Linear Programming Duality

The definition of the dual LP

Consider the following LP in vector-matrix from:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} \\ & \quad \mathbf{x} \geqslant 0 \end{array}$$

We can associate another LP with it, which is called the ***dual*** of the above, while the first is called the ***primal*** LP:

$$\begin{array}{llll} & \textit{Primal} & & \textit{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} & \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathrm{c}^T \\ & \quad \mathbf{x} \geqslant 0 & & \quad \boldsymbol{\lambda} \geqslant 0 \end{array} \tag{1}$$

Observe that this definition is symmetric, that is, the dual of the dual is the same as the primal. Therefore, it does not matter which of them we call the primal LP and which the dual LP. It turns out (by an important theorem) that they actually have the *same optimum value*. As we are going to see, this provides the importance of duality. The next example illustrates it in an economic setting.

Economic Interpretation of duality

- A resource allocation example. Dakota furniture company manufactures desks, tables and chairs. The manufacturing of each type of furniture requires three resources: Lumber, finishing labor, and carpentry labor. The amount of each resource needed to make one unit of a certain type of furniture is as follows.

	Desk	Table	Chair
Lumber	8 lft	6 lft	1 lf
Finishing hours	4 hrs	2 hrs	1.5 hrs
Carpentry hours	2 hrs	1.5 hrs	0.5 hrs

At present, 48 lft of lumber, 20 hours of finishing hours, and four hours of carpentry hours are available. A desk sells for $60, a table for $30, and a chair for $20. How many of each type of furniture should Dakota produce?

LP formulation

- Let x_1, x_2, and x_3 respectively denote the number of desks, tables and chairs produced. Dakota problem is solved with the following LP:

$$\begin{aligned}
\max \quad & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + 6x_2 + x_3 \le 48 \\
& 4x_1 + 2x_2 + 1.5\,x_3 \le 20 \\
& 2x_1 + 1.5\,x_2 + 0.5\,x_3 \le 8 \\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{aligned}$$

- Suppose that an entrepreneur wants to buy Dakota's resources. What are the *fair* prices, y_1, y_2, y_3, that the entrepreneur should pay for a lft of lumber, one hour of finishing and one hour of carpentry?

What is fairness here? The buyer, of course, wants to minimize the buying price. However, it cannot offer a lower price for the resources (material + labor) needed for a desk, than what a desk sells for. Otherwise the seller would say that he rather sells the desk than the resources. Similarly for the other products.

These provide constraints for the buyer's optimization. Let us formulate it as an LP.

- The entrepreneur wants to minimize buying cost. Then, his objective can be written as

$$\min \quad 48y_1 + 20y_2 + 8y_3$$

- In exchange for the resources that could make one desk, the entrepreneur is offering ($8y_1 + 4y_2 + 3y_3$) dollars. This amount should be larger than what Dakota could make out of manufacturing one desk (\$60). Therefore,

$$8y_1 + 4y_2 + 2y_3 \geq 60$$

- Similarly, by considering the "fair" amounts that the entrepreneur should pay for the combination of resources that are required to make one table and one chair, we conclude

$$6y_1 + 2\ y_2 + 1.5y_3 \geq 30$$
$$y_1 + 1.5y_2 +\ 0.5y_3 \geq 20$$

- Consequently, the entrepreneur should pay the prices y_1, y_2, y_3, solution to the following LP:

$$\begin{aligned}
\min \quad & 48y_1 + 20y_2 + 8y_3 \\
\text{s.t.} \quad & 8y_1 + \ 4y_2 + \ 2y_3 \geq 60 \\
& 6y_1 + 2\ y_2 + 1.5y_3 \geq 30 \\
& y_1 + 1.5y_2 + 0.5y_3 \geq 20 \\
& y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0
\end{aligned}$$

- The above LP is the dual to the manufacturer's LP.

Recall that the primal LP was:

(*Note:* this looks like the dual in (1), but the primal and dual roles are symmetric, so we can take either one as the primal, and the other as the dual.)

$$\begin{aligned}
\max \quad & 60x_1 + 30x_2 + 20x_3 \\
\text{s.t.} \quad & 8x_1 + \ 6x_2 + \ x_3 \leq 48 \\
& 4x_1 + \ 2x_2 + 1.5\ x_3 \leq 20 \\
& 2x_1 + 1.5\ x_2 + 0.5\ x_3 \leq 8 \\
& x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0
\end{aligned}$$

Discussion

The above example is quite interesting, as it shows that the following two tasks are dual to each other:

1. Maximize the profit under the resource constraints.
2. Minimize the buying price under the fairness constraints.

By the Duality Theorem of Linear Programming (see later), their optimum values are equal. This is reassuring if we consider the following economic reasoning:

If the buying price were higher than the maximum achievable profit, then the transaction would not make sense for the buyer. On the other hand, if the buying price were lower than the maximum achievable profit, then the transaction would not make sense for the seller. To make sense for both of them, they should be equal.

Quite interestingly, this equality indeed comes out from the LP formulation, using the Duality Theorem (see later), even though the latter is purely mathematical, no economic considerations were designed into it.

Finding the Dual of the Standard Form

The dual of any linear program can be found by converting the program to the form of the primal shown above. For example, given a linear program in standard form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{array}$$

we write it in the equivalent form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geqslant \mathbf{b} \\ & -\mathbf{A}\mathbf{x} \geqslant -\mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0}, \end{array}$$

which is in the form of the primal of (1) but with coefficient matrix $\begin{bmatrix} \mathbf{A} \\ --- \\ -\mathbf{A} \end{bmatrix}$. Using a dual vector partitioned as $(\mathbf{u}, \mathbf{v})$, the corresponding dual is

$$\begin{array}{ll} \text{maximize} & \mathbf{u}^T\mathbf{b} - \mathbf{v}^T\mathbf{b} \\ \text{subject to} & \mathbf{u}^T\mathbf{A} - \mathbf{v}^T\mathbf{A} \leqslant \mathbf{c}^T \\ & \mathbf{u} \geqslant \mathbf{0} \\ & \mathbf{v} \geqslant \mathbf{0}. \end{array}$$

Letting $\boldsymbol{\lambda} = \mathbf{u} - \mathbf{v}$ we may simplify the representation of the dual program so that we obtain the pair of problems displayed below:

$$\begin{array}{llcll} & \textit{Primal} & & & \textit{Dual} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} & & \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & & \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T. \\ & \mathbf{x} \geqslant \mathbf{0} & & & \end{array} \tag{2}$$

This is the *asymmetric form* of the duality relation. In this form the dual vector $\boldsymbol{\lambda}$ (which is really a composite of $\mathbf{u}$ and $\mathbf{v}$) is not restricted to be nonnegative.

The Duality Theorem

Throughout this section we consider the primal in standard form, so we have

$$\begin{array}{ll} & \textit{Primal} \\ \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{array} \tag{3}$$

and

$$\begin{array}{ll} & \textit{Dual} \\ \text{maximize} & \boldsymbol{\lambda}^T\mathbf{b} \\ \text{subject to} & \boldsymbol{\lambda}^T\mathbf{A} \leqslant \mathbf{c}^T. \end{array} \tag{4}$$

Lemma 1. (Weak Duality Lemma). *If* $\mathbf{x}$ *and* $\boldsymbol{\lambda}$ *are feasible for (3) and (4), respectively, then* $\mathbf{c}^T\mathbf{x} \geqslant \boldsymbol{\lambda}^T\mathbf{b}$.

What does this mean? It means that the objective function value of the primal is always greater than or equal to the objective function value of the dual. In other words any *feasible* solution of the primal minimization problem is an upper bound on the dual maximization optimum. Similarly, any *feasible* solution of the dual maximization task is a lower bound on the primal minimization optimum.

Fig 1. Relation of primal and dual objective function values

The Weak Duality Lemma is surprisingly easy to prove:

Proof. We have

$$\boldsymbol{\lambda}^T \mathbf{b} = \boldsymbol{\lambda}^T \mathbf{A}\mathbf{x} \leqslant \mathbf{c}^T \mathbf{x},$$

the last inequality being valid since $\mathbf{x} \geqslant \mathbf{0}$ and $\boldsymbol{\lambda}^T \mathbf{A} \leqslant \mathbf{c}^T$. ∎

What is the significance of all this? We can see that either task (primal and dual) yields a bound on the optimal value of the other. Therefore, we obtain the following consequence:

Corollary. *If $\mathbf{x}_0$ and $\boldsymbol{\lambda}_0$ are feasible for (3) and (4), respectively, and if $\mathbf{c}^T \mathbf{x}_0 = \boldsymbol{\lambda}_0^T \mathbf{b}$, then $\mathbf{x}_0$ and $\boldsymbol{\lambda}_0$ are optimal for their respective problems.*

This is very useful in the following situation. Assume we found a feasible solution to the primal problem and we claim it is optimal. The fact that it is *feasible* can be easily checked by substituting it into the constraints, and seeing that all constraints are satisfied. But how do we prove that it is *optimal*? How can we make sure that there was no error in the program that found it?

An opportunity is offered by the above Corollary. If we can exhibit a dual feasible solution, such that

$$\mathbf{c}^T \mathbf{x}_0 = \boldsymbol{\lambda}_0^T \mathbf{b}$$

holds, then this proves the optimality of the primal solution, since the left-hand side is the primal objective function value, which is always bounded from below by the dual objective function value that is on the right-hand side. That is, we always have

$$\mathbf{c}^T \mathbf{x} \geqslant \boldsymbol{\lambda}^T \mathbf{b}.$$

Now, if they are actually equal, then both must be optimal, as then the gap in Fig 1 disappears. In other words, if the primal objective function value is equal to the dual one, then this primal value must be the smallest possible, since the dual is always a lower bound. Thus, the dual solution can serve as a *certificate* of the optimality of the primal solution.

Note that in order to check this certificate, we only need to check that both the claimed primal and the dual solutions are *feasible*, we do not have to check their optimality!

But the whole thing only works if one can indeed guarantee that there is no gap between the set of primal and dual feasible solutions. In case there is a gap, then the above optimality certificate does not exist.

Therefore, it is an important question whether is it generally true that there is no such gap between the primal minimum and dual maximum, given that they are both finite? The answer is that it is indeed generally true, and it is guaranteed by the Duality Theorem of Linear Programming:

> ***Duality Theorem of Linear Programming.*** *If either of the problems (3) or (4) has a finite optimal solution, so does the other, and the corresponding values of the objective functions are equal. If either problem has an unbounded objective, the other problem has no feasible solution.*

The proof of this is significantly harder than that of the Weak Duality Lemma, so we do not study the proof details in this course.

LP duality is a fundamental result. The above "optimality certificate" application is only a very simple one. It is applied in much more sophisticated ways, for example in constructing LP algorithms.

Exercise

Find the dual of

$$\begin{array}{ll} \text{minimize} & 2x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + x_2 + 2x_3 = 3 \\ & 2x_1 + x_2 + 3x_3 = 5 \\ & x_i \geqslant 0, \quad x_2 \geqslant 0, \quad x_3 \geqslant 0. \end{array}$$