The Cutting Plane Algorithm

Consider the following ILP:

$$\max Z = \mathbf{cx}$$

subject to

$$\begin{aligned} \mathbf{Ax} &\leq \mathbf{b} \\ x_i &\in \mathbf{Z}, \qquad i = 1, \ldots, n \end{aligned}$$

where $\mathbf{Z}$ is the set of integers. Let us assume for simplicity that the coefficients of the $\mathbf{c}$ vector are integers. (If they are rationals, one can multiply them by a suficiently large number to make them integers).

The idea of the cutting plane method is this. Let us solve first the LP part, without the $\mathbf{x} \in \mathbf{Z}$ requirement. This can be done with well established LP techniques. Let the found optimum solution be $\mathbf{x}_0$. Then any *integer* vector that was among the feasible solutions of the LP satisfies the inequality

$$\mathbf{cx} \leq \lfloor \mathbf{cx}_0 \rfloor .$$

Why? Because, $\mathbf{x}_0$, being an optimum (maximum) solution of the LP, satisfies

$$\mathbf{cx} \leq \mathbf{cx}_0$$

for any feasible vector. Taking the integer part on the righthand-side cannot destroy the relationship for *integer* feasible vectors, since for them the lefthand side is already an integer (if $\mathbf{x}$ is an integer vector).

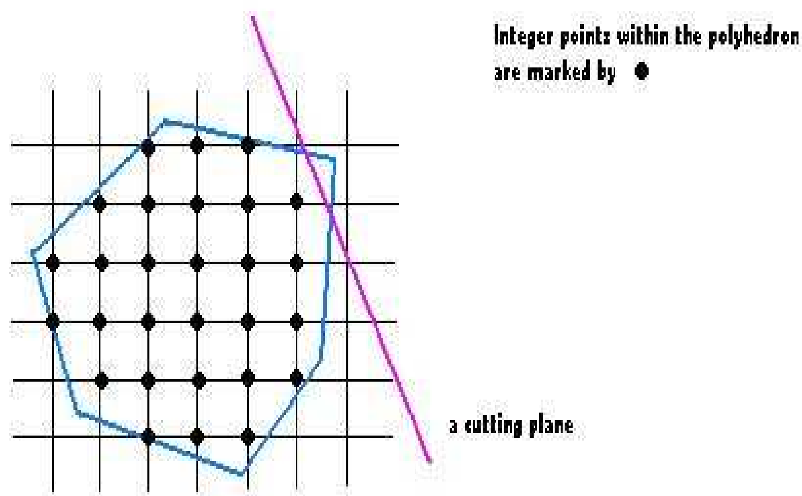
Now let us add the inequality $\mathbf{cx} \leq \lfloor \mathbf{cx}_0 \rfloor$ to the original system. (Note that $\lfloor \mathbf{cx}_0 \rfloor$ is a numerical constant, since $\mathbf{x}_0$ is already known, it is not a variable.) The we have a new extended system:

$$\min Z = \mathbf{cx}$$

subject to

$$\begin{aligned} \mathbf{Ax} &\leq \mathbf{b} \\ \mathbf{cx} &\leq \lfloor \mathbf{cx}_0 \rfloor \end{aligned}$$

The new inequality "cuts down" some part of the polyhedron, but guaranteed to keep the integer vectors.



Then we repeat the procedure to cut down more non-integer vectors. If with a given $\mathbf{c}$ we cannot cut anymore (because $\mathbf{c}\mathbf{x}_0$ is already integer), then we use another (in principle, arbitrary) integer vector $\mathbf{c}'$.

Without going into details, let us mention that in this way we cut down more and more from the original polyhedron, but never losing the integer vectors. Of course, we have more and more inequalities as the algorithm proceeds. One can formally prove that after finitely many iterations we arrive at a system in which all vertices of the polyhedron are integer vectors. Then solving the resulting extended LP we get an optimum solution of the ILP. The number of iterations, however, can be very large.