Chapter-2: Z $-$Transform

❑ Z $-$Transform ,Convergence of Z $-$ Transform and Region of Convergence (ROC):

1. Introduction:

- The $Z-$transform plays same role in the analysis of discrete-time signals and LTI systems as the Laplace transform does in the analysis of continuous-time signals and LTI systems.
- $Z-$transform may be used to solve LCCDE (Linear Constant Coefficient Difference Equations), evaluate the response of a LTI system to a given input and design linear filters.

2. Definition of Z $-$ Transform:

- The z-transform of a DT signal $x[n]$ is defined as the power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad \text{.....1}$$

where z is a complex variable.

- Equation (1) is called ***direct Z $-$ transform*** because it transforms the time-domain signal $x[n]$ into its complex plane representation $X(z)$. The inverse process, i. e. , obtaining $x[n]$ from $X(z)$ is called the ***inverse Z $-$ transform***.

- We can write ,

$$X(z) = Z\{x[n]\}$$

and $$x[n] = Z^{-1}\{X(z)\}$$

then the relationship between $x[n]$ and $X(z)$ is indicated as:

$$\mathrm{x[n]} \xleftrightarrow{Z} X(z) \qquad .....2$$

- Equation 2 shows the $\boldsymbol{Z}$ $-$ ***transform pair***.

3. Region of Convergence (ROC):

- The set of values of z in the Z $-$ plane for which the Z $-$ transform converges is called the region of convergence (ROC) or region of existence.
- We have, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \{x[n]r^{-n}\}e^{-j\omega n}$

  where, $z = re^{j\omega}$
- A necessary condition for convergence is absolute summability of $|x[n]z^{-n}|$. Since, $|x[n]z^{-n}| = |x[n]r^{-n}|$, we must have

$$\sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \qquad .....1$$

  the range of r for which this condition satisfied is treated as ROC of Z $-$ transform.

4. Properties of the ROC:

- The ROC depends on the nature of signal. Also, different signals have different ROCs. Thus, to find the inverse z-transform, we must specify the given $X(z)$ so as to determine the unique $x[n]$.
- The following are the properties of ROC.

1. The ROC of $X(z)$ consists of a *ring* or *disc* in the z-plane centered at origin.
2. The ROC does not contain any *pole*.
3. The z-transform $X(z)$ of $x[n]$ converges absolutely if and only if the ROC of the z-transform of $x[n]$ includes the *unit circle*.
4. If $x[n]$ is a *finite duration sequence* $(-\infty < N_1 \leq n \leq N_2 < \infty)$then the ROC is the entire z-plane , except possibly at $z = 0$ or $z = \infty$.
5. If $x[n]$ is a *right-sided sequence*, the ROC extends outward from the outermost (i.e., largest magnitude) finite pole in $X(z)$ to $z = \infty$.
6. If $x[n]$ is a *left-sided sequence*, the ROC extends inward from the innermost (i.e., smallest magnitude) finite pole in $X(z)$ to $z = 0$.

7. If the DT signal $x[n]$ is a *two-sided sequence*, the ROC will consists a ring in the z-plane, bounded on the interior and exterior by a pole, i. e. , the ROC will include the intersection of the ROC's of the components.
8. The ROC must be a connected region.

Examples:

1. Determine the z-transform of the signal

$$x[n] = \delta[n]$$

Solution:

- Given, $x[n] = \delta[n]$
- The z-transform is given by

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=-\infty}^{\infty} \delta[n]z^{-n} = 1\times z^{0} = 1$$

therefore, $X(z) = Z\{x[n]\} = 1$

or we can write, $\delta[n] \xleftrightarrow{z} 1$

- **ROC**: Entire z-plane including $z = 0$ and $z = \infty$ because there is no poles and zeros.

2. Determine the z-transform and ROC of the signal

$$\boldsymbol{x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}}$$ **and $a > 0$ (finite length signal)**

Solution:

- Given, $x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$ and $a > 0$
- The z-transform is given by

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n} = \sum_{n=0}^{N-1} a^n\ z^{-n} = \sum_{n=0}^{N-1}\ (a\, z^{-1})^n$$

$$= \frac{1-(az^{-1})^N}{1-az^{-1}} \times \frac{z^N}{z^N} = \frac{z^N-a^N}{z^N-az^{N-1}}$$

therefore, $X(z) = Z\{x[n]\} = \frac{1}{z^{N-1}}\,(\frac{z^N-a^N}{z-a})$

- **ROC:**
- Since there is pole of $(N-1)^{th}$ order at $z = 0$. therefore, ROC is $|z| > 0$ (i.,e., entire z-plane except at $z = 0$).
- **Zeros:** $z_k = ae^{j(\frac{2\pi k}{N})}$

➢ For example, at $N = 2$, $X(z) = \frac{1}{z^{2-1}}\left(\frac{z^2-a^2}{z-a}\right) = \frac{1}{z}\left(\frac{(z-a)(z+a)}{z-a}\right) = \frac{1}{z}(z+a)$

3. The discrete-time signal is given as

$$\boldsymbol{x[n] = a^n u[n]}$$ **(Right-sided sequence)**

Determine: a) z-transform of $x[n]$ and b) ROC

Solution:

a) The z-transform is given by

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n} = \sum_{n=-\infty}^{\infty} a^n u[n]\ z^{-n} = \sum_{n=0}^{\infty} (a\ z^{-1})^n$$

Therefore, $X(z) = \frac{1}{1-az^{-1}}$

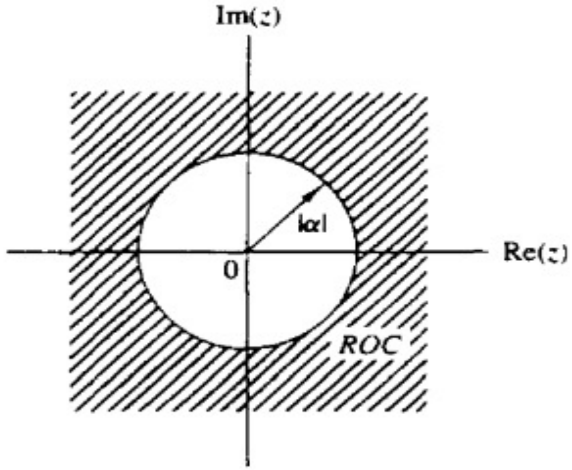
b) The ROC is obtained as

$$|az^{-1}| < 1$$

$$\left|\frac{a}{z}\right| < 1$$

$$|a| < |z|$$

therefore, **ROC:** $|\boldsymbol{z}| > |\boldsymbol{a}|$



4. Consider the discrete-time signal

$$\boldsymbol{x[n] = -a^n\, u[-n-1]}$$ **{ left-sided sequence }**

Determine: a) z-transform

b) ROC

c) Plot pole-zero diagram

Solution:

a. The z-transform is given by

$$X(z) = Z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]\ z^{-n} = \sum_{n=-\infty}^{\infty}\{-a^n u[-n-1]\}\ z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

➢ Let, $-n = m$ then we have

$$X(z) = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty}(a^{-1}z)^m = -\frac{a^{-1}z}{1-a^{-1}z}$$

$$= -\frac{1}{az^{-1}-1} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

b. ROC is obtained as

➢ For convergence, $|a^{-1}z| < 1$ or $\left|\frac{z}{a}\right| < 1$

therefore, ROC: $|z| < |a|$

c. Pole-zero plot:

5. Determine the z-transform of the signal

$$x[n] = a^n u[n] + b^n u[-n-1] \qquad \textbf{(two-sided sequence)}$$

Also determine its ROC and plot pole-zero diagram.

5. Properties of Z-Transform:

- The z-transform properties are useful in the study of discrete-time signals and systems. They are useful in the derivation of z-transforms of many discrete-time signals and also in the solution of ***Linear Constant Coefficient Difference Equations (LCCDE)***.

1. Linearity:

- If $x_1[n] \xleftrightarrow{z} X_1(z)$; ROC: R_1

  and $x_2[n] \xleftrightarrow{z} X_2(z)$; ROC: R_2

  then $x[n] = ax_1[n] + bx_2[n] \xleftrightarrow{z} X(z) = aX_1(z) + bX_2(z)$; ROC: $R_1 \cap R_2$

Proof:

- We have, $z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

$$= \sum_{n=-\infty}^{\infty}\{ax_1[n] + bx_1[n]\}z^{-n}$$

$$= a\{\sum_{n=-\infty}^{\infty} x_1[n]z^{-n}\} + b\{\sum_{n=-\infty}^{\infty} x_2[n]z^{-n}\}$$

$$= aX_1(z) + bX_2(z) = X(z)$$

- Therefore, $x[n] = ax_1[n] + bx_1[n] \xleftrightarrow{z} X(z) = aX_1(z) + bX_2(z)$

Examples:

1. Determine the z-transform and ROC of the signal

$$\boldsymbol{x[n] = [3\ (2)^n - 4(3)^n]u[n]}$$

Solution:

- Given , $x[n] = [3\ (2)^n - 4(3)^n]u[n]$

  or, $x[n] = 3\{(2)^n u[n]\} - 4\{(3)^n u[n]\}$

  or, $x[n] = 3\ x_1[n] - 4\ x_2[n]$
- From linearity property ,we have

$$x[n] = ax_1[n] + bx_1[n] \xleftrightarrow{z} X(z) = aX_1(z) + bX_2(z)$$

  where, $x_1[n] = (2)^n u[n]$ then $X_1(z) = \frac{1}{1-2z^{-1}}$ $\qquad \boldsymbol{\left(a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}\right)}$

  and $x_2[n] = (3)^n u[n]$ then $X_2(z) = \frac{1}{1-3z^{-1}}$
- therefore, $\mathrm{X(z)} = z\{x[n]\} = 3\,\frac{1}{1-2z^{-1}} - 4\,\frac{1}{1-3z^{-1}}$; ROC: $|z| > 3$

2. Determine the z-transform of the signals

a) $\boldsymbol{x[n] = cos\omega_0 n\, u[n]}$

b) $\boldsymbol{x[n] = sin\omega_0 n\, u[n]}$

Solution:

a) Given, $x[n] = cos\omega_0 n\, u[n] = \left\{\frac{e^{j\omega_0 n}+e^{-j\omega_0 n}}{2}\right\} u[n]$

Or, $x[n] = \frac{1}{2}\{e^{j\omega_0 n}\, u[n]\} + \frac{1}{2}\{e^{-j\omega_0 n}\, u[n]\}$

or, $x[n] = \frac{1}{2} x_1[n] + \frac{1}{2} x_2[n]$

where, $x_1[n] = e^{j\omega_0 n}\, u[n]$ then $X_1(z) = \frac{1}{1-e^{j\omega_0} z^{-1}}$ $\quad \left(\boldsymbol{a^n u[n]} \xleftrightarrow{\boldsymbol{z}} \frac{\boldsymbol{1}}{\boldsymbol{1-az^{-1}}}\right)$

and $x_2[n] = e^{-j\omega_0 n}\, u[n]$ then $X_2(z) = \frac{1}{1-e^{-j\omega_0} z^{-1}}$

also ROC is $|z| > 1$ in both cases.

➢ Then, $X(z) = \frac{1}{2}\frac{1}{1-e^{j\omega_0} z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0} z^{-1}}$; ROC: $|z| > 1$

or, $X(z) = \frac{1}{2}\left\{\frac{1-e^{-j\omega_0} z^{-1}+1-e^{j\omega_0} z^{-1}}{(1-e^{j\omega_0} z^{-1})(1-e^{-j\omega_0} z^{-1})}\right\}$

- Or, $$X(z) = \frac{1}{2}\,\frac{2-(e^{j\omega_0}+e^{-j\omega_0})z^{-1}}{1-e^{-j\omega_0}z^{-1}-e^{j\omega_0}z^{-1}+z^{-2}}$$
$$= \frac{1}{2}\,\frac{2-2(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2})z^{-1}}{1-2z^{-1}\left(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2}\right)+z^{-2}}$$
$$= \frac{1}{2}\;\frac{2-2cos\omega_0 z^{-1}}{1-2z^{-1}cos\omega_0+z^{-2}}$$
- Therefore, $X(z) = \frac{1-cos\omega_0 z^{-1}}{1-2z^{-1}cos\omega_0+z^{-2}}$; ROC: $|z| > 1$

2. Time-Shifting:

- If $\quad x[n] \xleftrightarrow{z} X(z)$

  then $\quad x[n-k] \xleftrightarrow{z} z^{-k}X(z)$
- The ROC of $z^{-k}\,X(z)$ is the same as that of $X(z)$ except for $z=0$ if $k>\infty$ and $z=\infty$ if $k<0$.

Proof:

- We have, $\quad X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

  then $\quad z\{x[n-k]\} = \sum_{n=-\infty}^{\infty} x[n-k]z^{-n}$
- Let, $n-k=l$ then $z\{x[n-k]\} = \sum_{l=-\infty}^{\infty} x[l]z^{-(l+k)} = \{\sum_{l=-\infty}^{\infty} x[l]z^{-l}\}z^{-k}$

  or, $\quad z\{x[n-k]\} = z^{-k}\,X(z)$
- Therefore, $\quad x[n-k] \xleftrightarrow{z} z^{-k}X(z)$

Examples:

1. **Consider the z-transform of a signal $x[n]$ is $X(z) = \frac{1}{z-\frac{1}{4}}$; ROC: $|z| > \frac{1}{4}$. Determine the signal $x[n]$ using time shifting property.**

Solution:

- Given, $X(z) = \frac{1}{z-\frac{1}{4}} = \frac{1}{z(1-\frac{1}{4}z^{-1})} = \frac{1/z}{1-\frac{1}{4}z^{-1}} = z^{-1}\frac{1}{1-\frac{1}{4}z^{-1}}$ i
- From time shifting property ,we have

$$x[n-k] \xleftrightarrow{z} z^{-k}X(z)$$

then we can write

$$\frac{\mathbf{1}}{\mathbf{1-\frac{1}{4}z^{-1}}} \xleftrightarrow{z} (\frac{\mathbf{1}}{\mathbf{4}})^{n}\, u[n]$$

and multiplying by z^{-1} shifts $(\frac{\mathbf{1}}{\mathbf{4}})^{n}\, u[n]$ by one sample to the right. Therefore,

$$x[n] = (\frac{\mathbf{1}}{\mathbf{4}})^{n-1}\, u[n-1]$$

3. Scaling in the z-domain(Multiplication by Exponential Sequence):

- If $x[n] \xleftrightarrow{z} X(z)$; ROC: $r_1 < |z| < r_2$

then $a^{n}x[n] \xleftrightarrow{z} X(a^{-1}z)$; ROC: $|a|\, r_1 < |z| < |a|r_2$

Proof:

- We have, $X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

  or, $z\{a^n x[n]\} = \sum_{n=-\infty}^{\infty} \{a^n x[n]\}z^{-n} = \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} = X(a^{-1}z)$
- Therefore, $a^n x[n] \overset{z}{\longleftrightarrow} X(a^{-1}z)$

- ROC: since the ROC of $X(z)$ is $r_1 < |z| < r_2$, then ROC of $X(a^{-1}z)$ is

$$r_1 < |a^{-1}z| < r_2$$

or $$|a|\, r_1 < |z| < |a| r_2$$

Example:

1. Determine the z-transform of the signal

a) $\boldsymbol{x[n] = a^n cos\omega_0 n\, u[n]}$

b) $\boldsymbol{x[n] = a^n sin\omega_0 n\, u[n]}$

Solution:

a) $x[n] = a^n cos\omega_0 n\, u[n]$

- Given, $x[n] = a^n cos\omega_0 n\, u[n] = a^n \left\{\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right\} = \frac{1}{2}\, a^n e^{j\omega_0 n}\, u[n] + \frac{1}{2}\, e^{-j\omega_0 n}\, u[n]$

- Or, $x[n] = \frac{1}{2}\ (ae^{j\omega_0})^n u[n] + \frac{1}{2}\ (ae^{-j\omega_0})^n u[n]$
- Using scaling in the z-domain property, we have

$$\mathrm{X}(z) = \frac{1}{2}\ z\{(ae^{j\omega_0})^n u[n]\} + \frac{1}{2}\ z\{(ae^{-j\omega_0})^n u[n]\}$$

or, $X(z) = \frac{1}{2}\ \frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-j\omega_0}z^{-1}}$; ROC: $|z| > 1$

$$= \frac{1}{2}\left\{\frac{1-ae^{-j\omega_0}z^{-1}+1-ae^{j\omega_0}z^{-1}}{(1-ae^{j\omega_0}z^{-1})(1-ae^{-j\omega_0}z^{-1})}\right\}$$

- $$= \frac{1}{2}\ \frac{2-(e^{j\omega_0}+e^{-j\omega_0})a\,z^{-1}}{1-ae^{-j\omega_0}z^{-1}-ae^{j\omega_0}z^{-1}+a^2z^{-2}}$$

$$= \frac{1}{2}\frac{2-2\left(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2}\right)az^{-1}}{1-2az^{-1}\left(\frac{e^{j\omega_0}+e^{-j\omega_0}}{2}\right)+a^2z^{-2}}$$

$$= \frac{1}{2}\ \frac{2-2az^{-1}cos\omega_0}{1-2az^{-1}cos\omega_0+a^2z^{-2}}$$

- Therefore, $X(z) = \frac{1-az^{-1}cos\omega_0}{1-2az^{-1}cos\omega_0+a^2z^{-2}}$; ROC: $|z| > a$

4. Time Reversal:

- If $\quad x[n] \xleftrightarrow{z} X(z)$; ROC: $r_1 < |z| < r_2$

  then $\quad x[-n] \xleftrightarrow{z} X(z^{-1})$; ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof:

- We have, $X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$

  then $\quad z\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$

- Put $-n = l$, then

$$z\{x[-n]\} = \sum_{l=-\infty}^{\infty} x[l]\ (z^{-1})^{-l} = X(z^{-1})$$

- Therefore, $\quad x[-n] \xleftrightarrow{z} X(z^{-1})$

Examples:

1. Determine the –transform of the signa

a) $\boldsymbol{x[n] = u[-n]}$

b) $\boldsymbol{x[n] = a^{-n}u[-n]}$

Solution:

a) Given, $x[n] = u[-n]$

- The –transform is given by

$$X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \text{ where } x[n] = u[-n]$$

and
$$u[-n] = \begin{cases} 1, n \leq 0 \\ 0, \; n > 0 \end{cases}$$

- Then $X(z) = z\{u[-n]\} = \sum_{n=-\infty}^{\infty} u[-n]z^{-n} = \sum_{n=-\infty}^{0} z^{-n}$
- Put $-n = l$ then $z\{u[-n]\} = \sum_{l=0}^{\infty} z^{l} = \frac{1}{1-z}$; ROC: $|z| < 1$
- Therefore, $X(z) = z\{u[-n]\} = \frac{1}{1-z}$

b) Given, $x[n] = a^{-n}u[-n]$

- The –transform is given by

$$X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \text{ where } x[n] = a^{-n}u[-n]$$

and
$$u[-n] = \begin{cases} 1, n \leq 0 \\ 0, \; n > 0 \end{cases}$$

- Then $X(z) = z\{a^{-n}u[-n]\} = \sum_{n=-\infty}^{\infty} a^{-n}u[-n]z^{-n} = \sum_{n=-\infty}^{0}(az)^{-n}$
- Put $-n = l$ then $z\{a^{-n}u[-n]\} = \sum_{l=0}^{\infty}(az)^{l} = \frac{1}{1-az}$; ROC: $|z| < \left|\frac{1}{a}\right|$
- Therefore, $X(z) = z\{a^{-n}u[-n]\} = \frac{1}{1-az}$

5. Differentiation in the z –Domain:

- If $x[n] \overset{z}{\longleftrightarrow} X(z)$

  then $n\,x[n] \overset{z}{\longleftrightarrow} -z\,\frac{d\{X(z)\}}{dz}$

Proof:

- We have, $X(z) = z\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$
- Differentiating on both sides with respect to z, we have

$$\frac{d\{X(z)\}}{dz} = \frac{d\{\sum_{n=-\infty}^{\infty} x[n]z^{-n}\}}{dz} = -n\sum_{n=-\infty}^{\infty} x[n]z^{-n-1} = -z^{-1}\sum_{n=-\infty}^{\infty}\{nx[n]\}z^{-n}$$

- Or, $\frac{d\{X(z)\}}{dz} = -\frac{1}{z}\,z\{nx[n]\}$
- Or, $z\{nx[n]\} = -z\,\frac{d\{X(z)\}}{dz}$

➢ Therefore, $n\,x[n] \xleftrightarrow{z} -z\,\frac{d\{X(z)\}}{dz}$

- Note that both transform have the same ROC.

Examples:

1. Determine the –transform and ROC of the signal

$$x[n] = n\,a^n u[n]$$

Solution:

➢ Given, $x[n] = n\,a^n u[n] = n x_1[n]$

where, $x_1[n] = a^n u[n]$

➢ We know, $a^n u[n] \xleftrightarrow{z} \frac{1}{1-az^{-1}}$; ROC: $|z| > |a|$

then from the differentiating property , we know

$$n x_1[n] \xleftrightarrow{z} -z\,\frac{d\{X_1(z)\}}{dz} = -z\,\frac{d\left\{\frac{1}{1-az^{-1}}\right\}}{dz} = -z\,\frac{0 - 1\times\frac{d\{1-az^{-1}\}}{dz}}{(1-az^{-1})^2} = z\,\frac{\frac{d\{1-az^{-1}\}}{dz}}{(1-az^{-1})^2}$$

$$= z\,\frac{0-(-1)\times az^{-2})}{(1-az^{-1})^2} = \frac{az^{-1}}{(1-az^{-1})^2}$$

- Therefore, $x[n] = na^n u[n] \xleftrightarrow{\quad z \quad} X(z) = \frac{az^{-1})}{(1-az^{-1})^2}$
- Note that if $a = 1$, then $x[n] = n\,u[n]$ then we can write

$$x[n] = nu[n] \xleftrightarrow{\quad z \quad} X(z) = \frac{z^{-1}}{(1-z^{-1})^2}$$

6. Convolution of Two Sequences:

- If $$x_1[n] \xleftrightarrow{z} X_1(z)$$

  and $$x_2[n] \xleftrightarrow{z} X_2(z)$$

  then $$x[n] = x_1[n] * x_2[n] \xleftrightarrow{z} X(z) = X_1(z)\ X_2(z)$$

- The ROC is, at least, the intersection between the ROCs of that of $X_1(z)$ and $X_2(z)$.

Proof:

- The convolution of $x_1[n]$ and $x_2[n]$ is defined as

$$x[n] = x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]$$

- The z–transform of $x[n]$ is

$$\begin{aligned} z\{x[n]\} &= \sum_{n=-\infty}^{\infty} x[n] z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \{x_1[n] * x_2[n]\} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} \{\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k]\} z^{-n} \\ &= \sum_{k=-\infty}^{\infty} x_1[k]\ \{\sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n}\} \qquad \text{.....i} \end{aligned}$$

- From time-shifting property , we have

$$x_2[n-k] \xleftrightarrow{z} z^{-k}\ X_2(z)$$

- Then, equation (i) becomes

$$z\{x[n]\} = \{\sum_{k=-\infty}^{\infty} x_1[k]z^{-k}\}X_2(z)$$
$$= X_1\ (z)X_2(z)$$
$$= X(z)$$

- Therefore, $x[n] = x_1[n] * x_2[n] \xleftrightarrow{z} X(z) = X_1(z)\ X_2(z)$

Example:

1. **Compute the convolution $\boldsymbol{x[n]}$ of the signals**

$$\boldsymbol{x_1[n] = \{1, -2, 1\}} \textbf{ and } \boldsymbol{x_2[n] = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & Otherwise \end{cases}}$$

Solution:

- We have, $X_1(z) = 1 - 2z^{-1} + z^{-2}$

  and $X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

- From the convolution property, we have

$$X(z) = X_1(z)\ X_2(z)$$

  Or, $X(z) = 1 - z^{-1} - z^{-6} + z^{-7}$

- Taking inverse –transform, we have

$$x[n] = \{\mathbf{1}, -1, 0, 0, 0, 0, -1, 1\}$$

- **Note:**

- The convolution property is one of the powerful properties of the z−transform because it converts the convolution of two time-domain signals in multiplication of their transforms.
- Computation of the convolution of two signals using z−transform requires the following steps:

1. Compute the z–transforms ($X_1(z)$ and $X_2(z)$) of the signals ($x_1[n]$ and $x_2[n]$) to be convolved.
2. Multiply the two z –transforms to obtain $X(z)$, where, $X(z) = X_1(z)\ X_2(z)$].
3. Find the inverse z-transform of $X(z)$.

❑ Inverse z–Transform by Long Division and Partial Fraction Expansion:

1. Methods for the inversion of z–transform:

- The procedure for transforming a signal from z–domain to the time-domain is called the inverse z–transform. The following are the methods for the inversion of the z–transform

a. Long division method (or power series expansion method)
b. Partial fraction expansion method
c. Contour integration method

a. Long Division Method (or power series expansion method):

- Consider a z–transform $X(z)$ with its corresponding ROC , then $X(z)$ can be expanded into a power series of the form

$$X(z) = \sum_{n=-\infty}^{\infty} C_n \, z^{-n} \qquad \text{.....i}$$

which converges into the given ROC, also $x[n] = C_n$ for all n.

- When $X(z)$ rational, i.,e., when $X(z) = \frac{N(z)}{D(z)}$, the expansion can be performed by ***long division*** (by dividing $N(z)$ by $D(z)$).

Example:

1. **Determine the inverse z-transform of** $\boldsymbol{X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}}$

   when: a) ROC: $|z| > 1$ **b) ROC:** $|z| < 0.5$

Solution:

a) Since, the ROC is exterior of a circle, $x[n]$ is ***causal signal***. Hence, we obtain a series in negative powers of z. Carrying out the long division, we have

$$
\begin{array}{r|l}
 & 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + \cdots \\
\hline
1 - 1.5z^{-1} + 0.5z^{-2} & 1 \\
 & 1 - 1.5z^{-1} + 0.5z^{-2} \\
 & - \quad + \quad - \\
\hline
 & \quad 1.5z^{-1} - 0.5z^{-2} \\
 & \quad 1.5z^{-1} - 2.25z^{-2} + 0.75z^{-3} \\
 & \quad - \quad + \quad - \\
\hline
 & \qquad 1.75z^{-2} - 0.75z^{-3} \\
 & \qquad 1.75z^{-2} - 2.625z^{-3} + 0.875z^{-4} \\
 & \qquad - \quad + \quad - \\
\hline
 & \qquad\quad 1.875z^{-3} - 0.875z^{-4} \\
 & \qquad\quad \cdots\cdots
\end{array}
$$

- Or, $X(z) = 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + \cdots$
- Therefore, $x[n] = \{\mathbf{1}, 1.5, 1.75, 1.875, \dots\}$

b) Here, the ROC is interior of the circle, the signal $x[n]$ is anti-causal signal. Thus, we divide so as to obtain a series in power of z as follows:

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + \cdots \\
\hline
0.5z^{-2} - 1.5z^{-1} + 1 & 1 \\
 & 1 - 3z + 2z^2 \\
 & -\quad +\quad - \\
\cline{2-2}
 & \quad 3z - 2z^2 \\
 & \quad 3z - 9z^2 + 6z^3 \\
 & \quad -\quad +\quad - \\
\cline{2-2}
 & \qquad 7z^2 - 6z^3 \\
 & \qquad 7z^2 - 21z^3 + 14z^4 \\
 & \qquad -\quad +\quad - \\
\cline{2-2}
 & \qquad \dots\dots\dots\dots\dots\dots
\end{array}
$$

- Therefore, $X(z) = 2z^2 + 6z^3 + 14z^4 + \cdots$
- Then the inverse –transform is

$$x[n] = \{\dots, 14, 6, 2, 0, \mathbf{0}\}$$

b. Partial Fraction Expansion Method:

- The z–transform is given by

$$X(z) = \frac{N(z)}{D(z)} = \frac{\sum_{k=0}^{M} b_k\, z^{-k}}{1+\sum_{k=0}^{N} a_k\, z^{-k}} \qquad \text{.....i}$$

where, $a_0 = 1$

Or, $$X(z) = \frac{b_0+b_1 z^{-1}+b_2\, z^{-2}+\cdots+b_M\, z^{-M}}{1+a_1\, z^{-1}+a_2\, z^{-2}+\cdots+a_N\, z^{-N}} \qquad \text{.....ii}$$

- If $a_0 \neq 1$, we can obtain equation (ii) from equation (i) by dividing both numerator and denominator by a_0.
- A **rational function** of the form of equation (ii) is called ***proper*** if $a_N \neq 0$ and $M < N$(i. e. , the number of finite zeros is less than the number of finite poles).

- An ***improper rational function*** $(\boldsymbol{M} \geq \boldsymbol{N})$ can always be written as the sum of a polynomial and a proper rational function as

$$\boldsymbol{X}(\boldsymbol{z}) = \frac{\boldsymbol{N}(\boldsymbol{z})}{\boldsymbol{D}(\boldsymbol{z})} = \boldsymbol{C_0} + \boldsymbol{C_1}\boldsymbol{z}^{-1} + \boldsymbol{C_2}\boldsymbol{z}^{-2} + \cdots + \boldsymbol{C_{M-N}}\boldsymbol{z}^{-(\boldsymbol{M}-\boldsymbol{N})} + \frac{\boldsymbol{N_1}(\boldsymbol{z})}{\boldsymbol{D_1}(\boldsymbol{z})} \quad \text{.....iii}$$

- To perform the partial fraction expansion, we first factor the denominator polynomial into factors that contain the poles $p_1,\ p_2,\ p_3$....., p_N of $X(z)$. We distinguishes two cases:

1. Distinct poles
2. Multiple order poles

1. Distinct Poles:

- Suppose that the poles p_1, p_2, p_3....., p_N are all distinct poles, we have

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \cdots + \frac{A_k}{z-p_k} + \cdots + \frac{A_N}{z-p_N} \quad \text{.....1}$$

- Multiplying equation (1) by $(z - p_k)$ on both sides, we get

$$(z - p_k)\frac{X(z)}{z} = (z - p_k)\frac{A_1}{z-p_1} + \cdots + (z - p_k)\frac{A_k}{z-p_k} + \cdots + (z - p_k)\frac{A_N}{z-p_N} \text{.....2}$$

- If $z = p_k$, then equation (2) becomes

$$\boldsymbol{A_k = (z - p_k)\left.\frac{X(z)}{z}\right|_{z=p_k}} \qquad \text{.....3}$$

Examples:

1. Determine the inverse z −transform of the function

$$\boldsymbol{X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}}$$

Solution:

- Given, $$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

- First, we eliminate the negative powers by multiplying numerator and denominator by z^2. Thus,

$$X(z) = \frac{z^2}{z^2-1.5z+0.5} = \frac{z^2}{(z-1)(z-0.5)}$$

- Now, $$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

- Now, $A_1 = (z - p_1)\ \frac{X(z)}{z}\Big|_{z=p_1}$ where, $p_1 = 1$

  $A_1 = (z - 1)\frac{z}{(z-1)(z-0.5)}\Big|_{z=1} = \frac{1}{1-0.5} = 2$

- Again, $A_2 = (z - p_2)\ \frac{X(z)}{z}\Big|_{z=p_2}$ where, $p_2 = 0.5$

  $A_2 = (z - 0.5)\frac{z}{(z-1)(z-0.5)}\Big|_{z=0.5} = \frac{0.5}{0.5-1} = -1$

- Thus, $\frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$

  Or, $X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5} = 2\frac{1}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$

- Taking inverse z-transform, we get

  $x[n] = 2\ (1)^n\ u[n] - (0.5)^n\ u[n]$

  Or, $x[n] = \{2\ (1)^n - (0.5)^n\}\ u[n]$

2. Multiple Order Poles:

- If $X(z)$ has a pole of multiplicity l, that is , it contains in its denominator the factor $(z - p_k)^l$. In such cases , different expansion is required as explained in the following example.

Example:

1. Determine the causal signal $x[n]$ having the z–transform

$$\boldsymbol{X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}}$$

Solution:

- We have, $$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2} \times \frac{z^3}{z^3} = \frac{z^3}{(z+1)(z-1)^2}$$

  Or, $$\frac{X(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A_1}{z+1} + \frac{A_2}{z-1} + \frac{A_3}{(z-1)^2}$$

- Now, $$A_1 = (z - p_1)\left.\frac{X(z)}{z}\right|_{z=p_1}$$ where, $p_1 = -1$

Or, $A_1 = (z+1)\frac{z^2}{(z+1)(z-1)^2}\Big|_{z=-1} = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$

- Similarly, $A_3 = (z-p_3)\ \frac{X(z)}{z}\Big|_{z=p_3}$ where, $p_3 = 1$

$$A_3 = (z-1)\frac{z^2}{(z+1)(z-1)^2}\Big|_{z=1} = \frac{(-1)^2}{(1+1)} = \frac{1}{2}$$

- Again, A_2 can be found by differentiating $(z-1)^2\times\frac{X(z)}{z}$with respect to z at $z=1$

$$A_2 = \frac{d\{\ (z-1)^2\times\frac{X(z)}{z}\}}{dz} = \frac{d\{\ (z-1)^2\times\frac{z^2}{(z+1)(z-1)^2}\}}{dz} = \frac{d\{\frac{z^2}{(z+1)}\}}{dz} = \frac{(z+1)\times 2z-z^2\times 1}{(z+1)^2}$$

Or, $A_2 = \frac{2z(z+1)-z^2}{(z+1)^2}\Big|_{z=1} = \frac{2\times 1(1+1)-(1)^2}{(1+1)^2} = \frac{3}{4}$

- Thus, $\frac{X(z)}{z} = \frac{1}{4}\ \frac{1}{z+1} + \frac{3}{4}\ \frac{1}{z-1} + \frac{1}{2}\ \frac{1}{(z-1)^2}$

or, $X(z) = \frac{1}{4}\ \frac{1}{1+z^{-1}} + \frac{3}{4}\ \frac{1}{1-z^{-1}} + \frac{1}{2}\ \frac{z^{-1}}{(1-z^{-1})^2}$

- Taking inverse z–transform , we get

$$x[n] = \frac{1}{4}(-1)^n\, u[n] + \frac{3}{4}(1)^n\, u[n] + \frac{1}{2}n(1)^n\, u[n]$$

$$x[n] = \frac{1}{4}(-1)^n\, u[n] + \frac{3}{4}(1)^n\, u[n] + \frac{1}{2}n\, u[n]$$