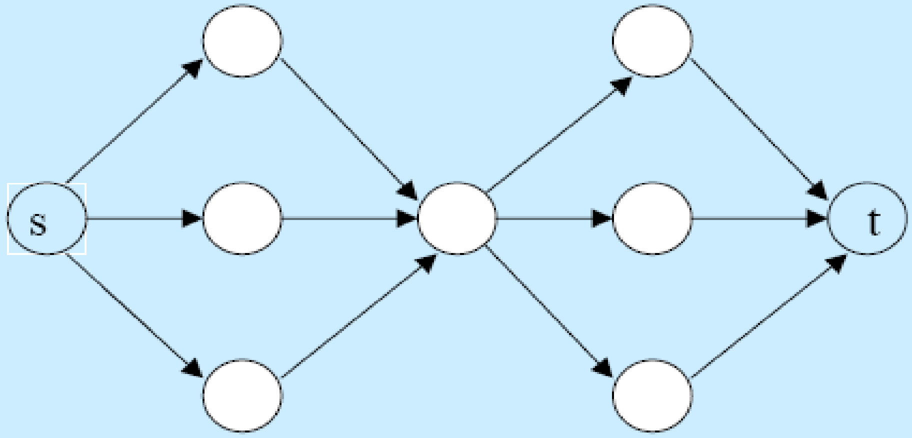
Proof. $\Leftarrow$

- **Suppose max flow value is k. By integrality theorem, there exists {0, 1} flow f of value k.**
- **Consider edge (s,v) with f(s,v) = 1.**
  - **by conservation, there exists an arc (v,w) with f(v,w) = 1**
  - **continue until reach t, always choosing a new edge**
- **Produces k (not necessarily simple) edge-disjoint paths.**

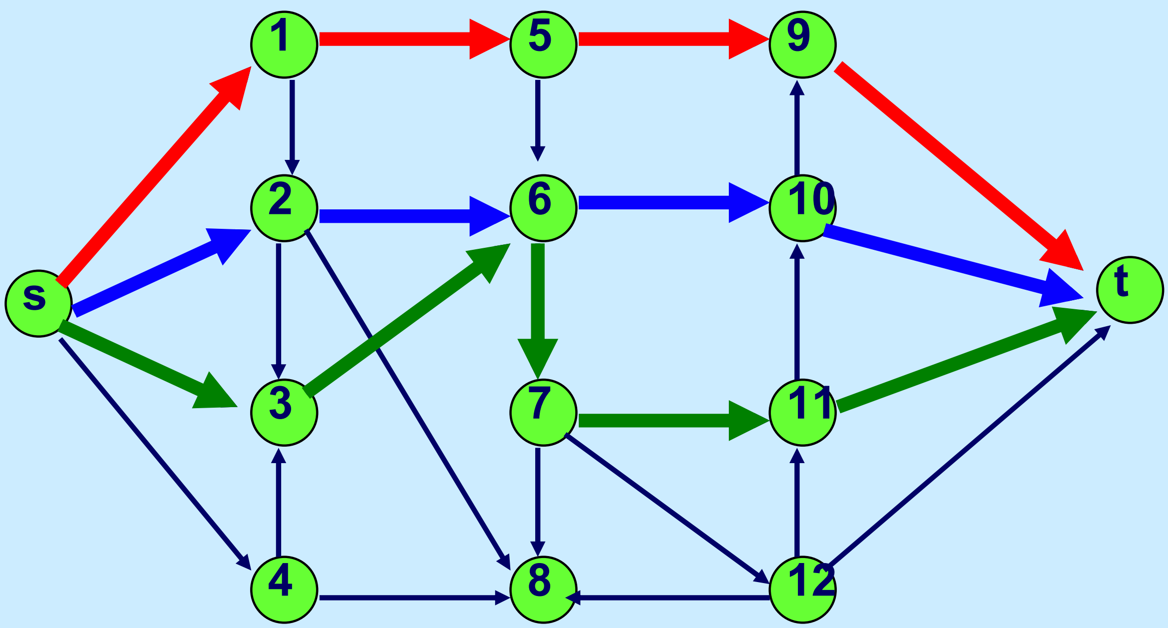
Link Disjoint Routes

- Communication Network
- What is the maximum number of arc disjoint paths from s to t?
  - How can we determine this number?

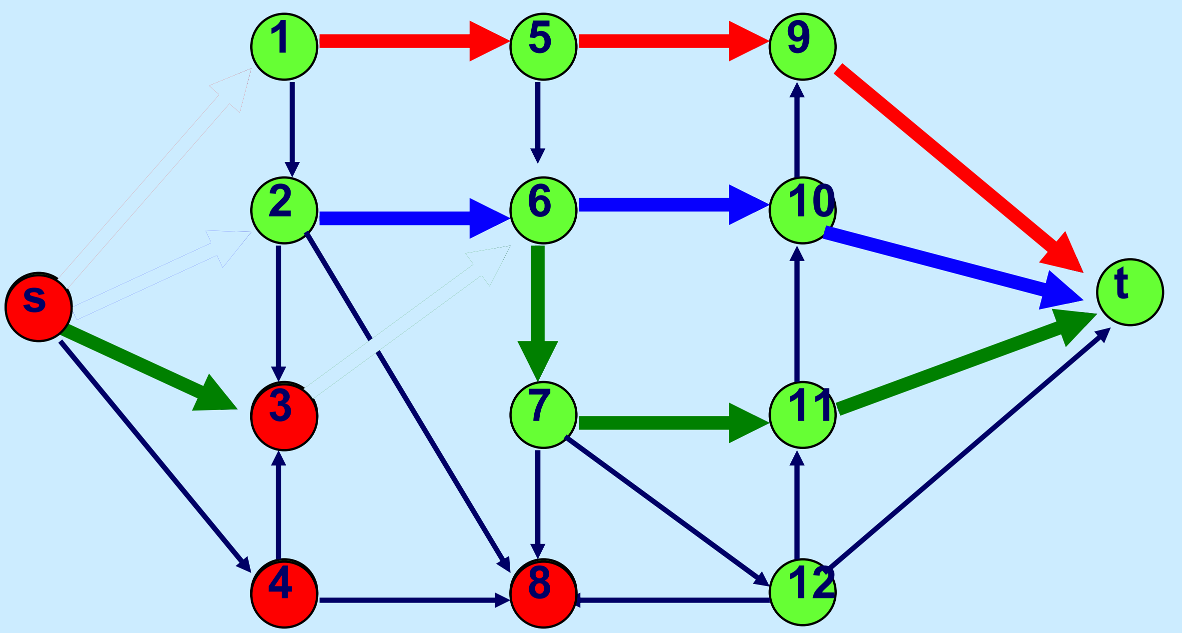
Theorem. *Let G = (N,A) be a directed graph. Then the maximum number of arc-disjoint paths from s to t is equal to the minimum number of arcs upon whose deletion there is no directed s-t path.*



There are 3 arc-disjoint s-t paths



Deleting 3 arcs disconnects s and t



Let S = {s, 3, 4, 8}. The only arcs from S to T = N\S are the 3 deleted arcs.

Node disjoint paths

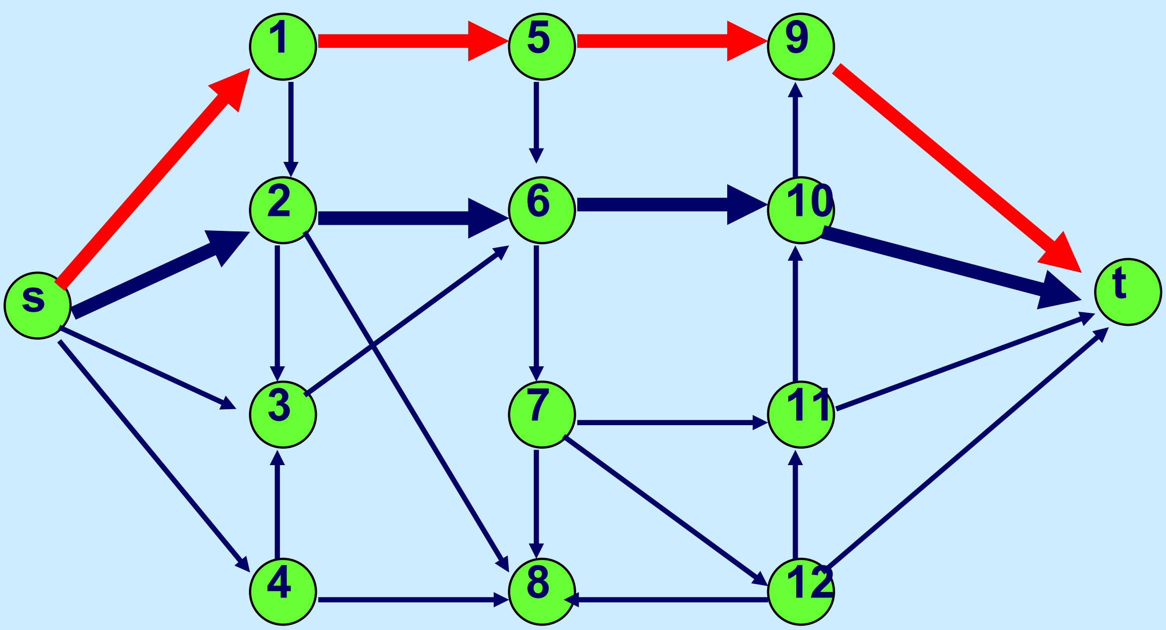
Two s-t paths P and P' f are said to be ***node-disjoint*** if the only nodes in common to P and P' are s and t).

How can one determine the maximum number of node disjoint s-t paths?

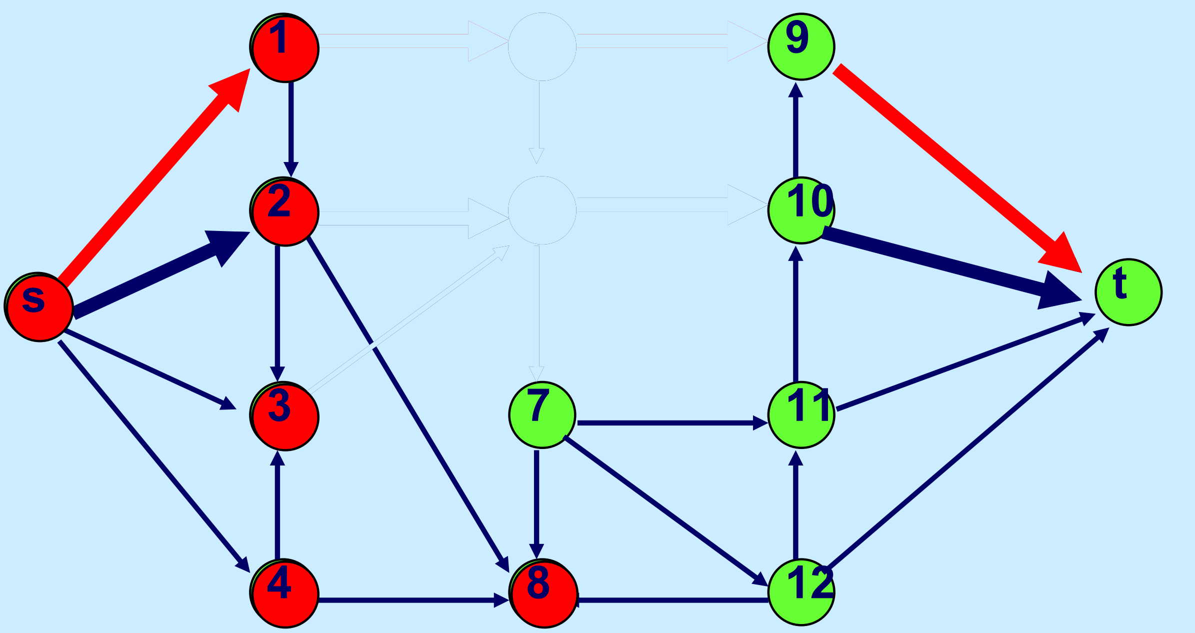
Answer: node splitting

Theorem. *Let G = (N,A) be a network with no arc from s to t. The maximum number of node-disjoint paths from s to t equals the minimum number of nodes whose removal from G disconnects all paths from nodes s to node t.*

There are 2 node disjoint s-t paths.



Deleting 5 and 6 disconnects t from s?



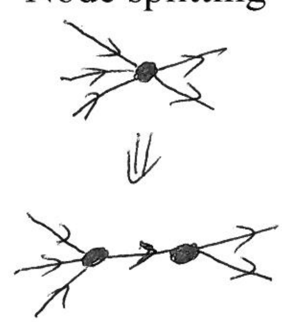
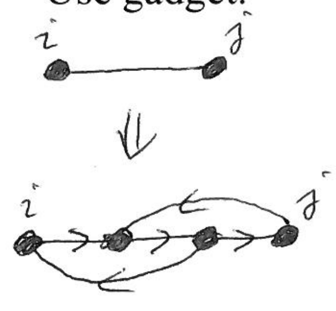
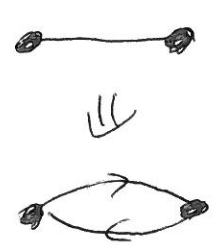
Let S = {s, 1, 2, 3, 4, 8}

Let T = {7, 9, 10, 11, 12, t}

There is no arc directed from S to T.

Variants of Finding a Maximum System of Disjoint Paths

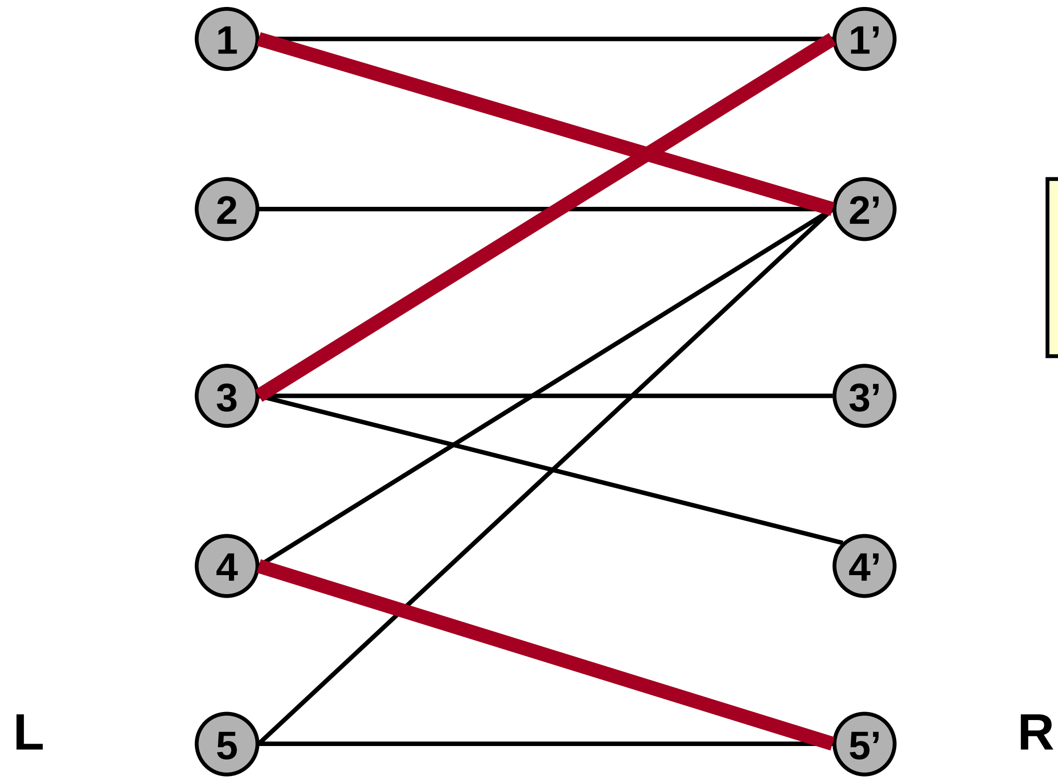
Notation: arrow means that after carrying out the transformation of the graph, call the case to which the arrow points.

	Edge-disjoint paths	Node-disjoint paths
Directed graph	Reduce to Max Flow with unit capacities	← Node splitting 
Undirected graph	↑ Use gadget: 	↑ Edge splitting 

Bipartite Matching

Bipartite matching.

- **Input: undirected, bipartite graph $G = (L \cup R, E)$.**
- **$M \subseteq E$ is a matching if each node appears in at most edge in M.**
- **Max matching: find a max cardinality matching.**



Matching

1-2', 3-1', 4-5'

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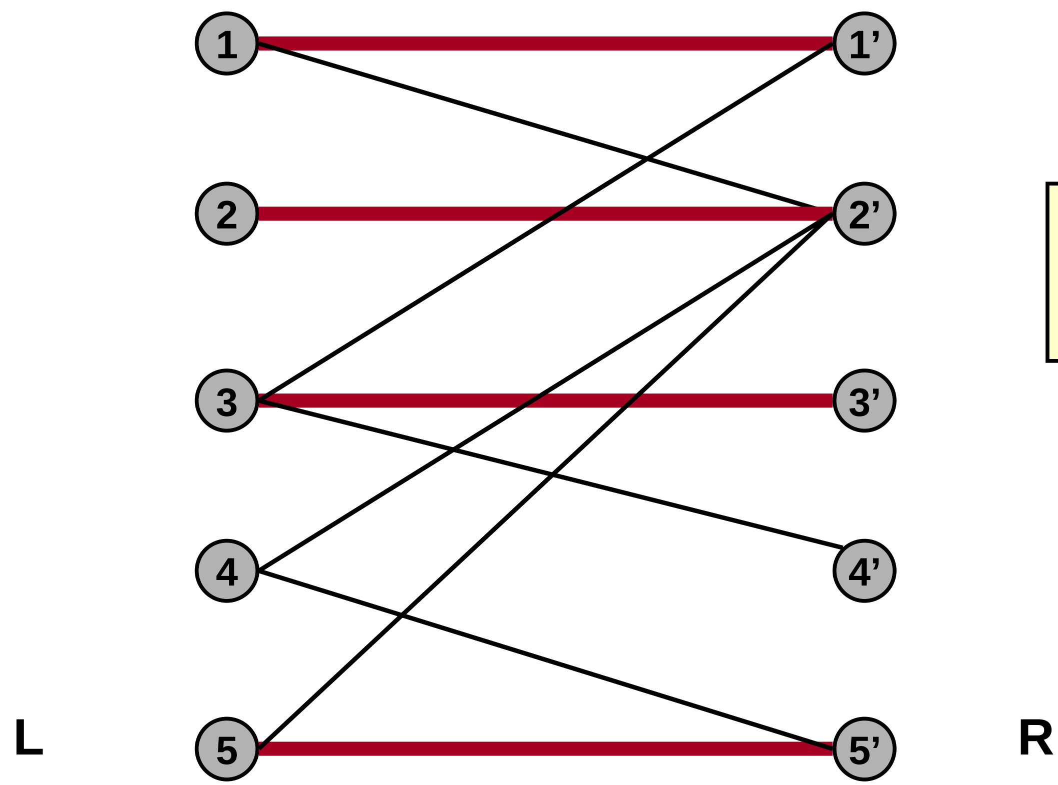

Matching

1-1', 2-2', 3-3', 4-4'

Bipartite Matching

Max flow formulation.

- **Create directed graph G' = (L ∪ R ∪ {s, t}, E').**
- **Direct all arcs from L to R, and give infinite (or unit) capacity.**
- **Add source s, and unit capacity arcs from s to each node in L.**
- **Add sink t, and unit capacity arcs from each node in R to t.**