Digital Signal Analysis And Processing

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5. IIR Filter Design 7 Hrs

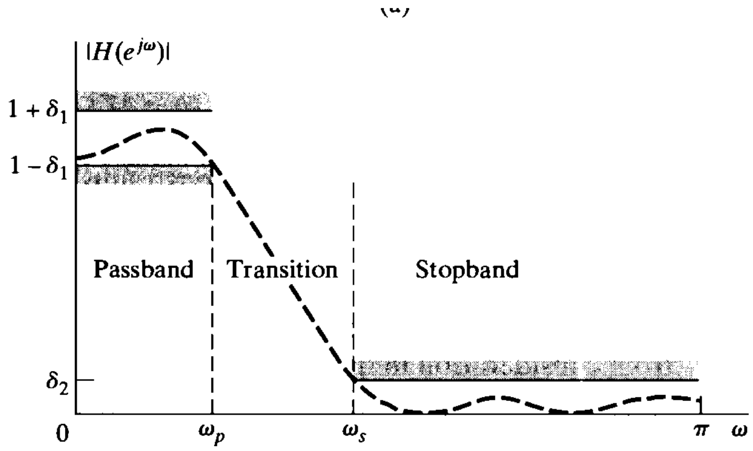
- 5.1 Filter Design using Impulse invariance method
- 5.2 Filter Design using Bilinear transformation
- 5.3 Design of digital low pass Butterworth filter
- 5.4 Properties of Chebyshev and Elliptic filters, properties of Bessel filter, spectral transformation.

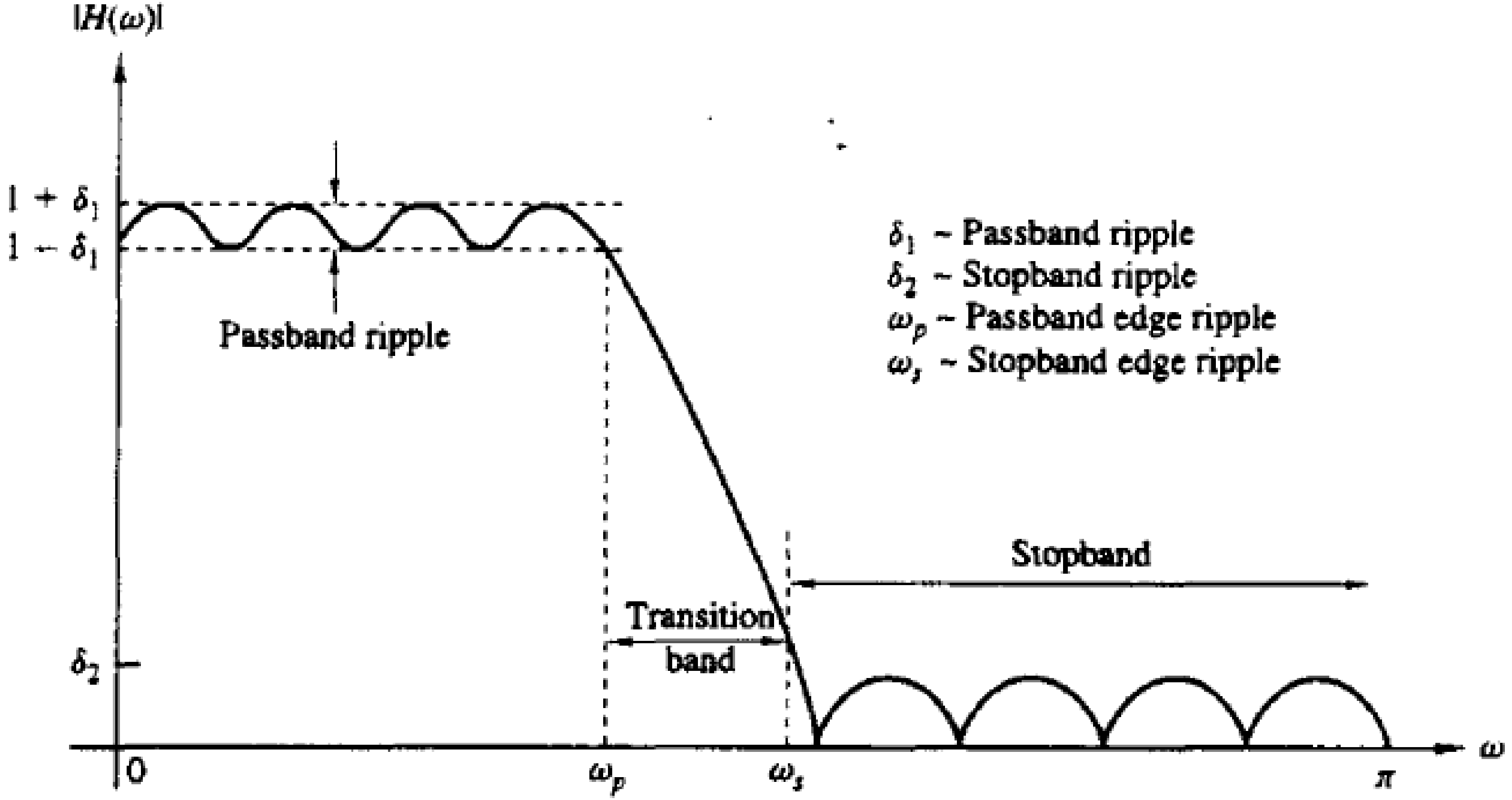
IIR Filter Design

- Filters are a particularly important class of linear time-invariant systems.
- The term frequency-selective filter suggests a system that passes certain frequency components and totally rejects all other, but in broader context any system that modifies certain frequencies to other is also called filter.
- The design of filters involves the following stages:

1. The specification of the desired properties of the system,
2. The approximation of the specifications using a causal discrete-time systems, and
3. The realization of the system.

- Given a set of specifications in the form of figure, we must determine the system functions of a discrete-time linear system whose frequency response falls within the prescribed tolerances.
- This is a problem in functional approximation.
- Designing IIR filters implies approximation by a rational function of z, while designing FIR filters implies polynomial approximation.





$|H(\omega)|$
$1+\delta_1$
$1-\delta_1$
Passband ripple
δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge ripple
ω_s ~ Stopband edge ripple
Stopband
Transition band
δ_2
0
ω_p
ω_s
π
ω

Filter Design using Analog filter

- The traditional approach to the design of discrete time IIR filters involves the transformation of a continuous-time filter into a discrete time filter meeting prescribed specification.
- Analog filter is a mature and well developed field, so we begin the design of a digital filter in the analog domain and then convert the design into the digital form.
- An analog linear time-invariant system with system function $H(s)$ is stable if all its poles lie in the left half of the s −plane.
- Consequently, if the conversion technique is to be effective, it should possess the following desirable properties:

a. The $j\Omega$ axis in the s −plane should map into the unit circle in the z −plane. Thus there will be a direct relationship between the two frequency variables in the two domains.

b. The left-half plane (LHP) of the s −plane should map into the inside of the unit circle in the z −plane. Thus a stable analog filter will be converted to a stable digital filter.

IIR Filter Design:

The IIR filter design process is as

$$H(s) \xrightarrow[inverse\ Laplace\ Trasform]{} h_a(t) \xrightarrow[sampling]{} h(n) \xrightarrow[z-transform]{} H(z)$$

The techniques for IIR filter design are

1. By approximation of derivative method
2. By Matched z-transform method
3. By Impulse Invariance method
4. By Bilinear transformation method

5.1 Filter Design using Impulse Invariance method

- The objective of impulse invariant transformation is to develop an IIR filter transfer function whose impulse repsonse is the sampled version of the impulse response of the analog filter.
- The main idea behind this technique is to preserve the frequency response characteristics of the analog filter.
- In the impulse invariance method, our objective is to design an IIR filter having a unit sample response $h(n)$ that is the sampled version of the impulse response of the analog filter.
- That is,

$$h(n) \equiv h(nT) \;\; n = 0,1,2,\ldots$$

where T is the sampling interval.

- Sampling the impulse response of an analog filter with frequency response $H_a(F)$, sampled at the rate $F_s = {}^{1}/_{T}$ samples per second, the digital filter with unit sample response $h(n) \equiv h_a(nT)$ has the frequency response

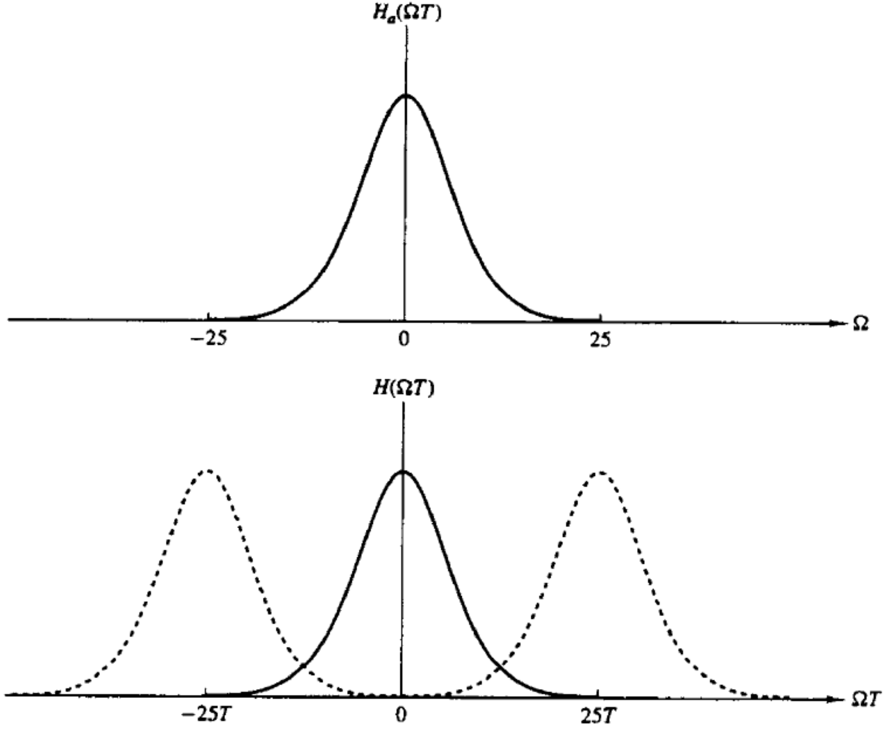
$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f-k)F_s]$$

- where $f = {}^{F}/_{F_s}$ is the normailzed frequency. Aliasing occurs if the sampling rate F_s is less than twice the highest frequency contained in $H_a(f)$.

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s]$$

$$H(\Omega T) = F_s \sum_{k=-\infty}^{\infty} H_a(\Omega - \frac{2\pi k}{T})$$

- Figure depicts the frequency response of a low pass analog filter and the frequency response of the corresponding digital filter.
- It is clear that the digital filter with frequency response $H(\omega)$ has the frequency response characteristics of the corresponding analog filter if the sampling interval T is selected sufficiently small to completely avoid or at least minimize the effects of aliasing.



Frequency response $H_a(\Omega)$ of the analog filter and frequency re-sponse of the corresponding digital filter with aliasing.

- It is also clear that the impulse invariance method is inappropriate for designing highpass filters due to the spectrum aliasing that results from the sampling process.
- The relationship that generalized the z-transform of $h(n)$ to the Laplace transform $h_a(t)$ is given as:

$$H(z)|_{z=e^{sT}} = F_s \sum_{k=-\infty}^{\infty} H_a(s - j\frac{2\pi k}{T})$$

where

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n}$$

$$H(z)|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n)e^{-sTn}$$

- Note when $s = j\Omega$, the above equation reduces to $H(\Omega)$ where the factor of j in $H_a(\Omega)$ is suppressed in our nototation.

Let us consider the mapping of points from the s −plane to the z −plane implied by the relation

$$z = e^{sT}$$

If we substitute $s = \sigma + j\Omega$ and expressed the complex variable z in polar form as $z = re^{j\omega}$ the above equation becomes

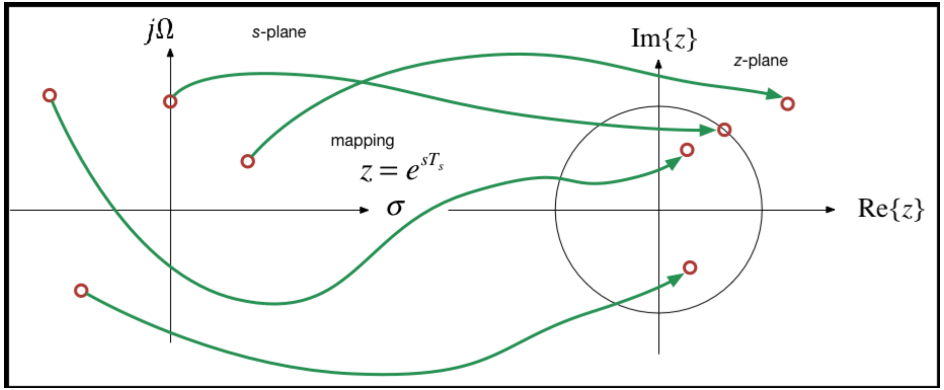
$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Thus $r = e^{\sigma T}$ and $\omega = \Omega T$

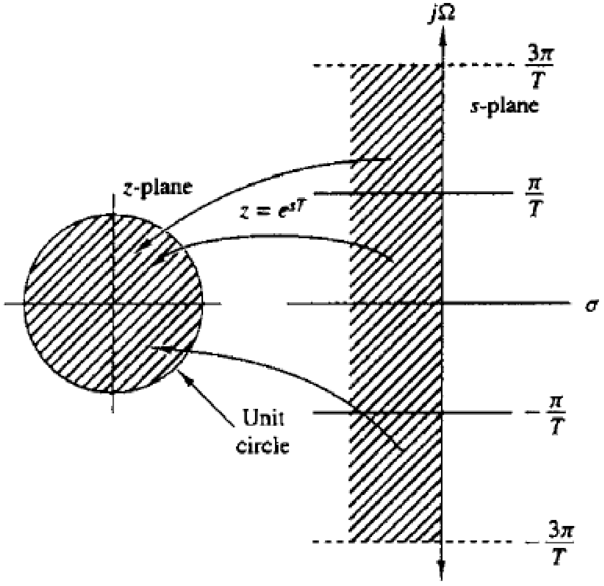
Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$.

When $\sigma = 0$ then $r = 1$.

Therefore, the LHS in s is mapped inside the unit circle in z and the RHS in s is mapped outside the unit circle in z as in figure.



- Also, the $j\Omega$ −axis is mapped into the unit circle in z as above.
- However, the mapping of $j\Omega$ −axis into the unit circle is not one-to-one.
- Since ω is unique over the range $(-\pi, \pi)$, the mapping $\omega = \Omega T$ implies that the interval $-{}^{\pi}/_{T} \leq \Omega \leq {}^{\pi}/_{T}$ maps into the corresponding values of $-\pi \leq \omega \leq \pi$.
- Furthermore, the frequency interval ${}^{\pi}/_{T} \leq \Omega \leq {}^{3\pi}/_{T}$ maps into the interval $-\pi \leq \omega \leq \pi$.
- In general, ${}^{(2k-1)\pi}/_{T} \leq \Omega \leq {}^{(2k+1)\pi}/_{T}$ where k is an integer, also maps into the corresponding values of $-\pi \leq \omega \leq \pi$.
- Thus mapping from the analog frequency Ω to the frequency variable ω in the digital domain is many-to-one, which simply reflects the effects of aliasing due to sampling.



Impulse Invariance Design method

Let, $h(n)$ be the Impulse response of analog filter and its Laplace transform $H(s)$ gives the transfer function of analog filter.

When $H(s)$ has N number of distinct poles $(-P_N)$, $N = 1,2, \dots, N$, it can be expressed as partial fraction expansion as

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s + P_i} = \frac{A_1}{s + P_1} + \frac{A_2}{s + P_2} + \cdots + \frac{A_N}{s + P_N}$$

On taking inverse Laplace transform we get

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$h(t) = \sum_{i=1}^{N} A_i e^{-P_i t} u(t) = A_1 e^{-P_1 t} u(t) + A_2 e^{-P_2 t} u(t) + \cdots + A_N e^{-P_N t} u(t)$$

Where $u(t) = \text{Continuous time unit step function}$

Let, $T = Sampling\ period, h(n) = Impulse\ response\ of\ digital\ filter$.
The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$$h(n) = h(t)\Big|_{t=nT} = h(nT) = \sum_{i=1}^{N} A_i e^{-P_i nT} u(nT)$$

$$= A_1 e^{-P_1 nT} u(nT) + A_2 e^{-P_2 nT} u(nT) + \cdots + A_N e^{-P_N nT} u(nT)$$

On taking z –transform we get,

$$H(z) = \frac{A_1}{1 - e^{-P_1 T} z^{-1}} + \frac{A_2}{1 - e^{-P_2 T} z^{-1}} + \cdots + \frac{A_N}{1 - e^{-P_N T} z^{-1}}$$

$$= \sum_{i=1}^{N} \frac{A_i}{1 - e^{-P_i T} z^{-1}}$$

$$\boxed{Z\left\{e^{-anT}\, u(nT)\right\} = \frac{1}{1 - e^{-aT} Z^{-1}}}$$

We observe that the digital filter has poles at

$$z_i = e^{-P_i T}, \qquad k = 1,2, \ldots, N$$

Comparing the expression of $H(s)$ and $H(z)$ we can say that

$$\frac{1}{s+P_i} \xrightarrow{\text{is transformed to}} \frac{1}{1-e^{-P_iT}z^{-1}}$$

$$\frac{1}{s-P_i} \xrightarrow{\text{is transformed to}} \frac{1}{1-e^{P_iT}z^{-1}}$$

Other useful Impulse Invariant Transformation are

$$\boxed{\frac{1}{(s+p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{dp_i^{m-1}} \frac{1}{1-e^{-p_iT}z^{-1}}}$$

$$\boxed{\frac{(s+a)}{(s+a)^2+b^2} \longrightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}}$$

$$\boxed{\frac{b}{(s+a)^2+b^2} \longrightarrow \frac{e^{-aT}(\sin bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}}$$

Example 1: Convert the analog filter with system function

$$H(s) = \frac{s + 0.1}{s^2 + 0.2s + 9.01}$$

into a digital IIR filter by means of the impulse invariance method.

Solution:

$$H(s) = \frac{s + 0.1}{s^2 + 0.2s + 9.01} = \frac{s + 0.1}{s^2 + 0.2s + 0.01 + 9} = \frac{s + 0.1}{(s + 0.1)^2 + 3^2}$$

We note that the analog filter has a zero at $s = -0.1$ and a pair of complex conjugate poles at $P_{1,2} = -0.1 \pm j3$

Now, by method of partial fraction expansion, we obtain

$$H(s) = \frac{s + 0.1}{(s + 0.1)^2 + 3^2} = \frac{A}{s + 0.1 + j3} + \frac{A^*}{s + 0.1 - j3}$$

$$A = \left.\frac{s + 0.1}{(s + 0.1 + j3)(s + 0.1 - j3)}\right|_{s=-0.1-j3} = \frac{1}{2} = A^*$$

$$H(s) = \frac{1/2}{s + 0.1 + j3} + \frac{1/2}{s + 0.1 - j3}$$

Mapping from analog to digital using impulse invariance method, we get

$$H(z) = \frac{1/2}{1 - e^{-0.1T - j3T} z^{-1}} + \frac{1/2}{1 - e^{-0.1T + j3T} z^{-1}}$$

$$= \frac{1/2}{1 - e^{-0.1T} e^{-j3T} z^{-1}} + \frac{1/2}{1 - e^{-0.1T} e^{j3T} z^{-1}}$$

$$= \frac{1/2\left(1 - e^{-0.1T} e^{j3T} z^{-1}\right) + 1/2\,(1 - e^{-0.1T} e^{-j3T} z^{-1})}{(1 - e^{-0.1T} e^{-j3T} z^{-1})(1 - e^{-0.1T} e^{j3T} z^{-1})}$$

$$= \frac{1 - 1/2\,(e^{j3T} + e^{-j3T}) e^{-0.1T} z^{-1}}{1 - e^{-0.1T}(e^{-j3T} + e^{j3T}) z^{-1} + e^{-0.2T} z^{-2}}$$

$$= \frac{1 - e^{-0.1T} \cos(3T)\, z^{-1}}{1 - 2e^{-0.1T} \cos(3T)\, z^{-1} + e^{-0.2T} z^{-2}}$$

Example 2: For the analog transfer function, $H(s) = \frac{2}{s^2+3s+2}$, determine $H(z)$ using impulse invariant transformation.

Soultion: $H(s) = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{2}{s+1} - \frac{2}{s+2}$

Using the impulse invariant transform we get

$$H(z) = \frac{2}{1 - e^{-T}z^{-1}} - \frac{2}{1 - e^{-2T}z^{-1}}$$

Example *Determine $H(z)$ using the impulse invariant technique for the analog system function*

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)}$$

Solution Using partial fractions, $H(s)$ can be written as

$$H(s) = \frac{1}{(s+0.5)(s^2+0.5s+2)} = \frac{A}{s+0.5} + \frac{Bs+C}{s^2+0.5s+2}$$

Solving the above simultaneous equations, we get $A = 0.5$, $B = -0.5$ and $C = 0$.
The system response can be written as,

$$\begin{aligned}
H(s) &= \frac{0.5}{s+0.5} - \frac{0.5s}{s^2+0.5s+2} \\
&= \frac{0.5}{s+0.5} - 0.5\left(\frac{s}{(s+0.25)^2+(1.3919)^2}\right) \\
&= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2} - \frac{0.25}{(s+0.25)^2+(1.3919)^2}\right) \\
&= \frac{0.5}{s+0.5} - 0.5\left(\frac{s+0.25}{(s+0.25)^2+(1.3919)^2}\right) \\
&\quad + 0.0898\left(\frac{1.3919}{(s+0.25)^2+(1.3919)^2}\right)
\end{aligned}$$

Using Eqs.

$$\begin{aligned}
H(z) &= \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right] \\
&\quad + 0.0898\left[\frac{e^{-0.25T}(\sin 1.3919T)z^{-1}}{1-2e^{-0.25T}(\cos 1.3919T)z^{-1}+e^{-0.5T}z^{-2}}\right]
\end{aligned}$$

Example 4: Using impulse invariant transformation convert the following analog filter transfer function to digital filter transfer function.

$$H(s) = \frac{2.8s^2 + 4.8s + 2.9}{(s+3)(s^2 + s + 0.85)}$$

Solution:

$$H(s) = \frac{2.8s^2 + 4.8s + 2.9}{(s+3)(s^2 + s + 0.85)} = \frac{A}{s+3} + \frac{Bs + C}{s^2 + s + 0.85} = \frac{2}{s+3} + \frac{0.8s + 0.4}{s^2 + s + 0.85}$$

$$= \frac{2}{s+3} + \frac{0.8(s+0.5)}{s^2 + s + 0.85} = \frac{2}{s+3} + \frac{0.8(s+0.5)}{s^2 + (2 \times 0.5)s + 0.5^2 - 0.5^2 + 0.85}$$

$$= \frac{2}{s+3} + \frac{0.8(s+0.5)}{(s+0.5)^2 + 0.7746^2}$$

Now using impulse invariant transformation,

$$H(z) = \frac{2}{1 - e^{-3T} z^{-1}} + 0.8 \frac{1 - e^{-0.5T} \cos(0.7746T) z^{-1}}{1 - 2e^{-0.5T} \cos(0.7746T) z^{-1} + e^{-2 \times 0.5T} z^{-2}}$$

5.2 Filter Design using Bilinear transformation

- The Impulse Invariant IIR filter design have a severe limitation in that they are appropriate only for low-pass and a limited class of band-pass filters. It is not suitable for high-pass and band-reject filters.
- This limitation is overcome in the mapping technique called bilinear transformation.
- This is one-to-one mapping from the s-domain to z-domain.
- That is, the bilinear transformation is a conformal mapping that transform the $j\Omega$ –axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.
- Also, the transformation of a stable analog filter results in a stable digital as all poles in the left of the s-plane are mapped onto the points inside the unit circle of z-domain and all points in the right of the s-plane are mapped into corresponding points outside the unit circle in the z-plane.
- The bilinear transformation is obtained by using the trapezoidal formula for numerical integration.

Consider the first order differential equation of an analog system as

$$\frac{dy(t)}{dt} = x(t)$$

And its Laplace transform is given as $sY(s) = X(s)$

On integrating the differential equation both sides from $(n-1)T\ to\ nT$, we get,

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt} = \int_{(n-1)T}^{nT} x(t)$$

$$[y(t)]_{(n-1)T}^{nT} = \int_{(n-1)T}^{nT} x(t)$$

$$y(nT) - y((n-1)T = \int_{(n-1)T}^{nT} x(t)$$

The trapezoidal rule when integration is approximated by two trapezoids is,

$$\int_a^b f(x)\,dx = \frac{b-a}{2}[f(a)+f(b)]$$

The integral on the right side of can be approximated by the trapezoidal rule, so that,

$$y(nT) - y[(n-1)T) = \frac{T}{2}[x(nT) + x\big((n-1)T\big)]$$

For discrete time system,

$$y(n) - y(n-1) = \frac{T}{2}[\mathrm{x(n) + x(n-1)}]$$

On taking z-transform we get,

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}[X(z) + z^{-1}X(z)]$$

$$[1 - z^{-1}]Y(z) = \frac{T}{2}[1 + z^{-1}]X(z)$$

$$\frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})}Y(z) = X(z)$$

Comparing this with the Laplace transform of the difference equation

$$sY(s) = X(s)$$

Clearly, the mapping from the s-plane to the z-plane is

$$s = \frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})}$$

This is called the bilinear transformation.

Although our derivation of the bilinear transformation was performed for a first-order differential equation, it holds, in general, for an N^{th}-order differential equation.

To investigate the characteristics of bilinear transformation, let

$$z = re^{j\omega}$$

$$s = \sigma + j\Omega$$

And we have

$$s = \frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})} = \frac{2}{T}\frac{(z - 1)}{(z + 1)} = \frac{2}{T}\frac{\left(re^{j\omega} - 1\right)}{(re^{j\omega} + 1)}$$

Substituting $e^{j\omega} = \cos\omega + j\sin\omega$, we get

$$s = \frac{2}{T}\frac{(r\cos\omega + rj\sin\omega - 1)}{(r\cos\omega + rj\sin\omega + 1)} = \frac{2}{T}\frac{(r\cos\omega + rj\sin\omega - 1)}{(r\cos\omega + rj\sin\omega + 1)} \times \frac{(r\cos\omega - rj\sin\omega + 1)}{(r\cos\omega + rj\sin\omega + 1)}$$

$$= \frac{2}{T}\frac{(r\cos\omega + (rj\sin\omega - 1))}{((1 + r\cos\omega) + rj\sin\omega)} \times \frac{(r\cos\omega - (rj\sin\omega - 1))}{((1 + r\cos\omega) - rj\sin\omega)}$$

$$= \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

Therefore

$$\sigma = \frac{2}{T}\left(\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right)$$

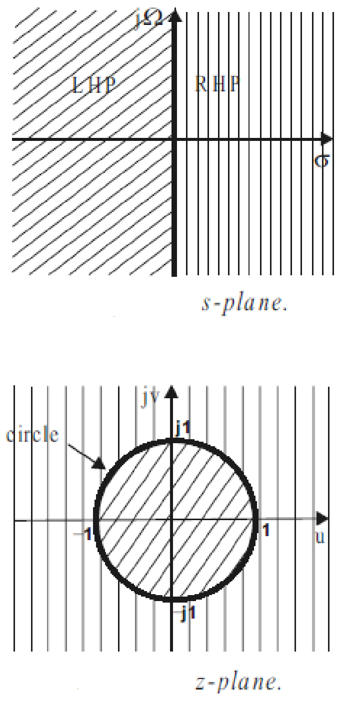
$$\Omega = \frac{2}{T}\left(\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right)$$

- ✓ If $r < 1$, then $\sigma < 0$.
- ✓ If $r > 1$, then $\sigma > 0$.
- ✓ That is, the LHS in s-plane maps into the inside of the unit circle in the z-plane
- ✓ The RHS in s-plane maps into the outside of the unit circle as shown in figure.
- ✓ When $r = 1$, then $\sigma = 0$ and

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{1 + \cos\omega}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

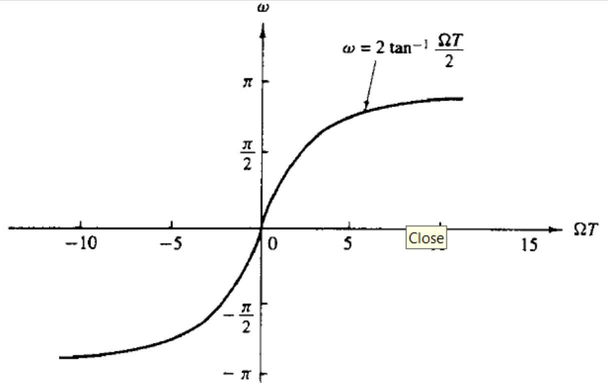
Thus

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

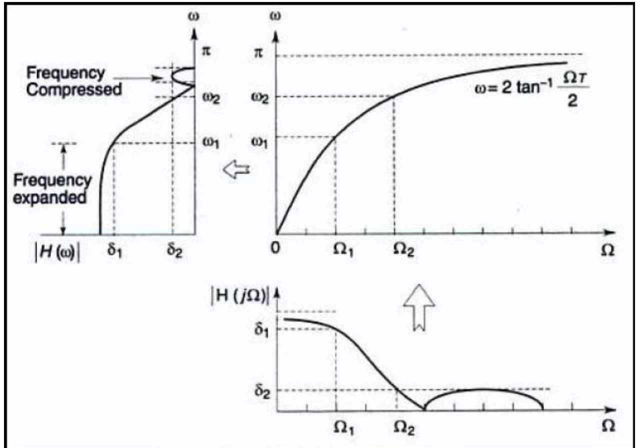


Mapping of s-plane into z-plane in bilinear transformation.

- The relationship between the frequency variables in the two domains is as in figure shown (up).
- The entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$.However, the mapping is highly nonlinear.
- We observe a frequency compression or frequency waping, due to the nonlinearity of the arctangent function.
- The lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed as shown if figure (down).
- Because of the frequency warping, the use of the bilinear transformation is restricted to the design of approximations to filters with piecewise constant frequency magnitude characteristics, such as highpass, lowpass and bandpass filters.



Mapping between the frequency variables ω and Ω resulting from the bilinear transformation.



Relationship between ω and Ω

Example 1: Convert the analog filter with system function

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 16}$$

Into a digital IIR filter by means of the bilinear transformation. The digital filter is to have a resonant frequency of $\omega_r = \frac{\pi}{2}$.

Solution: Comparing with

$$H_a(s) = \frac{s + \propto}{(s + \propto)^2 + {\Omega_r}^2}$$

The analog filter has a resonant frequency $\Omega_r = 4$. This frequency is to be mapped into $\omega_r = \frac{\pi}{2}$ by selecting the value of the parameter T. We must select $\mathrm{T} = \frac{1}{2}$ in order to have $\omega_r = \frac{\pi}{2}$.

Thus the desired mapping is

$$s = 4\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

The resulting digital filter has the system function

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.0006z^{-1} + 0.975z^{-2}}$$

Example 2: Design a single-pole low pass digital filter with a 3-dB bandwidth of 0.2π, using bilinear transformation applied to the analog filter

$$H(s) = \frac{\Omega_c}{s + \Omega_c}$$

Where Ω_c is the 3-dB bandwidth of the analog filter.

Solution: The digital filter is specified to have its -3-dB gain at $\omega_c = 0.2\pi$. In the frequency domain of the analog filter $\omega_c = 0.2\pi$ corresponding to

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{2}{T}\tan\frac{0.2\pi}{2} = \frac{0.65}{T}$$

Thus the analog filter has the system function, for simply let assume T=1,

$$\mathrm{H(s)} = \frac{0.65}{s + 0.65}$$

Using $s = 2\,(\frac{1-z^{-1}}{1+z^{-1}})$ we get the required digital filter

$$H(z) = \frac{0.245(1 + z^{-1})}{1 - 0.509z^{-1}}$$

The frequency response of the digital filter is $H(\omega) = \frac{0.245(1+e^{-j\omega})}{1-0.509e^{-j\omega}}$, at $\omega = 0, H(0)$ =1 and $\omega = 0.2\pi, |H(0.2\pi)| = 0.707$, which is the desired response.

Difference between IIM and BTM

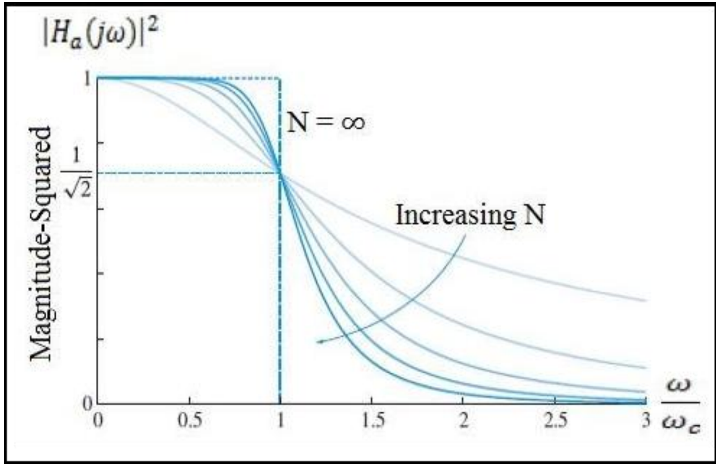
Impulse Invariant Method	Bilinear Transformation Method
Transformation of pole is done using the transformation equation, $$\frac{1}{s-P_i} \longrightarrow \frac{1}{1-e^{P_i T} z^{-1}}$$	Transformation of poles is done using the relation, $$s = \frac{2}{T}\frac{(1-z^{-1})}{(1+z^{-1})}$$
Mapping from s-plane to z-plane is many to one	Mapping from s-plane to z-plane is one to one
Presence of aliasing effect	Absence of aliasing effect
Only poles of the system can be mapped	Poles as well as zeros can be mapped
Frequency warping effect is absence	Frequency warping effect is present
Not suitable to design high pass and band reject filters	Suitable to design high pass and band reject filters

5.3 Filter Design using Low pass approximations Butterworth filter

- The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter.
- Hence to design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications.
- Then the analog filter transfer function is converted to a digital filter transfer function by using either impulse invariant transformation or bilinear transformation.

Analog Butterworth Filter

- The analog Butterworth filter is designed by approximating the ideal analog filter frequency response, $H(j\omega)$ using an error function.
- The error function is selected such that the magintude is maximally flat in the pass-band and monotonically decreasing in the stop-band as in figure. The magintude is maximally flat at the origin i.e., $\omega = 0$, and monotonically decreasing with increasing ω.



The magintude response of low-pass filter obtained by approximation is given by,

$$|H(\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2N}} = \frac{1}{1+\varepsilon^2(\frac{\Omega}{\Omega_p})^{2N}}$$

Where

N =the order of the filter,

Ω_c =is the -3dB frequency also called cutoff frequency,

Ω_p =the passband edge frequency,

$\frac{1}{1+\varepsilon^2}$ =is the band edge value of $|H(\Omega)|^2$

Since $H(s)H(-s)$ evaluated at $s = \mathrm{j}\Omega$ is simply equal to $|H(\Omega)|^2$, it follows that

$$H(s)H(-s) = \frac{1}{1+(-{s^2}/{\Omega_c^2})^N}$$

The poles of $H(s)H(-s)$ occurs on a circle of radius Ω_c at equally spaced points. That is, the poles occurs at

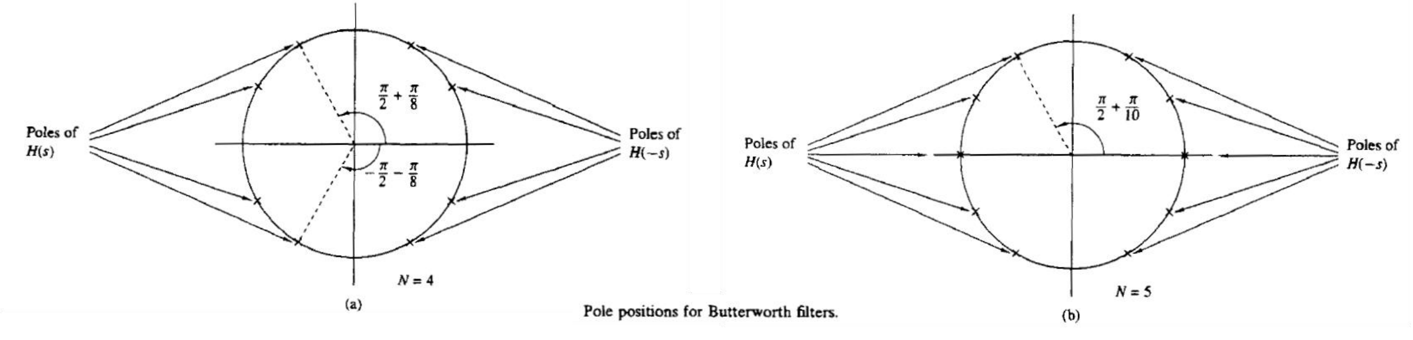
$$1 + \left(-{s^2}\big/{\Omega_c^2}\right)^N = 0$$

$$\frac{-s^2}{\Omega_c^2} = (-1)^{1/N} = e^{j(2k+1)\frac{\pi}{N}}, \qquad k = 0,1,\dots,N-1$$

Hence,

$$s_k = \Omega_c e^{j\frac{\pi}{2N}} e^{j(2k+1)\frac{\pi}{2N}}, \qquad k = 0,1,\dots,N-1$$

The pole positions for an N=4 and N=5 Butterworth filters are shown in figure.



Pole positions for Butterworth filters.

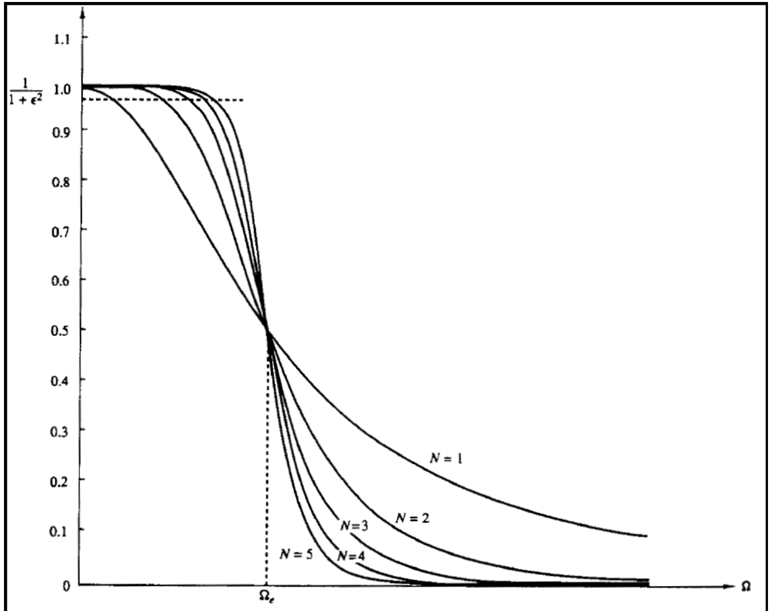
The frequency response charatceristics of the class Butterworth filters are shown in figure for several values of N. We note that $|H(\Omega)|^2$ is monotonic in both the passbabd and stopband. The order of the filter required to meet an attenuation δ_2 at the specified frequency Ω_s is easily determined as $\Omega = \Omega_s$, i.e.,

$$\frac{1}{1+\varepsilon^2(\frac{\Omega_s}{\Omega_p})^{2N}} = \delta_2^2$$

And hence

$$N = \frac{\log(\frac{1}{\delta_2^2} - 1)}{2\log(\frac{\Omega_s}{\Omega_c})} = \frac{\log(\frac{\delta}{\varepsilon})}{2\log(\frac{\Omega_s}{\Omega_p})}$$

Where, by defination, $\delta_2 = \frac{1}{\sqrt{1+\delta^2}}$.Thus the Butterworth filter is completely characterized by the parameters, N, δ_2, ε and the ratio $\frac{\Omega_s}{\Omega_p}$.



Frequency response of Butterworth filters.

Properties of Butterworth Filter

1. The Butterworth filters are all pole designs (i.e., the zeros of the filters exists at infinity).
2. At the cutoff frequency Ω_c the magnitude of normalized Butterworth filter is $\frac{1}{\sqrt{2}} = 0.707$ ($|H(\Omega)|^2 = \frac{1}{2}$). Hence the dB magnitude at the cutoff frequency will be 3dB less than the maximum value($20 \log 0.707$).
3. The filter order N completely specifies the filter.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of Ω.
6. The magintude response approaches the ideal response as the value of N increases.

Example : Determine the order and the poles of a lowpass Butterworth filter that has a -3dB bandwidth of 500Hz and an attenuation of 40dB at 1000Hz.

Solution: The critical frequency are the -3dB frequency Ω_c and the stopband frequency Ω_s which are

$$\Omega_c = 2\pi f_c = 1000\pi$$

$$\Omega_s = 2\pi f_s = 2000\pi$$

For an attenuation of 40dB $\quad -20\log\delta_2 = 40 \quad thus \quad \delta_2 = 0.01$

$$N = \frac{\log(\frac{1}{\delta_2^2} - 1)}{2\log(\frac{\Omega_s}{\Omega_c})} = \frac{\log(\frac{1}{0.01^2} - 1)}{2\log(\frac{2000\pi}{1000\pi})} = 6.64$$

To meet the desired specification, we select N=7.

The poles are located at $s_k = 1000\pi e^{j\frac{\pi}{14}} e^{j(2k+1)\frac{\pi}{14}}, \;\; k = 0,1,\dots,6$

Transfer function of Analog Butterworth Lowpass Filter

- For a stable and causal filter the poles should lie on the left hand of s-plane.
- Hence the desired filter transfer function is formed by choosing the N-number of left half poles.
- When N is even, all the poles are complex and exist as conjugate pair.
- When N is odd, one of the poles is real and all other poles are complex and exist as conjugate pair.
- The transfer function with Butterworth response is given by

$$H_a(s) = \frac{1}{B_N(s)}$$

Where,

$$B_N(s) = \begin{cases} \prod (s^2 + 2\cos\varphi_k\, s + 1)\,, & N\ is\ even \\ (\mathrm{s}+1) \times \prod (s^2 + 2\cos\varphi_k\, s + 1), & N\ is\ odd \end{cases}$$

- if N is odd, there is a pole at $\varphi = 0°$
- If N is even, there are poles at $\varphi = \pm\frac{90°}{N}$
- Poles are separated by $\varphi = \frac{180°}{N}$

For N=2, poles at $\varphi = \pm\frac{90°}{2} = \pm 45°$

Thus $B_N(s) = (s^2 + 2\cos 45°\, s + 1) = (s^2 + 1.414s + 1)$

The table of the noramlized transfer function for different order of the Butterworth filter is

Order (N)	Normalized Transfer function $B_N(s)$
1	(s+1)
2	$(s^2 + 1.414s + 1)$
3	(s+1)$(s^2 + s + 1)$
4	$(s^2 + 0.766s + 1)(s^2 + 1.848s + 1)$
5	(s+1))$(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + 1.414s + 1)\ )(s^2 + 1.932s + 1)$
7	(s+1)$(s^2 + 1.802s + 1)(s^2 + 1.247s + 1)(s^2 + 0.445s + 1)$

Order of the Lowpass Butterworth Filter

- The order N has to be esitmated to satisfy the given specifications.
- Usually the specifications of the filter are given in terms of gain at passband and stopband frequency.
- Let

  δ_p= Gain or Magnitude at a passband frequency ω_p

  δ_s= Gain or Magnitude at a stopband frequency ω_s
- Calculate N using (choose the nearest integer greater than calculated N)

$$N = \frac{1}{2} \frac{\log[\frac{1/\delta_s^2 - 1}{1/\delta_p^2 - 1}]}{\log[\frac{\Omega_s}{\Omega_p}]}$$

- Sometimes, the specification of the filter are given in terms of dB-attenuation at a passband and stopband frequency.
- Let, $\propto_p$ is the dB-attenuation at a passband frequency ω_p

  $\propto_s$ is the dB-attenuation at a stopband frequency ω_s
- Calculate a parameter N using equation (choose N as a integer greater than calculated N).

$$N = \frac{1}{2}\frac{\log(\frac{10^{0.1\propto_p} - 1}{10^{0.1\propto_s} - 1})}{\log(\frac{\Omega_s}{\Omega_p})}$$

Cutoff Frequency of Lowpass Butterworth Filter

- The IIR filter are designed to satisfy a prescribed gain or attenuation at a passband and stopband frequency.
- But practically the 3-dB cutoff frequency, Ω_c is used to decide the useful frequency range of the filter.
- Therefore, in Butterworth filter design the passband and stopband specification are used to estimate the order, N of the filter, and Nth order normalized Butterworth lowpass filter is designed.
- Then the normalized low pass filter is unnormalized using the cutoff frequency.

$$\Omega_c = \frac{\Omega_s}{[\frac{1}{\delta_s^2} - 1]^{\frac{1}{2N}}} = \frac{\Omega_p}{[\frac{1}{\delta_p^2} - 1]^{\frac{1}{2N}}}$$

$$\Omega_c = \frac{\Omega_s}{\left[10^{0.1\propto_s} - 1\right]^{\frac{1}{2N}}} = \frac{\Omega_p}{\left[10^{0.1\propto_p} - 1\right]^{\frac{1}{2N}}}$$

Where Ω_s and Ω_p are calculated as

- For impulse invariant transformation

$$\Omega_p = \frac{\omega_p}{T}; \qquad \Omega_s = \frac{\omega_s}{T};$$

- For bilinear transformation

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2}; \qquad \Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2}$$

Where T is the sampling time.

- To un-normalize the analog filter use transformation of s as

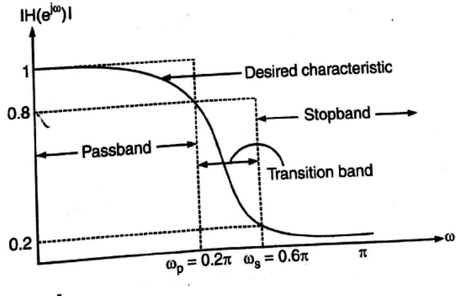
$$s = \frac{s}{\Omega_c}$$

- Realize the digital filter using the given transformation IIM or BTM.

Example: Design a digital Butterworth filter that satisfies the following constrait using bilinear transformation. Assume T=1s.

$$0.8 \le \left|H\left(e^{j\omega}\right)\right| \le 1; \;\; 0 \le \omega \le 0.2\pi$$

$$\left|H(e^{j\omega})\right| \le 0.2; \quad 0.6\pi \le \omega \le \pi$$



Solution: Here
Passband Frequency $\omega_p = 0.2\pi\, rad$
Stopband Frequency $\omega_s = 0.6\pi\, rad$
Passband ripple $\delta_p = 0.8$
Passband attenuation $\propto_p = -20\log\delta_p = -20\log 0.8 = 1.938dB$
Stopband attenuation $\propto_s = -20\log\delta_s = -20\log 0.2 = 13.44dB$
Assume T=1sec (if not given)
For bilinear transformation

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} = 2\tan\frac{0.2\pi}{2} = 0.65$$

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} = 2\tan\frac{0.6\pi}{2} = 2.75$$

$$N = \frac{1}{2}\frac{\log[\frac{{}^{1}/{\delta_s^2} - 1}{{}^{1}/{\delta_p^2} - 1}]}{\log[\frac{\Omega_s}{\Omega_p}]} = \frac{1}{2}\frac{\log[\frac{{}^{1}/{0.2^2} - 1}{{}^{1}/{0.8^2} - 1}]}{\log[\frac{2.75}{0.65}]} = 1.3$$

Thus, choose N=2

$$\Omega_c = \frac{0.65}{[\frac{1}{0.8^2} - 1]^{\frac{1}{2\times 2}}} = 0.75$$

For N=2 the normalized Butterworth filter transfer function is

$$H_a(s) = \frac{1}{(s^2 + 1.414s + 1)}$$

Let put $s = \frac{s}{\Omega_c}$ for un-normalized Butterworth filter

$$H_a(s) = \frac{1}{\left((\frac{s}{\Omega_c})^2 + 1.414\frac{s}{\Omega_c} + 1\right)} = \frac{1}{\left((\frac{s}{0.75})^2 + 1.414\frac{s}{0.75} + 1\right)}$$

$$= \frac{0.56}{(s^2 + 1.06s + 0.56)}$$

Using bilinear transformation method, $H(z)$ is obtained by putting

$$s = \frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})} = 2\frac{(1 - z^{-1})}{(1 + z^{-1})}$$

Thus

$$H(z) = \frac{0.56}{\left(4[\frac{(1 - z^{-1})}{(1 + z^{-1})}]^2 + 1.06 \times 2\frac{(1 - z^{-1})}{(1 + z^{-1})} + 0.56\right)}$$

$$H(z) = \frac{0.56}{4\frac{(z-1)^2}{(z+1)^2} + \frac{2.12z - 2.12}{Z+1} + 0.56}$$

$$= \frac{0.56(z+1)^2}{4(z^2 - 2z + 1) + (2.12z - 2.12)(z+1) + 0.56(z+1)^2}$$

$$= \frac{0.56(z+1)^2}{(6.68z^2 - 6.88z + 2.44)}$$

Which is required transfer function to design the digital filter.

Example : Design a second order discrete time Butterworth filter with cut-off frequency of 1KHz and sampling frequency of 10^4 samples/sec by using bilinear transformation. Also plot the poles of the filter.

Solution:

Order of filter, N=2

Cut-off frequency of analog filter, $F_c = 1kHz = 1000Hz$

Sampling frequency, $F_s = 10^4$ samples/sec=10,000Hz

$$f_c = \frac{F_c}{F_s} = \frac{1000}{10000} = 0.1 cycles/samples$$

Now $\omega_c = 2\pi f_c = 0.2\pi \; radian/samples$

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = 2 \times 10000 \tan\frac{0.2\pi}{2} = 6498.39 \; radian/sec$$

For N=2 the normalized Butterworth filter transfer function is

$$H_a(s) = \frac{1}{(s^2 + 1.414s + 1)}$$

Let put $s = \frac{s}{\Omega_c}$ for un-normalized Butterworth filter

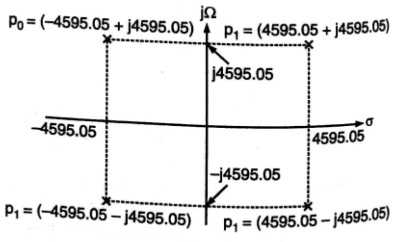
$$H_a(s) = \frac{1}{\left((\frac{s}{\Omega_c})^2 + 1.414\frac{s}{\Omega_c} + 1\right)} = \frac{1}{\left((\frac{s}{6498.39})^2 + 1.414\frac{s}{6498.39} + 1\right)}$$

$$= \frac{6498.39^2}{(s^2 + 9190.01s + 6498.39^2)}$$

The root of this equation are $P_0 = -4595.05 - j4595.05$ and $P_1 = -4595.05 + j4595.05$

These poles are of the left hand side of the s-plane.

Similar poles exists at right hand side of the s-plane which are not used because for stable poles should be on left hand side. The plot of poles is in figure.



Plot of poles, p_i.

Using bilinear transformation method, $H(z)$ is obtained by putting

$$s = \frac{2}{T}\frac{(1 - z^{-1})}{(1 + z^{-1})} = 20000\,\frac{(1 - z^{-1})}{(1 + z^{-1})}$$

Thus

$$H(z) = \frac{6498.39^2}{\left([20000\,\frac{(1 - z^{-1})}{(1 + z^{-1})}]^2 + 9190.01[20000\,\frac{(1 - z^{-1})}{(1 + z^{-1})}] + 6498.39^2\right)}$$

5.4 Properties of Chebyshev and Elliptic filters, properties of Bessel filter, spectral transformation.

1. **Chebyshev Filter**

- The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function.
- The approximation function is selected such that the error is minimized over a prescribed bands of frequencies.
- There are two types of Chebyshev approximation.
- In type-I approximation, the error function is selected such that, the magintude response is equiripple in the passband and monotonic in the stopband.
- Type-I Chebyshev filters are all-pole filters.
- In type-II approximation, the error function is selected such that, the magnitude reponse is monotonic in passband and equiripple in stopband.
- Type-II Chebyshev filters contains both poles and zeros.

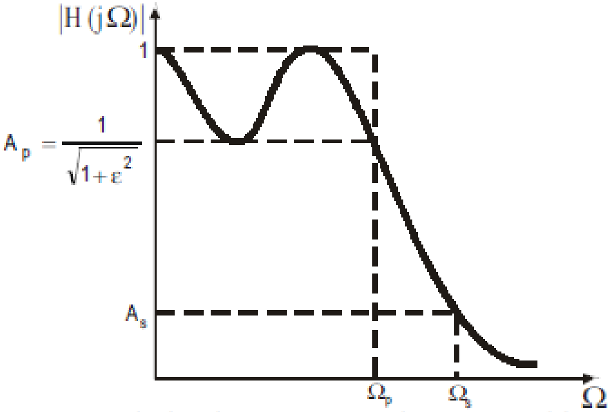


Fig a : Chebyshev type-1, when N is odd.

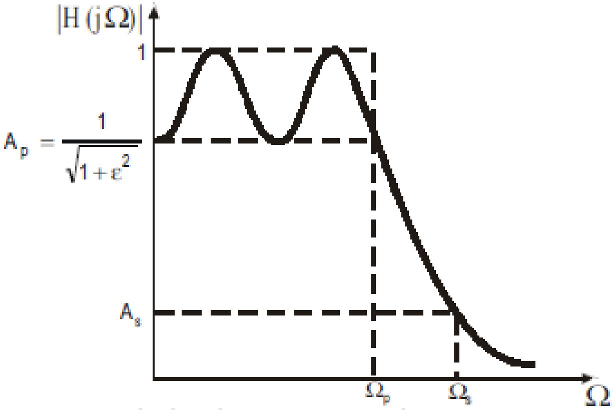


Fig b : Chebyshev type-1, when N is even.

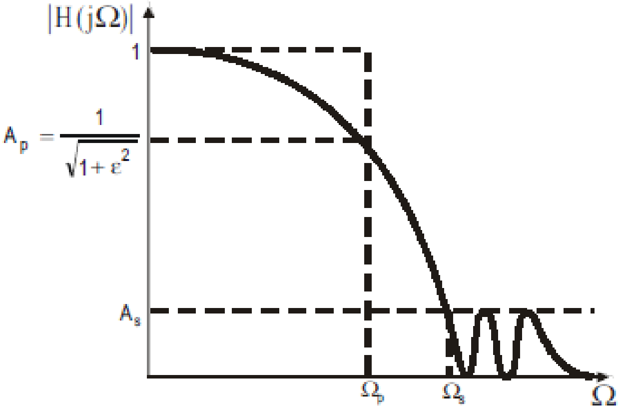


Fig c : Chebyshev type-2, when N is odd.

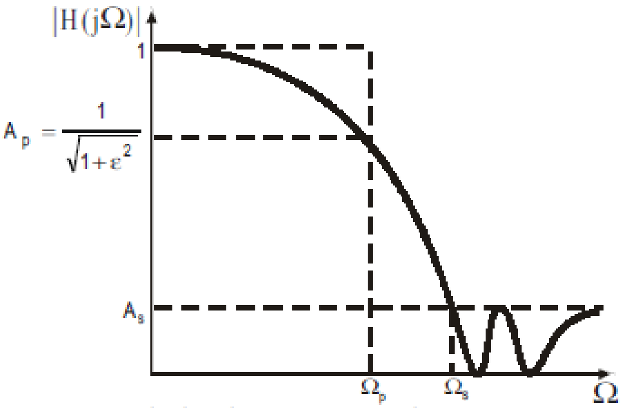


Fig d : Chebyshev type-2, when N is even.

Magnitude response of analog Chebyshev filters.

a. Chebyshev I

The magintude response of Chebyshev-I filter is

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\frac{\Omega}{\Omega_p})}$$

Where ε is a parameter of the filter related to the ripple in the passband and $T_N(x)$ is the N[th] order Chebyshev polynomial defined as

$$T_N(x) = \begin{cases} \cos(N\cos^{-1}x), & |x| \le 1 \\ \cosh(N\cosh^{-1}x), & |x| > 1 \end{cases}$$

The Chebyshev polynomials can be generated by the recursive equation

$$T_{N+1}(x) = 2xT_N(x) - T_{N-1}(x), \qquad N = 1,2,...$$

Where $T_0(x) = 1$ and $T_1(x) = x$. Thus $T_2(x) = 2x^2 - 1$, and $T_3(x) = 4x^3 - 3x$, and so on.

Some properties of these polynomials are as follows:

i. $|T_N(x)| \leq 1$ for all $|x| \leq 1$

ii. $T_N(1) = 1$ for all N.

iii. All the roots of the polynomila $T_N(x)$ occurs in the interval $-1 \leq x \leq 1$

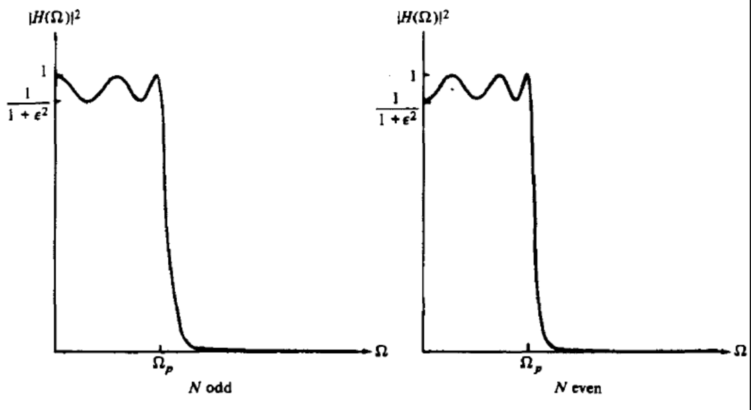
The filter parameter ε is related to the ripple in the passband as in figure for N odd and even.

For N odd, $T_N(0) = 0$ and hence $|H(0)|^2 = 1$.

For N even $T_N(0) = 1$ and hence $|H(0)|^2 = \frac{1}{1+\varepsilon^2}$

At the band edge frequency $\Omega = \Omega_c$,

we have $T_N(1) = 1$, so that



Type I Chebyshev filter characteristic.

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 1 - \delta_1$$

$$\varepsilon^2 = \frac{1}{(1-\delta_1)^2} - 1$$

Where δ_1 is the value of the passband ripple.

The pole of a type I Chebyshev filter lie on an ellipse in the s-plane with major axis

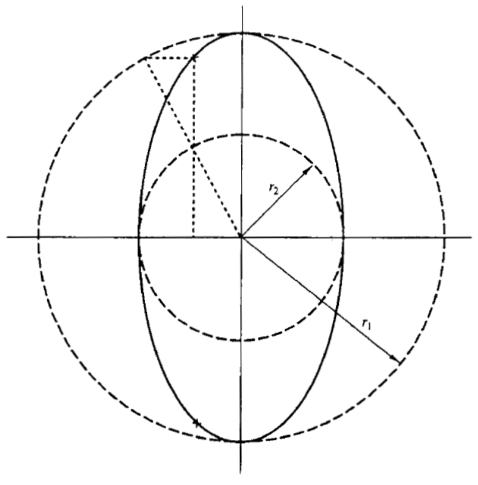
$$r_1 = \Omega_p \frac{\beta^2 + 1}{2\beta}$$

and minor axis

$$r_2 = \Omega_p \frac{\beta^2 - 1}{2\beta}$$

Where β is related to ε according to the equation

$$\beta = [\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}]^{1/N}$$



Determination of the pole locations for a Chebyshev filter.

The pole locations are most easily determined for a filter of order N by first locating the poles for an equivalent N[th] order Butterworth filter that lie on circle of radius r_1 or radius r_1 as in figure. If we denote the Butterworth filter as

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0,1,2,\ldots,N-1$$

then the positions of the poles for the Chebyshev filter lie on the ellipse at the coordinates $(x_k, y_k), k = 0,1,\ldots,N-1,$ where

$$x_k = r_2 \cos\phi_k, \qquad k = 0,1,\ldots,N-1$$
$$y_k = r_1 \sin\phi_k, \qquad k = 0,1,\ldots,N-1$$

Example : Design a Chebyshev analog filter with maximum passband attenuation of 2.5dB at $\Omega_p = 20\ rad/sec$ and stop band attenuation of 30 dB at $\Omega_s = 50\ rad/sec$.

Solution :

Passband attenuation $\alpha_p = 2.5dB$

Passband frequency $\Omega_p = 20\frac{rad}{sec}$

Stopband attenuation $\alpha_s = 30dB$

Passband frequency $\Omega_s = 50\frac{rad}{sec}$

Parameter ε represents ripple parameter in the passband. It is given by

$$\varepsilon = [10^{0.1\alpha_p} - 1]^{1/2}$$

$$\varepsilon = [\frac{1}{\delta_p^2} - 1]^{1/2}$$

Thus

$$\varepsilon = [10^{0.1\alpha_p} - 1]^{1/2} = [10^{0.25} - 1]^{1/2} = 0.882$$

Next we determine the order N of the filter as

$$|H(\Omega)|\ in\ dB = -20\log_{10}\varepsilon - 6(N-1) - 20N\log_{10}\Omega_s$$

$$-30 = -20\log_{10}0.882 - 6(N-1) - 20N\log_{10}50$$

$$-30 = 1.09 - 6N + 6 - 33.98N$$

$$N = 0.95$$

Thus, order of filter is $N \equiv 1$

$$\beta = [\frac{\sqrt{1+\varepsilon^2}+1}{\varepsilon}]^{\frac{1}{N}} = [\frac{\sqrt{1+0.882^2}+1}{0.882}]^{1} = 2.64$$

major axis

$$r_1 = \Omega_p \frac{\beta^2+1}{2\beta} = 20\frac{2.64^2+1}{2\times 2.64} = 30.19$$

and minor axis

$$r_2 = \Omega_p \frac{\beta^2-1}{2\beta} = 20\frac{2.64^2-1}{2\times 2.64} = 22.6$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0,1,2,\dots,N-1$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k+1)\pi}{2N}, \qquad k = 0$$

$$\phi_0 = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$x_0 = 22.6\cos\pi = -22.6 \text{ and } y_0 = 30.19\sin\pi = 0$$

$$s_0 = x_0 + jy_0 = -22.6$$

Thus system function for analog filter is

$$H_a(s) = \frac{i}{(s-s_0)(s-s_1)\dots} = \frac{i}{s^N + b_{N-1}s^{N-1} + \cdots + b_0}$$

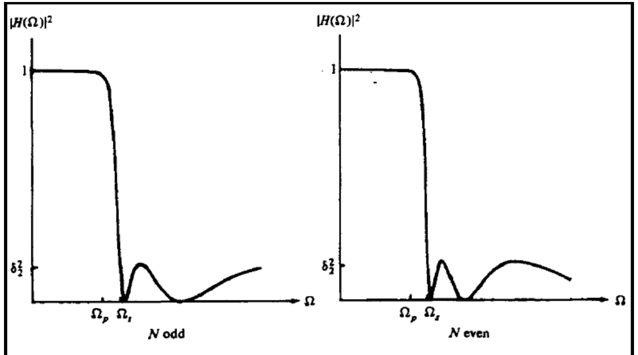
$$i = \begin{cases} b_0, & for\ N\ odd \\ \dfrac{b_0}{\sqrt{1+\varepsilon^2}}, & for\ N\ even \end{cases}$$

For N=1, $\mathrm{i} = b_0$

$$H_a(s) = \frac{i}{(s-(-22.6))} = -\frac{22.6}{(s+22.6)}$$

b. **Chebyshev Type II**

A type II Chebyshev filter contains zeros as well as poles.
The magnitude of its frequency response is given as



Type II Chebyshev filters.

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)}{T_N^2\left(\frac{\Omega_s}{\Omega}\right)}\right]}$$

Where $T_N(x)$ is the N[th] order Chebyshev polynomial and Ω_s is the stopband frequency. The zeros are located on the imaginary axis at the points

$$s_k = j\frac{\Omega_s}{\sin\phi_k}, \qquad k = 0,1,\ldots,N-1$$

The poles are located at the points (v_k, w_k), where

$$v_k = \frac{\Omega_s x_k}{\sqrt{x_k^2+y_k^2}}, and\ w_k = \frac{\Omega_s y_k}{\sqrt{x_k^2+y_k^2}}, \qquad k = 0,1,\ldots,N-1$$

Where x_k, y_k are defined in type I with β now related to the ripple in the stopband through the equation

$$\beta = \left[\frac{1+\sqrt{1-\delta_2^2}}{\delta_2}\right]^{1/N}$$

The Chebyshev filters are characterized by the parameters $N, \varepsilon, \delta_2, and\ \frac{\Omega_s}{\Omega_p}$.

For a given set of specifications on $\varepsilon, \delta_2, and\ \frac{\Omega_s}{\Omega_p}$, we can determine the order of the filter from the equation

$$N = \frac{\log\left[\left(\sqrt{1-\delta_2^2} + \sqrt{1-\delta_2^2(1+\epsilon^2)}\right)/\epsilon\delta_2\right]}{\log\left[(\Omega_s/\Omega_p) + \sqrt{(\Omega_s/\Omega_p)^2 - 1}\right]}$$

$$= \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

where, by definition, $\delta_2 = 1/\sqrt{1+\delta^2}$.

Comparison between Butterworth and Chebyshev Filter

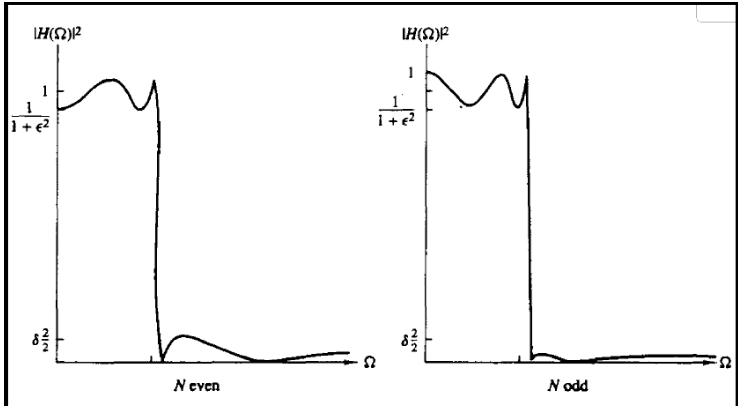
	Butterworth Filter	Chebyshev Filter
1	The magnitude frequency response is montonically decreasing	The magnitude frequency response has ripples in passband or stopband
2	The poles lie on a circle in the s plane	The poles lie on an ellipse in the s plane
3	For a given frequency specifications the number of poles are more	For a given frequency specifications the number of poles are less
4	For a given order N the width of the transition band is more	For a given order N the width of the transition band is less
5	Only few parameters has to be calculated to determine the transfer function	A large number of parameters has to be calculated to determine the transfer function

Eliptic Filters

- Elliptic (or Cauer) filters exhibit equiripple behavior in both the passband and the stopband as in figure for N odd and N even.
- This class of filters contains both poles and zeros and is characterized by the frequency response

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N(\frac{\Omega}{\Omega_p})}$$

Where $U_N(x)$ is the Jacobian elliptic Function of order N, and ε is a parameter Related to the passband ripple.
The zeros lie of the $j\Omega -$axis.



Magnitude-squared frequency characteristics of elliptic filters.

- Elliptic filters are most efficient from the viewpoint of yielding the smallest-order filter for a given set of specifications.
- Equivalently, we can say that for a given order and a given set of specifications, an elliptic filter has the smallest transition bandwidth.
- The filter order required to achieve a given set of specifications in passband ripple δ_1, stopband ripple δ_2 and transition ratio $\frac{\Omega_p}{\Omega_s}$ is given by

$$N = \frac{K(\Omega_p/\Omega_s)K\left(\sqrt{1-(\epsilon^2/\delta^2)}\right)}{K(\epsilon/\delta)K\left(\sqrt{1-(\Omega_p/\Omega_s)^2}\right)}$$

Where $K(x)$ is the complete elliptic integral of the first kind, defined as

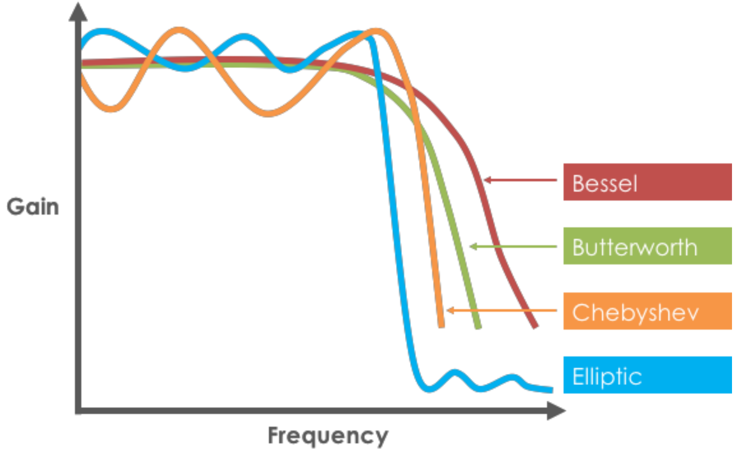
$$K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}}$$

and $\delta_2 = \frac{1}{\sqrt{1+\delta^2}}$

Properties of Bessel Filter

Bessel filter are a class of all-pole filters that are characterized by the system function

$$H(s) = \frac{1}{B_N(s)}$$



Where $B_N(s)$ is the Nth order Bessel polynomial.

These polynomials can be expressed in the form

$$B_N(s) = \sum_{k=0}^{N} a_k s^k$$

Where the coefficient $\{a_k\}$ are given as

$$a_k = \frac{(2N-k)!}{2^{N-k}k!\,(N-k)!}, \qquad k = 0,1, \dots, N$$

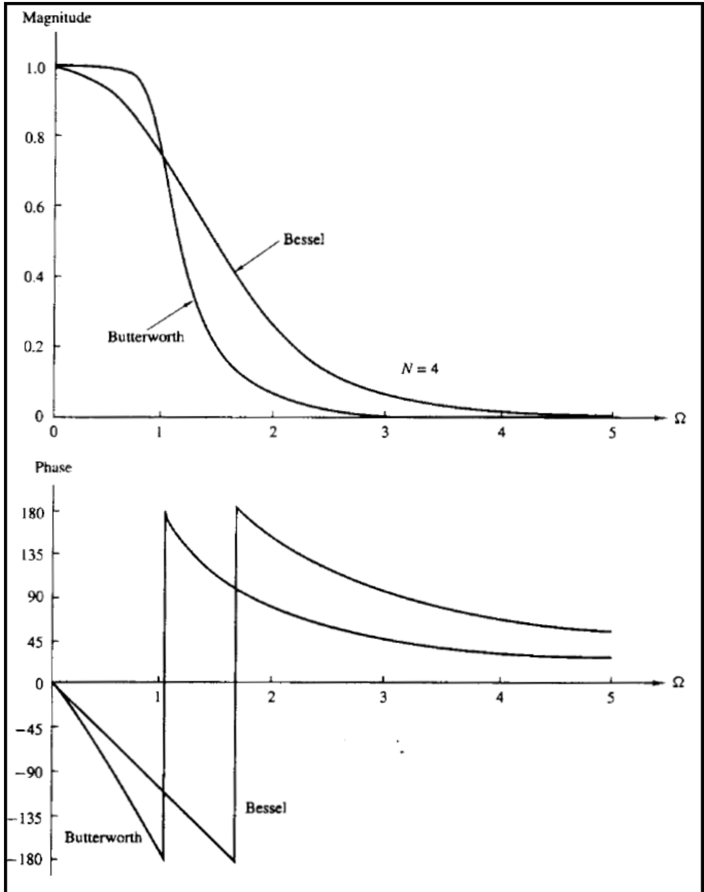
Alternatively, the Bessel polynomials may be generated recursively from the relation

$$B_N(s) = (2N-1)B_{N-1}(s) + s^2 B_{N-2}(s)$$

With $B_0(s) = 1$ and $B_1(s) = s + 1$ as initial conditions.

Properties of Bessel Filter

- An important characteristics of Bessel filters is the linear-phase response over the passband of the filter.
- Figure shows a comparison between of the magnitude and phase response of a Bessel and Butterworth filter of order N=4.
- The Bessel has a larger transition bandwidth, but its phase is linear with the passband.
- However, we should emphasize that the linear-phase characteristics of the analog filter are destroyed in the process of converting the filter into the digital domain by means of the transformations.



Magnitude and phase responses of Bessel and Butterworth filters of order $N = 4$.

High pass, Band pass and Notch Filters

- If we wish to design a highpass or a bandpass or a bandstop filter.
- It is a simple matter to take a lowpass prototype filter (Butterworth, Chebyshev, elliptic) and perform a frequency transformation.
- One possibility is to perform the frequency transformation in the analog domain and then to convert the analog filter into a corresponding digital filter by a mapping of the s-plane into the z-plane.
- An alternative approach is first to convert the anaolg lowpass filter into a lowpass digital filter and then to transform the lowpass digital filter into the desired digital filter by a digital transformation.
- In general, these two approaches yield different results, except for the bilinear transformation, in which the resulting filter design are identical.

1. Frequency Transformation in the Analog Domain

Filter Type	Transformation
Lowpass	$s_n \rightarrow \dfrac{s}{\Omega_c}$
Highpass	$s_n \rightarrow \dfrac{\Omega_c}{s}$
Bandpass	$s_n \rightarrow \dfrac{Q\,(s^2 + \Omega_0^2)}{\Omega_0 s}$
Bandstop	$s_n \rightarrow \dfrac{\Omega_0 s}{Q\,(s^2 + \Omega_0^2)}$

2. Frequency Transformations in Digital Domian

Filter Type	Transformation	Design Parameters
Lowpass	$z^{-1} \rightarrow \dfrac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \dfrac{\sin\left(\dfrac{\omega_c' + \omega_c}{2}\right)}{\sin\left(\dfrac{\omega_c' - \omega_c}{2}\right)}$
Highpass	$z^{-1} \rightarrow \dfrac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_c' + \omega_c}{2}\right)}{\cos\left(\dfrac{\omega_c' - \omega_c}{2}\right)}$
Bandpass	$z^{-1} \rightarrow -\dfrac{z^{-2} - \dfrac{2\alpha k}{k+1}z^{-1} + \dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1}z^{-2} - \dfrac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_s + \omega_p}{2}\right)}{\cos\left(\dfrac{\omega_s - \omega_p}{2}\right)} = \cos\omega_0$ $k = \cos\left(\dfrac{\omega_s - \omega_p}{2}\right)\tan\dfrac{\omega_c}{2}$
Bandstop	$z^{-1} \rightarrow \dfrac{z^{-2} - \dfrac{2\alpha k}{k+1}z^{-1} + \dfrac{k-1}{k+1}}{\dfrac{k-1}{k+1}z^{-2} - \dfrac{2\alpha k}{k+1}z^{-1} + 1}$	$\alpha = \dfrac{\cos\left(\dfrac{\omega_s + \omega_p}{2}\right)}{\cos\left(\dfrac{\omega_s - \omega_p}{2}\right)} = \cos\omega_0$ $k = \cos\left(\dfrac{\omega_s - \omega_p}{2}\right)\tan\dfrac{\omega_c'}{2}$

Questions:

1. Design a digital low pass Butterworth filter by applying bilinear transformation technique for the given sepcifications

   Pass band edge = 120Hz

   Pass band attenuation = 1dB

   Stop band edge = 170Hz

   stop band attenuation = 16 dB

   assume samping frequency of 512Hz

2. Obtain H(z) using the impulse invariant technique for an analog system function which is given by:

$$H_a(S) = \frac{1}{(S+0.5)(S^2+0.5S+2)}$$

3. Convert the analog filter with system function

$$H_a(S) = \frac{S+0.2}{(S+0.2)^2+9}$$

   Into a digital filter by means of the bilinear transformation. The digital filter is to have a resonant frequency of $\omega_r = \pi/2$.

4. Design a digital lowpass Butteworth filter using Impulse Invariance Method to meet the following specifications:

   Passband ripple $\leq$1.25dB

   Stopband ripple $\geq$ 15 dB

   Passband edge = 200 Hz

   Stopband edge = 300Hz

5. Derive Impulse Invariant Method for IIR filter design. Also illustrate the mapping from the s-plane to the z-plane while using IIM.
6. Design a second order discrete time Butterworth filter with cut-off frequency of 1KHz and sampling frequency of 10^4 samples/sec by using bilinear transformation. Also plot the poles of the filter.
7. Compare and contrast Impulse Invariance Method and Bilinear Transformation Method of designing IIR filter.
8. Design a Chebyshev analog filter with maximum passband attenuation of 2.5dB at $\Omega_p = 20\, rad/sec$ and stop band attenuation of 30 dB at $\Omega_s = 50\, rad/sec$.