Problem 1

1)

$$H(Y_1) = -\left(\frac{1}{3}\log\frac{1}{3}\right)$$
$$H(Y_2) = -\left(\frac{1}{3}\log\frac{1}{3}\right)$$
$$H(Y_3) = -\left(\frac{1}{3}\log\frac{1}{3}\right)$$

Entropy $H(Y)$ is the expected value of the probability of y:

$$H(Y) = -\sum_y p(y)\log p(y)$$

$$Therefore, H(y) = -\left(\frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3} + \frac{1}{3}\log\frac{1}{3}\right)$$

2)

$$E(S) = -\left(\frac{1}{6}\log_2\frac{1}{6} + \frac{2}{6}\log_2\frac{2}{6} + \frac{3}{6}\log_2\frac{3}{6}\right) = -(-0.43083 - 0.52857 - 0.5) = 1.45940$$

The entropy for X_1 split:

$$E(X_1 = 1) = -\left(\frac{1}{3}\log_2\frac{1}{3} + \frac{1}{3}\log_2\frac{1}{3} + \frac{1}{3}\log_2\frac{1}{3}\right) = 1.58496$$

$$E(X_1 = 0) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}\right) = 1$$

$$E_{\text{weighted}}(X_1) = \frac{4}{6}\cdot E(X_1 = 1) + \frac{2}{6}\cdot E(X_1 = 0) = 1.38997$$
$$IG(X_1) = E(S) - E_{\text{weighted}}(X_1) = 1.45940 - 0.33333 = 1.12617$$

The entropy for X_2 split:

$$E(X_2 = 1) = -\left(\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log_2\frac{1}{2}\right)$$
$$E(X_2 = 1) = -2\left(\frac{1}{2}\log_2\frac{1}{2}\right)$$
$$E(X_2 = 1) = 1$$

For $X_2 = 0$, with 4 instances, the entropy is:

$$E(X_2 = 0) = -\left(\frac{1}{4}\log_2\frac{1}{4} + \frac{1}{4}\log_2\frac{1}{4} + \frac{1}{2}\log_2\frac{1}{2}\right)$$
$$E(X_2 = 0) = 1.5$$

The weighted sum of these entropies is:

$$E_{\text{weighted}}(X_2) = \frac{2}{6}E(X_2 = 1) + \frac{4}{6}E(X_2 = 0)$$
$$E_{\text{weighted}}(X_2) = \frac{1}{3}\cdot(1) + \frac{2}{3}\cdot 1.5$$
$$E_{\text{weighted}}(X_2) = \frac{1}{3} + 1$$
$$E_{\text{weighted}}(X_2) = \frac{4}{3}$$

$$IG(X_2) = E(S) - E_{\text{weighted}}(X_2) = 1.45940 - 1.22221 = 0.23719$$

$$E_{\text{weighted}}(X_2) = \frac{2}{6}\cdot E(X_2 = 1) + \frac{4}{6}\cdot E(X_2 = 0) = 0.33333 + 0.88888 = 1.22221$$

$$IG(X_2) = E(S) - E_{\text{weighted}}(X_2) = 1.45940 - 1.22221 = 0.23719$$

4) It can be determined from the decision tree that when X_1 is equal to 0, Y will be 3 regardless of the value of X_2. Therefore, when X_1 is equal to 0 and X_2 is equal to 1, Y will be equal to 3.