# The Max Flow Problem

$G$ = (N,A)

$x_{ij}$ = flow on arc (i,j)

$u_{ij}$ = capacity of flow in arc (i,j)

$s$ = source node

$t$ = sink node

**Maximize** $v$

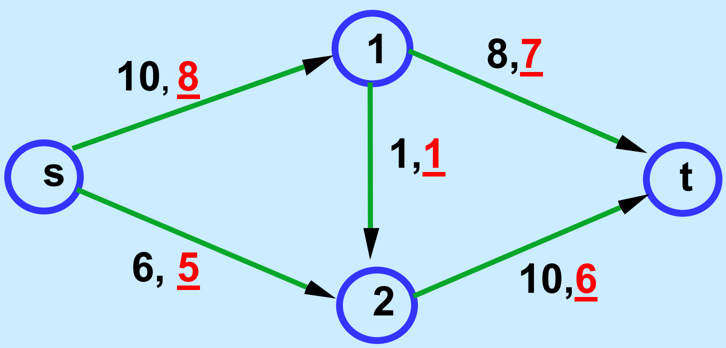
**Subject to** $\sum_j x_{ij} - \sum_k x_{ki} = 0$ for each $i \neq s,t$

$$\sum_j x_{sj} = v$$

$$0 \leq x_{ij} \leq u_{ij} \quad \text{for all } (i,j) \in A.$$

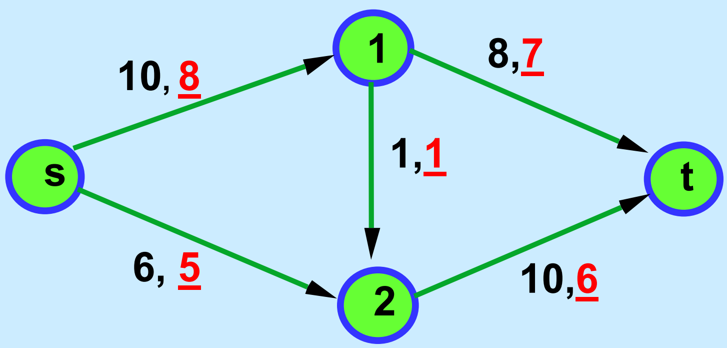
# Maximum Flows

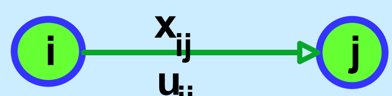
We refer to a flow x as *maximum* if it is feasible and maximizes v. Our objective in the max flow problem is to find a maximum flow.

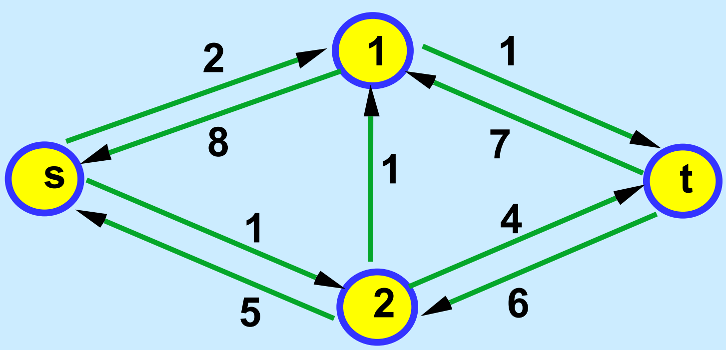



A max flow problem. Capacities and a non-optimum flow.

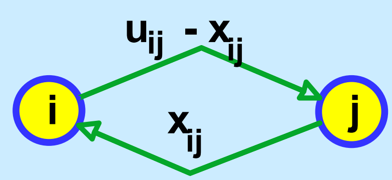
# The Residual Network




The ***Residual Network*** **G(x)**




**We let $r_{ij}$ denote the *residual capacity* of arc (i,j)**

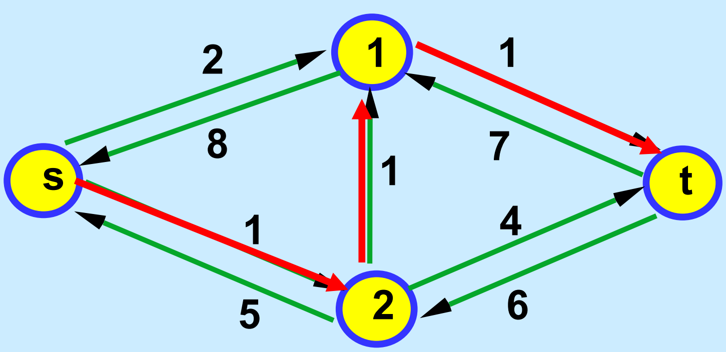
# A Useful Idea: Augmenting Paths

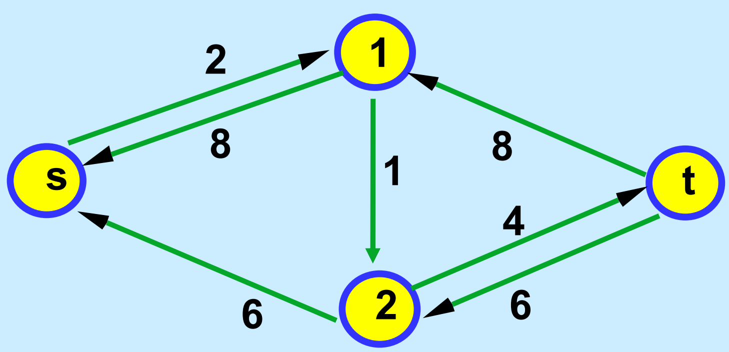
An ***augmenting path*** is a path from s to t in the residual network.

The ***residual capacity*** of the augmenting path P is $\delta(P) = \min\{r_{ij} : (i,j) \in P\}$.

To ***augment along P*** is to send d(P) units of flow along each arc of the path. We modify x and the residual capacities appropriately.

$r_{ij} := r_{ij} - \delta(P)$ and $r_{ji} := r_{ji} + \delta(P)$ for $(i,j) \in P$.



# The Ford Fulkerson Maximum Flow Algorithm

```
Begin
    x := 0;
    create the residual network G(x);
    while there is some directed path from
       s to t in G(x) do
    begin
        let P be a path from s to t in G(x);
        Δ := δ(P);
        send Δ units of flow along P;
        update the r's;
    end
end {the flow x is now maximum}.
```

**Ford-Fulkerson Algorithm Animation**

# Proof of Correctness of the Algorithm

Assume that all data are integral.

*Lemma: At each iteration all residual capacities are integral.*

Proof. It is true at the beginning. Assume it is true after the first k-1 augmentations, and consider augmentation k along path P.

The residual capacity $\Delta$ of P is the smallest residual capacity on P, which is integral.

After updating, we modify residual capacities by 0, or $\Delta$, and thus residual capacities stay integral.

## Theorem. The Ford-Fulkerson Algorithm is finite

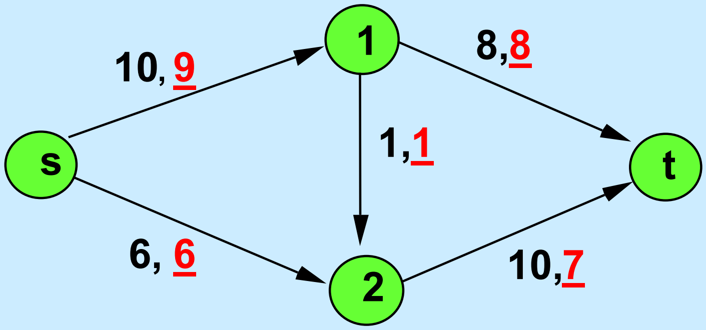
**Proof**. The capacity of each augmenting path is at least 1.

The augmentation reduces the residual capacity of some arc (s, j) and does not increase the residual capacity of (s, i) for any i.

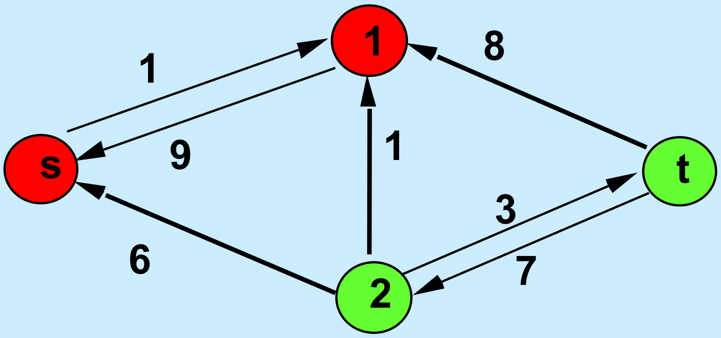
So, the sum of the residual capacities of arcs out of s keeps decreasing, and is bounded below by 0.

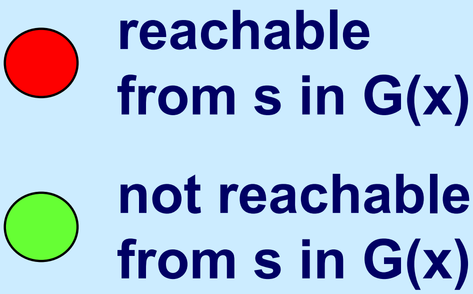
Number of augmentations is O(nU), where U is the largest capacity in the network.

# How do we know when a flow is optimal?

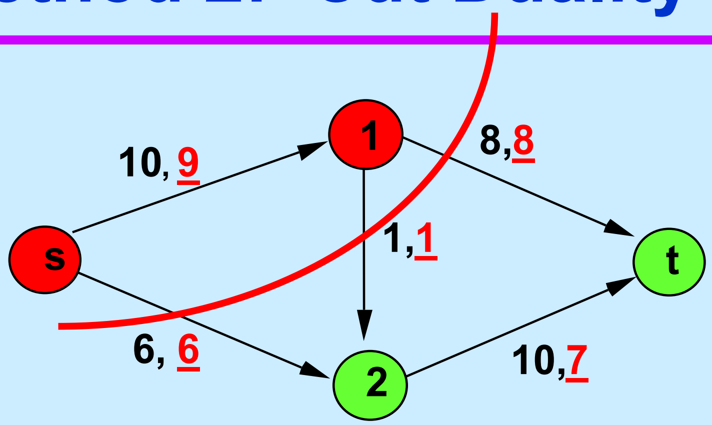



**METHOD 1. There is no augmenting path in the residual network.**



# Method 2: Cut Duality Theory




**An *(s,t)-cut* in a network G = (N,A) is a partition of N into two disjoint subsets S and T such that s $\in$ S and t $\in$ T, e.g., S = { s, 1 } and T = { 2, t }.**

**The *capacity* of a cut (S,T) is**

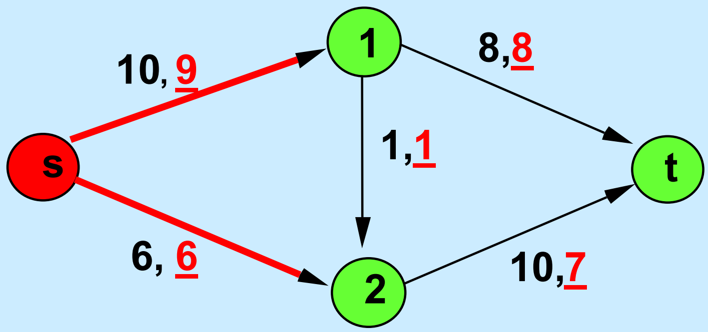
$$CAP(S,T) \quad = \quad \sum_{i \in S} \sum_{j \in T} u_{ij}$$

## Weak Duality Theorem for the Max Flow Problem

*Theorem*. *If x is any feasible flow and if (S,T) is an (s,t)-cut, then the flow v(x) from source to sink in x is at most CAP(S,T).*

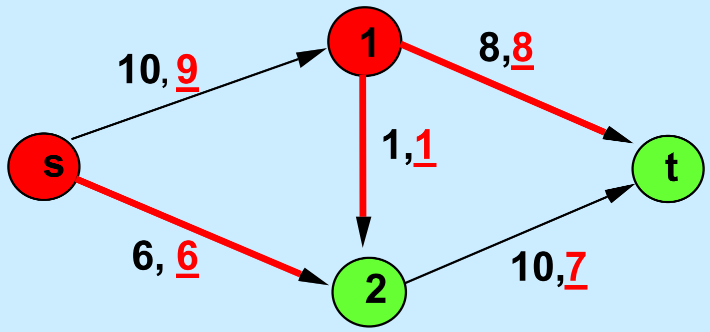
PROOF. We define the *flow across the cut* (S,T) to be

$$F_x(S,T) \quad = \quad \sum_{i\in S}\sum_{j\in T} x_{ij} \quad - \quad \sum_{i\in S}\sum_{j\in T} x_{ji}$$

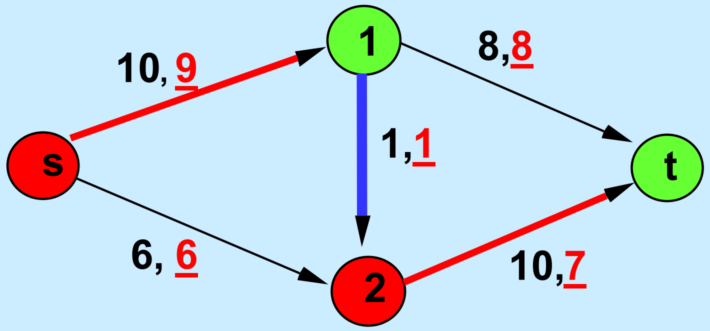



If S = {s}, then the flow across (S, T) is
9 + 6 = 15.

# Flows Across Cuts




If S = {s,1}, then the flow across (S, T) is 8+1+6 = 15.




If S = {s,2}, then the flow across (S, T) is 9 + 7 - 1 = 15.

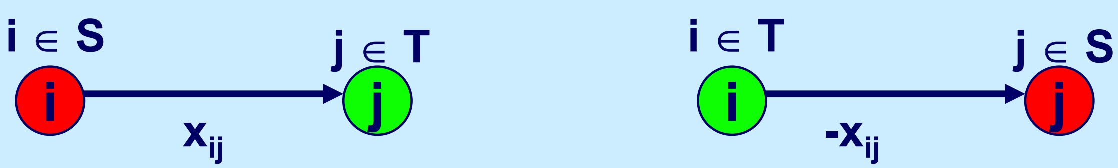
# More on Flows Across Cuts

***Claim**: Let (S,T) be any s-t cut. Then $F_x(S,T) = v$ = flow into t.*

**Proof**. Add the conservation of flow constraints for each node $i \in S - \{s\}$ to the constraint that the flow leaving s is v. The resulting equality is $F_x(S,T) = v$.

$$\sum_j x_{ij} - \sum_k x_{ki} = 0 \qquad \text{for each } i \in S \backslash s$$

$$\sum_j x_{sj} = v$$



# More on Flows Across Cuts

**Claim:** The flow across (S,T) is at most the capacity of a cut.

**Proof.** If $i \in S$, and $j \in T$, then $x_{ij} \leq u_{ij}$. If $i \in T$, and $j \in S$, then $x_{ij} \geq 0$.

$$F_x(S,T) = \sum_{i\in S}\sum_{j\in T} x_{ij} - \sum_{i\in S}\sum_{j\in T} x_{ji}$$

$$Cap(S,T) = \sum_{i\in S}\sum_{j\in T} u_{ij} - \sum_{i\in S}\sum_{j\in T} 0$$

# Max Flow Min Cut Theorem

*Theorem. (Optimality conditions for max flows).*
*The following are equivalent.*

1. *A flow x is maximum.*
2. *There is no augmenting path in G(x).*
3. *There is an s-t cutset (S, T) whose capacity is the flow value of x.*

*Corollary. (Max-flow Min-Cut). The maximum flow value is the minimum value of a cut.*

**Proof of Theorem.** $1 \Rightarrow 2$. (not $2 \Rightarrow$ not 1)

Suppose that there is an augmenting path in G(x). Then x is not maximum.

# Continuation of the proof.

**3 ⇒ 1.** Let $v = F_x(S, T)$ be the flow from s to t. By assumption, $v = CAP(S, T)$. By weak duality, the maximum flow is at most $CAP(S, T)$. Thus the flow is maximum.

**2 ⇒ 3.** Suppose there is no augmenting path in G(x).

**Claim**: Let S be the set of nodes reachable from s in G(x). Let $T = N \backslash S$. Then there is no arc in G(x) from S to T.

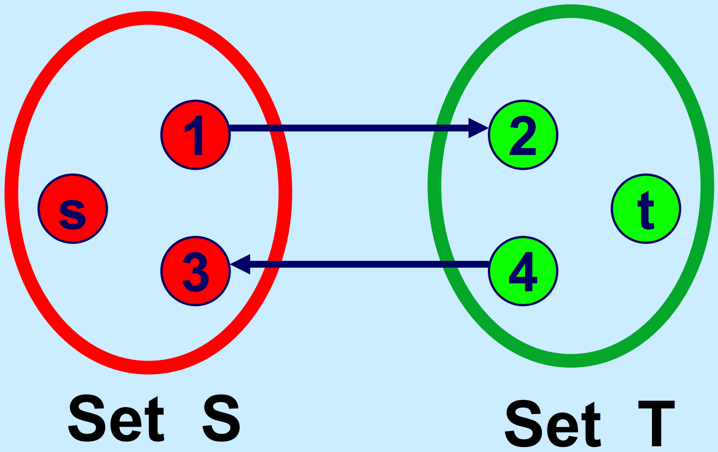
Thus $i \in S$ and $j \in T \Rightarrow x_{ij} = u_{ij}$

$i \in T$ and $j \in S \Rightarrow x_{ij} = 0$.

# Final steps of the proof

Thus $i \in S$ and $j \in T \Rightarrow x_{ij} = u_{ij}$

$i \in T$ and $j \in S \Rightarrow x_{ij} = 0.$




There is no arc from S to T in G(x)

$x_{12} = u_{12}$

$x_{43} = 0$

If follows that

$$F_x(S,T) = \sum_{i\in S}\sum_{j\in T} x_{ij} - \sum_{i\in S}\sum_{j\in T} x_{ji}$$

$$= \sum_{i\in S}\sum_{j\in T} u_{ij} - \sum_{i\in S}\sum_{j\in T} 0 = CAP(S,T)$$

# Review

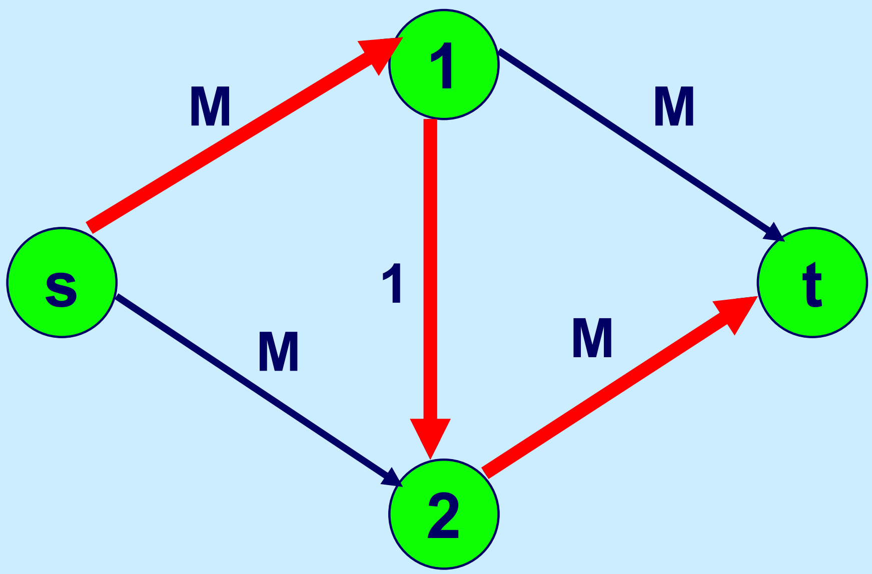
**Corollary.** If the capacities are finite integers, then the Ford-Fulkerson Augmenting Path Algorithm terminates in finite time with a maximum flow from s to t.

**Corollary.** If the capacities are finite rational numbers, then the Ford-Fulkerson Augmenting Path Algorithm terminates in finite time with a maximum flow from s to t. (why?)

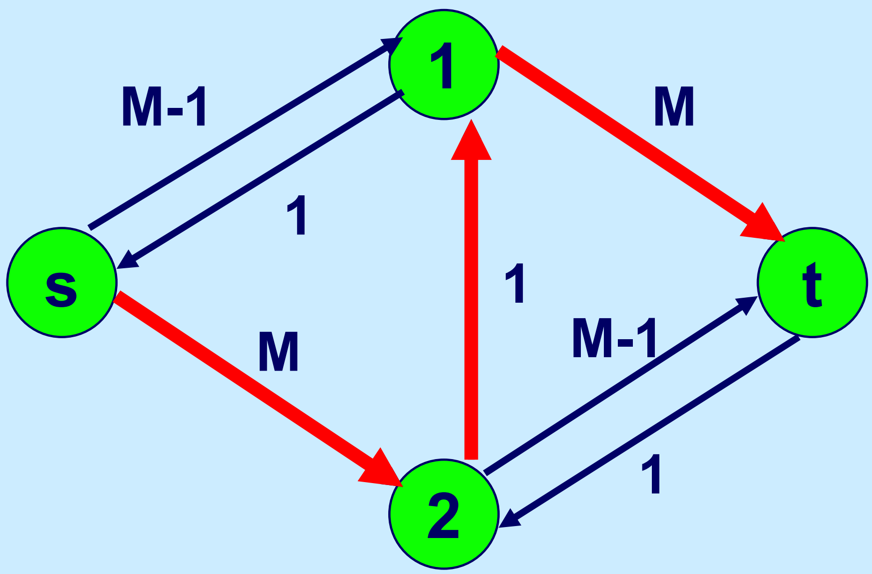
**Corollary.** To obtain a minimum cut from a maximum flow x, let S denote all nodes reachable from s in G(x).

**Remark.** This does not establish finiteness if $u_{ij} = \infty$ or if capacities may be irrational.

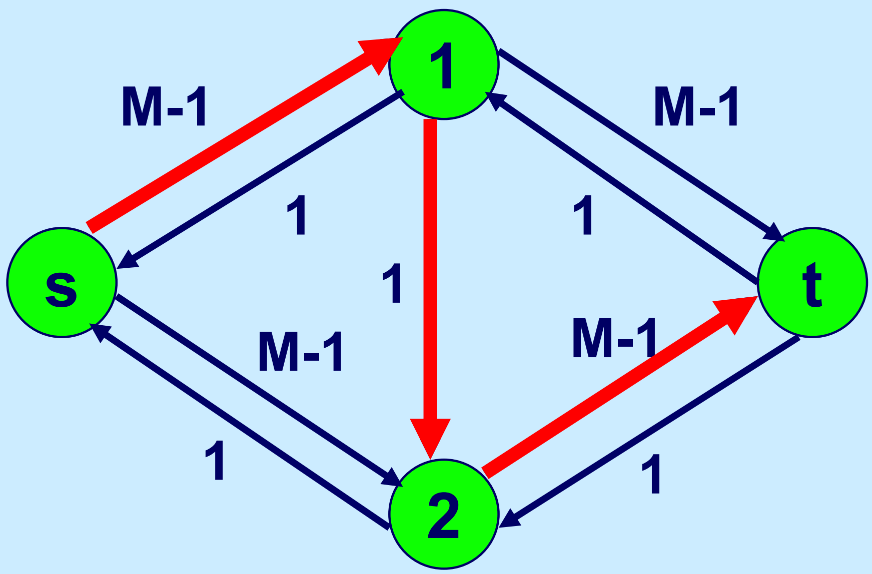
# A simple and very bad example



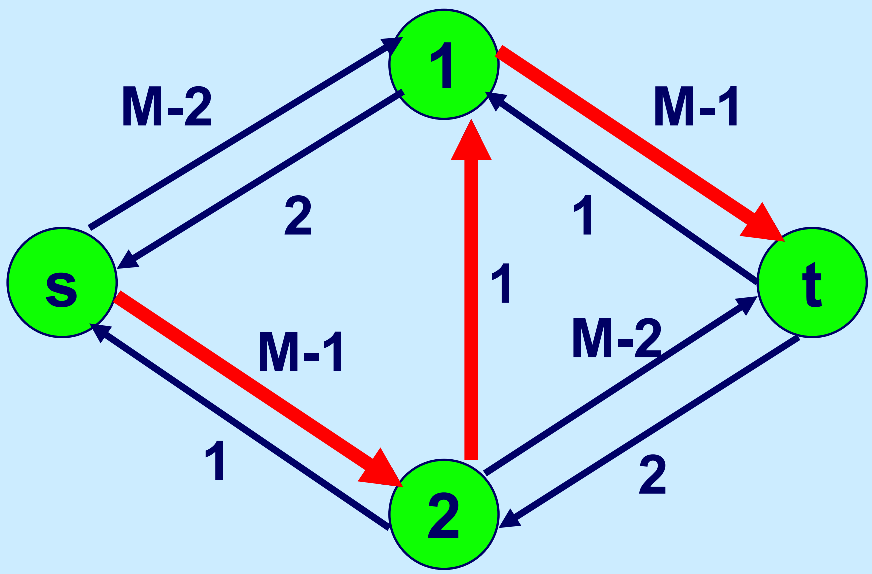
# After 1 augmentation



# After two augmentations



# After 3 augmentations



# And so on

• • •

# After 2M augmentations