On probabilities and conditional probabilities

Intuitively, a probability is a measure of likelihood of a Boolean outcome of an experiment. If the outcome of the experiment is the Boolean (random) variable x then we can discuss $\mathsf{Prob(x)}$. If x is real, we can still discuss probabilities of Boolean conditions such as:
$\mathsf{Prob(x > 2.4)}$, $\mathsf{Prob(x > 3 \wedge y < 2)}$, etc.

As a measure of of likelihood, probabilities satisfy a set of axioms that are not discussed here.

Conditional probabilities

The probability of p when given that q is true is written as: $\mathsf{Prob(p|q)}$. It is defined by:

$$\mathsf{Prob(p|q)} = \frac{\mathsf{Prob(p \wedge q)}}{\mathsf{Prob(q)}}$$

Conditional independence and the Bayes theorem for two variables

Since $p \wedge q \equiv q \wedge p$ it follows from the above definition that:

$$\mathsf{Prob(q|p)} = \frac{\mathsf{Prob(p|q)Prob(q)}}{\mathsf{Prob(p)}}$$

This formula is called *the Bayes theorem*.

We say that p is independent of q if the knowledge of q does not change our estimate for p. Specifically, this means that: $\mathsf{Prob(p|q)} = \mathsf{Prob(p)}$, so that:

$$\mathsf{Prob(p \wedge q)} = \mathsf{Prob(p)Prob(q)}$$

This shows that if p is independent of q then q is also independent of p.

Mutually independent variables

Consider the n variables $q_1, \ldots q_n$. From the above condition, q_1 is independent of $q_2, \ldots, q_n$ iff $\mathsf{Prob(q_1, \ldots, q_n)} = \mathsf{Prob(q_1)Prob(q_2, \ldots, q_n)}$. If each one of the n variables is independent of the others we say that $q_1, \ldots q_n$ are mutually independent. It follows by induction that in such case:

$$\mathsf{Prob(q_1, \ldots, q_n)} = \mathsf{Prob(q_1)} \cdot \ldots \cdot \mathsf{Prob(q_n)}$$