COMP 250

Lecture 16

big O …., big Omega Ω, big Theta Θ

Oct. 17, 2016

Recall motivation for O( )

The time it takes to perform instructions depends on:

- the size of the input (n, m, N, …)

- implementation details *that are unknown*
  (various constants c1, c2, ….)

Example: Grade School Addition

$carry = 0$ 1

for $i = 0$ to $N - 1$ **do**

$\quad r[i] \leftarrow (a[i] + b[i] + carry)\ \%\ 10$

$\quad carry \leftarrow (a[i] + b[i] + carry)/10$ N

end for

$r[N] \leftarrow carry$ 1

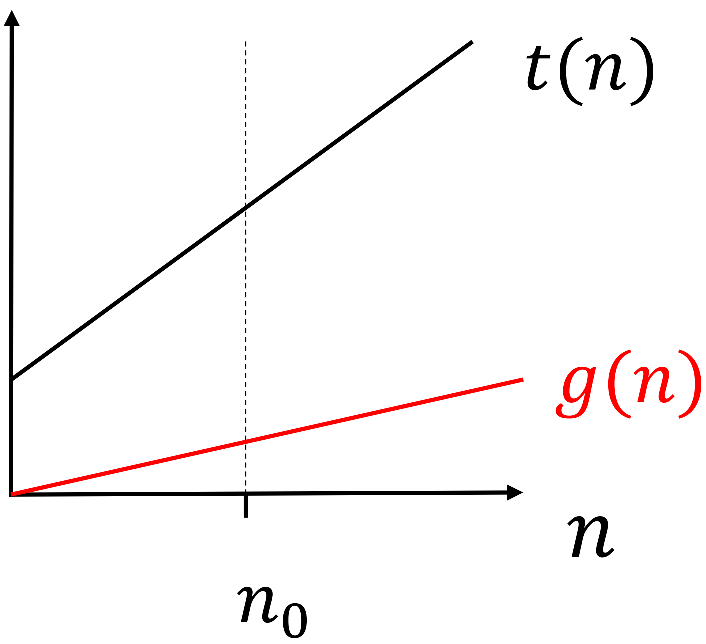
Example: Grade School Addition

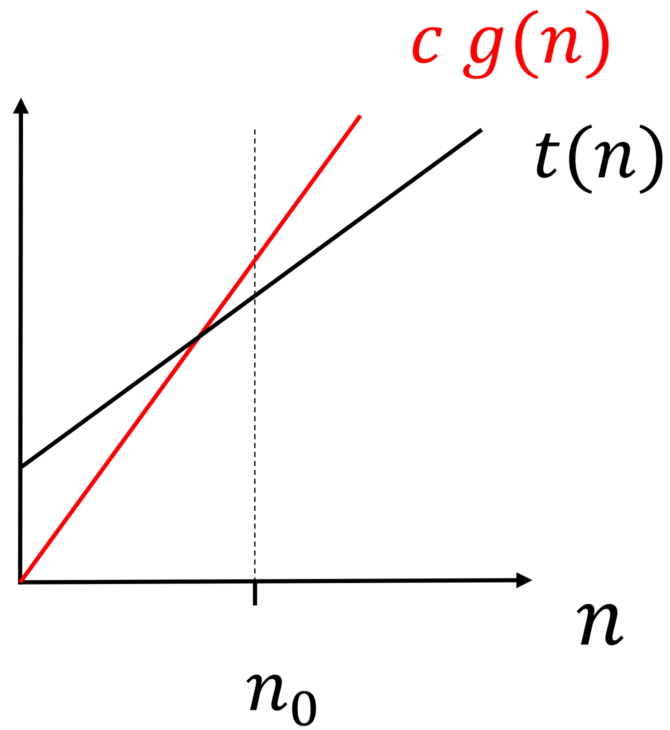
$$t(n) = \; c_0 \; + \; c_1 N$$

Constants c_0, c_1 … include time for :

- Primitive ops +,- , *, /, %
- Array address indexing
- Array get or set
- Assignment
- For loop administration
- Recursive call administration
- ……..

Recall last lecture: big O





$t(n)$ is $O\big(\ g(n)\big)$.

Formal Definition of Big O

Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.

We say $t(n)$ is $O(\ g(n)\ )$, if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\,n\,) \leq \; c \;\; g(\,n\,).$$

Never write $O(\, 3n \,)$, $O(\, 5\, log_2 n \,)$, etc.

The point of big O notation is to avoid dealing with constants.

It is still *technically* correct to write the above. We just don't do it.

Sets of O( ) functions

Each of the following holds for n sufficiently large:

$$1 < log_2 n < n < n\, log_2 n < n^2 < n^3 < \ldots < 2^n < n!$$

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Suppose $t(n)$ is $O(\, g(n)\,)$, and $g(n) < h(n)$ for $n \geq n_0$.

Then, $t(n)$ is $O(\, h(n)\,)$.

e.g. if $t(n)$ is $O(\, n\,)$, then $t(n)$ is $O(\, n^2\,), O(\, n^3\,), \ldots$

Sets of O( ) functions

If $t(n)$ is $O(\ g(n)\ )$, one often writes $t(n) \in O(\ g(n)\ )$,

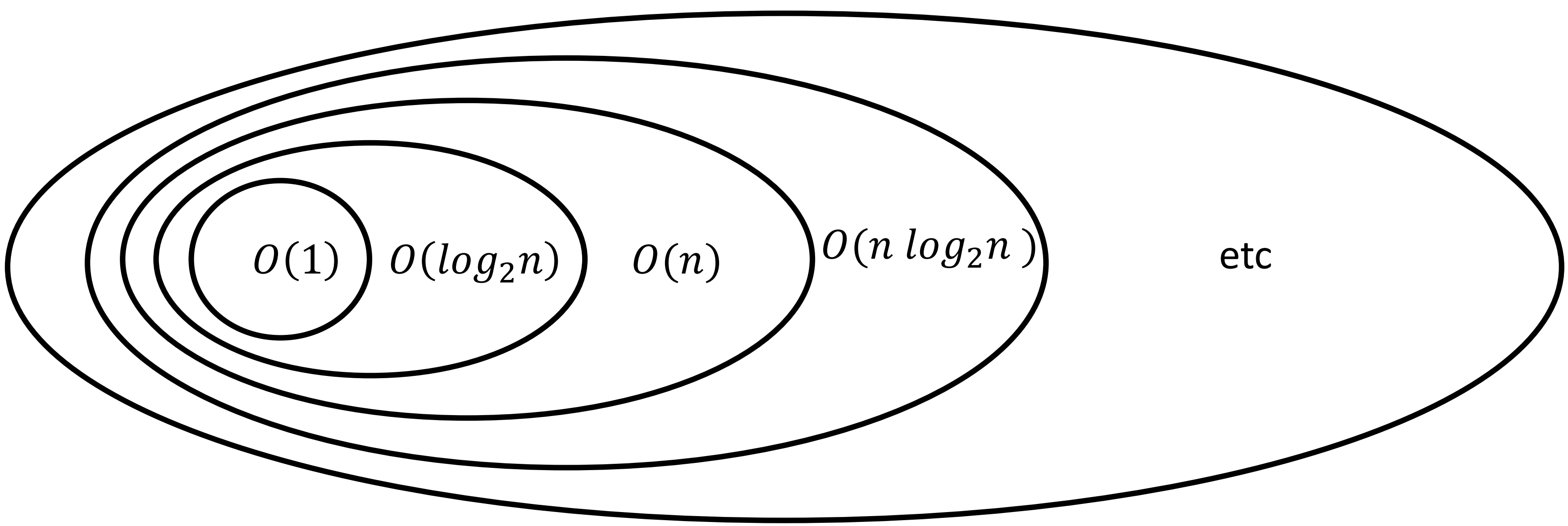
That is, $t(n)$ is a member of the set of functions that are $O(\ g(n)\ )$.

Sets of O( ) functions

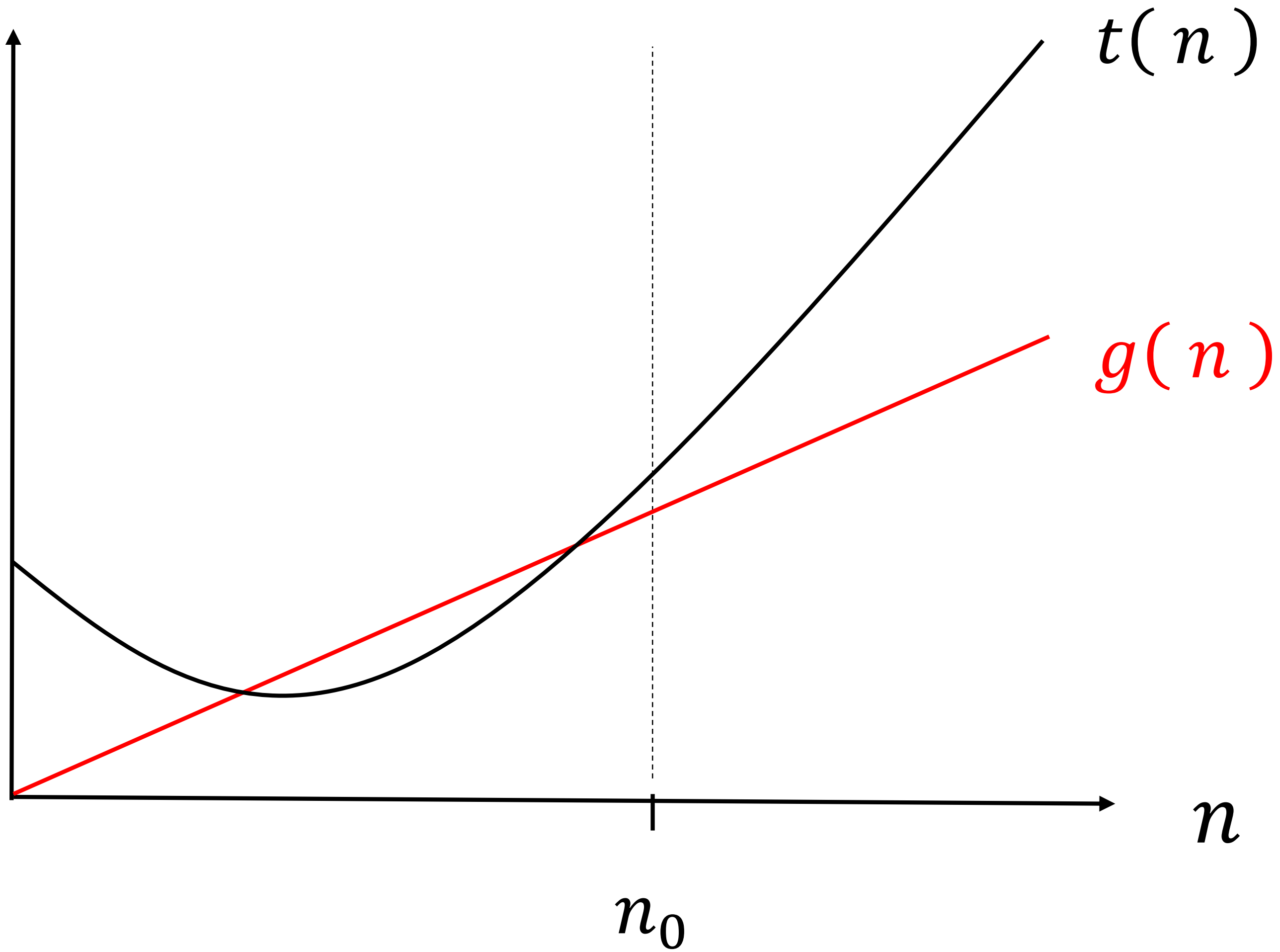
We have the following strict subset relationships:

$$O(1) \subset O(log_2 n) \subset O(n) \subset O(n\, log_2 n) \subset O(n^2) \ldots$$

$$\subset O(n^3) \subset \ldots \subset O(2^n) \subset O(n!)$$



Asymptotic *lower* bound



The constant n_0 is not unique.

Preliminary Definition

$t(n)$ is *asymptotically bounded* **below** by $g(n)$ if there exists an n_0 such that, for all $n \geq n_0$,

$$t(n) \geq g(n).$$

Example: $t(n) = \frac{n(n-1)}{2}$ is asymptotically bounded *below* by $g(n) = \frac{n^2}{4}$.

Proof:

$$\frac{n(n-1)}{2} \geq \frac{n^2}{4}$$

$\Leftrightarrow \quad 2n(n-1) \geq n^2$

$\Leftrightarrow \quad n^2 \geq 2n$

$\Leftrightarrow \quad n \geq 2$ So take $n_0 = 2$.

Definition of Big Omega (Ω)

Let $t(n)$ and $g(n)$ be two functions of $n \geq 0$.

We say $t(n)$ is $\Omega(g(n))$, if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(n) \geq c \; g(n).$$

Claim: $\frac{n(n-1)}{2}$ is $\Omega(n^2)$.

Proof (1): Use $c = \frac{1}{4}$ from two slides ago.

$$\frac{n(n-1)}{2} \geq \frac{n^2}{4}$$

$\Leftrightarrow$ $\vdots$

$\Leftrightarrow$ $n \geq 2$ So take $n_0 = 2, \quad c = \frac{1}{4}$.

Claim: $\frac{n(n-1)}{2}$ is $\Omega(n^2)$.

Proof (2): Try $c = \frac{1}{3}$

$$\frac{n(n-1)}{2} \geq \frac{n^2}{3}$$

$\vdots$ ← you can fill this in

$\Longleftrightarrow$ $n \geq 3$ So take $n_0 = 3$, $c = \frac{1}{3}$.

Sets of $\Omega(\,)$ functions

Claim : Suppose $t(n)$ is $\Omega(\, g(n) \,)$, and $g(n) > h(n)$ for $n \geq n_0$.

Then, $t(n)$ is $\Omega(\, h(n) \,)$. Proof follows straight from definition.

e.g. if $t(n)$ is $\Omega(n^3)$, then $t(n)$ is $\Omega(\, 1)$, $\Omega(\, n \,), \ldots, \Omega(\, n^2 \,)$

Sets of $\Omega(\ )$ functions

If $t(n)$ is $\Omega(\ g(n)\ )$, one often writes $t(n) \in \Omega(\ g(n)\ )$,

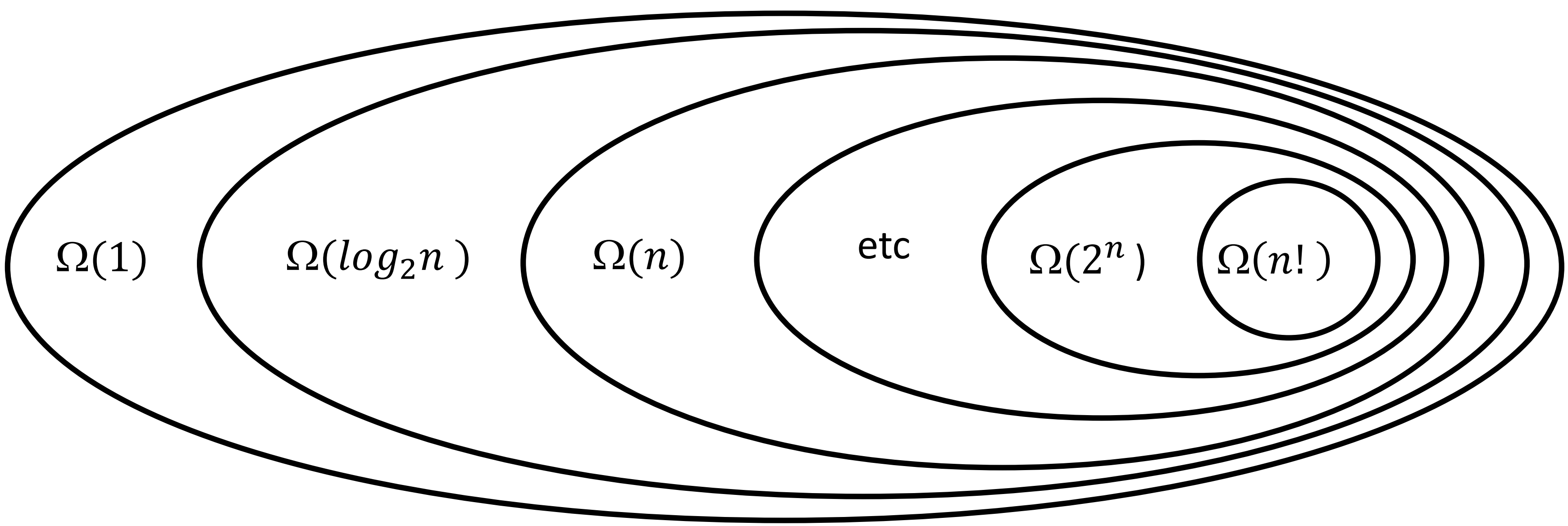
That is, $t(n)$ is a member of the set of functions that are $\Omega(\ g(n)\ )$.

Sets of $\Omega(\,)$ functions

Thus, we have the following strict subset relationships:

$$\Omega(1) \supset \Omega(log_2 n) \supset \Omega(n) \supset \Omega(n\, log_2 n) \supset \Omega(n^2) \ \ldots$$

$$\supset \Omega(n^3) \supset \ldots \ \Omega(2^n) \supset \Omega(n!)$$



Definition of Big Theta (Θ)

Let $t(n)$ and $g(n)$ be two functions of $n \geq 0$.

We say $t(n)$ is $\Theta(\ g(n)\ )$, if there exist three positive constants n_0, c_1, c_2 such that for all $n \geq n_0$,

$$c_1\ g(n) \leq t(n) \leq c_2\ g(n)$$

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$t(n)$ is $O(\ g(n)\ )$

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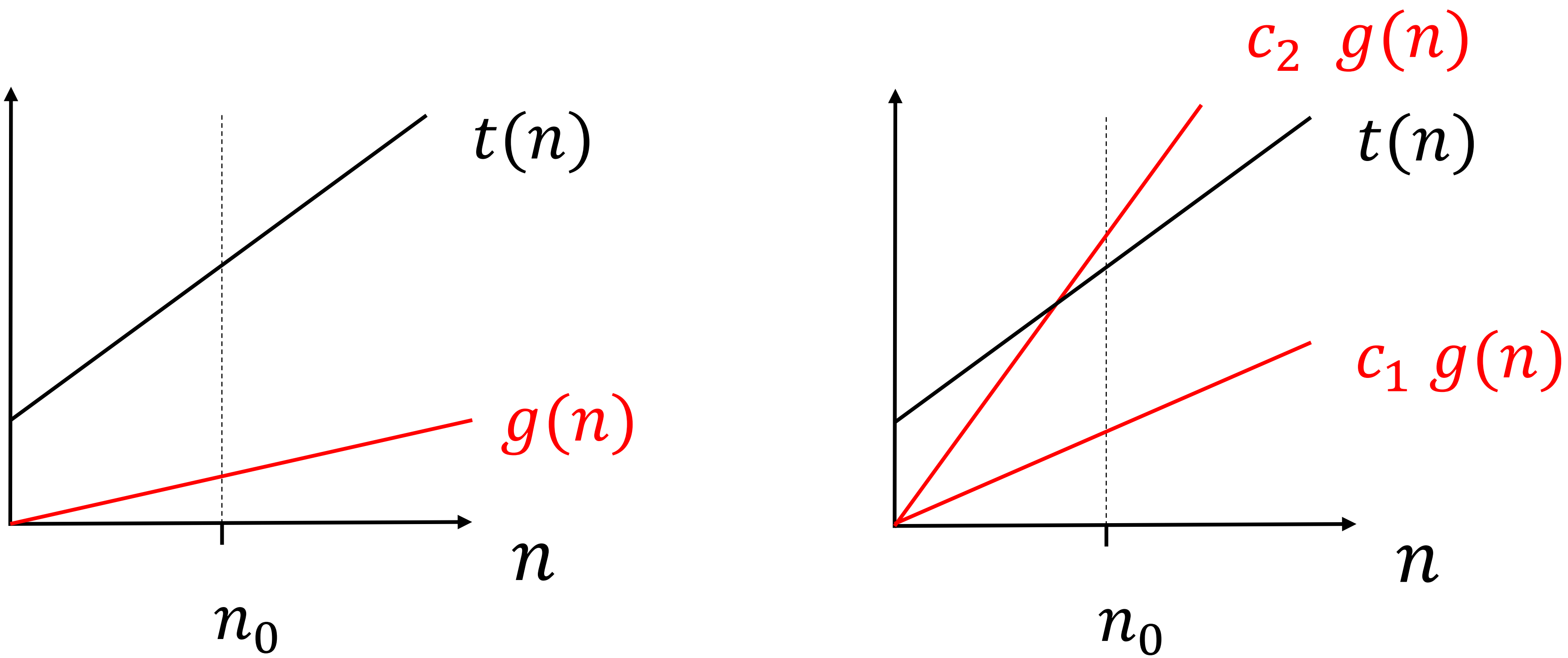
$$c_1\ g(\,n\,) \leq t(\,n\,) \leq c_2\ g(\,n\,)$$

$t(n)$ is $\Omega(\, g(n)\,)$

Example



$t(n)$ is $\Theta(\ g(n))$.

Example

Let $t(n) = 4 + 17\, log_2 n + 3n + 9\, n\, log_2 n + \frac{n(n-1)}{2}$

$t(n)$ is $\Theta($? $)$.

Example

Let $t(n)$ = $4 + 17\, log_2 n + 3n + 9\, n\, log_2 n + \frac{n(n-1)}{2}$

Claim: $t(n)$ is $\Theta(\ n^2\ )$.

Proof:

$$\frac{n^2}{4} \leq t(n) \leq (4 + 17 + 3 + 9 + \frac{1}{2})\ n^2$$

Can we write $\Theta(\,)$ for every $t(n)$?

No, as this contrived example shows:

Let $t(n) = \begin{cases} n, & n \text{ is odd} \\ n^2, & n \text{ is even.} \end{cases}$

$t(n)$ is $O(n^2)$ but not $O(n)$.

$t(n)$ is $\Omega(n)$ but not $\Omega(n^2)$.

There does not exist a $\Theta(\,)$ bound for this $t(n)$.

Algorithm best and worst cases

vs.

$O(\ ),\quad \Omega(\ ),\quad \Theta(\ ).$

What is relationship between these ideas?   See Exercises.

Sets of Θ ( ) functions

If $t(n)$ is $\Theta(\ g(n)\ )$, one often writes $t(n) \in \Theta(\ g(n)\ )$,

That is, $t(n)$ is a member of the set of functions that are $\Theta(\ g(n)\ )$.

$\Theta(1)$ $\Theta(log_2 n\ )$ $\Theta(n)$ … $\Theta(2^n\ )$ $\Theta(n!\ )$