Chapter-4: Discrete Filter Structures

❑ FIR Filter, Structures for FIR Filter:

1. Introduction:

- For the ***design of digital filters***, the ***system function H(z)*** or the ***impulse response h[n]*** must be specified. Then the ***digital filter structure*** can be implemented in ***hardware/software*** form by its ***difference equation*** obtained directly from H(z) or h[n].

- To implement the specified ***difference equation*** of the system, the required basic operations are ***addition, delay and multiplication by a constant***.

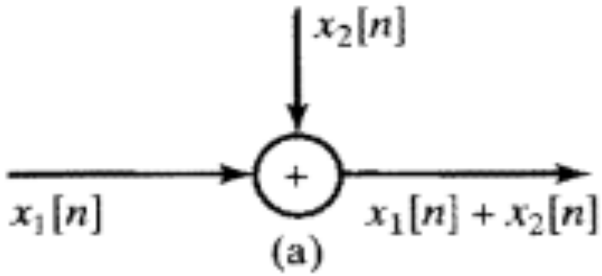
- The ***structures are derived*** on the basis of ***computational complexity***, ***ease of implementation of finite word length effect*** etc.

2. Block Diagram Representation:

- When the system function H(z) or the impulse response h[n] is specified then the digital filters can be implemented or realized using ***block diagram***. The following are the ***basic elements required for the implementation***.

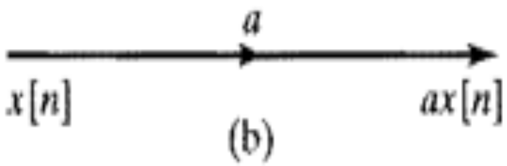
a. An adder:

- It performs the addition of two signals.



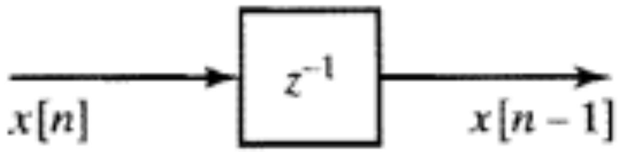
b. A Constant Multiplier:

- It multiplies a signal with a constant ***'a'***.



c. A Unit Delay Element:

➢ It delays the input sequence by one element.



(c)

3. Advantages of Representing the Digital System (i.e. Filters) in Block Diagram Form:

i. The computation algorithm can be easily written just by inspection.

ii. The hardware requirement can be easily determined.

iii. The relationship between input and output can be easily determined.

4. Canonic and Non-canonic Structures:

➢ If the ***number of delays*** in the structure or realization block diagram is ***equal*** to the ***order of the difference equation or the order of the system function*** of a digital filter, then the structure is canonic otherwise non-canonic.

4. Structures for FIR Systems:

- A ***causal FIR system*** can be described by the ***difference equation***

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] \quad \text{.....i}$$

Or, equivalently, by the ***system function***

$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \quad \text{.....ii}$$

- If we replace b_k by $h[k]$, we obtain

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k] \quad \text{.....iii}$$

(*convolution sum*)

Or, $$H(z) = \sum_{k=0}^{M-1} h[k]\, z^{-k} \quad \text{.....iv}$$

- Therefore, we can write

$$h[n] = \begin{cases} b_n, & 0 \le n \le M-1 \\ 0\,, & Otherwise \end{cases} \quad \text{.....v}$$

- **Note that FIR filter is called all-zero filter (or comb filter)**.

- There are several methods for implementation of FIR system. They are:

a. Direct form structures
b. Cascade form structures
c. Frequency sampling structures
d. Lattice structures
e. Linear phase structures

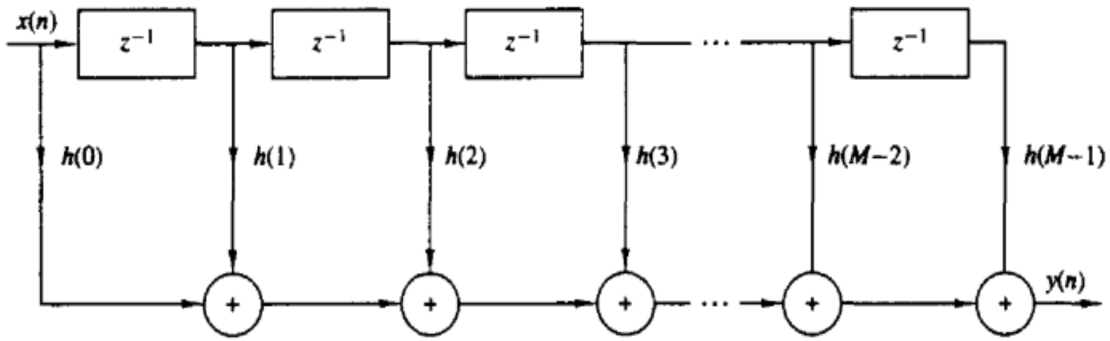
a. Direct Form Structures:

- The direct-form realization follows immediately from the non-recursive difference equation or, equivalently, by the convolution summation

$$y[n] = \sum_{k=0}^{M-1} h[k]\, x[n-k] \qquad \text{.....i}$$

Or, $y[n] = h[0]\, x[n] + h[1]\, x[n-1] + \cdots + h[M-1]\, x[n-(M-1)]$ ii

- The direct form structure can be realized as shown in figure below:



➢ Because of the ***chain of delay elements across the top of the diagram***, this structure is also referred to as a ***tapped delay line structure or a transversal filter structure***.

b. Cascade Form Structures:

➢ The ***cascade*** **form structure** of FIR system is obtained by ***factoring the polynomial system function*** as:

$$H(z) = \prod_{k=1}^{K} H_k(z)$$

where

$$H_k(z) = b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2} \qquad k = 1, 2, \ldots, K$$

where, K= integer part of $\frac{M+1}{2}$.

- If ***M is odd*** then $\boldsymbol{K = \frac{M+1}{2}}$.
- If ***M is even*** then ***K***= $\boldsymbol{\frac{M}{2}}$ with $\boldsymbol{b_{k2} = 0}$.

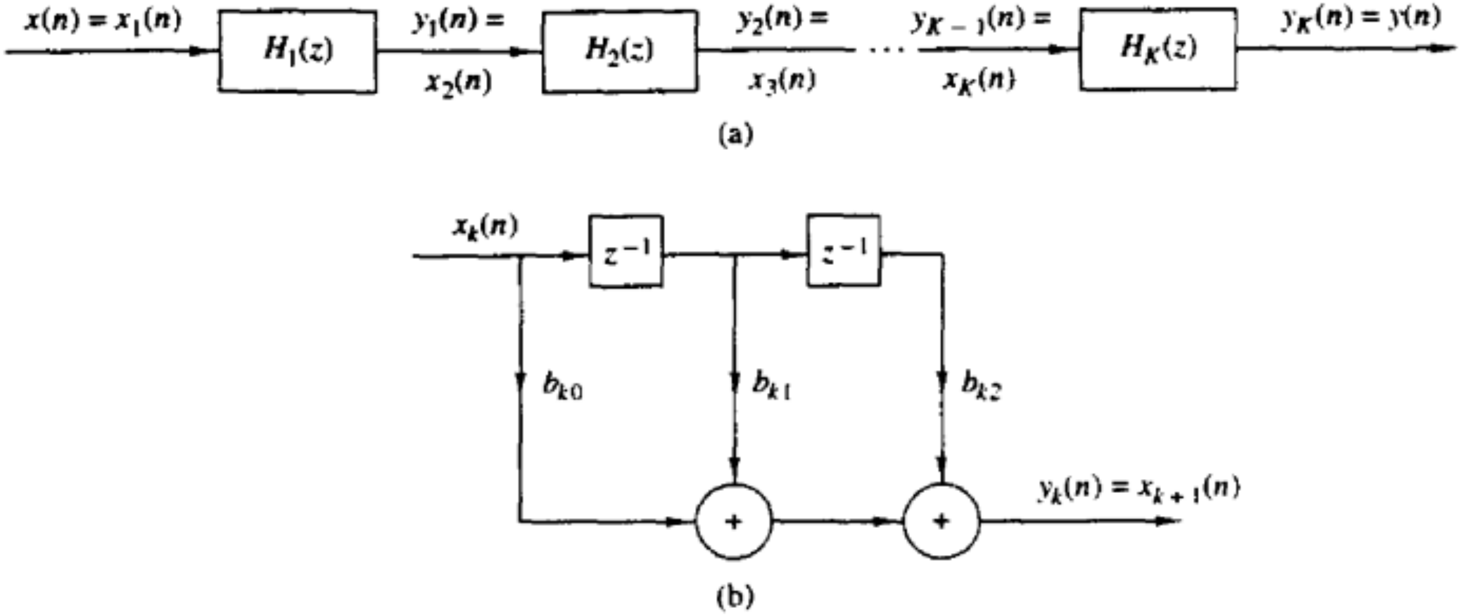


Fig: Cascade realization of an FIR system

Example 5.5: Obtain the direct-form and cascade-form realizations for the transfer function of an FIR system given by

(a) $H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$

(b) $H(z) = 1+\frac{5}{2}z^{-1}+2z^{-2}+2z^{-3}$

Solution:

(a) $H(z) = (1+2z^{-1}+z^{-2})(1+z^{-1}-z^{-2})$

(i) **Direct-form realization:**

→ For this, we have

$$H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$$

$$= 1+z^{-1}-z^{-2}+2z^{-1}+2z^{-2}-2z^{-3}-z^{-2}-z^{-3}+z^{-4}$$

$$\therefore H(z) = 1+3z^{-1}-3z^{-3}+z^{-4}$$

The direct-form realization is shown in fig 5.22.

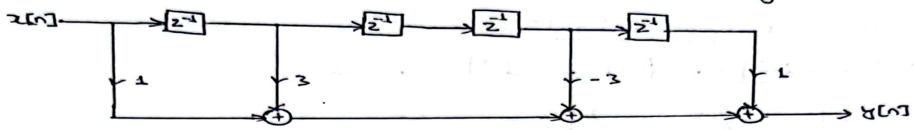


Fig 5.22: Direct-form realization

(ii) cascade-form realization:

→ For this, we have

$$H(z) = (1+2z^{-1}-z^{-2})(1+z^{-1}-z^{-2})$$

$$H(z) = H_1(z)\ H_2(z)$$

where, $H_1(z) = (1+2z^{-1}-z^{-2})$

and $H_2(z) = (1+z^{-1}-z^{-2})$

The cascade-form realization is shown in fig 5.23.

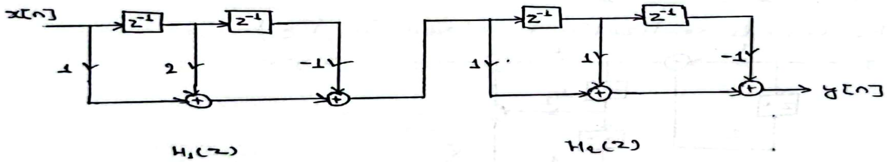


Fig 5.23: cascade-for realization

Note:
Fig 5.23 can be redrawn for cascade-form realization as shown in fig 5.24 below.

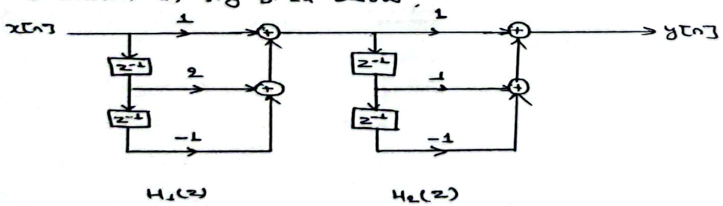


Fig 5.24

(24)

(b) $H(z) = 1 + \frac{5}{2} z^{-1} + 2z^{-2} + 2z^{-3}$

(i) Direct-Form realization:

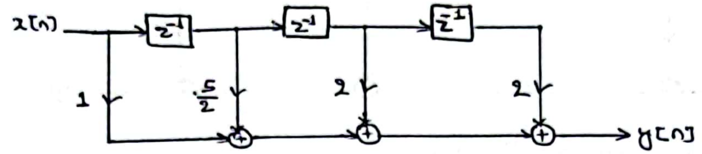


Fig 5.25: Direct-form realization

(ii) cascade-form realization:

→ For this, we have

$$H(z) = 1 + \frac{5}{2} z^{-1} + 2z^{-2} + 2z^{-3}$$

or $H(z) = (1+2z^{-1})(1+\frac{1}{2}z^{-1}+z^{-2}) = H_1(z)\,H_2(z)$

$\therefore$ $H_1(z) = (1+2z^{-1})$

and $H_2(z) = (1+\frac{1}{2}z^{-1}+z^{-2})$

The cascade-realization is shown in fig

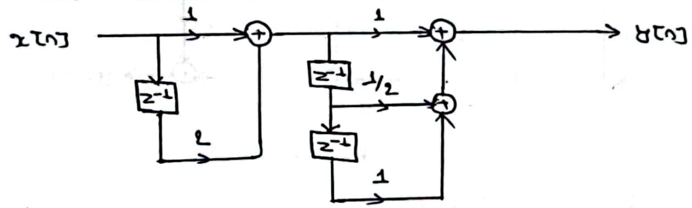


Fig 5.26: cascade-form realization

c. Frequency-Sampling Structures:

- The ***frequency-sampling realization*** is an alternative structure for an FIR filter in which the ***parameters that characterize the filter are the values of the desired frequency response instead of the impulse response h(n)***.

- To ***derive the frequency sampling structure***, we specify ***the desired frequency response at a set of equally spaced frequencies***, namely

$$\omega_k = \frac{2\pi}{M}(k+\alpha) \qquad k = 0,1,\ldots\ldots\frac{M-1}{2} \text{ for } \boldsymbol{M} \text{ odd.}$$

$$k = 0,1,\ldots\ldots\frac{M}{2}-1 \text{ for } \boldsymbol{M} \text{ even.}$$

$$\alpha = 0 \text{ or } \frac{1}{2}.$$

and ***solve for the*** $\boldsymbol{h[n]}$.

- The ***frequency response*** is

$$\mathbf{H}(\boldsymbol{e^{j\omega}}) = \boldsymbol{H(\omega)} = \sum_{\boldsymbol{n=0}}^{\boldsymbol{M-1}} \boldsymbol{h[n]\, e^{-j\omega n}}$$

and values of $\mathbf{H}(\boldsymbol{e^{j\omega}})$ at $\omega_k = \frac{2\pi}{M}(k+\alpha)$ are

$$H(k+\alpha) = H\left(\frac{2\pi}{M}(k+\alpha)\right)$$
$$= \sum_{n=0}^{M-1} h(n)e^{-j2\pi(k+\alpha)n/M} \qquad k = 0, 1, \ldots, M-1$$

- The set of values $\frac{2\pi}{M}(k+\alpha)$ are called the ***frequency samples of*** $\mathrm{H}(e^{j\omega})$. In the case, where $\alpha = 0$, $\{H(k)\}$ corresponds to the M **-point DFT of** $\{h[n]\}$.

- The ***impulse response*** $h[n]$ of above equation is

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha)e^{j2\pi(k+\alpha)n/M} \qquad n = 0, 1, \ldots, M-1$$

When $\alpha = 0$, above equation is simply the ***IDFT of*** $H(k)$.

- Now, if we use above equation to substitute for $h[n]$ in the z-transform $H(z)$, we have

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \right] z^{-n}$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \sum_{n=0}^{M-1} (e^{j2\pi(k+\alpha)/M} z^{-1})^n \right]$$

$$= \frac{1 - z^{-M} e^{j2\pi\alpha}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$$

- Thus, the ***system function*** $\boldsymbol{H(z)}$ ***is characterized by the set of frequency samples*** $\boldsymbol{H(k+\alpha)}$ ***instead of*** $\{\boldsymbol{h[n]}\}$.

- We view this FIR filter realization as a ***cascade of two filters*** **[**i.e., $\boldsymbol{H(z) = H_1(z).H_1(z)}$]

$$\boldsymbol{H_1(z) = \frac{1}{M}\left(1 - z^{-M} e^{j2\pi\alpha}\right)}$$ (***all zero filter or comb filter***)

Its ***zeros*** are located at equally spaced points on the unit circle at

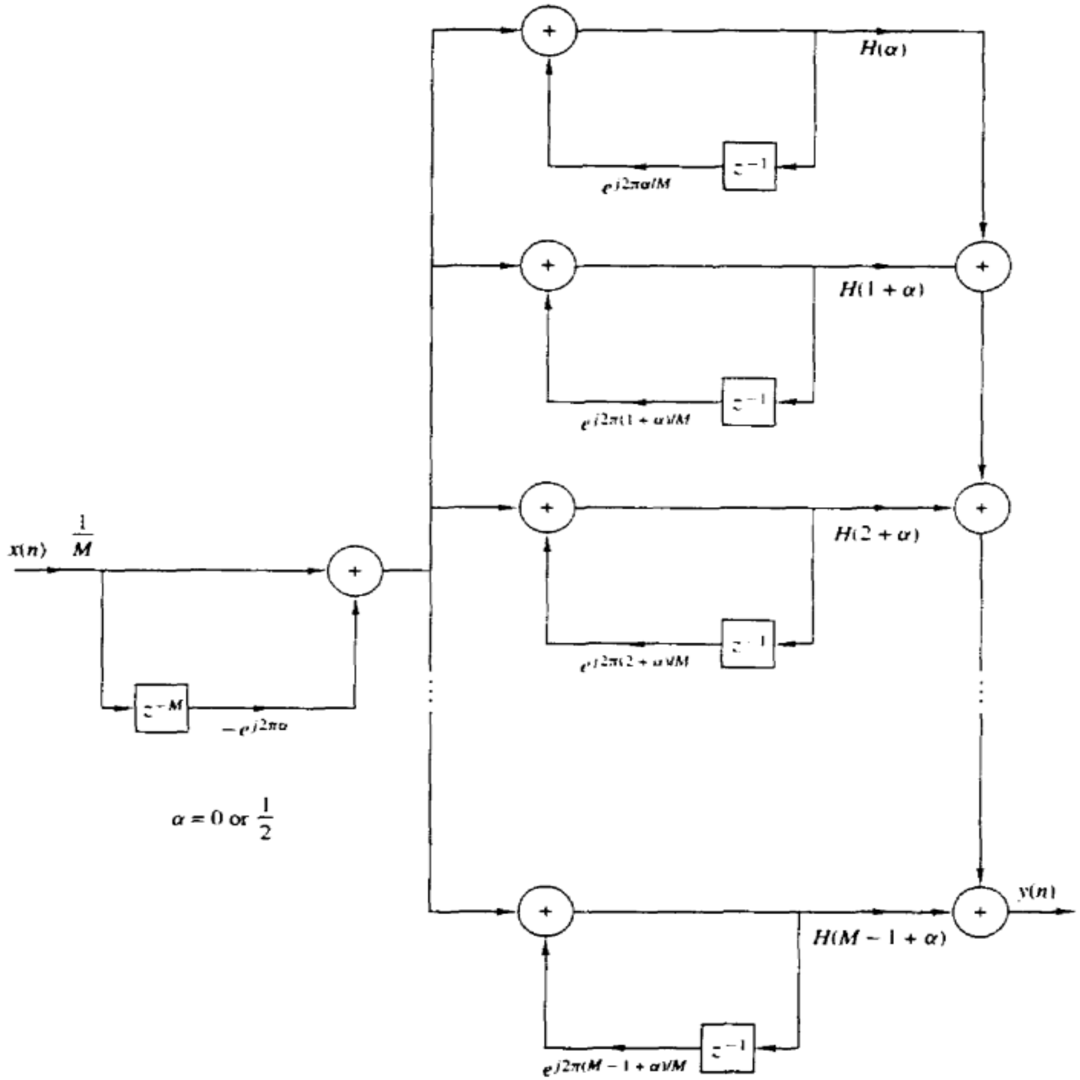
$$\boldsymbol{z_k = e^{j2\pi(k+\alpha)/M}\,, \quad k = 0, 1, \dots \dots M-1}$$

and $$\boldsymbol{H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M}\, z^{-1}}}$$

consists ***parallel banks of single-pole filters*** with resonant frequencies

$$\boldsymbol{p_k = e^{j2\pi(k+\alpha)/M}\,, \quad k = 0, 1, \dots \dots M-1}$$

- The cascade realization is shown in figure below:



x(n)
1/M
z^{-M}
−e^{j2πα}
α = 0 or 1/2
z^{-1}
e^{j2πα/M}
H(α)
e^{j2π(1+α)/M}
H(1 + α)
e^{j2π(2+α)/M}
H(2 + α)
e^{j2π(M−1+α)/M}
H(M − 1 + α)
y(n)

- For ***linear phase***, $H(k) = H^*(M-k)$ for $\alpha = 0$, and

$$H\left(k+\frac{1}{2}\right) = H^*\left(M-k-\frac{1}{2}\right) \quad \text{for } \alpha = \frac{1}{2}$$

As a result, ***a pair of single pole filters can be combined to form a single two-pole filter*** with ($\boldsymbol{\alpha = 0}$).

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{(M-1)/2} \frac{A(k)+B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}}$$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{(M/2)-1} \frac{A(k)+B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}}$$

where, by definition

$$A(k) = H(k) + H(M-k)$$

$$B(k) = H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M}$$

Similar expressions can be obtained for $\boldsymbol{\alpha = \frac{1}{2}}$.

d. Lattice Structures:

- ***Lattice filters*** are used extensively in ***digital speech processing*** and in the implementation of ***adaptive filters***. Let an FIR filter with system function

$$\boldsymbol{H(z) = A_m(z), m = 0, 1, \dots\dots, M-1}$$

$$\boldsymbol{H(z) = A_m(z) = 1 + \sum_{k=1}^{m} a_m(k) z^{-k}, \quad m \geq 1} \quad \text{.....1}$$

with $\boldsymbol{A_0(z) = 1}$, and impulse response

$$h[n] = \begin{cases} 1, & n = 0 \\ a_m(n), & n = 1, 2, \dots, m \end{cases}$$

OR,

$$Y(z) = X(z)[\,1 + \sum_{k=1}^{m} a_m(k) z^{-k}\,]$$

$$Y(z) = X(z) + X(z)\, \sum_{k=1}^{m} a_m(k) z^{-k}$$

- Taking inverse z-transform, we get

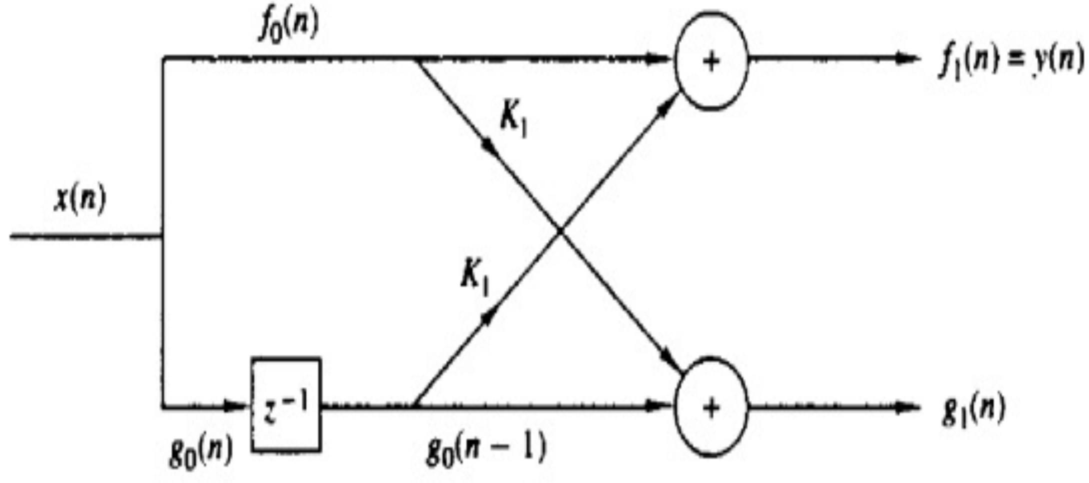
$$\boldsymbol{y[n] = x[n] + \sum_{k=1}^{m} a_m(k) x[n-k]} \quad \text{.....2}$$

($\boldsymbol{m}$ is the degree of polynomial)

- For $\boldsymbol{m = 1}$:
- Equation 2 reduces to

$$\boldsymbol{y[n] = x[n] + a_1(1) x[n-1]} \quad \text{.....3}$$

This ***single stage lattice filter*** can be realized as shown in figure below:



- From above figure, we have

$$x[n] = f_0[n] = g_0[n] \qquad .....4$$

$$y[n] = f_1[n] = f_0[n] + k_1\, g_0[n-1] = x[n] + k_1 x[n-1] \qquad .....5$$

$$g_1[n] = k_1 f_0[n] + g_0[n-1] = k_1 x[n] + x[n-1] \qquad .....6$$

- Comparing equations 3 and 4, we get

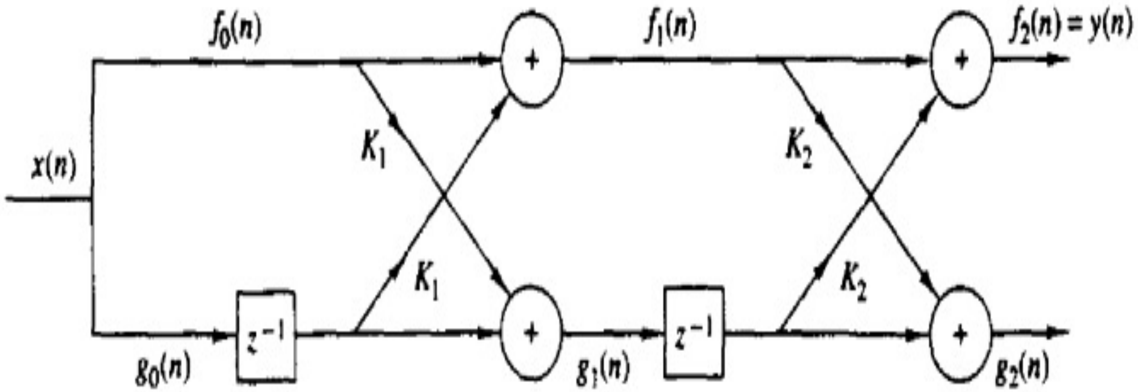
$$a_1(0) = 1, \quad a_1(1) = k_1 \qquad .....7$$

- Where, k_1 is ***lattice coefficient***. Similarly, $a_1(0)$ and $a_1(1)$ are ***direct form coefficients.***

- For $m = 2$:
- Equation 2 reduces to

$$y[n] = x[n] + a_2(1)x[n-1] + a_2(2)x[n-2] \qquad .....8$$

the ***two-stage lattice structure*** for this is shown in figure below:



- From above figure, we have

$$\boldsymbol{y[n] = f_2[n] = f_1[n] + k_2 g_1[n-1]}$$

But, $f_1[n] = f_0[n] + k_1 g_0[n-1] = x[n] + k_1 x[n-1]$

and, $g_1[n-1] = k_1 f_0[n-1] + g_0\,[n-2] = k_1 x[n-1] + x[n-2]$

therefore, $y[n] = f_2[n] = x[n] + k_1 x[n-1] + k_2\{k_1 x[n-1] + x[n-2]\}$

$$\boldsymbol{y[n] = f_2[n] = x[n] + k_1(1+k_2)x[n-1] + k_2 x[n-2]} \qquad .....9$$

- From equations 8 and 9, we get

$$\boldsymbol{a_2(0) = 1, \quad a_2(1) = k_1(1+k_2), \qquad a_2(2) = k_2}$$

- Also, from above figure

$$\boldsymbol{g_2[n] = k_2 f_1[n] + g_1[n-1]}$$

➢ Therefore,

$$g_2[n] = k_2 x[n] + k_1(1 + k_2)x[n-1] + x[n-2] \quad .....10$$

Note that two sets of filter coefficients in $f[n]$ and $g[n]$ are in reverse order.

- For $m = M - 1$:

➢ For M-1 stage filter, we have

$$f_0(n) = g_0(n) = x(n)$$

$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1) \qquad m = 1,2,......,M-1 \quad .....11$$

$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1) \qquad m = 1,2,......,M-1$$

➢ The output of M-1 stage is

$$y[n] = f_{M-1}[n] \quad .....12$$

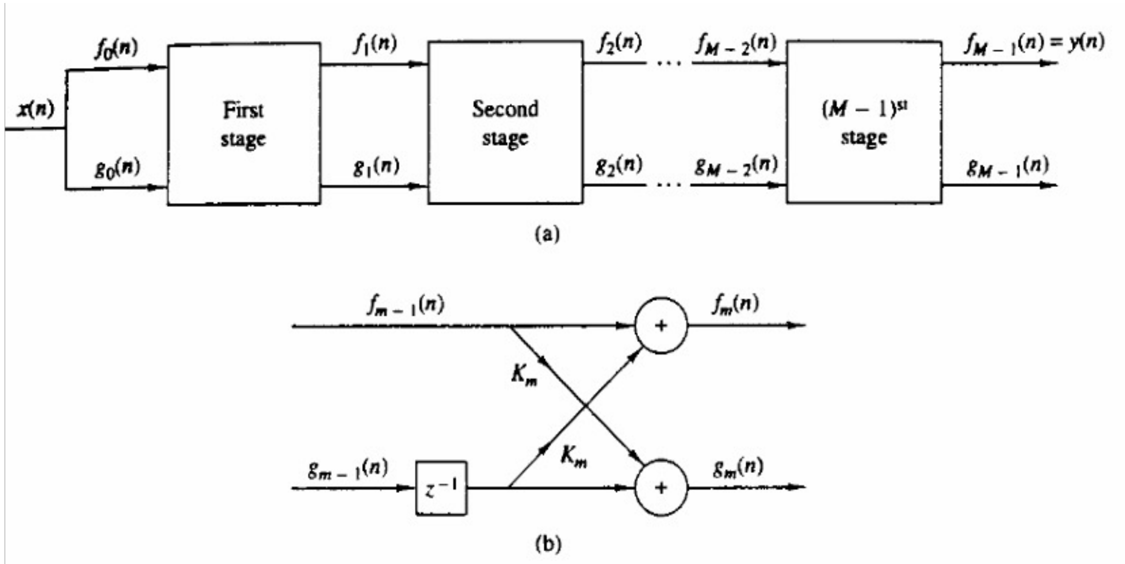


Fig: $(M-1)$-stage lattice filter.

- **Conversion of Lattice Coefficients to Direct Form Coefficients:**

➢ In general,

$$a_m(0) = 1$$
$$a_m(m) = k_m$$
$$a_m(k) = a_{m-1}(k) + a_m(m)\, a_{m-1}(m-k)$$

where, $1 \le k \le m-1$ and $m = 1,2,\ldots\ldots, M-1$

- **Conversion of Direct Form Coefficients to Lattice Coefficients :**

➢ In general,

$$a_m(0) = 1$$

$$k_m = a_m(m)$$

$$a_m(k) = a_{m-1}(k) + a_m(m)\, a_{m-1}(m-k)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - \{a_m(m)\}^2}$$

where, $1 \leq k \leq m-1$ and $m = 1,2, \dots \dots, M-1$

❑ IIR Filter, Structures for IIR Filter:

- Causal IIR systems are characterized by the difference equation as:

$$\boldsymbol{y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]} \qquad .....1$$

Or, $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]\,,\ a_0 = 1$$

- Taking z-transform on both sides of equation 1, we get

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)[1 + \sum_{k=1}^{N} a_k z^{-k}] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}} \qquad .....2$$

where, $H(z) =$ system function, $a_0 \neq 0$ and $M \leq N$

- From 1 and 2, we observe that the ***realization of IIR systems, i. e. filters , involves a recursive computational algorithm***.

- **Notes:**

A. Non-recursive and Recursive Systems:

- **Non-recursive System:**
- If the ***output of a system is the function of the present and past values of the inputs*** only then ***the system is known as non-recursive system***. Mathematically,

$$\boldsymbol{y[n] = F\{x[n], x[n-1], \dots \dots x[n-M]\}} \quad \text{.....1}$$

- A ***causal FIR system is non-recursive*** system. consider a causal FIR system,

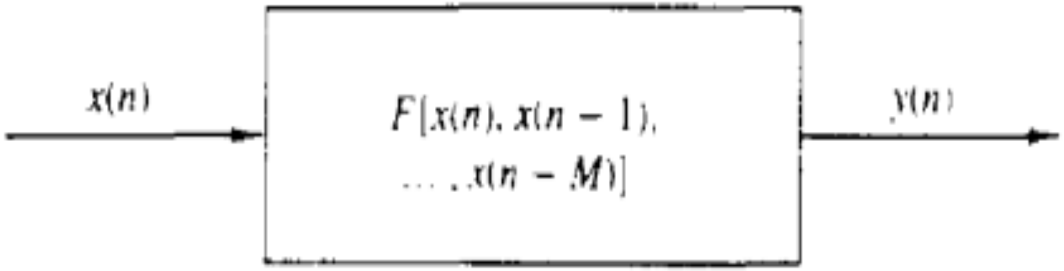
$$\boldsymbol{y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]} \quad \text{.....2}$$

where the output of the system is the function of present and past inputs. Thus, the causal FIR system is non-recursive.

➢ A ***non-recursive system*** can be represented in terms of ***difference equation*** as

$$\boldsymbol{y[n] = \sum_{k=0}^{M} b_k x[n-k]} \qquad \text{.....3}$$

➢ Non-recursive systems ***does not have feedback path***.



○ **Recursive System:**

➢ If the ***output of a system is the function of the present and past values of the inputs as well past outputs*** then ***the system is known as recursive system***. Mathematically,

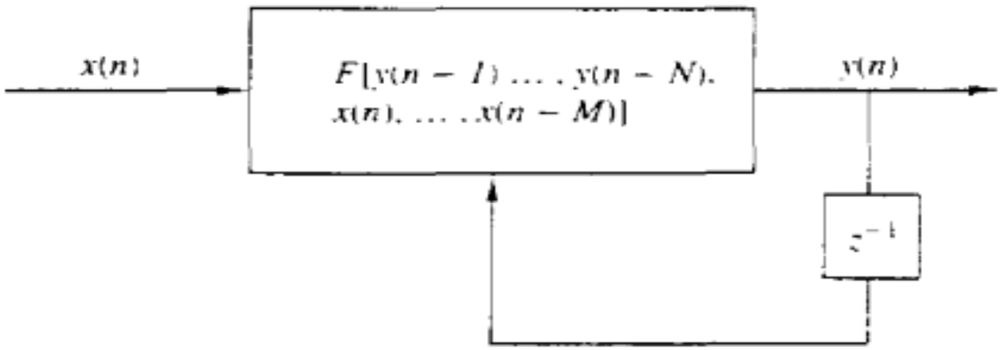
$$\boldsymbol{y[n] = F\{y[n-1], \ldots \ldots y[n-N], x[n], x[n-1], \ldots \ldots x[n-M]\}} \qquad \text{.....4}$$

(causal recursive system)

- A ***recursive LTI system*** is characterized by ***difference equation*** as

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \qquad .....5$$

- ***Recursive realization reduces*** the ***memory requirements, additions, and multiplications***.
- Recursive systems ***have feedback path***.



B. FIR (Finite-Duration Impulse Response) and IIR (Infinite-Duration Impulse Response) System:

- Let h[n] be the impulse response of a LTI system. Then LTI system can be subdivided into two types

1. **FIR (Finite-Duration Impulse Response) System, and**
2. **IIR (Infinite-Duration Impulse Response) System**

1. FIR System:

- For causal FIR system, the convolution sum formula is

$$\boldsymbol{y[n] = \sum_{k=0}^{M-1} h[k]x[n-k]} \qquad .....1$$

An ***FIR system has finite memory of length M and non-recursive***.

- The difference equation of FIR system is

$$\boldsymbol{y[n] = \sum_{k=0}^{M} b_k x[n-k]} \qquad .....2$$

2. IIR System:

- For causal IIR system, the **convolution sum** formula is

$$\boldsymbol{y[n]} = \sum_{\boldsymbol{k=0}}^{\infty} \boldsymbol{h[k]x[n-k]} \qquad .....1$$

An ***FIR system has finite memory of length M and non-recursive***.

- The difference equation of IIR system is

$$\mathbf{y[n]} = -\sum_{\boldsymbol{k=1}}^{\boldsymbol{N}} \boldsymbol{a_k y[n-k]} + \sum_{\boldsymbol{k=0}}^{\boldsymbol{M}} \boldsymbol{b_k x[n-k]} \qquad .....2$$

- An ***IIR system cannot be realized using convolution sum*** but it is ***realized using difference equation (LCCDE)***.

1. Methods for the Implementation of IIR Systems:

a. Direct form structures

b. Cascade form structures

c. Parallel form structures

d. Lattice and lattice-ladder structures

a. Direct Form Structures:

- The ***system function for IIR*** system is

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}\,,\ a_0 = 1} \qquad .....1$$

- Since, the ***multiplier coefficients*** ($\boldsymbol{a_k}$ and $\boldsymbol{b_k}$) in the structures are exactly the ***coefficients of the system function***, they are called ***direct form structures***.
- Direct form structures can be studied under:

i. Direct form I structure

ii. Direct form II structure

i. Direct form I structure:

- We know that the multiplier coefficients are the coefficients of system function

$$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}\,,\; a_0 = 1} \qquad \text{.....1}$$

Or, $$\boldsymbol{H(z) = H_1(z).H_2(z)}$$

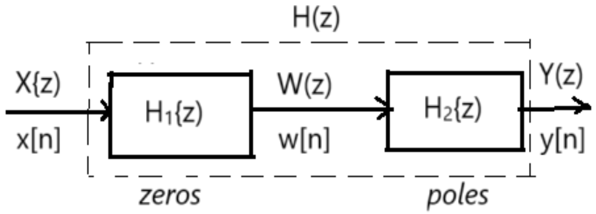
where, $$\boldsymbol{H_1(z) = \sum_{k=0}^{M} b_k z^{-k}}\text{ , all- zero system (non-recursive)} \qquad \text{.....2}$$

and $$\boldsymbol{H_2(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}}\text{ , all-pole system (recursive)} \qquad \text{.....3}$$

- Then the ***direct form I structure*** is obtained by ***cascading the structures for $H_1(z)$ and $H_2(z)$***.

- **Note:** $$\boldsymbol{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}\,,\; a_0 = 1}\quad \text{(in Oppenheim book)}$$

➢ From figure, we have



$$\frac{W(z)}{X(z)} = H_1(z)$$

$$\frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$W(z) = X(z) \sum_{k=0}^{M} b_k z^{-k}$$

➢ Or,

$$W(z) = b_0\, X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \cdots \; + b_M z^{-M} X(z) \qquad .....1$$

➢ Taking inverse z-transform, we get

$$w[n] = b_0\, x[n] + b_1\, x[n-1] + b_2\, x[n-2] + \cdots \; + b_M x[n-M]$$

$$\therefore \qquad w[n] = \sum_{k=0}^{M} b_k x[n-k]$$

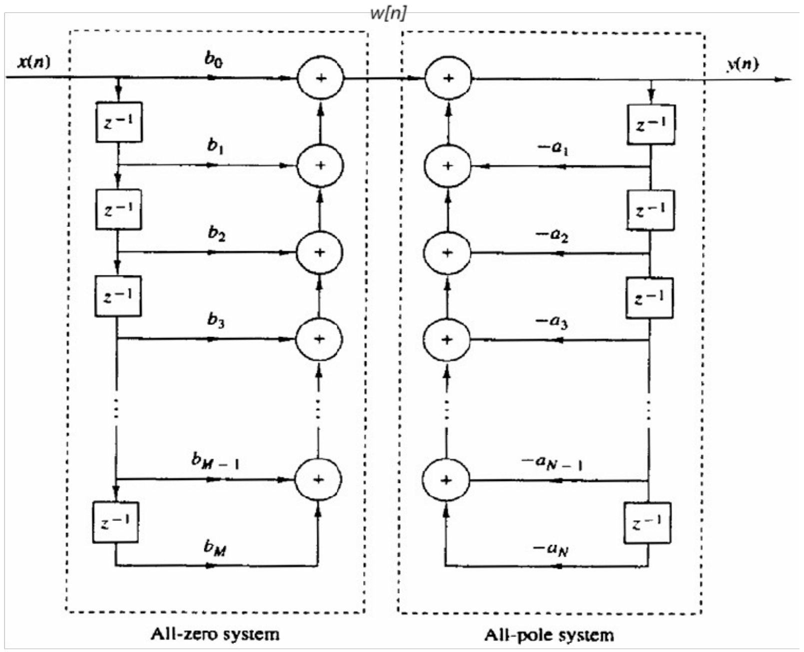
.....2

➢ Similarly,

$$\frac{Y(z)}{W(z)} = H_2(z)$$

$$\frac{Y(z)}{W(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

- Or, $$\mathrm{Y}(z) + \sum_{k=1}^{N} a_k z^{-k}\mathrm{Y}(z) = W(z)$$
$$\mathrm{Y}(z) = -\sum_{k=1}^{N} a_k z^{-k}\mathrm{Y}(z) + W(z) \qquad .....3$$
- Taking inverse z-transform, we get
$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + w[n] \qquad .....4$$
$$y[n] = -a_1\, y[n-1] - a_2\, y[n-2] - \cdots - a_N y[n-N] + w[n]$$
- Therefore, direct form-I realization becomes as shown in fig below:



- **Computational Complexity:**
- This realization requires:

i. Number of additions = M+N
ii. Number of multiplications = M+N+1
iii. Number of memory locations (delay elements) = M+N+1

ii. Direct form II structure:

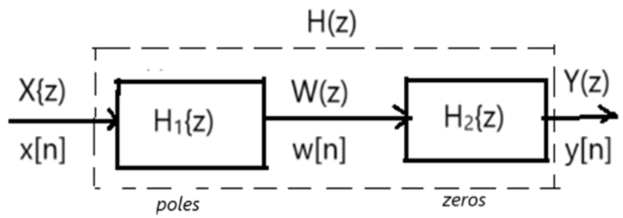
- Since, we are dealing with the LTI systems, we can interchange the positions of $H_1(z)$ and $H_2(z)$. This property gives the direct form-II structure.
- In direct form-II realization(or structure), poles of $H(z)$ is realized first and then the zeros second.

$$\boldsymbol{H(z) = H_1(z).H_2(z)}$$

where, $\boldsymbol{H_1(z) = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}}$, all−pole system (recursive)

and $\boldsymbol{H_2(z) = \sum_{k=0}^{M} b_k z^{-k}}$, all− zero system (non−recursive)

➢ From figure, we have



$$\frac{W(z)}{X(z)} = H_1(z)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

$$W(z) = X(z)\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

➢ Or,

$$W(z)\{1 + \sum_{k=1}^{N} a_k z^{-k}\} = X(z)$$

$$\boldsymbol{W(z) = -\sum_{k=1}^{N} a_k z^{-k} W(z) + X(z)} \qquad \textbf{.....1}$$

➢ Taking inverse z-transform, we get

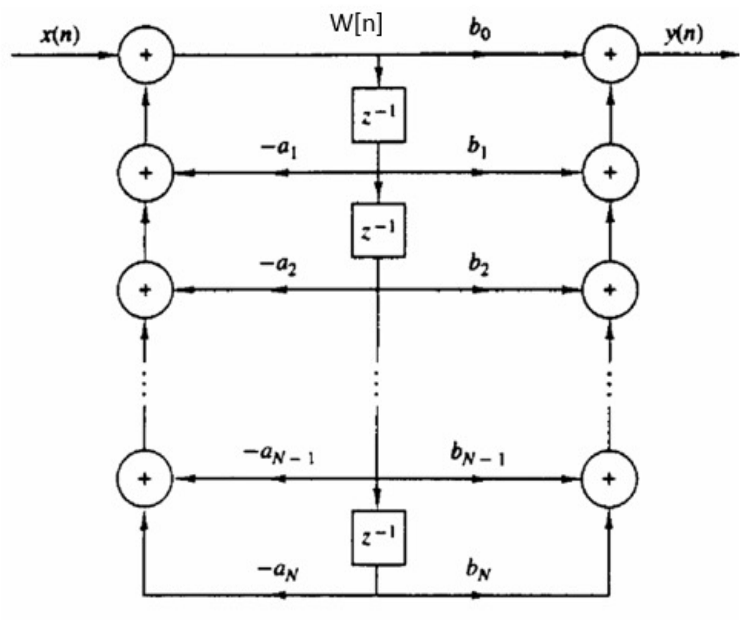
$$\boldsymbol{w[n] = -\sum_{k=1}^{N} a_k w[n-k] + x[n]} \qquad .....2$$

➢ Similarly,

$$\frac{Y(z)}{W(z)} = H_2(z)$$

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

- Or, $$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$
$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$$
- Taking inverse z-transform, we get
$$\boldsymbol{y[n] = \sum_{k=0}^{M} b_k w[n-k]} \quad .....3$$
- The direct form-II realization (or structure) is shown in figure below (for **N=M**):



- **Computational Complexity:**
- This realization requires:

i. Number of additions = M+N
ii. Number of multiplications = M+N+1
iii. Number of memory locations (delay elements) is equal to the order of the filter (or, order of the system function or difference equation), hence canonical structure

b. Cascade Form Structures:

➢ The cascade form realization of an IIR system or filter is obtained by decomposing the system function $H(z)$ into a product of simpler transfer functions as:

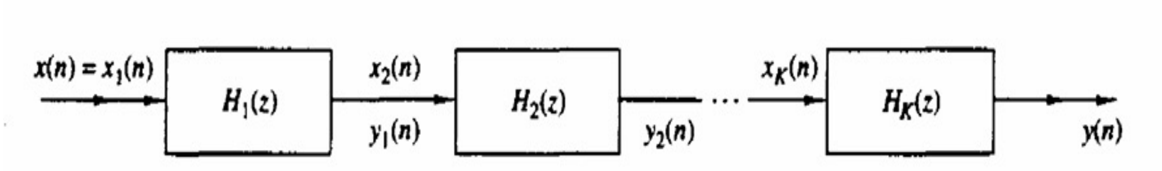
$$H(z) = AH_1(z)H_2(z) \dots \dots H_K(z)$$

$$H(z) = A\prod_{k=1}^{K} H_k(z) \qquad \text{.....1}$$

where, $A =$ a constant

$k =$ integer part of $\frac{N+1}{2}$

and it is assumed that $M \leq N$



c. Parallel Form Structures:

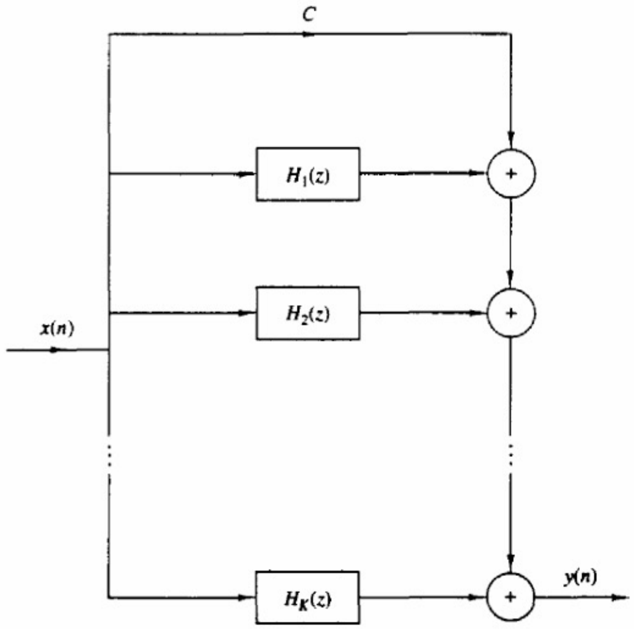
➢ By using partial fractionn epansion, the overall system function $H(z)$ can be expressed as:

$$\boldsymbol{H(z) = C + H_1(z) + H_2(z) + \cdots \ldots + H_K\,(z)} \qquad \ldots\ldots 1$$

where, C = a constant

$H_1(z), H_2(z), \ldots \ldots, H_K\,(z)$ = second order sub-systems

and, $$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}, a_0 = 0$$



- Application:

➢ For ***high-speed filtering***. Because the processing of filtering operation is performed parallelly.

d. Lattice and Lattice-ladder Structure:

➢ Lattice filters are used in ***digital speech processing*** and the implementing of ***adaptive filters***.

a. Lattice Structure:

➢ An all-pole system with system function

$$H(z) = \frac{1}{1+\sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{1}{A_N(z)} \quad .....1$$

and the difference equation is

$$y[n] = -\sum_{k=1}^{N} a_N(k)\, y[n-k] + x[n] \quad .....2$$

Or, $$x[n] = y[n] + \sum_{k=1}^{N} a_N(k)\, y[n-k] \quad .....3$$

- **For $N = 1$:**

➢ We have, $x[n] = y[n] + a_1(1)\, y[n-1] \qquad \because N = k = 1$

$$y[n] = x[n] - a_1(1)\, y[n-1] \quad .....4$$

- We have, $x[n] = y[n] + a_1(1)\, y[n-1] \quad \because N = k = 1$

$$\boldsymbol{y[n] = x[n] - a_1(1)\, y[n-1]} \qquad \textbf{.....4}$$

- From figure, we have

$$x[n] = f_1[n]$$
$$y[n] = f_0[n] = f_1[n] - k_1 g_0[n-1]$$

Or, $\boldsymbol{y[n] = x[n] - k_1 y[n-1]}$ **.....5**

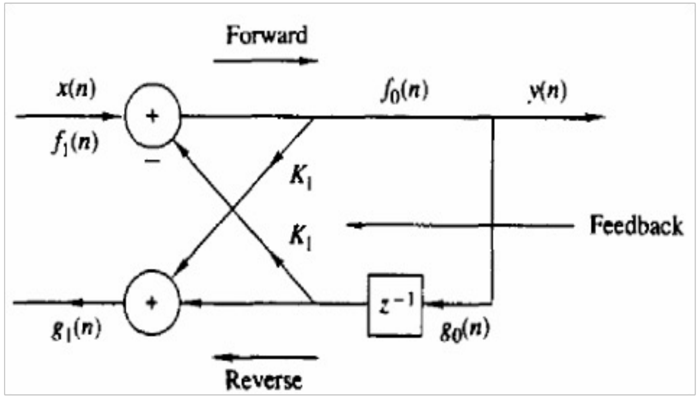


Fig: Single-stage all-pole lattice filter

- Also, $\boldsymbol{g_1[n] = k_1 y[n] + y[n-1]}$
- From equations (4) and (5), we know

$$\boldsymbol{k_1 = a_1(1)} \qquad \textbf{.....6}$$

- **For $N = 2$:**
- We have, $x[n] = y[n] + a_2(1)\, y[n-1] + a_2(2)\, y[n-2] \quad \because N = k = 1$

$$\boldsymbol{y[n] = x[n] - a_2(1)\, y[n-1] - a_2(2)\, y[n-2]} \qquad \textbf{.....7}$$

- This output can be achieved from two-stage lattice structure as shown in figure below:

- From figure, we have

$$x[n] = f_2[n]$$

$$f_1[n] = f_2[n] - k_2 g_1[n-1]$$
$$g_2[n] = k_2 f_1[n] + k_2 g_1[n-1]$$
$$f_0[n] = f_1[n] - k_1 g_0[n-1]$$
$$g_1[n] = k_1 f_0[n] + g_0[n-1]$$

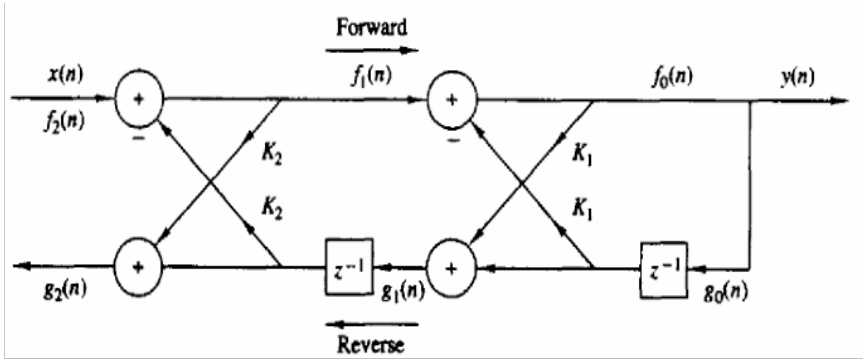


Fig: Two-stage all-pole lattice filter

- Therefore,

$$y[n] = f_0[n] = g_0[n]$$
$$y[n] = f_1[n] - k_1 g_0[n-1]$$
$$y[n] = \{f_2[n] - \ k_2 g_1[n-1]\} - k_1 g_0[n-1]$$
$$y[n] = f_2[n] - k_2\{k_1 f_0[n-1] + g_0[n-2]\} - k_1 g_0[n-1]$$
$$y[n] = f_2[n] - k_2 k_1 f_0[n-1] - k_2 g_0[n-2] - k_1 g_0[n-1]$$
$$y[n] = x[n] - k_2 k_1 y[n-1] - k_2 y[n-2] - k_1 y[n-1]$$

- Therefore, $\boldsymbol{y[n] = x[n] - k_1(1 + k_2)y[n-1] - k_2 y[n-2]}$ **.....8**

- In similar manner, we obtain

$$g_2[n] = k_2 y[n] + k_1(1 + k_2)y[n-1] + y[n-1] \qquad .....9$$

- From equations (7) and (8), we have

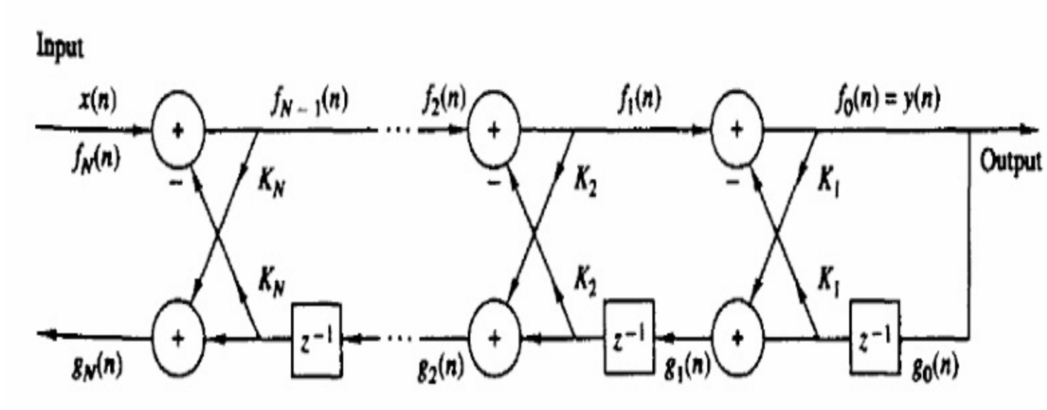
$$a_2(0) = 1$$

$$a_2(1) = k_1(1 + k_2)$$

and, $$k_1(1 + k_2)$$

$$a_2(2) = k_2$$

- Therefore, N-stage lattice structure of IIR filter is obtained as shown in figure below:



➢ From figure, we have

$$f_N[n] = x[n]$$
$$f_{m-1}[n] = f_m[n] - k_m g_{m-1}[n-1], \qquad m = N, N-1, \ldots, 1$$
$$g_m[n] = k_m f_{m-1}[n] + g_{m-1}[n-1], \qquad m = N, N-1, \ldots, 1$$
$$y[n] = f_0[n] = g_0[n]$$

b. Lattice-ladder Structure:

➢ A general IIR filter connecting both poles and zeros can be realized (or implemented) using all-pole lattice as building block. To develop appropriate structure, let us consider an IIR system with system function:

$$H(z) = \frac{\sum_{k=0}^{M} c_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{C_M(z)}{A_N(z)} \qquad \ldots..1$$

where, $N \geq M$

- The lattice structure for equation (1) is constructed first by realizing all-pole lattice coefficients $\boldsymbol{k_m}$, where, $\mathbf{1} \leq \boldsymbol{m} \leq \boldsymbol{N}$ for the denominator $\boldsymbol{A_N(z)}$, and then adding the ladder part by taking the output as a weighted linear combination of $\boldsymbol{g_m[m]}$.

- The result is the pole-zero IIR (lattice-ladder) structure.
- The lattice-ladder structure for $\mathrm{M} = \boldsymbol{N}$ is shown in figure below:

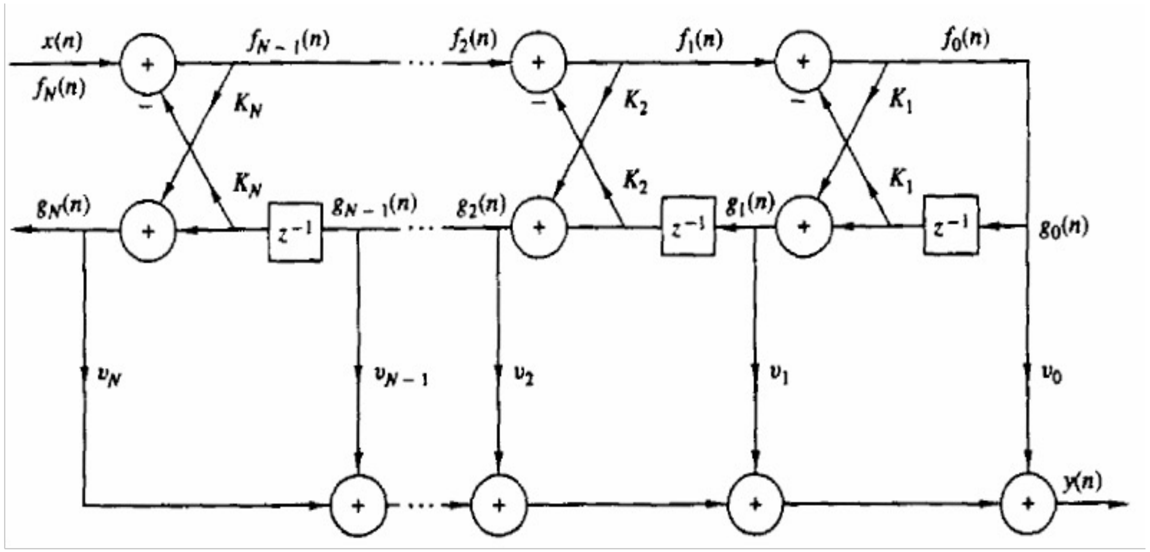


Fig: Lattice-ladder structure for the pole-zero system

➢ The output is given by

$$y(n) = \sum_{m=0}^{M} v_m g_m(n) \qquad .....2$$

where, v_m = ladder coefficients and obtained by the equation

$$\boldsymbol{v_m = c_m - \sum_{i=1+m}^{M} v_i a_i(i-m), \quad m = M, M-1, \dots .0} \qquad \textbf{.....3}$$

- **Conversion from Lattice Structure to Direct-form Structure:**

➢ In general,

$$\boldsymbol{a_m(0) = 1}$$
$$\boldsymbol{a_m(m) = k_m}$$
$$\boldsymbol{a_m(k) = a_{m-1}(k) + a_m(m)\, a_{m-1}(m-k)}$$

- **Conversion of Direct Form Coefficients to Lattice Coefficients :**

➢ In general,

$$a_m(0) = 1$$

$$k_m = a_m(m)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)a_m(m-k)}{1 - \{a_m(m)\}^2}$$

❑ Quantization Effect (Truncation and Rounding):