Chapter-5: FIR (Finite Impulse Response) Filter Design

- **Filter Design by Window Method, Commonly used Windows (Rectangular Window, Hanning Window, Hamming Window):**

1. Introduction:

- An ***FIR filter of length*** $\boldsymbol{M+1}$ is characterized by the system function

$$\boldsymbol{H(z)=\sum_{n=0}^{M} h[n]z^{-n}} \qquad .....1$$

where, $\boldsymbol{H(z)}=$ polynomial in $\boldsymbol{z^{-1}}$ of degree (or order) $\boldsymbol{M}$.

- The root of the polynomial $\boldsymbol{H(z)}$ constitute zeros of the filter. Thus, $\boldsymbol{H(z)}$ has:

a. $\mathbf{M}$ zeros that can be located anywhere in the finite z-plane, and

b. $\mathbf{M}$ poles which lie at $\boldsymbol{z=0}$.

- The frequency response is $H(e^{j\omega})=H(z)|_{z=e^{j\omega}}$.

- An FIR filter is a LTI system whose impulse response $\boldsymbol{h[n]}$ has a finite-duration.

2. Properties of FIR Filter:

- FIR filter has the following properties:

a. FIR filter is always ***stable***.

b. FIR filter can be designed to have exactly ***linear phase characteristic***. This is achieved by making the impulse response h[n] symmetric as

$$\boldsymbol{h[n] = \pm h[M - n]}$$

c. FIR filters with ***sharp cutoff frequencies require long sequences for*** $\boldsymbol{h[n]}$. Hence, ***long processing time*** is required.

d. The ***causal FIR filters are always non-recursive***. The non-recursive realization of FIR filter is always stable. However, ***recursive realization of FIR filter is also possible***.

3. Advantages of FIR Filters Over IIR Filters:

- FIR filters have the following advantages over IIR filters:

a. They can have ***exactly linear phase***, a characteristic which is very ***useful in speech signal processing***.
b. Since, the ***poles are at origin***, FIR filters are always ***stable***.
c. The ***design methods are generally linear***.
d. They can be ***realized efficiently in hardware***.
e. The filters s***tart-up transients have finite-duration*** in FIR filters.
f. **Quantization error is less** in FIR filters.
g. **Non-recursive FIR filters** ***have small round-off noise*** than IIR filters.

4. FIR Filter Design Techniques:

- FIR filter with linear phase can be designed by using any of the techniques mentioned below:

a. Window method

b. Frequency sampling method

c. Design of optimum equiripple linear phase FIR filter

5. Magnitude Characteristics of FIR Filter:

- The magnitude response of FIR filter can be expressed as:

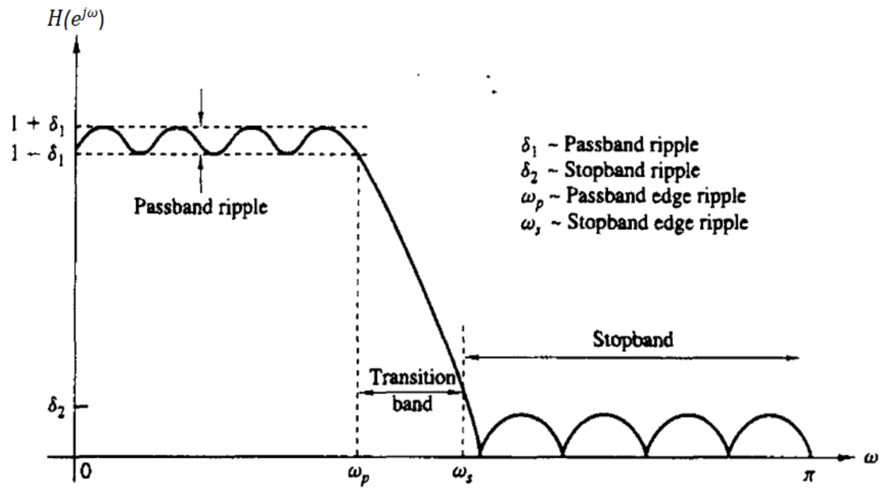
$$\begin{aligned} 1-\delta_1 \le \left|H(e^{j\omega})\right| \le 1+\delta_1, \qquad & 0 \le |\omega| \le \omega_p \\ 0 \le \left|H(e^{j\omega})\right| \le \delta_2, \qquad & \omega_s \le |\omega| \le \pi \end{aligned} \qquad .....3$$

Note that $H(e^{j\omega}) = H(\omega)$.

➤ The approximate formula for ***order N*** is

$$\boldsymbol{N = \frac{-10log_{10}(\delta_1\delta_2)-15}{14\,\Delta f}}$$

where, $\Delta \boldsymbol{f} = \frac{\boldsymbol{\omega_s-\omega_p}}{\boldsymbol{2\pi}} = \boldsymbol{f_s - f_p}$, is the ***transition band***



➤ Another approximate formula used to compute the order of the filter is

$$\boldsymbol{N = k\left(\frac{2\pi}{\omega_s-\omega_p}\right)}$$

where $\boldsymbol{k}$ can be obtained from the ***width of the main lobe.***

width of the main lobe$= \boldsymbol{k\,\frac{2\pi}{N}}$

$\boldsymbol{N}$ = order of the filter = $\boldsymbol{M}$, and $\boldsymbol{M+1}$ = length of the filter

- Note that ***for similar magnitude response the order of the FIR filter is higher than that of IIR*** because ***FIR filters do not use feedback, hence they require long sequences for impulse response $\boldsymbol{h[n]}$ to get sharp cutoff filters***. Long sequences of $h[n]$ indicate higher filter order.
- Because of ***increased filter coefficients***, they require ***long processing time***. The ***processing time can be reduced by using FFT algorithm***.

✓ ***NOTE:***

➢ For ***frequency sampling FIR filter design,*** length of the filter is $\boldsymbol{M}$, in this case order of the filter is $\boldsymbol{N = M - 1}$.

➢ For FIR filter design by windowing methods, ***order of the filter is denoted as $\boldsymbol{N = M}$ and length of the filter is $\boldsymbol{M + 1}$***.

6. Window Method:

- Note that ***IIR filters have evolved from applying transformations of continuous-time IIR systems into discrete-time IIR systems***. In contrast, ***the design techniques for FIR filters are based on directly approximating the desired frequency response or impulse response of the discrete-time system***.

- The ***simplest method of FIR filter design*** is called the ***window method***. This method generally ***begins with an ideal desired frequency response*** that can be represented as:

$$\boldsymbol{H_d\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_d[n]e^{-j\omega n}} \qquad .....1$$

where, $h_d[n]$ = corresponding impulse response sequence, and defined as

$$\boldsymbol{h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\,d\omega} \qquad .....2$$

- We can represent equation (1) as FIR filter by ***truncating the Fourier series***. In order to obtain ***realizable (causal) FIR filter***, we can truncate the impulse response sequence of length $\boldsymbol{M}$ as:

$$\boldsymbol{h[n]} = \begin{cases} \boldsymbol{h_d[n]}, & \boldsymbol{0 \leq n \leq M} \\ \boldsymbol{0}, & \boldsymbol{Otherwise} \end{cases} \quad .....3$$

That is, $\boldsymbol{h_d[n]}$ is truncated to length $\boldsymbol{M+1}$.

- The frequency response related to equation (3) is

$$\boldsymbol{H(e^{j\omega}) = \sum_{n=0}^{M} h[n]e^{-j\omega n}} \quad .....4$$

- Since, $\boldsymbol{0 \leq n \leq M}$, we can also think of obtaining equation (4) by ***looking through the window***. For this reason, the process of obtaining $\boldsymbol{H(e^{j\omega})}$ from $\boldsymbol{H_d(e^{j\omega})}$ is called ***windowing***.

- We can represent $\boldsymbol{h[n]}$ as the ***product of*** the desired impulse response $\boldsymbol{h_d[n]}$ and a finite-duration "window" $\boldsymbol{w[n]}$; i.e.,

$$h[n] = h_d[n]\, w[n] \quad \text{.....5}$$

where, $w[n]$ =window function

➢ There are several methods to design FIR filter using windows. They are:

1. **Rectangular window**
2. **Hanning window**
3. **Hamming window**
4. **Kaiser window**
5. Bartlett window (triangular window)
6. Blackman window
7. Lancoz window
8. Tukey window

a. Rectangular Window:

- We know for window technique to design FIR filter

$$\boldsymbol{h[n] = h_d[n]\, w[n]} \qquad \text{.....1}$$

- For ***rectangular window,*** the ***window function*** $\boldsymbol{w[n]}$ is defined as

$$\boldsymbol{w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & Otherwise \end{cases}} \qquad \text{.....2}$$

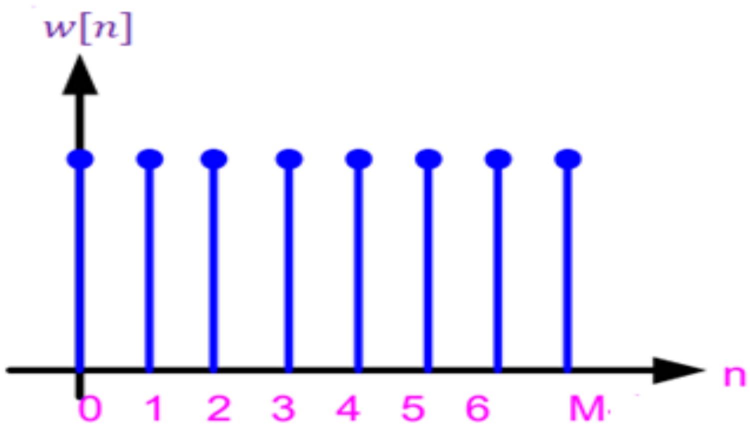


Fig: Rectangular window

- The Fourier transform of rectangular window is

$$\boldsymbol{W(\omega) = \sum_{n=0}^{M} w[n] e^{-j\omega n}} \qquad .....3$$

- But, form ***Modulation or windowing theorem of Fourier transform*** (multiplication in time domain results in convolution in frequency domain)

$$\boldsymbol{H\left(e^{j\omega}\right) = H_d\left(e^{j\omega}\right) * W\left(e^{j\omega}\right)}$$

Or, $$\boldsymbol{H\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta} \qquad .....4$$

- Thus, the ideal frequency response $\boldsymbol{H_d\left(e^{j\omega}\right)}$ is '***smoothed***' by the DTFT of the window, $\boldsymbol{W\left(e^{j\omega}\right)}$.
  (equation 4 is same for all types of windows, the window type depends on the $\boldsymbol{w[n]}$)

- The ***frequency response of rectangular window function*** $\boldsymbol{w[n]}$ is given by

$$W(\omega) = \sum_{n=0}^{M} 1\, e^{-j\omega n}$$

$$W(\omega) = \sum_{n=0}^{M} (e^{-j\omega})^n = \frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

➢ Or, $$W(e^{j\omega}) = \frac{e^{\frac{-j\omega(M+1)}{2}}\left(e^{\frac{j\omega(M+1)}{2}} - e^{\frac{-j\omega(M+1)}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}$$

Therefore, $$\boldsymbol{W(e^{j\omega}) = e^{-j\omega(\frac{M}{2})} \frac{\sin[\frac{\omega(M+1)}{2}]}{sin(\frac{\omega}{2})}, \omega \neq 0} \quad .....5$$

which is the required expression for frequency response.

- **Magnitude Response:**

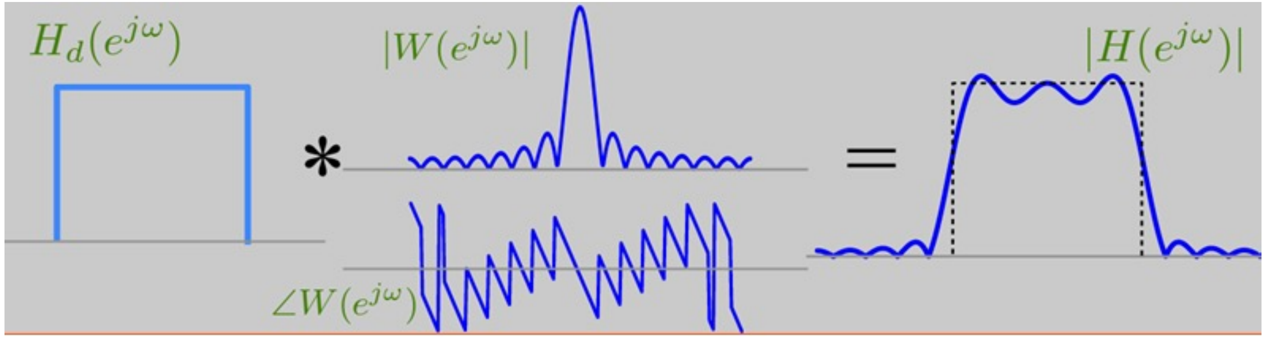
➢ From equation (5), it is obtained as

$$\boldsymbol{|W(e^{j\omega})| = \frac{\left|\sin[\frac{\omega(M+1)}{2}]\right|}{\left|sin(\frac{\omega}{2})\right|} \quad \text{for } -\pi \leq \omega \leq \pi} \quad .....6$$

- **Phase Response:**

➢ From equation (5), it is obtained as

$$\boldsymbol{\theta(e^{j\omega}) = \begin{cases} -\omega\left(\frac{M}{2}\right), & \sin(\frac{\omega M}{2}) \geq 0 \\ -\omega\left(\frac{M}{2}\right) + \pi, & \sin(\frac{\omega M}{2}) < 0 \end{cases}} \quad .....7$$



- **Discussions:**

➢ How well the frequency response, $H(e^{j\omega})$, of a filter designed with the ***window design method*** approximates a desired response ,$H_d(e^{j\omega})$, is determined by two factors:

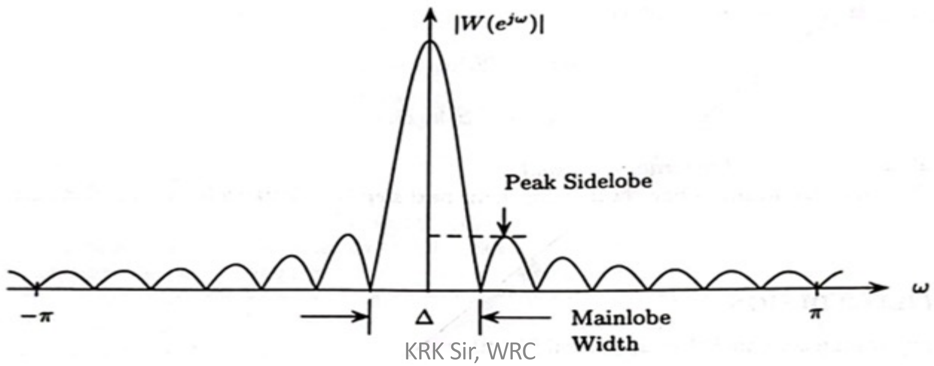
i. The width of the main-lobe of $W\left(e^{j\omega}\right)$.

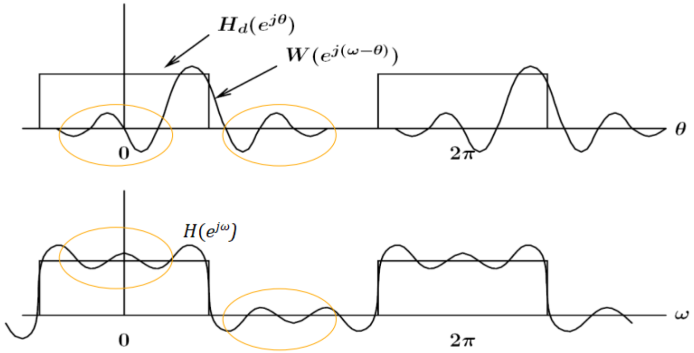
ii. The peak side-lobe amplitude of $H(e^{j\omega})$.

➢ Ideally, the ***main-lobe width should be narrow***, and the ***side-lobe amplitude should be small***.

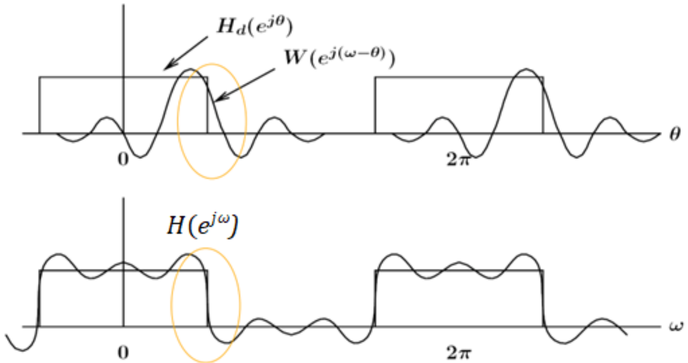
- However, for a fixed length window, these can not be minimized independently. Some ***general properties of windows*** are as follows:

i. As the length $M + 1$ ***of the window increases***, the ***width of the main-lobe decreases***, which results in a ***decrease in the transition width between passbands and stopbands***.
ii. The ***peak side-lobe amplitude of the window is determined by the shape of the window***, and it is essentially ***dependent of the window length***.
iii. If the ***window shape is changed to decrease the side-lobe amplitude***, the ***width of the main lobe will generally increase***.

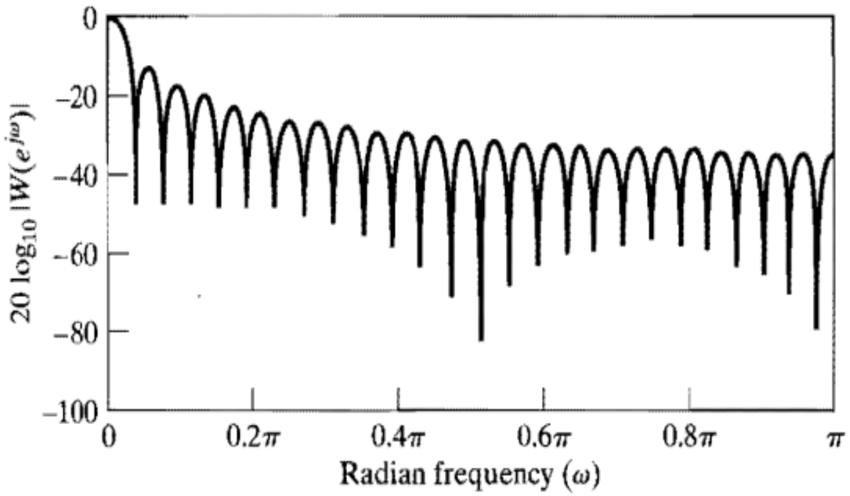




- The higher the areas under the sidelobes, the larger the ripples in the passband and stopband.



- The width of the transition region between passband and stopband in $H(e^{j\omega})$ increases with the width of the main-lobe of $W\left(e^{j\omega}\right)$.



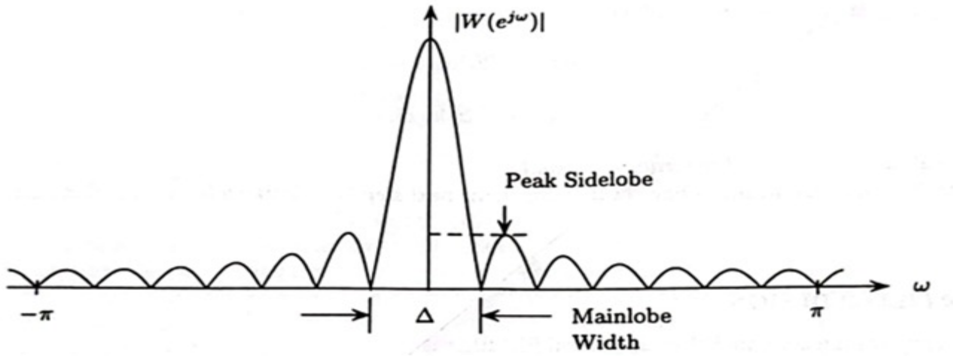
- Above figure shows the ***Fourier transforms (log magnitude) of rectangular window for order*** $\boldsymbol{M = 50\ or\ length\ M + 1 = 51}$.
- For the ***rectangular window***, the ***side lobes are large***, and in fact, as length $\boldsymbol{M + 1}$ (or order $\boldsymbol{M}$) increases, the peak amplitudes of the main lobe and the side lobes grow in a manner such that the area under each lobe is a constant while the width of each lobe decreases with $\boldsymbol{M + 1}$. This ***increases the number of ripples (oscillations or ringing) near the band edge***.

Gibbs Phenomenon:

- For the ***rectangular window***, the ***side lobes are large***, and in fact, as length of filter $\boldsymbol{M+1}$ increases, the peak amplitudes of the main lobe and the side lobes grow in a manner such that the area under each lobe is a constant while the width of each lobe decreases with $\boldsymbol{M+1}$.

- This increases the number of ripples (oscillations or ringing) near the band edge. This is due to the large side-lobes of $\boldsymbol{W\left(e^{j\omega}\right)}$. This oscillatory or ringing effect near the band edge (cutoff) is called ***Gibbs phenomenon***.

- Due to the ***abrupt discontinuity of the window function***, side-lobes are generated. In the ***rectangular window***, these side-lobes are larger in size, thus, ringing effect is maximum.

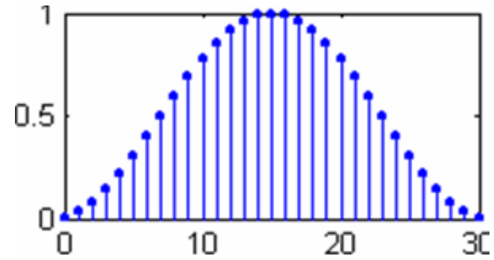


- Gibbs phenomenon or the ringing (oscillatory) effect can be reduced with the use of windows that do not contain abrupt discontinuities in their time-domain characteristics (Fourier series) , and have correspondingly low sidelobes in their frequency-domain characteristics.

b. Hanning Window:

- It is also known as ***raised cosine window***, relatively simple and the window function is given as:

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right), & 0 \le n \le M \\ 0, & Otherwise \end{cases}$$

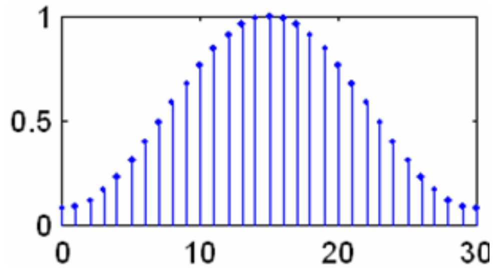


- Note that $w[n] = 0$ for $n = 0$ and $n = M$.
  Therefore, the non-zero ***length of Hanning filter*** is $M - 1$.

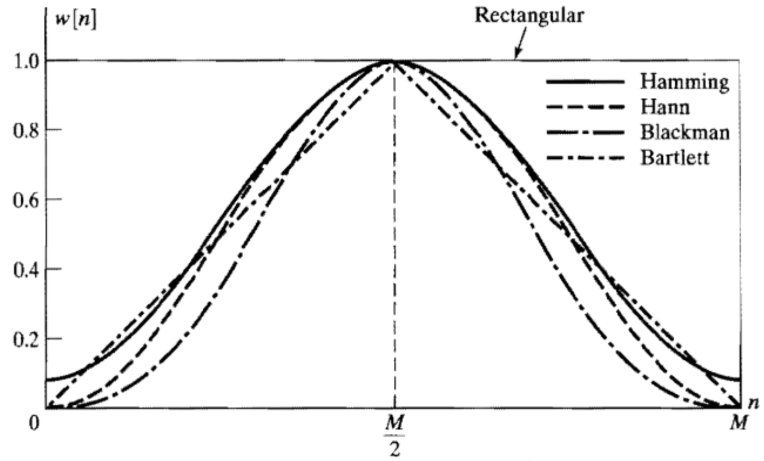
c. Hamming Window:

- It is similar to Hanning window but ***Hamming window*** has $M + 1$ non-zero terms instead of $(M - 1)$ as in the case of Hanning window.
- The window function is given by

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right), & 0 \le n \le M \\ 0, & Otherwise \end{cases}$$

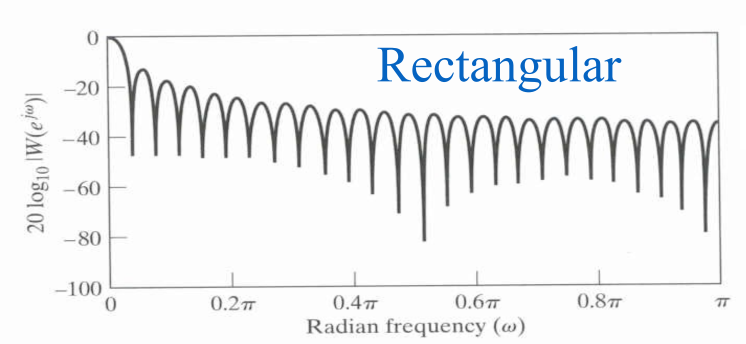


- **Properties of Commonly used Windows:**



COMPARISON OF COMMONLY USED WINDOWS

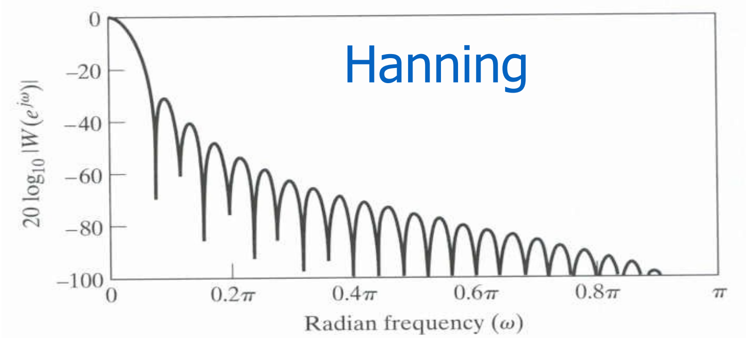
Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hann	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$



(a)

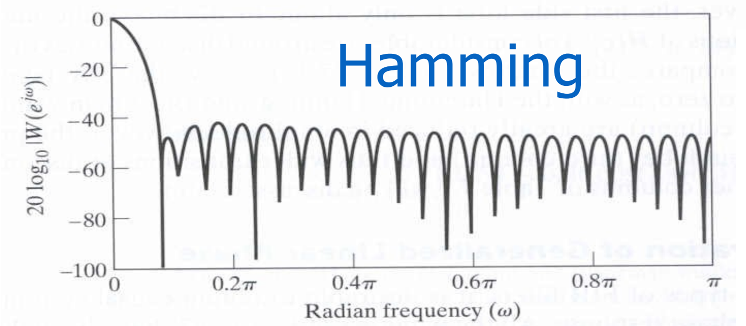
$$w[n] = \begin{cases} 1, & 0 \le n \le M \\ 0, & Otherwise \end{cases}$$

Fig(a)



$$w[n] = 0.5 - 0.5\cos(2\pi n/M)$$

Fig(b)



(d)

$$w[n] = 0.54 - 0.46\cos(2\pi n/M)$$

Frequency Spectrum of Windows

(a)- (c) attenuation of sidelobe increases, width of main-lobe increases.

Fig(c)

❑ Filter Design by Kaiser Window:

- We know that if window shape is changed to ***decrease the side-lobe amplitude***, the ***main lobe width will generally increase*** (trade-off between side-lobe amplitude and main-lobe width) . Also, increasing the main-lobe width will decrease the order of filter or length of the filter.

- In Kaiser window function, we can vary the side-lobe level with respect to main-lobe peak by varying shaping parameter $'\boldsymbol{\beta}'$. The main-lobe width can be varied by adjusting the length of order of the window which, in turn, changes the transition width for the desired filter.

- The Kaiser window is defined as

$$\boldsymbol{w[n]} = \begin{cases} \dfrac{\boldsymbol{I_0[\beta\{1-\left(\frac{n-\alpha}{\alpha}\right)^2\}^{(1/2)}]}}{\boldsymbol{I_0(\beta)}}, & \boldsymbol{0 \leq n \leq M} \\ \boldsymbol{0}, & \boldsymbol{otherwise} \end{cases} \qquad \text{.....1}$$

$$w[n] = \begin{cases} \frac{I_0[\beta\{1-\left(\frac{n-\alpha}{\alpha}\right)^2\}^{(1/2)}]}{I_0(\beta)}, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

- Where, $\alpha = \frac{M}{2}$ and $I_0(.)$ represents the zero-order modified Bessel function of the first kind.

- In contrast to the other windows, the Kaiser window has two parameters:

i. The length $(M + 1)$,
ii. A shape parameter β

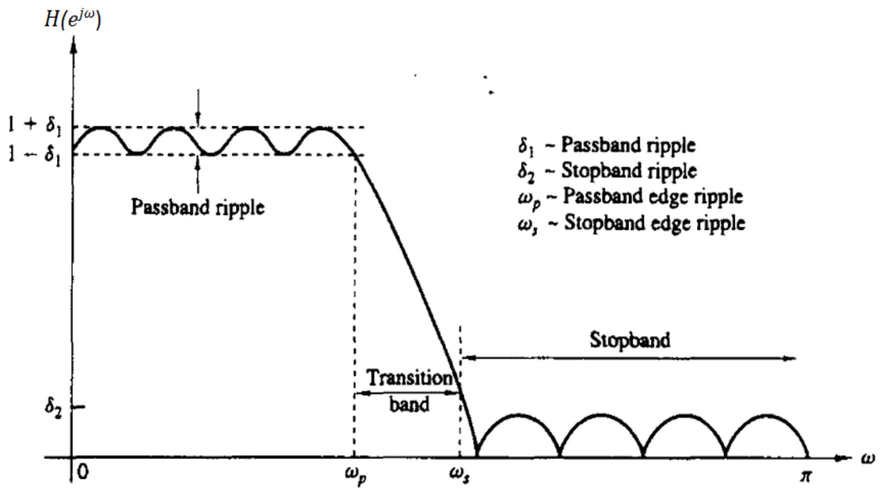
- By varying $(M + 1)$ and β, the window length and shape can be adjusted to trade side-lobe amplitude for main-lobe width.

- The modified Bessel function of the first kind is expressed by

$$I_0(x) = 1 + \sum_{r=1}^{\infty}\left(\frac{\left(\frac{x}{2}\right)^r}{r!}\right)^2 \qquad .....2$$

Or, $$I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots \qquad .....3$$

- Let the magnitude specifications of the required FIR filter be given as shown in figure.



- The ***ripple of the Kaiser window*** is given by

$$\delta = minimum(\delta_1, \delta_2) \qquad .....4$$

- The transition width $\Delta\boldsymbol{\omega}$ is given by

$$\Delta\boldsymbol{\omega} = \boldsymbol{\omega_s} - \boldsymbol{\omega_p} \qquad \textbf{.....5}$$

- Now, ***attenuation or peak approximation error*** is defined as:

$$\boldsymbol{A} = -\boldsymbol{20log_{10}(\delta)} \quad \boldsymbol{dB} \qquad \textbf{.....6}$$

- To achieve prescribed values of $\boldsymbol{A}$ and $\Delta\boldsymbol{\omega}$, $\boldsymbol{M}$ must satisfy

$$\boldsymbol{M} = \frac{\boldsymbol{A-8}}{\boldsymbol{2.285\,\Delta\omega}} \quad \pm \boldsymbol{2} \qquad .....7$$

over a wide range of values of $\boldsymbol{A}$ and $\Delta\boldsymbol{\omega}$.

- Kaiser determined empirically that the values of β (shape parameter) needed to achieve a specified value of $\boldsymbol{A}$ is given by

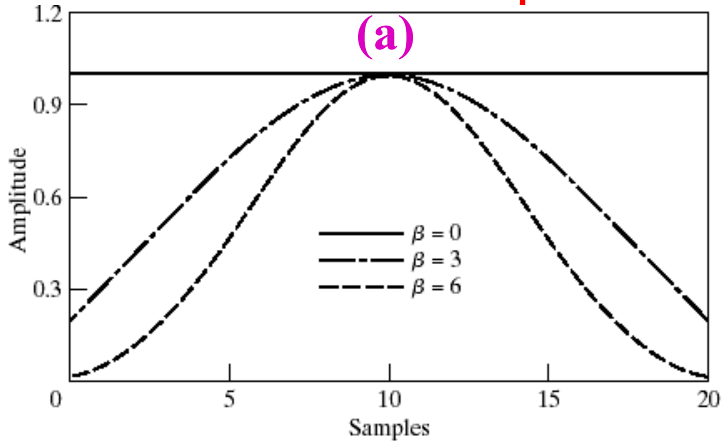
$$\boldsymbol{\beta} = \begin{cases} \mathbf{0.1102}(\boldsymbol{A}-\mathbf{8.7}), & \boldsymbol{A} > \mathbf{50} \\ \mathbf{0.5842}(\boldsymbol{A}-\mathbf{21})^{\mathbf{0.4}}+\mathbf{0.07886}(\boldsymbol{A}-\mathbf{21}), & \mathbf{21} \le \boldsymbol{A} \le \mathbf{50} \\ \mathbf{0.0}, & \boldsymbol{A} < \mathbf{21} \end{cases} \quad \text{.....8}$$

When $\boldsymbol{\beta} = \mathbf{0}$, equation (1) becomes ***rectangular window***.

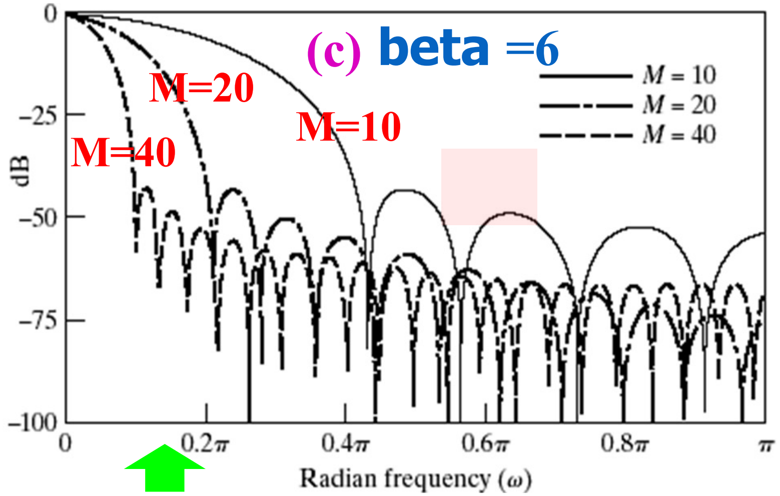
- **Continuous Envelopes of Kaiser Windows:**

- Figure (a) shown below ***continuous envelopes of Kaiser windows*** of length $\mathbf{M+1=21}$ for $\boldsymbol{\beta}=\mathbf{0,3,}$ and $\mathbf{6}$. Notice from equation (1) that the case $\boldsymbol{\beta}=\mathbf{0}$ reduces to the ***rectangular window***.
- Figure (b) shows the ***corresponding Fourier transforms of the Kaiser windows*** in figure (a).
- Figure (c) shows ***Fourier transforms of Kaiser windows*** with $\boldsymbol{\beta}=\mathbf{6}$ and $\boldsymbol{M}=\mathbf{10, 20,} \boldsymbol{and}\ \mathbf{40}$.

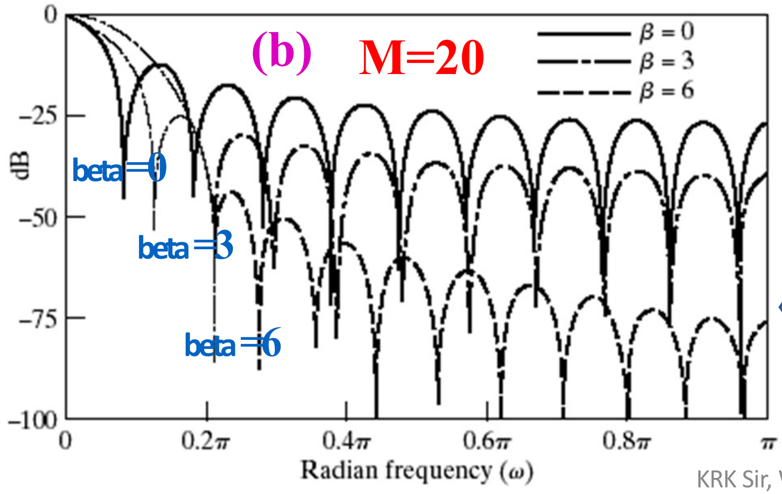
continuous envelopes of Kaiser windows(sequences)



(a) Window shape, M=20



(c) beta=6, M=10, 20, 40
As **M** increases, attenuation of sidelobe is preserved, **width of main-lobe** decreases.



(b) Frequency spectrum, M=20,
As **beta** increases, attenuation of side-lobe increases, **width of main-lobe** increases.

- If the ***window is tapered more***, the ***side lobes of the Fourier transform become smaller, but the main lobe becomes wider***.
- Figure (c) shows that increasing $\boldsymbol{M}$ while holding $\boldsymbol{\beta}$ constant causes the ***main lobe to decrease in width***, but it ***does not affect the peak amplitude of the side lobes***.

- **Examples:**

1. **Design the symmetric FIR lowpass filter for which desired response is expressed as**

$$\boldsymbol{H_d\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases}}$$

The length of the filter should be $\boldsymbol{7}$ and $\boldsymbol{\omega_c = 1\ rad/sample}$. Use rectangular window.

- **Solution:**

➢ Given, $$H_d\left(e^{j\omega}\right) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases}$$

Length of filter, $\boldsymbol{M + 1 = 7}$

Cutoff frequency, $\omega_c = 1\ rad/sample$

- **To obtain $\mathbf{h_d[n]}$:**

➢ The desired unit impulse response (unit sample response) is given by

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\, d\omega$$

Or, $$h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{-j\omega\tau}\, e^{j\omega n}\, d\omega$$

$$h_d[n] = \frac{1}{2\pi}\int_{-1}^{1}\ e^{j\omega(n-\tau)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j\,(n-\tau)}\right]_{-1}^{1}$$

$$\boldsymbol{h_d[n] = \frac{\sin(n-\tau)}{\pi(n-\tau)}\,,\ n \neq \tau}$$

For $\boldsymbol{n = \tau}$, $h_d[n] = \frac{1}{2\pi}\int_{-1}^{1}\ e^{j\omega(n-\tau)}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1}\ d\omega = \boldsymbol{\frac{1}{\pi}}$

- Therefore, $$\boldsymbol{h_d[n] = \begin{cases} \frac{\sin(n-\tau)}{\pi(n-\tau)}\,, & n \neq \tau \\ \frac{1}{\pi}, & n = \tau \end{cases}}$$

- **Determination of the value of $\boldsymbol{\tau}$**:
- ➢ The given filter is symmetric, then we have

$$h[n] = h[M-n]$$

- ➢ Also we know the impulse response of the filter is

$$h[n] = h_d[n]w[n]$$

then we can write,

$$h_d[n]w[n] = h_d[M-n]w[n]$$

- ➢ Or, $\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-n-\tau)}{\pi(M-n-\tau)}$ $\quad w[n] = 1,\ o \le n \le M$

Above condition is satisfied if

$$-(n-\tau) = M - n - \tau$$

Therefore, $\boldsymbol{\tau = \frac{M}{2}}$

Hence, $$h_d[n] = \begin{cases} \dfrac{\sin(n-\frac{M}{2})}{\pi(n-\frac{M}{2})}, & n \neq \dfrac{M}{2} \\ \dfrac{1}{\pi}, & n = \dfrac{M}{2} \end{cases}$$

- **To obtain ($h_d[n]$):**

➢ We know, $h[n] = h_d[n]w[n]$

where,

$$w[n] = \begin{cases} 1, & 0 \leq n \leq 6 \\ 0, & Otherwise \end{cases}$$

➢ Therefore,

$$h[n] = \begin{cases} h_d[n], & 0 \leq n \leq 6 \\ 0, & Otherwise \end{cases}$$

Table 8.4. Calculation of $h_d(n)$

S.No.	n	Value of Co-efficient h_d (n) According to Equation (vii)
1.	0	$h_d(0) = \frac{\sin(-3)}{-3\pi} = 0.01497$
2.	1	$h_d(1) = \frac{\sin(-2)}{-2\pi} = 0.14472$
3.	2	$h_d(2) = \frac{\sin(-1)}{-\pi} = 0.26785$
4.	3	$h_d(3) = \frac{1}{\pi} = 0.31831$
5.	4	$h_d(4) = \frac{\sin(1)}{\pi} = 0.26785$
6.	5	$h_d(5) = \frac{\sin(2)}{2\pi} = 0.14472$
7.	6	$h_d(6) = \frac{\sin(3)}{3\pi} = 0.01497$

Hence, from Table 8.4, co-efficients of FIR filter will become:

$h(0) = 0.01497$ $h(1) = 0.14472$
$h(2) = 0.26785$ $h(3) = 0.31831$
$h(4) = 0.26785$ $h(5) = 0.14472$
$h(6) = 0.01497$

- **Comments:**

➢ In the above example, we have $\tau = \frac{M}{2}$. Then

$$H_d(e^{j\omega}) = \begin{cases} 1\ e^{-j\omega\frac{M}{2}}, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases} \qquad .....1$$

➢ From the time-shifting property of DTFT, we know

$$FT\{x[n-k]\} = e^{-j\omega k} X(e^{j\omega}) \qquad .....2$$

which implies that multiplication $e^{-j\omega k}$ to the Fourier transform is equivalent to the delaying the time-domain sequence by k samples in time. Therefore,

$$|H_d(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases}$$

and $$\measuredangle H_d(e^{j\omega}) = \theta(e^{j\omega}) = -\omega\,\frac{M}{2}$$

2. **Design a lowpass filter to be used in an ADC-H(z)-DAC structure that will have a $-3\ dB$ cutoff of $30\pi\ rad/sec$ and an attenuation of $40\ dB$ at $45\pi\ rad/sec$. The filter is required to have a linear phase. Use a sampling rate of $100\ samples/sec$.**

- **Solution:**

➢ Given, $-3dB$ cutoff of $30\pi\ rad/sec$, i. e. , $\Omega_c = 30\pi\ rad/sec$
Stopband frequency, $\Omega_s = 45\pi\ rad/sec$
Passband attenuation, $\delta_1 = -3\ dB$
Stopband attenuation, $\delta_2 = -40\ dB$
Sampling frequency, $F_s = 100\ samples/sec$ or $H(z)$

▪ **Step-1: <u>Convert analog frequency specifications to equivalent digital</u>**

➢ We have, $f = \frac{F}{F_s}$, where, $F = \frac{2\pi}{\Omega}$ and $F_s = 100\ samples/sec$ or $H(z)$

Therefore, $\omega_p = \Omega_c\, T = \frac{\Omega_c}{F_s} = \frac{30\pi}{100} = 0.3\pi\ rad/sample$

And, $$\omega_s = \Omega_s\, T = \frac{\Omega_s}{F_s} = \frac{45\pi}{100} = 0.45\pi\, rad/sample$$

- Thus, the specifications of equivalent digital filter as:

$-3\ dB$ attenuation at $\omega_p = 0.3\pi\ rad/sample$
$-40\ dB$ attenuation at $\omega_s = 0.45\pi\ rad/sample$

- **Step-2: <u>Selection of the window type</u>**
- Since, the minimum stopband attenuation (i. e. , peak approximation error),

$$\delta_2 = -40\ dB$$

- But, we know ***Hanning window provides*** $\boldsymbol{-44\, dB}$ ***attenuation in stopband***. Therefore, ***we select Hanning window***.

- **Step-3: <u>Determination of the order of the filter (M)</u>**

➢ To find the order of the filter (or window), we know

$$M = k\,(\frac{2\pi}{\omega_s - \omega_p})$$

➢ But, we know from the table , for Hanning window,

$$\omega_s - \omega_p = \frac{8\pi}{M} \qquad (\frac{8\pi}{M} \text{ is the width of the main-lobe})$$

Therefore, $M = \frac{8\pi}{\omega_s - \omega_p} = \frac{8\pi}{0.45\pi - 0.3\pi} = 53.33 \approx 54$

Order of the filter, $\boldsymbol{M = 54}$

➢ Thus, the ***length of the filter*** is $\boldsymbol{M + 1 = 55}$

- **Step-4: <u>Select cutoff frequency $(\boldsymbol{\omega_s})$ and delay $(\boldsymbol{\tau})$</u>**

➢ We know, ***cutoff frequency***, $\omega_c = \omega_p = 0.3\pi\ \ rad/sample$, and

Delay, $\tau = \frac{M}{2} = \frac{54}{2} = 27$

- **Step-5: <u>Determination of desired (ideal) impulse response $h_d[n]$</u>**

➢ We know, the desired impulse response of LPF is given by

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}\, d\omega$$

where, $H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases} \quad = \begin{cases} e^{-j\omega\frac{M}{2}}, & |\omega| \le \omega_c \\ 0, & Otherwise \end{cases}$

➢ Then, $h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\frac{M}{2}}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\frac{M}{2})}\, d\omega$

$$h_d[n] = \frac{1}{2\pi}\ \frac{e^{j\omega(n-\frac{M}{2})}}{j\,(n-\frac{M}{2})}\Bigg|_{-\omega_c}^{\omega_c} = \frac{1}{2\pi}\ \left\{\frac{e^{j\omega_c\left(n-\frac{M}{2}\right)}-e^{-j\omega_c\left(n-\frac{M}{2}\right)}}{j\left(n-\frac{M}{2}\right)}\right\}$$

➢ Thus, $$h_d[n] = \frac{1}{\pi}\frac{\sin\{\omega_c\left(n-\frac{M}{2}\right)\}}{\left(n-\frac{M}{2}\right)}, n \ne \frac{M}{2}$$

- When, $n = \frac{M}{2}$, $h_d[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega(n-\frac{M}{2})}\ d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\ d\omega$

$$h_d[n] = \frac{\omega_c}{\pi}, n = \frac{M}{2}$$

- Therefore, $$\boldsymbol{h_d[n]} = \begin{cases} \dfrac{\boldsymbol{sin\{\omega_c\left(n-\frac{M}{2}\right)\}}}{\boldsymbol{\pi\left(n-\frac{M}{2}\right)}}, & \boldsymbol{n \neq \frac{M}{2}} \\ \dfrac{\boldsymbol{\omega_c}}{\boldsymbol{\pi}}, & \boldsymbol{n = \frac{M}{2}} \end{cases}$$

- **Step-6: <u>Determination of impulse response $h[n]$ by windowing</u>**

➢ We have,

$$h[n] = h_d[n]w[n]$$

where, $w[n]$ = window function (Hanning window in this case)

➢ For Hanning window, we know

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right), & 0 \le n \le M \\ 0, & Otherwise \end{cases}$$

➢ Then,

$$h[n] = h_d[n]w[n] = \begin{cases} \dfrac{\boldsymbol{sin\{\omega_c\left(n-\frac{M}{2}\right)\}}}{\boldsymbol{\pi\left(n-\frac{M}{2}\right)}}\{0.5 - 0.5\cos\left(\frac{2\pi n}{M}\right)\}, & n \ne \frac{M}{2} \\ \dfrac{\boldsymbol{\omega_c}}{\boldsymbol{\pi}}, & n = \frac{M}{2} \end{cases}$$

- By substituting the values of $\boldsymbol{\omega_c}$ and $\boldsymbol{M}$, we get

$$\boldsymbol{h[n] = h_d[n]w[n]} = \begin{cases} \frac{\boldsymbol{sin\{\omega_c(n-27)\}}}{\boldsymbol{\pi(n-27)}} \boldsymbol{\{0.5 - 0.5\, cos\left(\frac{\pi n}{27}\right)\}}, \boldsymbol{n \neq 27} \\ \boldsymbol{0.3\, \{0.5 - 0.5\, cos\left(\frac{\pi n}{27}\right)\}}, \qquad \boldsymbol{n = 27} \end{cases}$$

[for $(0 \leq n \leq 54)$]

which is the calculated (actual) impulse response using window.

3. **EXAMPLE 8.12** **Design a normalised linear phase FIR filter having the phase delay of $\tau = 4$ and at least 40 dB attenuation in the stopband. Also, obtain the magnitude/frequency response of the filter.**

Solution: First, let us obtain ω_c

The linear phase FIR filter is normalised. This means cut off frequency will be:

$$\omega_c = 1 \text{ rad/ sample}$$

Now, let us obtain length M of the filter

Phase delay, $\tau = 4$. We know that for a linear phase FIR filter, we have

$$\tau = \frac{M-1}{2}$$

Now, $\tau = 4$ gives, $\quad 4 = \frac{M-1}{2} \quad$ or $\quad M = 9$

Therefore, the length of the filter is $M = 9$.

Now, let us choose the type of the window.

- **Note:** In this example, ***length of the filter is*** $\boldsymbol{M}$ $(\boldsymbol{0\ to\ M-1})$ and the ***order of the filter is*** $\boldsymbol{M-1}$. $\boldsymbol{order\ of\ the\ filter = length\ of\ the\ filter - 1}$. **(John G. Proakis, Digital Signal Processing)** . We can also solve by defining the length and order of the filter as previous example.

From Table 8.3, Hanning window is given as under:

$$w(n) = \frac{1}{2}\left(1 - \cos\frac{2\pi n}{M-1}\right)$$

Substituting for M, above expression will take the following form:

$$w(n) = \frac{1}{2}\left(1 - \cos\frac{\pi n}{4}\right)$$

Also, h(n) may be obtained by windowing as under:

$$h(n) = h_d(n).w(n)$$

or

$$h(n) = \begin{cases} \dfrac{\sin(n-4).\left(1-\cos\frac{\pi n}{4}\right)}{2\pi(n-4)} & \text{for } n \neq 4 \\[2ex] \dfrac{1-\cos\frac{\pi n}{4}}{2\pi} & \text{for } n = 4 \end{cases} \qquad \text{...(ii)}$$

for n = 0 to 8

This is the unit sample response of the desired linear phase FIR filter.

Now, let us obtain the magnitude (frequency) response.

Here, order of the filter is, M = 9 i.e., odd. For odd M, the magnitude/frequency response of FIR filters is given by:

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n - \frac{M-1}{2}\right)$$

Substituting for M = 9, above expression becomes,

$$|H(\omega)| = h(4) + 2\sum_{n=0}^{3} h(n)\cos\omega(n-4)$$

Values of h (n) can be substituted in above expression from equation (ii). Then $|H(\omega)|$ may be obtained for various values of ω.

Therefore,

Window used: *Hanning window*

Now, let us obtain desired unit sample response $h_d(n)$.

In example 8.11, we have observed that h_d (n) has been obtained for an ideal low-pass filter of cut-off frequency ω_c. Such h_d (n) is given by equation (iv) in previous example as under :

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \dfrac{M-1}{2}\right)\right]}{\pi\left(n - \dfrac{M-1}{2}\right)} & \text{for } n \neq \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M-1}{2} \end{cases}$$

In above expression, we have to substitute $\omega_c = 1$ rad/sample and $\dfrac{M-1}{2} = \tau = 4$. This means that

$$h_d(n) = \begin{cases} \dfrac{\sin(n-4)}{\pi(n-4)} & \text{for } n \neq 4 \\ \dfrac{1}{\pi} & \text{for } n = 4 \end{cases} \qquad \text{...(i)}$$

Now, let us obtain h(n) by windowing.

In this example, we have selected Hanning window.

Now, let us obtain magnitude response

Here, length of the unit sample response is M = 7, i.e., odd.

For odd M, the magnitude response is given by:

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n - \frac{M-1}{2}\right)$$

For $M = 7$, $|H(\omega)| = h(3) + 2\sum_{n=0}^{2} h(n)\cos\omega(n-3)$

$$= h(3) + 2[h(0)\cos\omega(-3) + h(1)\cos\omega(1-3) + h(2)\cos\omega(2-3)]$$

$$= h(3) + 2[h(0)\cos(3\omega) + h(1)\cos(2\omega) + h(2)\cos(\omega)]$$

Substituting values in above expression, we shall have

$$|H(\omega)| = 0.31831 + 2[-0.04462\cos(3\omega) - 0.26517\cos(2\omega) + 0.02159\cos(\omega)]$$

$$= 0.31831 - 0.08924\cos(3\omega) - 0.53034\cos(2\omega) + 0.04318\cos(\omega)$$

The frequency response may be obtained by substituting $-\pi \le \omega \le \pi$ in the above expression.

4. **Design an FIR linear phase FIR filter using Kaiser window to meet the following specifications.**

$$\mathbf{0.99 \le \left|H(e^{j\omega})\right| \le 1.01, \qquad 0 \le |\omega| \le 0.4\pi}$$
$$\mathbf{0 \le \left|H(e^{j\omega})\right| \le 0.001, \qquad 0.6\pi \le |\omega| \le \pi}$$

- **Solution:**
- **Step-1:** **<u>Determination of the specifications</u>**
- Comparing the given equation with the equation

$$\mathbf{1-\delta_1 \le \left|H(e^{j\omega})\right| \le 1+\delta_1, \qquad 0 \le |\omega| \le \omega_p}$$
$$\mathbf{0 \le \left|H(e^{j\omega})\right| \le \delta_2, \qquad \omega_s \le |\omega| \le \pi}$$

we get, $\delta_1 = 0.01$ $\qquad \delta_2 = 0.001$

$\omega_p = 0.4\pi$ $\qquad \omega_s = 0.6\pi$

- Now, transition width, $\Delta\boldsymbol{\omega} = \omega_s - \omega_p = 0.6\pi - 0.4\pi = 0.2\pi$
  Minimum value of attenuation, δ = minimum (δ_1, δ_2)= $\delta_2 = 0.001$
  Then, the attenuation is $\mathrm{A} = -20\log_{10}(0.001) = 60\ dB$

- **Step-2: <u>Determination of cutoff frequency</u>**

- We know,

$$\omega_c = \frac{\omega_s + \omega_p}{2} = \frac{0.6\pi + 0.4\pi}{2} = 0.5\pi$$

- **Step-3: <u>Determination of Kaiser window parameters $\boldsymbol{\beta}$ and $\boldsymbol{M}$</u>**

- We know, the **order** of the filter $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \frac{\boldsymbol{A-8}}{\mathbf{2.285}\ \Delta\boldsymbol{\omega}} = \frac{\mathbf{60-8}}{\mathbf{2.285}\times 0.2\pi} = \mathbf{36.219} \approx \mathbf{37}$$

  Therefore, order $\boldsymbol{M} = \mathbf{37}$

➢ The shape parameter $\boldsymbol{\beta}$ is given by

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A - 21)^{0.4} + 0.07886(A - 21), & 21 \leq A \leq 50 \\ 0.0, & A < 21 \end{cases}$$

Since, $A = 60\ dB$

$$\beta = 0.1102(A - 8.7) = 0.1102(60 - 8.7) = 5.653$$

- **Step-4: <u>Determination of the expression of Kaiser window</u>**

➢ We know that, the Kaiser window expression is given by

$$w[n] = \begin{cases} \frac{I_0[\beta\{1-\left(\frac{n-\alpha}{\alpha}\right)^2\}^{(1/2)}]}{I_0(\beta)}, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

Putting $\alpha = \frac{M}{2} = \frac{37}{2} = 18.5,\ \beta = 5.653$ and $M = 37$, we get

Or,

$$w[n] = \begin{cases} \dfrac{I_0[5.653\{1-\left(\frac{n-18.5}{18.5}\right)^2\}^{(1/2)}]}{I_0(5.653)}, & 0 \leq n \leq 37 \\ 0, & otherwise \end{cases}$$

- **Step-5: <u>Determination of $h_d[n]$</u>**

➢ The desired impulse response $h_d[n]$ of ideal LPF is

$$h_d[n] = \begin{cases} \dfrac{sin\{\omega_c\left(n-\frac{M}{2}\right)\}}{\pi\left(n-\frac{M}{2}\right)}, & n \neq \dfrac{M}{2} \\ \dfrac{\omega_c}{\pi}, & n = \dfrac{M}{2} \end{cases}$$

where, $M = 37$ and $\frac{M}{2} = 18.5$. Also, $\omega_c = 0.5\pi$, then we obtain

➢ Therefore,

$$h_d[n] = \begin{cases} \frac{sin\{0.5\pi(n-18.5)\}}{\pi(n-18.5)}, & n \neq 18.5 \\ 0.5\ , & n = 18.5 \end{cases}$$

where, $M = 37$ and $\frac{M}{2} = 18.5$. Also, $\omega_c = 0.5\pi$, then we obtain

- **Step-6: <u>Determination of impulse response $\boldsymbol{h[n]}$ by windowing</u>**

➢ We have,

$$h[n] = h_d[n]w[n]$$

where, $w[n] =$ Kaiser window function

➢ Therefore, $\boldsymbol{h[n]} = \begin{cases} \left[\frac{\mathbf{sin\{0.5\pi(n-18.5)\}}}{\boldsymbol{\pi(n-18.5)}}\right]\left[\frac{\boldsymbol{I_0}[\mathbf{5.653}\{\mathbf{1}-\left(\frac{\boldsymbol{n}-\mathbf{18.5}}{\mathbf{18.5}}\right)^{\mathbf{2}}\}^{(\mathbf{1}/\mathbf{2})}]}{\boldsymbol{I_0}(\mathbf{5.653})}\right], & \mathbf{0 \leq \boldsymbol{n} \leq 37} \\ \mathbf{0}, & \boldsymbol{otherwise} \end{cases}$

Which is the required equation of $\boldsymbol{h[n]}$ of FIR filter using Kaiser window.

❑ Filter Design by Frequency Sampling Method:

- In the frequency sampling method for FIR filter design, we specify ***the desired frequency response at a set of equally spaced frequencies***, namely

$$\omega_k = \frac{2\pi}{M}(k+\alpha) \qquad k = 0,1,\ldots\ldots\frac{M-1}{2} \text{ for } \boldsymbol{M} \text{ odd.}$$
$$k = 0,1,\ldots\ldots\frac{M}{2}-1 \text{ for } \boldsymbol{M} \text{ even.}$$
$$\alpha = 0 \text{ or } \frac{1}{2}. \qquad \ldots..1$$

and ***solve for the*** $\boldsymbol{h}[\boldsymbol{n}]$ of the FIR filter. ($\boldsymbol{M}$ is the length of filter)

- To ***reduce sidelobes***, it is desirable to ***optimize the frequency specification in the transition band of the filter***. This optimization can be accomplished numerically on a digital computer by means of linear programming techniques.

- Let us begin with the ***desired frequency response of the FIR filter***, which is [for simplicity, we drop the subscript in $H_d(e^{j\omega})$],

$$\mathbf{H}(e^{j\omega}) = H(\omega) = \sum_{n=0}^{M-1} h[n]\, e^{-j\omega n} \qquad .....2$$

- Suppose that we specify the ***frequency response of the filter*** at the frequencies given by equation (1). Then from equation (2), we obtain

$$H[k+\alpha] = H[\frac{2\pi}{M}(k+\alpha)]$$

$$H[k+\alpha] = \sum_{n=0}^{M-1} h[n] e^{-j2\pi(k+\alpha)n/M}, k = 0, 1, \dots, M-1 \qquad .....3$$

- The set of values $\frac{2\pi}{M}(k+\alpha)$ are called the ***frequency samples of*** $\mathbf{H}(e^{j\omega})$. In the case, where $\alpha = 0$, $\{H(k)\}$ corresponds to the M **-point DFT of** $\{h[n]\}$.

➢ The ***impulse response*** $\boldsymbol{h[n]}$ of above equation is

$$\boldsymbol{h[n] = \frac{1}{M}\sum_{k=0}^{M-1} H[k+\alpha]e^{j2\pi(k+\alpha)n/M}, n = 0, 1, \dots, M-1} \qquad .....4$$

When $\boldsymbol{\alpha = 0}$, above equation is simply the ***IDFT of*** $\boldsymbol{H(k)}$.

➢ Thus, the ***unit sample response of FIR filter of length*** $\boldsymbol{M}$ ***is obtained by using frequency sampling technique***.

➢ We can realize the FIR filter if $\{\boldsymbol{h[n]}\}$ is real. Since, $\{\boldsymbol{h[n]}\}$ is real, we can easily show that the frequency samples $\{\mathbf{H}(\boldsymbol{k+\alpha})\}$ satisfy the ***symmetry condition***

$$\mathbf{H}(\boldsymbol{k+\alpha}) = \boldsymbol{H^*(M-k-\alpha)} \qquad .....5$$

➢ This symmetry condition, along with the symmetry conditions for $\{\boldsymbol{h(n)}\}$, can be used to reduce the frequency specifications from $\boldsymbol{M}$ **points to** $\boldsymbol{(M+1)/2}$ **points for** $\boldsymbol{M}$ **odd** and $\boldsymbol{M/2}$ **points for** $\boldsymbol{M}$ **even**. Thus the linear equations for determining $\{\boldsymbol{h(n)}\}$ from $\{\mathbf{H}(\boldsymbol{k+\alpha})\}$ are considerably simplified.

- For $\boldsymbol{\alpha} = \mathbf{0}$, equation (5) becomes

$$\mathbf{H}(\boldsymbol{k}) = \boldsymbol{H}^*(\boldsymbol{M} - \boldsymbol{k}) \qquad .....6$$

Or,
$$\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k}) = \boldsymbol{H}^*(\boldsymbol{k})$$

We know, $\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k})\boldsymbol{e}^{\boldsymbol{j2\pi n(M-k)}\ /\boldsymbol{M}} = \boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k})\boldsymbol{e}^{-\boldsymbol{j2\pi kn/M}}$ 7

- In terms of magnitude, we can write , $|\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k})| = |\mathbf{H}(\boldsymbol{k})|$. This can be verified from the theory of DFT. The magnitude of DFT from $\mathbf{0}$ to $\boldsymbol{\pi}$ is the same as $\boldsymbol{\pi}$ to $2\boldsymbol{\pi}$. Thus, we can write,

$$\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k})\boldsymbol{e}^{\boldsymbol{j2\pi n(M-k)}\ /\boldsymbol{M}} = \boldsymbol{H}(\boldsymbol{k})\boldsymbol{e}^{-\boldsymbol{j2\pi kn/M}} \qquad .....8$$

- The term $\boldsymbol{H}(\boldsymbol{k})\boldsymbol{e}^{-\boldsymbol{j2\pi kn/M}}$ is complex conjugate of $\boldsymbol{H}^*(\boldsymbol{k})\boldsymbol{e}^{\boldsymbol{j2\pi kn/M}}$.
Thus, $\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k})\boldsymbol{e}^{\boldsymbol{j2\pi n(M-k)}\ /\boldsymbol{M}}$ is also the complex conjugate of $\boldsymbol{H}^*(\boldsymbol{k})\boldsymbol{e}^{\boldsymbol{j2\pi kn/M}}$
That is,

$$\boldsymbol{H}(\boldsymbol{M} - \boldsymbol{k}) = \boldsymbol{H}^*(\boldsymbol{k})$$

- Using the relation of complex conjugate terms, the equation (4) is simplified to:

$$\boldsymbol{h}[\boldsymbol{n}] = \frac{\mathbf{1}}{\boldsymbol{M}}\left[\boldsymbol{H}[\mathbf{0}] + \mathbf{2}\sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{r}} \boldsymbol{Re}\left\{\boldsymbol{H}[\boldsymbol{k}]\boldsymbol{e}^{\boldsymbol{j2\pi kn}}\right\}\right] \qquad .....9$$

where,

$$\boldsymbol{r} = \begin{cases} \frac{\boldsymbol{M}-\mathbf{1}}{\mathbf{2}}, & \boldsymbol{if\ \ M\ \ odd} \\ \frac{\boldsymbol{M}}{\mathbf{2}} - \mathbf{1}, & \boldsymbol{if\ \ M\ \ even} \end{cases} \qquad .....10$$

- Equation (9) is obtained by combining complex conjugate terms in equation (4) and is used to compute coefficients of FIR filter.

- **Filter Design Using Optimum Approximation, Remez Exchange Algorithm: (Assignment)**

Different Methods of Optimum Filter Designing by Approximation

- Weighted-least-squares method
- Chebyshev method
- WLS-Chebyshev method
- Parks-Mcclellan algorithm

Optimum Filter Design

- Though ´windowing method is simple, it is not the most effective

- Rectangular windowing is optimum In one sense since they minimise the mean squared approximation error to desired response, but causes errors around discontinuities

- Most popular alternative method: Parks–McClellan Algorithm

- Uses minimax error method for function approximation

Parks- McClellan Method

- The resulting filters minimize the maximum error between the desired frequency response and the actual frequency response by spreading the approximation error uniformly over each band

- Such filters that exhibit equiripple behavior in both the passband and the stopband, and are sometimes called equiripple filters

Alternation Theorem

- The polynomial of degree L that minimizes the maximum error will have at least L+2 extrema.

- The optimal frequency response will just touch the maximum ripple bounds.

- Extrema must occur at the pass and stop band edges and at either $\omega=0$ or π or both.

- Extrema must occur at the pass and stop band edges and at either $\omega=0$ or π or both.

- The derivative of a polynomial of degree L is a polynomial of degree L-1, which can be zero in at most L-1 places. So the maximum number of local extrema is the L-1 local extrema plus the 4 band edges. That is L+3.

- The alternation theorem doesn't directly suggest a method for computing the optimal filter

Conclusion

- FIR filters allow the design of *linear phase* filters, which eliminate the possibility of signal phase distortion

- Two methods of linear phase FIR design were discussed:
  - -The ideal window method
  - -The optimal Parks-McClellan method

- FIR is advantageous due to linearity and stability

- The disadvantages of FIR include expensiveness and that the process is time consuming

Now, let us obtain magnitude response
Here, length of the unit sample response is M = 7, i.e., odd.
For odd M, the magnitude response is given by:

$$|H(\omega)| = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{\frac{M-3}{2}} h(n)\cos\omega\left(n - \frac{M-1}{2}\right)$$

For $M = 7$, $|H(\omega)| = h(3) + 2\sum_{n=0}^{2} h(n)\cos\omega(n-3)$

$$= h(3) + 2\,[h(0)\cos\omega(-3) + h(1)\cos\omega(1-3) + h(2)\cos\omega(2-3)]$$

$$= h(3) + 2\,[h(0)\cos(3\omega) + h(1)\cos(2\omega) + h(2)\cos(\omega)]$$

Substituting values in above expression, we shall have

$$|H(\omega)| = 0.31831 + 2[-0.04462\cos(3\omega) - 0.26517\cos(2\omega) + 0.02159\cos(\omega)]$$

$$= 0.31831 - 0.08924\cos(3\omega) - 0.53034\cos(2\omega) + 0.04318\cos(\omega)$$

The frequency response may be obtained by substituting $-\pi \le \omega \le \pi$ in the above expression.

EXAMPLE 8.14 **Design an FIR linear phase filter using Kaiser window to meet the following specifications:**

$$0.99 \le |H(e^{j\omega})| \le 1.01, \quad \text{for } 0 \le |\omega| \le 0.19\pi$$

$$|H(e^{j\omega})| \le 0.01, \quad \text{for } 0.21\pi \le |\omega| \le \pi \quad \text{...(i)}$$

Solution: The given specifications may be rewritten as under:

$$1 - 0.01 \le |H(e^{j\omega})| \le 1 + 0.01, \quad \text{for } 0 \le |\omega| \le 0.19\pi$$

$$|H(e^{j\omega})| \le 0.01, \text{ for } 0.21\pi \le |\omega| \le \pi$$

On comparing the above given specifications with equation (8.32), we find that

$$\delta_1 = 0.01, \quad \delta_2 = 0.01$$

$$\omega_p = 0.19\pi, \quad \omega_s = 0.21\pi$$

Therefore, transition width $\Delta\omega$ would be:

$$\Delta\omega = \omega_s - \omega_p$$

or $$\Delta\omega = 0.21\pi - 0.19\pi = 0.02\pi$$

And the minimum value of ripple is given by:

$$\delta = \text{minimum of } \delta_1 \text{ and } \delta_2$$

Therefore, $\delta = 0.01$

Using equation (8.51), the value of A may be calculated as under:

$$A = -20 \log_{10} \delta = -20 \log_{10} 0.01 = 40$$

Now, let us determine cut-off frequency ω_c

The cut-off frequency of $H_d(\omega)$ is required. We are given passband edge frequency ω_p and stopband edge frequency ω_s.

For such characteristic, cut-off frequency is given by:

$$\omega_c = \frac{\omega_p + \omega_s}{2} \quad \text{...(ii)}$$

Substituting the values in above expression, we get

$$\omega_c = \frac{0.19\,\pi + 0.21\,\pi}{2} = 0.2\pi$$

Now, let us obtain values of β and M.

Here, A = 40, which lies in the range of 21 to 50. Hence, β may be obtained as under:

$$\beta = 0.5842\,(A - 21)^{0.4} + 0.07886\,(A - 21)$$

or $$\beta = 0.5842\,(40 - 21)^{0.4} + 0.07886\,(40 - 21)$$

or $$\beta = 3.395$$

We know that 'M' is given by

$$M = \frac{A-8}{2.285\,\Delta\omega} = \frac{40-8}{2.285(0.02\pi)} = 222.88$$

Here, we select the value of M as 223. It may be noted that M is very large since transition band is very small, i.e., 0.02π.

Now, let us obtain the expression of Kaiser window.

We know that $\alpha = \frac{M}{2} = \frac{223}{2} = 111.5$

and $\beta = 3.395$

Substituting the values in equation (8.47), the window will be defined as under:

$$w(n) = \begin{cases} \dfrac{I_0\left\{3.395\left[1-\left(\dfrac{n-111.5}{111.5}\right)^2\right]^{\frac{1}{2}}\right\}}{I_0(3.395)} & \text{for } 0 \le n \le 223 \\ 0 & \text{elsewhere} \end{cases}$$

$w(n) = \dfrac{I_0(\beta[1.}{I_0(\beta)}$

The function I_0 ( ) in the above expression may be calculated with the help of equation (8.48). Since. it is an infinite series, maximum possible terms must be taken to reduce the error.

Now, let us obtain $h_d(n)$.

This is low-pass type of filter. The ideal desired frequency response is given by:

$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & \text{for} \quad -\omega_c \le \omega \le \omega_c \\ 0 & \text{otherwise} \end{cases}$$

Now, $h_d(n)$ may be obtained by taking inverse Fourier transform of above equation, i.e.,

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n-\dfrac{M-1}{2}\right)\right]}{\pi\left(n-\dfrac{M-1}{2}\right)} & \text{for} \quad n \ne \dfrac{M-1}{2} \\ \dfrac{\omega_c}{\pi} & \text{for} \quad n = \dfrac{M-1}{2} \end{cases}$$

At the beginning, it was stated that length of the filter is M + 1. The above equation is derived for filter length M. Hence, we should replace M by 'M + 1' in above equation.

This means that

$$h_d(n) = \begin{cases} \dfrac{\sin\left[\omega_c\left(n - \frac{M}{2}\right)\right]}{\pi\left(n - \frac{M}{2}\right)} & \text{for } n \neq \dfrac{M}{2} \\ \dfrac{\omega_c}{\pi} & \text{for } n = \dfrac{M}{2} \end{cases}$$

Here, M = 223, hence, $\frac{M}{2} = 111.5$. Therefore, $h_d(n)$ cannot be calculated for $n = \frac{M}{2}$ since 111.5 is not an integer.

In other words, filter length is M +1 = 223 + 1 = 224, i.e., even.

Therefore, $h_d(n)$ will be given by:

$$h_d(n) = \frac{\sin\left[\omega_c\left(n - \frac{M}{2}\right)\right]}{\pi\left(n - \frac{M}{2}\right)} \quad \text{for } 0 \leq n \leq m$$

Substituting for $\omega_c = 0.2\,\pi$ and $M = 223$, the last expression becomes,

$$h_d(n) = \frac{\sin\left[0.2\pi\left(n - \frac{223}{2}\right)\right]}{\pi\left(n - \frac{223}{2}\right)}$$

or $$h_d(n) = \frac{\sin\left[0.2\pi\,(n - 111.5)\right]}{\pi(n - 111.5)} \quad \text{for} \quad 0 \le n \le 223$$

Here, it may be noted that above expression also stands for linear phase requirement. This expression gives desired unit sample response of low-pass filter.

Lastly, let us obtain h(n).

The unit sample response of FIR filter may be obtained by windowing, i.e.,

$$h(n) = h_d(n)\, w(n)$$

or $$h(n) = \begin{cases} \dfrac{\sin[0.2\pi\,(n - 111.5)]}{\pi\,(n - 111.5)} \; \dfrac{I_0\left\{3.395\left[1 - \left(\dfrac{n - 111.5}{111.5}\right)^2\right]^{\frac{1}{2}}\right\}}{I_0(3.395)} & \text{for } 0 \le n \le 223 \\ 0 & \text{otherwise} \end{cases}$$

This is the required relationship for unit sample response of FIR filter using Kaiser window.