Double Integration :-

1] $\int_0^1 dx \int_0^x e^{y/x} dy$.

$= \int_0^1 dx \left[\frac{e^{y/x}}{1/x} \right]_0^x$

$= \int_0^1 x \left[e^{y/x} \right]_0^x dx$

$= \int_0^1 x \left[e^{x/x} - e^{0/x} \right] dx$

$= \int_0^1 x [e-1] dx$

$= (e-1) \int_0^1 x \, dx$

$= (e-1) \left[\frac{x^2}{2} \right]_0^1$

$\boxed{I = \frac{(e-1)}{2}}$

2] $\int_0^1 \int_0^x e^{x+y} dx dy$.

$= \int_0^1 \left[\int_0^x e^{x+y} dy \right] dx$

$= \int_0^1 \left[\frac{e^{x+y}}{(1)} \right]_0^x dx$

$= \int_0^1 \left[e^{x+x} - e^{x+0} \right] dx$

$= \int_0^1 \left[e^{2x} - e^x \right] dx$

$= \left[\frac{e^{2x}}{2} - e^x \right]_0^1$

$= \left[\frac{e^2}{2} - e \right] - \left[1/2 - 1 \right]$

$\boxed{I = \frac{e^2}{2} - e + 1/2}$

or

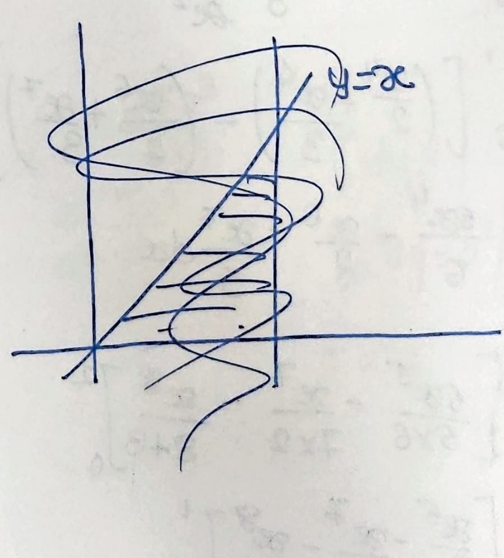
Q2] $\int_0^1 \int_0^x e^x \cdot e^y dy dx$

$= \int_0^1 e^x \left[\left(\int_0^x e^y dy \right) \right] dx$

$= \int_0^1 e^x \left[e^y \right]_0^x dx$

$= \int_0^1 e^x \left[e^x - 1 \right] dx$

$= \int_0^1 (e^{2x} - e^x) dx$



Q.3. $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$.

$= \int_0^1 \left[\int_0^1 \frac{1}{\sqrt{1-x^2}} \cdot \frac{1}{\sqrt{1-y^2}}\, dx\right] dy$.

$= \int_0^1 \frac{1}{\sqrt{1-y^2}}\, dy \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$.

$= \left[\sin^{-1} y\right]_0^1 \left[\sin^{-1} x\right]_0^1$

$= \left[\pi/2 - 0\right] \times \left[\pi/2 - 0\right]$.

$= \pi^2/4$.

Q.4 $\int_0^1 \int_{x^2}^{x} xy(x+y)\, dy\, dx$.

$= \int_0^1 \int_{x^2}^{x} (x^2 y + xy^2)\, dy\, dx$.

$= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{xy^3}{3}\right)_{x^2}^{x} dx$.

$= \int_0^1 \left[\left(\frac{x^4}{2} + \frac{x^4}{3}\right) - \left(\frac{x^6}{2} + \frac{x^7}{3}\right)\right] dx$.

$= \int_0^1 \frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\, dx$.

$= \left[\frac{5x^5}{5\times 6} - \frac{x^7}{7\times 2} - \frac{x^8}{8+3}\right]_0^1$.

$= \left[\frac{x^5}{6} - \frac{x^7}{14} - \frac{x^8}{24}\right]_0^1$

$= \frac{1}{6} - \frac{1}{14} - \frac{1}{24} = 3/56$.

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{b}\right)$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)dx.$$

Q.5 $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy.$

$\rightarrow \displaystyle\int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\right] dy$

$\rightarrow$ $\sqrt{a^2-y^2}$ is in y.

It is the Limit of x. solve first w.r.t. $x(dx)$

(a^2-y^2) is constant w.r.t x.

$(\sqrt{a^2-y^2})^2 = a^2-y^2.$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \left[\int \sqrt{\underbrace{(\sqrt{a^2-y^2})^2}_{b^2}-x^2}\,dx\right] dy.$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{(\sqrt{a^2-y^2})^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]^{\sqrt{a^2-y^2}} dy.$$

$$= \int_0^a \frac{\sqrt{a^2-y^2}}{2}\sqrt{a^2-y^2-(a^2-y^2)} + \frac{(\sqrt{a^2-y^2})^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right)dy.$$

$$= \int_0^a \frac{a^2-y^2}{2}\sin^{-1}(1)\,dy.$$

$$= \int_0^a \frac{\pi}{2}\left(\frac{a^2-y^2}{2}\right)dy.$$

$$= \frac{\pi}{4}\int_0^a (a^2-y^2)\,dy.$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3}\right]_0^a + c.$$

$$= \frac{\pi}{4}\left[a^2(a) - \frac{a^3}{3}\right].$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$\frac{2\pi a^3}{3\times 4}.$$

$$= \frac{\pi a^3}{6}.$$

$\int 1\,dy = y.$

$\int a^2\,dy = a^2\int dy.$

$= a^2 y.$

Q.2. find $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy\, dx$.

$$I = \int_0^1 \left(\int_0^{\sqrt{1+x^2}} \frac{1}{(1+x^2)+y^2} dy \right) dx.$$

$$= \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{\underbrace{(\sqrt{1+x^2})^2}_{a^2} + y^2} dy \right] dx.$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx.$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4} dx$$

$$= \frac{\pi}{4} \log\left(x + \sqrt{1+x^2}\right) \Big|_0^1 \quad \text{---} \quad \frac{1}{\sqrt{a^2+x^2}} = \log\left(x + \sqrt{a^2+x^2}\right) + c$$

$$= \frac{\pi}{4} \left[\log(1+\sqrt{1+1}) - \log(0+\sqrt{1+0}) \right] + c$$

$$= \frac{\pi}{4} \log(1+\sqrt{2}) - 0 + c$$

$$I = \frac{\pi}{4} \log(1+\sqrt{2}) + c.$$

Q.3. $\int_0^1 \int_{y^2}^1 \left[\int_0^{1-x} x\, dz \right) dx\, dy$.

$\rightarrow$

$$I = \int_0^1 \int_{y^2}^1 x \left[z \right]_0^{1-x} dx\, dy.$$

$$= \int_0^1 \int_{y^2}^1 x[1-x]\, dx\, dy.$$

$$= \int_0^1 \left[\int_{y^2}^1 (x - x^2)\, dx \right) dy.$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^{1} dy.$$

$$= \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right] dy.$$

$$= \int_0^1 \left[\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right] dy.$$

$$= \left[\frac{1}{6}y - \frac{y^5}{5\times2} + \frac{y^7}{7\times3}\right]_0^1.$$

$$= \left[\frac{1}{6}(1) - \frac{1}{10} + \frac{1}{21}\right].$$

$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21}.$$

$$= \frac{10-6}{60} + \frac{1}{21}.$$

$$= \frac{4}{60} + \frac{1}{21}$$

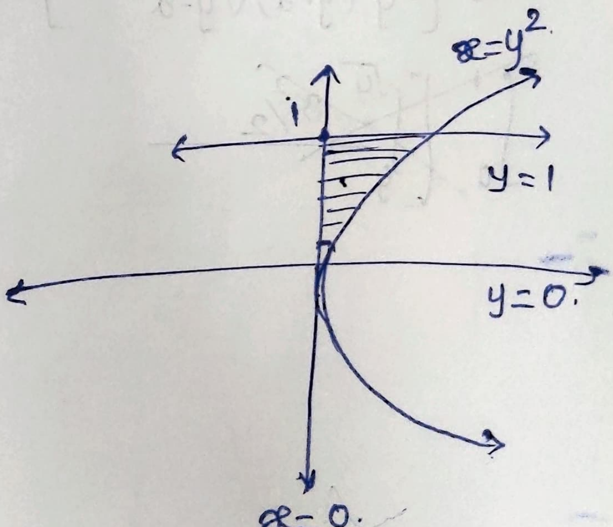
$$= \frac{1}{15} + \frac{1}{21}$$

$$= \frac{36}{315} = \frac{4}{35}$$

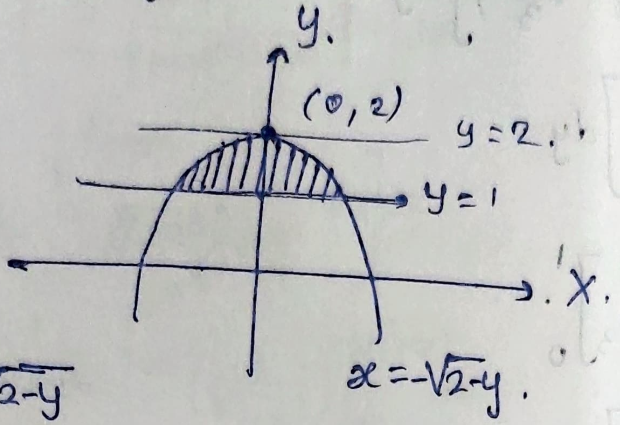
Q. 4. $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy.$ $y^2 \longrightarrow$ limits of x.

Draw the region of integratn.

$x=0,\ x=y^2.$
$y=0,\ y=1.$



Q.4. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} dx\,dy.$



$x = -\sqrt{2-y}$.
$x^2 = 2-y$.
$-y+2 = x^2$
$y-2 = -x^2$.
$y-k = -(x-h)^2$

$x = \sqrt{2-y}$
$x^2 = 2-y$
$-y+2 = x^2$
$y-2 = -x^2$.

26/2/24.

Double Integration :-

1] Evaluate. $\int_0^1 \int_y^{\sqrt{y}} \frac{x}{(y-x)\sqrt{y-x^2}}\,dx\,dy.$

$\rightarrow$ $\int_0^1 \left[\int_y^{\sqrt{y}} \frac{x}{(y-x)\sqrt{y-x^2}}\,dx\right] dy.$

$\cancel{\int_0^1 \left[\int_y^{\sqrt{y}} x^2/2\right]}$

1] Evaluate. $\int_0^1 \left[\int_y^{\sqrt{y}} \frac{x}{(1-y)\sqrt{y-x^2}} dx \right] dy$

→ put $y - x^2 = t$

Diff^r w.r.t. x only.

$-2x\,dx = dt$

$x\,dx = -dt/2$

when $x = y$, $t = y - y^2$ | $t: y-y^2 \to 0$

when $x = \sqrt{y}$, $t = 0$

$$= \int_0^1 \int_y^{\sqrt{y}} \frac{-1/2}{(1-y)\sqrt{t}} dt$$

$$= \int_0^1 \frac{1}{1-y}\left(-\tfrac{1}{2}\right) \left[\int_y^{\sqrt{y}} \frac{1}{\sqrt{t}} dt \right] dy$$

$$= \int_0^1 \frac{-1/2}{1-y} \left[\frac{t^{-1/2+1}}{-1/2+1} \right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{-1/2}{1-y} \left[\frac{\sqrt{t}}{(1/2)} \right]_y^{\sqrt{y}} dy$$

$$= -\int_0^1 \frac{1}{1-y} \left[\sqrt{y - x^2} \right]_y^{\sqrt{y}} dy$$

$$= -\int_0^1 \frac{1}{1-y} \left[\sqrt{y-y} - \sqrt{y-y^2} \right] dy$$

$$= \int_0^1 \frac{\sqrt{y}(\sqrt{1-y})}{(1-y)} dy \quad \to \sqrt{y(1-y)}$$

$$= \int_0^1 \frac{\sqrt{y}}{\sqrt{1-y}} dy$$

$$= \int_0^1 \sqrt{\frac{y'}{1-y'}} dy$$

$= \int$

$y = 0, \theta = 0$

$y = 1, \theta = \pi/2$

Put $y = \sin^2\theta$ $\theta: 0 \to \pi/2$

Diff^r

$dy = 2\sin\theta \cdot \cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\, 2\sin\theta\cdot\cos\theta\, d\theta$$

$\cos(2\theta) = 1 - 2\sin^2\theta$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} 2\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{1-\cos(2\theta)}{2} d\theta$$

$$= \int_0^{\pi/2} [1 - \cos(2\theta)]\, d\theta$$

$$= \left[\theta - \frac{\sin(2\theta)}{2} \right]_0^{\pi/2}$$

$$= [\pi/2 - 0] - [0 - 0]$$

$$= \pi/2$$

2] Evaluate $\int_0^{\pi/2}\int_0^{1-\sin\theta} r^2\cos\theta \, dr\, d\theta$.

→

$$I = \int_0^{\pi/2} \cos\theta \left(\int_0^{1-\sin\theta} r^2 \, dr \right) d\theta$$

$$= \int_0^{\pi/2} \cos\theta \left[\frac{r^3}{3} \right]_0^{1-\sin\theta} d\theta$$

$$= \frac{1}{3} \int_0^{\pi/2} \cos\theta \left[1-\sin\theta\right]^3 d\theta$$

$1-\sin\theta = t$

$-\cos\theta \, d\theta = dt$

$\cos\theta \, d\theta = -dt$

when $\theta = 0$, $t = 1$.

when $\theta = \pi/2$; $t = 0$.

$t : 1 \to 0$.

$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$

$$= \frac{1}{3} \int_1^0 (t^3)(-dt)$$

$$= \frac{1}{3} \int_1^0 -t^3 \, dt$$

$$= \frac{1}{3}\left[- \int_0^1 -t^3 \, dt \right] \quad --- \quad \int_a^b f\,dt = -\int_a^b f\,dt$$

$$= \frac{1}{3} \int_0^1 t^3 \, dt$$

$$= \frac{1}{3} \left[\frac{t^4}{4} \right]_0^1$$

$$= \frac{1}{12}.$$