Solution of Exercises 5 and 6

Exercise 5. (See the problem description in the previous lecture note.)

Let us introduce the following variables:

x_i: amount of regular production in month i

y_i: amount of overtime production in month i

u_i: amount used from storage in month i

v_i: amount put into storage in month i

w_i: amount available in storage at the beginning of month i

Then in each month we need to satisfy the following constraints:

$$x_i + y_i + u_i - v_i = d_i$$

$$w_{i+1} = w_i - u_i + v_i$$

Explanation: The first constraint expresses that the regular production, plus the overtime production, plus what we use from storage, should satisfy the demand (d_i), after subtracting the amount that we put into storage. The second constraint expresses that the storage content at the beginning of the next month will result by what is available in the current month, minus what we use from storage, plus what we add to it.

We have to include some further constraints:

- $x_i \le r \quad (\forall i)$
  (at most r units can be produced by regular production each month)
- $w_1 = 0$
  (there is nothing yet in storage at the beginning of the first month)
- $w_{n+1} = 0$
  (there is nothing left in storage after the last month)

- $u_i \le w_i \quad (\forall i)$
  (we cannot use more from storage than what is available there)

- All variables are non-negative.

The objective function is the total cost, summed over all months, including regular production, overtime, and storage. To make the expression more general, let us allow the cost coefficients vary month by month; it is expressed by indexing them with the month index i. Thus, the objective function is:

$$Z = \sum_{i=1}^{n} (b_i x_i + c_i y_i + s_i w_i).$$

Collecting the parts, we get the complete LP formulation:

$$\min Z = \sum_{i=1}^{n} (b_i x_i + c_i y_i + s_i w_i)$$

Subject to

$$x_i + y_i + u_i - v_i = d_i \qquad (\forall i)$$

$$w_{i+1} = w_i - u_i + v_i \qquad (\forall i)$$

$$x_i \le r, \quad u_i \le w_i \qquad (\forall i)$$

$$w_1 = 0, \quad w_{n+1} = 0$$

$$x_i, y_i, u_i, v_i, w_i \ge 0 \qquad (\forall i)$$

Exercise 6. (See the problem description in the previous lecture note.)

a.) Choosing notations

Let us index the links by $i = 1, \ldots, L$, and the routes by $j = 1, \ldots, R$. We use the following further constants, which are assumed known (given as input):

B_i: bandwidth available on link i

c_j: cost of unit bandwidth on route j

p_j: profit brought by unit capacity on route j

$A = [a_{ij}]$: link-route incidence matrix. Its size is $L \times R$. The matrix entries tell which link is used by which routes:

$$a_{ij} = \begin{cases} 1 & \text{if link } i \text{ is on route } j \\ 0 & \text{otherwise} \end{cases}$$

Note that this matrix is also known as input, since it is assumed that the route system is given.

Variables: $x_j,\ j = 1, \ldots, R$; it represents the capacity assigned to route j.

b.) Finding a mathematical programming formulation

The constraint we need to satisfy for each link i is this: the summed capacity of the routes that use the link cannot exceed the bandwidth available on the link. This can be expressed as

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{iR}x_R \leq B_i$$

Note that while all routes occur in this expression, only those contribute to the sum which use link i. For these we have $a_{ij} = 1$. For the rest, which do not use the link, we have $a_{ij} = 0$, so they do not contribute to the sum.

The objective function is the ratio of the total profit vs. the total cost, summed over all routes:

$$Z = \frac{\sum_{j=1}^{R} p_j x_j}{\sum_{j=1}^{R} c_j x_j}$$

Thus, the mathematical programming (but not yet linear programming!) formulation that directly follows the verbal description is this:

$$\max Z = \frac{\sum_{j=1}^{R} p_j x_j}{\sum_{j=1}^{R} c_j x_j}$$

Subject to

$$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{iR}x_R \leq B_i \qquad i = 1, \ldots, R$$

$$x_j \geq 0 \qquad i = 1, \ldots, R$$

Here the objective function is nonlinear, so we want to replace it with some linear formulation in the next step.

c.) Transforming into a linear programming task

Let us carry out this step in a slightly more general from, so that it becomes usable in other situations, as well. Let $\mathbf{x}$ be the variable vector, and $\mathbf{c}$, $\mathbf{d}$ be constant vectors. Further, let α, β be scalar constants, $\mathbf{A}$ be a constant matrix, and $\mathbf{b}$ a constant vector. They all have dimensions that make the operations below executable. Note that these notations are generic, not coming directly from the previous considerations.

Let us now consider the following nonlinear optimization task, in which the objective function is the *ratio* of two linear functions, subject to a system of linear constraints. The latter is in the LP standard form. Clearly, the formulation found in b.) can be put in this format.

$$\max Z = \frac{\mathbf{cx} + \alpha}{\mathbf{dx} + \beta}$$

Subject to

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

To avoid additional difficulties, we assume that the denominator $\mathbf{dx} + \beta$ is always positive in the feasible domain. Let us introduce a new variable t, which we call *scaling factor*. If we multiply both the numerator and the denominator in the objective function by t, then the value of the ratio remains the same:

$$Z = \frac{\mathbf{cx} + \alpha}{\mathbf{dx} + \beta} = \frac{(\mathbf{cx} + \alpha)t}{(\mathbf{dx} + \beta)t} = \frac{\mathbf{cx}t + \alpha t}{\mathbf{dx}t + \beta t}$$

Observe now that the arbitrary value of t allows it to be chosen such that the denominator becomes 1. This can be enforced by a new constraint. Then, the denominator being 1, it is enough to maximize the numerator. Thus, the following task will have the same optimum as the original:

$$\max\ \mathbf{cx}t + \alpha t$$

Subject to

$$\mathbf{dx}t + \beta t = 1$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

(A "sanity check" question: the usage of t assumed that $t \neq 0$. How do we know that it holds? Answer: the constraint $\mathbf{dx}t + \beta t = 1$ guarantees it, as it could not hold with $t = 0$. Note: it also could not hold with $\mathbf{dx} + \beta = 0$, but we assumed that the denominator $\mathbf{dx} + \beta$ is always positive.)

The above formulation is still nonlinear, since $\mathbf{x}t$ is a product of two variables. Let us introduce a new variable $\mathbf{y}$ by

$$\mathbf{y} = \mathbf{x}t. \tag{1}$$

Substituting it in the formulation, we get

$$\max\ \mathbf{cy} + \alpha t$$

Subject to

$$\mathbf{dy} + \beta t = 1$$

$$\mathbf{Ax} = \mathbf{b}$$

$$\mathbf{x} \geq 0$$

This is already a linear programming task, but the relationship between $\mathbf{x}$, $\mathbf{y}$ and t is expressed by the nonlinear formula (1). This relationship cannot be ignored, since it represents a dependence among the variables. If, however, we include (1) as a constraint, then the task becomes nonlinear again! We can avoid this problem by expressing $\mathbf{x}$ from (1) as

$$\mathbf{x} = \frac{1}{t}\mathbf{y}$$

and using it in place of $\mathbf{x}$. Then we get

$$\max\ \mathbf{cy} + \alpha t$$

Subject to

$$\mathbf{dy} + \beta t = 1$$

$$\mathbf{A}\frac{1}{t}\mathbf{y} = \mathbf{b}$$

$$\frac{1}{t}\mathbf{y} \geq 0$$

Multiplying both sides in the last two constraints by t, we get

$$\max\ \mathbf{cy} + \alpha t$$

Subject to

$$\mathbf{dy} + \beta t = 1$$

$$\mathbf{Ay} - \mathbf{b}t = 0$$

$$\mathbf{y} \geq 0$$

which is already a linear programming task. This is what we wanted! ♡

Two “sanity check” questions:

1. How do we know that the inequality $\frac{1}{t}\mathbf{y} \geq 0$ indeed transforms into $\mathbf{y} \geq 0$? After all, this only holds if $t > 0$, which was not explicitly required.

Answer: the constraint $\mathbf{dy} + \beta t = 1$ enforces $t > 0$, because it is equivalent to $\mathbf{dx}t + \beta t = 1$, which is in turn equivalent to $(\mathbf{dx} + \beta)t = 1$. As we assumed $\mathbf{dx} + \beta > 0$, the equation $(\mathbf{dx} + \beta)t = 1$ could not hold with $t \leq 0$.

2. Having obtained an LP formulation, assume we solve it by some off-the-shelf software, and get some optimal solution $\mathbf{y}^*, t^*$. But these variables come from a transformed task, they are not the original route capacity values that we were looking for! Then what will be the optimal route capacities?

Answer: the route capacities are the components of the $\mathbf{x}$ vector. Once $\mathbf{y}^*, t^*$ are numerically available, we can express the optimal value of $\mathbf{x}$ as

$$\mathbf{x}^* = \frac{1}{t^*}\mathbf{y}^*$$

giving us the optimal route capacities as the components of the $\mathbf{x}^*$ vector.