Digital Signal Analysis And Processing

Surendra K.C.

6. FIR Filter Design 7Hrs

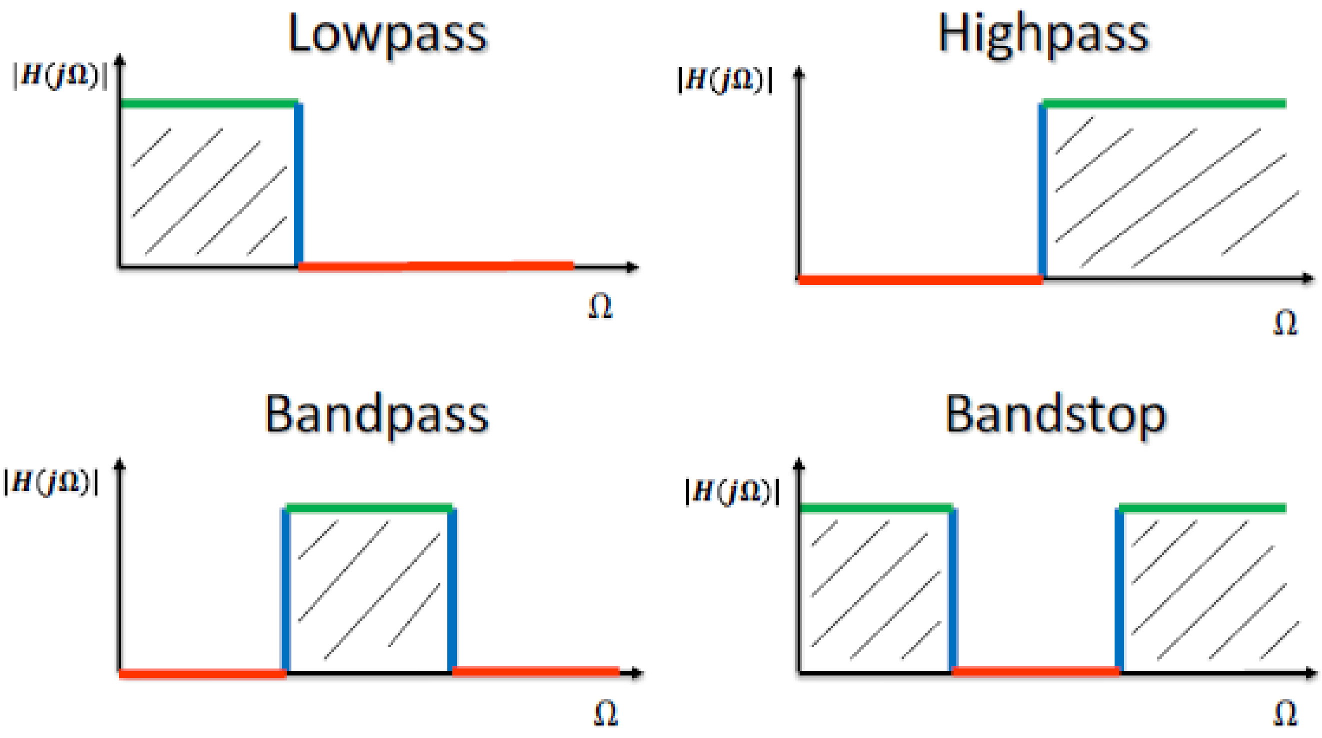
6.1 Filter Design by Window Method (Rectangular Window, Hanning Window, Hamming Window)

6.2 Filter Design by Kaiser Window

6.3 Filter Design by Frequency Sampling

6.4 Filter Design using optimum approximation, Remez exchange algorithm

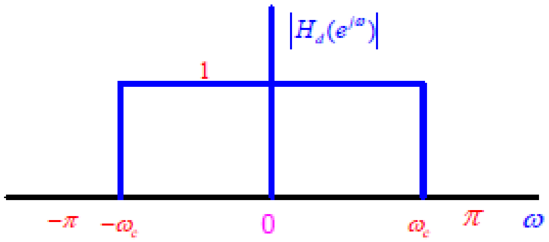
Four Types of Ideal Filters (Continuous-Time)



Ideal Discrete-Time Lowpass Filter

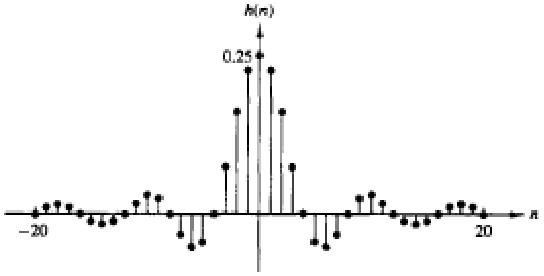
- An ideal lowpass filter is given by

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$



Ideal low pass filter

- Which specifies both magnitude and phase
  - Unity gain for the whole range of $\omega \in (0, \omega_c)$
  - Complete suppression for $\omega \in (\omega_c, \pi)$
  - Step change if the frequency response at $\omega = \omega_c$



Unit sample response

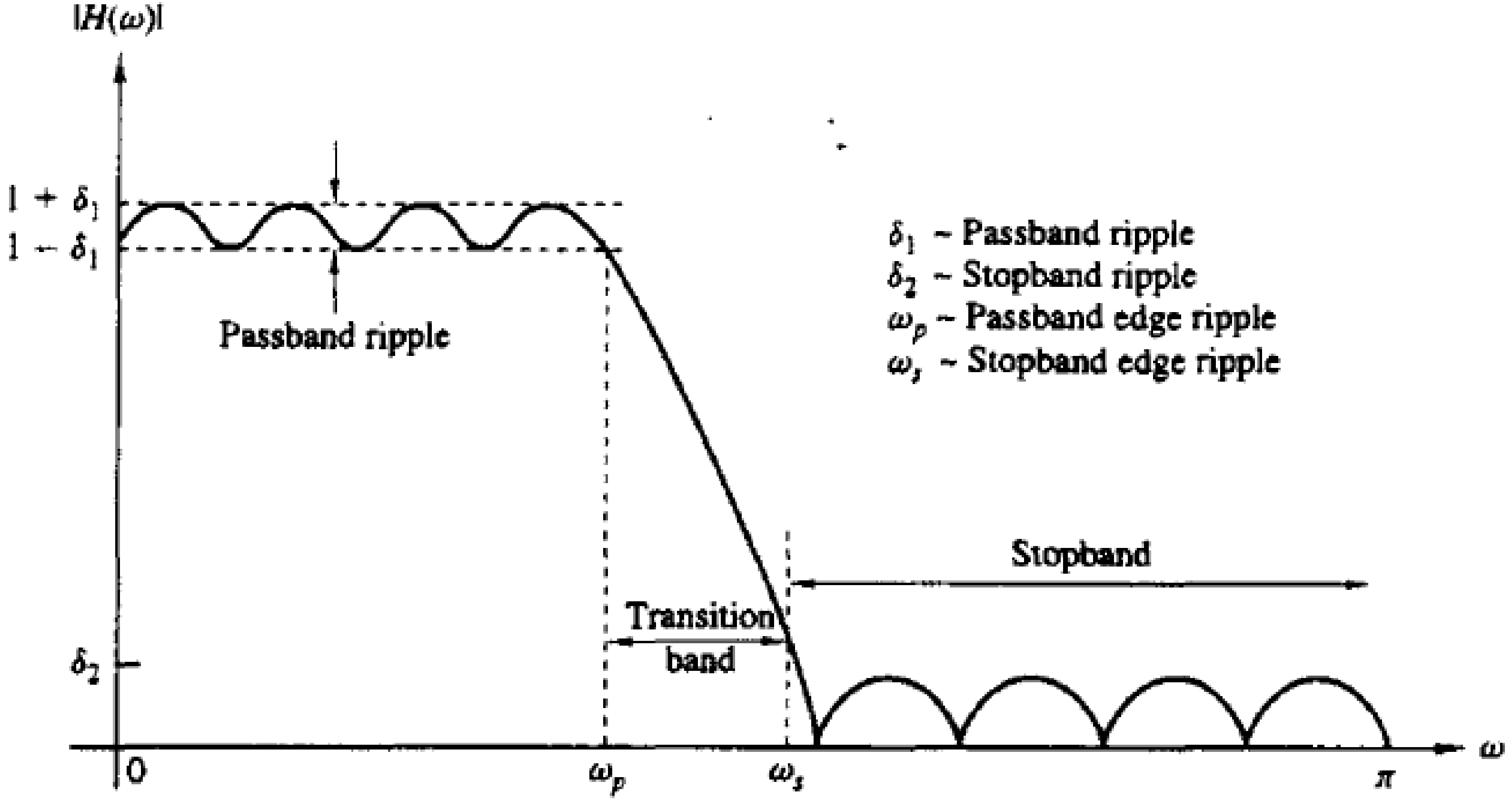
- The impulse response is given by

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H(\omega) e^{j\omega n} d\omega = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}, & n \ne 0 \end{cases}$$

- The plot of $h(n)$ for $\omega_c = {}^{\pi}/_{4}$ is shown in figure.
- It is clear that the ideal low-pass filter is non-causal and hence it cannot be realized in practice.

Practical Magnitude Response

- One possible solution is to introduce a large delay n_0 in $h(n)$ and arbitrarily to set $h(n) = 0$ for $n < n_0$.
- However, the resulting system no longer has an ideal frequency response characteristic.
- Although the frequency response characteristics possessed by ideal filters may be desirable, they are not absolutely necessary in most practical applications.
- Ideal filters are non-causal, hence physically unrealizable for real time signal processing applications.
- Causality implies that the frequency response characteristics $H(\omega)$ of the filter cannot be zero, except at finite set of points in the frequency range.
- Also $H(\omega)$ cannot have an infinitely sharp cutoff from passband to stopband, that is $H(\omega)$ cannot drop from unity to zero abruptly.
- It is not necessary to insist that the magintude be constant in the entire passband of the filter. A small amount of ripple in the passband is usually tolerable.
- The filter response may not be zero in the stopband. It may have samll nonzero value or ripple.
- The transition of the frequency response from passband to stopband defines transition band.



$|H(\omega)|$
$1+\delta_1$
$1-\delta_1$
Passband ripple
δ_1 ~ Passband ripple
δ_2 ~ Stopband ripple
ω_p ~ Passband edge ripple
ω_s ~ Stopband edge ripple
Stopband
Transition band
δ_2
0
ω_p
ω_s
π
ω

6.1 Design of FIR Filters by Windowing

- The simplest method of FIR filter design is called the Window method.
- In this method we begin with the desired frequency response sepcification $H_d(e^{j\omega})$ and determine the corresponding unit sample response $h_d(n)$.
- Indeed $h_d(n)$ is related to $H_d(e^{j\omega})$ by the Fourier transform relation as

$$H_d\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} h_d(n)e^{-j\omega n}$$

Where

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})e^{j\omega n}d\omega$$

- Thus, given $H_d(e^{j\omega})$, we can determine the unit sample response $h_d(n)$ by evaluation the above integral.
- In general, the unit sample response $h_d(n)$ obtained from this integration is infinite duration and must be truncated at some point, say $n = M$, to yield an FIR filter of range $M + 1$.

- Truncation of $h_d(n)$ to a length M is equivalent to multiplying $h_d(n)$ by a "rectangular window", defined as

$$w(n) = \begin{cases} 1, & n = 0,1, \dots M \\ 0, & otherwise \end{cases}$$

- Thus the unit sample response of the FIR filter becomes

$$h(n) = h_d(n)w(n)$$

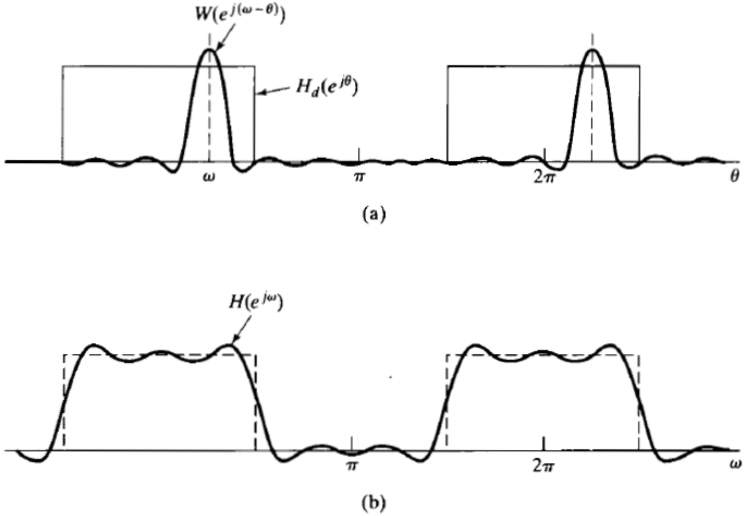
$$= \begin{cases} h_d(n), & n = 0,1, \dots, M \\ 0, & otherwise \end{cases}$$

- The multiplication of the window function $w(n)$ with $h_d(n)$ is equivalent to convolution of $H_d(e^{j\omega})$ with $W(e^{j\omega})$, where $W(e^{j\omega})$ is the frequency-domain representation (Forurier transform) of the window function, that is from modulatio or windowing therorem.

$$H\left(e^{j\omega}\right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta$$

- The windowed version is smeared version of desired response.

- Figure (a) depicts typical function $H_d(e^{j\theta})$ and $W(e^{j(\omega-\theta)})$. While figure (b) response from windowing the ideal impulse response.
- If $w(n) = 1$ for all n (i.e., if we do not truncate at all), $W(e^{j\omega})$ is a periodic impulse train with period 2π, and therefore, $H(e^{j\omega}) = H_d(e^{j\theta})$.
- This suggest that if $w(n)$ is chosen so that $W(e^{j\omega})$ is concentrated in narrow band of frequencies around $\omega = 0$, then $H(e^{j\omega})$ will "look like" $H_d(e^{j\theta})$, except where $H_d(e^{j\theta})$ changes abruptly.



a. Rectangular Window

- This is the simplest window function but provides the worst performance from the viewpoint of stopband attenuation.

$$w(n) = \begin{cases} 1, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

Thus

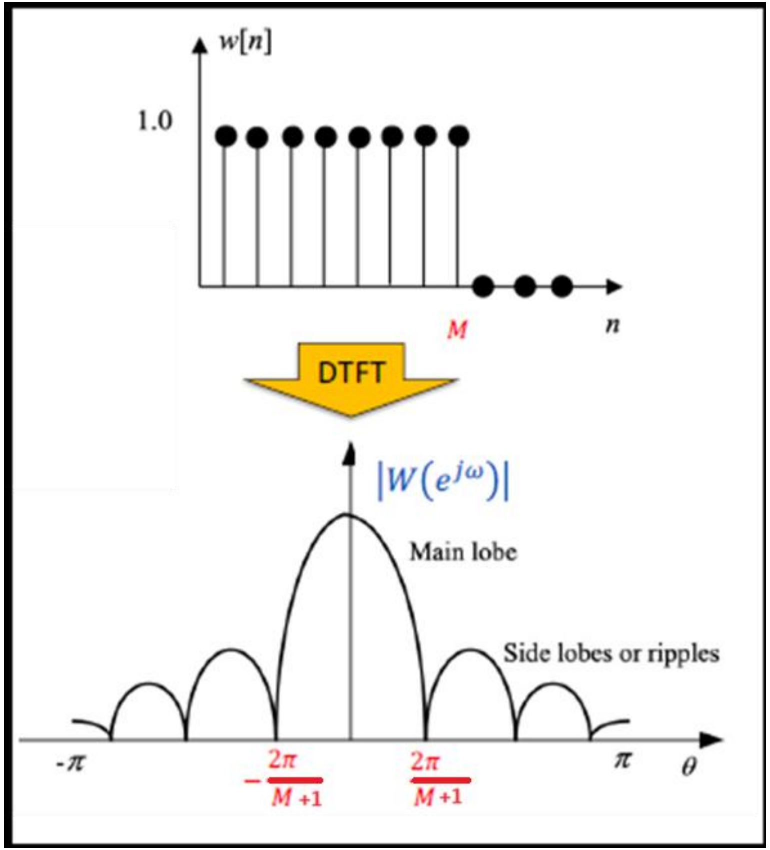
$$W\left(e^{j\omega}\right) = e^{-j\omega M/2} \frac{\sin(\omega\, {}^{(M+1)}/_{2})}{\sin({}^{\omega}/_{2})}$$

- Magintude response

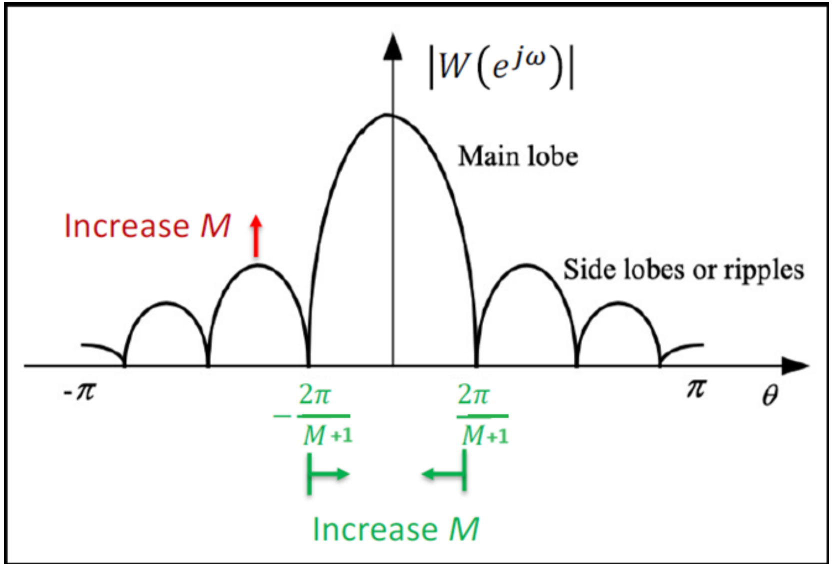
$$\left|W\left(e^{j\omega}\right)\right| = \left|\frac{\sin(\omega\, {}^{(M+1)}/_{2})}{\sin({}^{\omega}/_{2})}\right|$$

- The width of main lobe is $\frac{4\pi}{M+1}$
- Phase response

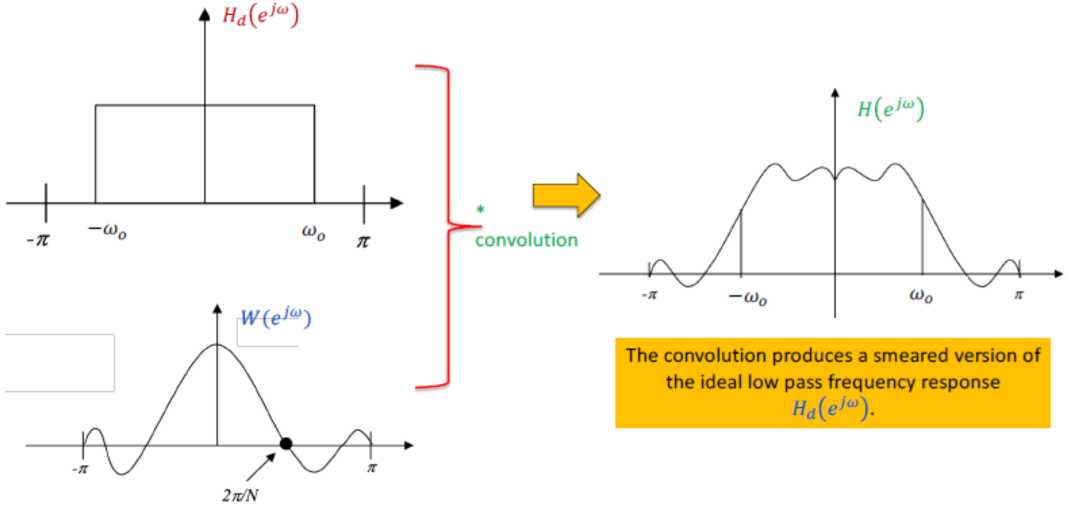
$$\angle W\left(e^{j\omega}\right) = -\omega(\frac{M}{2})$$



- As M increases, the width of the “main lobe” decreases.
- The main lobe is usually defined as the region between the first zero-crossing on the either side of the origin.
- For the rectangular window, the width of main lobe is $\frac{4\pi}{M+1}$.
- However, for the rectangular windown, the side lobes are large, and in fact, as M increases, the peak amplitudes of the main lobe and the side lobes grow in a manner such that the area under each lobe is a constant while the width of each lobes decreases with M.
- Since the area under each lobe remains constant with increasing M, the oscillations occurs more rapidly, but do not decrease in amplitude as M increases.



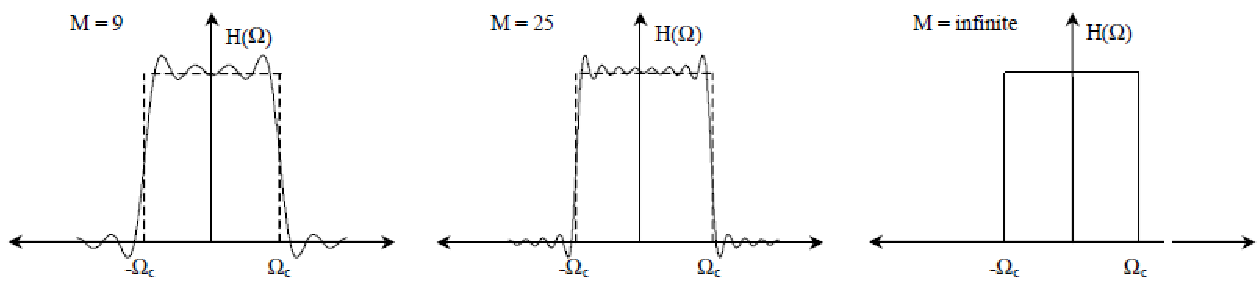
Effect of the Rectangular Window on Frequency Response



- Therefore, it is seen that the convolution produces a smeared version of the ideal low-pass frequency response $H_d(e^{j\theta})$.
- In general, the wider the main lobe of $W(e^{j\omega})$, the more spreading, whereas the narrow the main lobe (Larger M) the closer $|H(e^{j\omega})|$ comes to $|H_d(e^{j\theta})|$.
- In general, we are left with a trade-off on making M large enough so that smearing is minimized, yet small enough to allow reasonable implementation.

Gibbs Phenomena in FIR Filter Design

- In the figure below, oscillations or ringing takes places near band edge (ω_c) of the filter.
- Each oscillations or ringing is generated because of the side lobes in the frequency response of $W(e^{j\omega})$ of the window function.
- The truncation of the Fourier series is known to introduce ripples in the frequency response characteristics $H(e^{j\omega})$ due to the nonuniform convergence of the Fourier series at a discontiunity.
- This oscillatory behaviour (i.e., ringing effect) near the band edge filter is known as Gibbs Phenomenon.

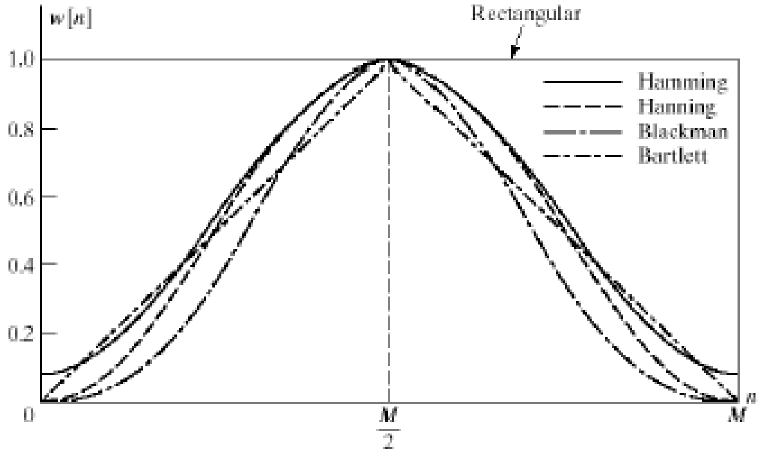


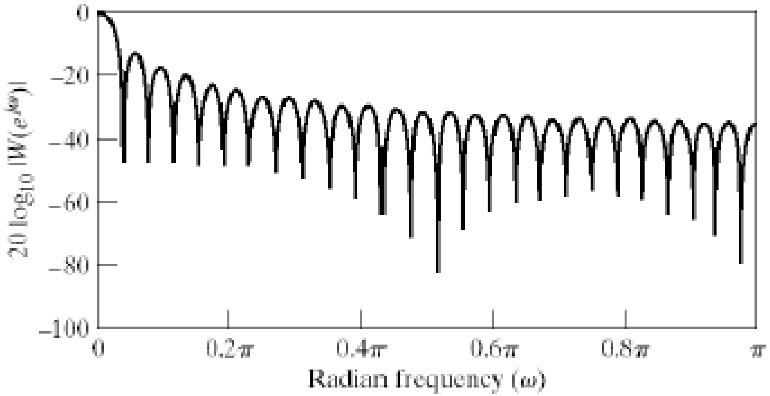
Effects on the frequency response of truncating the ideal impulse response.

Rectangular Window

- Narrowest main lob
  - $4\pi/(M+1)$
  - Sharpest transitions at discontinuities in frequency
- Large side lobs
  - **-13 dB**
  - Large oscillation around discontinuities
- Simplest window possible

$$w[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

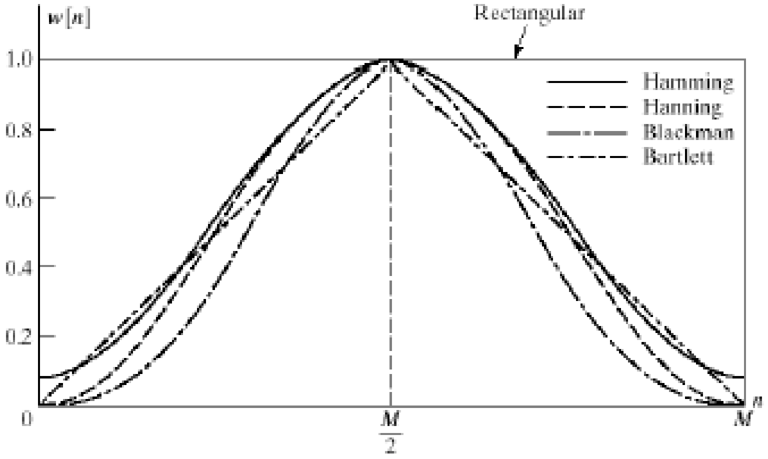


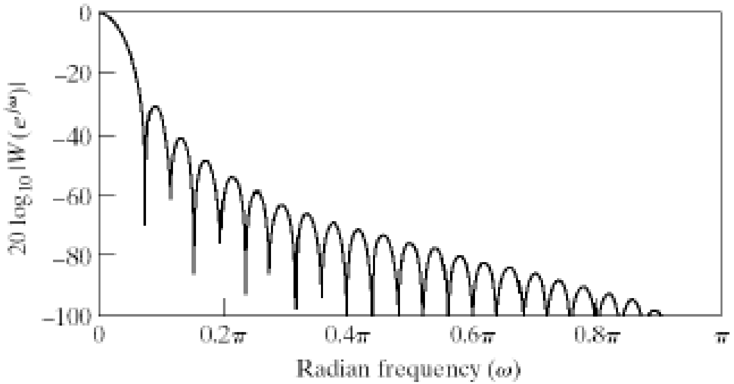


Hanning Window

- Medium main lob
  - $8\pi/M$
- Side lobs
  - **-31 dB**
- Hamming window performs better
- Same complexity as Hamming

$$w[n] = \begin{cases} \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{M}\right)\right], & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

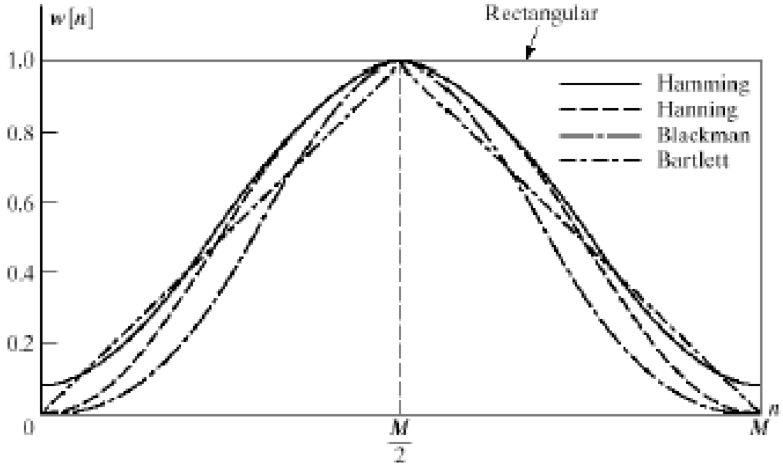


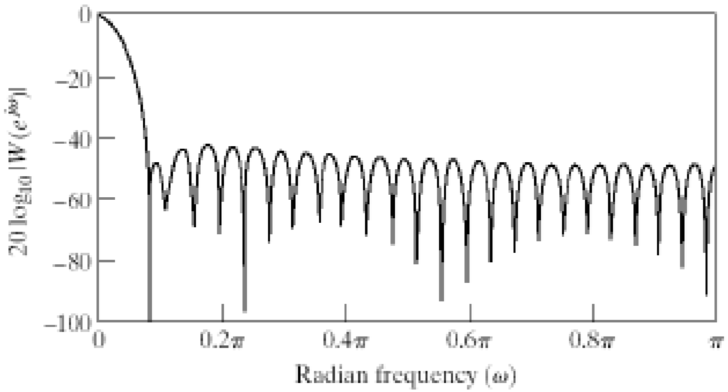


Hamming Window

- Medium main lob
  - $8\pi/M$
- Good side lobs
  - **-41 dB**

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{M}\right), & 0 \le n \le M \\ 0, & otherwise \end{cases}$$





Design Procedure of Window Method

1. An ideal low-pass filter with linear phase of slope or phase delay τ and cutoff ω_c can be characterized in the frequency domain by

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

2. The corresponding impulse respinse $h_d(n)$ can be obtainde by taking the inverse Fourier transform of $H_d(e^{j\omega})$ and is

$$h_d(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = \tau \\ \dfrac{\omega_c}{\pi}\dfrac{\sin(\omega_c(n-\tau))}{\omega_c(n-\tau)}, & n \ne \tau \end{cases}$$

3. A causal FIR filter with impulse response $h(n)$ can be obtainde by multiplying $h_d(n)$ by a window $\mathrm{w}(n)$
4. For $h(n)$ to be linear phase τ must be selected so that the resulting $h(n)$ is symmetic $\tau = \frac{M-1}{2}$
5. The order of the filter is $N = k\frac{2\pi}{\omega_s - \omega_p}$

   The value of k can be obtained from the width of the main lobes. The width of the main lobe $= k\frac{2\pi}{M}$

Table 1: Window and its functions

Window name	Window Function
Rectangular	$\omega_R(n) = \begin{cases} 1 & for\ 0 \leq n \leq M-1 \\ 0 & otherwise \end{cases}$
Triangular (Bartlet)	$\omega_T(n) = 1 - \frac{2\lvert n - \frac{M-1}{2} \rvert}{M-1}$
Hamming	$w(n) = \left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right]$
Hanning	$w(n) = \left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right]$
Blackman	$w(n) = \left[0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)\right]$

Table 2: Summary of window function characteristics

Window name	Transition width of main lobe	Min. stopband attenuation	Peak value of side lobe
Rectangular	$\frac{4\pi}{M+1}$	-21 dB	-21 dB
Hanning	$\frac{8\pi}{M}$	-44 dB	-31 dB
Hamming	$\frac{8\pi}{M}$	-53 dB	-41 dB
Bartlett	$\frac{8\pi}{M}$	-25 dB	-25 dB
Blackman	$\frac{12\pi}{M}$	-74 dB	-57 dB

Note:

1. The $H_d(\omega) = \begin{cases} e^{-j\omega\tau}, for\ \lvert\omega\rvert \leq \omega_c \\ 0, \quad elsewhere \end{cases}$
2. The order of the filter is

$$M = k\frac{2\pi}{\omega_s - \omega_p}$$

The value of k can be obtained from the width of the main lobes.

3. The width of the main lobe

$$= k\frac{2\pi}{M}$$

4. The phase delay

$$\tau = \frac{M-1}{2}$$

Example 1: Design a LPF FIR Filter which will have -3db cut off at $30\pi\ rad/sec$ and an attenuation of 50 dB at $45\pi\ rad/sec$. The filter is required to have a linear phase and the system uses a sampling of 100 samples/sec.

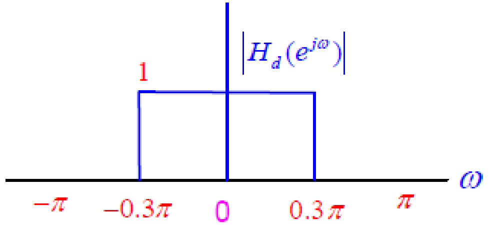
Solution: Here,

Cutoff Frequency $(\Omega_c) = 30\pi\ rad/sec$

Stop band Frequency $(\Omega_s) = 45\pi\ rad/sec$

Stop Band Attunuation $(A_s) = 50dB$

Now, after sampling



Frequency response of LPF

$$\omega_c = \frac{\Omega_c}{f_s} = \frac{30\pi}{100} = 0.3\pi \text{ rad/samples}$$

$$\omega_s = \frac{\Omega_s}{f_s} = \frac{45\pi}{100} = 0.45\pi\ rad/samples$$

Type of window is since the stopband attenuation is 50dB is provided by the Hamming window which of -53dB. Hence Hamming window is selected for the given specification.

To determine the order of the filter: The width of the main lobe in Hamming window is $\frac{8\pi}{M}$ thus,

$$k\frac{2\pi}{M} = \frac{8\pi}{M} \ thus\ k = 4$$

And $M = k\frac{2\pi}{\omega_s - \omega_p} = \frac{8\pi}{\omega_s - \omega_c} = \frac{8\pi}{0.45\pi - 0.3\pi} = 53.33$

Assume linear phase FIR filter of odd length. Hence select next odd integer of length of 55.

The filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau}, & for\ |\omega| \le \omega_c \\ 0, & elsewhere \end{cases}$$

The phase delay

$$\tau = \frac{M-1}{2} = \frac{55-1}{2} = 27$$

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H_d(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-0.3\pi}^{0.3\pi} e^{-j\omega\tau}e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-0.3\pi}^{0.3\pi} e^{-j\omega 27}e^{j\omega n}d\omega =$$

$$= \frac{1}{2\pi}\int_{-0.3\pi}^{0.3\pi} e^{j\omega(n-27)}d\omega = \begin{cases} \frac{\omega_c}{\pi}, & n = 27 \\ \frac{\omega_c}{\pi}\frac{\sin(\omega_c(n-27))}{\omega_c(n-27)}, & n \neq 27 \end{cases}$$

The selected window is Hamming $M = 27$

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{M-1}\right) = 0.54 - 0.46\cos\left(\frac{\pi n}{13}\right)$$

The value of $h(n) = h_d(n)w(n)$

For M=27, $h(n) = 0.3[0.54 - 0.46\cos\left(\frac{\pi n}{13}\right)]$

For M $\neq 27$, $h(n) = \frac{1}{\pi}\frac{\sin(0.3\pi(n-27))}{(n-27)}[0.54 - 0.46\cos\left(\frac{\pi n}{13}\right)]$

6.2 Filter Design by Kaiser Window

- The trade-off between the main-lobe width and side-lobe area can be quantified by seeking the window function that is maximally concentrated around $\omega = 0$ in the frequency domain.
- Kaiser found that a near-optimal window could be formed using the zeroth-order modified Bessel function of the first kind, a function that is much easier to compute.
- The Kaiser window is defined as

$$w(n) = \begin{cases} \dfrac{I_0[\beta\sqrt{(1-\left[\frac{n-\alpha}{\alpha}\right]^2}]}{I_0(\beta)}, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

where $\alpha = M/2$, and $I_0(.)$ represents the zeroth-order Bessel function of the first kind and $I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots$

- The Kaiser window has two parameters: the length $(M+1)$ and a shape parameter β.
- By varying $(M+1)$ and β, the window length and shape can be adjusted to trade side-lobe amplitude for main-lobe width.

Determining Kaiser Window Parameters

- Given δ $[\min(\delta_1, \delta_2)]$ is fixed and $A = -20\log_{10}\delta$
- The passband cutoff frequency ω_p of the low pass filter is defined to be the highest frequency such that $\left|H(e^{j\omega})\right| \geq 1 - \delta$.
- The stopband frequency ω_s is defined to be the lowest frequency such that $\left|H(e^{j\omega})\right| \leq \delta$.
- Therefore, the transition region has width $\Delta\omega = \omega_s - \omega_p$
- The value of β needed to achieve a specified value of A is given by

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \leq A \leq 50 \\ 0.0, & A < 21 \end{cases}$$

- To achieve prescribed values of A and $\Delta\omega$, M must satisfy

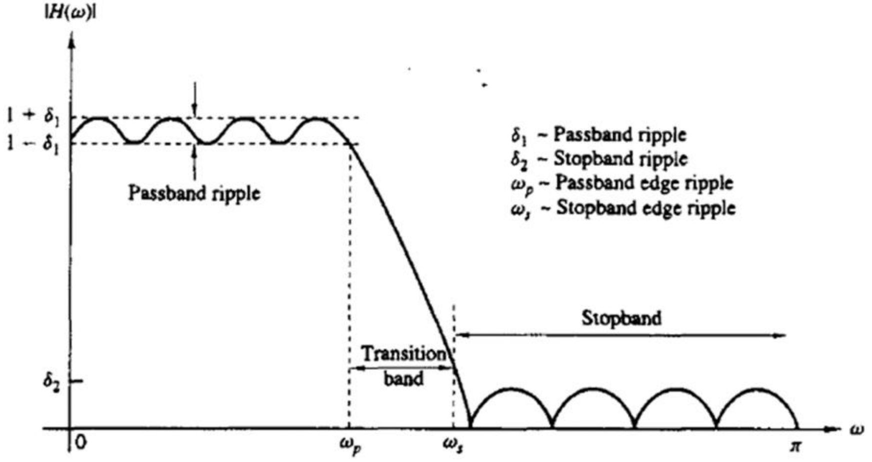
$$M = \frac{A - 8}{2.285\Delta\omega}$$

Example: Design a linear FIR filter using Kaiser window to meet the following specifications:

$$0.99 \le \left|H(e^{i\omega})\right| \le 1.01; for\ 0 \le |\omega| \le 0.19\pi$$

$$\left|H(e^{i\omega}\right| \le 0.01; for\ 0.21\pi \le |\omega| \le \pi$$

Solution:



$1+\delta_1$ and $1-\delta_1$ are ripple factors

Thus $1+\delta_1 = 1.01$ thus $\delta_1 = 0.01$

And $1-\delta_1 = 0.99$ thus $\delta_1 = 0.01$

$\delta_2 = 0.01 = \delta_1$

$\omega_s = 0.21\pi$ and $\omega_p = 0.19\pi$

Thus $\Delta\omega$=$\omega_s - \omega_p = 0.02\pi$

The mininum value of ripples is given by $\delta = \min(\delta_1, \delta_2) = 0.01$

And $A = -20\log_{10}\delta = A = -20\log_{10}(0.01) = 40dB$

Now, the cutoff frequency ω_c is given by

$$\omega_c = \frac{\omega_p + \omega_s}{2} = \frac{0.19\pi + 0.21\pi}{2} = 0.2\pi \frac{\text{rad}}{\text{samples}}$$

$$\beta = \begin{cases} 0.1102(A - 8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 \le A \le 50 \\ 0.0, & A < 21 \end{cases}$$

Since, $A = 40$, so

$$\beta = 0.5842(A-21)^{0.4} + 0.07886(A-21)$$

$$= 0.5842(40-21)^{0.4} + 0.07886(40-21) = 3.395$$

$$M = \frac{A-8}{2.285\Delta\omega} = M = \frac{40-8}{2.285 \times 0.02\pi} = 222.88$$

Thus $M = 223$ we have to make even so $M = 224$

$$\alpha = \frac{M}{2} = 112$$

$$w(n) = \begin{cases} \dfrac{I_0[\beta\sqrt{(1-\left[\frac{n-\alpha}{\alpha}\right]^2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

$$w(n) = \begin{cases} \dfrac{I_0[3.395\sqrt{(1-\left[\frac{n-112}{112}\right]^2}]}{I_0(3.395)}, & 0 \le n \le 224 \\ 0, & otherwise \end{cases}$$

Where $I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_d(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = \tau \\ \dfrac{\omega_c}{\pi}\dfrac{\sin(\omega_c(n-\tau))}{\omega_c(n-\tau)}, & n \ne \tau \end{cases}$$

$$h_d(n) = \begin{cases} \dfrac{0.2\pi}{\pi}, & n = 112 \\ \dfrac{1}{\pi}\dfrac{\sin(0.2\pi(n-112))}{(n-112)}, & n \neq 112 \end{cases}$$

$$h(n) = h_d(n)w(n)$$

$$= \begin{cases} 0.2\dfrac{I_0[3.395\sqrt{(1-\left[\frac{n-112}{112}\right]^2}]}{I_0(3.395)}, & n = 112 \\ \dfrac{1}{\pi}\dfrac{\sin(0.2\pi(n-112))}{(n-112)}\dfrac{I_0[3.395\sqrt{(1-\left[\frac{n-112}{112}\right]^2}]}{I_0(3.395)}, & n \neq 112 \end{cases}$$

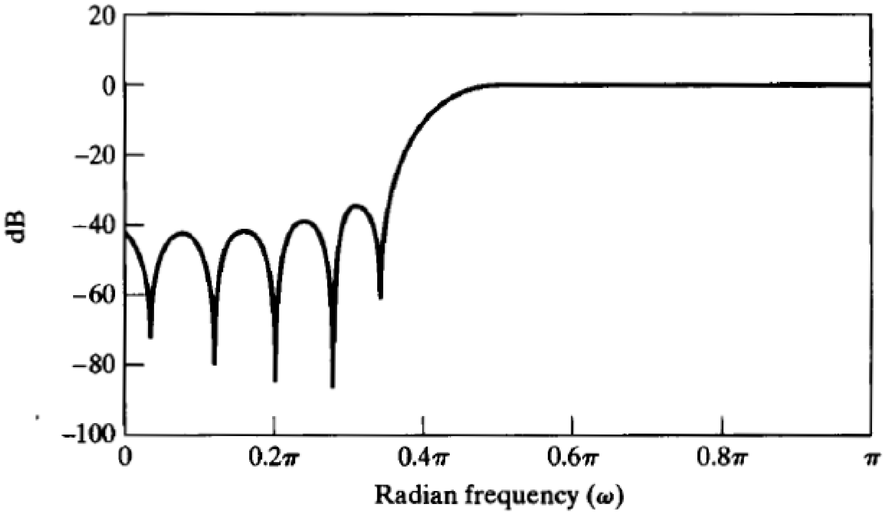
Kaiser Window Design of a HighPass Filter

Suppose we wish to design a filter to meet the highpass specifications

$$\left|H(e^{j\omega})\right| \leq \delta_2, \qquad |\omega| \leq \omega_s$$

$$1 - \delta_1 \leq \left|H(e^{j\omega})\right| \leq 1 - \delta_1, \omega_p \leq |\omega| \leq \pi$$

If $\omega_s = 0.35\pi$ and $\omega_p = 0.5\pi$ with $\delta_1 = \delta_2 = 0.021$, we can use Kaiser' formula to estimate the required values of $\beta = 2.6$ and M=24.



Example 2 : Find an expression for the impulse response $h(n)$ of a linear phase lowpass FIR filter using Kaiser window to satisyf the following magnitude response specification for the equivalent analog filter.

- Stopband attenuation: 40dB
- Passband ripple: 0.01dB
- Transition width: 1000 π rad/sec
- Ideal cutoff frequency: 2400 π rad/sec
- Sampling frequency: 10KHz

Solution:

Stopband attenuation = $-20\log\delta_s = 40dB$, thus $\delta_s = 0.01$

Passband ripple = $20\log\left(1+\delta_p\right) = 0.01$, thus $\delta_p = 0.00115$

$\delta = \min(\delta_s, \delta_p) = 0.00115$

Now $\mathrm{A} = -20\log_{10}\delta = -20\log_{10}(0.00115) = 58.8\,dB$

$$\beta = 0.1102(A - 8.7) = 5.5$$

$$\Delta\omega = \frac{\Delta\Omega}{f_s} = \frac{1000\pi}{10000} = 0.1\pi\ \mathrm{rad}$$

$$M = \frac{A-8}{2.285\Delta\omega} = \frac{58.8-8}{2.285 \times 0.1\pi} = 70.81$$

So taking M=72, we get $\alpha = \frac{M}{2} = 36$

$$w(n) = \begin{cases} \frac{I_0[\beta\sqrt{(1-\left[\frac{n-\alpha}{\alpha}\right]^2}]}{I_0(\beta)}, & 0 \le n \le M \\ 0, & otherwise \end{cases}$$

$$w(n) = \begin{cases} \frac{I_0[5.5\sqrt{(1-\left[\frac{n-36}{36}\right]^2}]}{I_0(5.5)}, & 0 \le n \le 36 \\ 0, & otherwise \end{cases}$$

Where $I_0(x) = 1 + \frac{0.25x^2}{(1!)^2} + \frac{(0.25x^2)^2}{(2!)^2} + \frac{(0.25x^2)^3}{(3!)^2} + \cdots$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\omega\tau}, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_d(n) = \begin{cases} \dfrac{\omega_c}{\pi}, & n = \tau \\ \dfrac{\omega_c}{\pi} \dfrac{\sin(\omega_c(n-\tau))}{\omega_c(n-\tau)}, & n \ne \tau \end{cases}$$

Ideal cutoff frequency: 2400 π rad/sec therefore

$$\omega_c = \frac{\Delta\omega}{2} + \frac{\Omega_{ip}}{f_s} = 0.05\pi + 0.24\pi = 0.29\pi$$

$$h_d(n) = \begin{cases} 0.29, & n = 36 \\ \dfrac{1}{\pi} \dfrac{\sin(0.29\pi\ (n-36))}{(n-36)}, & n \ne 36 \end{cases}$$

$$h(n) = h_d(n) w(n)$$

6.3 Filter Design by Frequency Sampling

- In this method a set of M equally spaced samples in the intervals $(0, 2\pi)$ are taken in the desired frequency response $H_d(\omega)$.
- The continuous frequency ω is replaced by

$$\omega = \omega_k = \frac{2\pi}{M}k, \qquad k = 0,1, \dots M-1$$

- The discrete time Fourier transform (DTFT) is

$$H(k) = H_d(\omega)\Big|_{\omega=\omega_k} = H_d(\frac{2\pi}{M}k), \qquad k = 0,1, \dots M-1$$

- The inverse M-point DFT (IDFT) is

$$h(n) = \frac{1}{M}\sum_{n=0}^{M-1} H(k)e^{j\omega n} = \frac{1}{M}\sum_{n=0}^{M-1} H(k)e^{j\frac{2\pi}{M}kn}, \qquad \mathrm{n} = 0,1, \dots \mathrm{M}-1$$

- For the FIR filter to be realizable the coefficients $h(n)$ must be real. This is possible if all complex terms appear in complex conjugate pairs.

$$H(M-k)e^{j2\pi n\frac{(M-k)}{M}} = H(M-k)e^{j2\pi n}e^{-j2\pi n\frac{k}{M}} = H(M-k)e^{-j2\pi n\frac{k}{M}}$$

- Since $|H(M-k)| = |H(k)|$

$$H(M-k)e^{-j2\pi n\frac{k}{M}} = H(k)e^{-j2\pi n\frac{k}{M}}$$

- The term $H(k)e^{-j2\pi n\frac{k}{M}}$ is complex conjugate of $H(k)e^{j2\pi n\frac{k}{M}}$
- Hence $H(M-k)e^{j2\pi n\frac{(M-k)}{M}}$ is complex conjugate of $H(k)e^{-j2\pi n\frac{k}{M}}$

$$H(M-k) = H^*(k)$$

- Thus

$$h(n) = \frac{1}{M}(H(0) + 2\sum_{k=1}^{P} Re\left[H(k)e^{j2\pi n\frac{k}{M}}\right])$$

where $P = \begin{cases} \frac{M-1}{2}, & if\ M\ is\ odd \\ \frac{M}{2} - 2, & if\ M\ is\ even \end{cases}$

Design a lowpass FIR filter using frequency sampling technique having cut-off frequency of $\pi/2$ rad/sample. The filter should have linear phase and length of 17.

Solution:

- The Ideal LPF frequency response $H_d(\omega)$ for the linear phase is

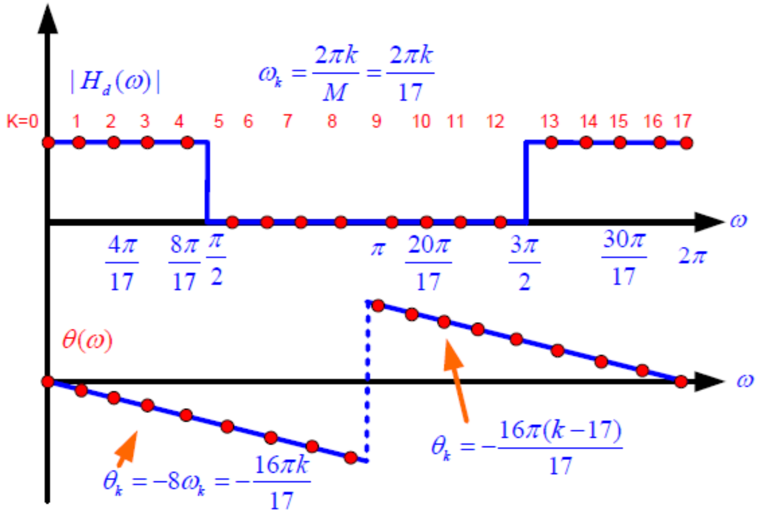
$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & 0 \leq \omega \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq \omega \leq \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j8\omega} & 0 \leq \omega \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq \omega \leq \pi \end{cases}$$

- To sample put $\omega = \frac{2\pi k}{M} = \frac{2\pi k}{17}$

$$H_d(\omega) = \begin{cases} e^{-j\frac{2\pi k}{17}8} & 0 \leq \frac{2\pi k}{17} \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq \frac{2\pi k}{17} \leq \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j\frac{16\pi k}{17}} & 0 \leq k \leq \frac{17}{4} \\ 0 & \frac{17}{4} \leq k \leq \frac{17}{2} \end{cases}$$



The range of k is

$$\frac{2\pi k}{17} = \frac{\pi}{2} \quad k = \frac{17}{4} \simeq 4$$

$$\frac{2\pi k}{17} = \pi \quad k = \frac{17}{2} \simeq 8$$

The range of k is $0 \leq k \leq \frac{17}{4}$
k is an integer.
Hence the range is $0 \leq k \leq 4$
Similarly $\frac{17}{4} \leq k \leq \frac{17}{2} = 4.25 \leq k \leq 8.5$
The range $5 \leq k \leq 8$

$$|H(k)| = \begin{cases} 1 & 0 \leq k \leq 4 \\ 0 & 5 \leq k \leq 8 \\ 1 & 13 \leq k \leq 16 \end{cases}$$

The value of h(n)is given by

$$\begin{aligned} h(n) &= \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{8} Re\left[H(k)e^{j2\pi kn/17}\right]\right) \end{aligned}$$

$|H(k)| = 1 \quad 0 \leq k \leq 4$

$$\begin{aligned} h(n) &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} Re\left[e^{-j\frac{16\pi k}{17}} e^{j2\pi kn/17}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} Re\left[e^{j2\pi k(n-8)/17}\right]\right) \\ &= \frac{1}{17}\left(1 + 2\sum_{k=1}^{4} cos\left[\frac{2\pi k(n-8)}{17}\right]\right) \end{aligned}$$

Determine the impulse response h(n) of a filter having desired frequency response

$$H_d(w) = \begin{cases} e^{-j\left(\frac{(M-1)\omega}{2}\right)} & 0 \le |\omega| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

M=7 use frequency sampling approach.

Solution:

- The Ideal LPF frequency response $H_d(\omega)$ is

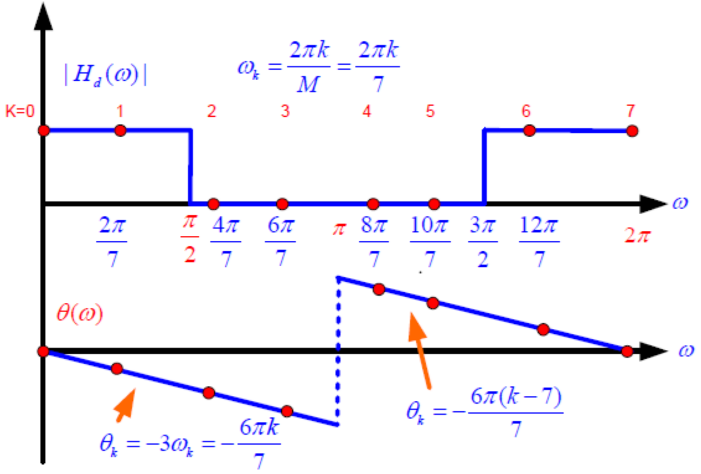
$$H_d(\omega) = \begin{cases} e^{-j\omega\left(\frac{M-1}{2}\right)} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j3\omega} & 0 \le \omega \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \omega \le \pi \end{cases}$$

- To sample put $\omega = \frac{2\pi k}{M} = \frac{2\pi k}{7}$

$$H_d(\omega) = \begin{cases} e^{-j\frac{2\pi k}{7}3} & 0 \le \frac{2\pi k}{7} \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le \frac{2\pi k}{7} \le \pi \end{cases}$$

$$H_d(\omega) = \begin{cases} e^{-j\frac{6\pi k}{7}} & 0 \le k \le \frac{7}{4} \\ 0 & \frac{7}{4} \le k \le \frac{7}{2} \end{cases}$$



The range of k is

$$\frac{2\pi k}{7} = \frac{\pi}{2} \quad k = \frac{7}{4} \simeq 1$$

$$\frac{2\pi k}{7} = \pi \quad k = \frac{7}{2} \simeq 3$$

The range of k is $0 \leq k \leq \frac{7}{4}$
k is an integer.
Hence the range is $0 \leq k \leq 1$
Similarly $\frac{7}{4} \leq k \leq \frac{7}{2} = 1.75 \leq k \leq 3.5$
The range $2 \leq k \leq 3$

$$|H(k)| = \begin{cases} 1 & 0 \leq k \leq 1 \\ 0 & 2 \leq k \leq 3 \\ 1 & k = 6 \end{cases}$$

n	$h(n)$	n	$h(n)$
0	-0.1146	4	321
1	0.0793	5	0.0793
2	0.321	6	-0.1146
3	0.4283		

The value of h(n)is given by

$$\begin{aligned} h(n) &= \frac{1}{M}\left(H(0) + 2\sum_{k=1}^{\frac{M-1}{2}} Re\left[H(k)e^{j2\pi kn/M}\right]\right) \\ &= \frac{1}{7}\left(1 + 2\sum_{k=1}^{3} Re\left[H(k)e^{j2\pi kn/7}\right]\right) \end{aligned}$$

$|H(k)| = 1 \quad 0 \leq k \leq 1$

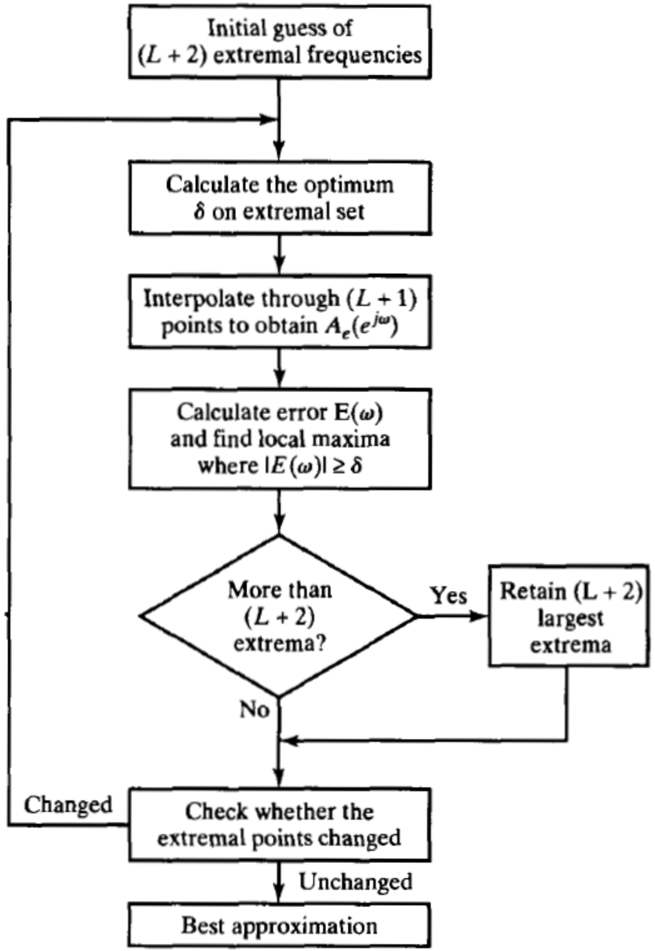
$$\begin{aligned} h(n) &= \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} Re\left[e^{-j\frac{6\pi k}{7}} e^{j2\pi kn/7}\right]\right) \\ &= \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} Re\left[e^{j2\pi k(n-3)/7}\right]\right) \\ &= \frac{1}{7}\left(1 + 2\sum_{k=1}^{1} cos\left[\frac{2\pi k(n-3)}{7}\right]\right) \end{aligned}$$

6.4 Filter Design using optimum approximation, Filter Design using the Remez exchange algorithm

- Parks-McClellan algorithm (Remez exhange algorithm) is a most popular optimal design menthod used in industry due to its efficiency and flexibility.
- The goal of the algorithm is to minimize the error in the pass and stop band bands by utilizing the Chesbyshev approximation.

The algorithm steps are

1. Make an initial guess of the L+2 external frequencies.
2. Compute δ using the equation given.
3. Using Lagrange Interpolation, we compute the dense set of samples of $A(\omega)$ over the passband and stopband.
4. Determine the new L+2 largest exterma.
5. If the alternation theorem is not satisfied, then go back to (2) and iterate unitl the alternation theorem is satisfied.
6. If the alternation theorem is satisifed, then we compute $h(n)$ and we are done.



Comparison between IIR and FIR Filter

	IIR Filter	FIR Filter
1	Linear characteristic cannot be achieved	Linear characteristic can be achieved
2	The impulse response cannot be directly converted to digital filter transfer function	The impulse response can be directly converted to digital filter transfer function
3	It is recursive filter and may be stable or unstable	It may be recursive or non recursive filter and recursive filter are stable
4	The specifications include the desired characteristics for magnitude response only	The specifications include the desired characteristics for both magnitude and phase response
5	The design involves design of analog filter and then transforming analog to digital filter	The digital filter can be directly designed to achieve the desired specifications.

Questions:

1. Design a linear FIR filter using Kaiser window to meet the following specifications:

   $0.99 \leq \left|H(e^{i\omega})\right| \leq 1.01; for\ 0 \leq |\omega| \leq 0.19\pi$

   $\left|H(e^{i\omega}\right| \leq 0.01; for\ 0.21\pi \leq |\omega| \leq \pi$

2. By using Hanning window, design a low pass filter to approximate the ideal response given by:

$$H\left(e^{i\omega}\right) = \begin{cases} 1, & for\ -\frac{\pi}{6} \leq \omega \leq \frac{\pi}{6} \\ 0, & otherwise \end{cases}$$

   Use the filter length of M=9 for your design.

3. How can you design FIR filter using rectangular window? Explain.
4. Design a lowpass FIR with 7 coefficients for the following specifications.

   Passband Frequency edge=300Hz

   Sampling Frequency =1 KHz

   Use Hanning window for your design.

5. Write short notes on Gibbs Phenomena.
6. Why ideal low pass filter cannot be realized in practice ? Explain how practical lowpass filter are realized in practice and also explain its effect.

7. Design a FIR system to meet the following specifications

   Pass band edge frequency = 2 KHZ

   Stop band edge frequency = 5 KHZ

   Stop band attenuation = 42db

   Sampling Frequency = 20KHz

8. Design a low pass filter which will have -3db cut off at $30\pi\, rad/sec$ and an attenuation of 50 db at $45\pi\, rad/sec$. The filter is required to have a linear phase and the system uses a sampling of 100 samples/sec.

9. With low pass specification $\Omega_p = 3.2KHz$ and $\Omega_s = 4.8KHz$ and sampling frequecny $\Omega_{fs} = 12KHz$ and $\alpha_s = 40db$, find the length and value of β for Kaiser Window.

10. Show that the Kaiser window includes the rectangular window as a special case.

11. Define symmetric and Anti-symmetric filter and discuss the appilcations.

12. Explain briefly the design of linear phase FIR filter by frequency sampling method with proper example.