1 Appendix

1.1 Proof of Theorem 1

Shown below are bounds on the warped length scale:

Lemma 1. *As defined by Theorem 6 of [3], for* $\delta \in (0, 1)$ *and under the assumtion that the noise* ϵ_t *is uniformly bounded by* σ*, with* $\beta_t = 2 \parallel f \parallel_k^2 + 300\gamma_t ln^3(t/\delta)$*, then:*

$$p\Big(\forall t, \forall x \in D, \mid \mu_{t-1}(x) - f(x) \mid \leq \sqrt{\beta_t}\sigma_{t-1}(x)\Big) \geq 1 - \delta$$

With this knowledge, proof of Theorem 1 is as follows:

Proof. The warped length scale is calculated as follows:

$$\boldsymbol{l}(\mathbf{x}_t) = \boldsymbol{l}_0 \log\Big(1 + \big(\frac{\mid \mu_{t-1}(x_t) - h \mid + \epsilon}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\big)^2\Big) + \boldsymbol{l}_1$$

Let:

$$\begin{aligned} \mu'_{t-1}(x) =& \mu_{t-1}(x_t) - h \\ f'(x_t) =& f(x_t) - h \end{aligned}$$

The length scale will be less than:

$$\begin{aligned} \boldsymbol{l}(\mathbf{x}) =& \boldsymbol{l}_0 \log\Big(1 + \big(\frac{\mid \mu'_{t-1}(x) \mid + \epsilon}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\big)^2\Big) + \boldsymbol{l}_1 \\ =& \boldsymbol{l}_0 \log\Big(1 + \big(\frac{\mid \mu_{t-1}(x_t) - f(x_t) + f'(x_t) \mid + \epsilon}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\big)^2\Big) \\ & + \boldsymbol{l}_1 \\ \leq& \boldsymbol{l}_0 \log\Big(1 + \big(\frac{\mid \mu_{t-1}(x_t) - f(x_t) \mid + \mid f'(x_t) \mid + \epsilon}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\big)^2\Big) \\ & + \boldsymbol{l}_1, \text{Triangle inequality} \\ \leq& \boldsymbol{l}_0 \log\Big(1 + \big(\frac{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid f'(x_t) \mid + \epsilon}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\big)^2\Big) \\ & + \boldsymbol{l}_1, \text{Lemma 1} \\ \leq& \boldsymbol{l}_0 \log\Big(1 + \Big(1 + \frac{\mid f'(x_t) \mid}{\sqrt{\beta_t}\sigma_{t-1}(x_t) + \epsilon}\Big)^2\Big) + \boldsymbol{l}_1 \\ \leq& \boldsymbol{l}_0 \log\Big(1 + \Big(1 + \frac{\mid f'(x_t) \mid}{\epsilon}\Big)^2\Big) + \boldsymbol{l}_1 \\ \leq& \boldsymbol{l}_0 \log\Big(1 + \Big(1 + \frac{\Delta_{fmax}}{\epsilon}\Big)^2\Big) + \boldsymbol{l}_1 \end{aligned}$$

where $\Delta_{fmax} = \max \mid f(\mathbf{x}) - h \mid$ with probability $\geq 1 - \delta$. Where $\delta \in (0, 1)$.

And intuitively the length scale is greater than:

$$\boldsymbol{l}(\mathbf{x}) \geq \boldsymbol{l}_1$$

With $\boldsymbol{l}_0 \geq 0$.

1.2 Proof of Theorem 2

Lemma 2. *As proven in Lemma 7 of [1], the sum of the predictive variances is bounded by the maximum information gain such that* $\forall x \in X, \sum_{t=1}^{T} \sigma_{t-1}^2(x) \leq \frac{2\gamma_t}{log(1+\sigma^{-2})}$

With this knowledge, proof of Theorem 2 is as follows:

Proof. The ambiguity acquisition function is defined in (3) with sampling done in each iteration according to (4).

Let:

$$\begin{aligned} \mu'_{t-1}(x) =& \mu_{t-1}(x_t) - h \\ \mu'_{t-1}(x^*) =& \mu_{t-1}(x^*) - h \\ f'(x_t) =& f(x_t) - h \\ f'(x^*) =& f(x^*) - h \end{aligned}$$

Regret is defined as:

$$\begin{aligned} r_t &= \mid f(x_t) - h \mid \\ &= \mid f(x_t) - \mu_{t-1}(x_t) + \mu'_{t-1}(x) \mid \\ &\leq \mid f(x_t) - \mu_{t-1}(x_t) \mid + \mid \mu'_{t-1}(x) \mid, && \text{Triangle inequality} \\ &\leq \sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu'_{t-1}(x) \mid, && \text{Lemma 1} \end{aligned}$$

$$r_t \leq \sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu_{t-1}(x_t) - h \mid \tag{1}$$

with probability $\geq 1 - \delta$. Where $\delta \in (0, 1)$.

Through the sampling definition in (4), know that $a_{t-1}(x^*) \leq a_{t-1}(x_t) = -\mid \mu'_{t-1}(x) \mid + \sqrt{\beta_t}\sigma_{t-1}(x_t)$. As such, it follows that

$$-\mid \mu'_{t-1}(x^*) \mid + \sqrt{\beta_t}\sigma_{t-1}(x^*) \leq -\mid \mu'_{t-1}(x) \mid + \sqrt{\beta_t}\sigma_{t-1}(x_t)$$

$$-\mid \mu'_{t-1}(x^*) \mid + \sqrt{\beta_t}\sigma_{t-1}(x^*) - \sqrt{\beta_t}\sigma_{t-1}(x_t) \leq -\mid \mu'_{t-1}(x) \mid$$

$$\mid \mu'_{t-1}(x) \mid \leq \mid \mu'_{t-1}(x^*) \mid - \sqrt{\beta_t}\sigma_{t-1}(x^*) + \sqrt{\beta_t}\sigma_{t-1}(x_t) \tag{2}$$

Substituting (2) into (1):

$$
\begin{aligned}
r_t \leq& \sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu'_{t-1}(x) \mid \\
\leq& \sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu'_{t-1}(x^*) \mid - \sqrt{\beta_t}\sigma_{t-1}(x^*) + \\
& \sqrt{\beta_t}\sigma_{t-1}(x_t) \\
=& 2\sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu_{t-1}(x^*) - f(x^*) + f(x^*) - h \mid - \\
& \sqrt{\beta_t}\sigma_{t-1}(x^*) \\
\leq& 2\sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid \mu_{t-1}(x^*) - f(x^*) \mid + \mid f'(x^*) \mid - \\
& \sqrt{\beta_t}\sigma_{t-1}(x^*) \\
\leq& 2\sqrt{\beta_t}\sigma_{t-1}(x_t) + \sqrt{\beta_t}\sigma_{t-1}(x^*) + \mid f'(x^*) \mid - \\
& \sqrt{\beta_t}\sigma_{t-1}(x^*) \\
=& 2\sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid f'(x^*) \mid \\
=& 2\sqrt{\beta_t}\sigma_{t-1}(x_t) + \mid f(x^*) - h \mid
\end{aligned}
$$

with probability $\geq 1 - 2\delta$ (By union bound and De Morgan's Law). Where $\delta \in (0, 1)$.

Now aim to determine cumulative regret:

$$R_T = \sum_{t=1}^{T} r_t$$

To do this:

$$r_t \leq \underbrace{2\sqrt{\beta_t}\sigma_{t-1}(x_t)}_{G_t} + \underbrace{\mid f(x^*) - h \mid}_{H_t}$$

First term:

$$
\begin{aligned}
\sum_{t=1}^{T} G_t^2 =& \sum_{t=1}^{T} [2\sqrt{\beta_t}\sigma_{t-1}(x_t)]^2 \\
=& \sum_{t=1}^{T} 4\beta_t\sigma_{t-1}(x_t)^2 \\
\leq& 4\beta_T \sum_{t=1}^{T} \sigma_{t-1}(x_t)^2, && \text{by } \beta_T \geq \beta_t, \forall t \leq T \\
\leq& \frac{8\beta_T\gamma_t}{log(1+\sigma^{-2})}, && \text{Lemma 2}
\end{aligned}
$$

Applying Cauchy-Schwartz (CS) Inequality:

$$\sum_{t=1}^{T} G_t \leq \sqrt{T}\sqrt{\sum_{t=1}^{T} G_t^2} \leq \sqrt{\frac{8T\beta_T\gamma_t}{log(1+\sigma^{-2})}}$$

Second term:

$$\sum_{t=1}^{T} H_t = \sum_{t=1}^{T} \mid f(x^*) - h \mid = T \mid f(x^*) - h \mid$$

Cumulative Regret $R_T \leq \sum_{t=1}^{T}(G_t + H_t)$:

$$R_T \leq \sqrt{\frac{8T\beta_T\gamma_t}{log(1+\sigma^{-2})}} + T \mid f(x^*) - h \mid$$

Now determining the average cumulative regret limit:

$$\begin{aligned} \lim_{T\to\infty} \frac{R_T}{T} &= \lim_{T\to\infty} \frac{\sqrt{\frac{8T\beta_T\gamma_t}{log(1+\sigma^{-2})}}}{T} + \mid f(x^*) - h \mid \\ &= \mid f(x^*) - h \mid \end{aligned}$$

The limit $\mid f(x^*) - h \mid$ accounts for situations where the function does not exist at the threshold h. In this case, the limit will tend to the function value closest to h.

1.3 Proof of Theorem 3

Proof. To prove Theorem 3, we will first develop a bound on the operator spectrum $\{\lambda_s\}$ of our proposed length scale varying covariance function described in (2) and (6). We then use this bound to generate a bound for the maximum information gain γ_t of our covariance function.

As shown in [2], for the square exponential $k(r) = \exp(-r^2/(2l^2))$, its operator spectrum $\{\lambda_s\}$ w.r.t the normal distribution $\mathcal{N}(0,(4a)^{-1}I)$ $(a > 0)$ will satisfy the following inequality:

$$\lambda_s \leq (2a/A)^{d/2} B^{s^{1/d}}, \tag{3}$$

where

$$\begin{aligned} A &= a + \frac{1}{2l^2} + \sqrt{a^2 + \frac{a}{l^2}}, \\ B &= \frac{1}{2l^2 A} = \frac{1}{2al^2 + 1 + 2l\sqrt{a^2l^2 + a}}. \end{aligned}$$

Note that for our proposed length scale varying covariance function, using Theorem 1, we have $L_1 \leq l \leq L_0 log(1 + (1 + f_{max}/\epsilon)^2) + L_1$. Hence, it is easily seen that,

$$\begin{aligned} (2a/A)^{d/2} &\leq (\frac{2a}{a + 1/(2L_2^2) + \sqrt{a^2 + a/L_2^2}})^{d/2}, \\ B^{s^{1/d}} &\leq (\frac{1}{2aL_1^2 + 1 + 2L_1\sqrt{a^2L_1^2 + a}})^{s^{1/d}} \end{aligned}$$

where $L_2 = L_0 log(1 + (1 + f_{max}/\epsilon)^2) + L_1$. Combining these inequalities and (3), we have a bound on the operator spectrum $\{\lambda_s\}$ of our proposed length scale varying covariance function,

$$\lambda_s \leq c_{new} B_{new}^{s^{1/d}},$$

where

$$c_{new} = (\frac{2a}{a + 1/(2L_2^2) + \sqrt{a^2 + a/L_2^2}})^{d/2}$$
$$B_{new} = \frac{1}{2aL_1^2 + 1 + 2L_1\sqrt{a^2L_1^2 + a}}.$$

Using the exactly same argument as in Section C.3 in [3] where the role of c_{new} is same as the role of c and the role of B_{new} is same as the role of B, we then have a bound for the maximum information gain for our proposed kernel,

$$\gamma_T = \mathcal{O}((\log T)^{d+1}).$$

References

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