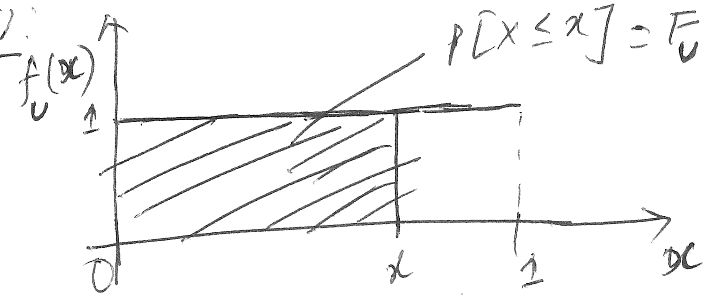
# Computer simulations and Monte Carlo Methods (chapter 5)

Assume that we can simulate $U \sim \text{Uniform}(0, 1)$. Recall that:

pdf of U:

$P[X \le x] = F_U(x) = x, \quad 0 \le x \le 1$




$X \sim \text{Uniform}(a,b)$

$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & o/w \end{cases}$

Every programming language has a *random number generator* that simulates a $U$. In R, this function is `runif()`. Subsequent calls to this function will give draws that are "independent" for all practical purposes.

# Simulating from discrete distributions

**Simulating $X \sim$ Bernoulli($p$):** , $0<p<1$.

Recall: If $X$ Bernoulli($p$), $P(X=1)=p$, $P(X=0)=1-p$.

1. Generate $U$.
2. If $U \leq p$; set $X=1$, else set $X=0$.

Verification:

$P[X=1] = P[U \leq p] = F_U(p) = p$

$P[X=0] = P[U > p] = 1 - P[U \leq p] = 1-p$.

Alternatively in R: rbinom(1, size=1, prob=p) — 1 draw from Bernoulli($p$)

rbinom(10, size=1, prob=p) — 10 indep. draws from Bernoulli($p$)

Simulating from Binomial($n,p$):

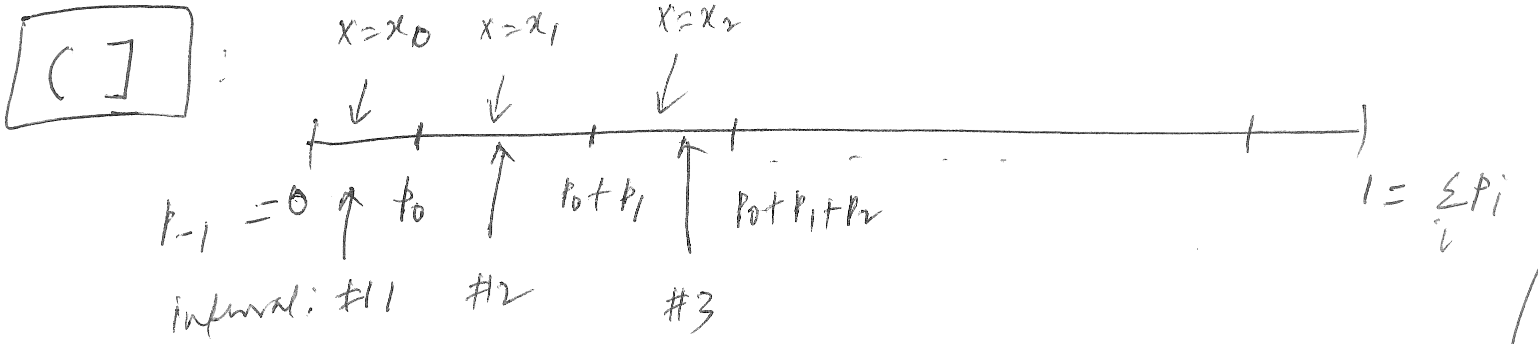
— rbinom(1, size=n, prob=p)

— simulate $X_1, X_2, \ldots, X_n$ as independent draws from a Bernoulli($p$), then $X = X_1 + \cdots + X_n$ ~~=~~ a draw from a Binomial($n,p$).

## Simulating $X \sim f(x)$, arbitrary discrete distribution:

Suppose $X$ takes values $x_0, x_1, \ldots$, with probabilities $p_0, p_1, \ldots$, where $p_i = P(X = x_i)$ and $\sum_i p_i = 1$.

1. Divide the interval $[0, 1]$ into subintervals as shown below.




Recall:
$F(x)$ — CDF of X

$P[a < X \le b] = P[X \le b] - P[X \le a]$
$= F(b) - F(a)$

2. Generate $U$.
3. If $U$ falls in subinterval $i$, take $X = x_i$.

Verification:

$P[X = x_i] = P[U \text{ falls in subinterval } i] = P[p_0 + p_1 + \cdots + p_{i-1} < U \le p_0 + p_1 + \cdots + p_i]$

$= F_U[p_0 + \cdots + p_i] - F_U[p_0 + \cdots + p_{i-1}]$

$= (p_0 + \cdots + p_i) - (p_0 + \cdots + p_{i-1}) = p_i$

— Always works, but may not be efficient.

# Simulating from continuous distributions

**Result**: If $X$ is a continuous rv with cdf $\underline{F}(x)$, then $U = F(X)$ follows Uniform(0, 1) distribution. – "probability integral transform"

<u>Inverse transform method</u>: To simulate a $\underline{X}$,

1. Generate $U$.
2. Set $U = F(X)$
3. Solve for $X$ (i.e., invert the cdf). $\Rightarrow X = F^{-1}(U)$

Often the equation cannot be solved explicitly or efficiently.
Alternatives are available.

<u>Ex</u>: Examples of dist. whose CDF is not available in a closed-form: Normal, Gamma, Cauchy, $\chi^2$ and so on.

rnorm rgamma rcauchy

**Simulating from Exponential($\lambda$) distribution:**

Recall: If $X \sim \text{Exponential}(\lambda)$, $F(x) = 1 - \exp(-\lambda x)$. $x > 0$, $\lambda > 0$.

Note: 'log' means natural log unless specified otherwise.

Solve for x:

$$U = F(X) = 1 - e^{-\lambda X}$$

$$\Rightarrow e^{-\lambda X} = 1 - U$$

$$\Rightarrow \log[e^{-\lambda X}] = \log[1-U]$$

$$\Rightarrow -\lambda X = \log[1-U]$$

$$\Rightarrow \boxed{X = -\frac{1}{\lambda}\log[1-U]}$$

Book:

$$X = -\frac{1}{\lambda}\log[U]$$

both work because ~~both~~ if $U \sim \text{Uniform}(0,1)$, then $1-U \sim \text{Uniform}(0,1)$.

# Solving problems by Monte Carlo methods

(Approximating)

**Estimating** $\mu = E(X)$ and $\sigma^2 = var(X) = E(X-\mu)^2$:

Simulate a large number $(N)$ of independent draws from the distribution of $X$, say, $X_1, X_2, \ldots, X_N$

*MC estimator of* $\mu$: random $\rightarrow$ $\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$ ← sample mean

[LLN: $\bar{X} \approx \mu$ since $N$ is large]

*MC estimator of* $E[g(X)]$ *where* $g$ *is a given function:*

$$E[g(X)] \approx \frac{1}{N}\sum_{i=1}^{N} g(X_i)$$

*MC estimator of* $\sigma^2$: $\sigma^2 = E[(X-\mu)^2]$, $\mu = E[X]$.

Approach 1: $\frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2$

Approach 2: $\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$ — more common

doesn't matter since $N$ is large.

## Estimating an integral $I = \int_a^b g(x)\,dx$:

— try to interpret as an expected value.

$$I = \int_a^b g(x)\,dx = (b-a)\boxed{\int_a^b \frac{1}{b-a}\, g(x)\,dx}$$

Recall: $X \sim \text{Uniform}(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

$$= (b-a)\boxed{\int_a^b f(x)\, g(x)\,dx}, \quad X \sim \text{Uniform}(a,b)$$

$$E[g(X)]$$

$$\approx (b-a)\,\frac{1}{N}\sum_{i=1}^{N} g(X_i),$$ where $X_1, \ldots, X_N$ are draws from $\text{Uniform}(a,b)$.

verify: If $U \sim \text{Uniform}(0,1)$, then $X = a + (b-a)U \sim \text{Uniform}(a,b)$.

## **Estimating** $\boxed{p = P(X \in A)}$ **for a given region** $A$:

Simulate a large number $(N)$ of independent draws from the distribution of $X$, say, $X_1, X_2, \ldots, X_N$

Define $Y_1, \ldots, Y_N$ as:

$$Y_i = I(X_i \in A)$$

$\Rightarrow Y_1, Y_2, \ldots, Y_N$ are draws from Bernoulli($p$)

$E(Y)$

*MC estimator of p:*

$$p \approx \frac{1}{N} \sum_{i=1}^{N} Y_i$$

*Properties of* $\hat{p}$*:*

Note:

Define: indicator fn.

$$Y = I(X \in A) = \begin{cases} 1, & \text{if } X \in A \\ 0, & \text{o/w.} \end{cases}$$

Then: $Y \sim$ Bernoulli($P(X \in A) = p$),

$E[Y] = p$,

$Var[Y] = p(1-p)$