Digital Signal Analysis And Processing

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1. Discrete-time Signals and System (8 Hrs)

1.1 Discrete-time signals, basic signal

1.2 Energy signal, Power signal

1.3 Periodicity of discrete time signal

1.4 Transformation of independent variables

1.5 Discrete time Fourier series and properties

1.6 Discrete time Fourier transform and properties

1.7 Discrete time system properties

1.8 Linear Time Invariant (LTI) system, convolution sum, Properties of Linear Time-Invariant system (LTI) system

1.9 Frequency Response of LTI systems

1.10 Sampling of continuous time signal, spectral properties of sampled signal

Signals

- The term signal is generally applied to something that conveys information. Examples :
  - Speech, is encountered in telephony, radio, TV, and everyday life
  - Biomedical signals, (heart signals, brain signals)
  - Video and image,
  - Radar signals, which are used to determine the range and bearing of distance targets.
- A Signal is defined as any physical quantity that varies with time, space, or any other independent variable or variables that carry some information. Examples :
  - $x(t) = t + 2t^2$ (signal of one independent variables i.e. time)
  - $s(x, y) = 3x + 5xy + 7y^2$ (signal of two independent variables i.e. x and y)
- Signal can be defined as a detectable physical quantity or impulse (voltage, current or magnetic field strength) by which messages or information can be transmitted.

1.1 Discrete-time Signals, basic signal

- The signal arising in digital signal processing are basically discrete-time signals, and discrete-time systems are used to process these signals.
- Discrete-time signal processing involves the processing of a discrete-time signal by a discrete-time system to develop another discrete –time signal with more desirable properties or to extract certain information about the original discrete-time signal.
- The continuous time signal is first converted into an equivalent discrete time signal by periodic sampling; the discrete-time signal is then processed by a discrete-time system to generate another discrete-time signal, and the latter is converted into an equivalent continuous-time signal, if necessary without any distortion.
- To understand the theory of digital signal processing and the design of discrete-time systems, we need to know the characterization of discrete-time signals and systems in the time-domain.

Elementary discrete-time signals: Sequences

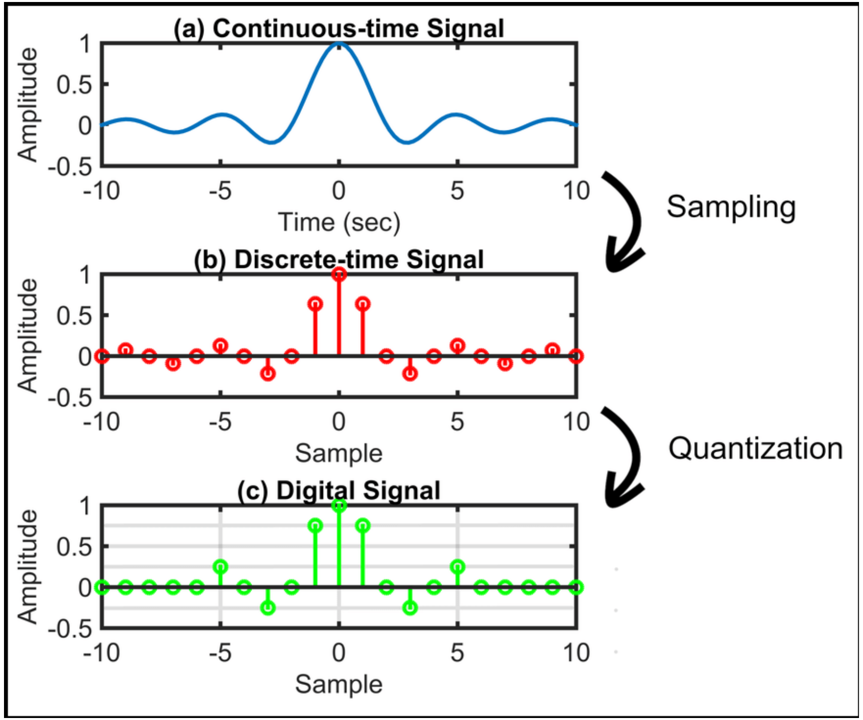
- ✓ Discrete-time signals are represented mathematically as sequences of numbers.
- ✓ A sequence of number x, in which the n^th number in the sequence is denoted x[n], is formally wiitten as

  $x = \{x[n]\}, -\infty < n < \infty,$ where n is an integer.
- ✓ Such sequences arise form periodic sampling of an anlaog signal. In this case, the numeric vlaue of the n^th number in the sequence is equal to the value of the analog signal, $x_a(t)$, at time nT; i.e.,

  $$x[n] = x_a(nT), -\infty < n < \infty$$

  Where T is called the sampling period, and its reciprocal is the sampling frequency.
- ✓ Digital signals are discrete in both time (the independent variable) and amplitude (the dependent variable).
- ✓ Signals that are discrete in time but continuous in amplitude are referred to as discrete time signals.

- ✓ Discrete-time signals are data sequences. The elements of the sequence are called samples.
- ✓ The index n associated with each sample is an integer.
- ✓ A discrete time signal x[n] is a function of an independent variable that is an integer.
- ✓ Discrete-time signal is not defined at instants between two successive samples.
- ✓ It is not correct to think of x[n] as being zero for n is not an integer.
- ✓ The signal x[n] is not defined for not-integer value of n.



Representation of Discrete-time Signals

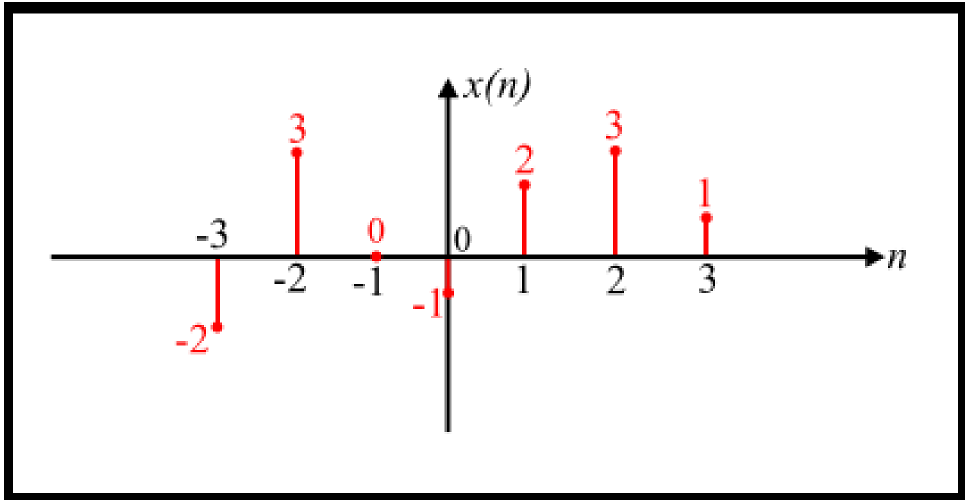
There are four ways of representing discrete-time signals.

i. Graphical representation

Consider a discrete time signal x[n] with the values,

x[-3]=-2, x[-2]=3, x[-1]=0, x[0]=-1, x[1]=2, x[2]=3, x[3]=1

The discrete time signal can be represented graphically as



ii. Functional representation

In functional representation of discrete time signals, the magnitude of the signal is written against the value of n. Therefore the discrete time sigal x[n] can be represented usnig functional represntation as

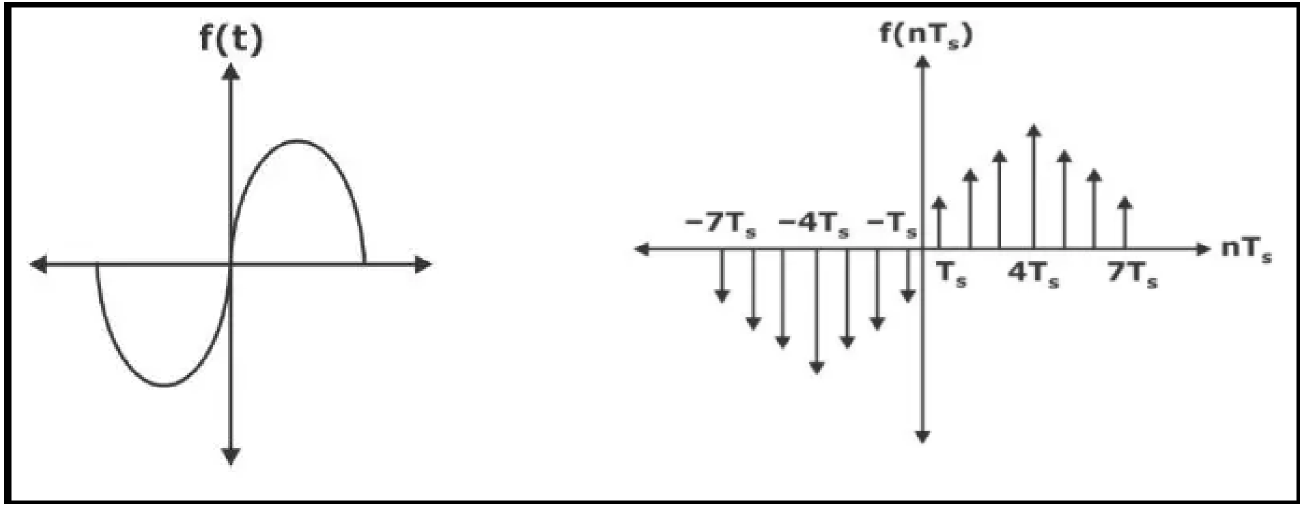
$$x[n]= \begin{cases} -2 \; for \; n = -3 \\ 3 \; for \; n = -2 \\ 3 \; for \; n = -2 \\ 0 \; for \; n = -1 \\ -1 \; for \; n = 0 \\ 2 \; for \; n = 1 \\ 3 \; for \; n = 2 \\ 1 \; for \; n = 1 \end{cases}$$

Continuous-Time and Discrete-Time Signal

If a signal value is defined for every instant of time in a considered interval however the small interval is, it is called a "continuous-time signal".

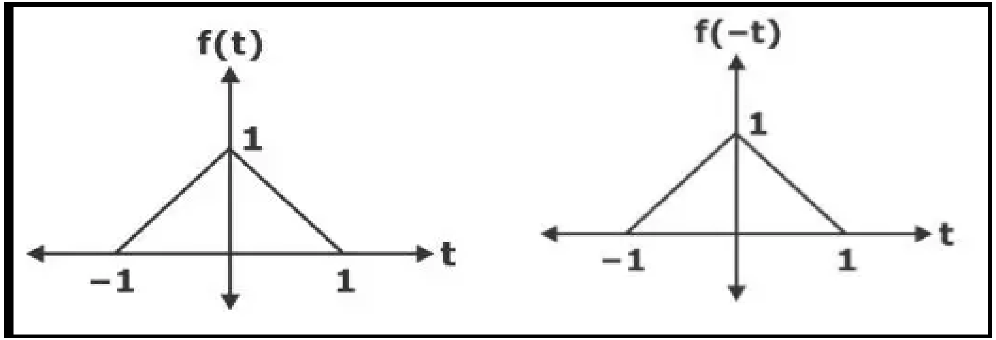
A discrete-time signal is derived from a continuous-time signal through a process called sampling, hence it is defined for discrete values of the time.

Figure below shows a continuous-time signal f(t) = $\sin t$ and its discrete-time version f(nT_s) = f(t), n = $\pm 1, \pm 2, \pm 3, \ldots\ldots$ Where T_s is the sampling period (sec/sample).



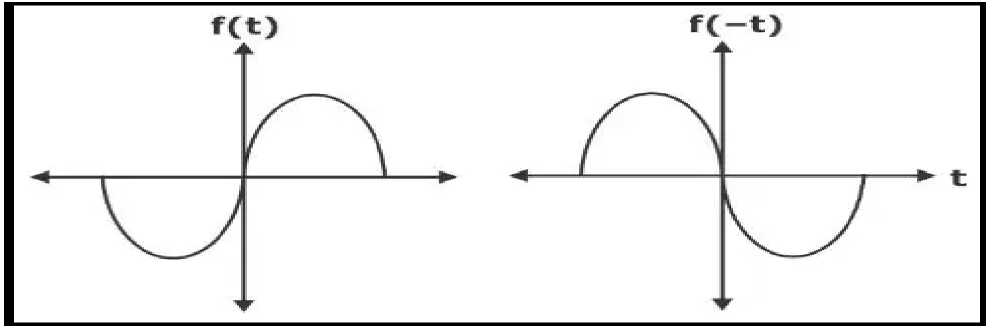
Even and Odd Signals

The signal that is symmetric about the vertical axis such that it appears visually identical to its time-reversal version, such signal is known as an even signal.



For continuous signal f(-t) = f(t),
For discrete signal f(-n) = f(n).

The signal is odd, if it is anti symmetric about its vertical axis.



For continuous signal f(-t) = -f(t),
For discrete signal f(-n) = -f(n).

A signal can be represented by sum of even component and odd component as
f(t) = $f_e(t) + \ f_o(t)$.

Periodic an Aperiodic Signals

If any signal has a definite pattern, that repeats itself at a regular interval, such signal is known as a periodic signal. If it does not have such a pattern, then it is known as an aperiodic signal.

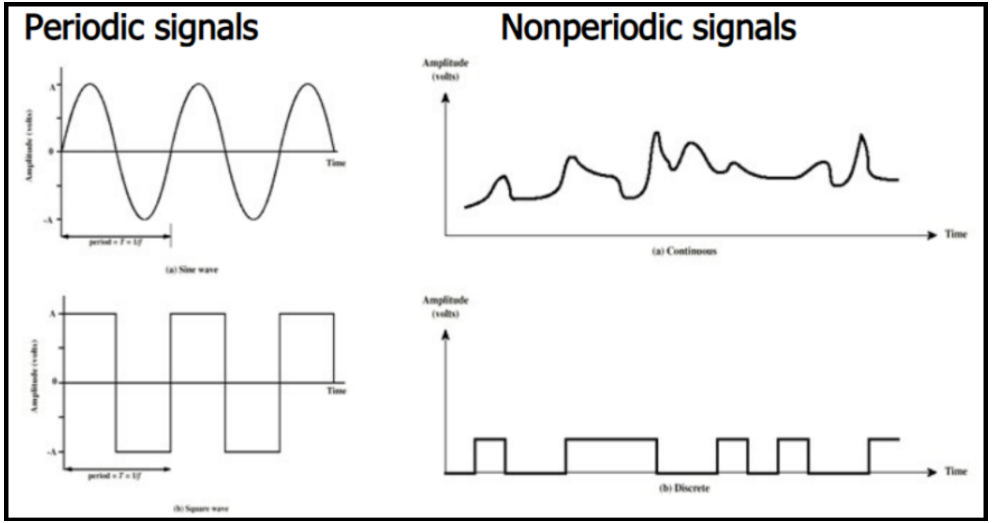
If f(t) is continuous-time periodic signal, then $f(t) = f(t \pm kT)$

Where 'T' is the fundamental time period of the signal f(t), and 'k' is an integer

If f(n) is a discrete time periodic signal,

Then $f(n)= f(n \pm N)$

Where N is the fundamental time period.



Deterministic and Random Signals

If a signal’s future value is known at present, such signal is called a deterministic signal. If the future value of the signal cannot be determined at present, such signals are called random signals, such signal can only be expected or estimated.

Example of deterministic signal $f(t) = 3t+6$

At $t = 0$, $f(0) = 6$,

At $t = 10$, $f(10) = 36$

Causal and Non-causal Signals

A signal which posses zero amplitude for all negative values of time, then the signal is known as a causal signal.

$X(t) > 0$, for $t \geq 0$ and

$X(t) = 0$ for $t<0$.

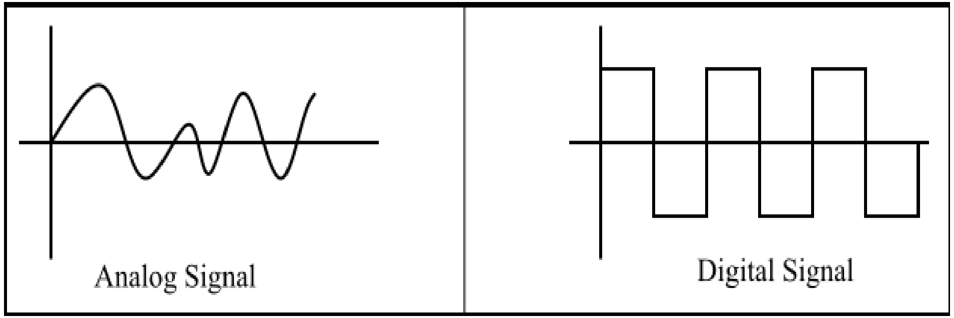
A signal that has positive value of amplitude for both positive and negative instances of time is a non causal signal.

A signal which posses zero value for all positive value of time, but has amplitude which is greater than zero for all negative value of time, then the signal is known as anti causal signal.

$X(t) > 0$, for $t \leq 0$ and

$X(t) = 0$, for $t>0$.

Analog Signal and Digital Signal



A signal which is a continuous function of time and used to carry the information is known as analog signal. All natural signals are analog signals.

A signal that is discrete function of time i.e. which is not continuous signal, is known as digital signal. The digital signals are represented in the binary form.

1.2 Energy and Power Signal

Signals may be classified as energy and power signals. However some signals which can neither be classified as energy signals nor power signals.

The energy signal is one which has finite energy and zero average power. x(t) is an energy signal if $0 < E < \infty,\ and\ P = 0$.

For any continuous time signal x(t) the energy is given as

$$E = \int_{-\infty}^{\infty} x^2(t)dt$$

For a complex value signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

For discrete time signal x(n) the energy is given as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Almost all non-periodic signals which are defined over finite time are energy signals.

A power signal is one which has finite power and infinite energy. X(t) is a power signal if $0 < P < \infty,\ and\ E = \infty$.

For a continuous time signal x(t) the average power is given as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt$$

For complex value signal

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |x(t)|^2 dt$$

Similarly for a discrete time signal x(n)

$$P = \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} |x(n)|^2$$

Clearly, if E is finite, P = 0 and if E is infinite, the average power P may be either finite or infinite. If P is finite (and nonzero), the signal is called power signal. However, if the signal does not satisfy any of the above conditions, then it is neither energy nor a power signal.

Energy Signal

i) $0 < E < \infty$ and $P_{av} = 0$

ii) For CT,

$$E = \lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

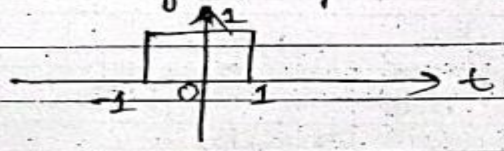
For DT,

$$E = \lim_{N \to \infty} \sum_{n=-N/2}^{N/2} |x(n)|^2 \cancel{dt}$$

iii) Most of non-periodic signals.

iv) Time limited.

v) eg, rectangular pulse



Power Signal

i) $0 < P_{av} < \infty$ and $E = \infty$

ii) For CT,

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt$$

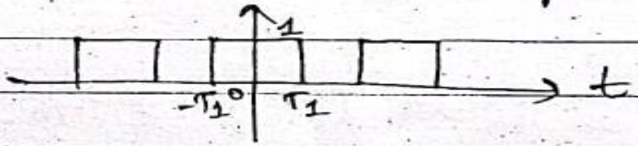
For DT,

$$P_{av} = \lim_{N \to \infty} \frac{1}{N+1} \sum_{n=-N/2}^{N/2} |x(n)|^2$$

iii) Most of periodic signals.

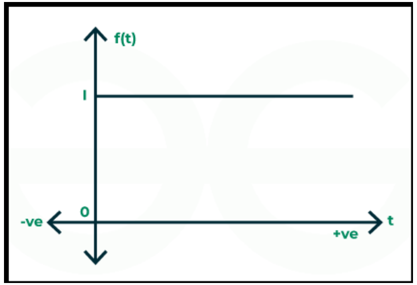
iv) exist over infinite time.

v) eg, periodic rectangular pulse



Examples:

1. Determine whether the following signals are energy or power signal or none.
   i. Continuous time unit step signal
   ii. Discrete time unit step signal



Solution :

i. Continuous time unit step signal is defined as u(t) = 1 for t ≥ 0

= 0 otherwise

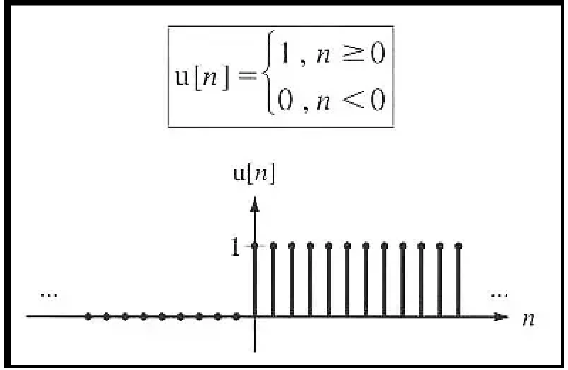
Now energy $E = \int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{0} x^2(t)dt + \int_{0}^{\infty} x^2(t)dt = \int_{-\infty}^{0} 0dt + \int_{0}^{\infty} 1^2 dt = \infty$

Power $P = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{0} x^2(t)dt + \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} x^2(t)dt$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{0} 0^2(t)dt + \lim_{T\to\infty} \frac{1}{T} \int_{0}^{\frac{T}{2}} 1^2(t)dt = 0 + \lim_{T\to\infty} \frac{1}{T} \times \frac{T}{2} = \frac{1}{2}$$

ii. Discrete time unit step signal u(n) = 1 for n$\geq$ 0

= 0 otherwise



Now $E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{0} |x(n)|^2 + \sum_{n=0}^{\infty} |x(n)|^2$

$= 0 + \sum_{n=0}^{\infty} 1^2 = \infty$

And $P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{0} |x(n)|^2 + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{0} 0^2 + \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} 1^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} (\mathrm{N} + 1)$$

$$= \lim_{N\to\infty} \frac{\left(1 + {}^{1}/_{N}\right)N}{\left(2 + {}^{1}/_{N}\right)N} = \frac{1}{2}$$

2. Check whether the given signal is energy or power signal $x(t) = A\sin(t)$.

Solution : since sine wave is periodic with period of T = 2π and therefore it is a power signal.

$$P = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt$$

$$P = \frac{1}{2\pi}\int_0^{2\pi} [A\sin(t)]^2(t)dt$$

$$= \frac{1}{4\pi}A^2 \int_0^{2\pi}(1 - \cos 2t)dt$$

$$= \frac{A^2}{2}$$

3. Check whether the given signal is energy or power signal $x(t) = A\cos(\omega_0 t + \theta)$.

Power $= \frac{A^2}{2}$

iii. Tabular representation

n	-3	-2	-1	0	1	2	3
x[n]	-2	3	0	-1	2	3	1

iv. Sequential representation

The discrete-time signal x[n] can be represented in the sequence as follows

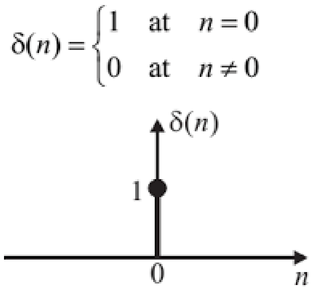
$$x[n] =\{ -2, 3, 0, \underset{\uparrow}{-1}, 2, 3, 1\}$$

Here, the arrow mark denotes the term correponding to n = 0. When no arrow is indicated in the sequence represention of the discrete signal, then the first term of the sequence corresponds to n = 0.

Some Elementary Discrete-Time Signals

1. The unit sample sequence is denoted by $\delta(n)$ and is defined as

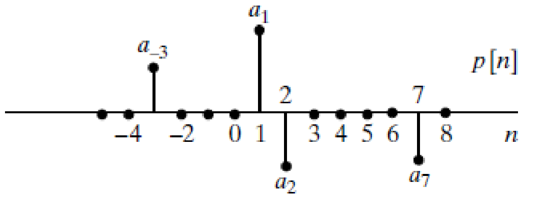
$$\delta(n) = \begin{cases} 1, for\ n = 0 \\ 0, for\ n \neq 0 \end{cases}$$



The unit sample sequence is a signal that is zero everywhere, except at n =0 where its unity. These signal is sometimes referred to as a unit impulse.

One of the important aspects of the impulse sequence is that an arbitrary sequence can be represented as a sum of scaled, delayed impuses.
For example, the sequence p(n) in figure can be expressed as

$$p(n) = a_{-3}\delta(n+3) + a_1\delta(n-1) + a_2\delta(n-2) + a_7\delta(n-7)$$



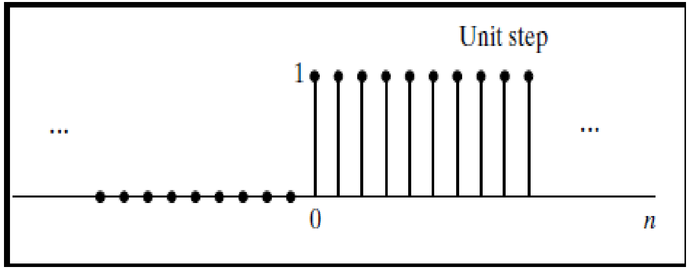
Example of a sequence to be represented as a sum of scaled, delayed impulses.

In general, any sequence can be expressed as

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

2. The unit step signal is denoted by u(n) and is defined as

$$u(n) = \begin{cases} 1, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$



The unit step is related to the unit impuse by

$$u(n) = \sum_{k=-\infty}^{n} \delta(k)$$

That is, the value of the unit step sequence at (time) index n is equal to the accumulated sum of the value at index n and all previous values of the impuse sequence.

An alternative representation of the unit step in terms of the impuse is

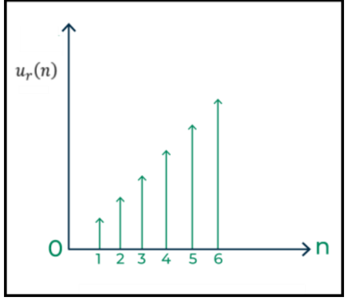
$$u(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

or $$u(n) = \sum_{k=0}^{\infty} \delta(n-k)$$

As yet another alternative, the impuse sequence can be expresses as the first backward difference of the unit step sequence, i.e.,

$$\delta(n) = u(n) - u(n-1)$$

3. The unit ramp signal is denoted as $u_r(n)$ and defined as

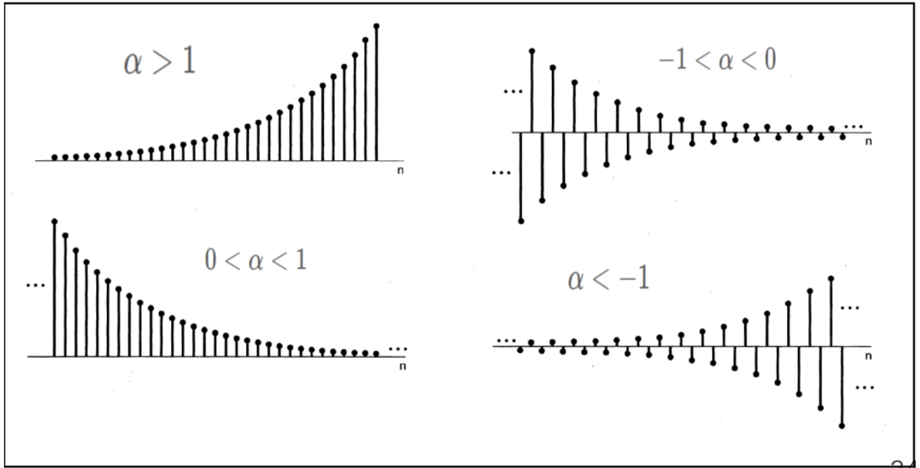


$$u_r(n) = \begin{cases} n, for\ n \geq 0 \\ 0, for\ n < 0 \end{cases}$$

4. Exponential sequences are another important class of basic signals. The general form of an exponential sequence is

$$x(n) = A\alpha^n$$

Where A and α are real numbers. Figure shows the graph for different value of α.



The exponential sequence $A\alpha^n$ with α complex has real and imaginary parts that are exponentially weighted sinusoids.

Specifically, if $\alpha = |\alpha|e^{jw_0}$ and $A = |A|e^{j\emptyset}$, the sequence $A\alpha^n$ can be expressed in any of the following ways:

$$\begin{aligned} x[n] = A\,\alpha^n &= |A\,|e^{j\phi}|\alpha|^n e^{j\omega_0 n} \\ &= |A\,|\,|\alpha|^n e^{j(\omega_0 n+\phi)} \\ &= |A\,|\,|\alpha|^n \cos(\omega_0 n+\phi) + j|A\,|\,|\alpha|^n \sin(\omega_0 n+\phi). \end{aligned}$$

The sequence oscillate with an exponentially growing envelope if $|\alpha| > 1$ or with an exponentially decaying envelope if $|\alpha| < 1$.

When $|\alpha| = 1$, the sequence has the form

$$x(n) = |A|e^{j(w_0 n+\emptyset)} = |A|\cos(w_0 n + \emptyset) + j|A|\sin(w_0 n + \emptyset)$$

That is, the real and imaginary parts of $e^{jw_0 n}$ vary with sinusoidally with n.

Let take frequency $(w_0 + 2\pi)$, in this case

$$x(n) = Ae^{j(w_0+2\pi)n}$$

$$x(n) = Ae^{jw_0n}e^{j2\pi n} = Ae^{jw_0n}$$

Also $x(n) = A\cos[(w_0 + 2\pi r)n + \emptyset] = A\cos(w_0 n + \emptyset)$, r is and integer.

Thus when discussing complex exponential signals of the form $x(n) = Ae^{jw_0n}$ or real sinusoidal signals of the form $x(n) = A\cos(w_0 n + \emptyset)$, we need only consider frequencies in an interval of length 2π, such as $-\pi \leq w_0 \leq \pi$ or $0 \leq w_0 \leq 2\pi$.

In discrete-time case, a periodic sequence is a sequence for which

$$x(n) = x(n + N), \text{for all n}$$

Where the period N is necessarily an integer. If this condition for periodicity is tested for the discrete-time sinsusoid, then

$$A\cos(w_0 n + \emptyset) = A\cos[w_0(n + N) + \emptyset] = A\cos(w_0 n + w_0 N + \emptyset)$$

Which requires that $w_0 N = 2\pi k$, where k is and integer.

A similar statement holds for the complex exponetial sequence $Ce^{-jw_0 n}$, that is, peridicity with period N requires that

$$e^{-jw_0(n+N)} = e^{-jw_0 n},$$

Which is true only for $w_0 N = 2\pi k$, as above. Consequenty, complex exponential and sinusoidal sequences are not necessarily periodic in n with period $({}^{2\pi}/_{w_0})$ and depending on the values of w_0, may not be periodic at all.

Consider the signal $x_1(n) = \cos({}^{\pi n}/_4) = \cos(\omega n)$.
Then $x_1(n+8) = \cos\left({}^{\pi(n+8)}/_4\right) = \cos({}^{\pi n}/_4 + 2\pi) = \cos({}^{\pi n}/_4)$
This signal has a period on N=8.
The time period is calculated as $N = {}^{2\pi}/_\omega$
$x_2(n) = \cos({}^{3\pi n}/_8)$,
$x_1(n+16) = \cos\left({}^{3\pi(n+16)}/_8\right) = \cos({}^{3\pi n}/_8 + 2\pi) = x_2(n)$
Thus, increasing the frequency from $w_0 = {}^{2\pi}/_8$ to $w_0 = {}^{3\pi}/_8$ also increases the period of the signal. This occurs because discrete-time signals are defined only for integer indices n.
The integer restriction on n causes some sinusoidal signals does not to be periodic at all.
This occurs because discrete-time signals are defined only for integer indices n.
The integer resticiton on n casues some sinusoidal signal not to be peridic at all.

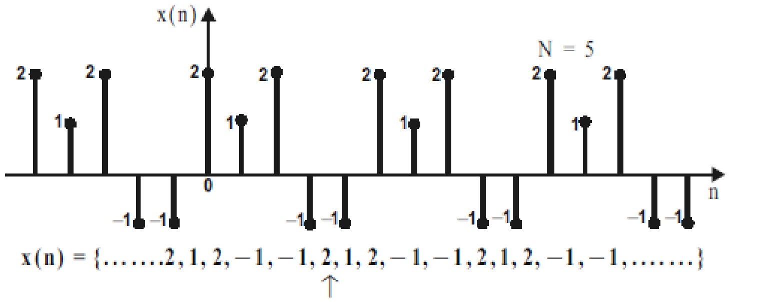
1.3 Periodicity of discrete time signal

- When a discrete time signal $x(n)$, satisfies the condition

$$x(n+N) = x(n)$$

for all integer values of N, then the discrete time signal $x(n)$ is called periodic signal. Here N is the number of samples of a period.

- The smallest value of N for which the above equation is true is called fundamental period.
- If there is no values of N that satisfies the above equation, then x(n) is called aperiodic or nonperiodic signal.
- The periodic signals are power signals.
- When a discrete time signal is a sum or product of two periodic signals with fundamental periods N_1 and N_2, then the discrete time signal will be periodic with period given by LCM of N_1 and N_2.



Periodic discrete time signal.

Examples: Determine whether following signals are periodic or not. If periodic find the fundamental period.

1. $x(n) = \cos(\frac{5}{9}\pi n + 1)$

Solution : Now $x(n+N) = \cos\left(\frac{5}{9}\pi(n+N)+1\right) = \cos(\frac{5}{9}\pi n + 1 + \frac{5}{9}\pi N)$

For periodicity $\frac{5}{9}\pi N$ should be integral multiple of 2π

i.e., $\frac{5}{9}\pi N = k2\pi$, thus $N = k \times 2\pi \frac{9}{5\pi} = \frac{18k}{5}$

Here N is a integer if k=5, 10, 15,… If k=5 then N=18

$$x(n+N) = \cos\left(\frac{5}{9}\pi n + 1 + \frac{5}{9}\pi 18\right) = \cos(\frac{5}{9}\pi n + 1 + 10\pi) = \cos(\frac{5}{9}\pi n + 1) = x(n)$$

Hence x(n) is periodic with fundamental period of 18 samples.

2. $x(n) = \sin(\frac{n}{9} - \pi)$

Solution: $x(n+N) = \sin(\frac{n+N}{9} - \pi) = \sin(\frac{n}{9} - \pi + \frac{N}{9})$

For periodicity $\frac{N}{9}$ should be equal to integral multiple of 2π, i.e., $\frac{N}{9} = 2\pi k$ or $\mathrm{N} = 18\pi k$

Which cannot be integer for any values of k. So, x(n) is aperiodic signal.

3. $x(n) = \sin(\frac{\pi}{8} n^2)$

4. $x(n) = e^{j\frac{7}{4}\pi n}$

5. $x(n) = 2\cos\left(\frac{5}{3}\pi n\right) + 3e^{j\frac{3}{4}\pi n}$

3. $x(n) = \sin(\frac{\pi}{8}n^2)$

Solution: $x(n+N) = \sin[\frac{\pi}{8}(n+N)^2]$

$$= \sin[\frac{\pi}{8}(n^2+N^2+2nN)] = \sin(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{\pi}{4}nN)$$

For periodicity $\frac{\pi}{8}N^2 and\ \frac{\pi}{4}N$ both should be equal to integral multiple of 2π.

i.e., $\frac{\pi}{8}N^2 = 2\pi k_1 \therefore N = 4\sqrt{k_1}$

Here N is integer only for $k_1 = 1^2, 2^2, 3^2, 4^2, \ldots$ where $N = 4, 8, 12, 16 \ldots.$

and $\frac{\pi}{4}N = 2\pi k_2 \therefore N = 8k_2$

Here N is integer only for $k_2 = 1, 2, 3, 4, \ldots$ where $N = 8, 16, 24, 32 \ldots.$

Taking $k_1 = 2^2$ and $k_2 = 1$ we get $N = 8$ as fundamental period.

When $N = 8$

$$x(n+N) = \sin(\frac{\pi}{8}n^2 + \frac{\pi}{8}N^2 + \frac{\pi}{4}nN) = \sin(\frac{\pi}{8}n^2 + \frac{\pi}{8}8^2 + \frac{\pi}{4}n8)$$

$$= \sin(\frac{\pi}{8}n^2 + 4 \times 2\pi + 2\pi n) = \sin(\frac{\pi}{8}n^2)$$

4. $x(n) = e^{j\frac{7}{4}\pi n}$

Solution: $x(n+N) = e^{j\frac{7}{4}\pi(n+N)} = e^{j\frac{7}{4}\pi n} e^{j\frac{7}{4}\pi N}$

Since $e^{j2\pi k} = 1$, for periodicity $\frac{7}{4}\pi N$ should be integral multiple of 2π.

i.e., $\frac{7}{4}\pi N = 2\pi k$ or $N = \frac{8}{7}k$

Here, N is integer, when $k = 7, 14, 21, \ldots$

Thus $x(n) = e^{j\frac{7}{4}\pi n}$ is periodic with fundamental period of 8 samples.

5. $x(n) = 2\cos\left(\frac{5}{3}\pi n\right) + 3e^{j\frac{3}{4}\pi n}$

Solution: Let take $x_1(n) = 2\cos\left(\frac{5}{3}\pi n\right)$ and $x_2(n) = 3e^{j\frac{3}{4}\pi n}$

$$x_1(n+N_1) = 2\cos\left(\frac{5}{3}\pi(n+N_1)\right) = x_1(n) = 2\cos\left(\frac{5}{3}\pi n + \frac{5}{3}\pi N_1\right)$$

This signal will be periodic if $\frac{5}{3}\pi N_1$ is an integral multiple of 2π,

i.e., $\frac{5}{3}\pi N_1 = 2\pi k_1$, or $N_1 = \frac{6}{5}k_1$. If $k_1 = 5, N_1 = 6$ which means the signal $x_1(n)$ is periodic with fundamental period, $N_1 = 6$ samples.

Similarly, $x_2(n+N_2) = 3e^{j\frac{3}{4}\pi(n+N_2)} = 3e^{j\frac{3}{4}\pi n}3e^{j\frac{3}{4}\pi N_2}$

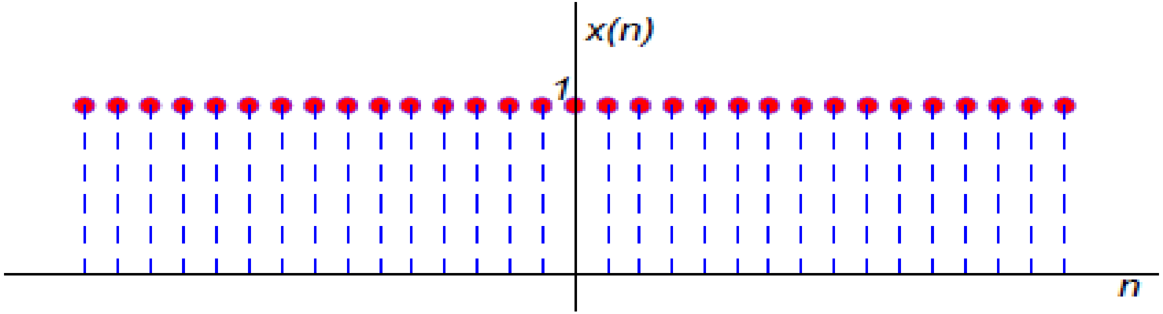
This signal will be periodic if $\frac{3}{4}\pi N_2$ is integral multiple of 2π,

i.e., $\frac{3}{4}\pi N_2 = 2\pi k_2,$ or $N_2 = \frac{8}{3}k_2$. If $k_2 = 3, N_2 = 8$ which means the signal $x_2(n)$ is periodic with fundamental period, $N_1 = 8$ samples.

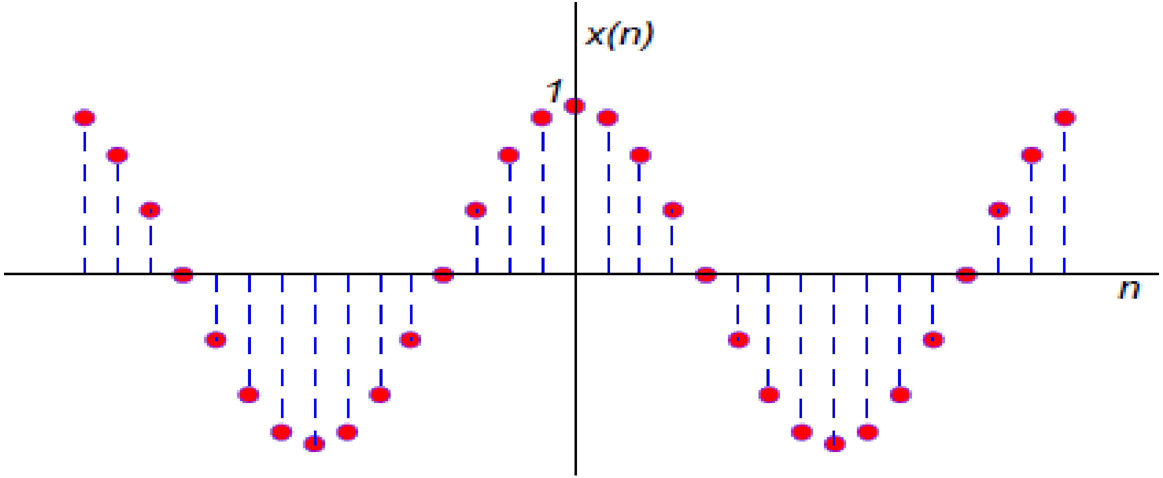
Here, $x(n) = x_1(n) + x_2(n)$, and $x_1(n)$ is periodic with period, $N_1 = 6$ and $x_2(n)$ is periodic with period, $N_2 = 8$.

Therefore, $x(n)$ is periodic with period N, where N is LCM of N_1 and N_2. And the LCM of 6 and 8 is 24, i.e., N=24. And $x(n)$ is periodic with fundamental period, $N = 24$.

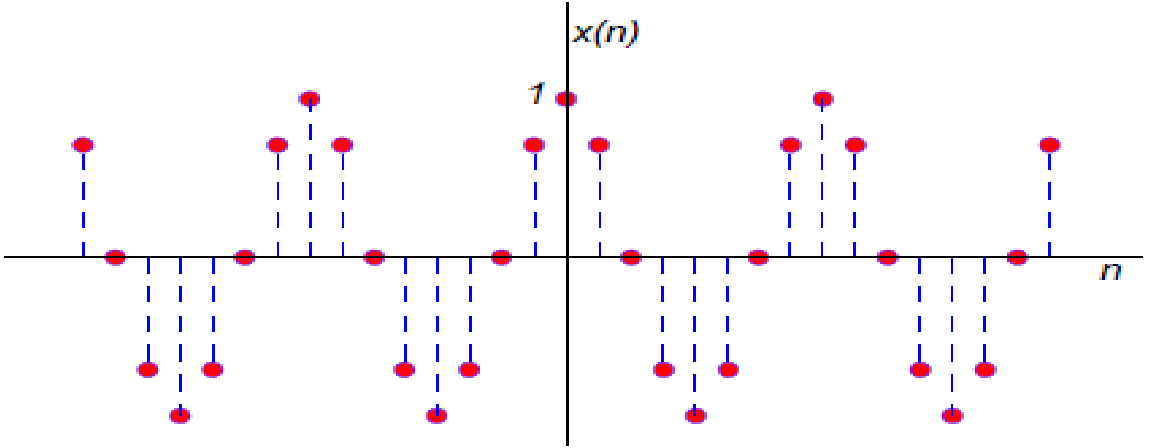
- $x(n) = \cos\omega_0 n,\ \omega_0 = 0 \Longrightarrow N = \infty$



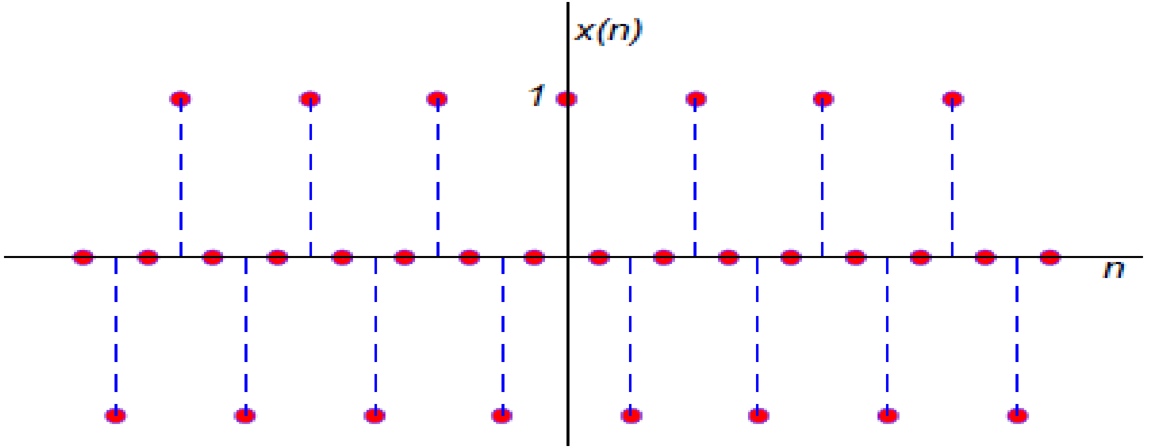
- $x(n) = \cos\omega_0 n,\ \omega_0 = \frac{\pi}{8} \Longrightarrow N = 16$



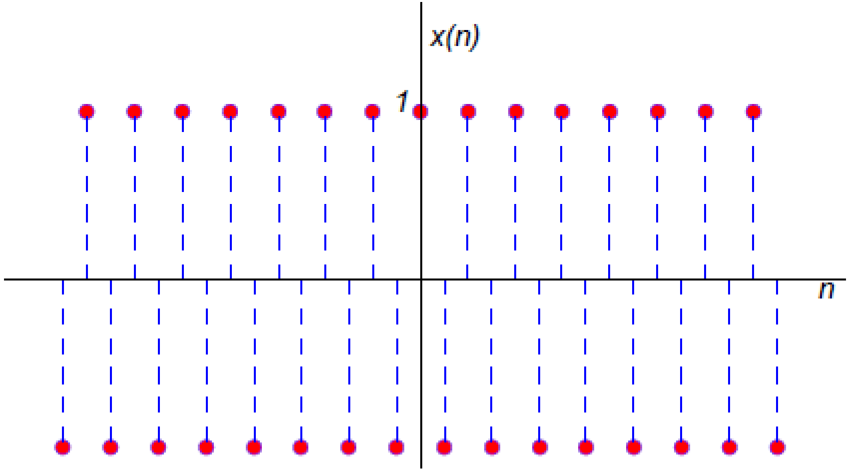
- $x(n) = \cos \omega_0 n, \ \omega_0 = \frac{\pi}{4} \Longrightarrow N = 8$



- $x(n) = \cos \omega_0 n, \ \omega_0 = \frac{\pi}{2} \Longrightarrow N = 4$



- $x(n) = \cos \omega_0 n,\ \omega_0 = \pi \Longrightarrow N = 2$



1.4 Transformation of independent variables (Basic Operations on Sequences)

When we process a sequence, this sequence may undergo several manipulations involving the independent variable or the amplitude of the signal. The basic operatios on sequences are as follows:

1. Time Shifting
2. Time Reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

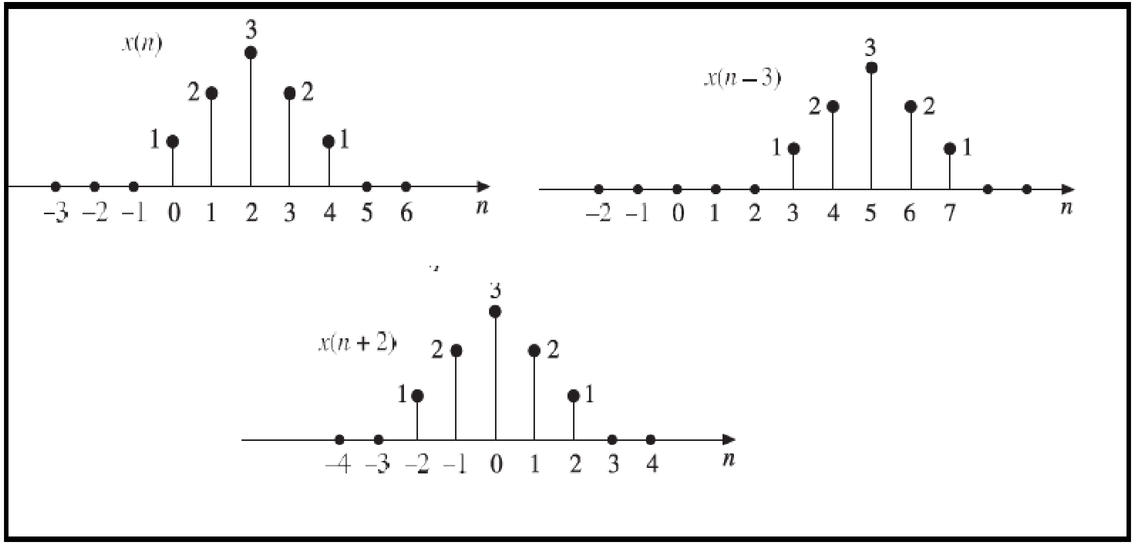
The first three operations correspond to transformation in independent variable n of the signal. The last three operations correspond to transformation on amplitude of a signal.

1. Time Shifting

The time shifting of a signal may result in time delay or time advance. The time shifting operation of a discrete-time signal $x(n)$ can be represented by

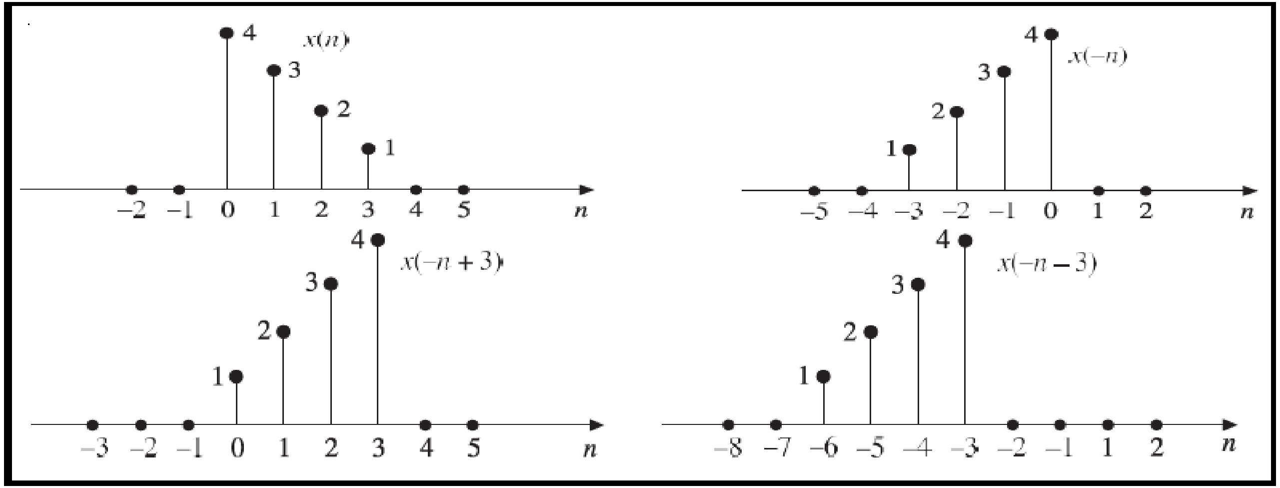
$$y(n) = x(n-k)$$

This shows that the signal $y(n)$ can be obtained by time shifting the signal $x(n)$ by k units. If k is positive, it is delay and the shift is to the right, and if k is negative, it is advance and the shift is to the left.



2. Time Reversal

The time reversal also called time folding of a discrete-time signal x(n) can be obtained by folding the sequence about n = 0. The time reversal signal is the reflection of the original signal. It is obtained by replacing the independent variable n by –n. The signal x(-n+3) is obtained by delaying (shifting to the right) the time reversed signal x(-n) by 3 units of time. The signal x(-n-3) is obtained by advancing (shifting to the left) the time reversed signal x(-n) by 3 units of time.

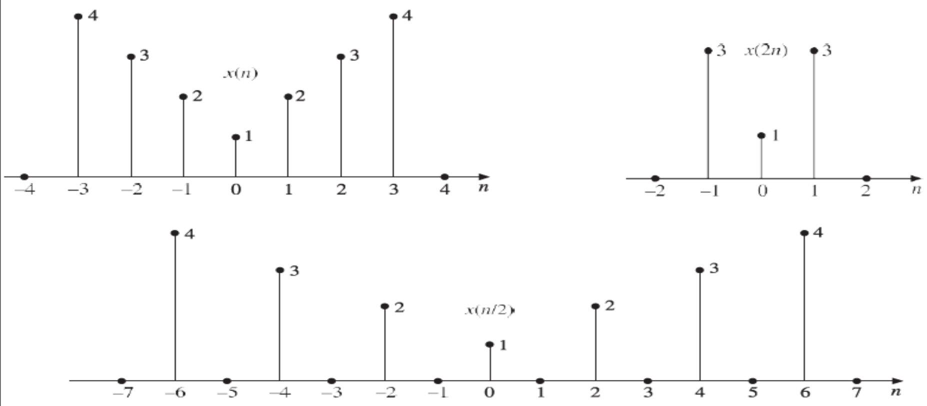


3. Time Scaling

Time scaling may be time expansion or time compression. The time scaling of a discrete-time signal $x(n)$ can be accomplished by replacing n by an in it. Mathematically, it can be expressed as:

$$y(n) = x(an)$$

When $a > 1$, it is time compression and when $a < 1$, it is a time expansion. Time scaling is very useful when data is to be fed at some rate and is to be taken out at a different rate.



Discrete-time scaling (a) Plot of x(n) (b) Plot of x(2n) (c) Plot of x(n/2)

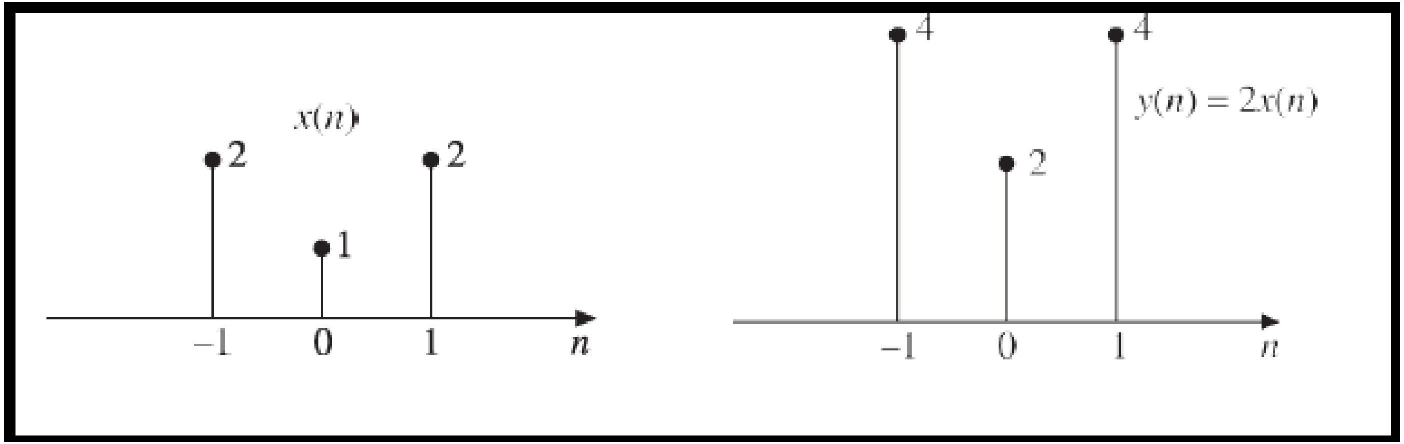
4. Amplitude Scaling

The amplitude scaling of a discrete-time signal can be represented by

$$y(n) = ax(n)$$

Where a is a constant.

The amplitude of $y(n)$ at any instant is equanl to a times the amplitude of $x(n)$ at that instant. If $a > 1$, it is amplification and if $a < 1$, it is attenuation. Hence the amplitude is rescaled. Hence the name amplitude scaling.



5. Signal Addition

In discrete time domain, the sum of two signals $x_1(n)$ and $x_2(n)$ can be obtained by adding the corresponding sample values and the subtraction of $x_2(n)$ from $x_1(n)$ can be obtained by subtracting each sampes of $x_2(n)$ from the corresponding smaples of $x_1(n)$.

If $x_1(n)$ = (1, 2, 3, 1, 5) and $x_2(n)$ = (2, 3, 4, 1, -2)

Then $x_1(n) + x_2(n) =$(1+2, 2+3, 3+4, 1+1, 5-2) = (3, 5, 7, 2, 3)

And $x_1(n) - x_2(n) =$ (1-2, 2-3, 3-4, 1-1, 5+2) = (-1, -1, -1, 0, 7)

6. Signal Multiplication

The multiplication of two discrete time sequences can be performed by multiplying their values at the sampling instants as shown below.

If $x_1(n)$ = (1, -3, 2, 4, 1.5) and $x_2(n) =$ (2, -1, 3, 1.5, 2)

Then $x_1(n) \times x_2(n) = (1 \times 2, -3 \times -1, 2 \times 3,\ 4 \times 1.5, 1.5 \times 2)$ = (2, 3, 6, 6, 3)

In some applications, the product operation is known as modulation. The device implementing the modulation operation is called a modulator.

An application of the product operation is in forming a finite-length sequence from an infinite length sequence by multiplying the latter with a finite length sequence called a window sequence. This process of forming the finite length sequence is usually called windowing, which plays an important role in the design of certain types of digital filters.

1.5 Discrete time Fourier series and properties

- ✓ The Fourier transform and Fourier series are mathematical tools that is useful in analysis and design of Linear time invariant systems.
- ✓ These signal representations basically involve the decomposition of the signals in terms of sinusoidal (or complex exponential) components.
- ✓ With such a decomposition, a signal is said to be represented in the frequency domain.
- ✓ For the class of periodic signals, such a decomposition is called Fourier series.
- ✓ For the class of finite energy signals, the decomposition is called the Fourier transform.
- ✓ The Fourier series is used for the spectrum analysis of periodic signals.
- ✓ The Fourier series is important because it allows one to model periodic signals as a sum of distinct harmonic component.
- ✓ Spectrum analysis of continuous time signal is continuous time Fourier series whereas of discrete time signal is called discrete time Fourier series.

Representation of Periodic Sequences: The Discrete Fourier Series

Condier a periodic sequence $x(n)$ with period N, that is $x(n) \; = \; x(n+N)$ for all n.

The Fourier series representation for $x(n)$ consists of N harmonically related exponential functions

$$e^{j2\pi k^{n}/_{N}}, \qquad k = 0, 1, \dots . N-1$$

and is expressed as

$$x(n) = \sum_{k=0}^{N-1} C_k \, e^{j2\pi k^{n}/_{N}} \qquad (synthesis\ equation)$$

Where C_k are the coefficients in the series represetation is given by

$$C_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \, e^{-j2\pi k^{n}/_{N}}, \qquad k = 0,1, \dots . , N-1 \quad (Analysis\ equation)$$

The synthesis equation is called discrete-time Fourier Series (DTFS). The Fourier coefficents C_k, $k = 0, 1, 2, \dots\ N-1$ provide the description of $x(n)$ in the frequency domain. The term in the summation for $k = K$ and $k = -K$ are called the K^{th} harmonic components, and have fundamental frequency $K(^{2\pi}/_{N})$.

Example: Determine the spectra of the signals

i. $x(n) = \cos\sqrt{2}\pi n$

ii. $x(n) = \cos \pi\, {}^{n}/_{3}$

iii. $x(n)$ is periodic with period $N = 4$ and $x(n)$ ={ <u>1</u>, 1, 0, 0 }

Solution:

i. For $\omega_0 = \sqrt{2}\pi$, we have $N = {}^{2\pi}/_{\omega_0} = \sqrt{2}$, since $x(n)$ is not defined for a non integer number which means the signal is not periodic. Consequently, this signal cannot be expanded in a Fourier series. Nevertheless, the signal does possess a spectrum. Its spectral content consists of the single frequency component at $\omega = \omega_0 = \sqrt{2}\pi$.

ii. In this case $f_0 = \frac{1}{6}$ and hence x(n) is periodic with fundamental period $N = 6$. Thus we have $C_k = \frac{1}{6}\sum_{n=0}^{5} x(n)\, e^{-j2\pi k^{n}/_{6}},\ k = 0,1, \ldots, 5$

However, x(n) can be expressed as $x(n) = \cos\frac{2\pi n}{6} = \frac{1}{2}e^{j2\pi^{n}/_{6}} + \frac{1}{2}e^{-j2\pi^{n}/_{6}}$ which is already in the form of the exponential Fourier series.

Comparing we get $C_1 = \frac{1}{2}$. The second exponential in $x(n)$ correponds to the term $k = -1$. However $e^{-j2\pi n/6} = e^{-j2\pi n(6-5)/6} = e^{j2\pi 5n/6}$

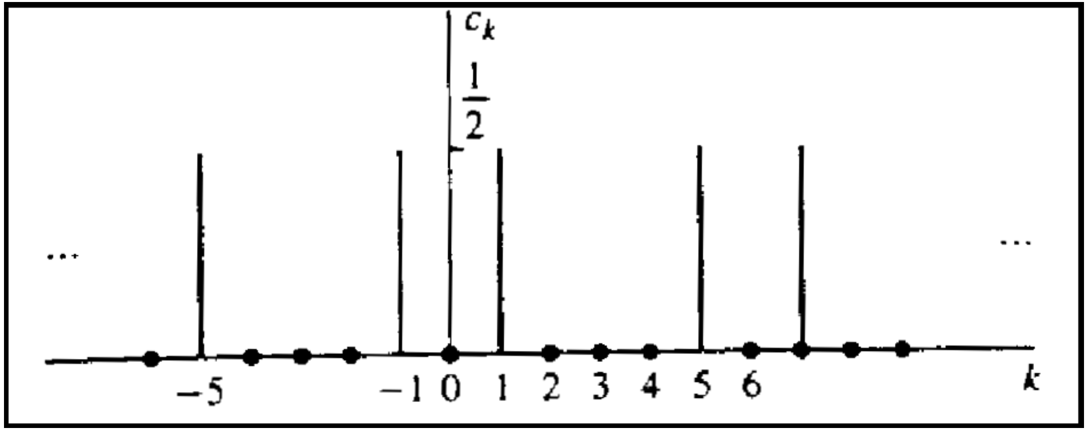
Which means $C_{-1} = C_5$

Therefore, the Fourier series coefficients C_k form a periodic sequence when extended outside of the range $k = 0, 1, 2, \ldots, N-1$, i.e. $C_{k+N} = C_k$.

Thus the spectrum of a signal x(n), which is periodic with period N, is a periodic sequence with period N.

$C_0 = C_2 = C_3 = 0$

$and\ C_1 = \frac{1}{2}, C_5 = \frac{1}{2}$



iii. $x(n)$ ={ $\underline{1}$, 1, 0, 0 }

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi k^{n}/_{N}}, \qquad k = 0,1,\dots,N-1$$

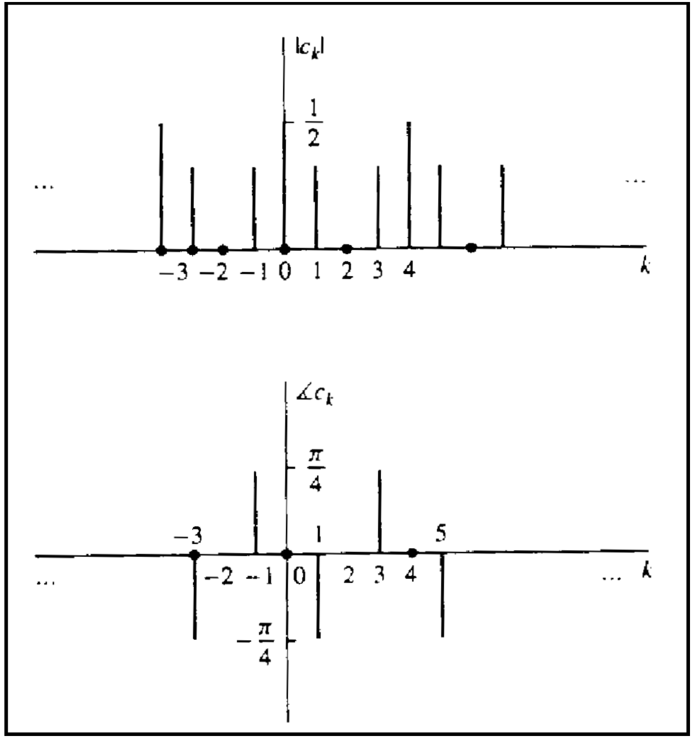
Or $C_k = \frac{1}{4}\left(1 + e^{-j\pi^{k}/_{2}}\right)\; k = 0,1,2,3$

Thus $C_0 = \frac{1}{2}$, $C_1 = \frac{1}{4}(1-j), C_2 = 0$, and $C_4 = \frac{1}{4}(1+j)$

The magnitude and phase spectra are

$|C_0| = \frac{1}{2}, \quad |C_1| = \frac{\sqrt{2}}{4}, \quad |C_2| = 0, \quad |C_3| = \frac{\sqrt{2}}{4}$

$\measuredangle C_0 = 0, \quad \measuredangle C_1 = -\frac{\pi}{4}, \quad \measuredangle C_2 = undefined, \quad \measuredangle C_3 = \frac{\pi}{4}$



Find the Fourier Series Coefficient of $x(n) = 1 + \sin(\frac{\pi n}{8} + \frac{3\pi}{8})$

Solution:

$$x(n) = 1 + \sin(\frac{\pi n}{8} + \frac{3\pi}{8}) = 1 + \frac{e^{\frac{j\pi n}{8}} e^{\frac{j3\pi}{8}}}{2j} - \frac{e^{\frac{-j\pi n}{8}} e^{\frac{-j3\pi}{8}}}{2j} = 1 + \frac{e^{\frac{j2\pi n}{16}} e^{\frac{j3\pi}{8}}}{2j} - \frac{e^{\frac{-j2\pi n}{16}} e^{\frac{-j3\pi}{8}}}{2j}$$

Comparing with DTFS,

$$x(n) = \sum_{k=0}^{N-1} C_k \, e^{j2\pi k {}^{n}/_{N}}$$

We get

$$C_0 = 1$$

$$C_1 = \frac{1}{2j} e^{j\frac{3\pi}{8}}$$

$$C_{-1} = -\frac{1}{2j} e^{-j\frac{3\pi}{8}}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \csc x = \frac{2i}{e^{ix} - e^{-ix}}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \sec x = \frac{2}{e^{ix} + e^{-ix}}$$

$$\tan x = \frac{e^{ix} - e^{-ix}}{i\left(e^{ix} + e^{-ix}\right)} \qquad \cot x = \frac{i\left(e^{ix} + e^{-ix}\right)}{e^{ix} - e^{-ix}}$$

Power Density Spectrum of Periodic Signals

The average power of a discrete-time periodic signal with period N was defined as

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \frac{1}{N}\sum_{n=0}^{N-1}x(n)\,x(n)^* = \frac{1}{N}\sum_{n=0}^{N-1}x(n)\,(\sum_{k=0}^{N-1}C_k^* e^{-j2\pi k\frac{n}{N}})$$

Now, interchanging the order of the two summations

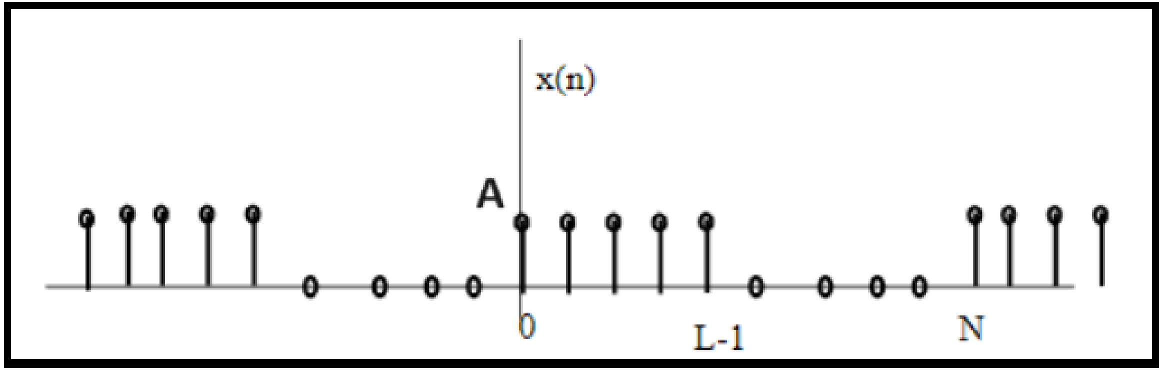
$$= \sum_{n=0}^{N-1}C_k^*\,(\frac{1}{N}\sum_{k=0}^{N-1}x(n)e^{-j2\pi k\frac{n}{N}}) = \sum_{n=0}^{N-1}C_k^*\,C_k = \sum_{n=0}^{N-1}|C_k|^2$$

Thus

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{n=0}^{N-1}|C_k|^2$$

The average power in the signal is the sum of the power of the individual frequency components. The sequence $|C_k|^2$ for $k = 0, 1, \ldots., N-1$ is the distribution of power as a function of frequency and is called the power density spectrum of the periodic signal. This is also called as Parseval's Relation for discrete-time periodic signals.

1. Example: Determine the Fourier series coefficients and the power density spectrum of the periodic signal as shown.



By applying the analysis equation to the signal, we obtained

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi k\frac{n}{N}} = \frac{1}{N}\sum_{n=0}^{L-1} Ae^{-j2\pi k\frac{n}{N}} \quad k = 0,1,\ldots,N-1$$

Which is a geometric summation. Thus

$$C_k = \frac{A}{N}\sum_{n=0}^{L-1}(e^{-j2\pi\frac{k}{N}})^n = \begin{cases} \dfrac{AL}{N}, k = 0 \\ \dfrac{A}{N}\dfrac{1-e^{-j2\pi k\frac{L}{N}}}{1-e^{-j2\pi\frac{k}{N}}}, k = 1,2,\ldots N-1 \end{cases}$$

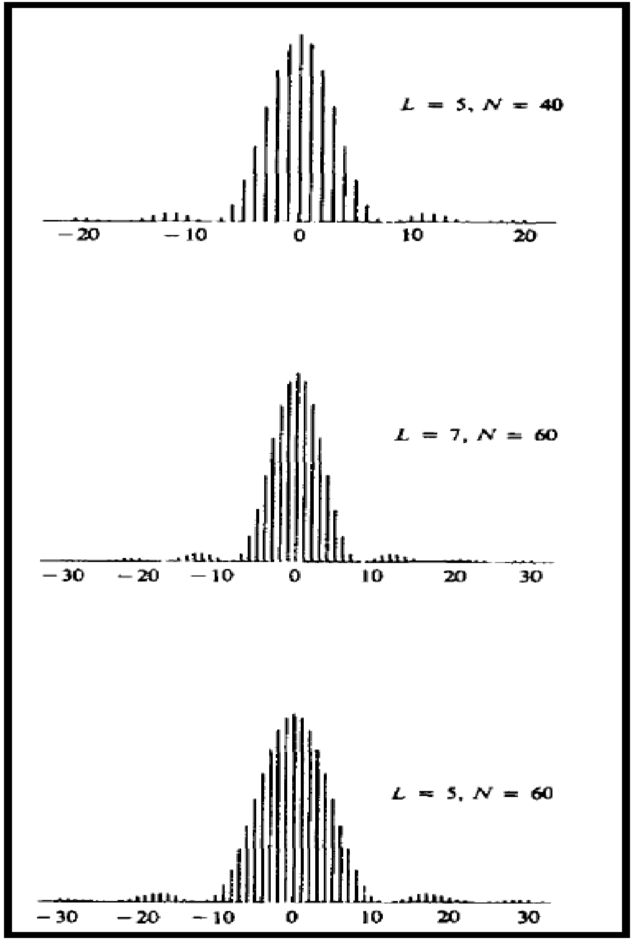
$$\frac{1-e^{-j2\pi k\frac{L}{N}}}{1-e^{-j2\pi\frac{k}{N}}} = \frac{e^{-j\pi k\frac{L}{N}}}{e^{-j\pi\frac{k}{N}}}\;\frac{e^{j\pi k\frac{L}{N}}-e^{-j\pi k\frac{L}{N}}}{e^{j\pi\frac{k}{N}}-e^{-j\pi\frac{k}{N}}} = e^{-j\pi k\frac{(L-1)}{N}}\;\frac{\sin(\pi k\, {}^{L}\!/_{N})}{\sin\pi\, {}^{k}\!/_{N}}$$

Therefore,

$$C_k = \begin{cases} \dfrac{AL}{N}, & k = 0, \pm N, \pm 2N \ldots. \\ \dfrac{A}{N} e^{-j2\pi k \frac{(L-1)}{N}} \dfrac{\sin(\pi k \, {}^{L}/_{N})}{\sin \pi \, {}^{k}/_{N}}, & otherwise \end{cases}$$

The power density spectrum of this periodic signal is

$$|C_k|^2 = \begin{cases} (\dfrac{AL}{N})^2, & k = 0, \pm N, \pm 2N \ldots. \\ \left(\dfrac{A}{N}\right)^2 (\dfrac{\sin\left(\pi k \, {}^{L}/_{N}\right)}{\sin \pi \, {}^{k}/_{N}})^2 & otherwise \end{cases}$$



Properties of The Discrete Fourier Series

If $x(n) \xleftrightarrow{\text{DTFS}} C_k$ and $y(n) \xleftrightarrow{\text{DTFS}} D_k$

Where $x(n)$ and $y(n)$ are periodic with period N and fundamental frequency $\omega_0 = \frac{2\pi}{N}$. Also C_k and D_k are periodic with period N.

Following are the properites of DTFS.

1. Linearity:

$$Ax(n) + By(n) \xleftrightarrow{\text{DTFS}} AC_k + BD_k$$

A linear combination of signals produces the same linear combination of their Fourier series coefficeints.

2. Time-shifting:

$$[x(n - n_0)] \xleftrightarrow{\text{DTFS}} C_k e^{-(jk\frac{2\pi}{N}n_0)}$$

Proof:

$$x(n - n_0) \xleftrightarrow{\text{DTFS}} \frac{1}{N}\sum_{n=0}^{N-1} x(n - n_0)e^{-j2\pi k\frac{n}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n-n_0)e^{-j2\pi k\frac{(n-n_0)}{N}}\ \ e^{-(jk\frac{2\pi}{N}n_0)}$$

Putting $\mathrm{m} = n - n_0$, we get

$$= \frac{1}{N}\sum_{m=n_0}^{N-n_0-1} x(\bar{n})e^{-j2\pi k\frac{(\bar{n})}{N}}\ \ e^{-(jk\frac{2\pi}{N}n_0)}$$

$$= C_k\ e^{-(jk\frac{2\pi}{N}n_0)}$$

The summation on a single full period give the same sum.

The shifting in time equals to a phase shift of Fourier coefficients.

3. Fourier series coefficients C_k form a periodic sequence when extended outside of the range k=0, 1, 2,….,N-1, i.e. $C_{k+N} = C_k$.

PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix} \text{viewed as periodic} \\ \text{with period } mN \end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix} \text{finite valued and periodic only} \\ \text{if } a_0 = 0 \end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$

PROPERTIES OF DISCRETE-TIME FOURIER SERIES

Property	Periodic Signal	Fourier Series Coefficients
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} \quad [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} \quad [x[n] \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Development of Fourier Transform

- Fourier series provide an extremely useful representation for periodic signals.
- Often, however, we need to deal with signals that are not periodic.
- A more general tool than the Fourier series is needed in this case.
- The Fourier transform can be used to represent both periodic and aperiodic signals.
- Since the Fourier transform is essentially derived form Fourier series through a limiting process, the Fourier transform has many similarities with Fourier series.
- By viewing an aperiodic sequence as the limiting case of an N-periodic sequence where $N \to \infty$, we can use the Fourier series to develop a more general signal representation that can be used for both aperiodic and periodic sequences.
- The more general signal representation is called Discrete Time Fourier Transform (DTFT).

1.6 Discrete time Fourier transform and properties

- The discrete-time Fourier transform (DTFT) or, simply the Fourier transform of a discrete-time sequence $x(n)$ is a representation of the sequence in terms of the complex exponential sequence $e^{-j\omega n}$ where ω is the real frequency variable.
- The DTFT representation of a sequence, if it exists, is unique and the original sequenece can be computed form its DTFT by an inverse transform operation.

The discrete-time Fourier transform $X\left(e^{j\omega}\right)$ of a sequence $x(n)$ is defined by

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad (Analysis\ equation\ direct\ transorm)$$

And the inverse discrete-time Fourier transform is defined by

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega n}\, d\omega \ (Synthesis\ equation\ direct\ transform)$$

In fact, due to the 2π periodicity of the DTFT, the integral can be computed over any 2π wide interval on the real line (i.e. between 0 to 2π)

In general, the Fourier transform is a complex-valued function of ω. As with the frequency response, we may either express $X(e^{j\omega})$ in rectangular form as

$$X\left(e^{j\omega}\right) = X_R\left(e^{j\omega}\right) + jX_I\left(e^{j\omega}\right)$$

or in polar form as

$$X\left(e^{j\omega}\right) = \left|X\left(e^{j\omega}\right)\right| e^{j\measuredangle X\left(e^{j\omega}\right)}$$

The quantities $\left|X\left(e^{j\omega}\right)\right|$ and $\measuredangle X\left(e^{j\omega}\right)$ are the magnitude and phase, repectively, of the Fourier transform. The Fourier transform is sometimes referred to as the Fourier spectrum or, simply, the spectrum.

If $x(n)$ is absolutely summable, i.e. $\sum_{n=-\infty}^{\infty}|x(n)| < \infty$, then $X\left(e^{j\omega}\right)$ exists. In this case, the series converge uniformaly to a continuous function of ω.

Some sequences are not absolutely summable, but are square summable, i.e. $\sum_{n=-\infty}^{\infty}|x(n)|^2 < \infty$, such sequences can be represented by a Fourier transform and have mean-square convergence.

Some DTFT pairs

$x[n] = \delta[n-k]$ $\quad$ $X(e^{j\omega}) = e^{-j\omega k}$

$x[n] = 1$ $\quad$ $X(e^{j\omega}) = \tilde{\delta}(\omega)$

$x[n] = u[n]$ $\quad$ $X(e^{j\omega}) = \dfrac{1}{1-e^{-j\omega}} + \dfrac{1}{2}\,\tilde{\delta}(\omega)$

$x[n] = a^n u[n], \quad |a| < 1$ $\quad$ $X(e^{j\omega}) = \dfrac{1}{1-ae^{-j\omega}}$

$x[n] = e^{j\omega_0 n}$ $\quad$ $X(e^{j\omega}) = \tilde{\delta}(\omega-\omega_0)$

$x[n] = \cos(\omega_0 n + \phi)$ $\quad$ $X(e^{j\omega}) = \dfrac{1}{2}\left[e^{j\phi}\,\tilde{\delta}(\omega-\omega_0) + e^{-j\phi}\,\tilde{\delta}(\omega+\omega_0)\right]$

$x[n] = \sin(\omega_0 n + \phi)$ $\quad$ $X(e^{j\omega}) = \dfrac{-j}{2}\left[e^{j\phi}\,\tilde{\delta}(\omega-\omega_0) - e^{-j\phi}\,\tilde{\delta}(\omega+\omega_0)\right]$

$x[n] = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise} \end{cases}$ $\quad$ $X(e^{j\omega}) = \dfrac{\sin((N/2)\omega)}{\sin(\omega/2)}\, e^{-j\frac{N-1}{2}\omega}$

- A conjugate-symmetric sequence $x_e(n)$ is defined as a sequence for which $x_e(n) = x_e^*(n)$.
- A conjugate-antisymmetric sequence $x_o(n)$ is defined as a sequence for which $x_o(n) = -x_o^*(-n)$.
- $X_e(e^{j\omega})$ is conjugate symmetric and $X_o(e^{j\omega})$ is conjugate antisymmetric if $X_e\left(e^{j\omega}\right) = X_e^*(e^{j\omega})$ and $X_o\left(e^{j\omega}\right) = -X_e^*(e^{-j\omega})$.
- If a real function of a continuous variable is conjugate symmetric, it is referred to as an even function, and a real conjugate-antisymmetric function of a continuous variable is referred to as an odd function.
- The symmetry properties of the Fourier transform are summarized as

SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}e\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)
4. $j\mathcal{I}m\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)	$X_R(e^{j\omega}) = \mathcal{R}e\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)	$jX_I(e^{j\omega}) = j\mathcal{I}m\{X(e^{j\omega})\}$
The following properties apply only when $x[n]$ is real:	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$ (magnitude is even)
11. Any real $x[n]$	$\sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$)	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$)	$jX_I(e^{j\omega})$

Properties of DTFT

1. Linearity

If

$$x_1[n] \xleftrightarrow{\mathcal{F}} X_1(e^{j\omega})$$

and

$$x_2[n] \xleftrightarrow{\mathcal{F}} X_2(e^{j\omega}),$$

then it follows by substitution into the definition of the discrete-time Fourier transform that

$$ax_1[n] + bx_2[n] \xleftrightarrow{\mathcal{F}} aX_1(e^{j\omega}) + bX_2(e^{j\omega}).$$

2. Time Shifting and Frequency Shifting

If

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}),$$

then, for the time-shifted sequence, a simple transformation of the index of summation in the discrete-time Fourier transform yields

$$x[n-n_d] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_d} X(e^{j\omega}).$$

Direct substitution proves the following result for the frequency-shifted Fourier transform:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)}).$$

3. Time Reversal

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then if the sequence is time reversed,

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}).$$

If $x[n]$ is real, this theorem becomes

$$x[-n] \stackrel{\mathcal{F}}{\longleftrightarrow} X^*(e^{j\omega}).$$

4. Differentiation in Frequency

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then, by differentiating the discrete-time Fourier transform, it is seen that

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}.$$

5. Parseval's Theorem

If

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}),$$

then

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

The function $\left|X(e^{j\omega})\right|^2$ is called the energy density spectrum, since it determines how the energy is distributed in the frequency domian. Necessarily, the energy density spectrum is defined only for finite-energy signals.

6. The Convolution Theorem

If

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

and

$$h[n] \xleftrightarrow{\mathcal{F}} H(e^{j\omega}),$$

and if

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n],$$

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$

7. The Modulation or Windowing Theorem

If

$$x[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

and

$$w[n] \stackrel{\mathcal{F}}{\longleftrightarrow} W(e^{j\omega}),$$

and if

$$y[n] = x[n]w[n],$$

then

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta.$$

FOURIER TRANSFORM THEOREMS

Sequence $x[n]$ $y[n]$	Fourier Transform $X(e^{j\omega})$ $Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n-n_d]$ (n_d an integer)	$e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\dfrac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$

Parseval's theorem:

8. $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

9. $$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$$

1.7 Discrete time system properties

Discrete-Time Systems

- In many applications of digital signal processing, a device or an algorithm that performs some prescribed operation on a discrete-time signal. Such device or algorithm is called a discrete-time system.
- A discrete-time system is a device or algorithm that operates on a discrete-time signal, called the input or excitation, according to some well-defined rule, to produce another discrete-time signal called the output or response of the system.
- In general, a discrete-time system is defined mathematically as a transformation or operator that maps an input sequence with values $x(n)$ into an output sequence with values $y(n)$. This is denoted as

$$y(n) = T\{x(n)\}$$

where T denotes the transformation (also called an operator), or processing performed by the system on $x(n)$ to produce $y(n)$.

Example: Determine the response of the following systems to the input signal

$$x(n) = \begin{cases} |n|, & -3 \le n \le 3 \\ 0, & otherwise \end{cases}$$

a. $y(n) = x(n-1)$

b. $y(n) = x(n+1)$

c. $y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$

d. $y(n) = \sum_{k=-\infty}^{n} x(k) = x(n) + x(n-1) + x(n-2) + \cdots$

Solution : x(n)= (….0, 3, 2, 1, $\underline{0}$, 1, 2, 3, 0, ….)

a. This system delays the input by one sample, thus y(n)= (..0, 3, 2, 1, 0, $\underline{1}$, 2, 3, 0,.)

b. This system advances the input by one sample, thus y(n)= (..0, 3, 2, $\underline{1}$, 0, 1, 2, 3, 0,.)

The ideal delay system is defined by the equation

$$y(n) = x(n - n_d), -\infty < n < \infty,$$

Where n_d is a fixed positive integer called the delay of the system. The ideal delay system simply shifts the input sequence to the right by n_d samples to form the output. If n_d is fixed negative integer, then the system would shift the input to the left by n_d samples, corresponding to a time advance.

c. The output of this system is any time the mean value of the present, the immediate past, and the immediate future samples.

thus for n=0, $y(n) = \frac{1}{3}[x(-1) + x(0) + x(1)] = \frac{1}{3}[1 + 0 + 1] = \frac{2}{3}$

$y(n)$ =[…,0, 1, $\frac{5}{3}$, 2, 1, $\underset{\uparrow}{\frac{2}{3}}$, 1, 2, $\frac{5}{3}$, 1, 0, …]

The general moving average system is defined by the equation

$$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} x(n-k)$$

$$= \frac{1}{M_1 + M_2 + 1} [x(\mathrm{n} + M_1) + x(\mathrm{n} + M_1 - 1) + \cdots + x(n) + x(n-1) + \cdots + x(\mathrm{n} - M_2)]$$

$x(n)= (\ldots.0, 3, 2, 1, \underline{0}, 1, 2, 3, 0, \ldots.)$

d. This system is basically an accumulator that computes the running sum of all the past input values up to present time. Thus

$y(n)=(\ldots 0, 3, 5, 6, \underline{6}, 7, 9, 12, \ldots.)$

The system defined by the input-output equation

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

is called the accumulator system, since the output at time n is just the sum of the present and all previous input samples. Also

$$y(n) = \sum_{k=-\infty}^{n} x(k) = y(n) = \sum_{k=-\infty}^{n-1} x(k) + x(n) = y(n-1) + x(n)$$

Indeed, the system computes the current value of the output by adding (accumulating) the current value of the input to the previous output value.

Memoryless Systems

A system is referred to as memoryless or static if the output $y(n)$ at every value of n depends only on the input $x(n)$ at the same value of n.

Example $y(n) = [x(n)]^2$, for each vlaue of n.

The ideal delay system is not memoryless or dynamic unless $n_d = 0$; in particular, this system is referred to as having “memory” whether n_d is positive (a time delay) or negative (a time advance). The moving average system is not memoryless unless $M_1 = M_2 = 0$.

Linear Sytems

The class of linear systems is defined by the principle of superposition. If $y_1(n)$ and $y_2(n)$ are the response of a system when $x_1(n)$ and $x_2(n)$ are the respective inputs, then the system is linear if and only if

$$T[x_1(n) + x_2(n)] = T[x_1(n)] + \mathrm{T}[x_2(n)] = y_1(n) + y_2(n)$$

and

$$T[ax(n)] = aT[x(n)] = ay(n),$$

Where a is an arbitrary constant. The first property is called the additivity property, and the second is called the homogeneity or scaling property.

Linear Sytems

These additivity and homogeneity or scaling properties can be combined into the principle of superposition, stated as

$$T[ax_1(n) + bx_2(n)] = aT[x_1(n)] + b\mathrm{T}[x_2(n)] = ay_1(n) + by_2(n)$$

For arbitraty constant a and b.

This equation can be generalized to the superposition of many inputs. Specifically, if

$$x(n) = \sum_k a_k x_k(n)\,,$$

Then the output of a linear system will be

$$y(n) = \sum_k a_k y_k(n)$$

Where $y_k(\mathrm{n})$ is the system response to the input $x_k(n)$.

The Ideal delay and Moving Average system are example of linear system.

The accumulator system is also a linear system. In order to prove this, we must show it satisfies the superposition principle for all inputs, not just any specific set of inputs.
Let two inputs $x_1(n)$ and $x_2(n)$ and their corresponding outputs.

$$y_1(n) = \sum_{k=-\infty}^{\infty} x_1(n)\,,$$

$$y_2(n) = \sum_{k=-\infty}^{\infty} x_2(n)\,.$$

When the input is $x_3(n) = ax_1(n) + bx_2(n)$, the superposition principle requires the output $y_3(n) = ay_1(n) + by_2(n)$ for all possible choices of a and b.

$$y_3(n) = \sum_{k=-\infty}^{\infty} x_3(n) = \sum_{k=-\infty}^{\infty} [ax_1(n) + bx_2(n)] = a \sum_{k=-\infty}^{\infty} x_1(n) + b \sum_{k=-\infty}^{\infty} x_2(n)$$
$$= ay_1(n) + by_2(n)$$

Thus the accumulator system satisfies the superposition principle for all inputs and is therefore linear.

Determine if the system described by the following input-output equation are linear or nonlinear.

a. $y(n) = x(n^2)$

b. $y(n) = x^2(n)$

c. $y(n) = Ax(n) + B$

Solution:

a. For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are $y_1(n) = x_1(n^2)$ and $y_2(n) = x_2(n^2)$

The output of the system to linear combination of two input sequences is

$$y_3(n) = T[ax_1(n) + bx_2(n)] = \mathrm{a}x_1(n^2) + \mathrm{b}x_2(n^2) = ay_1(n) + by_2(n)$$

Thus the system is linear.

b. $y(n) = x^2(n)$

For two input sequences $x_1(n)$ and $x_2(n)$, the corresponding outputs are $y_1(n) = x_1^2(n)$ and $y_2(n) = x_2^2(n)$

The output of the system to linear combination of two input sequences is

$$y_3(n) = T[ax_1(n) + bx_2(n)] = [ax_1(n) + bx_2(n)]^2$$
$$= a^2x_1^2(n) + 2abx_1(n)x_2(n) + b^2x_2^2(n)$$

On the other hand, if the system is linear,it would produce a linear combination of two outputs i.e. $ay_1(n) + by_2(n) = ax_1^2(n) + bx_2^2(n)$

Since the actual output of the system is different so the system is nonlinear.

The output of the system is the square of the input. Electronic devices that have such as input-output characteristic and are called square-law devices. Such system are memoryless and nonlinear.

c. $y(n) = Ax(n) + B$

Assuming that the system is excited by $x_1(n)$ and $x_2(n)$, the corresponding outputs are

$y_1(n) = Ax_1(n) + B$ and $y_2(n) = Ax_2(n) + B$

A linear combination of $x_1(n)$ and $x_2(n)$ produces the output

$$y_3(n) = T[ax_1(n) + bx_2(n)]$$
$$= A[ax_1(n) + bx_2(n)] + \mathrm{B}$$
$$= Aax_1(n) + Abx_2(n) + \mathrm{B}$$

On the other hand, if the system were linear, its output to the linear combination of $x_1(n)$ and $x_2(n)$ would be linear combination of $y_1(n)$ and $y_2(n)$, that is $ay_1(n) + by_2(n) = aAx_1(n) + \mathrm{aB} + \mathrm{b}Ax_2(n) + \mathrm{bB}$

Clearly the system fails to satisfy the linearity test. However if $B = 0$ the system will satisfy the linearity test and the system is now relaxed.

Time-Invariant Systems

- A time-invariant system (shift-invariant system) is a system for which a time shift or delay of the sequence causes a corresponding shift in the output sequence.
- Suppose that a system transforms the input sequence with values $x(n)$ into the output sequence with values $y(n)$. Then the system is said to be time invariant if, for all n_0, the input sequence with value $x_1(n) = x(n - n_0)$ produces the output sequence with values $y_1(n) = y(n - n_0)$.
- This time invariant property ensures that for a specified input, the output of the system is independent of the time the input is being applied.
- The Ideal Delay system, Moving average, accumulator system are time invariant system.

Consider the accumulator $y(n) = \sum_{k=-\infty}^{n} x(k)$
We define $x_1(n) = x(n - n_0)$. To show time invariance, we solve for both $y(n - n_0)$ and $y_1(n)$ and compare them to see whether they are equal.

$$y(n - n_0) = \sum_{k=-\infty}^{n-n_0} x(k)$$

Next, we find

$$y_1(n) = \sum_{k=-\infty}^{n} x_1(k) = \sum_{k=-\infty}^{n} x(k - n_0)$$

Substituting the change of variables $k_1 = k - n_0$ into the summation gives

$$y_1(n) = \sum_{k_1=-\infty}^{n-n_0} x(k_1) = y(n - n_0)$$

Thus, the accumulator is a time-invariant system.

Examples

a. $y(n) = x(n) - x(n-1)$

If the input is delayed by n_0 units in time i.e., $x_1(n) = x(n - n_0)$. and applied to the system, the output will be

$$y_1(n) = x(n - n_0) - x(n - n_0 - 1)$$

On other hand, if we delay $y(n)$ by n_0 units in time we obtain

$$y(n - n_0) = x(n - n_0) - x(n - n_0 - 1)$$

Thus both are identical, therefore the system is time invariant.

b. $y(n) = nx(n)$

If the input is delayed by n_0 units in time i.e., $x_1(n) = x(n - n_0)$. and applied to the system, the output will be

$$y_1(n) = \mathrm{n}x(n - n_0)$$

On other hand, if we delay $y(n)$ by n_0 units in time we obtain

$$y(n - n_0) = (n - n_0)x(n - n_0)$$

Thus this system is time variant.

c. $y(n) = x(-n)$

If the input is delayed by n_0 units in time i.e., $x_1(n) = x(n - n_0)$. and applied to the system, the output will be

$$y_1(n) = x(-n - n_0)$$

On other hand, if we delay $y(n)$ by n_0 units in time we obtain

$$y(n - n_0) = x\big(-(n - n_0)\big) = x(-n + n_0)$$

Thus this system is time variant.

d. $y(n) = x(Mn)$

If the input is delayed by n_0 units in time i.e., $x_1(n) = x(n - n_0)$. and applied to the system, the output will be

$$y_1(n) = x(Mn - n_0)$$

On other hand, if we delay $y(n)$ by n_0 units in time we obtain

$$y(n - n_0) = x\big(M(n - n_0)\big)$$

Thus this system is time variant.

Causality

- A system is said to be causal if the output of the system at any time n [i.e., $y(n)$] depends only on the present and past inputs [i.e., $x(n),\ x(n-1),\ x(n-2), \ldots$], but does not depend on future inputs [i.e., $x(n+1),\ x(n+2), \ldots$].
- In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \ldots]$$

Where $F[.]$ is some arbitraty function.

- If a system does not satisfy this defination, it is called noncausal. Such a system has an output that depends not only on present and past inputs but also on future inputs.
- It is apparent that in real-time signal processing applications we cannot observe future values of the signals, and hence a noncausal system is physical unrealizable (i.e., it cannot be implemented).

- On the other hand, if the signal is recorded so that the processing is done off-line (nonreal time), it is possible to implement a noncausal system, since all values of the signal are available at the time of processing.
- This is often the case in the case in the processing of geophysical signals and images.

Determine if the systems described by the following input-output are causal or noncausal

a. $y(n) = x(n) - x(n-1)$
b. $y(n) = \sum_{k=-\infty}^{n} x(k)$
c. $y(n) = ax(n)$
d. $y(n) = x(n) + 3x(n+4)$
e. $y(n) = x(n^2)$
f. $y(n) = x(2n)$
g. $y(n) = x(-n)$

Answer : Causal: a, b, c and noncausal: d, e, f, g.

For g if $n = -1$ then $y(-1) = x(1)$, thus the output depends on future inputs.

Stability

- Stability is an important property that must be considered in any practical application of a system
- Unstable systems usually exhibit erratic and extreme behavior and cause overflow in any practical implementation.
- A system is stable in the bounded-input, bounded-output (BIBO) sense if and only if every bounded input sequence produces a bounded output sequence.
- The input $x(n)$ is bounded if there exists a fixed positive finite value B_x such that $|x(n)| \leq B_x < \infty$, for all n.
- Stability requires that, for every bounded input, there exist a fixed positive finite value B_y such that $|y(n)| \leq B_y < \infty$, for all n.

Stability

- It is important that the properties are properties of system, not of the inputs to a system. That is, we may be able to find inputs for which the properties hold, but the existence of the property for some inputs does not mean that the system has the property.
- For the system to have the property, it must hold for all inputs.
- For example, an unstable system may have some bounded inputs for which the output is boudned, but for the system to have the property of stability, if must be true that for all bounded inputs, the output is bounded.
- If we can find just one input for which the system property does not hold, then we have shown that the system does not have that property.

Examples

1. The memoryless system $y(n) = [x(n)]^2$, for each values of n. The system is stable. To see this, assume that the input $x(n)$ is bounded such that $|x(n)| \leq B_x$ for all n. Then $|y(n)| = |x(n)|^2 \leq B_x^2$. Thus, we can choose $B_y = B_x^2$ and prove that $y(n)$ is bounded.
2. The accumulator is not stable. For example, consider the case when $x(n) = u(n)$, which is clearly bounded by $B_x = 1$. For this input, the output of the accumulator is

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \begin{cases} 0, & n < 0 \\ (n+1), & n \geq 0 \end{cases}$$

There is no finite choice for B_y such that $(n+1) \leq B_y < \infty$ for all n; thus the system is unstable.

3. $y(n) = \cos[x(n)]$ For every bounded value of x(n), it cosine value is always bounded (finite). So, the given system is stable.

1.8 Linear Time-Invariant systems, Convolution sum, Properties of Linear Time-Invariant system (LTI)

- We classified systems in accordance with a number of characteristics properties: Linearity, causality, stability, and time invariance.
- A particularly important class of systems consists of those that are linear and time invariant.
- This class of systems has significant signal-processing applications.
- If the linearity property is combined with the representation of a general sequence as a linear combination of delayed impulses (see slide 22), it follows that a linear system can be completely characterized by its impulse response.
- Let $h_k(n)$ be the response of the system to $\delta(n-k)$, an impulse occuring at $n=k$, then

$$y(n) = T\{\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\}$$

LTI…

From the principle of superposition, we can write

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)T\{\delta(n-k)\} = \sum_{k=-\infty}^{\infty} x(k)h_k(n)$$

The system response to any input can be expressed in terms of the responses of the system to the sequences $\delta(n-k)$. In deriving this equation we used the linearity property of the system but not its time invariance property. Thus this equation applies to any relaxed linear (time-variant) system.

If, in addition, the system is time invariant, the formula simplifies considerably.

In fact, if the response of the LTI system to the unit sample sequences $\delta(n)$ is denoted as $h(n)$, that is

$$h(\mathrm{n}) = \mathrm{T}[\delta(n)]$$

Then by the time-invariance property, the response if the system to the delayed unit sample sequence $\delta(n-k)$ is

$$h(n-k) = T[\delta(n-k)]$$

LTI

With this the equation becomes

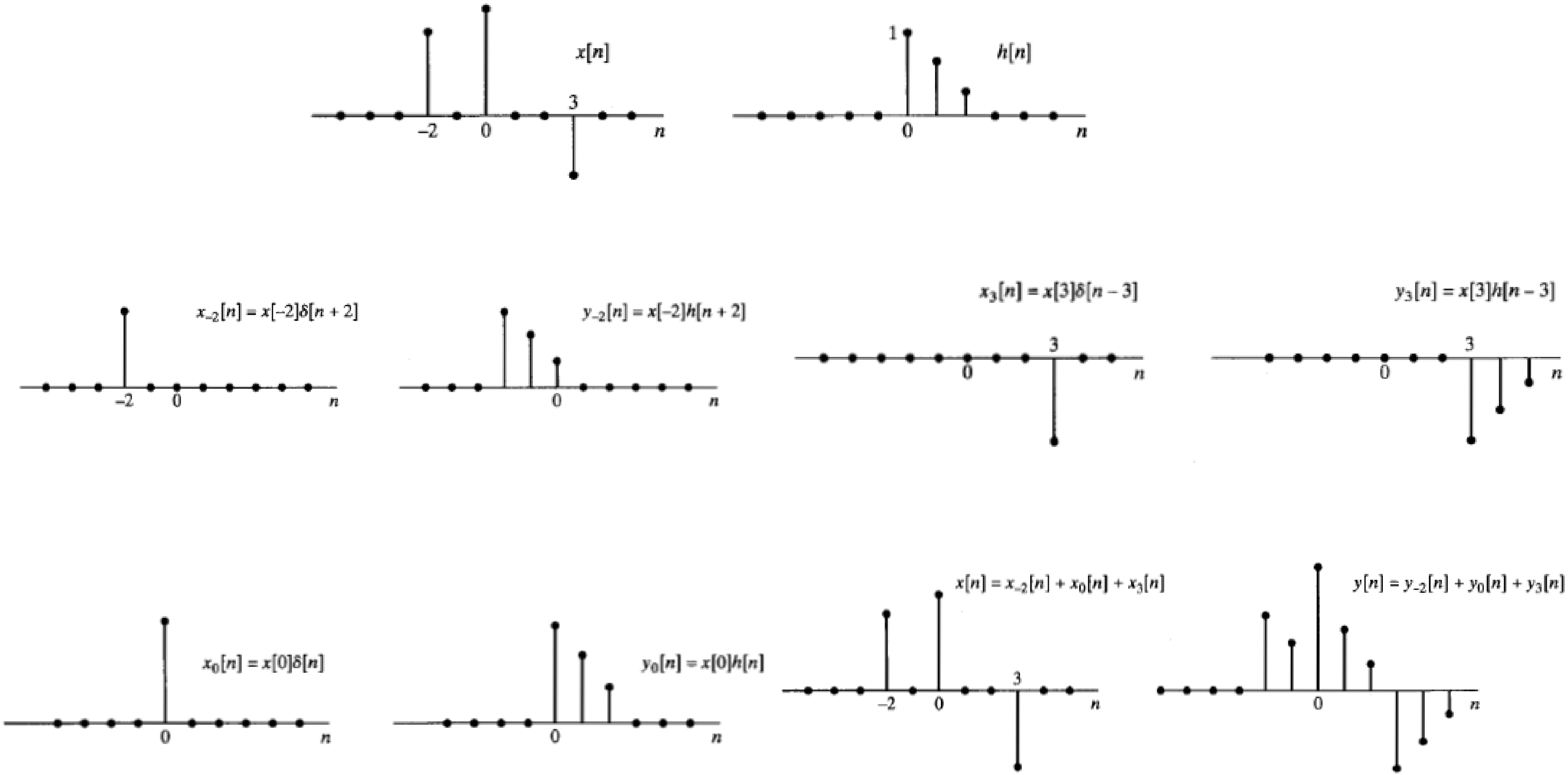
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h\,(n-k)$$

This equation is commonly called the convolution sum. We observe that the relaxed LTI system is completely characterized by a single function $h(n)$, namely, its response to the unit sample sequence $\delta(n)$.

If $y(n)$ is a sequence whose values are related to the values of two sequences $h(n)$ and $x(n)$ as in above equation we say that $y(n)$ is the convolution of $x(n)$ with $h(n)$ and represent this by the notaion

$$y(n) = x(n) * h(n).$$

The derivation suggests the interpretation that the input sample at n=k represented as $\mathrm{x}(k)\delta(n-k)$, for $-\infty < n < \infty$, and that, for each k, these sequences are superimposed to form the overall output sequence. The interpretation emphasizes that the convolution sum is a direct result of linearity and time invariance.


$x[n]$
-2
0
3
n
1
$h[n]$
$x_{-2}[n] = x[-2]\delta[n+2]$
$y_{-2}[n] = x[-2]h[n+2]$
$x_3[n] = x[3]\delta[n-3]$
$y_3[n] = x[3]h[n-3]$
$x_0[n] = x[0]\delta[n]$
$y_0[n] = x[0]h[n]$
$x[n] = x_{-2}[n] + x_0[n] + x_3[n]$
$y[n] = y_{-2}[n] + y_0[n] + y_3[n]$

Methods to calculate Convolution Sum

The different methods used for the computation of linear convolution are

a. Graphical method
b. Tabulation method
c. Multiplication method
d. Using mathematical equation

a. Graphical method: The procedure are:

i. Plotting: Plot $x(n)$ and $h(n)$ as $x(k)$ and $h(k)$.
ii. Folding : Fold $h(k)$ about $k = 0$ to obtain $h(-k)$.
iii. Shifting: Shift $h(-k)$ by n to right (left) if n is positive (negative), to obtain $h(n-k)$.
iv. Multiplication: Multiply x(k) by h(n-k) to obtain the product sequence $y_n(k) = x(k)h(n-k)$.
v. Summation: Sum all the values of the product sequence $y_n(k)$ to obtain the value of the output at time.
vi. Repetition: Repeat steps (iii) to (v) for all possible time shifts $-\infty < n < \infty$ to obtain overall response.

Example: Determing the response of the LTI system whose input $x(n)$ and impulse response $h(n)$ are given by

$x(n) = \{\underline{1}, 2, 0.5, 1\}$ and $h(n) = \{1, \underline{2}, 1, -1\}$

Solution: The response $y(n)$ of the system is given by convolution of $x(n)$ and $h(n)$.

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)h\,(n-m)$$

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

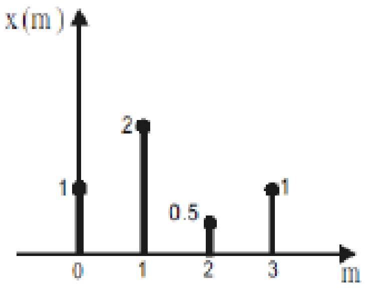


Fig 1 : Input sequence.

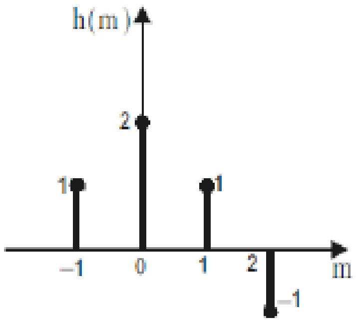


Fig 2 : Impulse response.

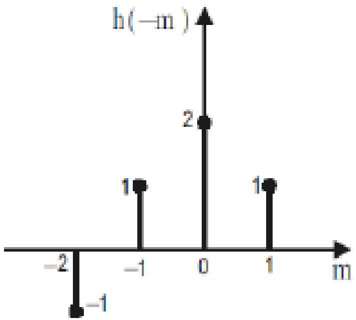


Fig 3 : Folded impulse response.

Lowest range of $y_l = x_l + h_l = 0 + (-1) = -1$, highest range of $y_h = x_h + h_h = 3 + 2 = 5$

When $n = -1$; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

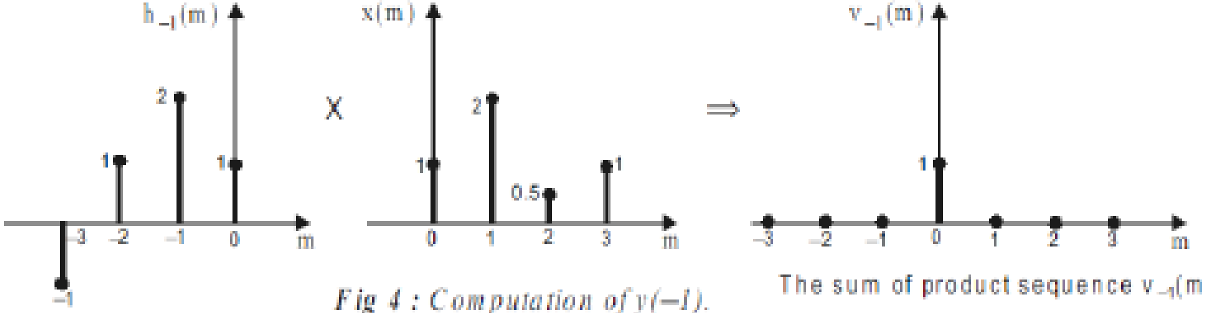


Fig 4 : Computation of y(−1).

The sum of product sequence $v_{-1}(m)$ gives $y(-1)$. $\therefore\ y(-1) = 1$

When $n = 0$; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

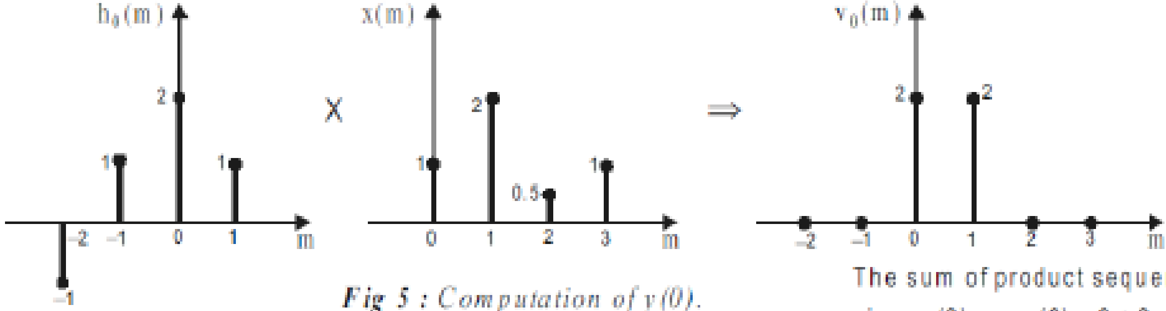
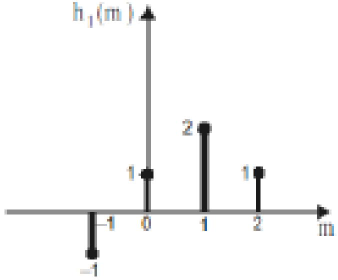


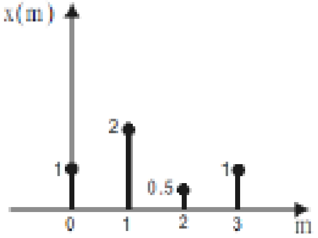
Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore\ y(0) = 2 + 2 = 4$

When n = 1 ; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$



X



$\Rightarrow$

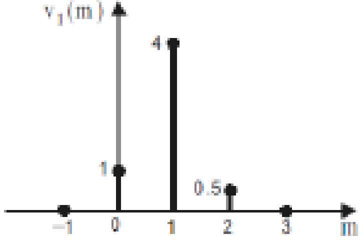
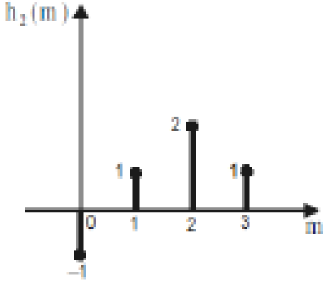


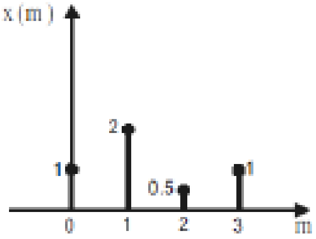
Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives y(1). $\therefore$ y(1) = 1 + 4 + 0.5 = 5.5

When n = 2 ; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$



X



$\Rightarrow$

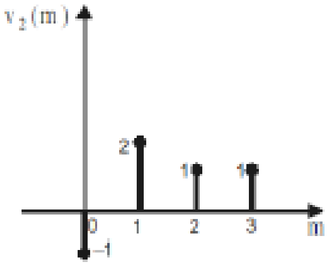


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives y(2). $\therefore$ **y(2)=-1+2+1+1=3**

When n = 3 ; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$

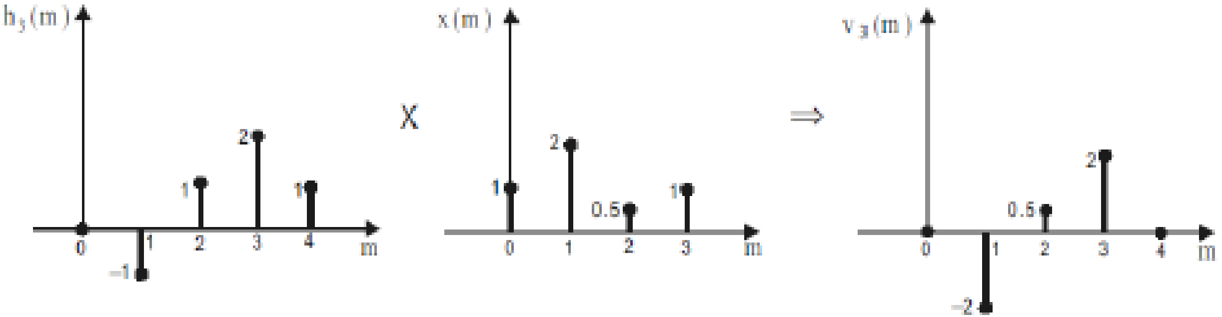


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives $y(3)$. $\therefore\ y(3) = -2 + 0.5 + 2 = 0.5$

When n = 4 ; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

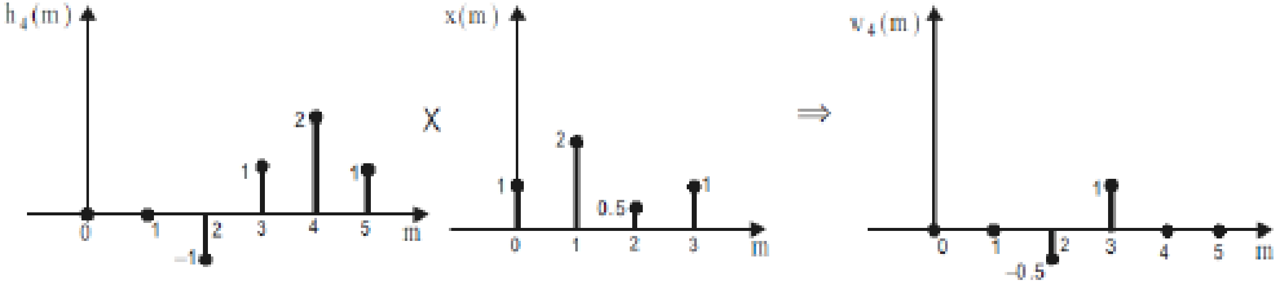


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives $y(4)$. $\therefore\ y(4) = -0.5 + 1 = 0.5$

When n = 5 ; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$

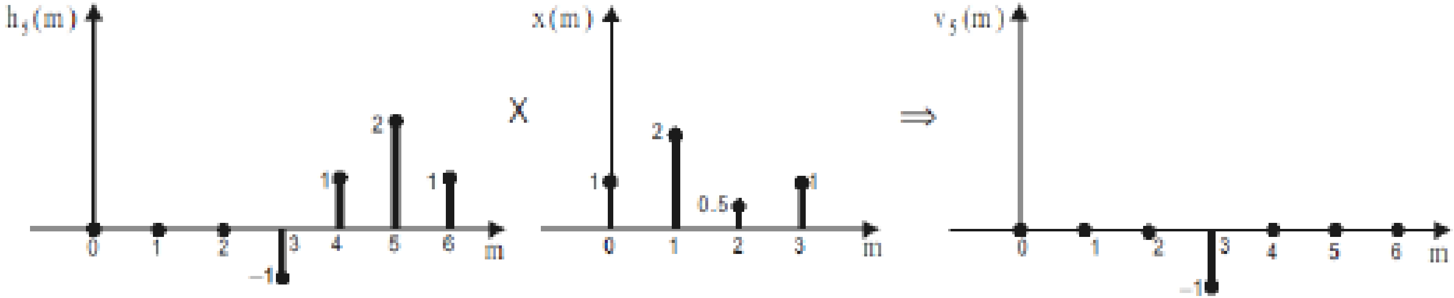


Fig 10 : Computation of y(5).

The sum of product sequence $v_5(m)$ gives y(5). $\therefore$ y(5) = –1

The output sequence, $y(n) = \{1,\ \underset{\uparrow}{4},\ 5.5,\ 3,\ 0.5,\ 0.5,\ -1\}$

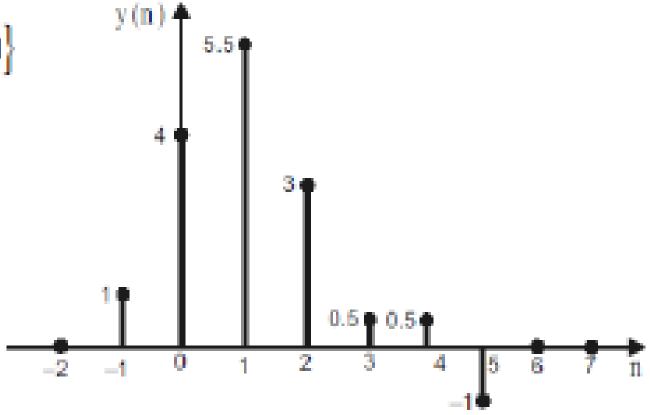


Fig 11 : Graphical representation of y(n).

b. Tabular Method

The tabular method is same as that of graphical method, except the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every sequence, folded and shifted sequenece is represented by a row in a table.

$x(n) = \{\underline{1}, 2, 0.5, 1\}$

and $h(n) = \{1, \underline{2}, 1, -1\}$

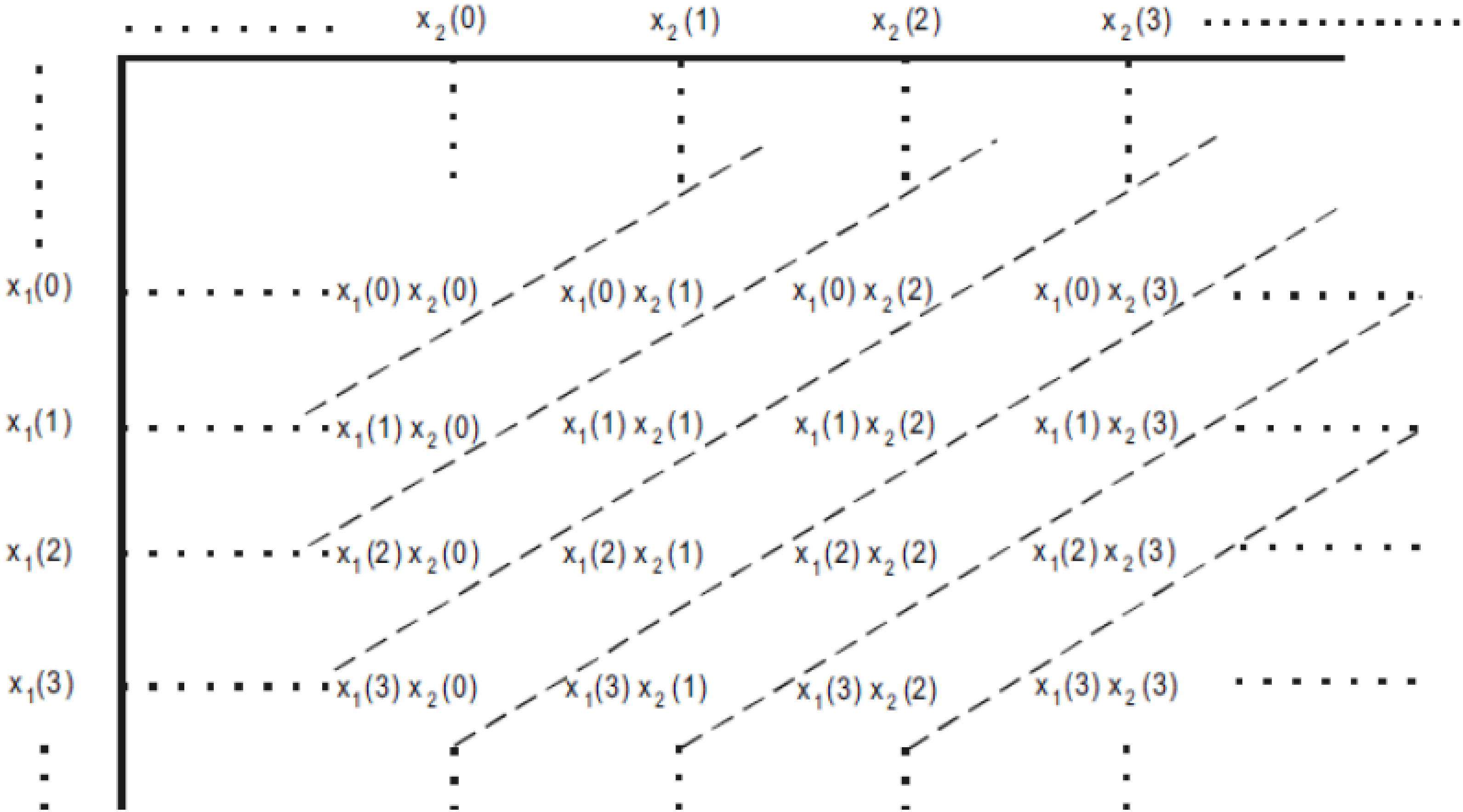
Each sample of y(n) is Calculated using Convolution formula.

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	−3	−2	−1	0	1	2	3	4	5	6
x(m)				1	2	0.5	1			
h(m)			1	2	1	−1				
h(−m)		−1	1	2	1					
$h(-1-m) = h_{-1}(m)$	−1	1	2	1						
$h(0-m) = h_0(m)$		−1	1	2	1					
$h(1-m) = h_1(m)$			−1	1	2	1				
$h(2-m) = h_2(m)$				−1	1	2	1			
$h(3-m) = h_3(m)$					−1	1	2	1		
$h(4-m) = h_4(m)$						−1	1	2	1	
$h(5-m) = h_5(m)$							−1	1	2	1

c. Matrix Method



$x(n) = \{\underline{1}, 2, 0.5, 1\}$ and $h(n) = \{1, \underline{2}, 1, -1\}$

x(n) ↓ \ h(n) →	1	2	1	−1
1	1×1	1×2	1×1	$1 \times (-1)$
2	2×1	2×2	2×1	$2 \times (-1)$
0.5	0.5×1	0.5×2	0.5×1	$0.5 \times (-1)$
1	1×1	1×2	1×1	$1 \times (-1)$

$\Rightarrow$

x(n) ↓ \ h(n) →	1	2	1	−1
1	1	2	1	−1
2	2	4	2	−2
0.5	0.5	1	0.5	−0.5
1	1	2	1	−1

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 0.5 + 4 + 1 = 5.5$

$y(2) = 1 + 1 + 2 + (-1) = 3$

$y(3) = 2 + 0.5 + (-2) = 0.5$

$y(4) = 1 + (-0.5) = 0.5$

$y(5) = -1$

\ $y(n) = \{1, \underline{4}, 5.5, 3, 0.5, 0.5, -1\}$

d. Mathematical equation

Compute convolution of x(n)= {$\underline{1}$, 1, 1, 1} and h(n)={$\underline{1}$, 1, 1, 1}

Lowest range of $y_l = x_l + h_l = 0 + 0 = 0$, highest range of $y_h = x_h + h_h = 3 + 3 = 6$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h\ (n-k)$$

Calculation for $y(0)$, putting $n = 0$ in this equation we get

$$y(0) = \sum_{k=0}^{3} x(k)h\ (0-k)$$

$$y(0) = \ x(0)h(-0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3)$$

$$= 1 \times 1 + 1 \times 0 + 1 \times 0 + 1 \times 0$$

$$y(0) = 1$$

Calculation for $y(1)$, putting $n = 1$, we get

$$y(1) = \sum_{k=0}^{3} x(k)h\,(1-k)$$

$$= x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2)$$

$$= 1 \times 1 + 1 \times 1 + 1 \times 0 + 1 \times 0 = 2$$

Calculation for $y(2)$, putting $n = 2$, we get

$$y(2) = \sum_{k=0}^{3} x(k)h\,(2-k)$$

$$= x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(-1)$$

$$= 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 0 = 3$$

Calculation for $y(3)$, putting $n = 3$, we get

$$y(3) = \sum_{k=0}^{3} x(k)h\,(3-k)$$

$$= x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$$

$$= 1 \times 1 + 1 \times 1 + 1 \times 1 + 1 \times 1 = 4$$

Calculation for $y(4)$, putting $n = 4$, we get

$$y(4) = \sum_{k=0}^{3} x(k)h\,(4-k)$$

$$= x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

$$= 1 \times 0 + 1 \times 1 + 1 \times 1 + 1 \times 1 = 3$$

Calculation for $y(5)$, putting $n = 5$, we get

$$y(5) = \sum_{k=0}^{3} x(k)h\,(5-k)$$

$$= x(0)h(5) + x(1)h(4) + x(2)h(3) + x(3)h(2)$$

$$= 1 \times 0 + 1 \times 0 + 1 \times 1 + 1 \times 1 = 2$$

Calculation for $y(6)$, putting $n = 6$, we get

$$y(6) = \sum_{k=0}^{3} x(k)h\,(6-k)$$

$$= x(0)h(6) + x(1)h(5) + x(2)h(4) + x(3)h(3)$$

$$= 1 \times 0 + 1 \times 0 + 1 \times 0 + 1 \times 1 = 1$$

Thus $y(n)$ ={$\underline{1}$, 2, 3, 4, 3, 2, 1}

Question: Obtain convolution of the given discrete time signals

$$x(n) = \sum_{k=0}^{2} \delta(n-k) \;\; and \; h(n) = 2^n\{u(n) - u(n-3)\}$$

Solution:

$x(n)$ =$\{\underline{1}, 1, 1\}$

$h(n) =$ $\{\underline{1}, 2, 4\}$

$y(n)$ =$\{\underline{1}, 3, 7, 6, 4\}$

Questions:

1. Find the convolution between two signals $x(n) = a^n$, for $0 \leq n \leq 6$ and $h(n) = 1$, for $0 \leq n \leq 4$.

Solution: see next slides.

Analytical Evaluation of the Convolution Sum

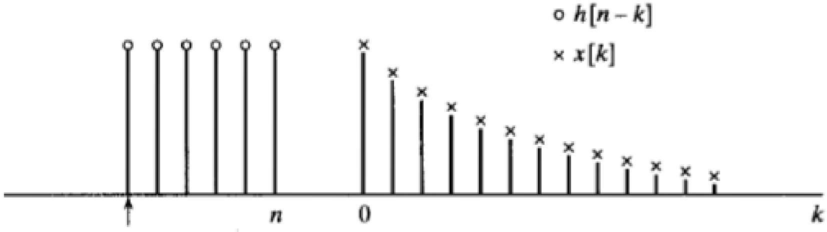
Consider a system with impulse response

$$h(n) = u(n) - u(n-N) = \begin{cases} 1, & 0 \le n \le N-1, \\ 0, & otherwise. \end{cases}$$

The input is $x(n) = a^n u(n)$.

To find the output at a particular index x, we must form the sums over all k of the prodcut $x(k)h(n-k)$. In this case, we can find formulas for $y(n)$ for different sets of values of n.

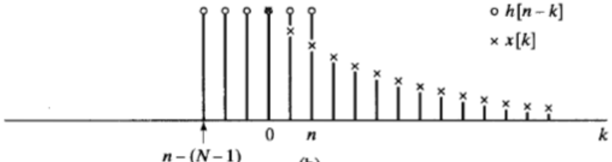
Figure (a) shows the sequences $x(k)$ and $h(n-k)$, plotted for n a negative integer. Clearly, all negative values of n give a similar picture, i.e., the nonzero portions of the sequences $x(k)$ and $h(n-k)$ do not overlap, so $y(n) = 0, n < 0$.



(a)

Figure (b) illustrates the two sequences when $0 \le n \; and \; n - N + 1 \le 0$. These two conditions can be combined into the single condition $0 \le n \le N - 1$.
From figure we see $x(k)h(n-k) = a^k$ So

$$y(n) = \sum_{k=0}^{n} a^k \,, for \; 0 \le n \le N - 1$$

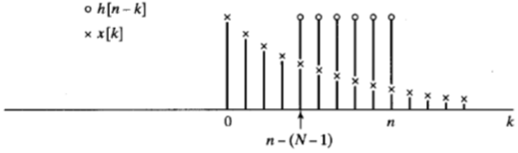


(b)

Which is geometric series so

$$y(n) = \frac{1-a^{n+1}}{1-a}, 0 \le n \le N - 1$$

Figure (c) shows the two sequences when $0 < n - N + 1 \; or \; N - 1 < n$



(c)

$$y(n) = \sum_{k=n-N+1}^{n} a^k \,, for \; N - 1 < n$$

$$\sum_{k=N_1}^{N_2} \alpha^k = \frac{\alpha^{N_1} - \alpha^{N_2+1}}{1-\alpha}, \qquad N_2 \ge N_1.$$

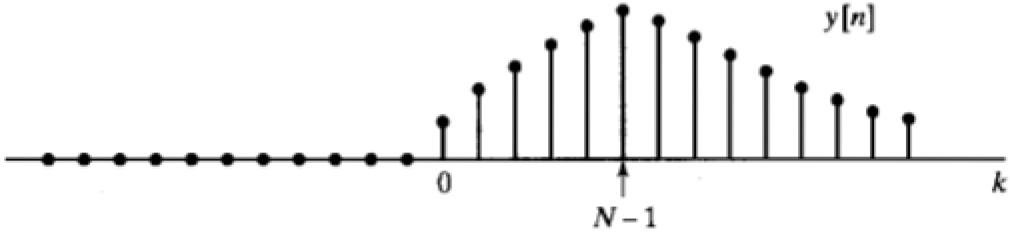
$$y(n) = \frac{a^{n-N+1} - a^{n+1}}{1-a}$$

$$= a^{n-N+1} \frac{1-a^N}{1-a}$$

Thus, because of the piecewise-exponential nature of both the input and the unit sample response, we have been able to obtain the following closed-form expression for y(n) as a function of the index n:

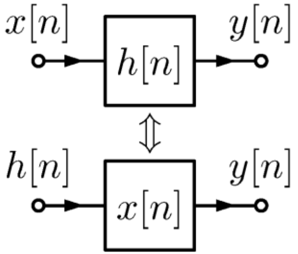
$$y(n) = \begin{cases} 0, & n < 0 \\ \dfrac{1 - a^{n+1}}{1 - a}, & 0 \le n \le N - 1 \\ a^{n-N+1}\dfrac{1 - a^{N}}{1 - a}, & N - 1 < n. \end{cases}$$

This sequence is shown in figure (d).



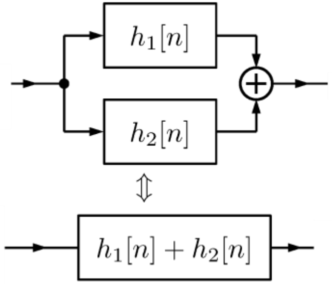
(d)

Properties of Linear Convolution



1. Commutative: $x(n) * h(n) = h(n) * x(n)$

The roles of $x(n)$ and $h(n)$ can be interchange. On other words, we may regard $x(n)$ as the impulse response of the system and $h(n)$ as the excitation or input signal as illustrates in figure.



2. Distributive:

$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

This law implies that if we have two linear time-invariant systems with impulse responses $h_1(n)$ and $h_2(n)$ exicted by the same input signal $x(n)$, the sum of the two responses is identical to the response of an overall system with impulse response

$$h(n) = h_1(n) + h_2(n)$$

Conversely, any linear time-invariant system can be decomposed into a parallel interconnection of subsystems.

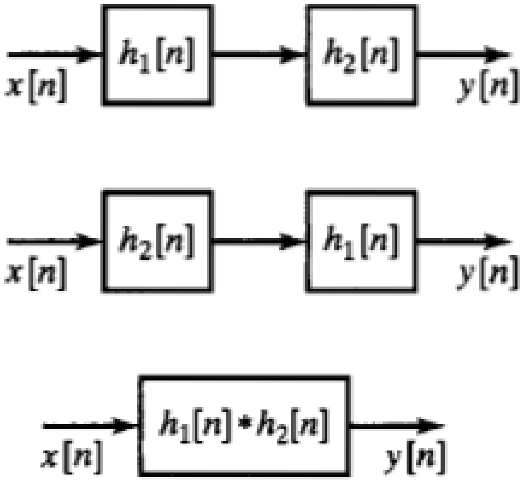
3. <u>Associative law</u>

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

The right hand side indicates that the input $x(n)$ is applied to an equivalent system having an impulse response, say $h(n)$, which is equal to the convolution of the two impulse responses.

That is $h(n) = h_1(n) * h_2(n)$

Furthermore, since the convolution operation satisfies the commutative property, one can interchange the order of the two systems with responses $h_1(n)$ and $h_2(n)$ without altering the overall input-output relationship.



Stability of LTI system

Stability is an important property that must be considered in any practical implementation of a system. We defined an arbitrary relaxed system as BIBO stable if and only if its output sequence $y(n)$ is bounded for every bounded input $x(n)$.

If $x(n)$ is bounded, there exists a constant M_x such that

$$|x(n)| \leq M_x < \infty$$

Similarly, if the output is bounded, there exists a constant M_y such that

$$|y(n)| \leq M_y < \infty$$

for all n.

Now, given such a bounded input sequence $x(n)$ to a linear time-invariant system, let us investigate the implications of the defination of stability on the characteristics of the system.

The convolution equation can be written as

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Now, taking absolute value of both sides, we obtain

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right|$$

The absolute value of the sum of terms is always less than or equal to the sum of the absolute values of the terms. Hence

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

Here, $x(n-k)$ is delayed input signal. If input is bounded then its delayed version is also bounded i.e., if $|x(n)| \leq M_x$ then $|x(n-k)| \leq M_x$.

Putting this values we get,

$$|y(n)| \leq \sum_{k=-\infty}^{\infty} |h(k)|\, M_x$$

$$|y(n)| \leq M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

We know that M_x is a finite number. We want output $y(n)$ to be bounded. That means $|y(n)|$ should be finite. So the output is bounded if the impulse response of the system satisfies the condition

$$S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

That is, **a linear time-invariant system is stable if its impulse response is absolutely summable**. This condition is not only sufficient but it is also necessary to ensure the stability of the system.

To show that it is also a necessary condition, we must show that if $S = \infty$, then a bounded input can be found that will cause an unbounded output.

Such an input is the sequence with values

$$x(n) = \begin{cases} \dfrac{h^*(-n)}{|h(-n)|}, & h(n) \neq 0, \\ 0, & h(n) = 0, \end{cases}$$

Where $h^*(-n)$ is the complex conjugate of $h(n)$. The sequence $x(n)$ is clearly bounded by unity. However, the value of the output at $n = 0$ is

$$y(0) = \sum_{k=-\infty}^{\infty} x(-k)h(k) = \sum_{k=-\infty}^{\infty} \frac{|h(k)|^2}{|h(k)|} = S$$

Therefore, if $S = \infty$, it is possible for a bounded input sequence to produce an unbounded output sequence.

Example1: Determine the range of values of the parameter a for which the linear time-invariant system with impulse response $h(n) = a^n u(n)$ is stable.

Solution: The system is causal. Now

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} \left|a^k\right| = 1 + |a| + |a|^2 + \cdots$$

This geometric series converges to

$$\sum_{k=-\infty}^{\infty} \left|a^k\right| = \frac{1}{1 - |a|}$$

provided that $|a| < 1$, otherwise it diverges. Therefore, the system is stable if $|a| < 1$, otherwis it is unstable.

Example2: Determine the range of values of a and b for which the linear time invariant system with impulse response

$$h(n) = \begin{cases} a^n, n \geq 0 \\ b^n, n < 0 \end{cases}$$

is stable.

Solution: This system is non-casual.

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} |b|^n + \sum_{n=0}^{\infty} |a|^n$$

The second term converges for $|a| < 1$, the first term can be manipulate as

$$\sum_{n=-\infty}^{-1} |b|^n = \sum_{n=1}^{\infty} \frac{1}{|b|^n} = \frac{1}{|b|}\left[1 + \frac{1}{|b|} + \frac{1}{|b|^2} + \cdots\right]$$

$$= \beta(1 + \beta^2 + \cdots) = \frac{\beta}{1-\beta}$$

where $\beta = \frac{1}{|b|}$ must be less than unity for the geometric series to converge. Cosnequently, the system is stable if both $|a| < 1\ and\ |b| > 1$ are satisfied.

Causal Linear Time invariant systems

A causal system is whose output at time n depends only on present and past inputs but does not depends on future inputs. In other words, the output of the system at some time instant n, say $\mathrm{n} = n_0$, depends only on values of $x(n)$ for $n \leq n_0$.

In case of a linear time-invariant system, causality can be translated to a condition on the impuse response. To determine this relationship, let us consider a linear time-invariant system having an output at time $n = n_0$ given by the convolution formula

$$y(n_0) = \sum_{k=-\infty}^{\infty} h(k)x(n_0 - k)$$

Suppose that we subdivide the sum into two sets of terms, one set involving present and past values of the input [i.e., $x(n)$ for$n \leq n_0$] and one set involving future values of the input [i.e., $x(n)$ for $n > n_0$].

Thus we obtain

$$y(n_0) = \sum_{k=-\infty}^{-1} h(k)x(n_0 - k) + \sum_{k=0}^{\infty} h(k)x(n_0 - k)$$

$$= [h(-1)x(n_0 + 1) + h(-2)x(n_0 + 2) + \cdots]$$

$$+[h(0)x(n_0) + h(1)x(n_0 - 1) + h(2)x(n_0 - 2) + \cdots]$$

We observe that the terms in the second sum involve $x(n_0)$, $x(n_0 - 1), x(n_0 -$

Hence, **an LTI system is causal if and only if its impulse response is zero for negative values of n.**

Since for a causal system, $h(n) = 0$ for $n < 0$, the limits on the summation of the convolution formula may be modified to reflect this restriction. Thus we have the equivalent forms

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=-\infty}^{n} x(k)h(n-k)$$

Causality is requried in any real-time signal processing application, since at any given time n we have no access to future values of the input signal. Only the present and past values of the input signal are available in computing the present output.

If the input to a causal linear time-invariant system is a causal sequence (i.e., if $x(n) = 0$ for $n < 0$), the limits on the convolution formula are further restricted. In this case the two equivalent forms of the convolution formaula become

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k)$$

$$= \sum_{k=0}^{n} x(k)h(n-k)$$

The response of a causal system to a causal input sequence is causal, since $y(n) = 0 \; for \; n < 0$.

To illustrate how the properties of LTI systems are reflected in the impulse response, let consider some linear and time invariant examples.

Ideal Delay

$$h(n) = \delta(n - n_d),$$

n_d a positive fixed integer

Moving Average

$$h(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta(n - k)$$

$$= \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2 \\ 0, & otherwise \end{cases},$$

Accumulator

$$h(n) = \sum_{k=-\infty}^{n} \delta(k) = \begin{cases} 1, n \geq 0, \\ 0, n < 0, \end{cases} = u(n)$$

Forward Difference

$$h(n) = \delta(n+1) - \delta(n)$$

Backward Difference

$$h(n) = \delta(n) - \delta(n-1)$$

Given the impulse response of these basic systems, we can test the stablilty of each one by computing the sum

$$S = \sum_{n=-\infty}^{\infty} |h(n)|$$

For the ideal delay, moving average, forward difference, and backward difference, it is clear that $S < \infty$, since the impulse response has only a finite number of nonzero samples. Such systems are called finite-duration impulse response (FIR) systems. Clearly, FIR systems will be always be stable, as long as each of the impulse response values is finite in magnitude.

The accumulator, however, is unstable because

$$S = \sum_{n=0}^{\infty} u(n) = \infty$$

Also accumulator is unstable when a bounded input (the unit step) is given (see slide 87) it produces an unbounded output.
The impulse of the accumulator is infinite in duration. This is an example of the class of systems reffered to as infinite-duration impulse response (IIR) systems.

An example of an IIR system that is stable is a system whose impulse response is $h(n) = a^n u(n)$ with $|a| < 1$. In this case,

$$S = \sum_{n=0}^{\infty} |a|^n$$

If $|a| < 1$, the formula for the sum of the terms of an infinite geometric series gives

$$S = \frac{1}{1 - |a|} < \infty$$

If, on other hand, $|a| \geq 1$, the sum is infinite and the system is unstable.

Note:

1. The ideal delay $n_d \geq 0$ is causal and $n_d < 0$ is noncausal.
2. The accumulator and backward difference systems are causal
3. The forward difference is noncausal.
4. For the moving average, causality requries that $-M_1 \geq 0\ and\ M_2 \geq 0$.

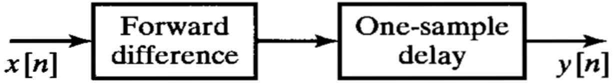
The concept of convolution as an operation between two sequences leads to the simplification of many problems involving systems.

A particular useful result can be stated for the ideal delay system. Since the output of the delay system is $y(n) = x(n - n_d)$, since the delay system has impulse response $h(n) = \delta(n - n_d)$, it follows that

$$x(n) * \delta(n - n_d) = \delta(n - n_d) * x(\mathrm{n}) = x(n - n_d)$$

That is, the convolution of a shifted impulse sequence with any signal $x(n)$ is easily evaluated by simplying shifting $x(n)$ by the displacement of the impulse.

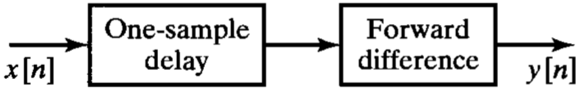
Consider the system which consists of a forward difference system cascaded with an ideal delay of one sample as in figure.



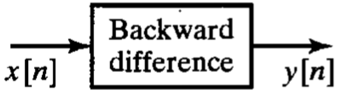
The impulse response of system is

$$h(n) = [\delta(n+1) - \delta(n)] * \delta(n-1)$$

According to cummulative property of convolution, the order in which system are cascaded does not matter, as long as they are linear and time invariant. Therefore we obtain same result when we delay the sequence first and the and compute forward difference as in figure.



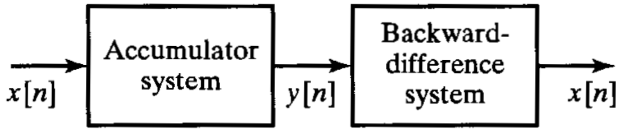
$$h(n) = \delta(n-1) * [\delta(n+1) - \delta(n)]$$
$$= \delta(n) - \delta(n-1)$$



Thus, $h(n)$ is identical to the impulse response of the backward difference system, i.e., ths cascade system can be repalced by a backward differnce system as in figure.

The noncausal forward difference systems have converted to causal systems by cascading them with a delay. In general, **any noncausal FIR system can be made causal by cascading it with a sufficient long delay**.

Consider another cascade system as in figure,
a cascade combination of accumulator followed by a backward difference (or vice versa).



The impulse response of the cascade system is

$$h(n) = u(n) * [\delta(n) - \delta(n-1)]$$
$$= u(n) - u(n-1)$$
$$= \delta(n)$$

The cascade combination of an accumulator followed by a backward difference (or vice versa) yields a system whose overall impulse response is the impulse.
Since $x(n) * \delta(n) = x(n)$, thus the backward difference system compensates exactly for (or inverts) the effect of the accumulator.
The backward difference system is the inverse system for the accumulator and from commutative property of convolution, the accumulator is likewise the inverse system for the backward difference system.
In general, if a linear time-invariant system has impulse response $h(n)$, then its inverse system, if it exists, has impulse respone $h_i(n)$ defined by the relation

$$h(n) * h_i(n) = h_i(n) * h(n) = \delta(n)$$

Discrete-time system described by Difference Equations

The convolution summation formula

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h\,(n-k)$$

This equation suggests a means for the realization of the system. In the case of FIR systems, such realization involves additions, multiplications, and a finite number of memory location which is readily implemented directly.

If the system is IIR, however, its practical implementation as given by convolution is clearly impossible, since it requires an infinite number of memory locations, multiplications, and additions.

There is a practical and computationally efficient means for implementing a family of IIR systems. This family of discrete-time systems is more conveniently described by difference equations. This family of IIR systems is very useful in a variety of practical applications, including the implementation of digital filters, and the modeling of physical phenomena and physical systems.

Recursive and Nonrecursive Discrete-time Systems

If the response of any discrete time systems depends only on the input signals then the system is known as non-recursive discrete time system.

If we express the output of the system not only in terms of the present and past values of the input, but also in terms of the past output values, then that system is known as recursive system.

Example the cumulative average of a signal $x(n)$ in the interval $0 \leq k \leq n$ defined as,

$$y(n) = \frac{1}{n+1} \sum_{k=0}^{n} x(k), \qquad n = 0,1, \dots$$

This equation implies the computation of $y(n)$ requires the storage of all the input samples $x(k)$ for $0 \leq k \leq n$. Since n is increasing, our memory requirements grow linearly with time. However, $y(n)$ can be computed more efficiently by utilizing the previous output values $y(n-1)$. By a simple algebraic rearrangement, we obtain,

$$(n+1)y(n) = \sum_{k=0}^{n-1} x(k) + x(n) = ny(n-1) + x(n)$$

$$y(n) = \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n)$$

Now the cumulative average y(n) can be computed recursively by multiplying the previous output value $y(n-1)$ by $n/(n+1)$, multiplying the present input $x(n)$ by $1/(n+1)$, and adding the two products as in figure. A system whose output $y(n)$ at time n depends on any number of past output values $y(n-1), y(n-2), \ldots$ is called recursive system.

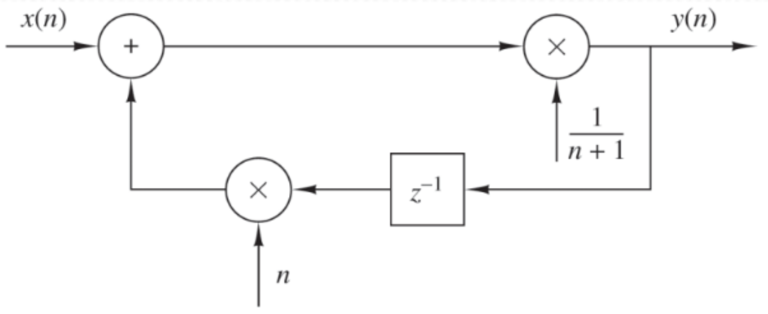
To determine the computation of recursive system, suppose we begin the process with $n = 0$ and proceed forward in time and we obtain

$y(0) = x(0)$

$y(1) = \frac{1}{2} y(0) + \frac{1}{2} x(0)$

$y(2) = \frac{2}{3} y(1) + \frac{1}{3} x(1)$

And so on.



Realization of a recursive cumulative averaging system.

1.9 Frequency Response of LTI systems

- Discrete time signals may be represented in a number of different ways.
- For example, sinusoidal and complex exponential sequences play a particularly important role in representing discrete time signals.
- This is because complex exponential sequences are eigenfunctions of linear time invariant systems and the response to a sinusoidal input is sinusoidal with the same frequency as the input an with amplitude and phase determined by the system.
- This fundamental property if linear time invariant systems makes representation of signals in terms of sinusoids or complex exponentials (i.e., Fourier representations) very useful in linear system theory.

Eigenfunctions for Linear Time-Invariant Systems

Consider an input sequence $x(n) = e^{j\omega n} \; for \; -\infty < n < \infty$, i.e., a complex exponential of radian frequency ω. The corresponding output of a linear time invariant system with impulse response $h(n)$ is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) e^{j\omega(n-k)}$$

$$= e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

If we define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k}$$

Thus

$$y(n) = H(e^{j\omega}) e^{j\omega n}$$

Consequently, $e^{j\omega n}$ is the eigenfunction of the system, and the associated eigenvalue is $H(e^{j\omega})$.

$H(e^{j\omega})$ describes the change in complex amplitude of a complex exponential input signal as a function of the frequency ω.

The eigenvalue $H(e^{j\omega})$ is called the frequency response of the system.

In general, $H(e^{j\omega})$ is complex and can be expressed in terms of its real and imaginary parts as

$$H\left(e^{j\omega}\right) = H_R\left(e^{j\omega}\right) + \mathrm{j}H_I(e^{j\omega})$$

Or in terms of magnitude and phase as

$$H\left(e^{j\omega}\right) = \left|H(e^{j\omega})\right| e^{j\measuredangle H(e^{j\omega})}$$

Example: Frequency Response of the Ideal Delay System

Consider the ideal delay system defined by

$$y(n) = x(n - n_d)$$

Where n_d is a fixed integer. If we consider $x(n) = e^{j\omega n}$ as input to this system, then, we have

$$y(n) = e^{j\omega(n-n_d)} = e^{j\omega n}e^{-j\omega n_d}$$

The frequency response of the ideal delay is therefore

$$H\left(e^{j\omega}\right) = e^{-j\omega n_d}$$

The real an imaginary parts of the frequency response are

$$H_R\left(e^{j\omega}\right) = \cos(\omega n_d)$$

$$H_I\left(e^{j\omega}\right) = -\sin(\omega n_d)$$

The magintude and phase are

$$\left|H(e^{j\omega})\right| = 1,$$

$$\sphericalangle H\left(e^{j\omega}\right) = -\omega n_d$$

Example 2: Sinusoidal Response of LTI Systems

Since it is simple to express a sinusoid as a linear combination of complex expoenetials, let us consider a sinusoidal input

$$x(n) = A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

Thus splitting the response as $x_1(n)$ and $x_2(n)$ as

$x_1(n) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n}$ and $x_2(n) = \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$ the response are

$$y_1(n) = H\left(e^{j\omega_0}\right) \frac{A}{2} e^{j\phi} e^{j\omega_0 n}$$

$$y_2(n) = H\left(e^{-j\omega_0}\right) \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

Thus, the total response is

$$y(n) = \frac{A}{2} [H\left(e^{j\omega_0}\right) e^{j\phi} e^{j\omega_0 n} + H\left(e^{-j\omega_0}\right) e^{-j\phi} e^{-j\omega_0 n}]$$

If $h(n)$ is real, it can be shown that $H\left(e^{-j\omega_0}\right) = H^*\left(e^{j\omega_0}\right)$ (See slide 43)

Hence,

$$y(n) = A\left|H\left(e^{j\omega_0}\right)\right| \cos(\omega_0 n + \phi + \theta)$$

Where $\theta = \sphericalangle H\left(e^{j\omega_0}\right)$ is the phase of the system function at frequency ω_0.

Let use ideal delay with

$$\left|H(e^{j\omega})\right| = 1, \qquad \sphericalangle H\left(e^{j\omega}\right) = -\omega n_d$$

therefore,

$$y(n) = A \cos(\omega_0 n + \phi - \omega n_d) = A \cos[\omega_0 (n - n_d) + \phi]$$

Which is consistent with what we would obtain directly usnig the definition of the ideal delay system.

The frequency response of discrete time linear time invariant system is always a periodic function of the frequency variable ω with period 2π.

To prove, we have

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h(k)e^{-j\omega n}$$

Let substitute $\omega\ by\ \omega + 2\pi$ we get

$$H\left(e^{j(\omega+2\pi)}\right) = \sum_{n=-\infty}^{\infty} h(k)e^{-j(\omega+2\pi)n}$$

Using the fact that $e^{\pm j(2\pi)n} = 1$

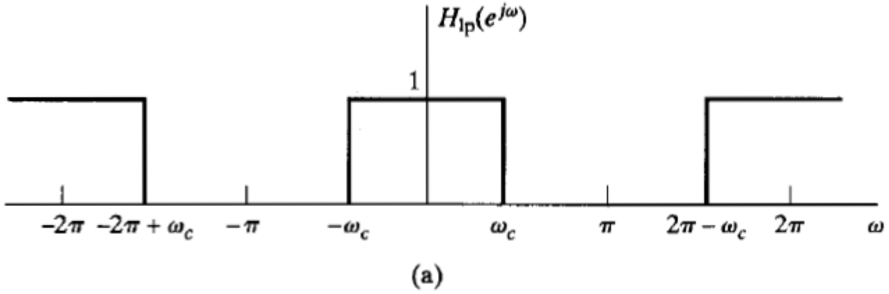
Therefore

$$H\left(e^{j(\omega+2\pi)}\right) = H\left(e^{j\omega}\right)$$

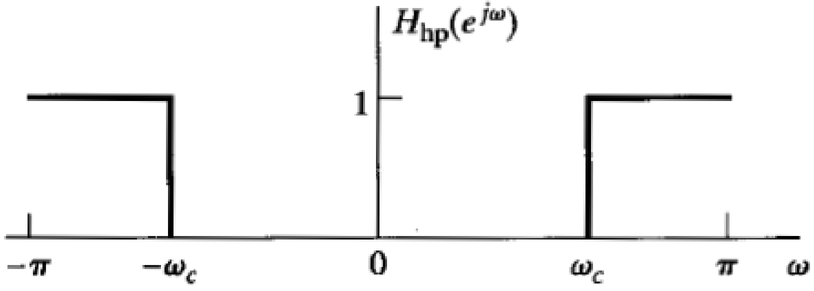
In general

$$H\left(e^{j(\omega+2\pi r)}\right) = H\left(e^{j\omega}\right), for\ r\ an\ integer$$

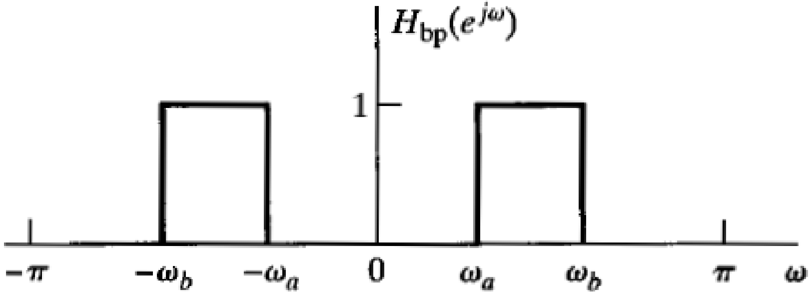
Since $H(e^{j\omega})$ is periodic with period 2π, and since the frequencies ω and $\omega + 2\pi$ are indistinguishable, it follows that we need only specify $H(e^{j\omega})$ over an interval of length 2π , e.g., $0 \leq \omega \leq 2\pi$ or $-\pi \leq \omega \leq \pi$.



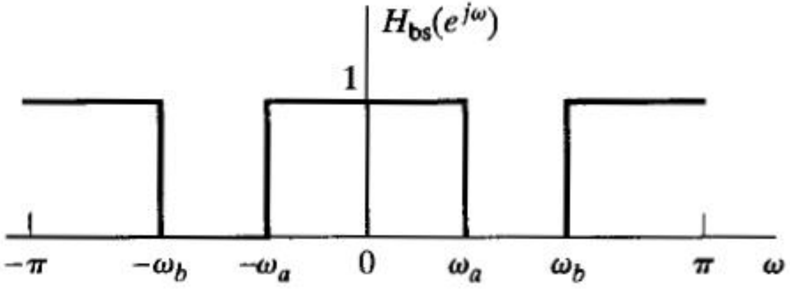
Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.



Highpass filter.



Bandpass filter.



Bandstop filter.

In each case, the frequency response is periodic with period 2π. Only one period is shown.

Frequency Response of the Moving Average System

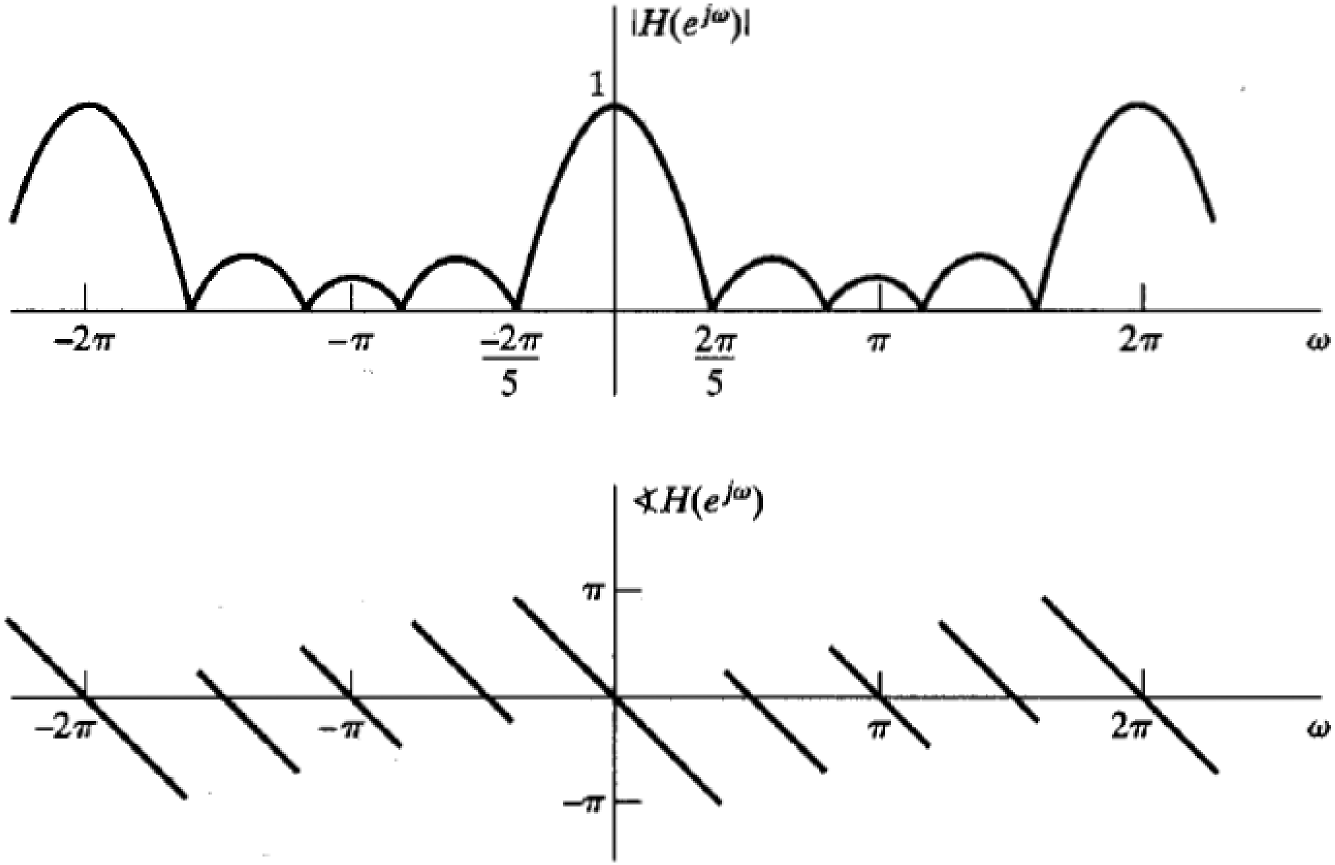
The impulse response of the moving average system is

$$h(n) = \begin{cases} \dfrac{1}{M_1 + M_2 + 1}, & -M_1 \leq n \leq M_2, \\ 0, & otherwise \end{cases}$$

The frequency response is

$$H\left(e^{j\omega}\right) = \frac{1}{M_1 + M_2 + 1} \sum_{n=-M_1}^{M_2} e^{-j\omega n}$$

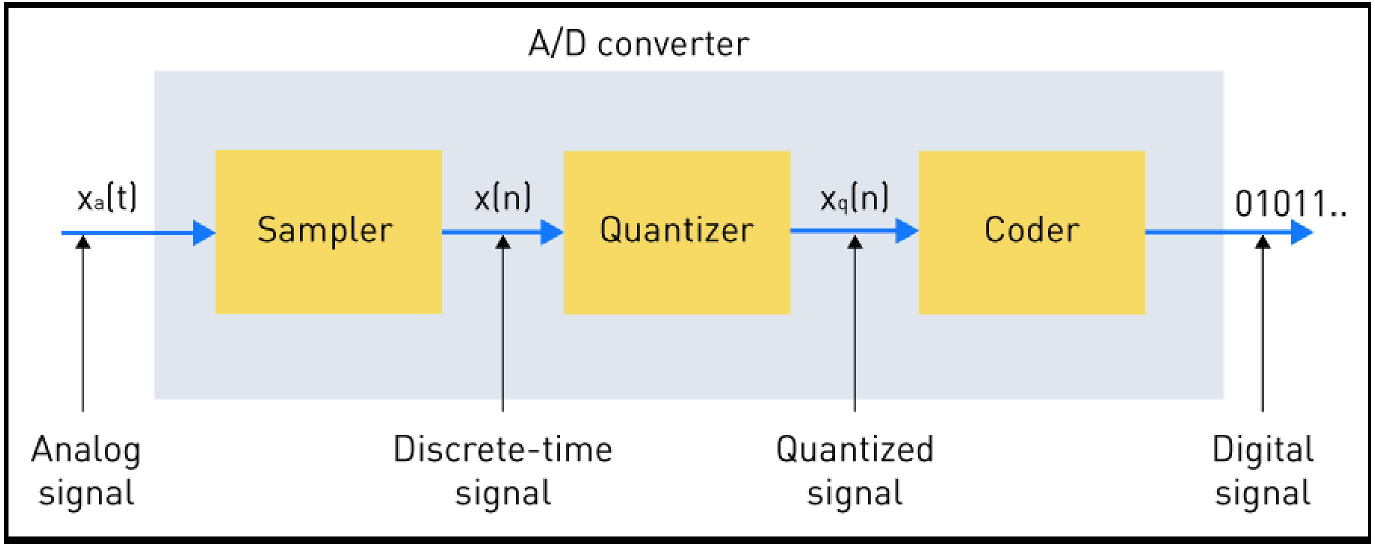
$$
\begin{aligned}
H(e^{j\omega}) &= \frac{1}{M_1+M_2+1}\frac{e^{j\omega M_1}-e^{-j\omega(M_2+1)}}{1-e^{-j\omega}} \\
&= \frac{1}{M_1+M_2+1}\frac{e^{j\omega(M_1+M_2+1)/2}-e^{-j\omega(M_1+M_2+1)/2}}{1-e^{-j\omega}}e^{-j\omega(M_2-M_1+1)/2} \\
&= \frac{1}{M_1+M_2+1}\frac{e^{j\omega(M_1+M_2+1)/2}-e^{-j\omega(M_1+M_2+1)/2}}{e^{j\omega/2}-e^{-j\omega/2}}e^{-j\omega(M_2-M_1)/2} \\
&= \frac{1}{M_1+M_2+1}\frac{\sin[\omega(M_1+M_2+1)/2]}{\sin(\omega/2)}e^{-j\omega(M_2-M_1)/2}.
\end{aligned}
$$



(a) Magnitude and (b) phase of the frequency response of the moving-average system for the case $M_1 = 0$ and $M_2 = 4$.

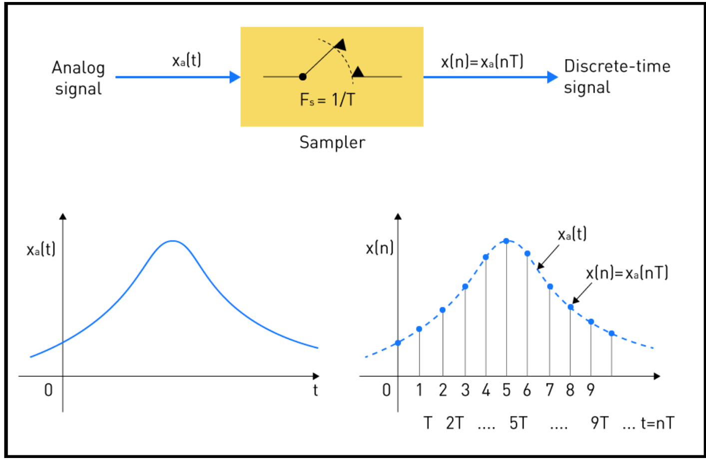
1.10 Sampling continuous signals and spectral properties of sampled signals

- Most of the signals of practical interest are analog in nature, such as voice signal, biological signal, communication signals, etc.
- To process analog signal by digital means, it is first necessary to convert them into digital form. This procedure is known as analog to digital conversion.
- Analog to digital conversion is a three step process as shown in the block diagram.



a. Sampling

- ✓ A continuous analog signal is first transformed into a discrete digital signal by sampling.
- ✓ The procedure entails periodic measurement or snapshots of the analog signal's amplitude at predetermined periods in time.
- ✓ The discrete set of data points that picture, or samples, reflect the original signal in the digital realm.
- ✓ The sampling rate or sampling frequency is the frequency at which these photographs or samples are taken.



In mathematical terms, if x(t) is a continuous-time analog signal, the sampled signal x(n) can be represented as

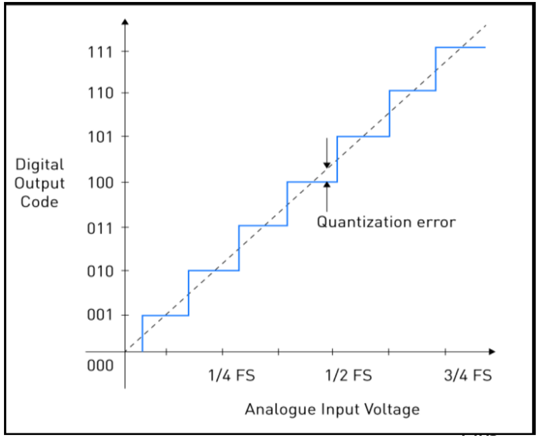
x(n) = x(nT_s)

Where T_s is the sampling period (the time period between the consecutive samples), n is an integer and x(n) denotes the value of the signal at the nth sample.

Sample and hold circuit are typically used to execute the sampling process. These circuit collects and hold the signal value for a brief period of time.

b. Quantization

- ✓ Quantization comes after sampling as a crucial step in converting continuous analog signals to digital signals.
- ✓ A continuous set of values is quantized into a discrete set of values.
- ✓ During the analog to digital conversion process, each sampled value is matched with the closest value among a limited number of discrete levels.
- ✓ Since quantization converts a continuous collection of values to a discrete set, it naturally involves an approximation inaccuracy.
- ✓ The discrepancy between the actual sampled value and the quantized value to which it is mapped is referred to as quantization error.
- ✓ Since the nature of quantization error is largely unpredictable, it may be though of as noise added to the signal.
- ✓ It can, however, be examined and its consequences recognized.
- ✓ The error is often constrained to $\pm\,^{1}/_{2}$ of the quantization step size, which is the separation between neighboring quantization levels.



Quantization Levels and Resolution

- ✓ The resolution of the quantizer is measured in bits that determines the number of quantization levels.
- ✓ The number of quantization levels increases with resolution, while the quantization error decreases.
- ✓ There are 2^N quantization levels for an N-bit quantizer. An 8-bit quantizer will have 256 quantization levels.
- ✓ The signal range divided by quantization levels yields the quantization step size. For example, the step size would be 10/256 = 0.039 volts for a signal that range from 0 to 10 volts and has 256 quantization levels.
- ✓ For high-fidelity applications, high resolution quantization is preferred because it lowers quantization error.
- ✓ It also needs extra bit for representation, which results in a trade off in terms of bandwidth and storage.

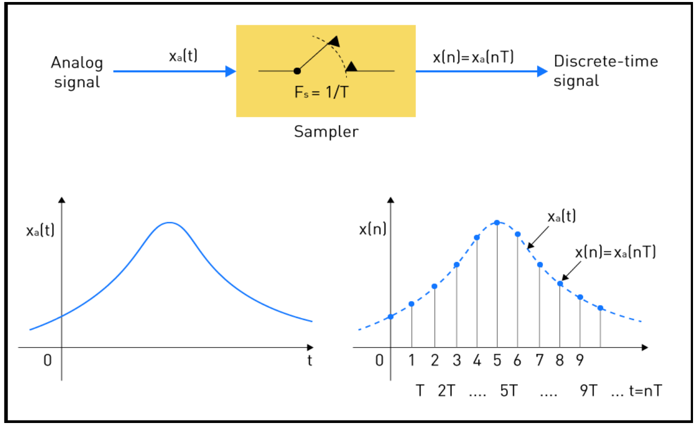
c. Encoding

- ✓ The process of converting an analog signal to a digital signal continues with encoding after the analog signal has been sampled and quantized.
- ✓ Each quantized value is represented using a binary code.
- ✓ A distinct binary code is assigned to each quantized samples.

Digital to Analog Conversion

- ➢ In many cases of practical interest it is desirable to convert the processed digital signals into analog form.
- ➢ The process of converting a digital signal to analog signal is known as digital to analog (D/A) conversion.
- ➢ Sampling does not result in a loss of information, nor does it introduce distortion in the signal if the signal bandwidth is finite.
- ➢ In principle, the analog signal can be reconstructed from the samples, provided that the sampling rate is sufficient high to avoid the problem commonly called aliasing.
- ➢ On the other hand, quantization is a noninvertible or irreversible process that results in signal distortion.

Sampling of Analog Signals



- There are many ways to sample an analog signal.
- The most often used sampling in practice is periodic or uniform sampling.
- The mathematical relationship is given as

$$x(n) = x_a(nT) \quad -\infty < n < \infty$$

Where x(n) is the discrete time signal obtained by taking samples of analog signal $x_a(t)$ every T seconds. This procedure is as in figure.

- The time interval T between successive samples is called the sampling period or sample interval and $1/_T = F_s$ is called the sampling rate (samples per second) or the sampling frequency (hertz).
- Periodic samples establishes a relationship between the time variables t and n of continuous time and discrete time signals, respectively as

$$t = nT = \frac{n}{F_s}$$

- There exists a relationship between the frequency variable F (or Ω) for analog signals and the frequency variable f (or ω) for discrete time signal.

Consider an analog sinusoidal signal of the form

$$x_a(t) = A\cos(2\pi F t + \theta)$$

Which, when sampled periodically at the rate $F_s = {}^1/_T$ samples per second, yields

$$x_a(nT) \equiv x_a(t) = A\cos(2\pi F n T + \theta)$$

$$= A\cos(\frac{2\pi F n}{F_s} + \theta)$$

A discrete time sinusoidal signal is expressed as

$$x(n) = A\cos(\omega n + \theta)\,, -\infty < n < \infty$$

$$= A\cos(2\pi f n + \theta)$$

Comparing we get that the frequency variables F and f are linearly related as

$$f = \frac{F}{F_s}\ or\ \omega = \Omega T$$

The range of frequency variable F (or Ω) for continuous time sinusoids are

$$-\infty < F \text{ (or } \Omega) < \infty$$

And for discrete time sinusoids $-\frac{1}{2} < f < \frac{1}{2}$ or $-\pi < \omega < \pi$

The frequency of the continuous time sinusoid when sampled at a rate $F_s = {}^1/_T$ must fall in the range

$$-\frac{1}{2T} = -\frac{F_s}{2} \leq F \leq \frac{F_s}{2} = \frac{1}{2T}$$

Or equivalently,

$$-\frac{\pi}{T} = -\pi F_s \leq \Omega \leq \pi F_s = \frac{\pi}{T}$$

From these relations the fundamental difference between continuous time and discrete time signals is in their range of values of the frequency variables.

Periodic sampling of a continuous time signal implies a mapping of the infinite frequency range for the variable F (or Ω) into a finite frequency range for the variable f (or ω).

Continuous time frequencies	Ω, F
Discrete time frequencies	ω, f
Sampling frequency	F_s or $\frac{1}{T}$
Conversion relations	$\omega = \frac{\Omega}{F_s} = \Omega T$
	$f = \frac{F}{F_s} = FT$
Range of continuous time frequencies	$-\infty < \Omega < \infty$
	$-\infty < F < \infty$
Range of discrete time frequencies	$-\frac{1}{2} \leq f \leq \frac{1}{2}$
	$-\pi \leq \omega \leq \pi$
	$-\frac{F_s}{2} \leq F \leq \frac{F_s}{2}$
	$-\pi F_s \leq F \leq \pi F_s$

Since the highest frequency in a discrete time signal is $\omega = \pi$ of $f = \frac{1}{2}$, it follows that, with a sampling rate F_s, the corresponding highest value of F and Ω are

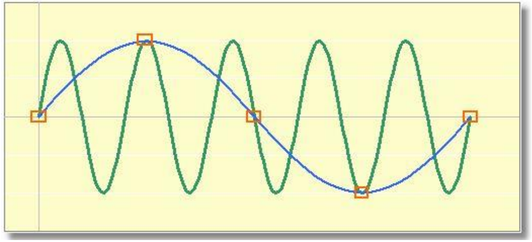
$$F_{max} = \frac{F_s}{2} = \frac{1}{2T}$$

$$\Omega_{max} = \pi F_s = \frac{\pi}{T}$$

Consider two analog sinusoidal signal $x_1(t) = \cos 2\pi(10)t$ and $x_2(t) = \cos 2\pi(50)t$ sampled at a rate $F_s = 40Hz$. The corresponding discrete time signals or sequences are

$$x_1(n) = \cos 2\pi(\frac{10}{40})\, n = \cos\frac{\pi}{2}\, n$$

$$x_2(n) = \cos 2\pi(\frac{50}{40})\, n = \cos\frac{5\pi}{2}\, n$$



However $\cos\frac{5\pi}{2}\, n = \cos(2\pi n + \frac{\pi}{2}\, n) = \cos\frac{\pi}{2}\, n$. Hence $x_1(n) = x_2(n)$.

Thus the sinusoidal signals are identical and, consequently, indistinguishable. The sampled value generated by $\cos\frac{\pi}{2}\, n$, there is some ambiguity as to whether these sampled values correspond to $x_1(t)$ or $x_2(t)$. Since $x_2(t)$ yields exactly the same value as $x_1(t)$ when the two are sampled at $F_s = 40Hz$.

The frequency $F_2 = 50Hz$ is an alias of the frequency $F_1 = 10Hz$ at the sampling rate of $40Hz$.

All of the sinusoids $\cos 2\pi(F_1 + 40k)t, k = 1, 2, 3, 4$ ... sampled at 40 samples per second yield identical values. Consequently, they are all aliases of $F_1 = 10Hz$.

In general, the sampling of a continuous time sinusoidal signal

$$x_a(t) = A\cos(2\pi F_0 t + \theta)$$

With a sampling rate $F_s = {}^{1}/_{T}$ results in a discrete time signal

$$x(n) = A\cos(2\pi f_0 n + \theta)$$

where $f_0 = {}^{F_0}/_{F_s}$, is the relative frequency of the sinusoid.

If we assume that $-\frac{F_s}{2} \leq F_0 \leq \frac{F_s}{2}$ the frequency f_0 of $x(n)$ is in the range $-\frac{1}{2} \leq f_0 \leq \frac{1}{2}$, which is the frequency range for discrete time signal.

In this case, the relationship between F_0 and f_0 is one-to-one, and hence it is possible to identify (or reconstruct) the analog signal $x_a(t)$ from the samples x(n).

If the sinusoids $x_a(t) = A\cos(2\pi F_k t + \theta)$ where $F_k = F_0 + kF_s,\ \mathrm{k} = \pm 1, \pm 2$ ...

are sampled at a rate F_s, it is clear that the frequency F_k is outside the fundamental frequency range $-\frac{F_s}{2} \leq F \leq \frac{F_s}{2}$. Consequently the sampled signal is

$$x_a(nT) \equiv x_a(t) = A\cos\left(2\pi \frac{F_0 + kF_s}{F_s} n + \theta\right) = A\cos\left(2\pi \frac{F_0}{F_s} n + \theta + 2\pi k n\right) = A\cos(2\pi f_0 n + \theta)$$

Thus an infinite number of continuous time sinusoids is represented by sample the same discrete time signal.

Consequently, if we are given the sequence x(n), an ambiguity exists as to which continuous time signal $x_a(t)$ these value represents.

The frequency $F_k = F_0 + kF_s, -\infty < k < \infty$ $(k\ integer)$ are indistinguishable from the frequency F_0 after sampling and hence they are aliases of F_0.

Since ${F_s}/{2}$, which corresponds to $\omega = \pi$, is the highest frequency that can be represented uniquely with a sampling rate F_s, it is a simple matter to determine the mapping of any (alias) frequency above ${F_s}/{2}$ or $\omega = \pi$ as the pivotal point and reflect or "fold" the alias frequency to the range $0 \leq \omega \leq \pi$.

Since the point of reflection is ${F_s}/{2}$ or $\omega = \pi$, the frequency ${F_s}/{2}$ or $\omega = \pi$ is called the folding frequency.

Example :Consider the analog signal $x_a(t) = 3\cos(100\pi t)$

i. Determine the minimum sampling rate required to avoid aliasing.

ii. If the signal is sampled at the rate $F_s = 200\ Hz$. What is the discrete-time signal obtain after sampling?

iii. What is the frequency $0 < F < {F_s}/{2}$ of the a sinusoid that yields samples identical to those obtained for $F_s = 200\ Hz$.

iv. Repeat (ii) and (iii) if the signal is sampled at the rate $F_s = 75\ Hz$.

Solution :

i. The frequency of the analog signal is $F = 50Hz$. Hence the minimum sampling rate required to avoid aliasing is $F_s = 100\ Hz$.

ii. If the signal is sampled at $F_s = 200\ Hz$, the discrete time signal is $x(n) = 3\cos\frac{100\pi}{200}n = 3\cos\frac{\pi}{2}n$

iii. For the sampling rate of $F_s = 200Hz,$ we have $f_0 = {F_0}/{F_s}$, thus $F_0 = f_0 F_s = 200 f_0$ from (ii) $f_0 = \frac{1}{4}$ thus $F_0 = 50Hz$,and $y_a(t) = 3\cos(2\pi F_0 t) = 3\cos(100\pi t)$

iv. If the signal is sampled at $F_s = 75\ Hz$, the discrete time signal is

$$x(n) = 3\cos\frac{100\pi}{75}n = 3\cos\frac{4\pi}{3}n = 3\cos\left(2\pi - \frac{2\pi}{3}\right)n = 3\cos\frac{2\pi}{3}n$$

For the sampling rate of $F_s = 75Hz$, we have $f_0 = {}^{F_0}/_{F_s}$, thus $F_0 = f_0 F_s = 75 f_0$

from (iv) $f_0 = \frac{1}{3}$ thus $F_0 = 25$Hz ,and $y_a(t) = 3\cos(2\pi F_0 t) = 3\cos(50\pi t)$

Sampled at $F_s = 75Hz$ identical samples. Hence F=50Hz is an alias of F=25Hz for the sampling rate $F_s = 75Hz$.

Note: To convert to discrete signal, use analog frequency F_0 and sampling frequency F_s and to convert back to analog signal from digital use digital frequency f_0 with sampling frequency F_s. Use the formula $f_0 = {}^{F_0}/_{F_s}$.

The Sampling Theorem

If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2F_{max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from its samples values using the interpolation function.

The sampling rate $F_N = 2B = 2F_{max}$ is called Nyquist rate. This is the boundary condition value required to avoid aliasing.

The sampling theorem states that a continuous time signal can be completely represented in its samples and recover that into an original form if the sampling frequency is greater than or equal to twice of the maximum frequency. In this situation aliasing does not occur.

Nyquist rate $F_N = 2F_{max} = F_s$

The folding frequency = $\frac{F_s}{2}$

Resolution $\Delta = \frac{X_{max} - X_{min}}{L-1}$

where X_{max} = $A_1 + A_2$ and X_{min} = $-X_{max}$ L = no. of levels

A_1 and A_2 are amplitude of signal.

Quantization

- ✓ Process of conveting a discrete time continuous amplitude signal into a digital signal

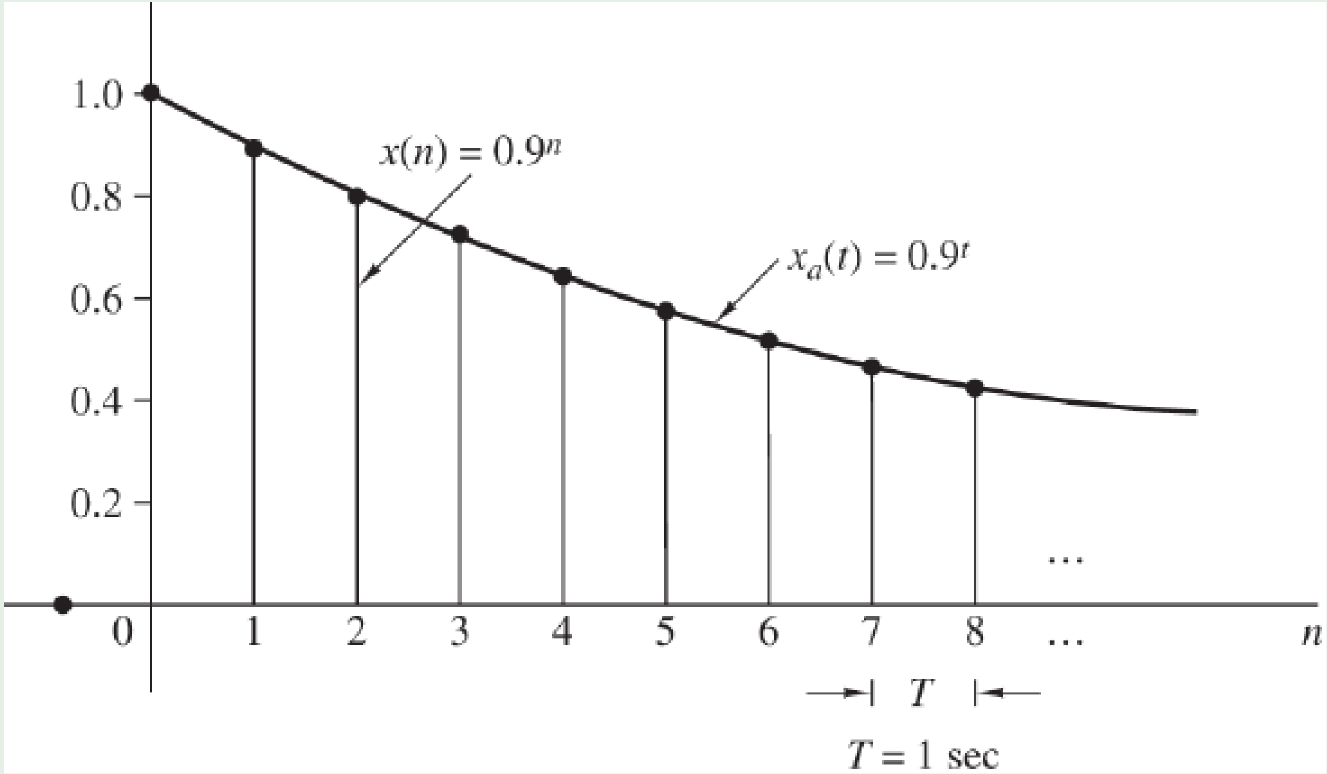
$$x_q(n) = Q[x(n)] = sequence\ of\ quantized\ samples$$

- ✓ Each sample value is expressed as a finite number of digits
- ✓ Error introduced is called quantization error or quantization noise.

$$e_q(n) = x_q(n) - x(n)$$

Example: Consider discrete time signal

$$x(n) = \begin{cases} 0.9^n, n \geq 0 \\ 0, n < 0 \end{cases}\ obtained\ by\ sampling\ x_a(t) = 0.9^t, t \geq 0\ with\ F_s = 1Hz$$



- Following table shows values of first 10 samples of $x(n)$
  - Description of sample value $x(n)$ requires n significant digits

n	$x(n)$ Discrete-time signal	$x_q(n)$ (Truncation)	$x_q(n)$ (Rounding)	$e_q(n) = x_q(n) - x(n)$ (Rounding)
0	1	1.0	1.0	0.0
1	0.9	0.9	0.9	0.0
2	0.81	0.8	0.8	-0.01
3	0.729	0.7	0.7	-0.029
4	0.6561	0.6	0.7	0.0439
5	0.59049	0.5	0.6	0.00951
6	0.531441	0.5	0.5	-0.031441
7	0.4782969	0.4	0.5	0.0217031
8	0.43046721	0.4	0.4	-0.03046721
9	0.387420489	0.3	0.4	0.012579511

- Assume using one significant digit. To eliminate excess digits
    - do **truncation**
    - or do **rounding**

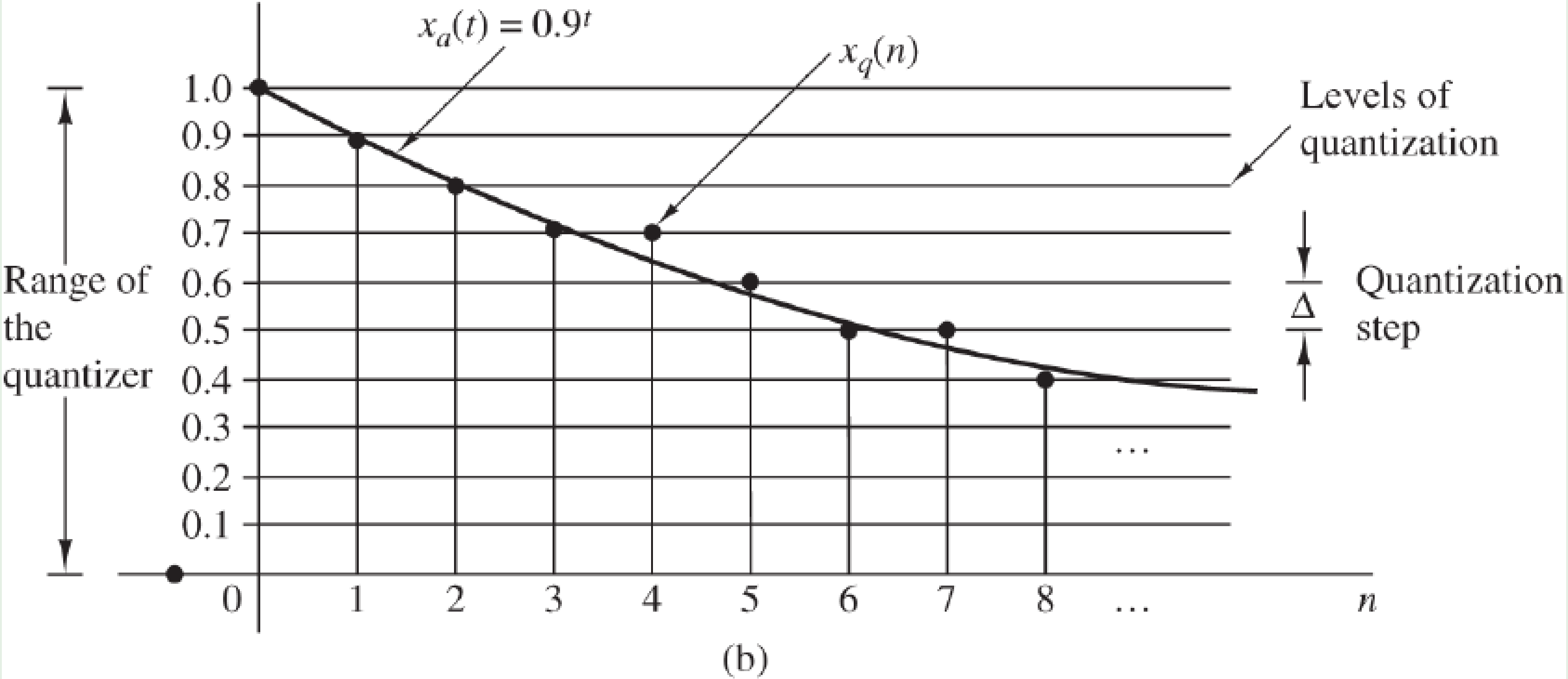


Figure 1.4.7 Illustration of quantization.

- Quantization step size (or resolution) $= \Delta = \frac{1-0}{11-1} = 0.1$

- Range of quantization error $e_q(n)$ in rounding

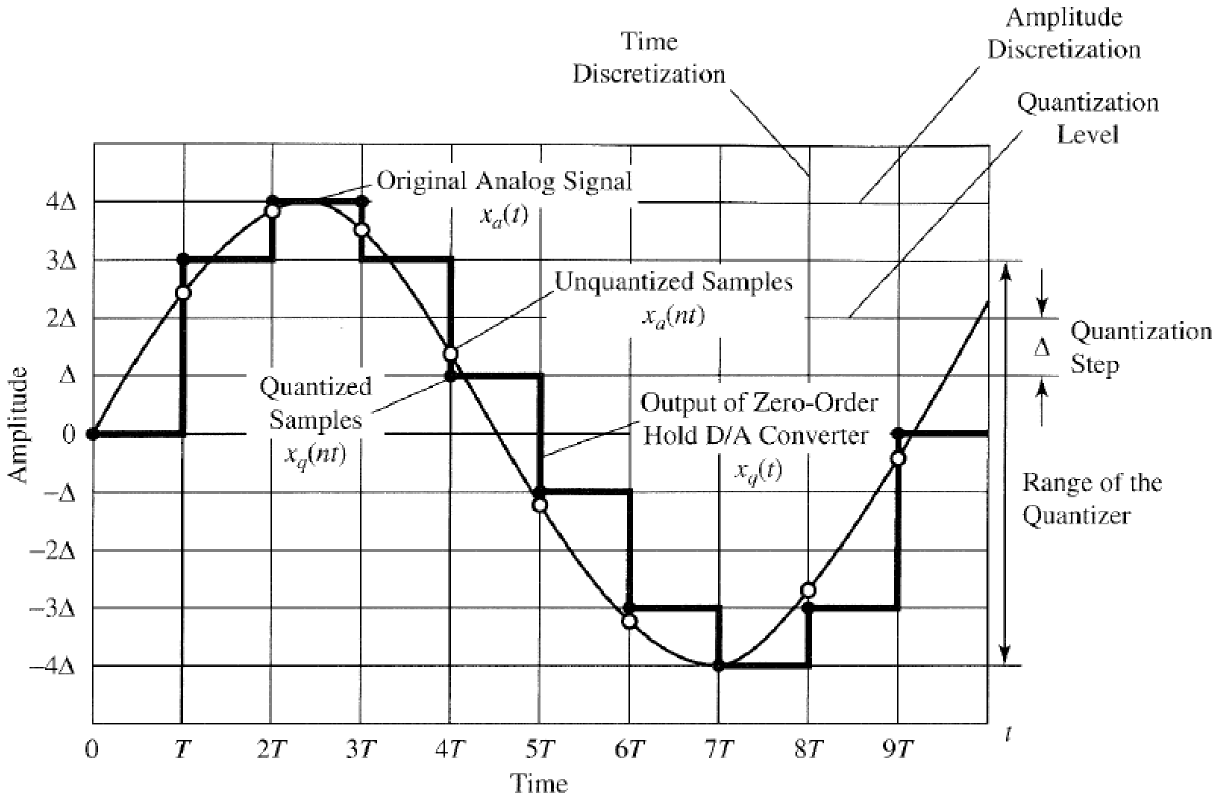
$$-\frac{\Delta}{2} \leq e_q(n) \leq \frac{\Delta}{2}$$

$$\Delta = \frac{x_{max} - x_{min}}{L - 1}$$

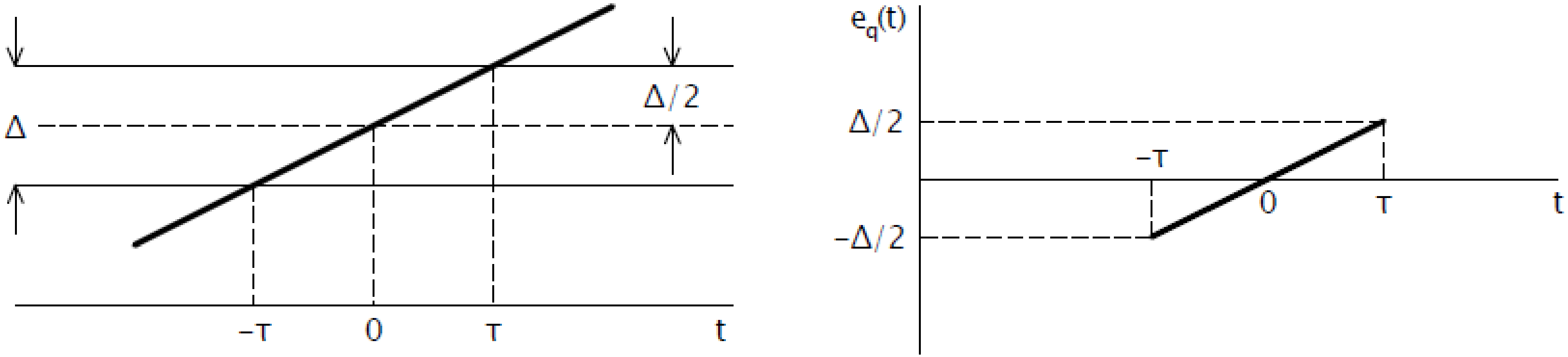
  - x_{min} and x_{max} = minimum and maximum value of $x(n)$
  - L = number of quantization levels
  - Dynamic range of signal $= x_{max} - x_{min}$
- Quantization of analog signals always results in a loss of information

Quantization of Sinusoidal Signals

- Sampling and quantization of $x_a(t) = A\cos\Omega_0 t$



- If F_s satisfies sampling theorem, quantization is the only error in A/D process
  - Thus we can evaluate quantization error by quantizing $x_a(t)$ instead of $x(n) = x_a(nT)$
- $x_a(t)$ is almost linear between quantization levels
  - Quantization error $= e_q(t) = x_a(t) - x_q(t)$



  - τ denotes time that $x_a(t)$ stays within quantization levels

Example: 1. Consider the analog signal $x_a(t) = 3\cos 50\pi t + 10\sin 300\pi t + \cos 100\pi t$

Solution: The frequency present in the signal above are $F_1 = 25Hz$, $F_2 = 150Hz$, and $F_3 = 50Hz$. Thus $F_{max} = 150Hz$ and the Nyquist rate is $F_N = 2F_{max} = 300Hz$.

Note: The signal component $10\sin 300\pi t$, sampled at the Nyquist rate $F_N = 300Hz$, results in the samples $10\sin \pi n$, which is identically zero. In other words, we are sampling the analog sinusoid at its zero-crossing points, and hence we miss this signal component completely.

This situation would not occur if the sinusoid is offset in phase by some amount θ. In such a case we have $10\sin(300\pi t + \theta)$ sampled at the Nyquist rate $F_N = 300Hz$ which yields the samples

$$10\sin(\pi n + \theta) = 10(\sin \pi n \cos\theta + \cos \pi n \sin\theta) = 10\cos \pi n \sin\theta = (-1)^n 10\sin\theta$$

Thus if $\theta \neq 0\ or\ \pi$, the samples of the sinusoid taken at the Nyquist rate are not all zero. However, we still cannot obtain the correct amplitude from the samples when the phase θ is unknown. A simple remedy that avoids this potentially troublesome situation is to sample the analog signal at a higher rate than Nyquist rate.

2. Consider the analog signal $x_a(t) = 3\cos 2000\pi t + 5\sin 6000\pi t + 10\cos 12000\pi t$

i) What is the Nyquist rate for this signal?

ii) Assume now that we sample this signal using a sampling rate $F_s = 5000$ samples/s. what is the discrete time signal obtained after sampling?

iii) What is the analog signal $y_a(t)$ we can reconstruct from the samples if we use ideal interpolation?

Solution:

i) The frequency present in the signal above are $F_1 = 1kHz$, $F_2 = 3kHz$, and $F_3 = 6kHz$. Thus $F_{max} = 6kHz$ and the Nyquist rate is $F_N = 2F_{max} = 12kHz$.

ii) Since we have chosen $F_s = 5kHz$ and the folding frequency is $^{F_s}/_2$ = 2.5kHz

And this is the maximum frequency that can be represented uniquely by the sampled signal.

$$x(n) = x_a(nT) = x_a\left(\frac{n}{F_s}\right) = 3\cos 2\pi\left(\frac{1000}{5000}\right)n + 5\sin 2\pi\left(\frac{3000}{5000}\right)n + 10\cos 2\pi\left(\frac{6000}{5000}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(\frac{3}{5}\right)n + 10\cos 2\pi\left(\frac{6}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(1-\frac{2}{5}\right)n + 10\cos 2\pi\left(1+\frac{1}{5}\right)n$$

$$= 3\cos 2\pi\left(\frac{1}{5}\right)n + 5\sin 2\pi\left(-\frac{2}{5}\right)n + 10\cos 2\pi\left(\frac{1}{5}\right)n$$

$$= 13\cos 2\pi\left(\frac{1}{5}\right)n - 5\sin 2\pi\left(\frac{2}{5}\right)n$$

Among the frequency $F_1 = 1kHz$, $F_2 = 3kHz$, and $F_3 = 6kHz$. F_1 is less than folding frequency thus it is not affected by aliasing. However the other two frequencies are above the folding frequency and they will be changed by the aliasing effect.

iii) Since only the frequency components at 1kHz and 2kHz are present in the sampled signal, the analog signal we can recover is

$$y_a(t) = 13\cos 2000\pi t - 5\sin 4000\pi t$$

Which is obviously different from the original signal $x_a(t)$. This distortion of the original analog signal was caused by the aliasing effect, due to the low sampling rate used.

3. A digital communication link carries binary coded words representing samples of an input signal $x_a(t) = 3\cos 600\pi t + 2\cos 800\pi t$. The link is operated at 10,000 bits/sec and each input sample is quantized into 1024 different voltage levels.

i. What is the sampling frequency and folding frequency?

ii. What is the Nyquist rate for the signal $x_a(t)$?

iii. What is the frequencies in the resulting discrete time signal x(n)?

iv. What is the resolution Δ?

Solution:

i. Each input is quantized into L =1024 levels. The number of bits/sample are denoted by ‘b’ and given by $b = \log_2 L = \log_2 1024 = \frac{\log_{10} 1024}{\log_{10} 2} = 10 bits$

The rate in the digital communication link is given as Bit rate = samples/sec×bits/sample

Samples/sec is sampling frequency F_s = $\frac{\text{Bit rate}}{\text{bits/sample}}$= 1kHz

folding frequency = $\frac{F_s}{2} = 500 Hz$

ii. The frequency present in the signal above are $F_1 = 300Hz$ and $F_2 = 400Hz$,

Thus $F_{max} = 400Hz$ and the Nyquist rate is $F_N = 2F_{max} = 800Hz$.

iii. The sampling frequency F_s = 1000Hz

The discrete time signal x(n) is given by putting t= nT =$\frac{n}{F_s}$

$$x(n) = 3\cos 2\pi \frac{300}{1000} n + 2\cos 2\pi \frac{400}{1000} n$$

$$= 3\cos 2\pi \frac{3}{10} n + 2\cos 2\pi \frac{4}{10} n$$

The frequencies in discrete time signal are $f_1 = 0.3\ and\ f_2 = 0.4$

iv. The two amplitudes are $A_1 = 3\ and\ A_2 = 2$ thus X_{max} = $A_1 + A_2$ = 5, X_{min} = $-X_{max}$= -5

$$\Delta = \frac{X_{max} - X_{min}}{L - 1}$$

$$\Delta = \frac{5 - (-5)}{1024 - 1} = 0.009775$$

Questions

1. Define energy and power signal with examples. Check whether the given signal is energy or power signal $x(t) = A\sin(t)$.
2. Define Signal Analysis and Signal Processing. What are the advantages of Digital signal processing over Analog signal processing?
3. Consider the analog signal $x_a(t) = 3\cos(100\pi t)$
   i. Determine the minimum sampling rate required to avoid aliasing.
   ii. What is the Nyquist rate for the signal $x_a(t)$?
   iii. If the signal is sampled at the rate $F_s = 200\ Hz$. What is the discrete-time signal obtain after sampling?
   iv. What is the frequency $0 < F < {}^{F_s}/_{2}$ of the a sinusoid that yields samples identical to those obtained for $F_s = 200\ Hz$.