Computer simulations and Monte Carlo Methods (chapter 5)

Assume that we can simulate $U \sim \text{Uniform}(0, 1)$. Recall that:

Every programming language has a *random number generator* that simulates a U. In `R`, this function is `runif()`. Subsequent calls to this function will give draws that are "independent" for all practical purposes.

Simulating from discrete distributions

Simulating $X \sim$ **Bernoulli**(p):

Recall: If X Bernoulli(p), $P(X = 1) = p$, $P(X = 0) = 1 - p$.

1. Generate U.
2. If $U \leq p$; set $X = 1$, else set $X = 0$.

Verification:

Simulating $X \sim f(x)$, arbitrary discrete distribution:

Suppose X takes values $x_0, x_1, \ldots$, with probabilities $p_0, p_1, \ldots$, where $p_i = P(X = x_i)$ and $\sum_i p_i = 1$.

1. Divide the interval $[0, 1]$ into subintervals as shown below.

2. Generate U.
3. If U falls in subinterval i, take $X = x_i$.

Verification:

Simulating from continuous distributions

Result: If X is a continuous rv with cdf $F(x)$, then $U = F(X)$ follows Uniform(0, 1) distribution.

Inverse transform method: To simulate a X,

1. Generate U.
2. Set $U = F(X)$
3. Solve for X (i.e., invert the cdf).

Often the equation cannot be solved explicitly or efficiently. Alternatives are available.

Simulating from Exponential(λ) distribution:
Recall: If $X \sim \text{Exponential}(\lambda)$, $F(x) = 1 - \exp(-\lambda x)$.

Solving problems by Monte Carlo methods

Estimating $\mu = E(X)$ and $\sigma^2 = var(X) = E(X - \mu)^2$:

Simulate a large number (N) of independent draws from the distribution of X, say, $X_1, X_2, \ldots, X_N$

MC estimator of μ*:*

MC estimator of $E[g(X)]$ *where* g *is a given function:*

MC estimator of σ^2*:*

Estimating an integral $I = \int_a^b g(x)dx$:

Estimating $p = P(X \in A)$ for a given region A:

Simulate a large number (N) of independent draws from the distribution of X, say, $X_1, X_2, \ldots, X_N$

Define $Y_1, \ldots, Y_N$ as:

MC estimator of p:

Properties of $\hat{p}$:

Accuracy of a Monte Carlo study:

Error in estimation:

Specify a small *margin of error* ϵ and a small probability α. Want N such that

$$P(|\hat{p} - p| > \epsilon) \leq \alpha \tag{1}$$

or equivalently

- Error exceeds ϵ with probability α or less
- Error is ϵ or less with probability more than $1 - \alpha$.

To derive a formula for N, suppose $Z \sim N(0, 1)$ and $z_{\alpha/2}$ is such that $P(Z > z_{\alpha/2}) = \alpha/2$.

From the symmetry,

$$P(|Z| > z_{\alpha/2}) = \tag{2}$$

Now, let's derive an expression for $P(|\hat{p} - p| > \epsilon)$:

Comparing (2) and (3), and noticing that $P(|Z| > x)$ is decreasing in x, we can conclude that (1) approximately holds if

A practical problem:

Alternative 1:

Alternative 2:

Note: This formula is valid only if N is large.

Ex: Suppose the desired accuracy is $(\epsilon, \alpha) = (0.03, 0.05)$. $N =?$