ILP Exercises

1.

Consider a U.S. company with separate east and west coast networks that are presently *not* connected. The west coast networks consists of switches (nodes) in Seattle (1), San Francisco (2), and Los Angeles (3). The east coast network consists of switches in New York (4), Washington DC, (5) and Atlanta (6).

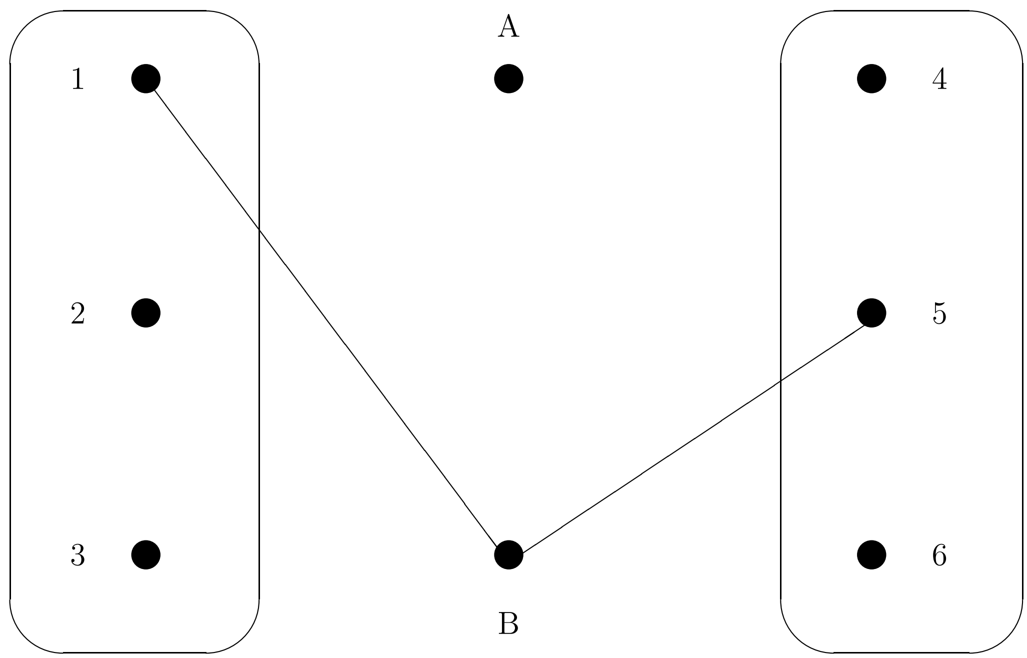
A decision is made to connect the two networks through a single transcontinental link that will originate in one of the west coast cities, pass through a hub in either Chicago (A) or Dallas (B), and continue on one of the east coast cities.

Draw a map of this situation. Let c_{ij} be the cost of a link from coast city i (note the ID numbers) to hub city j (j = A, B). Let x_{ij} be 1 if there is a link from coast city i ($i = 1, 2, \ldots, 6$) to hub city j (j = A, B) and 0 otherwise.

Write a mathematical program that minimizes the cost of the single transcontinental route and satisfies the constraints above.

Solution

First we draw a map of the situation:



A potential layout of the transcontinental link.

Let us now formulate the constraints.

- Exactly one of the West coast cities is connected to one of the hubs:

$$\sum_{i=1}^{3} \sum_{j=A}^{B} x_{ij} = 1$$

- Similarly, exactly one of the East coast cities is connected to one of the hubs:

$$\sum_{i=4}^{6} \sum_{j=A}^{B} x_{ij} = 1$$

- The same hub is connected to both coasts:

$$\sum_{i=1}^{3} x_{iA} = \sum_{i=4}^{6} x_{iA}$$

  Note that this already implies the same equation for hub B.

- The variables can only take 0-1 values:

$$x_{ij} \in \{0, 1\}, \qquad i = 1, \ldots, 6; \;\; j = A, B$$

The objective function is the total cost:

$$\sum_{i=1}^{6} \sum_{j=A}^{B} c_{ij} x_{ij}$$

Thus, the entire mathematical program will be an ILP with 0-1 valued variables:

$$\min Z = \sum_{i=1}^{6} \sum_{j=A}^{B} c_{ij} x_{ij}$$

Subject to:

$$\begin{aligned}
\sum_{i=1}^{3} \sum_{j=A}^{B} x_{ij} &= 1 \\
\sum_{i=4}^{6} \sum_{j=A}^{B} x_{ij} &= 1 \\
\sum_{i=1}^{3} x_{iA} &= \sum_{i=4}^{6} x_{iA} \\
\sum_{i=1}^{3} x_{iB} &= \sum_{i=4}^{6} x_{iB}
\end{aligned}$$

$$x_{ij} \in \{0, 1\}, \qquad i = 1, \ldots, 6; \;\; j = A, B$$

2. Assume that in an integer linear programming problem we have n variables. Each variable can only take the value 0 or 1. Express each of the conditions a.), b.), c.) given below, such that you can only use *linear* equations or inequalities, nothing else.

a) At least one variable must take the value 0.

Answer:

This means, they cannot be all 1, so their sum is at nost $n - 1$:

$$x_1 + x_2 + \ldots + x_n \leq n - 1$$

or

$$x_1 + x_2 + \ldots + x_n < n$$

b) At most one variable can take the value 0.

Answer:

This means, at least $n - 1$ of them is 1, so their sum is at least $n - 1$:

$$x_1 + x_2 + \ldots + x_n \geq n - 1$$

c) Either all variables are 0, or none of them.

Answer:

This means, they all must be equal:

$$x_1 = x_2 = \ldots = x_n$$

or, expressed via separate equations:

$$\begin{gathered} x_1 = x_2 \\ x_2 = x_3 \\ \vdots \\ x_{n-1} = x_n \end{gathered}$$

3. Let x, y, z be 0-1 valued variables. Express the following constraint via linear inequalities:

$$z = xy$$

Solution

If any of x, y is 0, then z must be 0, too. This can be expressed by two inequalities:

$$z \leq x$$

$$z \leq y$$

On the other hand, if x, y are *both* 1, then we have to force z to be 1. It can be done by

$$z \geq x + y - 1$$

Note that if at least one of x, y is 0, then the right-hand side is ≤ 0, so then nothing is forced on z.

Thus, the system

$$\begin{aligned} z &\leq x \\ z &\leq y \\ z &\geq x + y - 1 \end{aligned}$$

is *equivalent* to the nonlinear constraint

$$z = xy$$

for 0-1 valued variables. Note that the restriction $x, y, z \in \{0, 1\}$ is essential here, without it the equivalence would not hold.

4. Let us generalize the previous exercise to a longer product! Let $x_1, \ldots, x_n$ and z be 0-1 valued variables. Express the following constraint via linear inequalities:

$$z = x_1 \cdot \ldots \cdot x_n$$

Solution

Similarly to the previous exercise, it is easy to check that the following system of linear inequalities provides an equivalent expression for the case when $z, x_1, \ldots, x_n \in \{0, 1\}$:

$$\begin{aligned} z &\leq x_1 \\ &\vdots \\ z &\leq x_n \\ z &\geq x_1 + \ldots + x_n - n + 1 \end{aligned}$$