Reliability and Lifetimes

In the previous sections a high level of abstraction was used that ignored the *time dependence* of reliability. Now we look into some fundamental quantities that describe the *temporal behavior* of component reliability.

Basic Probabilistic Model

Let T be the *failure time* when the considered component brakes down. Naturally, T is a random variable.

A fundamental characteristics of any random variable is its *probability distribution function* (sometimes also called cumulative distribution function). For the failure time it is defined as

$$F(t) = \Pr(T < t).$$

In other words, $F(t)$ is the probability that the breakdown occurs before a given time t. Naturally, the value of this probability depends on what is this given time t, this dependence is described by $F(t)$.

Let us define now further important functions that characterize how the reliability of a component behaves in time.

Lifetime measures

- **Survivor function** (also called reliability function)

  The survivor function is defined by

  $$S(t) = 1 - F(t)$$

where $F(t)$ is the probability distribution function of the failure time. The meaning of the survivor function directly follows from the definition of $F(t)$:

$$S(t) = 1 - \Pr(T < t) = \Pr(T \geq t).$$

In other words, $S(t)$ is the probability that the component is still operational at time t, since $T \geq t$ means the component has not failed up to time t.

- **Probability density function (pdf)**

  If $F(t)$ is differentiable, then its derivative is called probability density function:

  $$f(t) = F'(t)$$

  With the pdf $f(t)$ we can easily express the probability that the failure occurs in a time interval $[t_1, t_2]$:

  $$\Pr(t_1 \leq T \leq t_2) = \int_{t_1}^{t_2} f(t)dt.$$

- **Hazard function** (also called failure rate or hazard rate)

  The goal of the hazard function $h(t)$ is to express the *risk* that the component fails at time t.

  How can we capture this risk with the already known quantities?

  Take a very small time Δt. Then the fact that the failure occurs at time t can be approximately expressed with the condition $t \leq T \leq t + \Delta t$. Of course, this can only happen if the component still has not broken down before t. Thus, the risk that the component brakes down at time t can be approximated by the conditional probability

  $$\Pr(t \leq T \leq t + \Delta t \mid T \geq t). \tag{1}$$

  Note that the *risk* may be quite different from the *probability* of failure. As an example, one can say that the probability that a person dies at age 120 is very small, since most people do not live that long. On the other hand, the *risk* that a person of age 120 dies is quite high, since the concept of risk assumes that the person is still alive (otherwise it would be meaningless to talk about risk).

To make the risk expression (1) exact and independent of Δt, we consider the limit $\Delta t \to 0$. Then, however, the probability would always tend to 0. To avoid this, we divide it by Δt, thus obtaining a quantity that is similar in spirit to the pdf. This gives the definition of the hazard function:

$$h(t) = \lim_{\Delta t \to 0} \frac{\Pr(t \leq T \leq t + \Delta t \mid T \geq t)}{\Delta t}$$

Thus, the hazard function gives the *risk density* for the failure to occur at time t.

Relationships between different lifetime measures

As we have seen there is a direct relationship between the survivor function and the probability density function:

$$S(t) = 1 - F(t)$$

$$F(t) = 1 - S(t)$$

By taking derivatives we can get a relationship between the survivior function and the pdf:

$$f(t) = -S'(t).$$

We can also express the survivior function with the pdf from the definition $S(t) = 1 - F(t)$. Using that 1-F(t)=$\int_t^\infty f(t)dt$, we obtain

$$S(t) = \int_t^\infty f(t)dt.$$

Relating the hazard function with the others is slightly more complex. One can prove the following formula:

$$h(t) = \frac{-S'(t)}{S(t)}.$$

Using the previous relationships, this implies

$$h(t) = \frac{f(t)}{\int_t^\infty f(t)dt}.$$

It is kown from basic calculus that the formula $h(t) = -S'(t)/S(t)$ is equivalent to

$$h(t) = -\frac{d}{dt} \ln S(t).$$

Using this we can express $S(t)$ by the hazard function as

$$S(t) = \mathrm{e}^{-\int_0^t h(t)dt}.$$

Typical hazard functions

If we look at the formula

$$h(t) = \frac{f(t)}{\int_t^\infty f(t)dt}$$

then we can see that there are two possible reasons for having high hazard (risk) at time t:

- either the numerator is large, that is, the probability of a failure is high around time t, and/or
- the denominator is small. The denominator $\int_t^\infty f(t)dt$ gives the probability that the failure has not occured before t. If this is small, that means the probability that the component is still alive is low. In other words, the component is old from the reliability point of view.

Since a typical pdf sooner or later starts decreasing with time, this effect tends to diminish the hazard as time advances. On the other hand, the denumerator decreases as the component gets older, which will increase the hazard. These two opposite effects can balance each other in different ways. A special case when they precisely balance each other is the exponential distribution, where the pdf is of the form

$$f(t) = \lambda e^{-\lambda t}.$$

In this case, if we compute the hazard function, we obtain $h(t) = \lambda$. That is, in this case the hazard function is constant.

In many practical cases the hazard function has a *bathtub* shape that consists three typical parts:

1. Initial "burn-in" period, when the hazard is relatively large, due to potential manufacturing defects that result in early failure.
2. Steady part with approximately constant hazard function.
3. Aging with increasing hazard.