# Quantitative Methods I Notes 2016

Quantitative Methods 1 (University of Melbourne)

**ECON10005**
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# 1 INTRODUCTION OF KEY TERMS AND CONCEPTS

**Basic terms**

- Population: Collection of everything in the universe with regards to the variable
- Parameter: Is a quantity that describes (or characterises) the shape of a probability density (or mas) function. Parameters are population objects and are unobservable.
- Estimator: An estimator is a function of random variables. It is a rule that tells us how to combine our data in order to produce an estimate of unknown population parameters. For instance, an estimator for the population mean $\mu$ is the sample mean $\bar{x}$.
- Estimate: An estimate is a particular realisation of the estimator (which is a random variable). It is a quantity that we compute from the data in order to infer the value of an unknown parameter.
- Sample: Small group of things taken from the population
- Statistic: A characteristic of the sample
- Variable: Is a characteristic that we get data on

**Types of variables**

- Discrete: cannot include every possible observation within sample space
- Continuous: can include all possible observations within sample space
- Qualitative: in terms of words, e.g. categories
- Quantitative: in terms of numbers
- Ordinal data: qualitative data with rankings
- Time series data: references data with time
- Descriptive statistics: summarise data in terms of numerical values
- Inferential statistics: gives inferences about the populations

**Types of data**

- Primary: first hand data.
- Secondary: data collected by someone else.
- Observational: measures behaviour and outcome.
- Experimental: use experiment conditions to determine the cause and effect of introducing a treatment variable.

**Types of sampling**

- Simple random sampling
- Stratified random sampling
- Systematic sampling
- Cluster sampling
- Convenience sampling
- Judgement sampling

# 2 DESCRIPTIVE STATISTICS

**Measures of central tendency**

- Sample mean: numerical centre of our data. Informative unless there are outliers.
- Median: the value that partitions the ordered set into two, not sensitive to extreme values.
- Mode: useful for categorical data or ordinal data, most frequently occurring value in the data set.

**Measures of spread**

- Sample variance: $s^2 = \frac{1}{n - 1} \sum_{i=1}^{n} ( x_i - \bar{x} )^2$
- Standard deviation: rescale of the sample variance back into the same units as the data. Square root of the sample variance.
- The interquartile range: $Q_1$, $Q_2$ (median), $Q_3$, $IQR = Q_3 - Q_1$

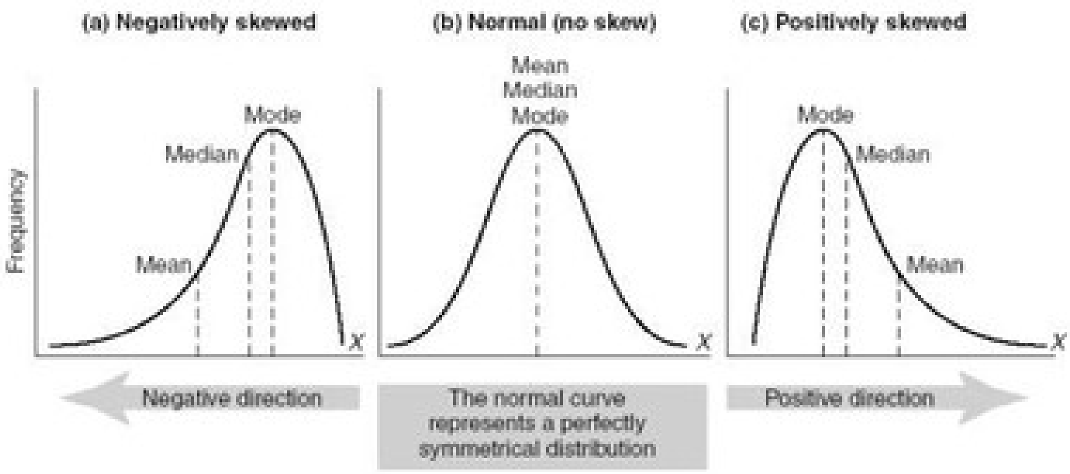
**Histograms**

- Sturge's formula: number of class intervals, $k = 1 + 3.3\log_{10}(n)$
- Class width = (largest value – smallest value) / k
- Depicts: location (range), spread, skew and kurtosis.

**Box and whisker plot**

- Depicts: median, upper and lower quartiles and range.
- Outliers: any value more than 1.5 x (IQR) above or below the upper and lower quartiles respectively.

**Measures of skew**




**Measures of kurtosis**

- Measures the relative proportion of extreme observations to 'regular' observations.
- Positive excess kurtosis is considered leptokurtic, while a negative excess kurtosis is considered platykurtic.

**Percentiles**

- Approximation formula: $L_P = (n + 1) \frac{P}{100}$
- Partition method: for median, $Q_1$ or $Q_3$.

## 3 BIVARIATE DATA

**The sample covariance**

- Cov(x,y) = $s_{xy} = \sum_{i=1}^{n} (x_i - \acute{x}$ ¿¿
- If x and y are positively related then small observations of x will tend to be associated with small observations of y.
- If x and y are negatively related, then small observations of x will be associated with large observations y and vice versa.

**Coefficient of correlation**

- Rescaled (just like z score [standardised]) covariance by dividing by the standard deviations of the two variables.
- $r = \frac{s_{xy}}{s_x s_y}$
- Takes values of $-1 \leq r \leq 1$
- Measures degree to which two variables are linearly related (sensitive to outliers).

## 4 SIMPLE LINEAR REGRESSIONS

**Linear relationships**

- If two variables are perfectly linearly related (i.e. $r = \pm 1$) then we can represent their relationship using a linear function, $y_i = b_0 + b_1x_i$
- y: dependent variable
- x: explanatory variable
- $b_0$: intercept
- $b_1$: slope coefficient (gradient)

**Line of best fit**

- $y_i = b_0 + b_1x_i + e_i$
- in order to fit a line to the data $\{x_i, y_i\}^n_{i=1}$, we need to choose $b_0$ and $b_1$. The fitted line: $\hat{y} = y_i = b_0 + b_1x_i$
- $b_0 = \overline{y} - b_1 \overline{x}$
- $b_1 = s_{xy} / s_x^2$
- Obtain the above values through ordinary least squares (OLS).

- OLS:

$$\min_{b_0,b_1} \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \min_{b_0,b_1} \sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2$$

The first step is to take the derivative of the above function with respect to $b_1$ and set it equal to zero. Let

$$SSE = \sum_{i=1}^{n}(y_i - b_0 - b_1 x_i)^2$$

$$\frac{\partial SSE}{\partial b_1} = -2\sum_{i=1}^{n} x_i(y_i - b_0 - b_1 x_i) = 0$$

Now we take the derivative of the $SSE$ function with respect to $b_0$ and also set it equal to zero

$$\frac{\partial SSE}{\partial b_0} = -2\sum_{i=1}^{n}(y_i - b_0 - b_1 x_i) = 0$$

Dividing out the $-2$, we have two equations and two unknowns

$$\sum_{i=1}^{n} x_i(y_i - b_0 - b_1 x_i) = 0 \qquad (1)$$
$$\sum_{i=1}^{n}(y_i - b_0 - b_1 x_i) = 0 \qquad (2)$$

To solve, take (2) and re-write

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} b_0 - \sum_{i=1}^{n} b_1 x_i = 0$$

Using the fact that $\sum_{i=1}^{n} b_0 = nb_0$, we can write

$$nb_0 = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} b_1 x_i$$

Which we can rewrite as

$$b_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} b_1 x_i = \overline{y} - b_1\overline{x} \qquad (3)$$

Now let's go back to equation (1)

$$\sum_{i=1}^{n} x_i(y_i - b_0 - b_1 x_i) = 0$$

Substituting (3) into (1) yields

$$\begin{aligned}\sum_{i=1}^{n} x_i(y_i - (\overline{y} - b_1\overline{x}) - b_1 x_i) &= 0\\ \sum_{i=1}^{n} x_i(y_i - \overline{y} - b_1(x_i - \overline{x})) &= 0\end{aligned}$$

All that's left to do is solve for $b_1$

To solve for $b_1$ rewrite the previous equation as

$$\sum_{i=1}^{n} x_i(y_i - \overline{y}) - b_1 \sum_{i=1}^{n} x_i(x_i - \overline{x}) = 0$$

Rearranging in terms of $b_1$ yields

$$b_1 = \frac{\sum_{i=1}^{n} x_i(y_i - \overline{y})}{\sum_{i=1}^{n} x_i(x_i - \overline{x})}$$

We can rewrite $b_1$ in terms of quantities that we already know.

Let's start with the numerator $\sum_{i=1}^{n} x_i(y_i - \overline{y})$, multiplying out we get

$$\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \overline{y} = \sum_{i=1}^{n} x_i y_i - n(\overline{y})(\overline{x})$$

Now the for the denominator $\sum_{i=1}^{n} x_i(x_i - \overline{x})$, multiplying out we get

$$\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \overline{x} = \sum_{i=1}^{n} x_i^2 - n\overline{x}^2$$

Therefore

$$b_1 = \frac{\sum_{i=1}^{n} x_i y_i - n(\overline{y})(\overline{x})}{\sum_{i=1}^{n} x_i^2 - n\overline{x}^2}$$

Finally, multiplying and dividing by $\frac{1}{n-1}$ we obtain

$$b_1 = \frac{\frac{1}{n-1}\left[\sum_{i=1}^{n} x_i y_i - n(\overline{y})(\overline{x})\right]}{\frac{1}{n-1}\left[\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right]} = \frac{s_{xy}}{s_x^2} \qquad (4)$$

Therefore we can always find the slope coefficient that minimises the sum of squared errors by dividing the sample covariance of $x$ and $y$ by the sample variance of $x$! Then to find the intercept:

$$b_0 = \overline{y} - b_1\overline{x} \qquad (5)$$

- So, given a set of n observations on variables x and y, we obtain a line of best fit by:
    1. Computing $\overline{x}$ and $\overline{y}$
    2. Then, computing $s_{xy}$ and $s_x^2$
    3. Finally, computing $b_1 = s_{xy} / s_x^2$ and $b_0 = \overline{y} - b_1 \overline{x}$
- Multiply $b_1$ by $s_x / s_y$ to get the coefficient of correlation, r.

**The degree of linear relationship**

- $\Sigma(y_i - \overline{y})^2 = \Sigma(y_i - \hat{y}_i)^2 + \Sigma(\hat{y}_i - \overline{y})^2$ [SST = SSE +SSR]
- SST = total variation of the dependent variable y
- SSE = sum of squares for errors (unexplained)
- SSR = sum of squares for regression (explained)

**Coefficient of determination $R^2$**

- $R^2$ = SSR/SST = 1 – SSE/SST = degree of linear relationship
- Lies within the range: $0 \leq R^2 \leq 1$
- $R^2 = s_{xy}^2/s_x^2 s_y^2 = r^2$

## 5 PROBABILITY

**Set operations**

- Union: $A \cup B$
- Intersection: $A \cap B$
- Difference: $A \backslash B$ = elements in $A$ that are not in $B$.
- Complement: $\overline{A} = \varphi \backslash A$ (where $\varphi$ = sample size).
- Subset: $A \subset B$, every element of $A$ is contained in $B$.
- Superset: $A \supset B$, every element in $B$ is contained in $A$.

**Rules for set operations**

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \backslash B = A \cap \overline{B}$

**Axioms of probability – implications**

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- Mutually exclusive events: $P(A \cap B) = 0$
- Independent events: $P(A \cap B) = P(A) \times P(B)$
  $P(A|B) = P(A)$

**Marginal probability**

- The probability of the event occurring e.g. $P(A)$

**Joint probability**

- $P(A \cap B)$

**Conditional probability**

- $P(A|B)$

**Independent events**

- $P(A \cap B) = P(A) \times P(B)$: the joint probability is equal to the product of its marginal.
- $P(A|B) = P(A)$: the conditional probability is equal to the marginal probability.

## 6 DISCRETE RANDOM VARIABLES

**Discrete bivariate distributions**

| X/Y | $y_1$ | $y_2$ | P(x) |
|---|---|---|---|
| $x_1$ | $(x_1, y_1)$ | $(x_1, y_2)$ | $P(x_1)$ |
| $x_2$ | $(x_2, y_1)$ | $(x_2, y_2)$ | $P(x_2)$ |
| P(Y) | $P(y_1)$ | $P(y_2)$ | 1 |

- Where $(x_1, y_1) = P(x_1 \cap y_1)$
- Expectation: $E[X + Y] = E[X] + E[Y]$
- Variance: $V(X + Y) = V(X) + V(Y) + 2COV(X,Y)$
  $V(X - Y) = V(X) + V(Y) - 2COV(X,Y)$
  For independent X and Y then: $V(X + Y) = V(X - Y) = V(X) + V(Y)$

**The Bernoulli distribution**

- Two possible outcomes: success or failure
- $B = \begin{cases} 1, & \text{with probability p} \\ 0, & \text{with probability q= 1 - p} \end{cases}$
- Mean, $E[B] = p$, and variance $\sigma_B^2 = p(1 – p)$

**Binomial distribution**

- Sequence of Bernoulli's trials with probability $p^x(1 – p)^{n – x}$
- ${}^nc_x = \frac{n!}{x!\,(n - x)!}$

- P(X = x) = $\left[\frac{n!}{x!\,(n-x)!}\right]$ x [ $p^x(1-p)^{n-x}$ ]
- E[X] = np, V[X] = np(1 – p)

**Poisson distribution**

- Let $\lambda$ be the average number of occurrences
- P(X = x) = $\frac{e^{-\lambda}\lambda^x}{x!}$
- E[X] = $\lambda$, V[X] = $\lambda$

## 7 CONTINOUS PROBABILITY

**Uniform distribution**

- $f_x(x)$¿ $\begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$
- Cumulative distribution: $F_x(x) = \frac{x-a}{b-a}$
- E[X] = $\frac{b+a}{2}$, V[X] = $\frac{(b-a)^2}{12}$

**Normal distribution**

- P( a ≤ x ≤ b) = $\int_a^b f_x(x)dx$ = P( $\frac{a-\mu}{\sigma} \le z \le \frac{b-\mu}{\sigma}$ )
- Z = $\frac{x-\mu}{\sigma}$

## 8 INFERENCE ON $\mu$

**The sample mean**

- Law of large numbers: $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mu$ as n $\to \infty$
- Sample mean = point estimate

**Sampling distribution**

- Central limit theorem: as n $\to \infty$

  $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \xrightarrow{\text{distribution}} N(\mu, \frac{\sigma^2}{n})$

**Confidence intervals**

- (1 – $\alpha$)% confidence interval
- Known $\sigma^2$: $\overline{x} \pm z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ (normal, z score)

- Unknown $\sigma^2$: $\bar{x} \pm t_{\frac{\alpha}{2}, (n-1)} \times \frac{s}{\sqrt{n}}$, where s = sample variance, when n ≥ 30 the student t converges to a normal: $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$
- Population proportion: $\hat{p} \pm z_{\frac{\alpha}{2}} \frac{\sqrt{\hat{p}(1-\hat{p})}}{n}$

**Estimating population proportion p**

- $\hat{p} = \frac{Y}{n}$, where Y = number of observations in the sample that belong to the category of interest.
- $E[\hat{p}] = E[\frac{Y}{n}] = p$ and $V[\hat{p}] = \frac{p(1-p)}{n}$, because Y is binomial, therefore $\frac{Y}{n} \to p$ as $n \to \infty$
- $\frac{Y}{n} \xrightarrow{\text{distribution}} N(p, \frac{p(1-p)}{n})$
- $\frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})/n}} \sim N(0,1)$

**Comparing two populations (known $\sigma^2$)**

- Distribution of difference of sample means
  $\bar{x}_1 - \bar{x}_2 \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$
- $Z = [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)] / \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
- Confidence interval: $(\bar{x}_1 - \bar{x}_2) \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

**Comparing two populations (unknown $\sigma^2$)**

- Replace with sample variance, student t
- Degrees of freedom: $\frac{(s_1^2/n_1 + s_2^2/n_2)^2}{\frac{(s_1^2/n_1)^2}{n_1 - 1} + \frac{(s_2^2/n_2)^2}{n_2 - 1}}$
- Confidence interval: $(\bar{x}_1 - \bar{x}_2) \pm t_{\frac{\alpha}{2}, \text{d.f.}} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

## 9 HYPOTHESIS TESTING

**Testing hypotheses on $\mu$**

- $H_0$: $\mu = k$
- $H_A$: $\mu \neq k$ (two tailed test)
- $H_A$: $\mu > k$ (one tailed test)
- $H_A$: $\mu < k$ (one tailed test)
- Decide on a significance level $\alpha$

- Convert relevant test statistic (e.g. $\bar{x}$ ) into test statistic e.g. z score or t score and then compare.
- Or use probability

**Type I and Type II errors**

- Type I: reject the null when the null is true (false positive, innocent person to jail)
- Type II: fail to reject the null when the null is false (false negative, guilty person walks free).

**Testing on a population proportion p**

- $H_0$: p = c
- $H_A$: p ≠ c
- Z = $\frac{\hat{p}-c}{\sqrt{\frac{c(1-c)}{n}}}$ ~ N (0,1)

**Testing hypotheses on $\sigma^2$**

- $H_0$: $\sigma^2 = \sigma_0^2$
- $H_A$: $\sigma^2 \neq \sigma_0^2$ (two tailed test)
- $H_A$: $\sigma^2 > \sigma_0^2$ (one tailed test)
- $H_A$: $\sigma^2 < \sigma_0^2$ (one tailed test)
- $\frac{(n-1)s^2}{\sigma_0^2}$ is a $\chi^2$ random variable
- chi-squared critical values: $\left[\frac{(n-1)s^2}{\chi^2_{k},\frac{\alpha}{2}}, \frac{(n-1)s^2}{\chi^2_{k},1-\frac{\alpha}{2}}\right]$

## 10 INFERENCE ON $\sigma^2$

**The sampling distribution of $s^2$**

- Chi-squared distribution with k = n – 1 degrees of freedom.
- $\sigma^2$: parameter of interest
- $s^2$: estimator
- Sampling distribution = $\frac{(n-1)s^2}{\sigma_0^2}$ ~ $\chi^2$(k)

**Confidence interval for $s^2$**

- Note: $\chi^2$(k) density function is not symmetric
- P($\chi^2$(k), 1 - $\frac{\alpha}{2}$ < $\frac{(n-1)s^2}{\sigma_0^2}$ < $\chi^2$(k), $\frac{\alpha}{2}$ ) = 1 – $\alpha$ %
- Re-arrange to centre solely around unknown $\sigma^2$
- P$\left[\frac{(n-1)s^2}{\chi^2_{k},\frac{\alpha}{2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{k},1-\frac{\alpha}{2}}\right]$ = 1 – $\alpha$ %

- Therefore, confidence interval for $\sigma^2$ is given by: $\left[\frac{(n-1)s^2}{\chi^2_{k}, \frac{\alpha}{2}}, \frac{(n-1)s^2}{\chi^2_{k}, 1-\frac{\alpha}{2}}\right]$

## 11 BIVARIATE DISTRIBUTION FUNCTIONS & CONDITIONAL MEANS

**Covariation and conditional means**

- Conditional mean of Y given a particular value of X is:

E[Y|X = $x_i$] = $f(x) = \begin{cases} \sum_{i=0}^{n} y_i \cdot P(Y = y_i | X = x_i), \wedge [\text{discrete}] \\ \int_{-\infty}^{\infty} y f(y|X = x_i) dy, \wedge\ [\text{continuous}] \end{cases}$

**The regression model**

- Y = $\beta_0 + \beta_1 X + \varepsilon$
- $\varepsilon \sim N(0, \sigma^2)$
- $\hat{\beta}_1 = s_{xy} / s_x^2$
- $\hat{\beta}_0 = \overline{Y} - \hat{\beta}_1 \overline{X}$

**Sampling distribution of $\hat{\beta}_1$**

- E[$\hat{\beta}_1$] = $\beta_1$
- $s_{\hat{\beta}_1} = \frac{s\varepsilon}{\sqrt{(n-1)s_x^2}}$ where $s\varepsilon = \sqrt{\frac{\Sigma(y_i - \hat{y})^2}{n - 2}}$
- $\frac{\hat{\beta}_1 - \beta_1}{s_{\hat{\beta}_1}} \sim t_{n-2}$

**Sampling distribution of $\hat{\beta}_0$**

- E[$\hat{\beta}_0$] = $\beta_0$
- $s_{\hat{\beta}_1} = \frac{s\varepsilon \sqrt{\frac{\Sigma x_i^2}{n}}}{\sqrt{(n-1)\ s_x^2}}$
- $\frac{\hat{\beta}_0 - \beta_0}{s_{\hat{\beta}_0}} \sim t_{n-2}$

**Interval estimators and hypotheses tests on $\beta_1$ and $\beta_0$**

- $(1 - \alpha)$ % interval estimator is given by:
  $\hat{\beta}_1 \pm t_{\frac{\alpha}{2}, n-2}\ s_{\hat{\beta}_1}$
  $\hat{\beta}_0 \pm t_{\frac{\alpha}{2}, n-2}\ s_{\hat{\beta}_0}$
- Hypothesis tests: whether or not a linear relationship exists between X and Y;
  $H_0$: $\beta_1 = 0$
  $H_A$: $\beta_1 \neq 0$ (two tailed test)

# SUMMATION NOTATION

1. Definition: $\sum_{i=1}^{n} R_i = R_1 + R_2 + \ldots + R_n$
2. $\sum_{i=1}^{n} bR_i = b\sum_{i=1}^{n} R_i$, where $b$ is any value that does not depend on the index $i$.
3. $\sum_{i=1}^{n} k = nk$, where $k$ is any value that does not depend on the index $i$.
4. $\sum_{i=1}^{n}(R_i \pm a) = \sum_{i=1}^{n} R_i \pm na$, where $a$ is any value that does not depend on the index $i$.
5. Using the above rules we can write:

$$\sum_{i=1}^{N}(X_i - \mu)^2 P(X_i) = \sum_{i=1}^{N} X_i^2 P(X_i) - \mu^2$$

$$\sum_{i=1}^{N_Y}\sum_{j=1}^{N_Y}(X - \mu_X)(Y - \mu_Y)P(X_i, Y_j) = \sum_{i=1}^{N_Y}\sum_{j=1}^{N_Y} X_i Y_j P(X_i, Y_j) - \mu_X \mu_Y$$

# DESCRIPTIVE STATISTICS

1. Sample mean (sample of size $n$): $\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$
2. Sample variance (sample of size $n$): $s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$
3. Sample skewness (sample of size $n$): $g = \frac{n^2}{(n-1)(n-2)} \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^3}{\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\overline{x})^2\right)^{\frac{3}{2}}}$
4. Sample excess kurtosis (sample of size $n$):

$$k = \frac{(n+1)n}{(n-1)(n-2)(n-3)} \frac{\sum_{i=1}^{n}(x_i - \overline{x})^4}{s^4} - 3\frac{(n-1)^2}{(n-2)(n-3)}$$

5. Approximate percentile location (sample of size $n$): $L_p = (n+1)\frac{p}{100}$

# POPULATION MOMENTS

1. Mean: $\mu = E[X]$
2. Variance: $\sigma^2 = E[(X - \mu)^2]$
3. Skewness: $\gamma_1 = E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$
4. Ex. Kurtosis: $\gamma_2 = E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] - 3$

| Variable | Density (Mass) Function | Mean | Variance | Skewness | Ex. Kurtosis |
|---|---|---|---|---|---|
| Binomial | $\frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$ | $np$ | $np(1-p)$ | $\frac{1-2p}{\sqrt{np(1-p)}}$ | $\frac{1-6p(1-p)}{np(1-p)}$ |
| Uniform | $\begin{matrix} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{matrix}$ | $\frac{1}{2}(a+b)$ | $\frac{1}{12}(b-a)^2$ | 0 | $-\frac{6}{5}$ |
| Bernoulli | $\begin{matrix} p & x=1 \\ (1-p) & x=0 \end{matrix}$ | $p$ | $p(1-p)$ | $\frac{1-2p}{\sqrt{p(1-p)}}$ | $\frac{1-6p(1-p)}{p(1-p)}$ |
| Poisson | $\frac{\lambda^x e^{-\lambda}}{x!}$ | $\lambda$ | $\lambda$ | $\frac{1}{\sqrt{\lambda}}$ | $\frac{1}{\lambda}$ |
| Normal | $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ | $\mu$ | $\sigma^2$ | 0 | 0 |
| Student $t$ | $\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$ | 0 | $\frac{\nu}{\nu-2}$ | 0 | $\frac{6}{\nu-4}$ |
| Chi-squared | $\begin{matrix} \frac{x^{\frac{k}{2}-1}e^{\frac{-x}{2}}}{2^{\frac{k}{2}}\Gamma(\frac{k}{2})} & x \geq 0 \\ 0 & \text{otherwise} \end{matrix}$ | $k$ | $2k$ | $\sqrt{\frac{8}{k}}$ | $\frac{12}{k}$ |

# PROBABILITY THEORY

1. Definition: If each simple event in a finite sample space $S$ has the same chance of occurring, then the probability that an event $A$ will occur is

$$P(A) = \frac{\text{number of simple events in } A}{\text{number of simple events in } S}$$

2. Addition Rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or, equivalently,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note that

$$P(A \cup B) = P(A) + P(B) \text{ if } A \text{ and } B \text{ are mutually exclusive}$$

3. Conditional Probability: $P(A|B) = \dfrac{P(A \text{ and } B)}{P(B)}$ provided that $P(B) > 0$.

4. Multiplication Rule:

$$P(A \text{ and } B) = P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

Note that

$$P(A \cap B) = P(A)P(B) \text{ if } A \text{ and } B \text{ are independent events}$$

5. Expectation: $\mu = E[X] = \sum_{i=1}^{K} X_i P(X_i)$ or $\mu = \int_{-\infty}^{\infty} x f(x) dx$

$$\mu = E[X] = \begin{cases} \displaystyle\sum_{i=1}^{N} X_i P(X_i), & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} x f(x) dx, & X \text{ continuous} \end{cases}$$

6. Variance:

$$\sigma^2 = E[(X-\mu)^2] = \begin{cases} \displaystyle\sum_{i=1}^{N} (X_i - \mu)^2 P(X_i), & X \text{ discrete} \\ \displaystyle\int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx & X \text{ continuous} \end{cases}$$

7. Covariance:

$$\sigma_{XY}^2 = COV[X,Y] = E[(X-\mu_X)(Y-\mu_Y)]$$

# EXPECTED VALUES

Let $X$ and $Y$ be random variables and let $a$, $b$ and $c$ denote constants with respect to the distributions of $X$ and $Y$. Then

1. $E[c] = c$
2. $E[cX] = cE[X]$
3. $E[X \pm Y] = E[X] \pm E[Y]$
4. $E[aX \pm bY] = aE[X] \pm bE[Y]$
5. $E[XY] = E[X]E[Y]$ if $X$ and $Y$ are independent
6. $V[c] = 0$
7. $V[cX] = c^2 V[X]$
8. $V[X + b] = V[X]$
9. $V[X \pm Y] = V[X] + V[Y] \pm 2COV[X,Y]$
10. $V[aX \pm bY] = a^2 V[X] + b^2 V[Y] \pm 2abCOV[X,Y]$

# MEASURING RELATIONSHIPS

Suppose that the exists two discrete random variables $X$ and $Y$ where $X$ can take on $k$ possible realisations and $Y$ can take on $l$ possible realisations. Then:

1. Population covariance: $\sigma_{XY}^2 = COV[X,Y] = \displaystyle\sum_{i=1}^{k} \sum_{j=1}^{l} (X_i - \mu_X)(Y_j - \mu_Y)$
2. Population correlation: $\rho = \dfrac{COV[X,Y]}{\sigma_X \sigma_Y}$

Suppose that a sample is comprised of $n$ pairs of observations on the variable $X, Y$. Then:

1. Sample covariance: $cov[x,y] = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})$
2. Sample correlation: $r = \frac{cov[x,y]}{s_x s_y}$

## Central Limit Theorem & Law of Large Numbers

Given a set or sequence of independent random variables $\{X_1, X_2, \ldots, X_n\}$ where for each $i$, $E[X_i] = \mu_X$ and $E[(X_i - \mu_X)^2] = \sigma_X^2$ the Central Limit Theorem states that

$$\frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X} \sim \mathcal{N}\left(\mu_X, \frac{\sigma_X^2}{n}\right)$$

As $n \to \infty$

Given a set of observations $\{x_1, x_2, \ldots, x_n\}$ where each $x_i$ is an observation from a random variable $X_i$ where for each $i$, $E[X_i] = \mu_X$ then the Law of Large Numbers states that

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \to \mu_X$$

As $n \to \infty$

# REGRESSION ANALYSIS

## Simple Linear Regression Model

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

so that

$$\mu_{Y|X} = E[Y|X] = \beta_0 + \beta_1 X$$

## Line of Best Fit

Suppose that a sample is comprised of $n$ pairs of observations on the variable $X, Y$, then the line of best fit is defined as

$$\hat{y}_i = b_0 + b_1 x_i$$

where for each $i$

$$y_i = b_0 + b_1 x_i + e_i$$

## Some Formulae

1. Total Sum of squares: $SST = \sum_{i=1}^{n}(y_i - \overline{y})^2$

2. Sum of squared errors: $SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n}(y_i - \widehat{y}_i)^2$

3. Sum of squares of regression: $SSR = \sum_{i=1}^{n}(\widehat{y}_i - \overline{y})^2$

4. $SST = SSR + SSE$

5. Standard error of the estimate: $s_e = \sqrt{\frac{SSE}{n-k-1}}$

6. Sampling distribution for $\hat{\beta}_j$: $\hat{\beta}_j \sim N\left(\beta_j, \sigma^2_{\hat{\beta}_j}\right)$, $j = 0, 1, \ldots, k$.

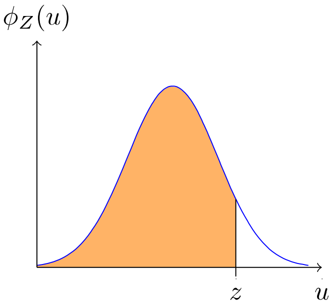
7. $t$ statistic for testing $H_0 : \beta_j = \beta_j^0$: $t = \frac{\hat{\beta}_j - \beta_j^0}{s_{\hat{\beta}_j}} \sim t_{n-k-1}$, $j = 0, 1, \ldots, k$.

8. Degrees of freedom term for the sampling distribution of the difference of means where the population variances are unknown:

$$d.f. = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{\left(s_1^2/n_1\right)^2}{n_1-1} + \frac{\left(s_2^2/n_2\right)^2}{n_2-1}}$$

9. Coefficient of Determination: $R^2 = 1 - \frac{SSE}{SST}$

Table 1: Values of the Standard Normal Distribution Function $\Phi_Z(z)$




Shaded Area: $\Phi_Z(z) = \int_{-\infty}^{z} \phi_Z(u)\,\mathrm{d}u$, where $\phi_Z(u)$ denotes a standard Normal density function.

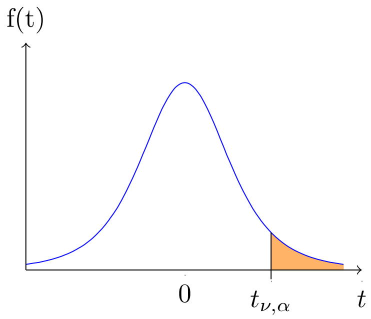
For example, $P(Z \leq 1.05) = 0.8531$.

By symmetry, if $z < 0$ then $\Phi_Z(z) = 1 - \Phi_Z(-z)$

Note that $z$ is the sum of terms in the first row and column.

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.00 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.10 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.20 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.30 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.40 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.50 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.60 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.70 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.80 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.90 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.00 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.10 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.20 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.30 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.40 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.50 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.60 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.70 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.80 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.90 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.00 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.10 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.20 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.30 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.40 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.50 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.60 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.70 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.80 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.90 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.00 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.10 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.20 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.30 | 0.9995 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.40 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 |
| 3.50 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.60 | 0.9998 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.70 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.80 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |
| 3.90 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |

Table 2: Critical Values for Upper-Tail Probabilities of Student's $t$ distribution



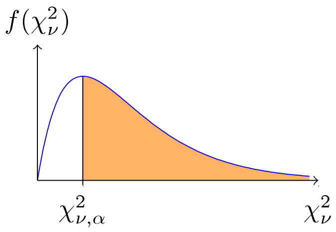

Shaded Area: $P(t \geq t_{\nu,\alpha}) = \alpha$, where $t \sim t_\nu$.

For example, $P(t \geq t_{3,0.05}) = P(t \geq 2.3534) = 0.05$.

Note that $\nu$ denotes the degrees of freedom of the distribution.

| $\nu$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 | $\nu$ | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 3.0777 | 6.3137 | 12.7062 | 31.8210 | 63.6559 | 28 | 1.3125 | 1.7011 | 2.0484 | 2.4671 | 2.7633 |
| 2 | 1.8856 | 2.9200 | 4.3027 | 6.9645 | 9.9250 | 29 | 1.3114 | 1.6991 | 2.0452 | 2.4620 | 2.7564 |
| 3 | 1.6377 | 2.3534 | 3.1824 | 4.5407 | 5.8408 | 30 | 1.3104 | 1.6973 | 2.0423 | 2.4573 | 2.7500 |
| 4 | 1.5332 | 2.1318 | 2.7765 | 3.7469 | 4.6041 | 31 | 1.3095 | 1.6955 | 2.0395 | 2.4528 | 2.7440 |
| 5 | 1.4759 | 2.0150 | 2.5706 | 3.3649 | 4.0321 | 32 | 1.3086 | 1.6939 | 2.0369 | 2.4487 | 2.7385 |
| 6 | 1.4398 | 1.9432 | 2.4469 | 3.1427 | 3.7074 | 33 | 1.3077 | 1.6924 | 2.0345 | 2.4448 | 2.7333 |
| 7 | 1.4149 | 1.8946 | 2.3646 | 2.9979 | 3.4995 | 34 | 1.3070 | 1.6909 | 2.0322 | 2.4411 | 2.7284 |
| 8 | 1.3968 | 1.8595 | 2.3060 | 2.8965 | 3.3554 | 35 | 1.3062 | 1.6896 | 2.0301 | 2.4377 | 2.7238 |
| 9 | 1.3830 | 1.8331 | 2.2622 | 2.8214 | 3.2498 | 36 | 1.3055 | 1.6883 | 2.0281 | 2.4345 | 2.7195 |
| 10 | 1.3722 | 1.8125 | 2.2281 | 2.7638 | 3.1693 | 37 | 1.3049 | 1.6871 | 2.0262 | 2.4314 | 2.7154 |
| 11 | 1.3634 | 1.7959 | 2.2010 | 2.7181 | 3.1058 | 38 | 1.3042 | 1.6860 | 2.0244 | 2.4286 | 2.7116 |
| 12 | 1.3562 | 1.7823 | 2.1788 | 2.6810 | 3.0545 | 39 | 1.3036 | 1.6849 | 2.0227 | 2.4258 | 2.7079 |
| 13 | 1.3502 | 1.7709 | 2.1604 | 2.6503 | 3.0123 | 40 | 1.3031 | 1.6839 | 2.0211 | 2.4233 | 2.7045 |
| 14 | 1.3450 | 1.7613 | 2.1448 | 2.6245 | 2.9768 | 45 | 1.3007 | 1.6794 | 2.0141 | 2.4121 | 2.6896 |
| 15 | 1.3406 | 1.7531 | 2.1315 | 2.6025 | 2.9467 | 50 | 1.2987 | 1.6759 | 2.0086 | 2.4033 | 2.6778 |
| 16 | 1.3368 | 1.7459 | 2.1199 | 2.5835 | 2.9208 | 60 | 1.2958 | 1.6706 | 2.0003 | 2.3901 | 2.6603 |
| 17 | 1.3334 | 1.7396 | 2.1098 | 2.5669 | 2.8982 | 70 | 1.2938 | 1.6669 | 1.9944 | 2.3808 | 2.6479 |
| 18 | 1.3304 | 1.7341 | 2.1009 | 2.5524 | 2.8784 | 80 | 1.2922 | 1.6641 | 1.9901 | 2.3739 | 2.6387 |
| 19 | 1.3277 | 1.7291 | 2.0930 | 2.5395 | 2.8609 | 90 | 1.2910 | 1.6620 | 1.9867 | 2.3685 | 2.6316 |
| 20 | 1.3253 | 1.7247 | 2.0860 | 2.5280 | 2.8453 | 100 | 1.2901 | 1.6602 | 1.9840 | 2.3642 | 2.6259 |
| 21 | 1.3232 | 1.7207 | 2.0796 | 2.5176 | 2.8314 | 120 | 1.2886 | 1.6576 | 1.9799 | 2.3578 | 2.6174 |
| 22 | 1.3212 | 1.7171 | 2.0739 | 2.5083 | 2.8188 | 140 | 1.2876 | 1.6558 | 1.9771 | 2.3533 | 2.6114 |
| 23 | 1.3195 | 1.7139 | 2.0687 | 2.4999 | 2.8073 | 160 | 1.2869 | 1.6544 | 1.9749 | 2.3499 | 2.6069 |
| 24 | 1.3178 | 1.7109 | 2.0639 | 2.4922 | 2.7970 | 180 | 1.2863 | 1.6534 | 1.9732 | 2.3472 | 2.6034 |
| 25 | 1.3163 | 1.7081 | 2.0595 | 2.4851 | 2.7874 | 200 | 1.2858 | 1.6525 | 1.9719 | 2.3451 | 2.6006 |
| 26 | 1.3150 | 1.7056 | 2.0555 | 2.4786 | 2.7787 | $\infty$ | 1.2816 | 1.6449 | 1.9600 | 2.3263 | 2.5758 |
| 27 | 1.3137 | 1.7033 | 2.0518 | 2.4727 | 2.7707 | | | | | | |

Table 3: Critical Values for Upper-Tail Probabilities of the Chi-square Distribution



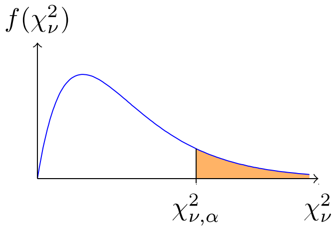

Shaded Area: $P(\chi^2_\nu \geq \chi^2_{\nu,\alpha}) = \alpha$.

For example, $P(\chi^2_3 \geq \chi^2_{3,0.95}) = P(\chi^2_\nu \geq 0.3518) = 0.95$.

Note that $\nu$ denotes the degrees of freedom of the distribution.

| | Upper-Tail Probabilities ($\alpha$) | | | | | | |
|---|---|---|---|---|---|---|---|
| $\nu$ | 0.9950 | 0.9900 | 0.9750 | 0.9500 | 0.9000 | 0.7500 | 0.5000 |
| 1 | $3.927 \times 10^{-5}$ | 0.0002 | 0.0010 | 0.0039 | 0.0158 | 0.1015 | 0.4549 |
| 2 | 0.0100 | 0.0201 | 0.0506 | 0.1026 | 0.2107 | 0.5754 | 1.3863 |
| 3 | 0.0717 | 0.1148 | 0.2158 | 0.3518 | 0.5844 | 1.2125 | 2.3660 |
| 4 | 0.2070 | 0.2971 | 0.4844 | 0.7107 | 1.0636 | 1.9226 | 3.3567 |
| 5 | 0.4117 | 0.5543 | 0.8312 | 1.1455 | 1.6103 | 2.6746 | 4.3515 |
| 6 | 0.6757 | 0.8721 | 1.2373 | 1.6354 | 2.2041 | 3.4546 | 5.3481 |
| 7 | 0.9893 | 1.2390 | 1.6899 | 2.1673 | 2.8331 | 4.2549 | 6.3458 |
| 8 | 1.3444 | 1.6465 | 2.1797 | 2.7326 | 3.4895 | 5.0706 | 7.3441 |
| 9 | 1.7349 | 2.0879 | 2.7004 | 3.3251 | 4.1682 | 5.8988 | 8.3428 |
| 10 | 2.1559 | 2.5582 | 3.2470 | 3.9403 | 4.8652 | 6.7372 | 9.3418 |
| 11 | 2.6032 | 3.0535 | 3.8157 | 4.5748 | 5.5778 | 7.5841 | 10.3410 |
| 12 | 3.0738 | 3.5706 | 4.4038 | 5.2260 | 6.3038 | 8.4384 | 11.3403 |
| 13 | 3.5650 | 4.1069 | 5.0088 | 5.8919 | 7.0415 | 9.2991 | 12.3398 |
| 14 | 4.0747 | 4.6604 | 5.6287 | 6.5706 | 7.7895 | 10.1653 | 13.3393 |
| 15 | 4.6009 | 5.2293 | 6.2621 | 7.2609 | 8.5468 | 11.0365 | 14.3389 |
| 16 | 5.1422 | 5.8122 | 6.9077 | 7.9616 | 9.3122 | 11.9122 | 15.3385 |
| 17 | 5.6972 | 6.4078 | 7.5642 | 8.6718 | 10.0852 | 12.7919 | 16.3382 |
| 18 | 6.2648 | 7.0149 | 8.2307 | 9.3905 | 10.8649 | 13.6753 | 17.3379 |
| 19 | 6.8440 | 7.6327 | 8.9065 | 10.1170 | 11.6509 | 14.5620 | 18.3377 |
| 20 | 7.4338 | 8.2604 | 9.5908 | 10.8508 | 12.4426 | 15.4518 | 19.3374 |
| 21 | 8.0337 | 8.8972 | 10.2829 | 11.5913 | 13.2396 | 16.3444 | 20.3372 |
| 22 | 8.6427 | 9.5425 | 10.9823 | 12.3380 | 14.0415 | 17.2396 | 21.3370 |
| 23 | 9.2604 | 10.1957 | 11.6886 | 13.0905 | 14.8480 | 18.1373 | 22.3369 |
| 24 | 9.8862 | 10.8564 | 12.4012 | 13.8484 | 15.6587 | 19.0373 | 23.3367 |
| 25 | 10.5197 | 11.5240 | 13.1197 | 14.6114 | 16.4734 | 19.9393 | 24.3366 |
| 26 | 11.1602 | 12.1981 | 13.8439 | 15.3792 | 17.2919 | 20.8434 | 25.3365 |
| 27 | 11.8076 | 12.8785 | 14.5734 | 16.1514 | 18.1139 | 21.7494 | 26.3363 |
| 28 | 12.4613 | 13.5647 | 15.3079 | 16.9279 | 18.9392 | 22.6572 | 27.3362 |
| 29 | 13.1211 | 14.2565 | 16.0471 | 17.7084 | 19.7677 | 23.5666 | 28.3361 |
| 30 | 13.7867 | 14.9535 | 16.7908 | 18.4927 | 20.5992 | 24.4776 | 29.3360 |
| 35 | 17.1918 | 18.5089 | 20.5694 | 22.4650 | 24.7967 | 29.0540 | 34.3356 |
| 40 | 20.7065 | 22.1643 | 24.4330 | 26.5093 | 29.0505 | 33.6603 | 39.3353 |
| 50 | 27.9907 | 29.7067 | 32.3574 | 34.7643 | 37.6886 | 42.9421 | 49.3349 |
| 60 | 35.5345 | 37.4849 | 40.4817 | 43.1880 | 46.4589 | 52.2938 | 59.3347 |
| 70 | 43.2752 | 45.4417 | 48.7576 | 51.7393 | 55.3289 | 61.6983 | 69.3345 |
| 80 | 51.1719 | 53.5401 | 57.1532 | 60.3915 | 64.2778 | 71.1445 | 79.3343 |
| 100 | 67.3276 | 70.0649 | 74.2219 | 77.9295 | 82.3581 | 90.1332 | 99.3341 |

Critical Values for Upper-Tail Probabilities of the $\chi^2_\nu$ Distribution (Table 3 continued)




Shaded Area: $P(\chi^2_\nu \geq \chi^2_{\nu,\alpha}) = \alpha$.

For example, $P(\chi^2_3 \geq \chi^2_{3,0.05}) = P(\chi^2_\nu \geq 7.8147) = 0.05$.

Note that $\nu$ denotes the degrees of freedom of the distribution.

| | Upper Tail Probabilities ($\alpha$) | | | | | |
|---|---|---|---|---|---|---|
| $\nu$ | 0.2500 | 0.1000 | 0.0500 | 0.0250 | 0.0100 | 0.0050 |
| 1 | 1.3233 | 2.7055 | 3.8415 | 5.0239 | 6.6349 | 7.8794 |
| 2 | 2.7726 | 4.6052 | 5.9915 | 7.3778 | 9.2103 | 10.5966 |
| 3 | 4.1083 | 6.2514 | 7.8147 | 9.3484 | 11.3449 | 12.8382 |
| 4 | 5.3853 | 7.7794 | 9.4877 | 11.1433 | 13.2767 | 14.8603 |
| 5 | 6.6257 | 9.2364 | 11.0705 | 12.8325 | 15.0863 | 16.7496 |
| 6 | 7.8408 | 10.6446 | 12.5916 | 14.4494 | 16.8119 | 18.5476 |
| 7 | 9.0371 | 12.0170 | 14.0671 | 16.0128 | 18.4753 | 20.2777 |
| 8 | 10.2189 | 13.3616 | 15.5073 | 17.5345 | 20.0902 | 21.9550 |
| 9 | 11.3888 | 14.6837 | 16.9190 | 19.0228 | 21.6660 | 23.5894 |
| 10 | 12.5489 | 15.9872 | 18.3070 | 20.4832 | 23.2093 | 25.1882 |
| 11 | 13.7007 | 17.2750 | 19.6751 | 21.9200 | 24.7250 | 26.7568 |
| 12 | 14.8454 | 18.5493 | 21.0261 | 23.3367 | 26.2170 | 28.2995 |
| 13 | 15.9839 | 19.8119 | 22.3620 | 24.7356 | 27.6882 | 29.8195 |
| 14 | 17.1169 | 21.0641 | 23.6848 | 26.1189 | 29.1412 | 31.3193 |
| 15 | 18.2451 | 22.3071 | 24.9958 | 27.4884 | 30.5779 | 32.8013 |
| 16 | 19.3689 | 23.5418 | 26.2962 | 28.8454 | 31.9999 | 34.2672 |
| 17 | 20.4887 | 24.7690 | 27.5871 | 30.1910 | 33.4087 | 35.7185 |
| 18 | 21.6049 | 25.9894 | 28.8693 | 31.5264 | 34.8053 | 37.1565 |
| 19 | 22.7178 | 27.2036 | 30.1435 | 32.8523 | 36.1909 | 38.5823 |
| 20 | 23.8277 | 28.4120 | 31.4104 | 34.1696 | 37.5662 | 39.9968 |
| 21 | 24.9348 | 29.6151 | 32.6706 | 35.4789 | 38.9322 | 41.4011 |
| 22 | 26.0393 | 30.8133 | 33.9244 | 36.7807 | 40.2894 | 42.7957 |
| 23 | 27.1413 | 32.0069 | 35.1725 | 38.0756 | 41.6384 | 44.1813 |
| 24 | 28.2412 | 33.1962 | 36.4150 | 39.3641 | 42.9798 | 45.5585 |
| 25 | 29.3389 | 34.3816 | 37.6525 | 40.6465 | 44.3141 | 46.9279 |
| 26 | 30.4346 | 35.5632 | 38.8851 | 41.9232 | 45.6417 | 48.2899 |
| 27 | 31.5284 | 36.7412 | 40.1133 | 43.1945 | 46.9629 | 49.6449 |
| 28 | 32.6205 | 37.9159 | 41.3371 | 44.4608 | 48.2782 | 50.9934 |
| 29 | 33.7109 | 39.0875 | 42.5570 | 45.7223 | 49.5879 | 52.3356 |
| 30 | 34.7997 | 40.2560 | 43.7730 | 46.9792 | 50.8922 | 53.6720 |
| 35 | 40.2228 | 46.0588 | 49.8018 | 53.2033 | 57.3421 | 60.2748 |
| 40 | 45.6160 | 51.8051 | 55.7585 | 59.3417 | 63.6907 | 66.7660 |
| 50 | 56.3336 | 63.1671 | 67.5048 | 71.4202 | 76.1539 | 79.4900 |
| 60 | 66.9815 | 74.3970 | 79.0819 | 83.2977 | 88.3794 | 91.9517 |
| 70 | 77.5767 | 85.5270 | 90.5312 | 95.0232 | 100.4252 | 104.2149 |
| 80 | 88.1303 | 96.5782 | 101.8795 | 106.6286 | 112.3288 | 116.3211 |
| 90 | 98.6499 | 107.5650 | 113.1453 | 118.1359 | 124.1163 | 128.2989 |
| 100 | 109.1412 | 118.4980 | 124.3421 | 129.5612 | 135.8067 | 140.1695 |