7 Ratio and Proportion

(Including Properties and Uses)

7.1 Introduction :

Basic *concepts of ratio and proportion* have already been studied in earlier classes, especially in classes 8 and 9. In this chapter of Class 10, we shall study ratio and proportion in more detail.

7.2 Ratio :

The ratio of two quantities of the same kind and in the same units is a comparison obtained by dividing the first quantity by the other.

If a and b are two quantities of the same kind and with the same units such that $b \neq 0$; then the quotient $\frac{a}{b}$ is called the **ratio** between a and b.

Remember :

1. Ratio $\frac{a}{b}$ has no unit and can be written as $a : b$ (read as a is to b).
2. The quantities a and b are called terms of the ratio. The first quantity a is called the *first term* or the *antecedent* and the second quantity b is called the *second term* or the *consequent* of the ratio $a : b$.

   The second term of a ratio cannot be zero.

   i.e. (i) In the ratio $a : b$, the second term $\boldsymbol{b}$ cannot be zero ($b \neq 0$).

   (ii) In the ratio $b : a$, the second term $\boldsymbol{a} \neq 0$.
3. If both the terms of a ratio are multiplied or divided by the same non-zero number, the ratio remains unchanged.
4. A *ratio must* always be expressed in its *lowest terms i.e.* both the terms of the ratio must be co-prime.

   The ratio is in its lowest terms, if the H.C.F. of its both the terms is 1 (unity).

   e.g. (i) The ratio 3 : 7 is in its lowest terms as the H.C.F. of its terms 3 and 7 is 1.

   (ii) The ratio 4 : 20 is not in its lowest terms as the H.C.F. of its terms 4 and 20 is 4 and not 1.
5. Ratios $a : b$ and $b : a$ cannot be equal unless $a = b$

   i.e., $a : b \neq b : a$, unless $a = b$.

   In other words, the order of the terms in a ratio is important.

1 (i) If $2x + 3y : 3x + 5y = 18 : 29$, find $x : y$.

(ii) If $x : y = 2 : 3$, find the value of $3x + 2y : 2x + 5y$.

Solution :

(i) $2x + 3y : 3x + 5y = 18 : 29$

$$\Rightarrow \quad \frac{2x+3y}{3x+5y} = \frac{18}{29}$$

$$\Rightarrow \quad 58x + 87y = 54x + 90y$$

$$\Rightarrow \quad 4x = 3y$$

$$\Rightarrow \quad \frac{x}{y} = \frac{3}{4} \quad \textit{i.e. } x : y = 3 : 4$$ **Ans.**

(ii) $x : y = 2 : 3 \Rightarrow \dfrac{x}{y} = \dfrac{2}{3}$

Now, $3x + 2y : 2x + 5y = \dfrac{3x+2y}{2x+5y}$

$$= \frac{3\left(\frac{x}{y}\right)+2}{2\left(\frac{x}{y}\right)+5}$$ [Dividing each term by y]

$$= \frac{3 \times \frac{2}{3} + 2}{2 \times \frac{2}{3} + 5}$$ [$\because \dfrac{x}{y} = \dfrac{2}{3}$]

$= 12 : 19$ **Ans.**

Alternative method :

$$\Rightarrow x : y = 2 : 3 \quad \Rightarrow \quad 3x = 2y \quad \Rightarrow \quad x = \frac{2y}{3}$$

$$\therefore \frac{3x+2y}{2x+5y} = \frac{3 \times \frac{2y}{3} + 2y}{2 \times \frac{2y}{3} + 5y} = \frac{4y}{\frac{19y}{3}} = \frac{4 \times 3}{19} = 12 : 19$$ **Ans.**

OR, $x : y = 2 : 3 \quad \Rightarrow \quad 3x = 2y \quad \Rightarrow \quad y = \dfrac{3x}{2}$

$$\therefore \frac{3x+2y}{2x+5y} = \frac{3x + 2 \times \frac{3x}{2}}{2x + 5 \times \frac{3x}{2}} = \frac{6x}{\frac{19x}{2}} = \frac{6 \times 2}{19} = 12 : 19$$ **Ans.**

3rd Method :

$x : y = 2 : 3 \quad \Rightarrow \quad$ if $x = 2\text{k}$ then $y = 3\text{k}$

And, $\dfrac{3x+2y}{2x+5y} = \dfrac{3 \times 2\text{k} + 2 \times 3\text{k}}{2 \times 2\text{k} + 5 \times 3\text{k}} = \dfrac{12\text{k}}{19\text{k}} = \dfrac{12}{19} = 12 : 19$ **Ans.**

Precaution :

For $x : y = 2 : 3$; if we take $x = 2$ and $y = 3$; then

$$\frac{3x + 2y}{2x + 5y} = \frac{3 \times 2 + 2 \times 3}{2 \times 2 + 5 \times 3}$$

$= \frac{12}{19} = 12 : 19$; which is the same as obtained in each solution given above.

But this solution is absolutely wrong and for this solution, a student will score no marks.

Reason : Let the age of Mohit = 15 yrs. and the age of his elder brother Rahul = 24 yrs. The ratio between the ages of Mohit and Rahul = 15 yrs : 24 yrs = 5 : 8. Now read it otherwise, that the ratio between the ages of Mohit and Rahul is 5 : 8. What does it mean ? Does it mean that Mohit's age is 5 years and Rahul's age is 8 years. The answer is simple, *i.e.* **No.**

In the same way, if $x : y = 2 : 3$, it does not mean $x = 2$ and $y = 3$.

2 If $a : b = 5 : 3$, find $(5a + 8b) : (6a - 7b)$. **[2002]**

Solution :

Let $a : b = 5 : 3 \quad \Rightarrow \quad$ if $a = 5x$, then $b = 3x$;

and $$\frac{5a + 8b}{6a - 7b} = \frac{5 \times 5x + 8 \times 3x}{6 \times 5x - 7 \times 3x} = \frac{49x}{9x} = 49 : 9$$ **Ans.**

3 Two numbers are in the ratio 3 : 5. If 8 is added to each number, the ratio becomes 2 : 3. Find the numbers.

Solution :

Since, the ratio between the numbers is 3 : 5

$\Rightarrow$ if one number is $3x$; the other number is $5x$

Given : $\frac{3x + 8}{5x + 8} = \frac{2}{3} \quad \Rightarrow \quad 10x + 16 = 9x + 24$

$\Rightarrow \quad x = 8$

$\therefore$ Nos. are $3x$ and $5x = 3 \times 8$ and $5 \times 8 = 24$ and 40 **Ans.**

4 (i) What quantity must be added to each term of the ratio 8 : 15 so that it becomes equal to 3 : 5 ?

(ii) What quantity must be subtracted from each term of the ratio $a : b$ so that it becomes $c : d$?

Solution :

(i) Let x be added to each term of the ratio 8 : 15.

Given : $\frac{8 + x}{15 + x} = \frac{3}{5}$

$\Rightarrow \quad 40 + 5x = 45 + 3x \quad \Rightarrow \quad x = 2\frac{1}{2}$ **Ans.**

(ii) Let x be subtracted, then :

$$\frac{a-x}{b-x} = \frac{c}{d}$$

$\Rightarrow \quad ad - dx = bc - cx$

$\Rightarrow \quad cx - dx = bc - ad$

$\Rightarrow \quad x(c - d) = bc - ad \quad \Rightarrow \quad x = \dfrac{bc - ad}{c - d}$ **Ans.**

5 The work done by $(x - 3)$ men in $(2x + 1)$ days and the work done by $(2x + 1)$ men in $(x + 4)$ days are in the ratio 3 : 10. Find the value of x. **[2003]**

Solution :

Assuming that all the men do the same amount of work in one day and one day work of each man = 1 unit; we get :

Amount of work done by $(x - 3)$ men in $(2x + 1)$ days

= amount of work done by $(x - 3)(2x + 1)$ men in one day

= $(x - 3)(2x + 1)$ units of work.

Similarly, amount of work done by $(2x + 1)$ men in $(x + 4)$ days.

= amount of work done by $(2x + 1)(x + 4)$ men in one day.

= $(2x + 1)(x + 4)$ units of work.

According to the given statement :

$$\frac{(x-3)(2x+1)}{(2x+1)(x+4)} = \frac{3}{10}$$

$\Rightarrow \quad \dfrac{2x^2 + x - 6x - 3}{2x^2 + 8x + x + 4} = \dfrac{3}{10} \qquad$ *i.e.* $\quad \dfrac{2x^2 - 5x - 3}{2x^2 + 9x + 4} = \dfrac{3}{10}$

$\Rightarrow \quad 20x^2 - 50x - 30 = 6x^2 + 27x + 12$

$\Rightarrow \quad 14x^2 - 77x - 42 = 0$

$\Rightarrow \quad 2x^2 - 11x - 6 = 0$

$\Rightarrow \quad (x - 6)(2x + 1) = 0$ [On factorising]

$\Rightarrow \quad x = 6, \text{ or } x = -\dfrac{1}{2}$

$x = -\dfrac{1}{2}$ is not possible as it will make no. of men $(x - 3)$ negative.

$\therefore \quad x = 6$ **Ans.**

7.3 Increase (or decrease) in a ratio :

1. Let the price of an article increases from ₹ 20 to ₹ 24; we say that the price has increased in the ratio 20 : 24 = 5 : 6.

   $\Rightarrow$ The original price of the article : Its increased price = 5 : 6

2. Let the price of an article decreases from ₹ 24 to ₹ 20; we say that the price has decreased in the ratio 24 : 20 = 6 : 5.

   $\Rightarrow$ The original price of the article : Its decreased price = 6 : 5

In general :

If a quantity increases or decreases in the ratio $a : b$.

$\Rightarrow$ The new (resulting) quantity $= \frac{b}{a}$ times of the original quantity.

6 When the fare of a certain journey by an airliner was increased in the ratio 5 : 7 the cost of the ticket for the journey became ₹ 1,421. Find the increase in the fare.

Solution :

According to the given statement :

The original fair : Increased fair = 5 : 7

$\Rightarrow$ 7 × The original fare = 5 × Increased fair

$\Rightarrow$ 7 × The original fare = 5 × ₹ 1,421

$\Rightarrow$ The original fare $= \frac{5 \times ₹1{,}421}{7}$ = ₹ 1,015

$\therefore$ **Increase in the fare** = ₹ (1,421 – 1,015) = **₹ 406** **Ans.**

7 In a regiment, the ratio of number of officers to the number of soldiers was 3 : 31 before a battle. In the battle 6 officers and 22 soldiers were killed. The ratio between the number of officers and the number of soldiers now is 1 : 13. Find the number of officers and soldiers in the regiment before the battle. **[1992]**

Solution :

Before the battle :

Let the number of officers be $3x$

$\Rightarrow$ the number of soldiers $= 31x$

After the battle :

The number of officers $= 3x - 6$

and, the number of soldiers $= 31x - 22$

Given : $\frac{3x-6}{31x-22} = \frac{1}{13} \Rightarrow x = 7$ [On solving]

$\therefore$ **The no. of officers before battle** $= 3x = 3 \times 7 =$ **21**

and **the no. of soldiers before battle** $= 31x = 31 \times 7 =$ **217** **Ans.**

8 If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$ and $a + b + c = 0$; show that each given ratio is equal to -1.

Solution :

Since, $a + b + c = 0 \Rightarrow a + b = -c$,

$b + c = -a$ and $c + a = -b$

$$\therefore \quad \frac{a}{b+c} = \frac{a}{-a} = -1; \quad \frac{b}{c+a} = \frac{b}{-b} = -1 \text{ and } \frac{c}{a+b} = \frac{c}{-c} = -1$$

Hence, **each of the given ratios is –1.** **Ans.**

9 If $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$ and $a + b + c \neq 0$; show that each given ratio is equal to $\frac{1}{2}$.

Solution :

For any two or more equal ratios, each ratio is equal to the ratio between sum of their antecedents and sum of their consequents.

$\therefore$ (i) $\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d}$

(ii) $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} \Rightarrow \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \frac{a+c+e}{b+d+f}$ and so on.

Given : $\frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b}$

$$\Rightarrow \quad \frac{a}{b+c} = \frac{b}{c+a} = \frac{c}{a+b} = \frac{\text{sum of antecedents}}{\text{sum of consequents}}$$

$$= \frac{a+b+c}{(b+c)+(c+a)+(a+b)}$$

$$= \frac{a+b+c}{2a+2b+2c} = \frac{a+b+c}{2(a+b+c)}$$

$$= \frac{1}{2}$$ **Ans**

7.4 Commensurable and incommensurable quantities :

If the ratio between any two quantities of the same kind and having the same unit can be expressed exactly by the ratio between two integers; the quantities are said to be *commensurable*; otherwise *incommensurable*,

e.g. (i) The ratio between $2\frac{1}{3}$ and $3\frac{1}{2}$

$= \frac{7}{3} : \frac{7}{2} = \frac{7}{3} \times \frac{2}{7} = 2 : 3$; which is the ratio between two integers 2 and 3.

Therefore, $2\frac{1}{3}$ and $3\frac{1}{2}$ are *commensurable quantities.*

(ii) The ratio between $\sqrt{3}$ and 5 is $\sqrt{3}$: 5; which can never be expressed as the ratio between two integers; therefore $\sqrt{3}$ and 5 are *incommensurable quantities.*

7.5 Composition of Ratios :

(i) Compound Ratio :

For two or more ratios, the ratio between the product of their antecedents to the product of their consequents is called ***compound ratio***.

e.g., For ratios $a : b$ and $c : d$; the compound ratio is $(a \times c) : (b \times d)$.

For ratios $a : b$, $c : d$ and $e : f$; the compound ratio is

$(a \times c \times e) : (b \times d \times f)$ and so on.

(ii) Duplicate Ratio :

It is the compound ratio of two equal ratios.

e.g., **Duplicate ratio of $a : b$** = Compound ratio of $a : b$ and $a : b$

$= (a \times a) : (b \times b) = a^2 : b^2$

Thus, duplicate ratio of $2 : 3 = 2^2 : 3^2$

$= 4 : 9$

(iii) Triplicate Ratio :

It is the compound ratio of three equal ratios.

e.g., **Triplicate ratio of $a : b$** = Compound ratio of $a : b$, $a : b$ and $a : b$

$= (a \times a \times a) : (b \times b \times b) = a^3 : b^3$

Thus, triplicate ratio of $2 : 3 = 2^3 : 3^3$

$= 8 : 27$

(iv) Sub-duplicate Ratio :

For any ratio $a : b$, its sub-duplicate ratio is $\sqrt{a} : \sqrt{b}$

Thus, sub-duplicate ratio of $9 : 16 = \sqrt{9} : \sqrt{16}$

$= 3 : 4.$

(v) Sub-triplicate Ratio :

For any ratio $a : b$, its sub-triplicate ratio is $\sqrt[3]{a} : \sqrt[3]{b}$

Thus, sub-triplicate ratio of $27 : 64 = \sqrt[3]{27} : \sqrt[3]{64}$

$= 3 : 4$

(vi) Reciprocal Ratio :

For any ratio $a : b$; where $a, b \neq 0$, **its reciprocal ratio** $= \frac{1}{a} : \frac{1}{b} = b : a.$

Thus, reciprocal ratio of $3 : 5 = \frac{1}{3} : \frac{1}{5} = 5 : 3.$

> **10** Find the compound ratio of :
>
> (i) $3a : 2b$, $2m : n$ and $4x : 3y$
>
> (ii) $a - b : a + b$, $(a + b)^2 : a^2 + b^2$ and $a^4 - b^4 : (a^2 - b^2)^2$.

Solution :

(i) **Required compound ratio** $= (3a \times 2m \times 4x) : (2b \times n \times 3y)$

$= \frac{24\,a\,m\,x}{6\,b\,n\,y} = 4\,amx : bny$ **Ans.**

(ii) **Required compound ratio** $= [(a-b).(a+b)^2.(a^4-b^4)] : [(a+b).(a^2+b^2).(a^2-b^2)^2]$

$$= \frac{(a-b)(a+b)^2(a^2+b^2)(a^2-b^2)}{(a+b)(a^2+b^2)(a^2-b^2)(a+b)(a-b)}$$

$= \mathbf{1 : 1}$ **Ans.**

11 Find the ratio compounded of the duplicate ratio of 5 : 6, the reciprocal ratio of 25 : 42 and the sub-triplicate ratio of 216 : 343.

Solution :

Since, duplicate ratio of 5 : 6 $= 5^2 : 6^2 = 25 : 36$,

reciprocal ratio of 25 : 42 $= \frac{1}{25} : \frac{1}{42} = 42 : 25$

and, sub-triplicate ratio of 216 : 343 $= \sqrt[3]{216} : \sqrt[3]{343} = 6 : 7$.

Therefore, **the required compounded ratio** $= (25 \times 42 \times 6) : (36 \times 25 \times 7)$

$$= \frac{25 \times 42 \times 6}{36 \times 25 \times 7} = \mathbf{1 : 1}$$ **Ans.**

EXERCISE 7(A)

1. If $a : b = 5 : 3$, find : $\frac{5a-3b}{5a+3b}$.

2. If $x : y = 4 : 7$, find the value of $(3x + 2y) : (5x + y)$.

3. If $a : b = 3 : 8$, find the value of $\frac{4a+3b}{6a-b}$.

4. If $(a - b) : (a + b) = 1 : 11$, find the ratio $(5a + 4b + 15) : (5a - 4b + 3)$.

5. Find the number which bears the same ratio to $\frac{7}{33}$ that $\frac{8}{21}$ does to $\frac{4}{9}$.

6. If $\frac{m+n}{m+3n} = \frac{2}{3}$, find : $\frac{2n^2}{3m^2+mn}$.

7. Find $\frac{x}{y}$; when $x^2 + 6y^2 = 5xy$.

8. If the ratio between 8 and 11 is the same as the ratio of $2x - y$ to $x + 2y$, find the value of $\frac{7x}{9y}$.

9. Divide ₹ 1,290 into A, B and C such that A is $\frac{2}{5}$ of B and B : C = 4 : 3.

10. A school has 630 students. The ratio of the number of boys to the number of girls is 3 : 2. This ratio changes to 7 : 5 after the admission of 90 new students. Find the number of newly admitted boys.

11. What quantity must be subtracted from each term of the ratio 9 : 17 to make it equal to 1 : 3 ?

12. The monthly pocket money of Ravi and Sanjeev are in the ratio 5 : 7. Their expenditures are in the ratio 3 : 5. If each saves ₹ 80 every month, find their monthly pocket money. **[2012]**

13. The work done by $(x - 2)$ men in $(4x + 1)$ days and the work done by $(4x + 1)$ men in $(2x - 3)$ days are in the ratio 3 : 8. Find the value of x.

14. The bus fare between two cities is increased in the ratio 7 : 9. Find the increase in the fare, if :
    (i) the original fare is ₹ 245;
    (ii) the increased fare is ₹ 207.

15. By increasing the cost of entry ticket to a fair in the ratio 10 : 13, the number of visitors to the fair has decreased in the ratio 6 : 5. In

what ratio has the total collection increased or decreased ?

16. In a basket, the ratio between the number of oranges and the number of apples is 7 : 13. If 8 oranges and 11 apples are eaten, the ratio between the number of oranges and the number of apples becomes 1 : 2. Find the original number of oranges and the original number of apples in the basket.

17. In a mixture of 126 kg of milk and water, milk and water are in ratio 5 : 2. How much water must be added to the mixture to make this ratio 3 : 2 ?

18. (a) If A : B = 3 : 4 and B : C = 6 : 7, find :
 (i) A : B : C
 (ii) A : C
 (b) If A : B = 2 : 5 and A : C = 3 : 4, find : A : B : C.

19. (i) If 3A = 4B = 6C; find : A : B : C.
 (ii) If $2a = 3b$ and $4b = 5c$, find : $a : c$.

20. Find the compound ratio of :
 (i) 2 : 3, 9 : 14 and 14 : 27.
 (ii) $2a : 3b$, $mn : x^2$ and $x : n$.
 (iii) $\sqrt{2} : 1$, $3 : \sqrt{5}$ and $\sqrt{20} : 9$.

21. Find duplicate ratio of :
 (i) 3 : 4 (ii) $3\sqrt{3} : 2\sqrt{5}$

22. Find triplicate ratio of :
 (i) 1 : 3 (ii) $\frac{m}{2} : \frac{n}{3}$

23. Find sub-duplicate ratio of :
 (i) 9 : 16 (ii) $(x - y)^4 : (x + y)^6$

24. Find sub-triplicate ratio of :
 (i) 64 : 27 (ii) $x^3 : 125y^3$

25. Find the reciprocal ratio of :
 (i) 5 : 8 (ii) $\frac{x}{3} : \frac{y}{7}$

26. If $(x + 3) : (4x + 1)$ is the duplicate ratio of 3 : 5, find the value of x.

27. If $m : n$ is the duplicate ratio of $m + x : n + x$; show that : $x^2 = mn$.

28. If $(3x - 9) : (5x + 4)$ is the triplicate ratio of 3 : 4, find the value of x.

29. Find the ratio compounded of the reciprocal ratio of 15 : 28, the sub-duplicate ratio of 36 : 49 and the triplicate ratio of 5 : 4.

30. (a) If $r^2 = pq$, show that $p : q$ is the duplicate ratio of $(p + r) : (q + r)$.
 (b) If $(p - x) : (q - x)$ be the duplicate ratio of $p : q$ then show that : $\frac{1}{p} + \frac{1}{q} = \frac{1}{x}$.

7.6 Proportion :

Four non-zero quantities, a, b, c and d are said to be in proportion (or, are proportional), if $a : b = c : d$.

This is often expressed as $a : b :: c : d$ and is read as "*a is to b as c is to d*".

1. In $a : b = c : d$,
 (i) a, b, c and d are called the terms of the proportion; where a = first term, b = second term, c = third term and d = fourth term.
 (ii) 'a' and 'd' are called *extremes* (end-terms) whereas 'b' and 'c' are called ***means*** (middle terms).
2. $a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d} \Rightarrow a \times d = b \times c$
 $\Rightarrow$ **product of extremes = *product of means.***
3. In $a : b = c : d$, the fourth term 'd' is called the *fourth proportional*.
4. In $a : b = c : d$, quantities a and b must be of the same kind with the same units, whereas; c and d may separately be of the same kind with the same units.
 e.g. 5kg : 15 kg = ₹ 75 : ₹ 225

7.7 Continued proportion :

Three non-zero quantities of the same kind and in the same unit are said to be in ***continued proportion***, if the ratio of the first to the second is the same as the ratio of the second to the third.

Thus, a, b and c are in the continued proportion if $a : b = b : c$.

In general, the non-zero quantities a, b, c, d, e ......, etc. (all of the same kind and in the same unit) are in continued proportion $\Leftrightarrow \dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = \dfrac{d}{e}$ ....

> a, b and c are in continued proportion
>
> $\Leftrightarrow a : b = b : c$,
>
> Here, the *second quantity i.e.* 'b' is called the *mean proportional* between 'a' and 'c'; whereas the *third quantity i.e.* 'c' is called the *third proportional* to 'a' and 'b'.

12 Find : (i) the fourth proportional to 3, 6 and 4·5.

(ii) the mean proportional between 6·25 and 0·16.

(iii) the third proportional to 1·2 and 1·8.

Solution :

(i) Let the fourth proportional to 3, 6 and 4·5 be x.

$\Rightarrow \quad 3 : 6 = 4{\cdot}5 : x$

$\Rightarrow \quad 3 \times x = 6 \times 4{\cdot}5 \quad \Rightarrow \quad \boldsymbol{x = 9}$ **Ans.**

(ii) Let the mean proportional between 6·25 and 0·16 be x.

$\Rightarrow$ 6·25, x and 0·16 are in continued proportion.

$\Rightarrow 6{\cdot}25 : x = x : 0{\cdot}16$

$\Rightarrow \quad x \times x = 6{\cdot}25 \times 0{\cdot}16 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad \boldsymbol{x = 1}$ **Ans.**

(iii) Let the third proportional to 1·2 and 1·8 be x

$\Rightarrow$ 1·2, 1·8 and x are in continued proportion.

$\Rightarrow 1{\cdot}2 : 1{\cdot}8 = 1{\cdot}8 : x \quad \Rightarrow \quad x = \dfrac{1{\cdot}8 \times 1{\cdot}8}{1{\cdot}2} = \boldsymbol{2{\cdot}7}$ **Ans.**

13 Quantities a, 2, 10 and b are in continued proportion; find the values of a and b.

Solution :

a, 2, 10 and b are in continued proportion

$\Rightarrow \quad \dfrac{a}{2} = \dfrac{2}{10} = \dfrac{10}{b} \quad \Rightarrow \quad \dfrac{a}{2} = \dfrac{2}{10}$ and $\dfrac{2}{10} = \dfrac{10}{b}$

$\Rightarrow \quad \boldsymbol{a = 0{\cdot}4}$ **and** $\boldsymbol{b = 50}$ **Ans.**

14 What number should be subtracted from each of the numbers 23, 30, 57 and 78; so that the remainders are in proportion ? [2004]

Solution :

Let the number subtracted be x.

$\therefore$ $(23 - x) : (30 - x) : : (57 - x) : (78 - x)$

$$\Rightarrow \quad \frac{23-x}{30-x} = \frac{57-x}{78-x}$$

$\Rightarrow$ $1794 - 101x + x^2 = 1710 - 87x + x^2 \Rightarrow$ **$14x = 84$** and **$x = 6$** **Ans.**

15 What should be added to each of the numbers 13, 17 and 22 so that the resulting numbers are in continued proportion ?

Solution :

Let the required number to be added is x.

$\therefore$ $13 + x$, $17 + x$ and $22 + x$ are in continued proportion.

$$\Rightarrow \quad \frac{13+x}{17+x} = \frac{17+x}{22+x} \quad \text{i.e.} \quad (13 + x)(22 + x) = (17 + x)^2$$

$\Rightarrow$ $286 + 35x + x^2 = 289 + 34x + x^2$ *i.e.* $x = 3$

$\therefore$ **Required number = 3** **Ans.**

16 If $(a^2 + c^2)$, $(ab + cd)$ and $(b^2 + d^2)$ are in continued proportion; prove that a, b, c and d are in proportion.

Solution :

Given, $(a^2 + c^2)$, $(ab + cd)$ and $(b^2 + d^2)$ are in continued proportion.

$$\therefore \quad \frac{a^2+c^2}{ab+cd} = \frac{ab+cd}{b^2+d^2} \quad \Rightarrow (a^2 + c^2)(b^2 + d^2) = (ab + cd)^2$$

i.e. $a^2b^2 + a^2d^2 + b^2c^2 + c^2d^2 = a^2b^2 + 2abcd + c^2d^2$

$\Rightarrow$ $a^2d^2 + b^2c^2 - 2abcd = 0$ *i.e.* $(ad - bc)^2 = 0$

$\Rightarrow$ $ad - bc = 0$ *i.e.* $ad = bc$

$\Rightarrow$ $\dfrac{a}{b} = \dfrac{c}{d}$ *i.e.* **a, b, c and d are in proportion**

Hence Proved.

17 If $p : q :: q : r$, prove that $p : r = p^2 : q^2$.

Solution :

$p : q :: q : r \quad \Rightarrow \quad q^2 = pr$

$$\therefore \; p^2 : q^2 = \frac{p^2}{q^2} = \frac{p^2}{pr} \qquad [\because q^2 = pr]$$

$$= \frac{p}{r} = p : r.$$

Hence Proved.

18 If $a \neq b$ and $a : b$ is the duplicate ratio of $a + c$ and $b + c$, prove that 'c' is the mean proportional between 'a' and 'b'.

Solution :

'c' will be mean proportional between 'a' and 'b', if $a : c = c : b$ *i.e.*, if $c^2 = ab$.

Given : $\dfrac{a}{b} = \dfrac{(a+c)^2}{(b+c)^2}$

$\Rightarrow \quad a(b^2 + c^2 + 2bc) = b(a^2 + c^2 + 2ac)$

$\Rightarrow \quad ab^2 + ac^2 + 2abc = a^2b + bc^2 + 2abc$

$\Rightarrow \quad ac^2 - bc^2 = a^2b - ab^2$

$\Rightarrow \quad c^2(a - b) = ab(a - b)$

$\Rightarrow \quad c^2 = ab$ [As $a \neq b$]

$\Rightarrow$ **'c' is mean proportional between 'a' and 'b'.** **Hence Proved.**

19 If $a + c = mb$ and $\dfrac{1}{b} + \dfrac{1}{d} = \dfrac{m}{c}$, prove that a, b, c and d are in proportion.

Solution :

$\dfrac{1}{b} + \dfrac{1}{d} = \dfrac{m}{c} \quad \Rightarrow \quad \dfrac{d+b}{bd} = \dfrac{m}{c}$

i.e. $\quad cd + cb = mbd$

$\Rightarrow \quad cd + bc = (a + c)d$ $\quad [\because a + c = mb]$

$\Rightarrow \quad cd + bc = ad + cd$

$\Rightarrow \quad bc = ad \quad$ *i.e.* $\quad \dfrac{a}{b} = \dfrac{c}{d}$

$\Rightarrow$ **a, b, c and d are in proportion** **Hence Proved.**

Alternative method :

$a + c = mb \quad \Rightarrow \quad m = \dfrac{a+c}{b}$

Substituting the value of m in the other given equation, we get :

$\dfrac{1}{b} + \dfrac{1}{d} = \dfrac{a+c}{bc} \quad \Rightarrow \quad \dfrac{d+b}{bd} = \dfrac{a+c}{bc}$

i.e. $\quad \dfrac{d+b}{d} = \dfrac{a+c}{c} \quad \Rightarrow \quad cd + bc = ad + cd$

i.e. $bc = ad \Rightarrow \dfrac{a}{b} = \dfrac{c}{d}$

i.e. **a, b, c and d are in proportion** **Hence Proved.**

20 If q is the mean proportional between p and r, prove that :

$$p^2 - q^2 + r^2 = q^4\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right).$$

Solution :

$\because$ q is the mean proportional between p and r $\Rightarrow$ $q^2 = pr$

$$\therefore \quad \text{R.H.S.} = q^4\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right)$$

$$= \frac{q^4}{p^2} - q^2 + \frac{q^4}{r^2}$$

$$= \frac{p^2r^2}{p^2} - q^2 + \frac{p^2r^2}{r^2} \qquad [q^2 = pr \Rightarrow q^4 = p^2r^2]$$

$$= r^2 - q^2 + p^2 = \text{L.H.S.}$$

Hence Proved.

***Alternative method* ('k' method) :**

Step : 1. Put each given ratios equal to k.
2. Obtain the antecedent of each ratio in terms of k.
3. Substitute the values, obtained in step 2 in terms of k.
4. Simplify.

Given : q is the mean proportional between p and r

$\Rightarrow \quad p : q = q : r$

$\Rightarrow \quad \dfrac{p}{q} = \dfrac{q}{r} = k$ (say) $\qquad \Rightarrow \dfrac{p}{q} = k$ and $\dfrac{q}{r} = k$

i.e. $p = qk$, $q = rk$ and $p = qk = (rk)k = rk^2$

$$\therefore \quad \text{R.H.S.} = q^4\left(\frac{1}{p^2} - \frac{1}{q^2} + \frac{1}{r^2}\right)$$

$$= r^4k^4\left(\frac{1}{r^2k^4} - \frac{1}{r^2k^2} + \frac{1}{r^2}\right) \qquad [\because q = rk \text{ and } p = rk^2]$$

$$= r^2 - r^2k^2 + r^2k^4$$

$$= r^2 - (rk)^2 + (rk^2)^2$$

$$= r^2 - q^2 + p^2 \qquad [\because q = rk \text{ and } p = rk^2]$$

$$= \text{L.H.S.}$$

Hence Proved.

21 If $\frac{a}{b} = \frac{c}{d}$, prove that each given ratio ($\frac{a}{b}$ and $\frac{c}{d}$) is equal to :

(i) $\frac{3a-5c}{3b-5d}$ (ii) $\sqrt{\frac{2a^2+9c^2}{2b^2+9d^2}}$ (iii) $\left(\frac{5a^3-13c^3}{5b^3-13d^3}\right)^{\frac{1}{3}}$

Solution :

Let $\frac{a}{b} = \frac{c}{d} = k \Rightarrow \frac{a}{b} = k$ and $\frac{c}{d} = k \Rightarrow a = bk$ and $c = dk$

(i) $\frac{3a-5c}{3b-5d} = \frac{3(bk)-5(dk)}{3b-5d} = \frac{k(3b-5d)}{3b-5d} = k =$ **each given ratio**

Hence Proved.

(ii) $\sqrt{\frac{2a^2+9c^2}{2b^2+9d^2}} = \sqrt{\frac{2(bk)^2+9(dk)^2}{2b^2+9d^2}} = \sqrt{\frac{k^2(2b^2+9d^2)}{2b^2+9d^2}}$

$= \sqrt{k^2} = k =$ **each given ratio**

Hence Proved.

(iii) $\left(\frac{5a^3-13c^3}{5b^3-13d^3}\right)^{\frac{1}{3}} = \left[\frac{5(bk)^3-13(dk)^3}{5b^3-13d^3}\right]^{\frac{1}{3}} = \left[\frac{k^3(5b^3-13d^3)}{5b^3-13d^3}\right]^{\frac{1}{3}}$

$= \left[k^3\right]^{\frac{1}{3}} = k =$ **each given ratio**

Hence Proved.

22 If a, b, c and d are in proportion, prove that :

(i) $\frac{a-b}{c-d} = \sqrt{\frac{3a^2+8b^2}{3c^2+8d^2}}$ (ii) $\frac{5a^2+12c^2}{5b^2+12d^2} = \sqrt{\frac{3a^4-7c^4}{3b^4-7d^4}}$

Solution :

a, b, c and d are in proportion

$\Rightarrow \frac{a}{b} = \frac{c}{d} = k$ (let)

$\Rightarrow \frac{a}{b} = k$ and $\frac{c}{d} = k$ i.e. $a = bk$ and $c = dk$

(i) L.H.S. $= \frac{a-b}{c-d} = \frac{bk-b}{dk-d} = \frac{b(k-1)}{d(k-1)} = \frac{b}{d}$ I

R.H.S. $= \sqrt{\frac{3a^2+8b^2}{3c^2+8d^2}} = \sqrt{\frac{3(bk)^2+8b^2}{3(dk)^2+8d^2}}$

$= \sqrt{\frac{b^2(3k^2+8)}{d^2(3k^2+8)}} = \sqrt{\frac{b^2}{d^2}} = \frac{b}{d}$ II

From equations I and II, we get : **L.H.S. = R.H.S.** **Hence Proved.**

(ii) L.H.S. $= \dfrac{5a^2+12c^2}{5b^2+12d^2} = \dfrac{5(bk)^2+12(dk)^2}{5b^2+12d^2}$

$$= \frac{k^2(5b^2+12d^2)}{5b^2+12d^2} = k^2 \qquad \text{..... I}$$

R.H.S. $= \sqrt{\dfrac{3a^4-7c^4}{3b^4-7d^4}} = \sqrt{\dfrac{3(bk)^4-7(dk)^4}{3b^4-7d^4}}$

$$= \sqrt{\frac{k^4(3b^4-7d^4)}{3b^4-7d^4}} = \sqrt{k^4} = k^2 \qquad \text{..... II}$$

From equations I and II, we get : **L.H.S. = R.H.S.** **Hence Proved.**

23 6 is the mean proportion between two numbers x and y and 48 is third proportion to x and y. Find the numbers. [2011]

Solution :

Since, 6 is mean proportional between x and y.

$\Rightarrow \quad x : 6 = 6 : y \quad \Rightarrow \quad xy = 36$ I

and, 48 is third proportional to x and y

$\Rightarrow \quad x : y = y : 48 \quad \Rightarrow \quad y^2 = 48\,x$ II

From eq. (I); $xy = 36 \quad \Rightarrow \quad x = \dfrac{36}{y}$.

Substituting $x = \dfrac{36}{y}$ in eq. II, we get :

$$y^2 = 48 \times \frac{36}{y} \quad \Rightarrow \quad y^3 = 36 \times 48 \text{ and, } y = 12$$

$$\therefore \quad x = \frac{36}{y} = \frac{36}{12} = 3$$

$\therefore$ **The required nos. are 3 and 12.** **Ans.**

EXERCISE 7(B)

1. Find the fourth proportional to :
   (i) 1·5, 4·5 and 3·5
   (ii) $3a$, $6a^2$ and $2ab^2$
2. Find the third proportional to :
   (i) $2\frac{2}{3}$ and 4
   (ii) $a - b$ and $a^2 - b^2$
3. Find the mean proportional between :
   (i) $6 + 3\sqrt{3}$ and $8 - 4\sqrt{3}$
   (ii) $a - b$ and $a^3 - a^2b$.
4. If $x + 5$ is the mean proportion between $x + 2$ and $x + 9$; find the value of x.
5. If x^2, 4 and 9 are in continued proprotion, find x.
6. What least number must be added to each of the numbers 6, 15, 20 and 43 to make them proportional ? **(2005, 2013)**
7. (i) If a, b, c are in continued proportion, show that : $\dfrac{a^2+b^2}{b(a+c)} = \dfrac{b(a+c)}{b^2+c^2}$.

(ii) If a, b, c are in continued proportion and $a(b - c) = 2b$, prove that :

$$a - c = \frac{2(a+b)}{a}.$$

(iii) If $\frac{a}{b} = \frac{c}{d}$, show that :

$$\frac{a^3c + ac^3}{b^3d + bd^3} = \frac{(a+c)^4}{(b+d)^4}.$$

8. What least number must be subtracted from each of the numbers 7, 17 and 47 so that the remainders are in continued proportion ?

9. If y is the mean proportional between x and z; show that $xy + yz$ is the mean proportional between $x^2 + y^2$ and $y^2 + z^2$.

10. If q is the mean proportional between p and r, show that :

$pqr\,(p + q + r)^3 = (pq + qr + pr)^3.$

> q is the mean proportional between p and $r \Rightarrow q^2 = pr.$
>
> **L.H.S.** $= pqr\,(p + q + r)^3$
> $= q \cdot q^2\,(p + q + r)^3$ $\quad \because pr = q^2$
> $= q^3(p + q + r)^3$
> $= [q(p + q + r)]^3$
> $= (pq + q^2 + qr)^3$
> $= (pq + pr + qr)^3 =$ **R.H.S.**

11. If three quantities are in continued proportion; show that the ratio of the first to the third is the duplicate ratio of the first to the second.

> If the three quantities be x, y and z; then $x : y = y : z$ and to prove that $x : z = x^2 : y^2$

12. If y is the mean proportional between x and z, prove that : $\dfrac{x^2 - y^2 + z^2}{x^{-2} - y^{-2} + z^{-2}} = y^4.$

13. Given four quantities a, b, c and d are in proportion. Show that :

$(a - c)\,b^2 : (b - d)\,cd$
$= (a^2 - b^2 - ab) : (c^2 - d^2 - cd)$

> Given : $\frac{a}{b} = \frac{c}{d} = k$ (let)
>
> $\Rightarrow\; a = bk$ and $c = dk$
>
> Now, find the values of L.H.S. and R.H.S. of the required result by substituting $a = bk$ and $c = dk$; and show **L.H.S. = R.H.S.**

14. Find two numbers such that the mean proportional between them is 12 and the third proportional to them is 96.

15. Find the third proportional to $\frac{x}{y} + \frac{y}{x}$ and $\sqrt{x^2 + y^2}$

16. If $p : q = r : s$; then show that :

$mp + nq : q = mr + ns : s.$

> $\frac{p}{q} = \frac{r}{s} \Rightarrow \frac{mp}{q} = \frac{mr}{s}$
>
> $\Rightarrow \frac{mp}{q} + n = \frac{mr}{s} + n$ and so on.

17. If $p + r = mq$ and $\frac{1}{q} + \frac{1}{s} = \frac{m}{r}$; then prove that : $p : q = r : s$.

18. If $\frac{a}{b} = \frac{c}{d}$, prove that each of the given ratio is equal to :

(i) $\dfrac{5a+4c}{5b+4d}$ (ii) $\dfrac{13a-8c}{13b-8d}$

(iii) $\sqrt{\dfrac{3a^2-10c^2}{3b^2-10d^2}}$ (iv) $\left(\dfrac{8a^3 + 15c^3}{8b^3 + 15d^3}\right)^{\frac{1}{3}}$

19. If a, b, c and d are in proportion, prove that:

(i) $\dfrac{13a + 17b}{13c + 17d} = \sqrt{\dfrac{2ma^2 - 3nb^2}{2mc^2 - 3nd^2}}$

(ii) $\sqrt{\dfrac{4a^2+9b^2}{4c^2+9d^2}} = \left(\dfrac{xa^3-5yb^3}{xc^3-5yd^3}\right)^{\frac{1}{3}}$

20. If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that :

$$\frac{2x^3 - 3y^3 + 4z^3}{2a^3 - 3b^3 + 4c^3} = \left(\frac{2x-3y+4z}{2a-3b+4c}\right)^3$$

7.8 Some Important Properties of Proportion :

If four quantities a, b, c and d form a proportion

i.e. if $a : b :: c : d$, many other proportions may be obtained using the properties of fractions. Some of these proportions are given below :

1. Invertendo :

According to this property of proportions :

If $a : b = c : d$, then $b : a = d : c$.

Proof : $a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{b}{a} = \frac{d}{c}$ [Taking reciprocal of both the sides]

$\Rightarrow b : a = d : c.$

2. Alternendo :

According to this property of proportions :

If $a : b = c : d$, then $a : c = b : d$.

Proof : $a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d}$

$\Rightarrow a \times d = b \times c$ [By cross-multiplication]

$\Rightarrow \frac{a}{c} = \frac{b}{d} \Rightarrow a : c = b : d.$

3. Componendo :

If $a : b = c : d$, then $a + b : b = c + d : d$.

Proof : $a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{a}{b} + 1 = \frac{c}{d} + 1$ [Adding 1 on each side]

$\Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$

$\Rightarrow a + b : b = c + d : d$

4. Dividendo :

If $a : b = c : d$, then $a - b : b = c - d : d$.

Proof : $a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d}$

$\Rightarrow \frac{a}{b} - 1 = \frac{c}{d} - 1$ [Subtracting 1 from each side]

$\Rightarrow \frac{a-b}{b} = \frac{c-d}{d}$

$\Rightarrow a - b : b = c - d : d.$

5. Componendo and Dividendo :

If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

By componendo, $a : b = c : d \Rightarrow a + b : b = c + d : d$

$\Rightarrow \frac{a+b}{b} = \frac{c+d}{d}$ I

By dividendo, $a : b = c : d \Rightarrow \quad \frac{a-b}{b} = \frac{c-d}{d}$ II

Dividing I by II, we get : $\frac{a+b}{a-b} = \frac{c+d}{c-d}$

Thus, $\boldsymbol{a : b = c : d \Rightarrow a + b : a - b = c + d : c - d.}$

Thus; $\frac{a}{b} = \frac{c}{d} \Rightarrow$ (i) $\frac{b}{a} = \frac{d}{c}$ **By Invertendo**

(ii) $\frac{a}{c} = \frac{b}{d}$ **By Alternendo**

(iii) $\frac{a+b}{b} = \frac{c+d}{d}$ **By Componendo**

(iv) $\frac{a-b}{b} = \frac{c-d}{d}$ **By Dividendo**

(v) $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ **By Componendo and Dividendo**

7.9 Direct Applications :

24 If $\frac{8x+13y}{8x-13y} = \frac{9}{7}$, find $x : y$.

Solution :

Applying componendo and dividendo :

$$\frac{8x+13y}{8x-13y} = \frac{9}{7} \text{ gives } \frac{8x+13y+8x-13y}{8x+13y-8x+13y} = \frac{9+7}{9-7}$$

i.e. $\frac{16x}{26y} = \frac{16}{2} \Rightarrow \frac{x}{y} = \frac{16}{2} \times \frac{26}{16} = \frac{13}{1}$ *i.e.* $\boldsymbol{x : y = 13 : 1}$ **Ans.**

Alternative method :

$$\frac{8x+13y}{8x-13y} = \frac{9}{7} \Rightarrow 72x - 117y = 56x + 91y$$

$$\Rightarrow 16x = 208y$$

$\Rightarrow \frac{x}{y} = \frac{208}{16} = \frac{13}{1}$ *i.e.* $\boldsymbol{x : y = 13 : 1}$ **Ans.**

25 If $a : b = c : d$, show that : $3a + 2b : 3a - 2b = 3c + 2d : 3c - 2d$.

Solution :

$a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d}$

$$\Rightarrow \quad \frac{3a}{2b} = \frac{3c}{2d} \quad \text{[Multiplying each side by } \frac{3}{2}\text{]}$$

$$\Rightarrow \quad \frac{3a+2b}{3a-2b} = \frac{3c+2d}{3c-2d} \quad \text{[By componendo and dividendo]}$$

$\Rightarrow$ **$3a + 2b : 3a - 2b = 3c + 2d : 3c - 2d$** **Ans.**

Alternative method :

$$a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d} = k \text{ (let)}$$

$$\Rightarrow \quad a = bk \text{ and } c = dk$$

$$\therefore 3a + 2b : 3a - 2b = \frac{3a+2b}{3a-2b} = \frac{3bk+2b}{3bk-2b} \quad \text{[As } a = bk\text{]}$$

$$= \frac{b(3k+2)}{b(3k-2)} = \frac{3k+2}{3k-2} \quad \text{.....I}$$

$$\text{and } 3c + 2d : 3c - 2d = \frac{3c+2d}{3c-2d} = \frac{3dk+2d}{3dk-2d} \quad \text{[As } c = dk\text{]}$$

$$= \frac{d(3k+2)}{d(3k-2)} = \frac{3k+2}{3k-2} \quad \text{.....II}$$

From I and II, we get :

$3a + 2b : 3a - 2b = 3c + 2d : 3c - 2d$ **Ans.**

26 If $\frac{8a-5b}{8c-5d} = \frac{8a+5b}{8c+5d}$, prove that $\frac{a}{b} = \frac{c}{d}$. **[2008]**

Solution :

$$\text{Given : } \frac{8a-5b}{8c-5d} = \frac{8a+5b}{8c+5d}$$

$$\Rightarrow \quad \frac{8a-5b}{8a+5b} = \frac{8c-5d}{8c+5d} \quad \text{[Applying alternendo]}$$

$$\Rightarrow \quad \frac{8a-5b+8a+5b}{8a-5b-8a-5b} = \frac{8c-5d+8c+5d}{8c-5d-8c-5d} \quad \text{[Applying componendo and dividendo]}$$

$$\Rightarrow \quad \frac{16a}{-10b} = \frac{16c}{-10d}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{c}{d}$$

Hence Proved.

27 If $p = \frac{4xy}{x+y}$, find the value of $\frac{p+2x}{p-2x} + \frac{p+2y}{p-2y}$.

Solution :

$$p = \frac{4xy}{x+y} \Rightarrow \frac{P}{2x} = \frac{2y}{x+y} \qquad \text{[Now apply componendo and dividendo]}$$

$$\Rightarrow \frac{p+2x}{p-2x} = \frac{2y+x+y}{2y-x-y} = \frac{x+3y}{y-x}$$

Again, $p = \dfrac{4xy}{x+y} \Rightarrow \dfrac{P}{2y} = \dfrac{2x}{x+y}$ [Now apply componendo and dividendo]

$$\Rightarrow \frac{p+2y}{p-2y} = \frac{2x+x+y}{2x-x-y} = \frac{3x+y}{x-y}$$

$$\therefore \frac{p+2x}{p-2x} + \frac{p+2y}{p-2y} = \frac{x+3y}{y-x} + \frac{3x+y}{x-y}$$

$$= \frac{x+3y}{y-x} - \frac{3x+y}{y-x} = \frac{x+3y-3x-y}{y-x} = 2 \qquad \textbf{Ans.}$$

Alternative method :

$$\frac{p+2x}{p-2x} + \frac{p+2y}{p-2y} = \frac{\frac{4xy}{x+y}+2x}{\frac{4xy}{x+y}-2x} + \frac{\frac{4xy}{x+y}+2y}{\frac{4xy}{x+y}-2y}$$

$$= \frac{4xy+2x(x+y)}{4xy-2x(x+y)} + \frac{4xy+2y(x+y)}{4xy-2y(x+y)}$$

$$= \frac{4xy+2x^2+2xy}{4xy-2x^2-2xy} + \frac{4xy+2xy+2y^2}{4xy-2xy-2y^2}$$

$$= \frac{6xy+2x^2}{2xy-2x^2} + \frac{6xy+2y^2}{2xy-2y^2}$$

$$= \frac{2x(3y+x)}{2x(y-x)} + \frac{2y(3x+y)}{2y(x-y)}$$

$$= \frac{3y+x}{y-x} - \frac{3x+y}{y-x} \qquad \left[\because \frac{3x+y}{x-y} = -\frac{3x+y}{y-x}\right]$$

$$= \frac{3y+x-3x-y}{y-x} = \frac{2y-2x}{y-x} = \frac{2(y-x)}{y-x} = 2 \qquad \textbf{Ans.}$$

28 If $a : b = c : d$; prove that :

$(a^2 + ac + c^2) : (a^2 - ac + c^2) = (b^2 + bd + d^2) : (b^2 - bd + d^2)$

Solution :

$$a : b = c : d \Rightarrow \frac{a}{b} = \frac{c}{d} = k \text{ (say)} \Rightarrow a = bk \text{ and } c = dk$$

$$\therefore (a^2+ac+c^2) : (a^2-ac+c^2) = \frac{a^2+ac+c^2}{a^2-ac+c^2}$$

$$= \frac{b^2k^2 + (bk)(dk) + d^2k^2}{b^2k^2 - (bk)(dk) + d^2k^2} \quad [\because a = bk \text{ and } c = dk]$$

$$= \frac{k^2(b^2 + bd + d^2)}{k^2(b^2 - bd + d^2)}$$

$$= (b^2 + bd + d^2) : (b^2 - bd + d^2)$$

Hence Proved.

29 If x, y and z are in continued proportion, prove that :
$x^2 - y^2 : x^2 + y^2 = x - z : x + z$.

Solution :

x, y and z are in continued proportion

$$\Rightarrow \quad \frac{x}{y} = \frac{y}{z} = k \text{ (say)} \quad \Rightarrow \quad x = yk,\ y = zk \text{ and } x = yk = (zk)\,k = zk^2$$

$$\therefore \quad x^2 - y^2 : x^2 + y^2 = \frac{x^2 - y^2}{x^2 + y^2} = \frac{y^2k^2 - y^2}{y^2k^2 + y^2} \quad [\because x = yk]$$

$$= \frac{y^2(k^2 - 1)}{y^2(k^2 + 1)} = \frac{k^2 - 1}{k^2 + 1} \quad \text{....I}$$

$$\text{Also, } x - z : x + z = \frac{x - z}{x + z} = \frac{zk^2 - z}{zk^2 + z} \quad [\because x = zk^2]$$

$$= \frac{z(k^2 - 1)}{z(k^2 + 1)} = \frac{k^2 - 1}{k^2 + 1} \quad \text{.....II}$$

From I and II, we get :

$$x^2 - y^2 : x^2 + y^2 = x - z : x + z$$ **Hence Proved.**

Alternative method :

x, y and z are in continued proportion $\Rightarrow \frac{x}{y} = \frac{y}{z} \Rightarrow y^2 = xz$.

$$\therefore x^2 - y^2 : x^2 + y^2 = \frac{x^2 - y^2}{x^2 + y^2} = \frac{x^2 - xz}{x^2 + xz} \quad [\because y^2 = xz]$$

$$= \frac{x(x - z)}{x(x + z)} = \frac{x - z}{x + z} = x - z : x + z$$ **Hence Proved.**

30 Using the properties of proportion, solve the following equation for x :

$$\frac{x^3 + 3x}{3x^2 + 1} = \frac{341}{91}$$

Solution :

Applying componendo and dividendo, we get :

$$\frac{x^3 + 3x + 3x^2 + 1}{x^3 + 3x - 3x^2 - 1} = \frac{341 + 91}{341 - 91}$$

$$\Rightarrow \quad \frac{(x+1)^3}{(x-1)^3} = \frac{432}{250} = \frac{216}{125} = \left(\frac{6}{5}\right)^3$$

$$\Rightarrow \quad \frac{x+1}{x-1} = \frac{6}{5}$$

Again, applying componendo and dividendo, we get :

$$\frac{x+1+x-1}{x+1-x+1} = \frac{6+5}{6-5} \quad i.e. \quad \frac{2x}{2} = \frac{11}{1} \Rightarrow x = 11$$ **Ans.**

31 If $x = \dfrac{\sqrt{3a+2b}+\sqrt{3a-2b}}{\sqrt{3a+2b}-\sqrt{3a-2b}}$, prove that : $bx^2 - 3ax + b = 0$

Solution :

Given : $$\frac{x}{1} = \frac{\sqrt{3a+2b}+\sqrt{3a-2b}}{\sqrt{3a+2b}-\sqrt{3a-2b}}$$

$$\Rightarrow \quad \frac{x+1}{x-1} = \frac{\sqrt{3a+2b}+\sqrt{3a-2b}+\sqrt{3a+2b}-\sqrt{3a-2b}}{\sqrt{3a+2b}+\sqrt{3a-2b}-\sqrt{3a+2b}+\sqrt{3a-2b}}$$

[Applying componendo and dividendo]

$$\Rightarrow \quad \frac{x+1}{x-1} = \frac{2\sqrt{3a+2b}}{2\sqrt{3a-2b}}$$

$$\Rightarrow \quad \frac{x+1}{x-1} = \frac{\sqrt{3a+2b}}{\sqrt{3a-2b}}$$

$$\Rightarrow \quad \frac{x^2+2x+1}{x^2-2x+1} = \frac{3a+2b}{3a-2b}$$ [Squaring both the sides]

$$\Rightarrow \quad \frac{x^2+2x+1+x^2-2x+1}{x^2+2x+1-x^2+2x-1} = \frac{3a+2b+3a-2b}{3a+2b-3a+2b}$$ [By componendo and dividendo]

$$\Rightarrow \quad \frac{2(x^2+1)}{4x} = \frac{6a}{4b}$$

$$\Rightarrow \quad \frac{x^2+1}{2x} = \frac{3a}{2b} \quad i.e., \; 2bx^2 + 2b = 6ax$$

$$\Rightarrow \quad bx^2 - 3ax + b = 0$$ **Hence Proved.**

Alternative method :

Given : $$x = \frac{\sqrt{3a+2b}+\sqrt{3a-2b}}{\sqrt{3a+2b}-\sqrt{3a-2b}}$$

$$\Rightarrow \quad x = \frac{\sqrt{3a+2b}+\sqrt{3a-2b}}{\sqrt{3a+2b}-\sqrt{3a-2b}} \times \frac{\sqrt{3a+2b}+\sqrt{3a-2b}}{\sqrt{3a+2b}+\sqrt{3a-2b}}$$

$$= \frac{3a+2b+3a-2b+2\sqrt{(3a+2b)(3a-2b)}}{3a+2b-3a+2b}$$

$$= \frac{6a+2\sqrt{9a^2-4b^2}}{4b} = \frac{3a+\sqrt{9a^2-4b^2}}{2b}$$

$$\Rightarrow \quad 2bx = 3a + \sqrt{9a^2 - 4b^2}$$

$$\Rightarrow \quad 2bx - 3a = \sqrt{9a^2 - 4b^2}$$

$$\Rightarrow \quad 4b^2x^2 + 9a^2 - 12abx = 9a^2 - 4b^2 \qquad \text{[Squaring]}$$

$$\Rightarrow \quad 4b^2x^2 - 12abx + 4b^2 = 0 \Rightarrow \mathbf{bx^2 - 3ax + b = 0}$$

Hence Proved.

EXERCISE 7(C)

1. If $a : b = c : d$, prove that :
   (i) $5a + 7b : 5a - 7b = 5c + 7d : 5c - 7d$.
   (ii) $(9a + 13b)(9c - 13d) = (9c + 13d)(9a - 13b)$.
   (iii) $xa + yb : xc + yd = b : d$

2. If $a : b = c : d$, prove that :
   $(6a + 7b)(3c - 4d) = (6c + 7d)(3a - 4b)$.

3. Given, $\frac{a}{b} = \frac{c}{d}$, prove that :
   $\frac{3a - 5b}{3a + 5b} = \frac{3c - 5d}{3c + 5d}$ **[2000]**

4. If $\frac{5x + 6y}{5u + 6v} = \frac{5x - 6y}{5u - 6v}$;
   then prove that $x : y = u : v$.

5. If $(7a + 8b)(7c - 8d) = (7a - 8b)(7c + 8d)$;
   prove that $a : b = c : d$.

6. (i) If $x = \frac{6ab}{a + b}$, find the value of :
   $\frac{x + 3a}{x - 3a} + \frac{x + 3b}{x - 3b}$.
   (ii) If $a = \frac{4\sqrt{6}}{\sqrt{2} + \sqrt{3}}$, find the value of :
   $\frac{a + 2\sqrt{2}}{a - 2\sqrt{2}} + \frac{a + 2\sqrt{3}}{a - 2\sqrt{3}}$.

7. If $(a + b + c + d)(a - b - c + d) = (a + b - c - d)(a - b + c - d)$;
   prove that : $a : b = c : d$.

8. If $\frac{a - 2b - 3c + 4d}{a + 2b - 3c - 4d} = \frac{a - 2b + 3c - 4d}{a + 2b + 3c + 4d}$,
   show that : $2ad = 3bc$.

9. If $(a^2 + b^2)(x^2 + y^2) = (ax + by)^2$;
   prove that : $\frac{a}{x} = \frac{b}{y}$.

10. If a, b and c are in continued proportion, prove that :
    (i) $\frac{a^2 + ab + b^2}{b^2 + bc + c^2} = \frac{a}{c}$
    (ii) $\frac{a^2 + b^2 + c^2}{(a + b + c)^2} = \frac{a - b + c}{a + b + c}$.

11. Using properties of proportion, solve for x :
    (i) $\frac{\sqrt{x + 5} + \sqrt{x - 16}}{\sqrt{x + 5} - \sqrt{x - 16}} = \frac{7}{3}$.
    (ii) $\frac{\sqrt{x + 1} + \sqrt{x - 1}}{\sqrt{x + 1} - \sqrt{x - 1}} = \frac{4x - 1}{2}$.
    (iii) $\frac{3x + \sqrt{9x^2 - 5}}{3x - \sqrt{9x^2 - 5}} = 5$.

12. If $x = \frac{\sqrt{a + 3b} + \sqrt{a - 3b}}{\sqrt{a + 3b} - \sqrt{a - 3b}}$, prove that :
    $3bx^2 - 2ax + 3b = 0$. **[2007]**

13. Using the properties of proportion, solve for x, given $\frac{x^4 + 1}{2x^2} = \frac{17}{8}$. **[2013]**

14. If $x = \dfrac{\sqrt{m+n}+\sqrt{m-n}}{\sqrt{m+n}-\sqrt{m-n}}$, express n in terms of x and m.

15. If $\dfrac{x^3+3xy^2}{3x^2y+y^3} = \dfrac{m^3+3mn^2}{3m^2n+n^3}$, show that : $nx = my$.

EXERCISE 7(D)

1. If $a : b = 3 : 5$, find :
   $(10a + 3b) : (5a + 2b)$
2. If $5x + 6y : 8x + 5y = 8 : 9$, find : $x : y$.
3. If $(3x - 4y) : (2x - 3y)$
   $= (5x - 6y) : (4x - 5y)$, find : $x : y$.
4. Find the :
   (i) duplicate ratio of $2\sqrt{2} : 3\sqrt{5}$
   (ii) triplicate ratio of $2a : 3b$,
   (iii) sub-duplicate ratio of $9x^2a^4 : 25y^6b^2$
   (iv) sub-triplicate ratio of $216 : 343$
   (v) reciprocal ratio of $3 : 5$
   (vi) ratio compounded of the duplicate ratio of $5 : 6$, the reciprocal ratio of $25 : 42$ and the sub-duplicate ratio of $36 : 49$.
5. Find the value of x, if :
   (i) $(2x + 3) : (5x - 38)$ is the duplicate ratio of $\sqrt{5} : \sqrt{6}$.
   (ii) $(2x + 1) : (3x + 13)$ is the sub-duplicate ratio of $9 : 25$.
   (iii) $(3x - 7) : (4x + 3)$ is the sub-triplicate ratio of $8 : 27$.
6. What quantity must be added to each term of the ratio $x : y$ so that it may become equal to $c : d$?
7. A woman reduces her weight in the ratio $7 : 5$. What does her weight become if originally it was 84 kg ?
8. If $15(2x^2 - y^2) = 7xy$, find $x : y$; if x and y both are positive.
9. Find the :
   (i) fourth proportional to $2xy$, x^2 and y^2.
   (ii) third proportional to $a^2 - b^2$ and $a + b$.
   (iii) mean proportion to $(x - y)$ and $(x^3 - x^2y)$
10. Find two numbers such that the mean proportional between them is 14 and third proportional to them is 112.
11. If x and y be unequal and $x : y$ is the duplicate ratio of $x + z$ and $y + z$, prove that z is mean proportional between x and y.
12. If q is the mean proportional between p and r, prove that : $\dfrac{p^3+q^3+r^3}{p^2q^2r^2} = \dfrac{1}{p^3} + \dfrac{1}{q^3} + \dfrac{1}{r^3}$.
13. If a, b and c are in continued proportion, prove that : $a : c = (a^2 + b^2) : (b^2 + c^2)$.
14. If $x = \dfrac{2ab}{a+b}$, find the value of : $\dfrac{x+a}{x-a} + \dfrac{x+b}{x-b}$.
15. If $(4a + 9b)(4c - 9d) = (4a - 9b)(4c + 9d)$, prove that : $a : b = c : d$.
16. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that :
   $(a + b) : (c + d) = \sqrt{a^2+b^2} : \sqrt{c^2+d^2}$
17. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that :
   $$\frac{ax-by}{(a+b)(x-y)} + \frac{by-cz}{(b+c)(y-z)} + \frac{cz-ax}{(c+a)(z-x)} = 3$$
18. There are 36 members in a student council in a school and the ratio of the number of boys to the number of girls is $3 : 1$. How many more girls should be added to the council so that the ratio of number of boys to the number of girls may be $9 : 5$?
19. If $7x - 15y = 4x + y$, find the value of $x : y$. Hence, use componendo and dividendo to find the values of :
   (i) $\dfrac{9x+5y}{9x-5y}$ (ii) $\dfrac{3x^2+2y^2}{3x^2-2y^2}$
20. If $\dfrac{4m+3n}{4m-3n} = \dfrac{7}{4}$, use properties of proportion to find :
   (i) $m : n$ (ii) $\dfrac{2m^2-11n^2}{2m^2+11n^2}$.

21. If x, y, z are in continued proportion, prove that : $\dfrac{(x+y)^2}{(y+z)^2} = \dfrac{x}{z}$. **[2010]**

22. Given $x = \dfrac{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-\sqrt{a^2-b^2}}$.

Use componendo and dividendo to prove that :

$b^2 = \dfrac{2a^2x}{x^2+1}$. **[2010]**

23. If $\dfrac{x^2+y^2}{x^2-y^2} = 2\dfrac{1}{8}$, find :

(i) $\dfrac{x}{y}$ (ii) $\dfrac{x^3+y^3}{x^3-y^3}$ **[2014]**

24. Using componendo and dividendo, find the value of x :

$\dfrac{\sqrt{3x+4}+\sqrt{3x-5}}{\sqrt{3x+4}-\sqrt{3x-5}} = 9$. **[2011]**

25. If $x = \dfrac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}$, using properties of proportion show that :

$x^2 - 2ax + 1 = 0$. **[2012]**

26. Given $\dfrac{x^3+12x}{6x^2+8} = \dfrac{y^3+27y}{9y^2+27}$. Using componendo and dividendo, find $x : y$.

[2015]