# Chapter-3: Analysis of LTI Systems in Frequency Domain

## ❑ Frequency Response of LTI System, Response to Complex Exponential:

### 1. Frequency Response of LTI System:

➢ If $x[n]$ is the arbitrary input and $h[n]$ is the unit impulse response of LTI system then the response or output $y[n]$ of the LTI system is expressed in terms of convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \qquad \text{.....i}$$
$$= x[n] * h[n]$$

➢ To analyze LTI system, it is convenient to utilize the frequency domain because difference equation and convolution operation in the time domain become algebraic operation in frequency domain.

➢ Applying convolution property of DTFT in above equation, we get

$$Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

and
$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad \text{.....i}$$

where $H(e^{j\omega})$ is the **frequency response** of LTI system. KRK,WRC

- **Magnitude and Phase Representation of Frequency Response:**

➢ The frequency response $H(e^{j\omega})$ can be written in polar form as

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{\,j\{\theta(e^{j\omega})\}} \qquad \text{.....i}$$

where, $|H(e^{j\omega})|$ = amplitude (or magnitude) response

and $\theta(e^{j\omega})$ = phase response

➢ Note that phase response does not affect the amplitude of the individual frequency components but only provides information concerning the relative phases of exponentials that make up $h[n]$.

➢ $H(e^{j\omega})$ exhibits conjugate symmetry. That is,

$$|H(e^{j\omega})| = |H(-\,e^{j\omega})| \qquad \text{, symmetric about origin.}$$

and

$$\theta(-e^{j\omega}) = -\theta(e^{j\omega}) \qquad \text{, antisymmetric about origin.}$$

○ **Frequency response is the measure of magnitude and phase of the output as a function of frequency, in comparison to the input.**

## 2. Frequency Response of LTI System to Complex Exponential Signal:

- An LTI system is characterized in time-domain by its impulse response. The output of the LTI system is given by convolution sum as

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n] \qquad \text{.....i}$$

- Let, the input be the complex exponential defined as

$$x[n] = Ae^{j\omega n} \qquad -\infty < n < \infty \qquad \text{.....ii}$$

where, $A =$ amplitude, and

$\omega =$ arbitrary frequency confined to the interval $[-\pi, \pi]$

- From (i) and (ii), we get

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]Ae^{j\omega(n-k)}$$

$$y[n] = A\ \left[\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right] e^{j\omega n}$$

But, $\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} = H(e^{j\omega})$ .....iii

and $H(e^{j\omega})$ is the Fourier transform of the unit impulse response $h[k]$

- Then, $\boldsymbol{y[n] = A\,H(e^{j\omega})\,e^{j\omega n}}$ ....iv

Equation (iii) is the response of LTI system to the complex exponential input signal.

- **Note:**

- The $x[n] = Ae^{j\omega n}$ is the **eigenfunction** of the LTI system, and $H(e^{j\omega})$ is the corresponding **eigenvalue**. $H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential input signal as a function of the frequency $\omega$ and is the **frequency response** of the system.

- In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$\boldsymbol{H\left(e^{j\omega}\right) = H_R\left(e^{j\omega}\right) + jH_I\left(e^{j\omega}\right)} \qquad \text{.....i}$$

In polar form, $$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\theta\left(e^{j\omega}\right)} \qquad \text{.....ii}$$

where, $$\theta\left(e^{j\omega}\right) = \measuredangle H\left(e^{j\omega}\right)$$

- We know, $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]cos\omega k - j\sum_{k=-\infty}^{\infty} h[k]sin\omega k$$

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

- The function $\mathbf{H(e^{j\omega})}$ **exists** when the system is ***BIBO stable***. i., e.,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- The **impulse response** is the **inverse Fourier transform of** $\boldsymbol{H(e^{j\omega})}$ given by the equation $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$

## 3. Frequency Response, Phase and Group Delay:

- The **Fourier transforms** of the system input and output are related by

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) H\left(e^{j\omega}\right) \qquad \text{.....i}$$

and $$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)}$$

where $H\left(e^{j\omega}\right)$ is the **frequency response** of LTI system.

- The frequency response is in general a complex number at each frequency. In polar form, equation (i) can be written as

$$\left|Y\left(e^{j\omega}\right)\right| = \left|X\left(e^{j\omega}\right)\right|\left|H\left(e^{j\omega}\right)\right| \qquad \text{.....ii}$$

$$\measuredangle Y\left(e^{j\omega}\right) = \measuredangle X\left(e^{j\omega}\right) + \measuredangle H\left(e^{j\omega}\right) \qquad \text{.....iii}$$

where, $\left|H\left(e^{j\omega}\right)\right|$ = magnitude response or gain of the system, and

$\measuredangle H\left(e^{j\omega}\right)$ = phase response or phase angle of the system

- The magnitude and phase effects represented by Eqs. (ii) and (iii) can be:

i. Desirable, if the input signal is modified in a useful way, or
ii. Undesirable, if the input signal is changed in a deleterious manner (magnitude and phase distortion occurs)

- The phase angle of any complex number is not uniquely defined, since any integer multiple of $2\pi$ ( $i.e., 2\pi r$) can be added without affecting the complex number.

$$H\left(e^{j\omega}\right) = \left|H\left(e^{j\omega}\right)\right| e^{j\left(\measuredangle H\left(e^{j\omega}\right)+2\pi r\right)} = \left|H\left(e^{j\omega}\right)\right| e^{j\measuredangle H\left(e^{j\omega}\right)}$$

- We denote the **principal value** of the phase of $H\left(e^{j\omega}\right)$ as $\mathbf{ARG}[\boldsymbol{H}(\boldsymbol{e^{j\omega}})]$, where

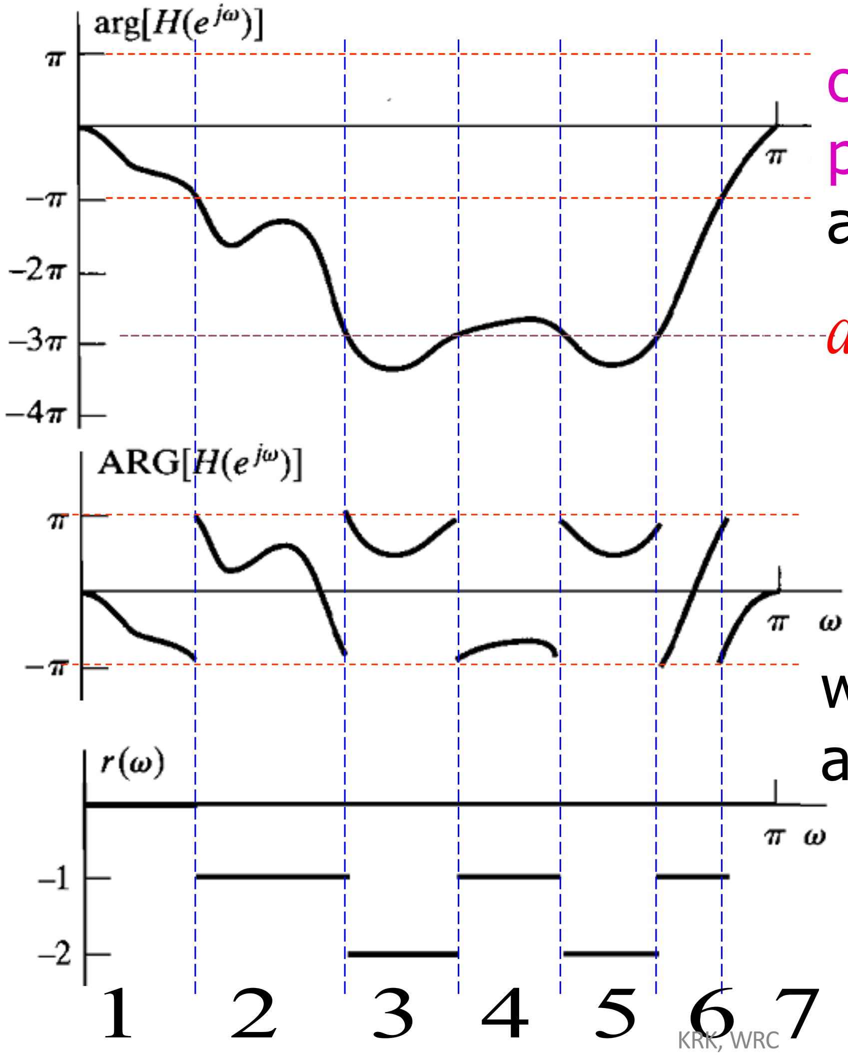
$$-\pi < \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] < \pi \qquad \text{.....iv}$$

and the **ambiguous phase** is given by

$$\measuredangle H\left(e^{j\omega}\right) = \mathrm{ARG}\left[H\left(e^{j\omega}\right)\right] + 2\pi r(\omega) \qquad \text{.....v}$$

where, $r(\omega) =$ positive or negative integer that can be different at each value of $\omega$. **(**$r(\omega)$ is somewhat arbitrary**)**

- We refer to $\mathrm{ARG}\left[H\left(e^{j\omega}\right)\right]$ as the "**wrapped**" phase.




continuous (unwrapped) phase curve is denoted as $arg\,[H\,(e^{\,jw})]$

$$arg\left[H\left(e^{jw}\right)\right] = ARG\left[H\left(e^{jw}\right)\right] + 2\pi\, r\left(w\right)$$

Principal Value

we refer to $ARG\,[H(e^{jw})]$ as the "wrapped" phase,

$r\left(w\right)$

- Another particularly useful representation of phase is through the **group delay** $\tau(\omega)$ defined by

$$\tau(\omega) = \mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left[\arg\{H\left(e^{j\omega}\right)\}\right] \qquad \text{.....vi}$$

- Similarly, we can express the **group delay in terms of the ambiguous phase** $\measuredangle H\left(e^{j\omega}\right)$ as

$$\tau(\omega) = \mathrm{grd}\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left[\measuredangle H\left(e^{j\omega}\right)\right] \qquad \text{.....vii}$$

with the interpretation that impulses caused by discontinuities of size $2\pi$ in $\measuredangle H\left(e^{j\omega}\right)$ are ignored.

## ❑ Linear Constant Coefficient Difference Equation (LCDDE) and Corresponding System Function:

- Let us consider the linear time-invariant (LTI) discrete-time systems characterized by the general **linear constant-coefficient difference equation (LCCDE)** by

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad \text{.....i}$$

- Taking z–transform on both sides, we get

$$\sum_{k=0}^{N} a_k Y[z]\, z^{-k} = \sum_{k=0}^{M} b_k X[z]\, z^{-k}$$

Or, $$H[z] = \frac{Y[z]}{X[z]} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad \text{.....ii}$$

where, $\boldsymbol{H[z]}$ = **system function** and takes the form of a ratio of polynomials in $z^{-1}$.

- In factored form, equation (ii) can be written as

$$H[z] = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1-c_k z^{-1})}{\prod_{k=1}^{N}(1-d_k z^{-1})} \qquad \text{.....iii}$$

- **Poles and Zeros:**

a. The factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$.

b. The factors $(1 - d_k z^{-1})$ in the denominator contributes a pole at $z = d_k$ and a zero at $z = 0$.

- **Example-1: Determine the difference equation of the system function given by**

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$$

**Solution:**

➢ The given system function is

$$H(z) = \frac{Y[z]}{X[z]} = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$$

$$H(z) = \frac{Y[z]}{X[z]} = \frac{1+2z^{-1}+z^{-2}}{(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})}$$

- Or, $Y[z] =$

## 1. Causality and Stability:

- The difference equation does not uniquely specify the impulse response of an LTI system. For the system function of equation (i) or (ii), there are a number of choices for the ROC.
- For a given ratio of polynomials, each of the ROC will lead to a different impulse response, but they will all correspond to the same difference equation.

### a. Causality:

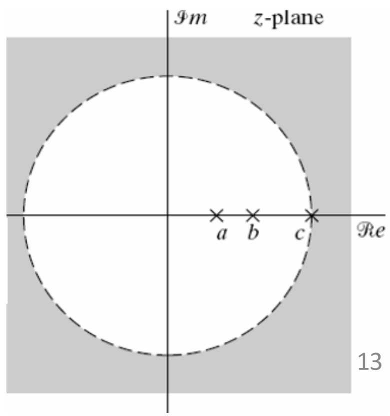
- For a **causal** system the **impulse response** $h[n]$ must be **right-sided sequence**.

$$h[n] = 0,\ n < 0$$

then the **region of convergence (ROC)** of $\boldsymbol{H(z)}$ must be **outside the outermost pole**.

$$H[z] = z\{h[n]\}$$

$$H[z] = \frac{Y[z]}{X[z]} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}\ , ROC: |z| > r_R$$



**b. Stability:**

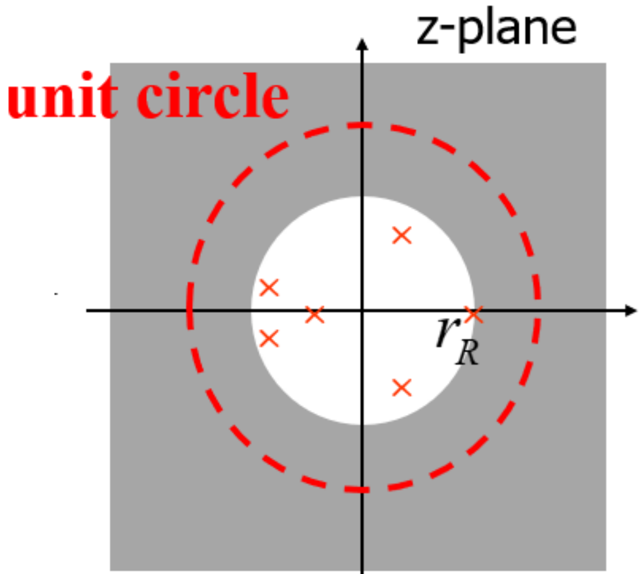
- For a **stable system**, the **impulse response** must be **absolutely summable**. That is,

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

this is identical to the condition that

$$\sum_{n=-\infty}^{\infty} |h[n] z^{-n}| < \infty \text{ for } |z| = 1$$

and the **ROC** of $\boldsymbol{H(z)}$ include the **unit circle**.



- **Causality and Stability Conditions:**

- **Causal**: ROC must be outside the outermost pole.
- **Stable**: ROC includes the unit circle.
- **Causal and stable**: All the **poles** of the system function are **inside the unit circle;**

  **ROC** is outside the outermost pole, and includes the unit circle.

**Example:**

**Example 5.2 Determining the ROC**

Consider the LTI system with input and output related through the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]. \tag{5.27}$$

From the previous discussions, the algebraic expression for $H(z)$ is given by

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - 2z^{-1})}. \tag{5.28}$$

The corresponding pole–zero plot for $H(z)$ is indicated in Figure 5.7. There are three possible choices for the ROC. If the system is assumed to be causal, then the ROC



is outside the outermost pole, i.e., $|z| > 2$. In this case, the system will not be stable, since the ROC does not include the unit circle. If we assume that the system is stable, then the ROC will be $\frac{1}{2} < |z| < 2$, and $h[n]$ will be a two-sided sequence. For the third possible choice of ROC, $|z| < \frac{1}{2}$, the system will be neither stable nor causal.

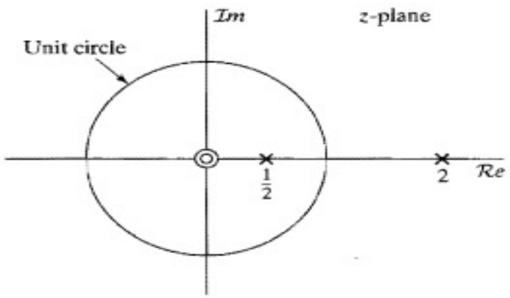



**Figure 5.7** Pole–zero plot for Example 5.2.

## 2. Impulse Response for Rational System Functions:

- A **system function** that takes the form of a ratio of polynomials in $z^{-1}$ is expressed as:

$$\boldsymbol{H[z] = \frac{Y[z]}{X[z]} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad , M \geq N} \qquad \text{.....i}$$

- Any **rational function of $z^{-1}$ with only $1^{st}$-order poles** can be expressed in the form

$$\boldsymbol{H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-k}}} \qquad \text{.....ii}$$

- where the **terms in the first summation would be obtained by long division** of the denominator into the numerator and would be present only **if $M \geq N$.**
- If the system is assumed to be **causal**, then the **ROC is outside all of the poles** in Eq. (ii), and it follows that

$$\boldsymbol{h[n] = \sum_{r=0}^{M-N} B_r\, \delta[n-r] + \sum_{k=1}^{N} A_k\, (d_k)^n u[n]} \qquad \text{.....ii}$$

where the first summation is included only if $\boldsymbol{M \geq N}$.
- For a **LTI system**, it is useful to classify **two classes**:

a. **Infinite Impulse Response (IIR):**

- For **IIR class**, at least **one nonzero pole** of $H(z)$ is **not canceled by a zero**. In this case, $h[n]$ will have at least one term of the form $A_k\ (dk)^n\, u[n]$, and $h[n]$ will not be of finite length, i.e., will not be zero outside a finite interval.

b. **Finite Impulse Response (FIR):**

- For a second class of systems, $H(z)$ has no poles except at $z = 0$; i.e., $N = 0$. Thus , a partial fraction expansion is not possible, and $H(z)$ is simply a polynomial in $z^{-1}$ of the form

$$\boldsymbol{H(z) = \sum_{k=0}^{M} b_k\, z^{-k}} \qquad \text{.....iii}$$

we assume, without loss of generality, that $a_0 = 1$.

- The **impulse response** of equation (iii) is

$$\boldsymbol{h[n] = \sum_{k=0}^{M} b_k \delta[n-k] = \begin{cases} b_n, & 0 \le n \le M \\ 0, & Otherwise \end{cases}} \qquad \text{.....iv}$$

- From **convolution sum**

$$h[n] = \sum_{k=0}^{M} b[k]\delta[n-k]$$

➢ The difference equation of equation (iii) is

$$\mathbf{h[n]} = \sum_{k=0}^{M} b_k\, x[n-k] \qquad \text{....iv}$$

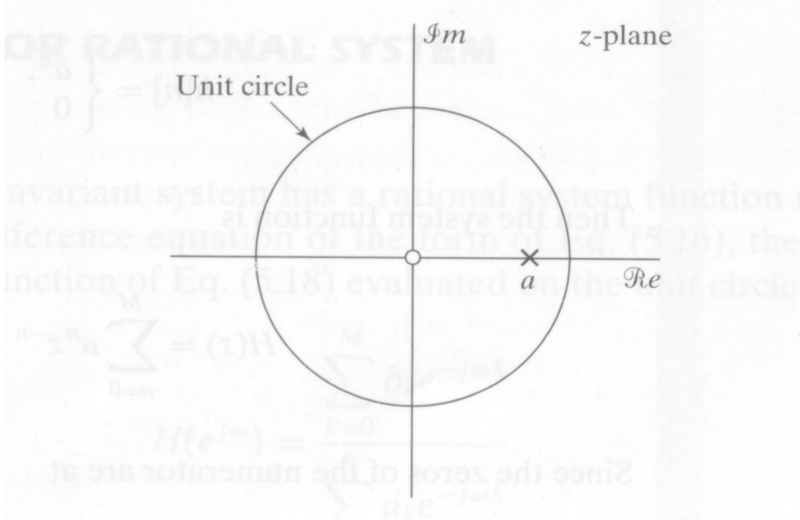
- **Examples:**

1) A first order IIR system defined by the difference equation

$$y[n] - ay[n-1] = x[n]$$

Find:

i. System function
ii. Condition for stability
iii. Impulse response




**Solution:**

➢ Given, $y[n] - ay[n-1] = x[n]$

i. Taking $z$-transform on both sides, we have

$$Y(z) - az^{-1}Y(z) = X[z]$$

$$H(z) = \frac{Y[z]}{X[z]} = \frac{1}{1-az^{-1}}, \quad \textbf{ROC: } |z| > |a|$$

which is the required expression for system function.

ii. For stable system, $|a| < 1$

iii. The impulse response is

$$h[n] = a^n u[n]$$

**Example 5.5 A Simple FIR System**

Consider an impulse response that is a truncation of the impulse response of an IIR system with system function

$$G(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

i.e.,

$$h[n] = \begin{cases} a^n, & 0 \leq n \leq M, \\ 0 & \text{otherwise.} \end{cases}$$

Then, the system function is

$$H(z) = \sum_{n=0}^{M} a^n z^{-n} = \frac{1 - a^{M+1} z^{-M-1}}{1 - az^{-1}}. \tag{5.41}$$

Since the zeros of the numerator are at $z$-plane locations

$$z_k = ae^{j2\pi k/(M+1)}, \qquad k = 0, 1, \dots, M, \tag{5.42}$$

where $a$ is assumed real and positive, the pole at $z = a$ is canceled by the zero denoted $z_0$. The pole–zero plot for the case $M = 7$ is shown in Figure 5.8.

The difference equation satisfied by the input and output of the LTI system is the discrete convolution

$$y[n] = \sum_{k=0}^{M} a^k x[n-k]. \tag{5.43}$$

However, Eq. (5.41) suggests that the input and output also satisfy the difference equation

$$y[n] - ay[n-1] = x[n] - a^{M+1}x[n-M-1]. \tag{5.44}$$

These two equivalent difference equations result from the two equivalent forms of $H(z)$ in Eq. (5.41).

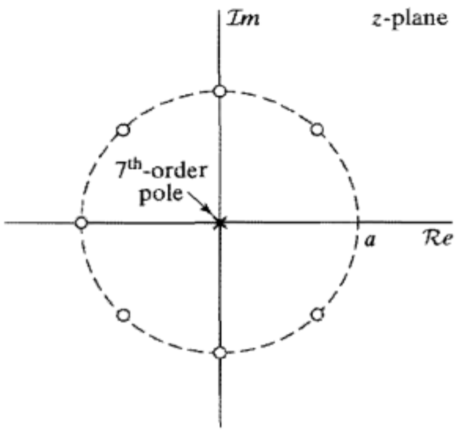



**Figure 5.8** Pole–zero plot for Example 5.5.

## ❑ Relationship of Frequency Response to Pole Zero of System:

- A stable LTI system has a **rational system function** as

$$\boldsymbol{H[z] = \frac{Y[z]}{X[z]} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}} \qquad \text{.....i}$$

then its **frequency response** (evaluated in the unit circle) has the form

$$\boldsymbol{H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k\, e^{-j\omega k}}{\sum_{k=0}^{N} a_k\, e^{-j\omega k}}} \qquad \text{.....ii}$$

that is, we obtain frequency response from system function with $z = e^{j\omega}$.

- In factored form, equation (i) can be written as

$$\boldsymbol{H[z] = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}} \qquad \text{.....iii}$$

then the frequency response of (iii) is

$$\boldsymbol{H\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{k=1}^{M}(1 - c_k\, e^{-j\omega})}{\prod_{k=1}^{N}(1 - d_k\, e^{-j\omega})}} \qquad \text{.....iv}$$

- **Magnitude:**

$$\left|H(e^{j\omega})\right| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M}\left|1-c_k\, e^{-j\omega}\right|}{\prod_{k=1}^{N}\left|1-d_k\, e^{-j\omega}\right|} \qquad \text{.....v}$$

- **Magnitude Squared Frequency Response (Function):**

$$\left|H(e^{j\omega})\right|^2 = H\left(e^{j\omega}\right)H^*\left(e^{j\omega}\right) = \left(\frac{b_0}{a_0}\right)^2 \frac{\prod_{k=1}^{M}\left(1-c_k\, e^{-j\omega}\right)\left(1-c_k^* e^{j\omega}\right)}{\prod_{k=1}^{N}\left(1-d_k\, e^{-j\omega}\right)\left(1-d_k^* e^{j\omega}\right)} \qquad \text{.....vi}$$

➢ From equation (v), we note that $\left|H(e^{j\omega})\right|$ is the ***product of the magnitudes of all the zero factors of $H[z]$ evaluated on the unit circle, divided by the product of the magnitudes of all the pole factors evaluated on the unit circle***.

- **Log Magnitude Gain (Gain in dB) of $H\left(e^{j\omega}\right)$**:

➢ Gain in dB is expressed as

$$20\, log_{10}\left|H(e^{j\omega})\right| = 20log_{10}\left|\frac{b_0}{a_0}\right| + \sum_{k=1}^{M} 20log_{10}\left|1 - c_k\, e^{-j\omega}\right| - \sum_{k=1}^{N} 20log_{10}\left|1 - d_k\, e^{-j\omega}\right| \qquad \text{.....vii}$$

where,

$$\text{Gain in dB} = 20\ log_{10}\left|H(e^{j\omega})\right|$$

attenuation in $dB = -20\ log_{10}\left|H(e^{j\omega})\right|$

- **Log Magnitude Output, Phase:**

➢ The output of the frequency response of equation (ii) is

$$Y\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)X\left(e^{j\omega}\right) \qquad \text{.....viii}$$

$$\left|Y(e^{j\omega})\right| = \left|H(e^{j\omega})\right| \left|X\left(e^{j\omega}\right)\right|$$

taking log on both sides, we get

$$\boldsymbol{20\ log_{10}\left|Y\left(e^{j\omega}\right)\right| = 20\ log_{10}\left|H\left(e^{j\omega}\right)\right| 20\ log_{10}\left|X\left(e^{j\omega}\right)\right|} \qquad \text{.....i}x$$

which is the log magnitude output.

➢ And, the **phase** is

$$\boldsymbol{\measuredangle Y\left(e^{j\omega}\right) = \measuredangle H\left(e^{j\omega}\right) + \measuredangle X\left(e^{j\omega}\right)} \qquad .....x$$

- **Phase Response and Group Delay:**

➢ The **phase** response for a rational system function has the form

$$\arg \boldsymbol{H}(e^{j\omega}) = \arg\left(\frac{b_0}{a_0}\right) + \sum_{k=0}^{M} arg\left(1 - c_k\, e^{-j\omega}\right) - \sum_{k=0}^{N} arg\left(1 - d_k\, e^{-j\omega}\right)$$

➢ And, the corresponding **group delay** is

$$grd[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{d}{d\omega}\left(arg\left[1 - d_k\, e^{-j\omega}\right]\right) - \sum_{k=1}^{M} \frac{d}{d\omega}\left(arg\left[1 - c_k\, e^{-j\omega}\right]\right)$$

where, $grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\{\arg H\left(e^{j\omega}\right)\}$

here, $\boldsymbol{arg}[\ ]$ represents the **continuous phase**.

➢ An equivalent expression is

$$\mathrm{grd}[H(e^{j\omega})] = \sum_{k=1}^{N} \frac{|d_k|^2 - \mathcal{R}e\{d_k e^{-j\omega}\}}{1 + |d_k|^2 - 2\mathcal{R}e\{d_k e^{-j\omega}\}} - \sum_{k=1}^{M} \frac{|c_k|^2 - \mathcal{R}e\{c_k e^{-j\omega}\}}{1 + |c_k|^2 - 2\mathcal{R}e\{c_k e^{-j\omega}\}}.$$

## 1. Frequency Response of a Single Zero or Pole: First Order System

- To study the detail properties of **frequency response**, first we examine the properties of a **single factor of the form** $(\mathbf{1} - \boldsymbol{r}\boldsymbol{e}^{\boldsymbol{j\theta}}\boldsymbol{e}^{-\boldsymbol{j\omega}})$, where $\boldsymbol{r}$ is the radius and $\boldsymbol{\theta}$ is the angle of the **pole or zero in the z-plane**. This factor is typical of **either a pole or a zero at a radius $\boldsymbol{r}$ and angle $\boldsymbol{\theta}$ in the z-plane**. That is,

$$\left|(1 - re^{j\theta}e^{-j\omega})\right|^2 = (1 - re^{j\theta}e^{-j\omega})\left(1 - re^{j\theta}e^{-j\omega}\right)^*$$

$$\left|(1 - re^{j\theta}e^{-j\omega})\right|^2 = \left(1 - re^{j\theta}e^{-j\omega}\right)\left(1 - re^{-j\theta}e^{j\omega}\right)$$

here, $\boldsymbol{re^{j\theta}} = \boldsymbol{d_k}\ (\boldsymbol{or}\ \boldsymbol{c_k})$

$$\left|(1 - re^{j\theta}e^{-j\omega})\right|^2 = 1 - re^{-j\theta}e^{j\omega} - re^{j\theta}e^{-j\omega} + r^2$$

$$\left|(1 - re^{j\theta}e^{-j\omega})\right|^2 = 1 - r(e^{j(\omega-\theta)}) + e^{-j(\omega-\theta)} + r^2$$

$$\left|(1 - re^{j\theta}e^{-j\omega})\right|^2 = 1 + r^2 - 2rcos(\omega - \theta) \qquad .....i$$

which is the **magnitude squared frequency response**.

- **Log Magnitude in dB:**

➢ Taking 10log on both sides of equation i , we get

$$\pm 20log_{10}\left|(1 - re^{j\theta}e^{-j\omega})\right| = \pm 10log_{10}[1 + r^2 - 2rcos(\omega - \theta)] \qquad .....ii$$

$$"+" : for\ zero\ factor, \quad "-": for\ pole\ factor$$

- **Phase response:**

➢ We know, $1 - re^{j\theta}e^{-j\omega} = 1 - re^{-j(\omega-\theta)}$

$$1 - re^{j\theta}e^{-j\omega} = 1 - rcos(\omega - \theta) + jrsin(\omega - \theta)$$

then the phase is

$$\pm ARG[1 - re^{j\theta}e^{-j\omega}] = \pm tan^{-1}\left[\frac{rsin(\omega-\theta)}{1-rcos(\omega-\theta)}\right] \qquad .....iii$$

- **Group Delay:**

➢ Group delay is obtained by differentiating the right hand side of equation iii as

$$(+/-)\text{grd}[1 - re^{j\theta}e^{-j\omega}] = (+/-)\frac{r^2 - r\cos(\omega-\theta)}{1 + r^2 - 2r\cos(\omega-\theta)} = (+/-)\frac{r^2 - r\cos(\omega-\theta)}{|1 - re^{j\theta}e^{-j\omega}|^2}. \qquad .....iv$$

- The functions in equations ii, iii, and iv are **periodic with period $2\pi$**.
- Note that if we plot above functions for fixed value of $r$ and variable $\omega$ with different values of $\theta$, we obtain the magnitude, phase and group delay.

**1. Example:** Plot the magnitude and phase response of the system which has $zeros$ at

a. $r = 0.9\ and\ \theta = 0$

b. $r = 0.9\ and\ \theta = \frac{\pi}{2}$ **(assignment)**

c. $r = 0.9\ and\ \theta = \pi$ **(assignment)**

**Solution:**

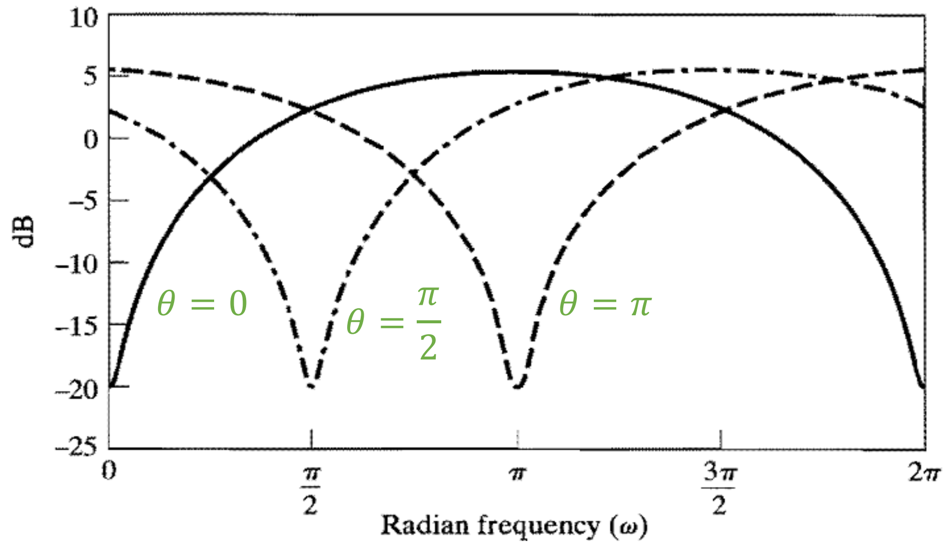
**a. For $r = 0.9$ and $\theta = 0$**

- We know, **magnitude** = $10log_{10}[1 + r^2 - 2rcos(\omega - \theta)]$

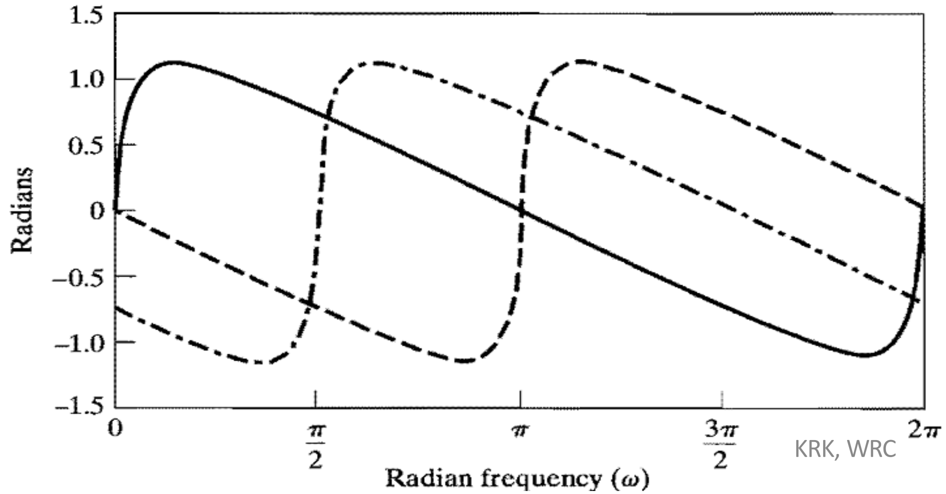
  = $10log_{10}[1 + 0.9^2 - 2\times0.9\ cos\omega]$ .....i

- Similarly, **phase** = $tan^{-1}\left[\frac{rsin(\omega-\theta)}{1-rcos(\omega-\theta)}\right] = tan^{-1}[\frac{0.9\ sin\omega}{1-0.9\ cos\omega}]$ .....ii

| $\omega$ | $Magnitude\ (dB)$ | $Phase\ (rad)$ |
|---|---|---|
| $0$ | $-20$ | $0$ |
| $\pi/_4$ | $-2.6969$ | $1.051$ |
| $\pi/_2$ | $2.576$ | $0.732$ |
| $\pi$ | $5.575$ | $0.20035$ |
| $5\pi/_4$ | $4.889$ | $-0.3709$ |
| $3\pi/_2$ | $2.576$ | $-0.732$ |
| $7\pi/_4$ | $-2.6969$ | $-1.051$ |
| $2\pi$ | $-20$ | $0$ |




Magnitude response




Phase response

## 2. Frequency Response of Multiple Poles and Zeros:

➢ Let, there are ***pole pair and zero pair at*** $\boldsymbol{r_1}$ ***and*** $\boldsymbol{r_2}$ respectively, then

- **Magnitude:**

➢ We know, $magnitude = 20log\left|H(e^{j\omega})\right|$

$$\begin{aligned}\boldsymbol{magnitude} = \mathbf{10\,log}\left[\mathbf{1} + \boldsymbol{r_2^2} - \mathbf{2}\boldsymbol{r_2}\,\mathbf{cos}(\boldsymbol{\omega} - \boldsymbol{\theta})\right] \\ +\mathbf{10\,log}\left[\mathbf{1} + \boldsymbol{r_2^2} - \mathbf{2}\boldsymbol{r_2}\,\mathbf{cos}(\boldsymbol{\omega} + \boldsymbol{\theta})\right] \\ -\mathbf{10\,log}\left[\mathbf{1} + \boldsymbol{r_1^2} - \mathbf{2}\boldsymbol{r_1}\,\mathbf{cos}(\boldsymbol{\omega} - \boldsymbol{\theta})\right] \\ -\mathbf{10log}[\mathbf{1} + \boldsymbol{r_1^2} - \mathbf{2}\boldsymbol{r_1}\,\mathbf{cos}(\boldsymbol{\omega} + \boldsymbol{\theta})]\end{aligned} \qquad .....i$$

$$\{" + " : \boldsymbol{for\ zero\ factor}, \quad " - ": \boldsymbol{for\ pole\ factor}\}$$

- **Phase:**

➢ Phase= $\measuredangle H\left(e^{j\omega}\right) = \arg[H\left(e^{j\omega}\right)]$

➢ The **phase** is given by the equation

$$\measuredangle H(e^{j\omega}) = -tan^{-1}\left[\frac{r_1 \sin(\omega-\theta)}{1-r_1\, cos(\omega-\theta)}\right] - tan^{-1}\left[\frac{r_1\, sin(\omega+\theta)}{1-r_1\, cos(\omega+\theta)}\right] +$$

$$tan^{-1}\left[\frac{r_2 \sin(\omega-\theta)}{1-r_2\, cos(\omega-\theta)}\right] + tan^{-1}\left[\frac{r_2\, sin(\omega+\theta)}{1-r_2\, cos(\omega+\theta)}\right] \qquad .....ii$$

- **Group Delay:**

➢ It is given by

$$grd[H(e^{j\omega})] = \frac{r_2^2 - r_2 cos(\omega-\theta)}{1+r_2^2-2r_2 cos(\omega-\theta)} + \frac{r_2^2 - r_2 cos(\omega+\theta)}{1+r_2^2-2r_2 cos(\omega+\theta)} +$$

$$\frac{r_1^2 - r_1 cos(\omega-\theta)}{1+r_1^2-2r_1 cos(\omega-\theta)} + \frac{r_1^2 - r_1 cos(\omega+\theta)}{1+r_1^2-2r_1 cos(\omega+\theta)} \qquad .....iii$$

- **Examples:**

1. Plot the magnitude and phase response of the system which has pole pair at $r = 0.9$ and $\theta = \frac{\pi}{4}$.

   **Solution:**

- Given, $r = 0.9$ and $\theta = \pm\frac{\pi}{4}$ ( for pole pair take "$\pm$" )

**i. Magnitude Response:**

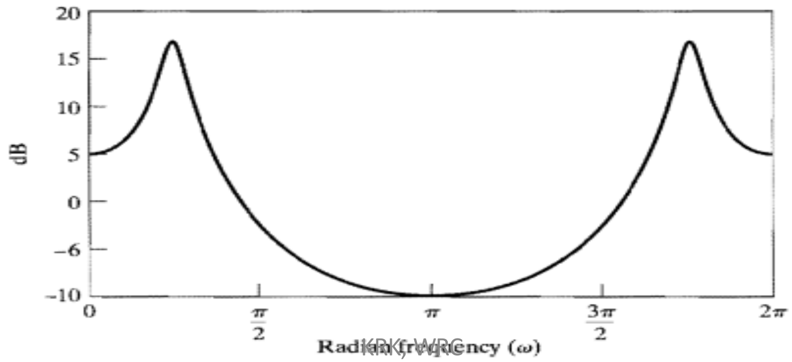
$$magnitude = -10log[1 + r^2 - 2rcos(\omega \pm \theta)] \qquad \text{\{"$-$" sign for poles\}}$$

$$= -10log\left[1 + (0.9)^2 - 2\times 0.9cos\left(\omega - \frac{\pi}{4}\right)\right]$$

$$-10log\left[1 + (0.9)^2 - 2\times 0.9cos\left(\omega + \frac{\pi}{4}\right)\right]$$

The magnitude response is given by the given table.

| $\omega$ | $Magnitude\ (dB)$ |
|---|---|
| $0$ | $5.39$ |
| $\pi/_4$ | $17.42$ |
| $\pi/_2$ | $-2.19$ |
| $\pi$ | $-9.77$ |
| $3\pi/_2$ | $-2.19$ |
| $7\pi/_4$ | $17.42$ |
| $2\pi$ | $5.39$ |



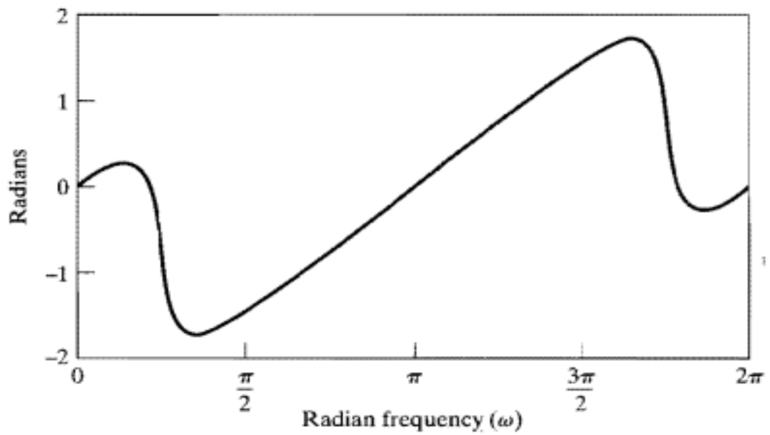

Magnitude response

## ii. Phase response:

$$-\boldsymbol{tan^{-1}}\left[\frac{\boldsymbol{r_1 \sin(\omega-\theta)}}{\boldsymbol{1-r_1 \cos(\omega-\theta)}}\right]-\boldsymbol{tan^{-1}}\left[\frac{\boldsymbol{r_1 \sin(\omega+\theta)}}{\boldsymbol{1-r_1 \cos(\omega+\theta)}}\right]$$

$$\measuredangle H(e^{j\omega}) = -tan^{-1}\left[\frac{0.9\sin\left(\omega-\frac{\pi}{4}\right)}{1-0.9\cos\left(\omega-\frac{\pi}{4}\right)}\right]-tan^{-1}\left[\frac{0.9\sin\left(\omega+\frac{\pi}{4}\right)}{1-0.9\cos\left(\omega+\frac{\pi}{4}\right)}\right]$$

| $\omega$ | $Phase(rad)$ |
|---|---|
| $0$ | $0$ |
| $\pi/4$ | $-0.73$ |
| $\pi/2$ | $-1.42$ |
| $\pi$ | $0$ |
| $3\pi/2$ | $1.42$ |
| $7\pi/4$ | $0.73$ |
| $2\pi$ | $0$ |




Phase response

## ❑ Linear Phase of LTI System and its Relationship to Causality:

### 1. Linear Phase:

- A system has ***linear phase if its phase response*** $\theta(e^{j\omega})[or \measuredangle H(e^{j\omega})]$ ***is linear function of frequency*** $\omega$. In general, a linear phase system has frequency response

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{-j\omega\alpha} \qquad .....1$$

### 2. Generalized Linear Phase (GLP):

- A system ***has generalized linear phase (GLP)*** if its frequency response can be written as

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta} \qquad .....2$$

where, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ are constants and $A(e^{j\omega})$ is a real (possibly bipolar) function of $\boldsymbol{\omega}$.

- It is called a generalized linear-phase system because the phase of such a system consists of constant terms added to the linear function $-\omega\alpha$ ; i.e., $-\omega\alpha + \beta$ is the equation of a straight line.

- GLP systems have ***constant group delay*** except at discontinuities in the phase response.

## 3. Causal Generalized Linear-Phase Systems:

- A ***causal FIR systems have generalized linear phase*** if its impulse response satisfies the condition

$$\boldsymbol{h[n]} = \begin{cases} \boldsymbol{h[M-n]}, & \boldsymbol{0 \le n \le M} \\ \boldsymbol{0}, & \boldsymbol{Otherwise} \end{cases} \qquad \text{.....1}$$

then $$H(e^{j\omega}) = A_e(e^{j\omega})e^{-j\omega M/2} \qquad \text{.....2}$$

where, $\boldsymbol{A_e(e^{j\omega})}$ ***is a real, even, periodic function of $\boldsymbol{\omega}$***. (***Symmetric FIR filters***)

- Similarly, if $$\boldsymbol{h[n]} = \begin{cases} \boldsymbol{-h[M-n]}, & \boldsymbol{0 \le n \le M} \\ \boldsymbol{0}, & \boldsymbol{Otherwise} \end{cases} \qquad \text{.....3}$$

then it follows that

- Then it follows that

$$H(e^{j\omega}) = jA_0\left(e^{j\omega}\right)e^{-\omega M/2} = jA_0\left(e^{j\omega}\right)e^{-j\omega M/2+j\pi/2} \qquad .....4$$

where $jA_0\left(e^{j\omega}\right)$ is a real, odd, periodic function of $\omega$. ***(Antisymmetric FIR systems)***

- Note that in both cases the length of the impulse response is $(M + 1)$ samples.

- The conditions in equations (1) and (3) are sufficient to guarantee a causal system with generalized linear phase. However, they are not necessary conditions.

**a. Type I Causal FIR Generalized Linear Phase Systems:**

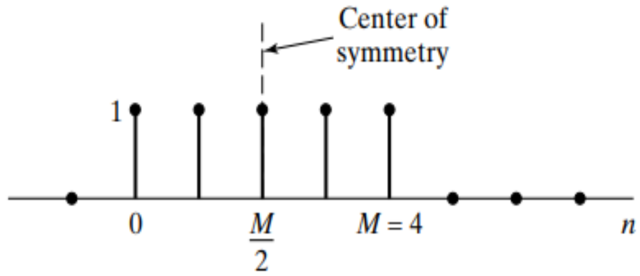
- A ***type I*** system is defined as a system that has a ***symmetric impulse response***

$$h[n] = h[M - n], 0 \le n \le M$$

with ***M an even integer***. The delay $M/2$ is an integer.

- The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$




$$H(e^{j\omega}) = h[0]\left(e^{-j\omega 0} + e^{-j\omega M}\right) + h[1]\left(e^{-j\omega 1} + e^{-j\omega(M-1)}\right) + \cdots + h[\tfrac{M}{2}]e^{-j\omega M/2}$$

$$H(e^{j\omega}) = e^{-\frac{j\omega M}{2}}\{h[0]\,(e^{\frac{j\omega M}{2}} + e^{-\frac{j\omega M}{2}}) + h[1]\,(e^{j\omega\left(\frac{M}{2}-1\right)} + e^{-j\omega\left(\frac{M}{2}-1\right)}) + \ldots\ldots + h[\tfrac{M}{2}]\}$$

$$H(e^{j\omega}) = e^{-\frac{j\omega M}{2}}\{h[0]2\cos(\omega M/2) + h[1]2\cos\left(\omega(\tfrac{M}{2} - 1\right) + \cdots + h[\tfrac{M}{2}]$$

- Therefore, $$\boldsymbol{H(e^{j\omega}) = e^{-\frac{j\omega M}{2}} \sum_{k=0}^{\frac{M}{2}} a[k]\cos(\omega k)}$$

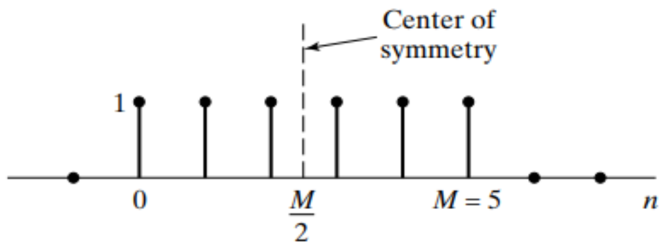
where, $$a[k] = \begin{cases} h\left[\frac{M}{2}\right], & k = 0 \\ 2h\left[\frac{M}{2} - k\right], & k = 1,2,\ldots,\frac{M}{2} - 1, M/2 \end{cases}$$

## b. Type II Causal FIR Generalized Linear Phase Systems:

- A ***type I*** system is defined as a system that has a ***symmetric impulse response***

$$\boldsymbol{h[n] = h[M - n], 0 \leq n \leq M}$$

with ***M an odd integer***.




- The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$\boldsymbol{H(e^{j\omega}) = \{h[0]e^{-j\omega 0} + h[1]e^{-j\omega 1} + \cdots + h[(M-1)/2]e^{-j\omega(M-1)/2}\}}$$
$$\boldsymbol{+\{h[(M+1)/2]\, e^{-j\omega(M\_1)/2} + \cdots + h[M]e^{-j\omega M}}$$

- Since, $h[n] = h[M - n], 0 \le n \le M$

  then $h[0] = h[M]$

  $h[1] = h[M-1]$

  $\cdot \qquad \cdot$

  $\cdot \qquad \cdot$

  $h[(M-1)/2 = h[(M+1)/2$

- Now, $H(e^{j\omega}) = e^{-j\omega M/2}\,\{h[0](\,e^{j\omega M/2} + e^{-j\omega M/2}\,) + \cdots$

  $\ldots + h[(M-1)/2](\,e^{j\omega/2} + e^{-j\omega/2}\,)$

$$H(e^{j\omega}) = e^{-j\omega M/2}\,\{h[0]2\cos\left(\frac{\omega M}{2}\right) + \cdots + h[(M-1)/2]2\cos\left(\frac{\omega}{2}\right)\}$$

- Therefore, $\boldsymbol{H(e^{j\omega}) = e^{-j\omega M/2}\sum_{k=0}^{(M-1)/2} b[k]\cos\{\omega\left(k+\frac{1}{2}\right)\}}$

  where $\boldsymbol{b[k] = 2h[\frac{M-1}{2} - k], \quad k = 1, 2, \ldots, (M-1)/2}$

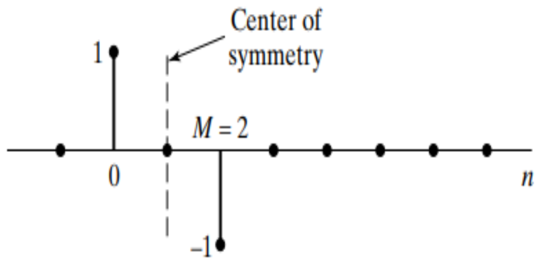
## c. Type III Causal FIR Generalized Linear Phase Systems:

- A ***type I*** system is defined as a system that has a ***antisymmetric impulse response***

$$\boldsymbol{h[n] = -h[M - n], 0 \leq n \leq M}$$

with ***M an even integer***.

Also, $\boldsymbol{h\left[\frac{M}{2}\right] = 0}$




- The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$H(e^{j\omega}) = h[0]\left(e^{-j\omega 0} - e^{-j\omega M}\right) + h[1]\left(e^{-j\omega 1} - e^{-j\omega(M-1)}\right) + \cdots + h[\tfrac{M}{2}]e^{-j\omega M/2}$$

$$H(e^{j\omega}) = e^{-\frac{j\omega M}{2}}\{h[0]\ (e^{\frac{j\omega M}{2}} - e^{-\frac{j\omega M}{2}}) + h[1]\ (e^{j\omega\left(\frac{M}{2}-1\right)} - e^{-j\omega\left(\frac{M}{2}-1\right)}) + \cdots\}$$

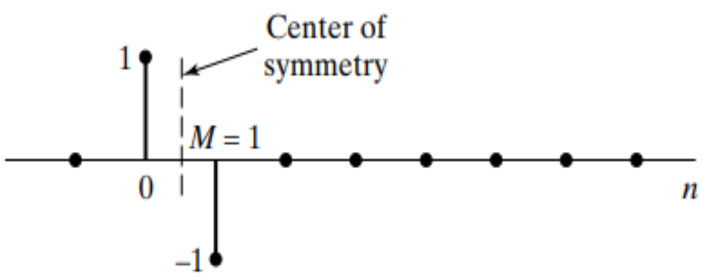
$$H(e^{j\omega}) = e^{-\frac{j\omega M}{2}}\{h[0]2j\sin(\omega M/2) + h[1]2j\sin\left(\omega(\tfrac{M}{2} - 1\right) + \cdots + h[\tfrac{M}{2}]$$

- Therefore, $\boldsymbol{H(e^{j\omega}) = je^{-\frac{j\omega M}{2}} \sum_{k=0}^{\frac{M}{2}-1} c[k]\sin[\omega(k+1)]}$

- Therefore, $H(e^{j\omega}) = je^{-\frac{j\omega M}{2}} \sum_{k=0}^{\frac{M}{2}-1} c[k]\sin[\omega(k+1)]$
  where,
  $$c[k] = 2h\left[\frac{M}{2} - k - 1\right]$$

## d. Type IV Causal FIR Generalized Linear Phase Systems:

- A ***type I*** system is defined as a system that has a ***antisymmetric impulse response***
  $$h[n] = -h[M - n], 0 \le n \le M$$
  with ***M an odd integer***.



- The frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}$$

$$\boldsymbol{H(e^{j\omega}) = \{h[0]e^{-j\omega 0} + h[1]e^{-j\omega 1} + \cdots + h[(M-1)/2]e^{-j\omega(M-1)/2}\}}$$
$$\boldsymbol{+\{h[(M+1)/2]\, e^{-j\omega(M\_1)/2} + \cdots + h[M]e^{-j\omega M}}$$

- Since, $h[n] = -\,h[M\ -\ n], 0\ \leq\ n\ \leq\ M$

  then
$$h[0] = -\,h[M]$$
$$h[1] = -\,h[M-1]$$
$$.\qquad\qquad .$$
$$.\qquad\qquad .$$
$$h[(M-1)/2 = -h[(M+1)/2$$

- Now, $H(e^{j\omega}) = e^{-j\omega M/2}\,\{h[0](\,e^{j\omega M/2} - e^{-j\omega M/2}\,) + \cdots$

$$\ldots + h[(M-1)/2](\,e^{j\omega/2} - e^{-j\omega/2})$$

$$H(e^{j\omega}) = e^{-j\omega M/2}\,\{h[0]2\,jsin\left(\frac{\omega M}{2}\right) + \cdots + h[(M-1)/2]2\,jsin\left(\frac{\omega}{2}\right)\}$$

- Therefore, $$\boldsymbol{H(e^{j\omega}) = je^{-j\omega M/2}\sum_{k=0}^{(M-1)/2} d[k]\sin\{\omega\left(k+\frac{1}{2}\right)\}}$$

$$\boldsymbol{d[k] = 2h[\frac{M-1}{2} - k], \quad k = 1, 2, \ldots, (M-1)/2}$$