first order differential equation

The differential equation of the form $Mdx + Ndy = 0$ is called first order differential equation.

* Method of solving first order differential equation

(i) variable seperable
(ii) Reducable in ~~symb~~ seperable form
(iii) Homogenous diff eqn
(iv) Exact diff eqn
(v) Linear diff eqn
(vi) Reducable to linear form.

(i) variable seperable form.

The differential equation $Mdx + Ndy = 0$ is said to be separable if it can be reduce in the form of $f(x)dx + f(y)dy = 0$

Ex (1) $y\,dx + x\,dy = 0$

$$\frac{1}{x}dx + \frac{1}{y}dy = 0$$

$$\log x + \log y = \log c$$

$$\log(xy) = \log c$$

$$xy = c$$

general soln.

(ii) $x(y^2+1)\,dx + (x^2+1)\,dy = 0$

$$\frac{dx}{x^2+1} + \frac{dy}{y^2+1} = 0$$

soln $\tan^{-1}x + \tan^{-1}y = c$

(ii) Reducable to separable form.

(1) $(x+y+1)\dfrac{dy}{dx} = 1$ --- (1)

soln

put $x+y+1 = v$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

eqn (1) becomes

$$v\left(\frac{dv}{dx} - 1\right) = 1$$

$$\frac{dv}{dx} = \frac{1}{v} + 1 \Rightarrow \frac{1+v}{v}$$

$$dv = \frac{1+v}{v}dx$$

Integrating,

$$\int \frac{v+1-1}{1+v}dv = x + C$$

$$\int dv - \int \frac{1}{1+v}dv = x + C$$

$$v - \log(1+v) = x + C$$

$$\cancel{x}+y+1 - \log(x+y+2) = \cancel{x}+C$$

$$y+1 - \log(x+y+2) = C$$

* If $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

Sol^n,

Here, $u=\sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

$$\sin u=\frac{x^2+y^2}{x+y}$$

$$z=\sin u=\frac{x^2\left[1+(y/x)^2\right]}{x\left[1+(y/x)\right]}$$

$$z=\sin u=x\,\phi(y/x)$$

z is homogenous function of two variables of degree 1

so By Euler's Theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$$

$$x\cos u\frac{du}{dx}+y\cos u\frac{du}{dy}=\sin u$$

$$x\frac{du}{dx}+y\frac{du}{dy}=\tan u \text{ proved}$$

$z=\sin u$

$\frac{dz}{dx}=\frac{d}{dx}(\sin u)$

$=\frac{d\sin u}{du}\cdot\frac{du}{dx}$

$=\cos u\,\frac{du}{dx}$

① If $u = \log \frac{x^2+y^2}{x+y}$ prove $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$

Here,

$$u = \log \frac{x^2+y^2}{x+y}$$

$$\therefore e^u = \frac{x^2+y^2}{x+y}$$

$$z = e^u = \frac{x^2[1+(y/x)^2]}{x[1+(y/x)]}$$

$$z = e^u = x^1 \phi(y/x)$$

$\therefore$ z is homogenous of two variables of degree 1.

So By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$$

proved

$$z = e^u$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial u} e^u \frac{\partial u}{\partial x}$$

$$= e^u\frac{\partial u}{\partial x}$$

② If $u = \log\left(\frac{x^4+y^4}{x+y}\right)$ prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Soln

$$u = \log\left(\frac{x^4+y^4}{x+y}\right)$$

$$\therefore e^u = \frac{x^4+y^4}{x+y}$$

$$z = e^u = \frac{x^4[1+(y/x)^4]}{x[1+(y/x)]}$$

$$z = e^u = x^3\phi(y/x)$$

$\therefore$ z is homogenous function of two variable of degree 3. So By Euler's theorem.

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$$

$$xe^u\frac{\partial u}{\partial x} + ye^u\frac{\partial u}{\partial y} = 3e^u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$$

proved

$$z = e^u$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial u} e^u \frac{\partial u}{\partial x}$$

$$= e^u\frac{\partial u}{\partial x}$$

(2) $\frac{dy}{dx} + 1 = e^{x+y}$

soln

put $x+y = a$

* Homogenous differential equation

A differential equation of the form $M\,dx + N\,dy = 0$ is said to be first order Homogenous differential equation if M and N are homogenous function of x and y.

Ex $(x^2+y^2)\,dx - 2xy\,dy = 0$

or $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ --- (1)

put $y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$

eqn (1) becomes,

or $v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x \cdot vx} = \frac{1+v^2}{2v}$

or $x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v}$

$= \frac{1-v^2}{2v}$

or $x\,dv = \frac{1-v^2}{2v}\,dx$

$\frac{2v}{1-v^2}\,dv = \frac{1}{x}\,dx$

Int.

$-\log|1-v^2| = \log x + \log c$

$-\log(1-v^2) = \log(cx)$

$\frac{1}{1-v^2} = cx$

$\frac{1}{1-\frac{y^2}{x^2}} = cx$

$\frac{x^2}{x^2-y^2} = cx$

(i) Euler's Theorem for $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}u$

when $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Here,

$$u = \frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$$

$$= \frac{x^{1/4}\left[1 + (y/x)^{1/4}\right]}{x^{1/5}\left[1 + (y/x)^{1/5}\right]}$$

$$x^{1/20}\ \phi(y/x)$$

$\therefore$ u is homogenous function of degree 1/20 so

By Euler's theorem then,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}u$$

Now, $\dfrac{\partial u}{\partial x} = \dfrac{\partial}{\partial u}\left\{x^{1/20}\phi(y/x)\right\}$

Diff w.r. to x partially, we get

$$= \frac{1}{20}x^{-19/20}\phi(y/x) + x^{1/20}\phi'(y/x)\cdot(-y/x^2)$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{1}{20}x^{1/20}\phi(y/x) - x^{-19/20}y\,\phi'(y/x) \quad \text{(i)}$$

Again

diff w.r. to y partially, we get

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{x^{1/20}\phi(y/x)\right\}$$

$$= x^{1/20}\frac{\partial}{\partial y}\phi(y/x)$$

$$= x^{1/20}\phi'(y/x)\cdot\frac{1}{x}$$

$$= x^{-19/20}\phi'(y/x) \quad \text{(ii)}$$

$$\therefore y\frac{\partial u}{\partial y} = x^{-19/20}y\,\phi'(y/x)$$

Adding eq^n (i) and (ii), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{20}x^{1/20}\phi(y/x) - x^{-19/20}y\,\phi'(y/x) + x^{-19/20}y\,\phi'(y/x)$$

$$= \frac{1}{20}x^{1/20}\phi(y/x)$$

$$= \frac{1}{20}u$$

Hence, $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{20}u$ $\therefore$ Euler's Theorem verified.

(ii) If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = -\dfrac{\sin u \cos 2u}{4\cos^3 u}$

$x = (x^2+y^2)c$ #

Exact differential

A differential equation of form $Mdx + Ndy = 0$ is said to be exact diff equation if there exist a function U such that $Mdx + Ndy = dU$

Ex. $xdy + ydx = 0$

$d(xy) = 0$

init $xy = c$

NOTE:

condition of exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

soln of exact eqn is

$$\int M dx + \int (\text{term of } N \text{ not containing } x) dy = 0$$

Q) $(x^2 + 2xy^2)dx + (2x^2y + y^2)dy = 0$

soln

comparing with $Mdx + Ndy = 0$

$M = x^2 + 2xy^2$

$N = 2x^2y + y^2$

Now, $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2 + 2xy^2) = 4xy$

$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x^2y + y^2) = 4xy$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

given eqn (1) is exact equation.

soln is

$$\int M dx + \int (\text{term of } N \text{ not containing } x)\, dy = C$$

$$\int (x^2 + 2xy^2)dx + \int y^2 dy = C$$

$$\frac{x^3}{3} + 2y^2\frac{x^2}{2} + \frac{y^3}{3} + C$$ #

Q) $(x^2+y^2)dx = 2xydy$

soln

$(x^2+y^2)dx - 2xydy = 0$ — (1)

comparing with $Mdx + Ndy = 0$

$M = x^2+y^2$

$N = -2xy$

Now, $\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = -2y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

~~multiplying~~ multiplying eqn (1) by $1/y^2$, we get

$$\frac{(x^2+y^2)}{y^2}dx - \frac{2xy}{y^2}dy = 0$$

$$\left(\frac{x^2}{y^2} + 1\right)dx - \frac{2x}{y}dy = 0$$

Here, $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

$$\sin u = \left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

Let $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})}$

$$= x^{1/2}\,\phi(y/x) \quad \text{— (I)}$$

∵ z is homogenous function of degree $1/2$ so by euler's Theorem then,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \quad \text{— (II)}$$

Now, diff (II) w.r.to x particaly, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\cdot 1 + y\frac{\partial^2 u}{\partial y\partial x} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

Multiply by x,

$$x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial y\partial x} = \frac{1}{2}\sec^2 u\cdot x\frac{\partial u}{\partial x} \quad \text{— (III)}$$

diff (II) w.r.to y partially, we get

$$x\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\cdot 1 = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

multiply by y

$$xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = \frac{1}{2}\sec^2 u\; y\frac{\partial u}{\partial y} \quad \text{— (IV)}$$

Adding (III) and (IV), we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$= \frac{1}{2}\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\tan u = \frac{1}{2}\sec^2 u\cdot\frac{1}{2}\tan u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\tan u\left(\frac{1}{2}\sec^2 u - 1\right)$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{1}{2}\frac{1}{\cos^2 u} - 1\right)$$

$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{\sin u}{\cos u}\left(\frac{1 - 2\cos^2 u}{2\cos^2 u}\right)$$

$$x\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\frac{\sin u - 2\sin u\cos^2 u}{2\cos^3 u}$$

Now,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

$$= \frac{1}{-2xy}(2y+2y) = \frac{-4y}{2xy} = \frac{-1}{2x} = f(x)$$

$$\therefore I.F = e^{\int f(x)\,dx}$$
$$= e^{-\int \frac{2}{x}dx}$$
$$= e^{-2\log x}$$
$$= e^{\log x^{-2}}$$
$$= 1/x^2$$

Multiplying eq^n ① by $1/x^2$, we get

$$1 + \frac{y^2}{x^2}dx - \frac{2y}{x}dy = 0$$

$$M = \frac{1+y^2}{x^2}$$

$$N = \frac{-2y}{x}$$

Now $\frac{\partial M}{\partial y} = \frac{2y}{x^2}$

$\frac{\partial N}{\partial x} =$

Integrating factor (I.F)

A factor which when multiply the differential equation become exact is called Integrating factor

finding of I.F

① $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x ~~only i.e. and y~~ only i.e. $f(x)$

$$\therefore I.F = e^{\int f(x)dx}$$

⑪ $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y only i.e. $f(y)$,

$$I.F = e^{\int f(y)dy}$$

Linear diff eq^n and Bernoulli's eq^n

A diff eq^n of the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x or constant is called first order linear differential eq^n,

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The differential eq^n of the form $\frac{dy}{dx} + Py = Qy^n$ where P and Q are function of x or constant is called Bernoulli's eq^n

Total derivative.

If $u = f(x,y)$ then Total derivative of u denoted by du and define by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

If $u = f(x,y,z)$ then Total derivative of u is denoted by du and define by

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Derivative of compositive function.

~~If $u = f(x,y)$~~

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

If $u = f(x,y)$ and $x = \phi(t)$ and $y = \psi(t)$

Example

(1) $u = x^3 + 3x^2y + 3xy^2 + y^3$

$x = t^2 - 1$, $y = 2t + 1$

find total derivative of u.

Soln

we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{du}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$= (3x^2 + 6xy + 3y^2)2t + (3x^2 + 6xy + 3y^2)2$

$= \{3(t^2-1)^2 + 6(t^2-1)(2t+1) + 3(2t+1)^2\} 2t +$
$2\{3(t^2-1)^2 + 6(t^2-1)(2t+1) + 3(2t+1)^2\}$

$=$

Method of solving linear eqn

$$\frac{dy}{dx} + Py = Q$$

1) find I.F by using, $IF = e^{\int P dx}$

(II) solⁿ of linear eqⁿ is $y \times I.F = \int Q \cdot I.F \, dx + c$

Eg:

(I) $y' + 2y = 4x$

comparing it with $\frac{dy}{dx} + Py = Q$

where $P = 2$, $Q = 4x$

$\therefore$ $I.F = e^{\int P dx} \Rightarrow e^{\int 2 dx} = e^{2x}$

solⁿ is, $y \times IF = \int Q \cdot IF \, dx + C$

$$y \times e^{2x} = \int 4x \, e^{2x} dx + C$$

$$y \times e^{2x} = \int 4x e^{2x} dx + C$$

$$= 4\left\{ x \cdot \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2} \right\} + C$$

$$= y = 2xe^{2x} - e^{2x} + C$$

(II) $(1-x^2)\frac{dy}{dx} - xy = 1$

$$\Rightarrow \frac{dy}{dx} - \frac{xy}{(1-x^2)} = \frac{1}{1-x^2}$$

comparing it with $\frac{dy}{dx} + Py = Q$ then

$$P = \frac{-x}{1-x^2}, \quad Q = \frac{1}{1-x^2}$$

$\therefore$ $I.F = e^{\int P dx} \Rightarrow e^{\int \frac{x}{1-x^2} dx} = e^{\frac{1}{2}\log(1-x^2)}$

$$= \sqrt{1-x^2}$$

solⁿ is

$$y \times I.F = \int Q \times I.F \, dx + C$$

$$y \times \sqrt{1-x^2} = \int \frac{\sqrt{1-x^2}}{1-x^2} dx + C$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + C$$

$$= \cancel{\log(1-x^2)}$$

(III) $\frac{dy}{dx} = \frac{y}{2y\log y + y - x}$

$$\frac{dx}{dy} = \frac{2y\log y + y - x}{y}$$

$$\frac{dx}{dy} + \frac{1}{y}x = 2\log y + 1$$

$$P = \frac{1}{y} \quad Q = 2\log y + 1$$

$$I.F = e^{\int P dy} = e^{\int 1/y \, dy} = e^{\log y} = y$$

Application of partial derivative

Extreme value (Maxima or Minima)

(1) for two variable

Let $z = f(x,y)$

Condition	for max	for min	saddle point	undetermined information
(i) first order partial derivative	$f_x = 0$, $f_y = 0$	$f_x = 0$, $f_y = 0$	$f_x = 0$, $f_y = 0$	$f_x = 0$, $f_y = 0$
(ii) second order partially derivative	$f_{xx} < 0$, $f_{xx}f_{yy} - f_{xy}^2 < 0$	$f_{xx} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$	$f_{xx}f_{yy} - f_{xy}^2 < 0$	$f_{xx}f_{yy} - f_{xy}^2$ [illegible]

* find maximum and minimum (extreme value) of function.

$$f(x,y) = x^3 + 3x^2 + 4xy + y^2 \quad \text{--- (1)}$$

diff w.r.to x and y partially, we get

$$f_x = 3x^2 + 6x + 4y$$
$$f_y = 4x + 2y$$
$$f_{xx} = 6x + 6$$
$$f_{yy} = 2$$
$$f_{xy} = 4$$

for extreme value (for max^m or min^m)

$f_x = 0$ and $f_y = 0$ Then,

$$3x^2 + 6x + 4y = 0$$
$$4x + 2y = 0 \Rightarrow y = (-2x)$$

solving, we get

$$3x^2 + 6x + 4(-2x) = 0$$
$$3x^2 + 6x - 8x = 0$$
$$3x^2 - 2x = 0$$
$$x(3x - 2) = 0$$

either, $x = 0$

or $x = 2/3$

And $y = 0, -4/3$

Points are $(0,0)$ and $(2/3, -4/3)$

At $(0,0)$

$f_{xx} = 6x + 6 = 6 > 0$

$f_{xx}f_{yy} - f_{xy}^2 = 6 \cdot 2 - 16$

$= -4 < 0$ saddle point.

Neither max^m nor min^m at $(0,0)$

putting $(0,0)$ in eq^n (1) for saddle value.

$\therefore (0,0)$ is saddle value.

At $(2/3, -4/3)$

$$f_{xx} = 6x + 6 = 6 \cdot \frac{2}{3} + 6 = 10 > 0$$

$$f_{xx}f_{yy} - f_{xy}^2 = 10 \cdot 2 - 16 = 4 > 0$$

$\therefore$ The given function gives minima.

min^m value of $\left(\frac{2}{3}, -\frac{4}{3}\right)$

$$f_{min} = \frac{8}{27} + 3 \times \frac{4}{9} + 4 \cdot \frac{2}{3}\left(-\frac{4}{3}\right) + \frac{16}{9}$$

$$= -4/27 \ \#$$

$n = \frac{x^{n+1}}{n+1}$ $-3/1$

Soln is $n \times I.F = \int Q \times I.F\, dy + C$

$$n \times y = \int (2\log n + 1) \times y\, dy + C$$

Beronsoll's eqn reduce to linear form

(I) $\frac{dy}{dn} + \frac{y}{n} = y^2$

(II) $\frac{dy}{dn} + \frac{y \cdot \log y}{n} = \frac{y(\log y)^2}{n^2}$

(I)

Soln

Here,

$$\frac{dy}{dn} + \frac{y}{n} = y^2$$

$$\frac{1}{y^2}\frac{dy}{dn} + \frac{1}{y}\cdot\frac{1}{n} = 1 \quad \text{--- (i)}$$

put $\frac{1}{y} = u$

$$-\frac{1}{y^2}\frac{dy}{dn} = \frac{du}{dn}$$

eqn (i) becomes

$$-\frac{du}{dn} + \frac{1}{n}\cdot u = 1$$

$$\frac{du}{dn} - \frac{u}{n} = -1$$

$$I.F = e^{\int -\frac{1}{n}dn} = e^{-\log n} = e^{\log n^{-1}} = \frac{1}{n}$$

Soln

$$u \times I.F = \int Q \times I.F\, dn + C$$

$$\text{or } u \cdot \frac{1}{n} = \int -1 \cdot \frac{1}{n}\, dn + C$$

$$\text{or } \frac{u}{n} = -\log n + C$$

$$\text{or } \frac{1}{ny} = -\log n + C$$

(II) Soln

Here, $\frac{1}{y(\log y)^2}\frac{dy}{dn} + \frac{1}{\log y}\cdot\frac{1}{n} = \frac{1}{n^2}$

put $\frac{1}{\log y} = u$

$$-\frac{1}{y^2(\log y)^2}\frac{dy}{dn} = \frac{du}{dn}$$

$$\frac{d}{dn}(\log y)^{-1} = \frac{d}{d(\log y)}(\log y)^{-1}\ \frac{d(\log y)}{dy}$$

(11) $f(x,y) = x^3+y^3-3x-12y+10$

Soln

Diff w.r.to x and y partially, we get

$f_x = 3x^2-3$

$f_y = 3y^2-12$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = 0$

for extreme values (for max^m and min^m)

$f_x=0$ and $f_y=0$

$3x^2-3=0$	$3y^2-12=0$
$3x^2=3$	$3y^2=12$
$x=\pm1$	$y=\pm2$

Points are $(1,2)$, $(1,-2)$, $(-1,2)$ and $(1,2)$

At $(1,2)$

$f_{xx} = 6x \Rightarrow 6>0$ $\quad f_{yy}=6y \Rightarrow 12$

$f_{xx}\cdot f_{yy} - f_{xy}^2 \Rightarrow 6\cdot 12 - 0 \Rightarrow 72>0$

∴ The given function gives minima.

minimum value of $(1,2)$

$= 1+8-3-24+10$

$= -8$

At $(1,-2)$

$f_{xx} = 6x \Rightarrow 6>0$ $\quad f_{yy} = 6y \Rightarrow -12$

$f_{xx}f_{yy}-f_{xy}^2 = 6\cdot -12 - 0 \Rightarrow -72<0 \rightarrow$ saddle point

The function neither maxima nor minima.

saddle value is,

$= 1-8-3+24+10$

$= 24$

At $(-1,2)$

$f_{xx} = 6x \Rightarrow -6<0$ $\quad f_{yy} = 6y \Rightarrow 12$

$f_{xx}f_{yy}-f_{xy}^2 \Rightarrow -72<0$

∴ The function gives maxima

max^m value of $(-1,2)$,

$= -1+8+3-24+10$

$= -4$

At $(-1,-2)$

$f_{xx} = 6x \Rightarrow -6<0$ $\quad f_{yy} = 6y \Rightarrow -12$

$f_{xx}\cdot f_{yy} - f_{xy}^2 = -6\cdot -12 - 0 \Rightarrow 72>0 \rightarrow$ saddle point

The function neigther maxima nor minima.

Riccati's eqn

$\frac{dy}{dx} + Py + Ry^2 = Q$ where P, Q, R are f(x) — A non-linear eqn of the form is called Riccati's eqn.

Note:-

(I) If $R=0$, the Riccati's eqn becomes linear equation.

(II) If $Q=0$, the Riccati's eqn becomes Bernoulli's equation.

Q.N (I) Show that the substitution $y = y_1 + u$ where y_1 is soln of Riccati's equation. Reduce the Riccati's eqn to Bernoulli's eqn.

(II) Show that the substitution $y = y_1 + \frac{1}{u}$ where y_1 is soln of Riccati's eqn. Reduce the Riccati's eqn into first order linear eqn.

(I) soln

Prove

we have the Riccati's eqn is.

$$\frac{dy}{dx} + Py + Ry^2 = Q \quad --- (1)$$

put $y = y_1 + u$

diff w.r.to x, we get

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

Now eqn (1) becomes,

$$\left(\frac{dy_1}{dx} + \frac{du}{dx}\right) + P(y_1+u) + R(y_1+u)^2 = Q$$

$$\frac{dy_1}{dx} + \frac{du}{dx} + Py_1 + Pu + Ry_1^2 + 2Ry_1u + Ru^2 = Q$$

$$\left(\frac{dy_1}{dx} + Py_1 + Ry_1^2\right) + \frac{du}{dx} + (P + 2Ry_1)u + Ru^2 = Q$$

$$\cancel{\left(\frac{dy_1}{dx} + Py_1 + Ry_1^2\right) + du}$$

$$Q + \frac{du}{dx} + (P + 2Ry_1)u + Ru^2 = Q$$

$$\frac{du}{dx} + (P + 2Ry_1)u + Ru^2 = 0$$

$$\frac{du}{dx} + (P + 2Ry_1)u = -Ru^2$$

which is Bernoulli's eqn //

(II) soln

we have the Riccati's eqn is

$$\frac{dy}{dx} + Py + Ry^2 = Q \quad (1)$$

put $y = y_1 + \frac{1}{u}$

diff w.r.to x, we get

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{d}{dx}\left(\frac{1}{u}\right)$$

Saddle value is,

$= -1 - 8 + 3 + 24 + 10$

$= 28$ #

For maxima for three variable.

(I) $f_n = 0, f_y = 0, f_z = 0$

(II) $f_{nn} < 0$, $f_{nn} f_{yy} - f_{ny}^2 > 0$, $\begin{vmatrix} f_{nn} & f_{yn} & f_{zn} \\ f_{ny} & f_{yy} & f_{zy} \\ f_{nz} & f_{yz} & f_{zz} \end{vmatrix} < 0$

for minima

(I) $f_n = 0, f_y = 0, f_z = 0$

(II) $f_{nn} > 0$, $f_{nn} f_{yy} - f_{ny}^2 > 0$, $\begin{vmatrix} f_{nn} & f_{yn} & f_{zn} \\ f_{ny} & f_{yy} & f_{zy} \\ f_{nz} & f_{yz} & f_{zz} \end{vmatrix} > 0$

* find Extreme values,

(I) $f(n,y,z) = 2n^2 + ny + 4y^2 + nz + z^2 + 2$

(II) $f(n,y,z) = (n+y+z)^2 - 3(n+y+z) - 24nyz + 1$

$f_n = 4n + y + z$

$f_y = n + 8y$

$f_z = n + 2z$

$f_{nn} = 4$

$f_{yy} = 8$ $\qquad f_{yz} = 0$

$f_{zz} = 2$ $\qquad f_{zn} = 1$

f_{ny}

for extreme value (max or min)

$4n + y + z = 0$

$n + 8y = 0$

$n + 2z = 0$

solving, we get

$4n + \left(\frac{-n}{8}\right) + \left(\frac{-n}{2}\right) = 0$

$n = 0,\ y = 0,\ z = 0$

points are $(0,0,0)$

At $(0,0,0)$

$f_{nn} = 4 > 0$

$f_{nn} \cdot f_{yy} - f_{ny}^2 = 4 \cdot 8 - 1 = 31 > 0$

Also,

$$\begin{vmatrix} f_{nn} & f_{yn} & f_{zn} \\ f_{ny} & f_{yy} & f_{zy} \\ f_{nz} & f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 54 > 0$$

$\therefore$ The function gives minimum value at $(0,0,0)$.

$f_{min} = 2$ #

(II) Soln,

$f(n,y,z) = (n+y+z)^3 - 3(n+y+z) - 24nyz + 1$

$f_n = 3(n+y+z)^2 - 3 - 24yz$

$f_n = 3(n+y+z)^2 - 24yz - 3$

$f_y = 3(x+y+z)^2 - 3 - 24xz$
$f_z = 3(x+y+z)^2 - 3 - 24xy$

$f_{xx} = 6(x+y+z)$
$f_{yy} = 6(x+y+z)$
$f_{zz} = 6(x+y+z)$

$f_{xy} = 6(x+y+z) - 24z$
$f_{yz} = 6(x+y+z) - 24x$
$f_{zx} = 6(x+y+z) - 24y$

aw

for extreme value.

$3(x+y+z)^2 - 3 - 24yz = 0$ —— (I)
$3(x+y+z)^2 - 3 - 24xy = 0$ —— (II)
$3(x+y+z)^2 - 3 - 24xz = 0$ —— (III)

from eq^n (I) and (II), $24yz = 24xz \Rightarrow x = y$
from eq^n (II) and (III), $24xz = 24xy \Rightarrow z = y$
from eq^n (III) and (I), $24yz = 24xy \Rightarrow x = z$

from eq^n (I),

$3(x+x+x)^2 - 3 - 24 \cdot x \cdot x = 0$
$27x^2 - 3 - 24x^2$
$3x^2 - 3 = 0$
$3(x^2 - 1) = 0$
$x = \pm 1$
$y = \pm 1$
$z = \pm 1$

points are $(1,1,1)$ and $(-1,-1,-1)$
At $(1,1,1)$, $f_{xx} = 18$ $f_{yy} = 18$ $f_{zz} = 18$
$f_{xy} = -6$ $f_{yz} = -6$ $f_{zx} = -6$

Now:
$f_{xx} = 18 > 0$
$f_{xx} \cdot f_{yy} - f_{xy}^2 = 18 \cdot 18 - (-6)^2 > 0$

Also, $\begin{vmatrix} f_{xx} & f_{yx} & f_{zx} \\ f_{xy} & f_{yy} & f_{zy} \\ f_{xz} & f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix}$ $3456 > 0$

$f_{min} = 3^3 - 9 - 24 + 1 = -5$

At $(-1,-1,-1)$
$f_{xx} = -18$
$f_{yy} = -18$
$f_{zz} = -18$
$f_{xy} = 6$
$f_{xz} = 6$

Ex $\frac{dy}{dx} + x^2y - xy^2 = 1$ — (1)

$y = x$ is a sol^n of eq^n (1)

put $y = x + \frac{1}{v}$

Now $\frac{dy}{dx} = 1 + \left(-\frac{1}{v^2}\right)\frac{dv}{dx}$

eq^n (1) becomes

$$1 - \frac{1}{v^2}\frac{dv}{dx} + x^2\left(x + \frac{1}{v}\right) - x\left(x + \frac{1}{v}\right)^2 = 1$$

$$-\frac{1}{v^2}\frac{dv}{dx} + x^3 + \frac{x^2}{v} - x^3 - 2x^2\frac{1}{v} - \frac{x}{v^2} = 0$$

$$-\frac{1}{v^2}\frac{dy}{dx} - \frac{x^2}{v} = \frac{x}{v^2}$$

$\therefore \frac{dv}{dx} + x^2 v = -x$ — (2) linear in v

$$I \cdot F = e^{\int x^2 dx} = e^{x^3/3}$$

sol^n is $v \cdot e^{x^3/3} = -\int x e^{x^3/3} + c$

$m = \pm 5$

Roots are real and diff

General soln is,

$$y = Ae^{5x} + Be^{-5x}.$$

(iii) $y'' - 4y' + y = 0$

soln

$m^2 - 4m + 1 = 0$

solving

$m = 2+\sqrt{3},\ 2-\sqrt{3}$

Roots are real and different

General soln is,

$$y = Ae^{(2+\sqrt{3})x} + Be^{(2-\sqrt{3})x}$$

(iv) $4y'' + 4y' - 3y = 0$

soln

$4m^2 + 4m - 3 = 0$

solving

$m = \frac{1}{2}, \frac{-3}{2}$

Roots are real

2nd order non-homogenous diff eqn with constant coeff

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad (R \neq 0)$$

General eqn is $y = y_h + y_p$

where,

y_h = General eqn soln of homogenous eqn.

y_p = particular integral

	Term of R	Choice of y_p	Roots
1)	Ke^{px}	Ce^{px}.	p
2	Kx^2	$K_0 + K_1x^1 + \dots K_nx^n$	0
3	$K\cos qx$ $K\sin qx$	$\cos qx + K\sin qx$	iq
4	$Ke^{\alpha x}\cos \omega x$ $Ke^{\alpha x}\sin \omega x$	$e^{\alpha x}(A\cos\omega x + B\sin\omega x)$	-

Ex

1) $y'' - y = e^x$

Here,

$y'' - y = e^x$

d.F is $m^2 - 1 = 0$

$m = \pm 1$ (Real and different)

$m = 1, -1$

$\therefore y_h = C_1e^x + C_2e^{-x}$

To find y_p,

(I) fonction of the two variable with one subsidiary condition

(II) function of three variable with one subsidiary

(III) function of ~~two~~ three variable with ~~one~~ two subsidiary

(IV) ~~function of three~~

Ex (I) $f = 48 - (x-5)^2 - 3(y-4)^2$ such that $x + 3y = 9$

(II) $f(x,y,z) = xyz$ given $x + y + z$

(I)

Soln

Here, $f(x,y) = 48 - (x-5)^2 - 3(y-4)^2$ such that $x + 3y = 9$

$$\therefore f(y) = 48 - (9-3y-5)^2 - 3(y-4)^2$$
$$= 48 - \cancel{81}6 - 24y + 9y^2 - 3(y^2 - 8y + 16)$$
$$= 48 - 16 + 24y - 9y^2 - 3y^2 + 24y - 48$$
$$= -16 + 48y - 12y^2$$

diff w.r.to y partially, we get

$$f_y = -24y + 48$$
$$f_{yy} = -24 < 0 \text{ (max)}$$

for extreme value, $f_y = 0$

$$-24y + 48 = 0 \Rightarrow y = 2$$
$$x = 3$$

$\therefore$ The given funct^n gives max value at $(3, 2)$

$2+y+z=24$

$f_{max} = 32$

(ii) Here,

$f(x,y,z) = xyz$ gives $x+y+z = 24$

or $f(x,y) = xy(24-x-y) = 24xy - x^2y - xy^2$

diff w.r.to x and y partially,

$f_x = 24y - 2xy - y^2$

$f_y = 24x - x^2 - 2xy$

$f_{xx} = -2y$

$f_{yy} = -2x$

$f_{xy} = 24 - 2x - 2y$

for extreme values,

$f_x = 0 \quad f_y = 0$

$24y - 2xy - y^2 = 0 \;\cdots (i) \quad y(24 - 2x - y) = 0$

$24x - x^2 - 2xy = 0 \;\cdots (ii) \quad x(24 - x - 2y) = 0$

$x = 0, y = 0$

Solving (i) and (ii) $\quad x(24 - x - 2y) = y(24 - 2x - y)$

$x = 8$

$y = 8$

$\therefore$ points are $(0,0,24)$ and $(8,8,8)$

At $(8,8,8)$

$f_{xx} = -2 \times 8 = -16 > 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = (-16)\cdot(-16) - \{24 - 16 - 16\}^2 > 0$

$\therefore$ The give funt^n gives max value at $(8,8,8)$.

$f_{max} = 512$

At $(0,0,24)$

$f_{xx} = 0 \quad$ neither max^m nor min^m

(iii) $f(x,y,z) = x^2 + y^2 + z^2$ such that $x + y + z = 1$ $\cancel{x + y + z = 4}$

$2y + z = 2$

Here,

$f(z) = (1-z)^2 + \left(\frac{2-z}{2}\right) + z^2$

$f(z) = 1 - 2z$

$f(z) = 1 - 2z + 2z^2 + \frac{1}{4}(4 - 4z + z^2)$

diff w.r.to z partially

$f_z = -2 + 4z + \frac{1}{4}(4 + 2z)$

$f_z = 4 + \frac{1}{4}(\cdot 0 + 2)$

$= 4 + \frac{1}{2} = \frac{9}{2} > 0$ (minimum)

for extreme value,

$f_z = 0$

or $-2 + 4z - 1 + \frac{1}{2}z = 0$

or $-3 + 4z + \frac{1}{2}z = 0$

Here, $R = e^x$

Let us choose $y_p = ce^x x$

diff w.r.to x, we get

$y' = ce^x + cxe^x$

$y'' = ce^x + ce^x + cxe^x$

substituting the values of y, y', y'' in ①: $2ce^x + cxe^x - cxe^x - e^x \Rightarrow \therefore 2c = 1$

$c = 1/2$

$\therefore y_p = \frac{1}{2} xe^x$

Req general soln as

$y = y_h + y_p$

or, $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$

2) $y'' + y' - 2y = 14$

Here, $y'' + y' - 2y = 14$ ---- ①

$\therefore m = 1, -2$

$y_h = c_1 e^x + c_2 e^{-2x}$

To find y_p

Here, $R = 14 + 2x - 2x^2 \rightarrow$ highest term

Let us chose $y_p = K_0 + K_1 x + K_2 x^2$

diff w.r.to x, we get

$y' = K_1 + 2K_2 x$

$y'' = 2K_2$

$2K_2 + K_1 + K_2 x - 2(K_0 + K_1 x + K_2 x^2) = 14 + 2x - 2x^2$

Eq. coeff of x^2 $-2K_2 = -2 \Rightarrow K_2 = 1$

Eq. coeff of x, $2K_2 - 2K_1 = 2 \Rightarrow K_1 = 1$

Eq. coeff of constant term, $2K_2 + K_1 - 2K_0 = 14$

$2 + 0 - 2K_0 = 14 \Rightarrow K_0 = -6$

$\therefore y_p = -6 + x^2 = x^2 - 6$

Req general soln is

$y = y_h + y_p$

$= c_1 e^x + c_2 e^{-2x} + (-6) + x^2 - x^2 + 6$

$c_1 e^x + c_2 e^{2x}$ //

3) $y'' - y - 2y = 10\cos x$

Here,

$y'' - y - 2y = 10\cos x$ ---- ①

A.E. is $m^2 - m - 2 = 0$

or $m^2 - 2m + m - 1 = 0$

or $m(m-2) + 1(m-2) = 0$

$m = -1, 2$

$y_h = c_1 e^{-x} + c_2 e^{2x}$

$R = 10\cos x$

Let us choose, $y_p = L\cos x + M\sin x$

Diff w.r.to x, $y' = -L\sin x + M\cos x$

$y'' = -L\cos x - M\sin x$

Substituting the values of y, y', y'' in ①

$-L\cos x - M\sin x + L\sin x - M\cos x - 2L\cos x - 2M\sin x$

$= 10\cos x$

$\frac{9z}{2} = 3$

$z = \frac{6}{9} = \frac{2}{3}$

Also $x = 1 - z = 1 - \frac{2}{3} = \frac{1}{3}$

$y = \frac{2-z}{2} = \frac{2 - \frac{2}{3}}{2} = \frac{2}{3}$

point $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

$f_{min} = 1$ #

(iv) $f(x,y,z) = x^2+y^2+z^2$ such that $x+y+z = 1$, $xyz = -1$

$f(x,y,z) = x^2 + (y+z)^2 - 2yz$

$= x^2 + (1-x)^2 - 2yz$

$= x^2 + 1 - 2x + x^2 + \frac{2}{x}$

$f(x) = 2x^2 - 2x + \frac{2}{x} + 1$

diff w.r.t. to x partially,

$f_x = 4x - 2 - \frac{2}{x^2}$

$f_{xx} = 4 + \frac{4}{x^3}$

for extreme value

$f_x = 0$

$\therefore 4x - 2 - \frac{2}{x^2} = 0$

$\Rightarrow 4x^3 - 2x^2 - 2 = 0$

$\Rightarrow 2x^3 - x^2 - 1 = 0$

$\Rightarrow 2x^3 - 2x^2 + x^2 - 1 = 0$

$\Rightarrow 2x^2(x-1) + (x-1)(x+1) = 0$

$(x-1)(2x^2+x+1) = 0$

$\therefore x = 1$

for $x = 1$, $y+z = 0$, $yz = -1$

Solving we get

$y(-y) = -1$

$y^2 = 1 \Rightarrow y = \pm 1$

$z = \pm 1$

Point $(1, \pm 1, \pm 1)$

$(1, 1, -1)$, $(1, -1, 1)$

~~At~~ only this points satisfy the $x+y+z=1$ and $xyz=-1$ equation.

At $(1, 1, -1)$

$f_{xx} = 4 + \frac{4}{x^3} = 8 > 0$

$\therefore$ The given fun gives min value at $(1, 1, -1)$

Q) find the dimension of rectangular box open at the top with volm 32 cc requiring lest material for the construction

soln,

Let x, y, z be length, breath and height of the rectangular box. Let V be the volume and S be surface area of the box. By question

$$V = xyz = 32 \Rightarrow y = 32/xz$$

and $S = xz + xz + yz + yz + xy$

$$\therefore S(x,z) = xz + xz + 2z \cdot \frac{32}{xz} + x \cdot \frac{32}{xz}$$

$$= \cancel{2xz + 2yz + xy}$$

$$= 2xz + \frac{64}{x} + \frac{32}{z}$$

diff w.r.to x and z partially, we get

$S_x = 2z - \frac{64}{x^2}$ $S_z = 2x - \frac{32}{z^2}$

$S_{xx} = \frac{128}{x^3}$ $S_{zz} = \frac{64}{z^3}$

$S_{xz} = 2$

for extreme values

$S_x = 0$ and $S_z = 0$

$2z - \frac{64}{x^2} = 0$ and $2x - \frac{32}{x2} = 0$

solving we get

$2x^2 - 32 = 0$ — (I)

$2xz^2 - 32 = 0$ — (II)

$2x^2z^2 = 2zx^2$

$x = 2z$

from eqn (II)

$2 \cdot 4x^2 = 32$

$z^3 = 8$

$z = 2$

$x = 4$

Also, $xyz = 32$

$y = \frac{32}{xz} = \frac{32}{2\times4} = 4$

$\therefore (x, y, z) = (4, 4, 2)$

$S_{xx} = \frac{128}{4^3} = 2 > 0$

and

$S_{xx}S_{yy} - S_{xz}^2 = 2 \cdot 8 - 4 = 12 > 0$

$\therefore$ Surface area is Minimum when $x = 4, y = 4, z = 2$

$(-3l-m)\cos x+(l-3m)\sin x=10\cos x$

Eq coeff $\cos x$, $-3l-m=10$

Eq coeff $\sin x$, $l-3m=0$

$l=3m$

$-9m-m=10$

$8m=10$

$m=5/4$

$l=15/4$

$y_p=\frac{15}{4}\cos x+\frac{5}{4}\sin x$

General eqn is

$y=y_h+y_p$

$=c_1e^{-x}+c_2e^{2x}+\frac{15}{4}\cos x+\frac{5}{4}\sin x$

Method of variable of parameter

General soln $y=y_h+y_p$

$y_p=y_1\frac{y_2R}{w}dx+y_2\int\frac{y_1R}{w}R$

Wronskian $(w)=y_1y_2'-y_2y_1'$

Rolle's Theorem, Lar

Q) If the sum of dimension of rectangular swimming pool is given. prove that the amount of water in the pool is max^m when it is cube.

Sol^n

Let x, y, z be length, breath and height of rectangular swimming pool. Let v be the volume of water in swimming pool.

$$x+y+z = p \text{ (given)}$$

Also,

$$V = xyz$$

$$\therefore v(x,y) = xy(p-x-y)$$

$$v(x,y) = pxy - x^2y - xy^2$$

diff w.r.to x and y partially, we get

$$V_x = py - 2xy - y^2$$

$$V_{xx} = -2y$$

$$V_y = px - x^2 - 2xy$$

$$V_{yy} = -2x$$

$$V_{xy} = p - 2x - 2y$$

for extreme values,

$$V_x = 0 \quad V_y = 0$$

$$py - 2xy - y^2 = 0 \quad px - x^2 - 2xy = 0$$

solving

$$py - y^2 = 2xy \quad \text{--- (I)}$$

$$px - x^2 = 2xy \quad \text{--- (II)}$$

$$p - y = 2x \quad \text{--- (I)}$$

$$p - x = 2x \quad \text{--- (II)}$$

Date

solving we get

$$2x+y = x+2y$$

$$x = y$$

from eq^n (I) $p - 2x - x = 0$

$$p = 3x$$

$$x = p/3$$

$$y = p/3$$

Also, $x+y+z = p$

$$z = p - x - y = p - p/3 - p/3 = p/3$$

$$\therefore x = z = y = p/3$$

Now,

$$V_{xx} = -2y = -2p/3 < 0$$

$$V_{xx} \cdot V_{yy} - V_{xy}^2 = (-2p/3)(-2p/3) - (p - 2p/3 - 2p/3)^2$$

$$= \frac{4p^2}{9} - (-p/3)^2$$

$$= \frac{4p^2}{9} - p^2/9$$

$$= \frac{3p^2}{9} = \frac{p^2}{3} > 0$$

$\therefore$ value of max^m when $x = y = z$

(8) $y'' - 2y' + y = \frac{12e^{x}}{x^3}$

Soln

A.E is $m^2 - 2m + 1 = 0$

or $(m-1)^2 = 0$

$\therefore m = 1, 1$

$y_h = (C_1 + C_2 x)e^{x}$

$y_1 = e^{x}, \quad y_2 = xe^{x}$

$y_1' = e^{x}, \quad y_2' = e^{x} + xe^{x}$

we have, $w = y_1 y_2' - y_2 y_1'$

$= e^{x}(e^{x} + xe^{x}) - xe^{x} \times e^{x}$

$= e^{2x} + xe^{2x} - xe^{2x}$

$= e^{2x}$

The second order homogenous differential eqn with variable coeff of the form $x^2y'' + axy' + by = 0$ is called Euler Cauchy eqn and soln

A.E is $m^2 + (a-1)m + b = 0$

Roots	General soln
(I) Real and diff ie α, β	$y = C_1 x^{\alpha} + C_2 x^{\beta}$
(II) Real and equal ie α, α	$y = (C_1 + C_2 \log x) x^{\alpha}$
(III) Complex roots $\alpha + i\beta$	$y = x^{\alpha}\{A\cos(\beta \log x) + B \sin(\beta \log x)\}$

(1) $x^2y'' - \frac{5}{2}xy' - 2y = 0$

Soln

A.E is $m^2 + \left(-\frac{5}{2} - 1\right)m + (-2) = 0$

$m^2 - \frac{7}{2}m - 2 = 0$

$2m^2 - 7m - 4 = 0$

$2m^2 - 8m + m - 4 = 0$

$2m(m-4) + 1(m-4) = 0$

$m = 4, -1/2$

General eqn is $y = C_1 x^4 + C_2 x^{-1/2}$

(iii) $x^2 y'' + 7xy' + 13y = 0$

Soln

A.E is $m^2 + (7-1)m + 13 = 0$

or $m^2 + 6m + 13 = 0$

$m = -3 \pm 2i$

$y = x^{-3} \{ A\cos(2\log x) + B\sin(2\log x) \}$ Ans

Rolle's Theorem, Lagrange's mean value Theorem

Statement: If a function $f(x)$ is

(i) continuous on closed interval $[a,b]$

(ii) differentiable on open interval (a,b)

(iii) $f(a) = f(b)$

Then there exist at least one point $c \in (a,b)$ such that $f'(c) = 0$

$f(x)$ at continous $[a,b]$

max (M) — min (m)

$F(c) = M$ — $f(a) = m$

(i) $M = m$ then $f'(x) = 0$ for all x.

(ii) $M \neq m$

$M \neq f(a)$ or $f(b)$

$c \neq a$, $c \neq b$

$c \in (a,b)$

Prove

Suppose a function $f(x)$ is conti on close interval $[a,b]$ than it has greatest value say (M) and lest value say (m) in the interval $[a,b]$. Let $f(c) = M$ (max value) and $f(a) = m$ (minimum value) then two cases arise.

Case I

(i) If $M = m$ then the function is ~~continous~~ constant

$\therefore f'(n) = 0$ for all n in this case. Theorem proved.

(ii) If $M \neq m$ then at least one of the point differential from $f(a)$ or $f(b)$

$c \neq a,\ c \neq b$

$c \in (a,b)$

Since, the function $f(n)$ is differentiable on open interval (a,b) and $c \in (a,b)$.
The function $f(n)$ is differentiable at $n=c$

R.H.D at $n=c$

$$\lim_{n \to c^+} \frac{f(n)-f(c)}{n-c} \leq 0 \text{ —— (i)}$$

L.H.D at $n=c$

$$\lim_{n \to c^-} \frac{f(n)-f(c)}{n-c} \geq 0 \text{ —— (ii)}$$

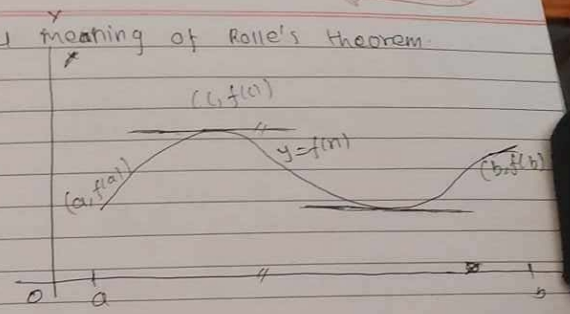
Since the function $f(n)$ is differentiable at $n=c$ for this left hand derivative must be equal to Right hand derivative. This will be possible only when (1) and (2) equal to zero. Therefore $f'(c)=0$ will $c \in (a,b)$

IIIly,

If minimum value (m) is different from $f(a)$ or $f(b)$ it can be shown that $f'(d)=0$ with $c \in (a,b)$

~~unconditi~~

Gometrical meaning of Rolle's theorem.



If the function $y=f(n)$ satisfy all condition of Rolle's theorem Then there exist at least one point $(c,f(c))$ between $(a,f(a))$ and $(b,f(b))$ where the tangent parallel to x-axis.

Q. State and prove Lagrange's Mean value Theorem

→

Statement:- If a function $f(n)$ is
(i) continous or closed interval $[a,b]$
(ii) differentiable or open interval (a,b)

Then there exist at least one point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

prove:

Suppose the function $f(n)$ is continous on closed interval $[a,b]$ and differentiable on open interval (a,b) not define the ϕ function

$$\phi(n) = f(n) + A \cdot n \text{ —— (1).}$$

Q) If the sum of dimension of rectangular swimming pool is given. prove that the [illegible] of water in the pool is maxm when it is [illegible]

soln

Let x, y, z be length, breath and height of rectangular swimming pool. Let v be the [illegible] of water in swimming pool.

$$x+y+z = p \text{ (given)}$$

Also,

$$v = xyz$$

$$\therefore v(x,y) = xy(p-x-y)$$

$$\therefore v(x,y) = pxy - x^2y - xy^2$$

diff w.r.to x and y partially, we get

$$v_x = py - 2xy - y^2$$

$$v_{xx} = -2y$$

$$v_y = px - x^2 - 2xy$$

$$v_{yy} = -2x$$

$$v_{xy} = p - 2x - 2y$$

for extreme values,

$$v_x = 0 \quad v_y = 0$$

$$py - 2xy - y^2 = 0 \qquad px - x^2 - 2xy = 0$$

solving

$$py - y^2 = 2xy \quad \text{--- (I)}$$

$$px - x^2 = 2xy \quad \text{--- (II)}$$

$$p - y = 2x \quad \text{--- (I)}$$

$$p - x = 2x \quad \text{--- (II)}$$

solving, we get

$$2x + y = x + 2y$$

$$x = y$$

from eqn (I) $p - 2x - x = 0$

$$p = 3x$$

$$x = p/3$$

$$y = p/3$$

Also, $x + y + z = p$

$$z = p - x - y = p - p/3 - p/3 = p/3$$

$$\therefore x = z = y = p/3$$

Now,

$$v_{xx} = -2y = -2p/3 < 0$$

$$v_{xx} \cdot v_{yy} - v_{xy}^2 = (-2p/3)(-2p/3) - (p - 2p/3 - 2p/3)^2$$

$$= \frac{4p^2}{9} - (-p/3)^2$$

$$= \frac{4p^2}{9} - p^2/9$$

$$= \frac{3p^2}{9} = \frac{p^2}{3} > 0$$

$\therefore$ value of maxm when $x = y = z$

where A is constant suitablely choosen that $\phi(a) = \phi(b)$

$$\therefore f(a) + A\cdot a = f(b) + Ab$$

$$\therefore A = \frac{f(b) - f(a)}{a-b}$$

$$\therefore \phi(x) = f(x) + \frac{f(b)-f(a)}{a-b}\cdot x \quad \text{---} (2)$$

Now $\phi(x)$ is

(i) continuous on $[a,b]$ because $f(x)$ and $A\cdot x$ are continuous on $[a,b]$ and sum of two continous function is again continous.

(ii) differentiable on (a,b) because both $f(x)$ and x are differentiable on (a,b) and sum of two differentiable function is again differentiable

(iii) $\phi(a) = \phi(b)$

$\therefore \phi(x)$ satisfies all condition of rolle's Theorem. So by rolle's theorem there exist at least one point $c \in (a,b)$ such that $\phi'(c) = 0$

Now diff eqn (2) w.r.to x

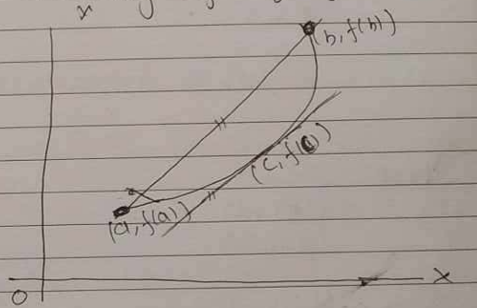
$$\phi'(x) = f'(x) = \frac{f(b)-f(a)}{a-b}$$

$$\phi'(c) = f'(c) + \frac{f(b)-f(a)}{a-b}$$

$$0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \underline{\text{proved}}$$

Geometrical meaning of Lagrange's theorem



If the $y = f(x)$ satisfies all condition of Lagrange's mean value theorem then there exist at least one point ~~$c \in (a,b)$~~ $(c, f(c))$ between $(a, f(a))$ and $(b, f(b))$ where the tangent is parall to cord joining the end point.

verify rolle's theorem

(i) $f(x) = x^2 - 4x + 3$ for $1 \le x \le 3$

(ii) $f(x) = (x-2)(x-3)(x-4)$ for $[2,4]$

(iii) $f(x) = \frac{\sin x}{e^x}$ for $[0, \pi]$

Then

from eq^n (ii)

$$9-3x^2-x^2=0$$
$$9-4x^2=0$$
$$x^2=9/4$$
$$x=3/2$$
$$y=3/2$$

now from eq^n (i) given,

$$\frac{9}{4}+\frac{9}{4}+\cancel{\frac{4}{9}+\frac{4}{9}}+z^2=9$$

$$z^2=9-\cancel{\frac{4}{9}-\frac{4}{9}}-9/4-9/4$$

$$z=9/2$$

The largest

now

$$V_{xx}=-6x$$
$$=-6\times\frac{9}{2}$$
$$=-9<0$$

$$V_{xy}=9-3\times\frac{3}{2}-3\times\frac{3}{2}$$
$$=\frac{18-9-9}{2}=0$$

$$V_{yy}=-6y$$
$$=-9<0$$

now

$$V_{xx}\cdot V_{yy}-V_{xy}^2=-9\times-9-(0)^2$$
$$=81>0$$

The max^m value at $x=y\neq z$

* Lagrange's Method

function of two variable $f(x,y)$ under condition $\phi(x,y)$ is equal to zero.

* For max^m value

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} > 0$$

* for min^m value,

$$\begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} < 0$$

function of three variable $f(x,y,z)$ under condition $\phi(x,y,z) = 0$

Method of finding max and min

(i) $F(x,y,z,\lambda) = f + \lambda\phi$

(ii) $F_x\, F_y\,,\, F_z\, F_\lambda$ (partial derivative)

(iii) for extreme value

$F_x = 0 \quad F_y = 0 \quad F_z = 0\,,\, F_\lambda = 0$

(iv) $H_1 = \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{xy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix}$ $\quad H_2 = \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{yx} & f_{zx} \\ \phi_y & f_{xy} & f_{yy} & f_{zy} \\ \phi_z & f_{xz} & f_{yz} & f_{zz} \end{vmatrix}$

$\exists$ - There exist
$\forall$ - for all

(i) $f(x) = x^2 - 4x + 3$ for $1 \le x \le 3$

Here,

$f(x) = x^2 - 4x + 3$ in $[1, 3]$.

This function $f(x)$ is ploynomial function. so it is continous on $[1,3]$. also

$f'(x) = 2x - 4$ exist for all values of $(1, 3)$.

$\therefore f(x)$ is differential on $(1,3)$

Also,

$f(1) = 0$

$f(3) = 0$

$f(1) = f(3)$

$\therefore$ The given $f(x)$ satisfy all condition of Rolle's theorem. so by rolle's theorm then,

$\exists$ $c \in (1,3)$ such that

$f'(c) = 0$

$2c - 4 = 0$

$c = 2 \in (1,3)$

Rolle's theorem is verified.

(b) verify Lagrange's mean value theorem

(i) $f(x) = x(x-1)(x-2)$ on $[0, 1/2]$

soln

$f(x) = x^3 - 3x^2 + 2x$ in $[0, 1/2]$

This function $f(x)$ is polynomial function so. It is continuous on $[0, 1/2]$. Also

$f'(x) = 3x^2 - 6x + 2$. which exist for all values of $x \in (0, 1/2)$

$\therefore f(x)$ satisfies all condition of L.M.V Theorem. so By L.M.V Theorem $\exists$ at least one point $c \in (0, 1/2)$ such that

$$f'(c) = \frac{f(1/2) - f(0)}{\frac{1}{2} - 0}$$

$$3c^2 - 6c + 2 = \frac{3}{4}$$

$$12c^2 - 24c + 8 = 3$$

solving

$c = 1.76$ or 0.23

$\therefore c = 0.23 \in (0, 1)$

Hence L.M.V Theorem is verified

* can rolle's theorem be verified for function
(i) $f(x) = \log x$ in $[-2,2]$
(ii) $f(x) = \frac{1}{x^2}$ in $[-1,1]$

Sol^n

$f(x) = \log x$ in $[-2,2]$ Rolle's theorem cannot be applicable for this function because value of log of -ve number can't be define. Therefore, given function is not continous in $[-2,2]$

$f(x) = \frac{1}{x^2}$ in $[-1,1]$

Rolle's theorem can't be applicable for the function $f(x) = \frac{1}{x^2}$ in $[-1,1]$. Therefore $f(x)$ is not continous at $x=0$ which belong to $[-1,1]$.

* can Lagrange's mean value theorem verified for the function
$f(x) = (x-2)^{2/3}$ in $[1,4]$

Here, $f(x) = (x-2)^{2/3}$ in $[1,4]$
continous for all value of x.

$f'(x) = \frac{2}{3} \frac{1}{(x-2)^{1/3}}$ doesn't diff at $x=2$.

* Taylor's series and maclaurin's series
If $f(x)$ is first a derivative $f', f'', \dots f^n$ are all such that

(i) continuous on $[a,b]$
(ii) f^n is differentiable in (a,b). $\exists$ $c \in (a,b)$

such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!} f''(a) + \frac{(b-a)^3}{3!} f'''(a) + \dots + \frac{(b-a)^n}{n!} f^n(a) + \frac{(b-a)^{n+1}}{(n+1)!} f^{n+1}(a) \quad \text{--- (i)}$$

put $b = a+h$ Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^n}{n!} f^n(a) + \frac{h^{n+1}}{(n+1)!} f^{n+1}(a) \quad \text{--- (ii)}$$

Taylor's series in infinite Term.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a) + \dots \quad \text{--- (iii)}$$

put $a=0$ and $h=x$ Then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) \dots + \frac{x^n}{n!} f^n(0) + \dots \quad \text{--- (iv)}$$

* Show that $y = \sqrt{x}$ cannot be expand by maclaurin's infinite series.

* $f(x) = (x-2)(x-3)(x-4)$ for $[2,4]$

Soln

$$f(x) = x(x-3) - 2(x-3)(x-4)$$
$$= (x^2 - 3x - 2x + 6)(x-4)$$
$$= x(x^2 - 5x + 6) - 4(x^2 - 5x + 6)$$
$$= x^3 - 5x^2 + 6x - 4x^2 + 20x - 24$$
$$= x^3 - 9x^2 + 26x - 24 = 0$$

This function $f(x)$ is ploynomial function so it is continous on $[2,4]$ also.

$f'(x) = 3x^2 - 18x + 26$ exist for all values of $(2,4)$

$\therefore f(x)$ is differential on $(2,4)$

Also

for Max^m: $H_1 > 0$ and $H_2 < 0$
for min^m: $H_1 < 0$ and $H_2 < 0$

* find minimum value of $x^2+y^2+z^2$ when $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$
here,

$f(x,y,z) = x^2+y^2+z^2$, $\phi(x,y,z) = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$

define

$$F(x,y,z) = f + \lambda\phi$$
$$= x^2+y^2+z^2 + \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

diff w.r.to x,y,z and λ partially, we get

$$F_x = 2x - \frac{\lambda}{x^2}$$

$$F_y = 2y - \lambda/y^2$$

$$F_z = 2y - \lambda/z^2$$

$$F_\lambda = \frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1$$

for extreme value,

$F_x = 0 \Rightarrow 2x - \lambda/x^2 = 0 \Rightarrow x = (\lambda/2)^{1/3}$

$F_y = 0 \Rightarrow 2y - \lambda/y^2 = 0 \Rightarrow y = (\lambda/2)^{1/3}$

$F_z = 0 \Rightarrow 2z - \lambda/y^2 = 0 \Rightarrow z = (\lambda/2)^{1/3}$

solving, we get

$$3(\lambda/2)^{-1/3} = 1$$

$$(\lambda/2)^{-1/3} = 1/3$$

$$(\lambda/2)^{1/3} = 3$$

$$\lambda/2 = 27$$

$$\lambda = 54$$

$\therefore x = 3,\ y = 3,\ z = 3$

Now,

$$H_1 = \begin{vmatrix} 0 & \phi_x & \phi_y \\ \phi_x & f_{xx} & f_{yy} \\ \phi_y & f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 \\ -1/9 & 2 & 0 \\ -1/9 & 0 & 2 \end{vmatrix}$$

$$= (-1/9)(-1/9)\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{81} \times (2 \times -2)$$

$$= -\frac{4}{81} < 0$$

$$H_2 = \begin{vmatrix} 0 & \phi_x & \phi_y & \phi_z \\ \phi_x & f_{xx} & f_{yx} & f_{zx} \\ \phi_y & f_{xy} & f_{yy} & f_{zy} \\ \phi_z & f_{xz} & f_{yz} & f_{zz} \end{vmatrix} = \begin{vmatrix} 0 & -1/9 & -1/9 & -1/9 \\ -1/9 & 2 & 0 & 0 \\ -1/9 & 0 & 2 & 0 \\ -1/9 & 0 & 0 & 2 \end{vmatrix}$$

$$= (-1/9)(-1/9)\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{vmatrix}$$

$$= \frac{1}{81}\left\{ -1\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{vmatrix} \right\}$$

$\frac{-4}{27} < 0$

$\therefore H_1 < 0$ and $H_2 < 0$.

$\therefore$ The given function gives minimum.

$$f_{min} = 3^2+3^2+3^3 = 27$$

[C - orbitary const]

* Differential equation

An equation which involves derivative is called differential equation.

Order and degree of diff eqⁿ.

The order of differential equation is the order of highest order derivative term involve in the equation.

The degree of differential equation is the power of highest order derivative term in the equation after it has been made free from fraction and radical signs.

* Solution of differential equation.

Any relation between variables which doesn't contain their derivative and satisfyies the differential equation is called solution of differential equation.

(i) General solution

(ii) particular solution.

If the no. of orbitary const in the solution is equal to order of diff equation then this type of solution is called general solution.