Rosen, Discrete Mathematics and Its Applications, 6th edition
Extra Examples

Section 2.1—Sets

Extra Examples — *Page references correspond to locations of Extra Examples icons in the textbook.*

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p.112, icon at Example 4

#1. Write the set $\{2, 3, 4\}$ (given in list notation) in set builder notation.

Solution:
Here are three ways of writing the set in set builder notation:

$\{\, x \mid x \in \mathbf{N},\ 1 < x < 5 \,\}$,
$\{\, x \mid x \in \mathbf{N},\ 2 \le x \le 4 \,\}$, or
$\{\, x \mid x \in \mathbf{R},\ x^3 - 9x^2 + 26x - 24 = 0 \,\}$.

(This last set was obtained by taking the equation $(x - 2)(x - 3)(x - 4) = 0$ and multiplying out the left side.)

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#2. Write the set $\{\, x \mid x \in \mathbf{R},\ x^2 = 4 \text{ or } x^2 = 9 \,\}$ in list form.

Solution:
The set can be written in list form as $\{-3, -2, 2, 3\}$ because the two equations each have two real number solutions.

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#3. Write the set $\{\, x \mid x \in \mathbf{R}, x \text{ is a solution to } x^2 = -1 \,\}$ in list form.

Solution:
The set is the empty set because $x^2 = -1$ has no real number solutions; we can write it as $\{\ \}$.

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#4. The same set can be written as a list in different ways.
For example,

$$\{\, 1, 2, 3, \ldots, 99 \,\} = \{\, 1, 2, 3, \ldots, 98, 99 \,\} = \{\, 1, 2, 3, 4, 5, \ldots, 97, 98, 99 \,\}.$$

These three sets all describe the set of positive integers less than 100. As long as the pattern is clear, you can use any number of terms before and after the ellipsis.

Write this set in set-builder notation.

Solution:

We can write this set in set-builder notation by specifying the property that the elements must have: each x satisfies $x \geq 1$ and $x \leq 99$. Therefore, we can write this set as $\{\, x \mid x \in \mathbf{N}, 1 \leq x \leq 99 \,\}$.

We can also write the set as $\{\, y \mid y \in \mathbf{N},\ 1 \leq y \leq 99 \,\}$. You can name the variable however you wish, but you must use the same letter to name the general element as you use in the description of the property.

The same set can also be written as $\{\, x \mid x \in \mathbf{Z},\ 1 \leq x \leq 99 \,\}$, using $\mathbf{Z}$ instead of $\mathbf{N}$, because the property restricts membership to the same collection of integers.

We can also write the set as $\{\, x \mid x \in \mathbf{N},\ 0 < x < 100 \,\}$, $\{\, x \mid x$ a positive integer with one or two digits $\}$, or $\{\, x \mid x$ a positive integer less than 100 $\}$ by altering the property or by describing the property in English.

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#5. Write the set $S = \{x \mid x$ is an even positive integer and $x \leq 64\}$ in list notation.

Solution:
The integers in S are the even integers beginning with 2 and ending with 64. We can write $S = \{2, 4, 6, 8, \ldots, 64\}$, $S = \{2, 4, 6, \ldots, 64\}$, or $S = \{2, 4, 6, \ldots, 62, 64\}$, for example. Note than writing $S = \{2, 4, \ldots, 64\}$ would not be wise because the pattern in the list is not obvious — it could be $2, 4, 8, 16, 32, 64$.

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p.116, icon at Example 12

#1. Determine whether each set is finite or infinite:

(a) $\{1, 10, 100, 1000, 10000, \ldots\}$.

(b) $\{1, 3, 5, 7, 9, \ldots, 599\}$.

(c) The set of all real number solutions to $x + 3 + 2x = 3(x + 1)$.

(d) The set of telephone numbers of the form "(XXX) XXX-XXXX" in the United States.

(e) The set of real number solutions to the equation $x^2 = -4$.

(f) $\{x \mid x$ an integer, $x^2 - 9x + 14 < 0\}$.

Solution:
(a) The ellipsis indicates that the pattern continues forever, so the set is infinite.

(b) The set consists of the first 300 odd positive integers, stopping at 599. Therefore the set is finite.

(c) The equation can be rewritten as $3x + 3 = 3x + 3$; hence all real numbers are solutions. The set is therefore infinite.

(d) The number is quite large, but the number of possibilities is still finite.

(e) There are no real numbers that are solutions to this equation. Therefore the set is empty, that is, it has size zero, and hence is finite.

(f) The polynomial factors as $(x - 7)(x - 2)$. This product is positive if $x > 7$ or $x < 2$, and is negative if $2 < x < 7$. Therefore, the set is finite — it is equal to $\{3, 4, 5, 6\}$.

#1. Let $S = \{\emptyset, a, \{a\}\}$. Determine whether each of these is an element of S, a subset of S, neither, or both.

    (a) $\{a\}$

    (b) $\{\{a\}\}$

    (c) $\emptyset$

    (d) $\{\{\emptyset\}, a\}$

    (e) $\{\emptyset\}$

    (f) $\{\emptyset, a\}$

Solution:
(a) $\{a\}$ is the third element in the list of elements of S. Therefore $\{a\} \in S$. The set $\{a\}$ is also a subset of S: $\{a\}$ has one element, a, which is also an element of S.

(b) $\{\{a\}\}$ is not an element of S, because it does not appear in the list of elements of S. However $\{\{a\}\} \subseteq S$ because every element of $\{\{a\}\}$ belongs to S. (The only element of $\{\{a\}\}$ is $\{a\}$, which is an element of S.)

(c) $\emptyset \in S$ (it is the first element in the list for S) and $\emptyset \subseteq S$ (the empty set is a subset of all sets).

(d) $\{\{\emptyset\}, a\}$ is neither a subset of S nor an element of S. It is not a subset of S because $\{\emptyset\}$ is not an element of S; it is not an element of S because it does not appear in the list of elements of S.

(e) $\{\emptyset\} \subseteq S$. The set $\{\emptyset\}$ has one element, $\emptyset$, which is the first element in the list for S. However, $\{\emptyset\}$ is not an element of S.

(f) $\{\emptyset, a\} \subseteq S$. The set $\{\emptyset, a\}$ has two elements, each of which is an element of S. However $\{\emptyset, a\}$ is not an element of S because it is not one of the three elements in the list for S.

#2.
(a) Prove that $\mathbf{P}(A) \cup \mathbf{P}(B) \subseteq \mathbf{P}(A \cup B)$ is true for all sets A and B.

(b) Prove that the converse of (a) is not true. That is, prove that $\mathbf{P}(A \cup B) \subseteq \mathbf{P}(A) \cup \mathbf{P}(B)$ is false for some sets A and B.

Solution:
(a) The statement $\mathbf{P}(A) \cup \mathbf{P}(B) \subseteq \mathbf{P}(A \cup B)$ is true. Let $S \in \mathbf{P}(A) \cup \mathbf{P}(B)$. Therefore $S \subseteq A$ or $S \subseteq B$. Therefore S is a subset of $A \cup B$, and hence $S \in \mathbf{P}(A \cup B)$.

(b) The statement $\mathbf{P}(A \cup B) \subseteq \mathbf{P}(A) \cup \mathbf{P}(B)$ is false. You can have a subset $S \subseteq A \cup B$ without having S be a subset of either A or B. For example, let $A = \{1, 2\}$, $B = \{2, 3\}$, and $S = \{1, 3\}$. Then $S \subseteq A \cup B$ is true, so $S \in \mathbf{P}(A \cup B)$, but $S \subseteq A$ and $S \subseteq B$ are both false, so $S \notin \mathbf{P}(A)$ and $S \notin \mathbf{P}(B)$.

#3. Suppose that A and B are sets such that $\mathbf{P}(A \cup B) \subseteq \mathbf{P}(A) \cup \mathbf{P}(B)$. Prove that $A \subseteq B$ or $B \subseteq A$.

Solution:
We give a proof by contraposition. That is, we will give a direct proof of the following: If $A \subseteq B$ and $B \subseteq A$ are false, then $\mathbf{P}(A \cup B) \subseteq \mathbf{P}(A) \cup \mathbf{P}(B)$ is false.

Suppose $A \subseteq B$ and $B \subseteq A$ are false. Then there is an element $a \in A - B$ and an element $b \in B - A$. Consider the set $S = \{a, b\}$. Then $S \subseteq A \cup B$, but S is not a subset of A and S is not a subset of B. Therefore, $S \in \mathbf{P}(A \cup B)$ but $S \notin \mathbf{P}(A)$ and $S \notin \mathbf{P}(B)$.

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p.118, icon at Example 15

#1. The set $\mathbf{R} \times \mathbf{R} = \{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}\} = \mathbf{R}^2$ is the xy-plane. Of particular interest is any subset of $\mathbf{R}^2$ defined by $\{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}, y = f(x)\}$ where $f: \mathbf{R} \to \mathbf{R}$ is a function. This set of points is the graph of the function. For example, $\{(x, y) \mid x \in \mathbf{R}, y = x^2\} = \{(x, x^2) \mid x \in \mathbf{R}\}$ is the graph of a parabola.

More generally, any "relation" between real numbers x and y can be described as a subset of the Cartesian product $\mathbf{R} \times \mathbf{R}$. For example, $\{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}, x^2 + y^2 = 1\}$ is the graph of the circle of radius 1 with center at the origin. The set $\{(x, y) \mid x \in \mathbf{R}, y \in \mathbf{R}, x < y\}$ is the portion of the plane above the diagonal line $y = x$.