Randomized Rounding

The randomized rounding technique works the best when the ILP is of combinatorial nature, that is, a 0-1 programming problem. The principle is shown via the following example. Consider the problem

$$\max Z = \mathbf{cx}$$

subject to

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ x_i &\in \{0,1\}, \qquad i = 1, \ldots, n \end{aligned}$$

Replace the last constraint with $0 \leq x_i \leq 1$. The new problem is already an LP, called the *LP relaxation* of the 0-1 program.

We can solve the LP relaxation by any LP algorithm, which generally results in variable assignments that are between 0 and 1. The next step is to *round* the variables to 0/1 values. This could be done deterministically as usual rounding, but then in a pessimistic case too much violation of the constraints can occur. For example, assume the problem contains the following constraint:

$$x_1 + x_2 + \ldots + x_{1000} = 510 \tag{1}$$

and the LP solution is $x_1 = \ldots = x_{1000} = 0.51$, which satisfies the constraint. If we round the variables in the usual way, then they all will be rounded to one, so the lefthand-side becomes 1000.

One the other hand, we can have much smaller violation of the constraints by rounding *randomly,* according to the following rule. The randomly rounded value $\widetilde{x}_i$ of x_i will be:

$$\widetilde{x}_i = \begin{cases} 1 & \text{with probability } x_i \\ 0 & \text{with probability } 1 - x_i \end{cases}$$

This can be easily implemented by drawing a uniformly distributed random number $y_i \in [0, 1]$ and setting $x_i = 1$ if $x_i \geq y_i$, otherwise $x_i = 0$.

What is the advantage of the random rounding? We can observe that the *expected value* of $\tilde{x}_1 + \ldots + \tilde{x}_{1000}$ will be exactly 510, thus it satisfies the constraint. Of course, the *actual* value may still violate it, but one can expect that the actual values fluctuate around the average, so the errors of different signs largely cancel out if there are many variables.

One can quantitatively estimate the error that can still occur with a certain small probability. The way of this estimation is shown in the following theorem (informal explanation follows after the theorem).

Theorem 1. *Let* $\mathbf{x}$ *be an* n*-dimensional vector with* $0 \leq x_i \leq 1$ *for all* i*. Assume* $\mathbf{x}$ *satisfies the constraint* $\mathbf{ax} = b$*. Let* $\tilde{\mathbf{x}}$ *be the rounded version of* $\mathbf{x}$*, obtained by randomized rounding. Then the following inequality holds*

$$b - a_{max}\sqrt{\alpha n \log n} \leq \mathbf{a}\tilde{\mathbf{x}} \leq b + a_{max}\sqrt{\alpha n \log n}$$

with probability at least $1 - n^{-\alpha}$*, where* $\alpha > 0$ *is any constant and* $a_{max} = \max_i |a_i|$.

The Theorem says that the error, which is the deviation of the actual value of $\mathbf{a}\tilde{\mathbf{x}}$ from the expected value b, is bounded by $a_{max}\sqrt{\alpha n \log n}$. This may be much smaller than b.

Observe the effect of the α parameter. The bound on the error holds with probability at least $1 - n^{-\alpha}$. If α is large, then this probability is very close to 1. Then, however, the error $a_{max}\sqrt{\alpha n \log n}$ gets also larger. If α is small, then the error is small, but it holds with smaller probability. Thus, there is trade-off between providing a tighter bound less surely or a looser bound more surely. Note, however, that this plays a role only in the analysis, not in the actual algorithm.

Let us look at example (1) we used. Choose $\alpha = 1$. The other parameters are:

$$n = 1000, \quad a_{max} = 1, \quad b = 510$$

The error term will be:

$$a_{max}\sqrt{\alpha n \log n} = \sqrt{1000 \log 1000} \approx 83$$

and the probability is

$$1 - n^{-\alpha} = 1 - 1000^{-1} = 0.999 = 99.9\%.$$

Thus we obtain that the deviation from the required $\mathbf{ax} = 510$ is bounded as

$$427 \leq \mathbf{a\tilde{x}} \leq 593$$

with 99.9% probability, thus, almost surely. In contrast, with deterministic rounding we could only say that the value is between 0 and 1000.

What if we are satisfied with less certainty, say, with 90% probability? Then we can choose $\alpha = 1/3$, since $1 - 1000^{-1/3} = 0.9$ and then we get the error bound of

$$a_{max}\sqrt{\alpha n \log n} = \sqrt{\frac{1}{3} \cdot 1000 \log 1000} \approx 48.$$

This yields the tighter estimation

$$462 \leq \mathbf{a\tilde{x}} \leq 558 \tag{2}$$

which holds, however, only with 90% probability.

What if we would like to have the tighter bound (2) but also the higher 99.9% probability? There is a way to achieve that, too. Repeat the randomized rounding 3 times independently. Then in each trial the probability of violating (2) will be at most 10%. Hence the probability that *all* the 3 independent trials violate (2) is $0.1^3 = 0.001 = 0.1\%$. Thus, among the 3 trials there must be one with 99.9% probability that satisfies the bound (2), so we can choose this one. The moral is that we can "amplify" the power of the method by repeating it several times independently and then choosing the best result.

The Theorem can directly be extended to the case that involves repeated independent trials. With r repetitions the probability is amplified to $1-n^{-\alpha r}$, while the error bound remains the same.

Remark: The approach is most useful if the problem contains "soft" constraints. What are these? It is customary to differentiate two types of constriants:

- *Hard constraints:* these must be obeyed. For example, they may represent some physical law which is impossible to violate.
- *Soft constraints:* these can possibly be violated if there is no other way to solve the problem, but then we have to pay a penalty, so we would like to minimize the violation. For example, budget constraints often behave this way.

Since randomized rounding may result in constraint violation, it is typically good for soft constraints.

Exercises

1. If we can guarantee an error bound B with probability p, then how many repetitions are needed if we want to decrease the error bound by a factor of 10, while keeping the same probability?

Answer: If we use r repetitions, then the probability bound is amplified to $1 - n^{-\alpha r}$. For keeping the original probability we can choose a new α as $\alpha' = \alpha/r$. Then the error will decrease by a factor of $\sqrt{r}$, as α is under the square root in the expression of the error. Thus, if we want to decrease the error by a factor of 10, then we need 100 repetitions.

better solution via randomized rounding than with naive deterministic rounding, especially if there are many variables.

2. Generalize Theorem 1 for the case when there are more constraints.

Answer: Theorem 1, when applied to several constraints, implies the theorem below, via the union bound of probabilities. The union bound says that for any events $A_1, A_2, \ldots A_m$, the probability of their union (i.e., the probability that at least one of them occurs) is bounded by the sum of their individual probabilities, no matter whether the events are independent or not. In formula:

$$\Pr(A_1 \cup \ldots \cup A_m) \leq \Pr(A_1) + \ldots + \Pr(A_m).$$

Theorem 2. *Let* $\mathbf{x}$ *be an* n*-dimensional vector with* $0 \leq x_i \leq 1$ *for all* i*. Assume* $\mathbf{x}$ *satisfies* $\mathbf{Ax} = \mathbf{b}$*, representing* m *constraints. Let* $\tilde{\mathbf{x}}$ *be the rounded version of* $\mathbf{x}$*, obtained by randomized rounding. Then the following inequality* *holds:*

$$\mathbf{b} - \left(a_{max}\sqrt{\alpha n \log n}\right)\mathbf{1} \leq \mathbf{A}\tilde{\mathbf{x}} \leq \mathbf{b} + \left(a_{max}\sqrt{\alpha n \log n}\right)\mathbf{1}$$

with probability at least $1 - mn^{-\alpha}$*, where* $\alpha > 0$ *is any constant,* a_{max} *is the largest absolute value of any entry in* $\mathbf{A}$*, and* $\mathbf{1}$ *is a vector in which all coordinates are 1.*