Example 2: Fixed Charge Problem

A telecom company wants to offer new services to its customers. Each service can be offered in different amounts (e.g., with different bandwidth). The goal is to decide how much of each service is to be offered to maximize the total profit, such that the total cost remains within a given budget.

The following information is available to formulate the problem:

- There are N potential services.
- If service i is offered, then it incurs a cost of c_i per unit amount of service, plus a fixed charge k_i, which is independent of the amount.
- Service i brings a profit of p_i per unit amount of offered service.
- The total available budget is C.

To formulate a mathematical program, let us introduce variables:

$$x_i = \text{the amount of service } i \text{ offered}$$

Note that x_i is a continuous variable. To handle the fixed charge we also introduce indicator variables for the services, to indicate which ones are actually offered:

$$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

How much is the cost of service i? The proportional cost is clearly $c_i x_i$. It would be wrong, however, to say that the cost with fixed charge is $c_i x_i + k_i$, since k_i is incurred *only* if $x_i > 0$. Thus, we have to distinguish the cases when $x_i > 0$ and $x_i = 0$. This is why we need the y_i variables. Thus, the correct expression is $c_i x_i + k_i y_i$ and the total cost is:

$$\sum_{i=1}^{N}(c_i x_i + k_i y_i).$$

The total profit is:

$$\sum_{i=1}^{N} p_i x_i.$$

The complete formulation then can be expressed as

$$\max\ Z = \sum_{i=1}^{N} p_i x_i$$

Subject to

$$\begin{array}{rcll} \sum_{i=1}^{N}(c_i x_i + k_i y_i) & \leq & C & \\ x_i & \geq & 0, & i = 1, \ldots, N \\ y_i & \in & \{0, 1\}, & i = 1, \ldots, N \end{array}$$

$$y_i = \begin{cases} 1 & \text{if } x_i > 0 \\ 0 & \text{if } x_i = 0 \end{cases}$$

This is a mixed integer program, since some of the variables are continuous. On the other hand, this formulation is *not linear*, since the last constraint (the one that enforces the relationship between y_i and x_i) is not linear. Often it is desirable, however, to have a linear formulation which is useful when we want to use commercial ILP software.

Exercises

1. Replace the last constraint with a linear formulation.

Solution:

Observe that the constraint

$$\sum_{i=1}^{N}(c_i x_i + k_i y_i) \le C$$

implies that $c_i x_i \le C$ must hold for every x. Therefore, if we introduce a new constant value by

$$M = \max_i \frac{C}{c_i}$$

then $x_i \le M$ holds for every x_i. Then we may replace the nonlinear constraint by the following linear one:

$$y_i \ge \frac{1}{M} x_i.$$

Why does the new (linear) formulation remains equivalent with the original one? If $x_i > 0$, then $0 < \frac{1}{M} x_i \le 1$, so then $y_i \ge \frac{1}{M} x_i$ forces $y_i = 1$. That is exactly what we want if $x_i > 0$. If $x_i = 0$, then, according to the new formulation, y_i can be both 0 or 1. In this case, however, whenever it is 1, changing it to 0 does not cause any new constraint violation and it also does not change the the objective function. Therefore, the new formulation serves the purpose just as well as the original one, it does not change the result. It is, however, linear, which is useful when we want to use commercial ILP software.

2. What will be the optimal solution, if the company decides to offer only a single service? (But we do not know in advance, which one will be the offered service.)

Solution:

Let i be the service that the company offers. Then the total profit will be $p_i x_i$, as for all $j \neq i$ we have $x_j = 0$. For any given i, the profit will be maximum, if x_i is as large as possible. Since there is no other service in the considered case, therefore, service i will use the entire available budget C. This means

$$c_i x_i + k_i = C$$

(Since obviously $x_i > 0$ in this case, therefore, the fixed charge will be there.) From this we get

$$x_i = \frac{C - k_i}{c_i}$$

and the profit will be

$$p_i x_i = \frac{p_i(C - k_i)}{c_i}.$$

How do we know the value of i? We simply have to check which i makes the expression

$$\frac{p_i(C - k_i)}{c_i}$$

the largest. (Since the expression contains only known constants, we can easily check which i makes it the largest.)

3*. Show that the special case discussed in **2** is not as special as it looks: even in the general case the optimum is always attained by a solution in which only one x_i differs from 0. Using this, devise an algorithm that solves the problem optimally in the general case without ILP. This gives a clever shortcut, using the special features of the problem, making the task efficiently solvable even in the general case.