Some common rejection regions

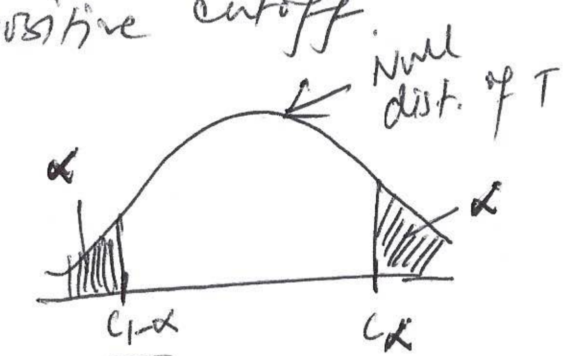
Suppose $T = \dfrac{\hat{\theta} - \theta_0}{SE(\hat{\theta})}$. When H_0 is true, we expect T to be close to zero.

T_{obs} = Observed value of T.

In this case, it is often easy to guess $\mathcal{R}$.

Case 1: $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$

Reject H_0 if: $|T|$ large $\Rightarrow$ $|T| > c$, where c is some positive cutoff.



Case 2: $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$

Reject H_0 if $T > c$
some positive cutoff

Take $c = c_\alpha$, then
$P[\text{Type I error}]$
$= P[T > c_\alpha \mid H_0 \text{ is true}]$
$= \alpha$

Case 3: $H_0 : \theta = \theta_0$ against $H_1 : \theta < \theta_0$

Reject H_0 if $T < c$
some negative cutoff.

Compute the critical point in a way that ensures that the level of the test equals the prescribed α.

Take $c = c_{1-\alpha}$ then $P[\text{Type I error}] = P[T < c_{1-\alpha} \mid H_0 \text{ is true}] = \alpha$

Note: If the null dist. of T is symmetric, then $c_{1-\alpha} = -c_\alpha$

The corresponding level α tests:

Suppose c_α is such that $P(T > c_\alpha | \theta = \theta_0) = \alpha$.

Suppose:

$\|$ $(1-\alpha)$ th percentile of null dist.

Recall: $T = \dfrac{\hat{\theta} - \theta_0}{\hat{SE}(\hat{\theta})}$ $\Rightarrow$ Intuitively: reject H_0 if $|T| >$ cutoff. positive

Case 1: $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$

$\mathcal{R} = \{|T| > c_{\alpha/2}\}$, i.e., $\boxed{\text{reject } H_0 \text{ when } |T| > c_{\alpha/2},}$ otherwise accept it.

$P[\,|T| > c_{\alpha/2} \,|\, H_0 \text{ is true}] = P[\text{TYPE I error}] = \alpha$

— see next page

Case 2: $H_0 : \theta = \theta_0$ against $H_1 : \theta > \theta_0$

$\mathcal{R} = \{T > c_\alpha\}$, i.e., reject H_0 when $T > c_\alpha$, otherwise accept it.

Case 3: $H_0 : \theta = \theta_0$ against $H_1 : \theta < \theta_0$

$\mathcal{R} = \{T < \overset{c_{1-\alpha}}{\cancel{c_\alpha}}\}$, i.e., reject H_0 when $T < \overset{c_{1-\alpha}}{\cancel{c_\alpha}}$, otherwise accept it.

Hypothesis testing (continued)

We can perform a level α test by comparing T_{obs} with the critical point. But how strong is the evidence against the null? This is formally measured by *p-value*. Let's play a game to motivate its definition.

My bag has 10 small balls. I claim that 8 are red and 2 are blue. I will bet 3 people a candy bar that a blue ball will come up. My chances are not very good but I will take them anyway.

trial #	PKC vs ?	color drawn	winner
1	Bijont	blue	pkc
2	Neetu	blue	pkc
3	Nisha	blue	pkc

Q. Does is it seem reasonable that I would win 3 ... times in 3 trials if the bag contained 2 blue balls? → Not.

⇒ data indicates that perhaps my claim is true.

Let's cast this problem as a test of hypothesis. $X = \begin{cases} 1, & \text{if 'blue'} \\ 0, & \text{o/w} \end{cases}$

Hypotheses: $p = P[X=1]$

$H_0: p = 0.2$

$H_1: p > 0.2$

↑ For the sake of simplicity [more appropriate: $H_1: p \neq p_0$].

***T* and T_{obs}**

$n=3$, $T = X_1 + X_2 + X_3$ = # blue balls drawn in 3 trials

$T_{obs} = 3$

Null distribution *T*: $\boxed{T \sim \text{Binomial}(n=3, p=0.2)}$

Reject H_0 if $T > c$ large.

Q. What is the actual chance of getting T_{obs} if H_0 is true? What does it indicate about H_0?

$= P[T = T_{obs} \mid H_0 \text{ is true}] = P[T=3 \mid H_0 \text{ is true}]$

$= (0.2)(0.2)(0.2) = 0.008$

very small

p-value $= P[T \geq T_{obs} \mid H_0] = P[T \geq 3 \mid H_0] = P[T=3 \mid H_0] = 0.008$

⇒ The data that we observed are quite rare if H_0 is true ⇒ Perhaps H_0 is not true

⇒ Reject H_0.

p-value: The probability of getting a T that is *as extreme or more extreme* than T_{obs} assuming that H_0 is true.

Meaning of "extreme" depends on H_1

- Small p-value implies $\Rightarrow$ the observed data are not consistent with H_0
  $\Rightarrow$ Reject H_0.
- Smaller the p-value, stronger the evidence against H_0.
- Level α test: Reject H_0 if p-value $\leq \alpha$.
- Another interpretation of p-value: The smallest level of significance at which H_0 is rejected.
- Advantage of p-value over critical point: p-value summarizes the strength of evidence against H_0 in a prob.

Q. Is p-value $= P(H_0 \text{ is true})$?

$\rightarrow$ NO. not a random quantity.

- H_0 is either true or not true, but we don't know the truth. Certainly, H_0 is not a random quantity.
- p-value tells us how likely our T_{obs} is (or something more extreme) if H_0 is true.

	<u>As extreme or more extreme</u>
If $H_1: \theta > \theta_0 \Rightarrow$	$T \geq T_{obs}$.
$H_1: \theta < \theta_0 \Rightarrow$	$T \leq T_{obs}$
$H_1: \theta \neq \theta_0 \Rightarrow$	$\lvert T \rvert \geq \lvert T_{obs} \rvert$

p-value is the prob. of one of these possibilities assuming H_0 is true.

Summary of steps in a hypothesis test: $T = \dfrac{\hat{\theta} - \theta_0}{SE(\hat{\theta})}$

- Formulate H_0 and H_1 → usual form:
- Find a test statistic T and get its null distribution
- Compute T_{obs}
- Use the null distribution to compute either the critical point or the p-value for the test.
- State your conclusion.

in layman terms w/to, not just accept or reject H_0

Some specific tests

One-sample tests for μ where $\underline{X} \sim N(\mu, \sigma^2)$

Case 1: z**-test (known** σ^2**):** $H_0 : \mu = \mu_0$

$SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$ — known

Test statistic: $Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$

$\bar{X} \sim N\left[\mu, \frac{\sigma^2}{n}\right]$.

Null dist. : $\bar{X} \sim N\left[\mu_0, \frac{\sigma^2}{n}\right]$ under H_0

$\Rightarrow Z \sim N(0,1)$

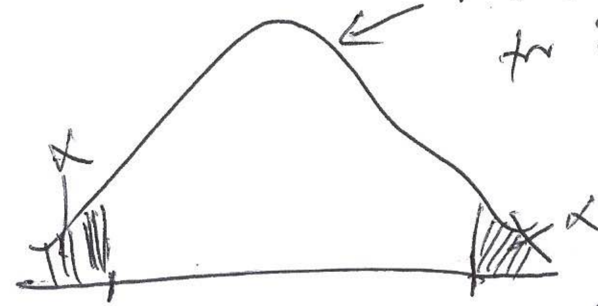
Critical point for the level α **test:** if H_0 is true.

$N(0,1)$ dist. for Z

One-sided alternative:

$H_1 : \theta > \theta_0 \Rightarrow$ critical pt. is z_α

Two-sided alternative:



$z_{1-\alpha} = \text{qnorm}(\alpha)$

$z_\alpha = \text{qnorm}(1-\alpha)$

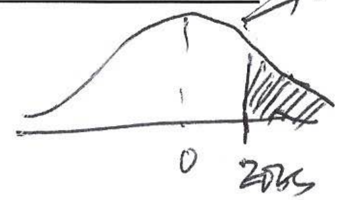
$H_1 : \theta < \theta_0 \Rightarrow$ critical pt. is $z_{1-\alpha} = -z_\alpha$

$H_1 : \theta \neq \theta_0 \Rightarrow$ critical pt. is $z_{\alpha/2}$

p-value:

H_1	reject when	p-value	computing p-value
$\mu \neq \mu_0$			
$\mu > \mu_0$			
$\mu < \mu_0$			

$\mu > \mu_0 \rightarrow$ p-value $= P[Z \geq z_{obs} \mid H_0] =$



$= 1 - F_Z(z_{obs})$

$= 1 - \text{pnorm}(z_{obs})$

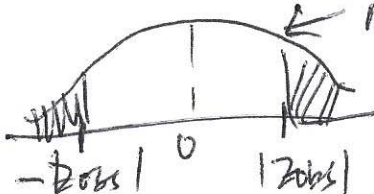
$\mu < \mu_0$: p-value $= P[Z \leq z_{obs} \mid H_0] = F_Z(z_{obs}) = \text{pnorm}(z_{obs})$

$\mu \neq \mu_0$: p-value $= P[\,|Z| \geq |z_{obs}| \mid H_0]$

$= P[Z \geq |z_{obs}| \text{ or } Z \leq -|z_{obs}| \mid H_0]$

$= P[Z \geq |z_{obs}| \mid H_0] + P[Z \leq -|z_{obs}| \mid H_0]$

$=$



$= 2\,P[Z \geq |z_{obs}| \mid H_0] = 2(1 - \text{pnorm}(|z_{obs}|))$

$= 2[1 - F_Z(|z_{obs}|)]$