COMP 250

Lecture 15

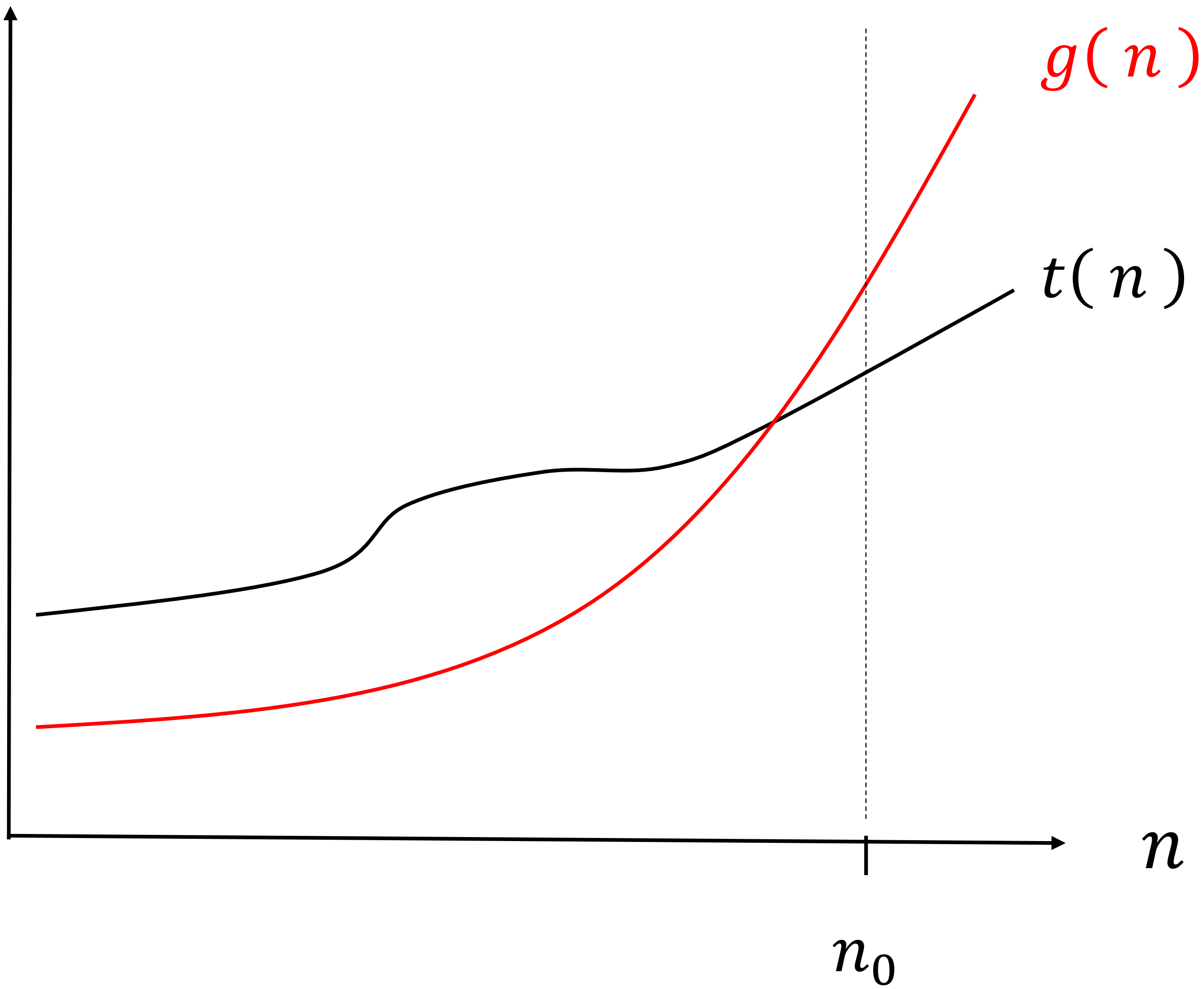
big O

Oct. 14, 2016

Motivation

We want to express how some function, $t(n)$, grows with n, as n becomes large.

We want to compare the function with *simpler* functions such as $log_2 n$, n, n^2, 2^n , etc.



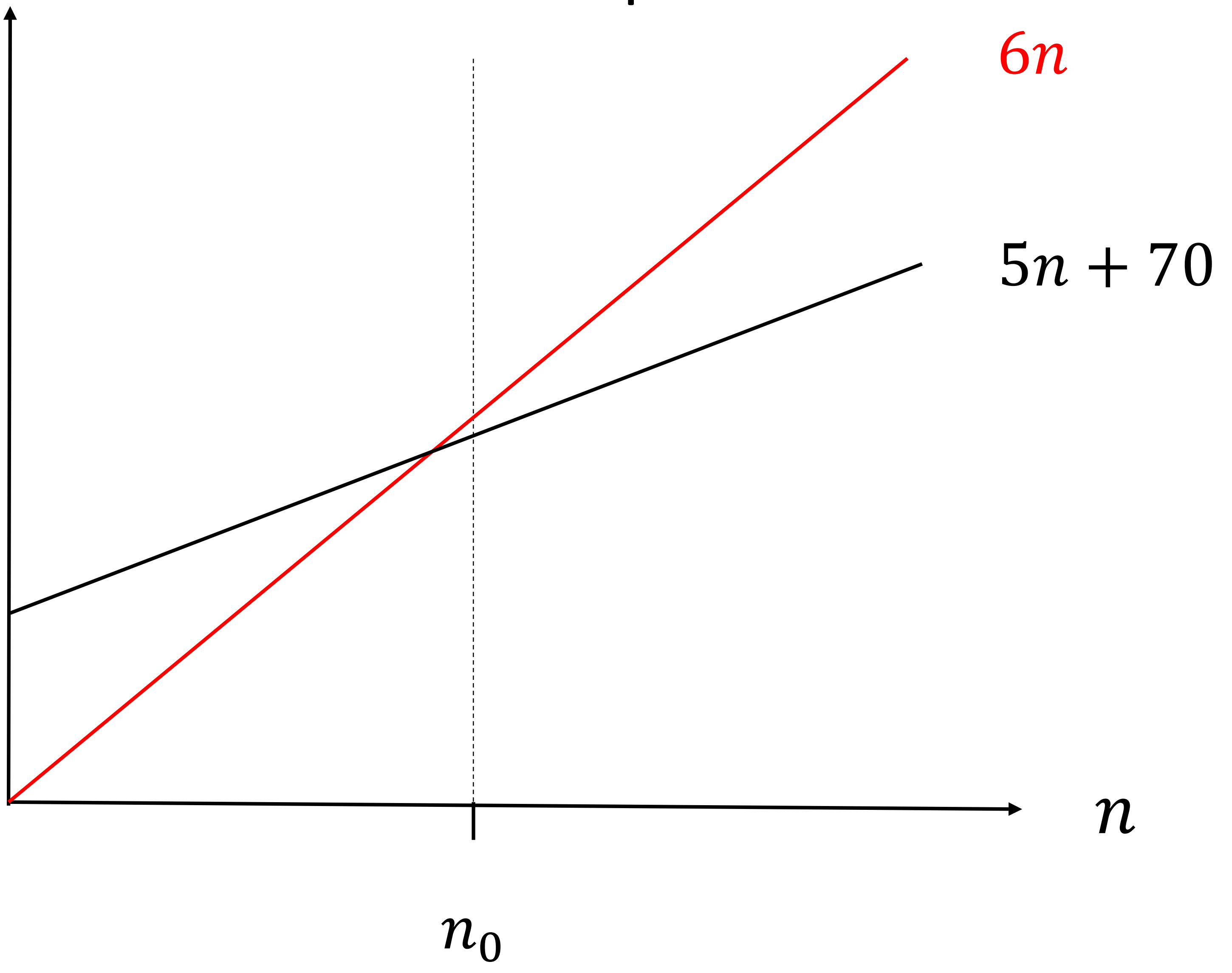
The constant n_0 is not unique.

Preliminary Definition

Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.
We say $t(n)$ is *asymptotically bounded* above by $g(n)$
if there exists an n_0 such that, for all $n \geq n_0$,

$$t(n) \leq g(n).$$

Example



Claim: $5n + 70$ is asymptotically bounded by $6n$.

Proof:

We want to show there exists an n_0 such that,
for all $n \geq n_0$, $5n + 70 \leq 6n$.

$$5n + 70 \leq 6n$$

$$\Longleftrightarrow \quad 70 \leq n$$

Thus, we can use $n_0 = 70$.

Means "if and only if" i.e. logically equivalent

Motivation for big O definition

Express how some function $t(n)$ grows with n as n becomes large.

Compare the function with *simpler* functions $g(n)$, such as $log_2 n$, n, $n \, log_2 n$, n^2, 2^n, etc.

Ignore constant factors, which depend on implementation.

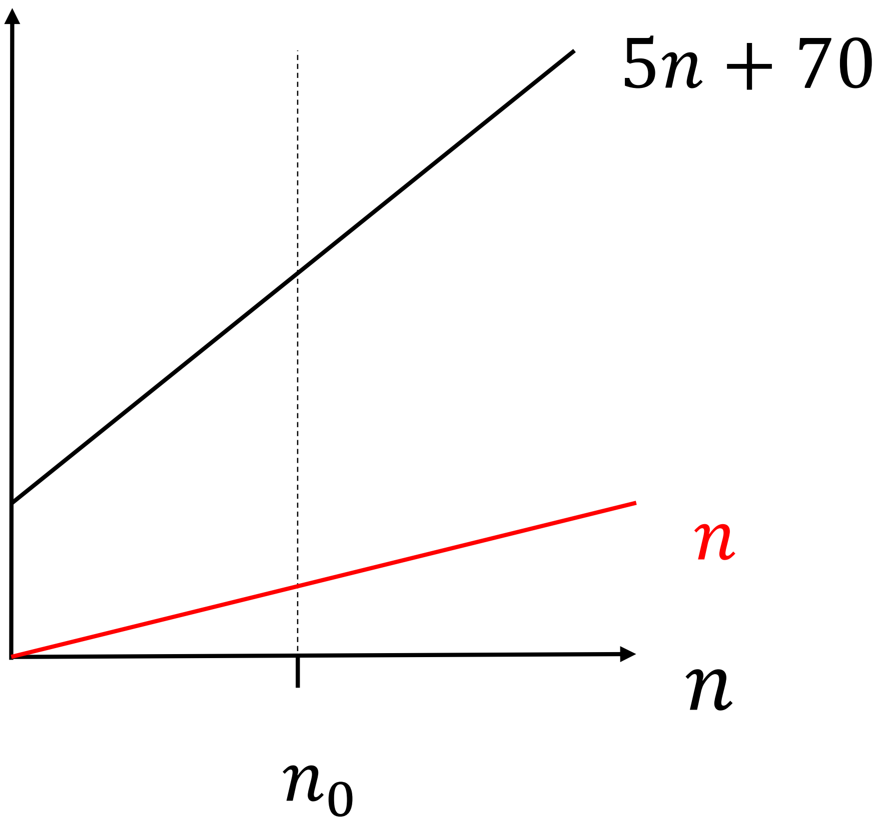
Definition of Big O

Let $t(n)$ and $g(n)$ be two functions, where $n \geq 0$.

We say $t(n)$ is $O(\, g(n)\,)$, if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(\, n\,) \leq c\ g(\, n\,).$$

Claim: $5\,n + 70$ is $O(\,n\,)$.



Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 1:

$$5\,n + 70 \;\le\; 5\,n + 70n, \quad n \ge 1$$

$$= \; 75\,n$$

So take $c = 75, \;\; n_0 = 1$.

Claim: $5\,n + 70$ is $O(\,n\,)$.

Proof 2:

$$5\,n + 70 \;\leq\; 5\,n + 6n, \quad n \geq 12$$

$$= \;11\,n$$

So take $c = 11$, $n_0 = 12$.

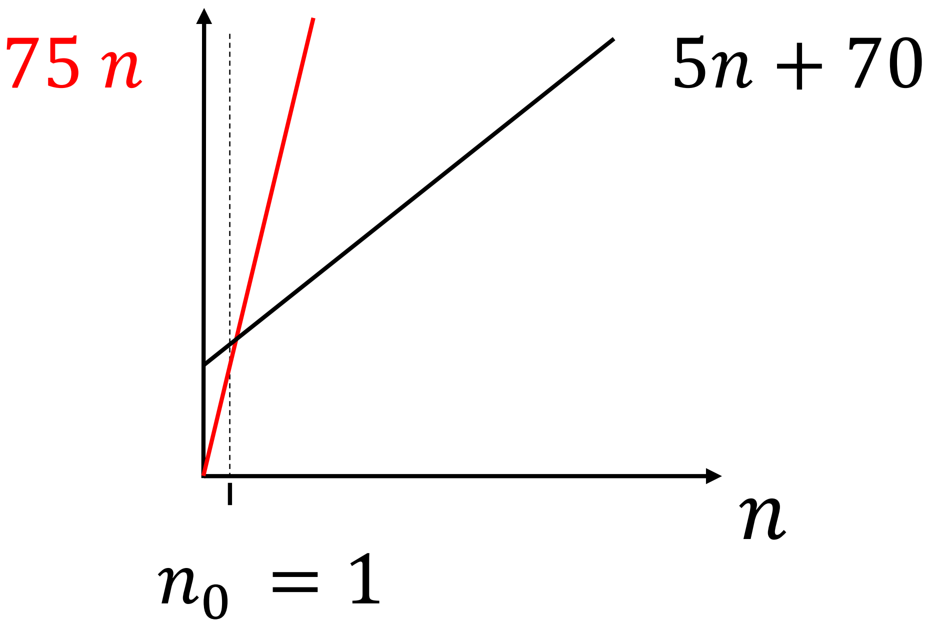
Claim: $5\,n + 70$ is $O(\,n\,)$.

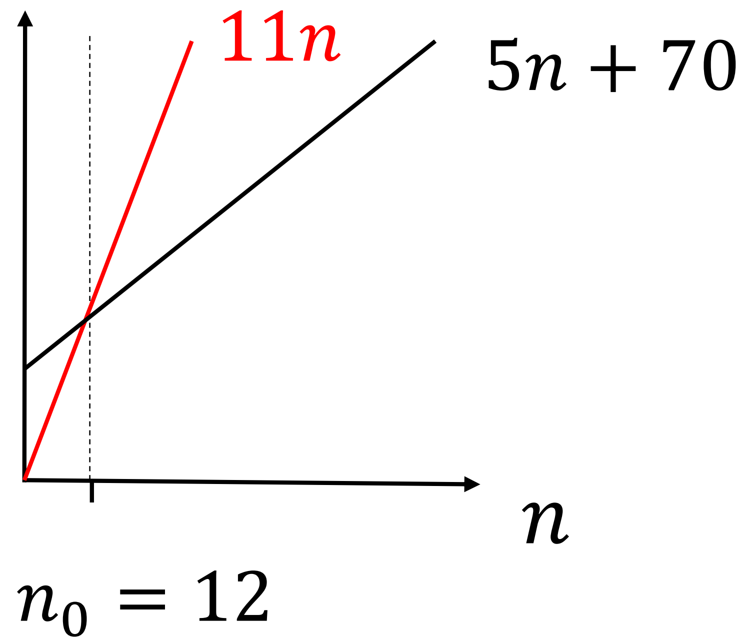
Proof 3:

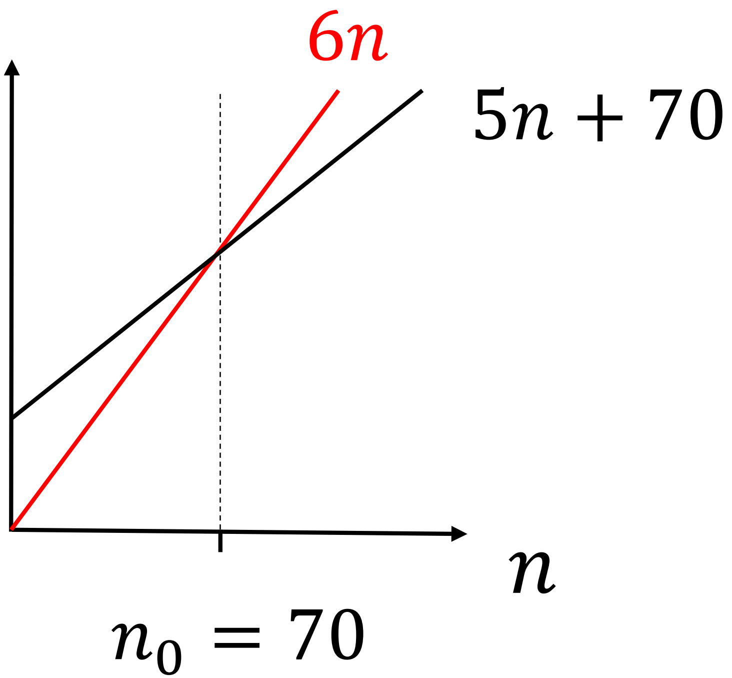
$$5\,n + 70 \;\leq\; 5\,n + n, \quad n \geq 70$$

$$= \;6\,n$$

So take $c = 6,\;\; n_0 = 70.$


$75\,n$
$5n + 70$
n
$n_0 = 1$


$11n$
$5n + 70$
n
$n_0 = 12$


$6n$
$5n + 70$
n
$n_0 = 70$

Claim: $8\,n^2 \; - 46n + 17$ is $O(n^2)$.

Proof (1):

$$8\,n^2 \; - 46n + 17$$

Claim: $8\,n^2 - 46n + 17$ is $O(n^2)$.

Proof (1):

$$8\,n^2 - 46n + 17$$

$$\leq 8\,n^2 + 17\,n^2, \qquad n \geq 1$$

$$\leq 25\,n^2$$

So take $c = 25,\ \ n_0 = 1.$

Claim: $8\,n^2 - 46n + 17$ is $O(n^2)$.

Proof (2):

$$8\,n^2 - 46n + 17$$

$$\leq 8\,n^2 \qquad + n^2, \qquad n \geq 5$$

$$\leq 9\,n^2$$

So take $c = 9$, $n_0 = 5$.

What does $O(1)$ mean?

We say $t(n)$ is $O(1)$, if there exist two positive constants n_0 and c such that, for all $n \geq n_0$,

$$t(n) \leq c.$$

Assignment 2 Question 2d

d) (10 points)

Let $t(n) = \sqrt{n^2 + 100\,n} - n$.

Use the formal definition of $O(\,)$ from lecture ~~14~~ to show that $t(n)\ is\ O(1)$.

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Correct vs. incorrect vs. incomplete proofs

Example of incorrect proof

Claim: for all $n > 0$, $2n^2 \leq (n+1)^2$.

Proof:

$$\begin{aligned} 2n^2 &\leq (n+1)^2 \\ &\leq (n+n)^2, \quad \text{when } n > 0 \\ &= 4n^2 \end{aligned}$$

Since $2n^2 \leq 4n^2$, we are done.

But the claim is false!

Claim: for all $n > 0$, $2n^2 \leq (n+1)^2$.

Proof:

$$
\begin{aligned}
2n^2 &\leq (n+1)^2 \\
&\leq (n+n)^2, \quad \text{when } n > 0 \\
&= 4n^2
\end{aligned}
$$

It is incorrect to assume what you are trying to prove.

Example: midterm exam Q7

7. **(2 points)**

Use mathematical induction to prove that, for all $n \geq 4$, $2^n \leq n!$.

<u>**SOLUTION:**</u>

```
                                                        GRADING POINTS
Base case....                                                 0.5

induction step....
   2^{k+1} =       2 * 2^k                                      0.5
           <   (k + 1)* 2^k,   when k >=  4                     0.5
           <=  (k + 1)* k! ,  by induction hypothesis           0.5
           <=  (k + 1)!
```

Proof : Q: What's wrong with it?

Base case: $16 \leq 24.$

Induction step:

$$2^{k+1} \leq (k+1)\,!$$

$$2 \; * 2^{\;k} \leq \; (k+1) \; * k!$$

(induction hypothesis)

$$2 \; * 2^{k} \leq (k+1) \; * 2^{k}$$

$2 \; \leq \; k+1$ which is true for $k \geq 4$

Proof :

Q: What's wrong with it?

Base case: $16 \leq 24.$

A: The proof is **incomplete.**

Induction step:

It isn't clear which statement implies which.

$$2^{k+1} \leq (k+1)\,!$$

(We gave either 0 or .5 / 1)

$$2 \, * 2^{\,k} \leq \; (k+1) \; * k!$$

(induction hypothesis)

$$2 \; * 2^{k} \leq (k+1) \; * 2^{k}$$

$$2 \; \leq \; \; k+1$$ which is true for $k \geq 4$

Claim: $5n + 70$ is $O(n)$.

We want to show there exists two constants n_0, c such that, for all $n \geq n_0$, $5n + 70 \leq cn$.

Proof : What's wrong with it?

$$5n + 70 \leq cn$$

$$5n + 70n \leq cn, \qquad n \geq 1$$

$$75n \leq cn$$

So take $c = 75$, $n_0 = 1$.

Claim: $5n + 70$ is $O(n)$.

We want to show there exists two constants n_0, c such that, for all $n \geq n_0$, $5n + 70 \leq cn$.

Proof : The proof is incomplete.

$$5n + 70 \leq cn$$

$$5n + 70n \leq cn, \qquad n \geq 1$$

$$75n \leq cn$$

Which inequality implies which ?
Does the first imply the second, or vice-versa ?
Does the first inequality refer to "some n" or "all n" or "some c" or "all c" ?

So take $c = 75$, $n_0 = 1$.

Announcements

Next week: one more big O lecture.

Then onto trees and graphs !