SIGNALS AND SYSTEMS

About the Author

A Nagoor Kani is a multifaceted personality with efficient technical expertise and management skills. He obtained his BE degree in Electrical and Electronics Engineering from Thiagarajar College of Engineering, Madurai, and MS (Electronics and Control) through Distance Learning program of BITS, Pilani. He is a life member of ISTE and IETE.

He started his career as a self-employed industrialist (1986–1989) and then changed over to teaching in 1989. He has worked as Lecturer in Dr M G R Engineering College (1989–1990) and as Asst. Professor in Satyabhama Engineering College (1990–1997). In 1993, he started a teaching centre for BE students named Institute of Electrical Engineering, which was renamed RBA Tutorials in 2005.

A Nagoor Kani launched his own organisation in 1997. The ventures currently run by him are RBA engineering (involved in manufacturing of lab equipments, microprocessor trainer kits and undertake Electrical contracts and provide electrical consultancy), RBA Innovations (involved in developing projects for engineering students and industries), RBA Tutorials (conducting tutorial classes for engineering students and coaching for GATE, IES, IAS) and RBA Publications (publishing of engineering books), RBA Software (involved in web-design and maintenance). His optimistic and innovative ideas have made the RBA Group a very successful venture.

A Nagoor Kani is a well-known name in major engineering colleges in India. He is an eminent writer and till now he has authored several engineering books (published by RBA Publications) which are very popular among engineering students. He has written books in the areas of Control Systems, Microprocessors, Microcontrollers, Digital Signal Processing, Electric Circuits, Electrical Machines and Power Systems.

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Dedicated to my father

Late N Allaudeen

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Preface

The main objective of this book is to explore the basic concepts of signals and systems in a simple and easy-to-understand manner.

This text on signals and system has been crafted and designed to meet students' requirements. Considering the highly mathematical nature of this subject, more emphasis has been given on the problem-solving methodology. Considerable effort has been made to elucidate mathematical derivations in a step-by-step manner. Exercise problems with varied difficulty levels are given in the text to help students get an intuitive grasp on the subject.

This book with its lucid writing style and germane pedagogical features will prove to be a master text for engineering students and practitioners.

Salient Features

The salient features of this book are

- Separate discussions on continuous time and discrete time signals for thorough understanding of the concepts
- Proof of properties of transforms clearly highlighted by shaded boxes for quick review
- Additional explanations for solutions and proofs provided in separate boxes
- Different types of fonts used for text, proof and solved problems providing better clarity and user-friendliness

Organization

In this book, the concepts of continuous time signals and systems are organized in four chapters. The concepts of discrete time signals and systems are organized in six chapters, and one chapter is devoted to a general discussion on signals and systems. Each chapter provides the foundations and practical implications with a large number of solved numerical examples for better understanding.
The important concepts are summarized at the end of each chapter which help in quick reference. Another significant aspect of this book is MATLAB based computer exercises with complete explanations given in each chapter. This will be of great assistance to both instructors and students.

Chapter 1 deals with a general introduction to various types of signals, systems and their importance in real life. Basic definitions of signals, their mathematical representation, significance of their frequency domain analysis and usage of MATLAB in this course are presented in a brief manner.

Chapter 2 introduces analysis of continuous time signals and systems in depth. It explores various classifications of continuous time signals, systems, and possible mathematical operations such as scaling, folding, time shifting, addition, multiplication, differentiation and integration on them. Then, it discusses block diagrams and signal flow graph representation of continuous time systems, LTI systems characterized by linear differential equations and methods to solve those equations.
Another vital aspect of Chapter 2 is the discussion on graphical convolution operation of continuous-time signals by clearly separating the shift index and the time index. This will aid in clear understanding of the concepts.

Chapter 3 discusses the analysis of continuous time systems using Laplace transform. The rational functions of 's' and their representation in terms of poles and zeros, region of convergence of Laplace transform and its properties are presented in a crisp and easy manner. The stability of the LTI systems and their response via Laplace transforms are dealt with lucidly.

Further, the inverse Laplace transform using partial-fraction method and convolution theorem are discussed. The convolution and deconvolution operations are explained with simple numerical examples. The realization structures for continuous-time systems characterized by differential equations are also presented in this chapter.

Chapter 4 is concerned with Fourier analysis of continuous-time systems which forms the basis for frequency domain analysis. The first half of this chapter is dedicated to Fourier series in both trigonometric and exponential forms, Fourier coefficients of various signals with symmetry, properties of Fourier series and the Gibbs phenomenon.
The second half of the chapter explains the development of Fourier transform from Fourier series, frequency spectrum, various properties of Fourier transform, and Fourier transform of some standard signals. It also covers the computation of frequency responses of LTI systems using Fourier transform explained with examples. The chapter also talks about the relation between Fourier transform and Laplace transform of continuous-time signals.

Chapter 5 deals with the concepts of state space analysis of continuous-time systems. In this chapter, the development of state model, solutions to state equations and response of continuous-time systems to state models are discussed.

Chapter 6 is devoted to concepts of discrete-time signals and systems and is more concerned with the generation, representation, classification, mathematical operations of discrete time signals and systems, block diagrams and signal-flow graph notations.
The chapter also presents the methods of obtaining responses of LTI discrete-time systems and various convolution methods. The deconvolution, correlation techniques and the inverse systems are clearly explained with solved numerical examples. In addition, the concept of sampling and its importance are dealt with briefly.

Chapter 7 explains Z-transform and its application to signals and systems. The concepts are similar to Laplace transform except as applied to discrete-time signals and systems. All the important properties of Z-transform are presented explicitly. Inverse Z-transform and solution of difference equations describing the discrete-time systems are demonstrated with numerical examples. Also given are the systems interconnections and standard system realization using structures.

Chapter 8 is dedicated to discrete-time Fourier series and Fourier transform which forms the basis for frequency domain analysis of discrete-time signals and systems. In the first half of this chapter, the discrete-time Fourier series and the frequency spectrum using discrete-time Fourier series are discussed with relevant examples.
The second half of the chapter details the development of discrete-time Fourier transform from discrete-time Fourier series, frequency spectrum, various properties of Fourier transform, and Fourier transform of some standard discrete-time signals. In addition, the computation of frequency responses of LTI discrete-time systems using Fourier transform are also explained with numerous examples. The relation between Fourier transform and Z-transform of discrete-time signals is also discussed in the chapter.

Chapter 9 extends the understanding of the concepts of Discrete-Time Fourier Transform (DTFT) to DFT (Discrete Fourier Transform) and FFT (Fast Fourier Transform). Development of DFT from DTFT, properties of DFT, relation between DFT and Z-transform, analysis of the LTI systems using DFT and FFT are extensively discussed.

Chapter 10 focuses on structures for realization of discrete-time systems with special attention to IIR and FIR systems.

Chapter 11 presents the concepts of state space analysis of discrete-time systems. In this chapter, the development of state model of discrete-time systems, solutions to state equations and response of discrete-time systems from state models are discussed in an easy manner.

Web Supplements

This book is accompanied by a comprehensive website which can be accessed at http://www.mhhe.com/nagoorkani/signals1e. It has been designed to provide valuable resources for students, instructors and professionals.

- Students can access Interactive Quiz, Objective-Type Questions and Short-Answer-Type Questions on the website.
- Supplementary teaching material for Instructors includes chapterwise PowerPoint slides for effective lecture presentation and an on-request Solution Manual.

Feedback

I have taken care to present the concepts of signals and systems in a user-friendly manner and hope that the teaching and student community will welcome the book. The readers may feel free to convey their criticism and suggestions for further improvement of the book. The feedback is welcome at my email address: kani@vsnl.com

A Nagoor Kani

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Tata McGraw Hill Education looks forward to receiving views, comments and suggestions for improvement from teachers and students, all of which may be sent to tmh.ecefeedback@gmail.com, mentioning the title and author's name in the subject line. Piracy-related issues may also be reported.

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A Nagoor Kani

List of Symbols and Abbreviations

Symbols

a_o, a_n, b_n	-	Fourier coefficients of trigonometric form of Fourier series of x(t)
B	-	Bandwidth in Hz
C_n	-	Fourier coefficients of exponential form of Fourier series of x(t)
c_k	-	Fourier coefficients of discrete time signal x(n)
E	-	Energy of a signal
f	-	Frequency of discrete time signal in Hz/sample
F	-	Frequency of continuous time signal in Hz
F_o	-	Fundamental frequency of continuous time signal in Hz
F_m	-	Maximum frequency of continuous time signal
F_s	-	Sampling frequency of continuous time signal in Hz
$\mathcal{H}$	-	System operator
j	-	Complex operator, $\sqrt{-1}$
L	-	Inductance
$n\Omega_o$	-	Harmonic angular frequency, where n = 1,2,3.......
P	-	Power of a signal
p	-	Pole
R	-	Resistor
s	-	Complex frequency ($s = \sigma + j\Omega$)
t	-	Time in seconds
T	-	Time period in seconds
W	-	Phase factor or Twiddle factor
z	-	Complex variable ($z = u + jv$)
z	-	Unit advance operator or zero
z^{-1}	-	Unit delay operator
Ω	-	Angular frequency of continuous time signal in rad/sec
Ω_o	-	Fundamental angular frequency
Ω_{max}	-	Maximum angular frequency in rad/sec
ω	-	Angular frequency of discrete time signal
ω_k	-	Sampling frequency point
σ	-	Neper frequency (Real part of s)
*	-	Convolution operator
*	-	Circular convolution operator
$\oint$	-	Integration operator
$\frac{d}{dt}$	-	Differentiation operator

Standard/Input/Output Signals

h(n)	-	Impulse response of discrete time system
h(t)	-	Impulse response of continuous time system
$r_{xy}(m)$	-	Cross-correlation sequence of x(n) and y(n)
$r_{xx}(m)$	-	Auto-correlation sequence of x(n)
$\bar{r}_{xy}(m)$	-	Circular cross-correlation sequence of x(n) and y(n)
$\bar{r}_{xx}(m)$	-	Circular auto-correlation sequence of x(n)
sgn(t)	-	Signum signal
sinc (t)	-	Sinc signal
u(n)	-	Discrete time unit step signal
u(t)	-	Continuous time unit step signal
x(n)	-	Discrete time signal
x(n)	-	Input of discrete time system
$x_o(n)$	-	Odd part of discrete time signal x(n)
$x_e(n)$	-	Even part of discrete-time signal x(n)
x(n–m)	-	Delayed or linearly shifted x(n) by m units
$x((n-m))_N$	-	Circularly shifted x(n) by m units, where N is period
x(t)	-	Continuous time signal or Input of continuous time system
$x_o(t)$	-	Odd part of continuous time signal x(t)
$x_e(t)$	-	Even part of continuous time signal x(t)
x(t–m)	-	Delayed or linearly shifted x(t) by m units
y(n)	-	Output / Response of discrete time system
$y_{zs}(n)$	-	Zero state response of discrete time system
$y_{zi}(n)$	-	Zero input response of discrete time time system
y(t)	-	Output / Response of continuous time system
$y_{zs}(t)$	-	Zero state response of continuous time system
$y_{zi}(t)$	-	Zero input response of continuous time system
$\delta(t)$	-	Continuous time impulse signal
$\delta(n)$	-	Discrete time impulse signal
$\Pi(t)$	-	Unit pulse signal

Transform Operators and Functions

$\mathcal{DFT}$	-	Discrete Fourier Transform (DFT)
$\mathcal{DFT}^{-1}$	-	Inverse DFT
$\mathcal{F}$	-	Fourier Transform
$\mathcal{F}^{-1}$	-	Inverse Fourier Transform
H(s)	-	Laplace Transform of h(t)
$\mathcal{L}$	-	Laplace Transform
$\mathcal{L}^{-1}$	-	Inverse Laplace Transform
$X(e^{j\omega})$	-	Discrete Time Fourier Transform of x(n)
$X_r(e^{j\omega})$	-	Real part of $X(e^{j\omega})$
$X_i(e^{j\omega})$	-	Imaginary part of $X(e^{j\omega})$

$X(j\Omega)$	-	Fourier Transform of $x(t)$
$X(k)$	-	Discrete Fourier Transform of $x(k)$
$X_r(k)$	-	Real part of $X(k)$
$X_i(k)$	-	Imaginary part of $X(k)$
$X(s)$	-	Laplace Transform of $x(t)$
$X(z)$	-	$\mathcal{Z}$-transform of $x(n)$
$\mathcal{Z}$	-	$\mathcal{Z}$-transform
$\mathcal{Z}^{-1}$	-	Inverse $\mathcal{Z}$-transform

Matrices and Vectors

$\mathbf{A}$	-	System matrix
$\mathbf{A}^n$	-	State transition matrix of discrete time state model
$\mathbf{B}$	-	Input matrix
$\mathbf{C}$	-	Output matrix
$\mathbf{D}$	-	Transmission matrix
$e^{\mathbf{A}t}$	-	State transition matrix of continuous time state model
$\mathbf{I}$	-	Identity / Unit matrix
$\mathbf{Q}(t)$	-	State vector of continuous time state model
$\mathbf{Q}(n)$	-	State vector of discrete time state model
$\dot{\mathbf{Q}}(t)$	-	First derivative of continuous time state vector
$\dot{\mathbf{Q}}(n)$	-	First derivative of discrete time state vector
$\mathbf{X}(t)$	-	Input vector of continuous time state model
$\mathbf{X}(n)$	-	Input vector of discrete time state model
$\mathbf{Y}(t)$	-	Output vector of continuous time state model
$\mathbf{Y}(n)$	-	Output vector of discrete time state model

Abbreviations

BIBO	-	**B**ounded **I**nput **B**ounded **O**utput
CT	-	**C**ontinuous **T**ime
CTFS	-	**C**ontinuous **T**ime **F**ourier **S**eries
CTFT	-	**C**ontinuous **T**ime **F**ourier **T**ransform
DFT	-	**D**iscrete **F**ourier **T**ransform
DIF	-	**D**ecimation **I**n **F**requency
DIT	-	**D**ecimation **I**n **T**ime
DT	-	**D**iscrete **T**ime
DTFS	-	**D**iscrete **T**ime **F**ourier **S**eries
DTFT	-	**D**iscrete **T**ime **F**ourier **T**ransform
FFT	-	**F**ast **F**ourier **T**ransform
FIR	-	**F**inite **I**mpulse **R**esponse
IIR	-	**I**nfinite **I**mpulse **R**esponse
LHP	-	**L**eft **H**alf **P**lane
LTI	-	**L**inear **T**ime **I**nvariant
RHP	-	**R**ight **H**alf **P**lane
ROC	-	**R**egion **O**f **C**onvergence

CHAPTER 1

Introduction to Signals and Systems

1.1 Signal

Any physical phenomenon that conveys or carries some information can be called a signal. The music, speech, motion pictures, still photos, heart beat, etc., are examples of signals that we normally encounter in day to day life.

Usually, the information carried by a signal will be a function of an independent variable. The independent variable can be time, spatial coordinates, intensity of colours, pressure, temperature, etc.,. The most popular independent variable in signals is time and it is represented by the letter "t".

The value of a signal at any specified value of the independent variable is called its ***amplitude***. The sketch or plot of the amplitude of a signal as a function of independent variable is called its ***waveform***.

Mathematically, any signal can be represented as a function of one or more independent variables.Therefore, *a **signal** is defined as any physical quantity that varies with one or more independent variables*.

For example, the functions $x_1(t)$ and $x_2(t)$ as defined by the equations (1.1) and (1.2) represents two signals: one that varies linearly with time "t" and the other varies quadratically with time "t". The equation (1.3) represents a signal which is a function of two independent variables "p" and "q".

$$x_1(t) = 0.7t \quad .....(1.1)$$

$$x_2(t) = 1.8t^2 \quad .....(1.2)$$

$$x(p,q) = 0.6p + 0.5q + 1.1q^2 \quad .....(1.3)$$

The signals can be classified in number of ways. Some way of classifying the signals are,

I. Depending on the number of sources for the signals.

1.One-channel signals

2. Multichannel signals

II. Depending on the number of dependent variables.

1.One-dimensional signals

2. Multidimensional signals

III. Depending on whether the dependent variable is continuous or discrete.

1.Analog or Continuous signals

2. Discrete signals

1. One-channel signals

Signals that are generated by a single source or sensor are called one-channel signals.

The record of room temperature with respect to time, the audio output of a mono speaker, etc., are examples of one-channel signals.

2. Multichannel signals

Signals that are generated by multiple sources or sensors are called multichannel signals.

The audio output of two stereo speakers is an example of two-channel signal.The record of ECG (Electro Cardio Graph) at eight different places in a human body is an example of eight-channel signal.

3. One-dimensional signals

A signal which is a function of single independent variable is called one-dimensional signal.

The signals represented by equation (1.1) and (1.2) are examples of one-dimensional signals.

The music, speech, heart beat, etc., are examples of one-dimensional signals where the single independent variable is time.

4. Multidimensional signals

A signal which is a function of two or more independent variables is called multidimensional signal.

The equation (1.3) represents a two dimensional signal.

A photograph is an example of a two-dimensional signal. The intensity or brightness at each point of a photograph is a function of two spatial coordinates "x" and "y", (and so the spatial coordinates are independent variables). Hence, the intensity or brightness of a photograph can be denoted by b(x, y).

The motion picture of a black and white TV is an example of a three-dimensional signal. The intensity or brightness at each point of a black and white motion picture is a function of two spatial coordinates "x" and "y", and time "t". Hence, the intensity or brightness of a black and white motion picture can be denoted by b(x, y, t).

5. Analog or Continuous signals

When a signal is defined continuously for any value of independent variable, it is called analog or continuous signal. Most of the signals encountered in science and engineering are analog in nature. When the dependent variable of an analog signal is time, it is called continuous time signal.

6. Discrete signals

When a signal is defined for discrete intervals of independent variable, it is called discrete signal . When the dependent variable of a discrete signal is time, it is called discrete time signal. Most of the discrete signals are either sampled version of analog signals for processing by digital systems or output of digital systems.

1.1.1 Continuous Time Signal

In a signal with time as independent variable, if the signal is defined continuously for any value of the independent variable time "t", then the signal is called ***continuous time signal***. The continuous time signal is denoted as "x(t)".

The continuous time signal is defined for every instant of the independent variable time and so the magnitude (or the value) of continuous time signal is continuous in the specified range of time. Here both the magnitude of the signal and the independent variable are continuous.

1.1.2 Discrete Time Signal

In a signal with time as independent variable, if the signal is defined only for discrete instants of the independent variable time, then the signal is called ***discrete time signal***.

In discrete time signal the independent variable time "t" is uniformely divided into discrete intervals of time and each interval of time is denoted by an integer "n", where "n" stands for discrete interval of time and "n" can take any integer value in the range $-\infty$ to $+\infty$. Therefore, for a discrete time signal the independent variable is "n" and the magnitude of the discrete time signal is defined only for integer values of independent variable "n". The discrete time signal is denoted by "x(n)".

1.1.3 Digital Signal

The quantized and coded version of the discrete time signals are called ***digital signals***. In digital signals the value of the signal for every discrete time "n" is represented in binary codes. The process of conversion of a discrete time signal to digital signal involves quantization and coding.

Normally, for binary representation, a standard size of binary is chosen. In m-bit binary representation we can have 2^m binary codes. The possible range of values of the discrete time signals are usually divided into 2^m steps called ***quantization levels***, and a binary code is attached to each quantization level. The values of the discrete time signals are approximated by rounding or truncation in order to match the nearest quantization level.

1.2 System

Any process that exhibits cause and effect relation can be called a ***system***. A system will have an input signal and an output signal. The output signal will be a processed version of the input signal. A system is either interconnection of hardware devices or software / algorithm.

A system is denoted by letter $\mathcal{H}$. The diagrammatic representation of a system is shown in fig 1.1.

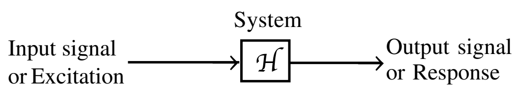


Fig 1.1 : *Representation of a system.*

The operation performed by a system on input signal to produce output signal can be expressed as,

$$\text{Output} = \mathcal{H}\{\text{Input}\}$$

where $\mathcal{H}$ denotes the system operation (also called ***system operator***).

The systems can be classified in many ways.

Depending on type of energy used to operate the systems, the systems can be classified into Electrical systems, Mechanical systems, Thermal systems, Hydraulic systems, etc.

Depending on the type of input and output signals, the systems can be classified into Continuous time systems and Discrete time systems.

1.2.1 Continuous Time System

A system which can process continuous time signal is called ***continuous time system***, and so the input and output signals of a continuous time system are continuous time signals.

A continuous time system is denoted by letter $\mathcal{H}$. The input of continuous time system is denoted as x(t) and the output of continuous time system is denoted as y(t). The diagrammatic representation of a continuous time system is shown in fig 1.2

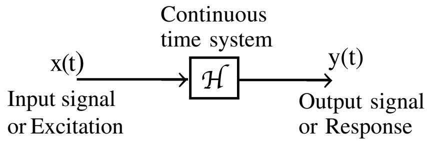


Fig 1.2 : *Representation of continuous time system.*

The operation performed by a continuous time system on input to produce output or response can be expressed as,

Response, $y(t) = \mathcal{H}\{x(t)\}$

where, $\mathcal{H}$ denotes the system operation (also called system operator).

When a continuous time system satisfies the properties of linearity and time invariance then it is called ***LTI (Linear Time Invariant) continuous time system*** . Most of the practical systems that we encounter in science and engineering are LTI systems.

The input-output relation of an LTI continuous time system is represented by constant coefficient differential equation shown below(equation (1.4)).

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + ... + a_{N-1} \frac{d}{dt} y(t) + a_N\, y(t) = b_0 \frac{d^M}{dt^M} x(t)$$
$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... + b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \qquad .....(1.4)$$

where, N = Order of the system, $M \le N$, and $a_0 = 1$.

The solution of the above differential equation is the response y(t) of the continuous time system, for the input x(t).

1.2.2 Discrete Time System

A system which can process discrete time signal is called ***discrete time system***, and so the input and output signals of a discrete time system are discrete time signals.

A discrete time system is denoted by the letter $\mathcal{H}$. The input of discrete time system is denoted as "x(n)" and the output of discrete time system is denoted as "y(n)". The diagrammatic representation of a discrete time system is shown in fig 1.3.

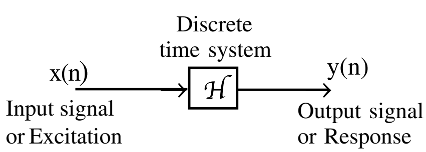


Fig 1.3 : *Representation of discrete time system.*

The operation performed by a discrete time system on input to produce output or response can be expressed as,

Response, $y(n) = \mathcal{H}\{x(n)\}$

where, $\mathcal{H}$ denotes the system operation (also called system operator).

When a discrete time system satisfies the properties of linearity and time invariance then it is called ***LTI (Linear Time Invariant) discrete time system*** .

The input-output relation of an LTI discrete time system is represented by constant coefficient difference equation shown below(equation (1.5)).

$$y(n) = -\sum_{m=1}^{N} a_m \, y(n-m) + \sum_{m=0}^{M} b_m \, x(n-m) \qquad .....(1.5)$$

where, N = Order of the system, and $M \leq N$.

The solution of the above difference equation is the response y(n) of the discrete time system, for the input x(n).

1.3 Frequency Domain Analysis of Continuous Time Signals and Systems

Physically, we realize any signal or system in time domain. In time domain, the continuous time systems are governed by differential equations. The analysis of continuous time signals and systems in time domain involves solution of differential equations. The solution of differential equations are difficult due to assumption of a solution and then solving the constants using initial conditions.

In order to simplify the task of analysis, the signals can be transformed to some other domain, where the analysis is easier. One such transform exists for continuous time signals is Laplace transform. The ***Laplace transform***, will transform a function of time "t" into a function of complex frequency "s" where $s = \sigma + j\Omega$. Therefore, Laplace transform of a continuous time signal will transform the time domain signal into s-domain signal.

On taking Laplace transform of the differential equation governing the system, it becomes algebraic equation in "s" and the solution of algebraic equation will give the response of the system as a function of "s" and it is called s-domain response. The inverse Laplace transform of the s-domain response, will give the time domain response of the continuous time system. Also, the stability analysis of the continuous time systems are much easier in s-domain.

Another important characteristics of any signal is frequency, and for most of the applications the frequency content of the signal is an important criteria. The frequency contents of a signal can be

Table 1.1 : Frequency Range of Some Electromagnetic Signals

Type of signal	Wavelength (m)	Frequency range (Hz)
Radio broadcast	10^4 to 10^2	3×10^4 to 3×10^6
Shortwave radio signals	10^2 to 10^{-2}	3×10^6 to 3×10^{10}
Radar / Space communications	1 to 10^{-2}	3×10^8 to 3×10^{10}
Common-carrier microwave	1 to 10^{-2}	3×10^8 to 3×10^{10}
Infrared	10^{-3} to 10^{-6}	3×10^{11} to 3×10^{14}
Visible light	3.9×10^{-7} to 8.1×10^{-7}	3.7×10^{14} to 7.7×10^{14}
Ultraviolet	10^{-7} to 10^{-8}	3×10^{15} to 3×10^{16}
Gamma rays and x-rays	10^{-9} to 10^{-10}	3×10^{17} to 3×10^{18}

Table 1.2 : Frequency Range of Some Biological and Seismic Signals

Type of Signal	Frequency Range (Hz)
Electroretinogram	0 to 20
Electronystagmogram	0 to 20
Pneumogram	0 to 40
Electrocardiogram (ECG)	0 to 100
Electroencephalogram (EEG)	0 to 100
Electromyogram	10 to 200
Sphygmomanogram	0 to 200
Speech	100 to 4000
Wind noise	100 to 1000
Seismic exploration signals	10 to 100
Earthquake and nuclear explosion signals	0.01 to 10
Seismic noise	0.1 to 1

studied by taking Fourier transform of a signal. The Fourier transform of a signal is a particular class of Laplace transform in which $s = j\Omega$, where "Ω" is real frequency.

The ***Fourier transform***, will transform a function of time "t" into a function of real frequency "Ω". Therefore, Fourier transform of a continuous time signal will transform the time domain signal into frequency domain signal. From the Fourier transform of a continuous time signal, the frequency spectrum of the signal can be obtained which is used to study the frequency content of a signal. The frequency range of some of the signals are listed in table 1.1 and 1.2.

1.4 Frequency Domain Analysis of Discrete Time Signals and Systems

Mostly, the discrete time systems are designed for analysis of discrete time signals. Physically, the discrete time systems are also realized in time domain. In time domain, the discrete time systems are governed by difference equations. The analysis of discrete time signals and systems in time domain involves solution of difference equations. The solution of difference equations are difficult due to assumption of a solution and then solving the constants using initial conditions.

In order to simplify the task of analysis, the discrete time signals can be transformed to some other domain, where the analysis may be easier. One such transform exists for discrete time signals is $\mathcal{Z}$-transform.The $\mathcal{Z}$-transform, will transform a function of discrete time "n" into a function of complex variable "z" and it is expressed as, $z = re^{j\omega}$. Therefore, ***$\mathcal{Z}$-transform*** of a discrete time signal will transform the time domain signal into z-domain signal.

On taking $\mathcal{Z}$ -transform of the difference equation governing the discrete time system, it becomes algebraic equation in "z" and the solution of algebraic equation will give the response of the system as a function of "z" and it is called z-domain response. The inverse $\mathcal{Z}$ -transform of the z-domain response, will give the time domain response of the discrete time system. Also, the stability analysis of the discrete systems are much easier in z-domain.

The frequency contents of a discrete time signal can be studied by taking Fourier transform of the discrete time signal. The Fourier transform of discrete time signal is a particular class of $\mathcal{Z}$-transform in which $z = e^{j\omega}$, where "ω" is the frequency of the discrete time signals.

The Fourier transform, will transform a function of discrete time "n" into a function of frequency "ω". Therefore, Fourier transform of a discrete time signal will transform the discrete time signal into frequency domain signal. From the Fourier transform of the discrete time signal, the frequency spectrum of the discrete time signal can be obtained which is used to study the frequency content of the discrete time signal.

1.5 Importance of Signals and Systems

Every part of the universe is a system which generates or processes some type of signal [Of course the universe itself is a system and said to be controlled by signals (or commands) issued by God].

The signals and systems play a vital role in almost every field of Science and Engineering. Some of the applications of signals and systems in various field of Science and Engineering are listed here.

1. Biomedical

- ECG is used to predict heart diseases.
- EEG is used to study normal and abnormal behaviour of the brain.
- EMG is used to study the condition of muscles.
- X-ray images are used to predict the bone fractures and tuberclosis.
- Ultrasonic scan images of kidney and gall bladder is used to predict stones.
- Ultrasonic scan images of foetus is used to predict abnormalities in a baby.
- MRI scan is used to study minute inner details of any part of the human body.

2. Speech Processing

- Speech compression and decompression to reduce memory requirement of storage systems.
- Speech compression and decompression for effective use of transmission channels.
- Speech recognization for voice operated systems and voice based security systems.
- Speech recognization for conversion of voice to text.
- Speech synthesis for various voice based warnings or annoucements.

3. Audio and Video Equipments

- The analysis of audio signals will be useful to design systems for special effects in audio systems like stereo, woofer, karoke, equalizer, attenuator, etc.
- Music synthesis and composing using music keyboards.
- Audio and video compression for storage in DVDs.

4. Communication

- The spectrum analysis of modulated signals helps to identify the information bearing frequency component that can be used for transmission.

- The analysis of signals received from radars are used to detect flying objects and thier velocity.
- Generation and detection of DTMF signals in telephones.
- Echo and noise cancellation in transmission channels.

5. Power electronics

- The spectrum analysis of the output of coverters and inverters will reveal the harmonics present in the output, which in turn helps to design suitable filter to eliminate the harmonics.
- The analysis of switching currents and voltages in power devices will help to reduce losses.

6. Image processing

- Image compression and decompression to reduce memory requirement of storage systems.
- Image compression and decompression for effective use of transmission channels.
- Image recognization for security systems.
- Filtering operations on images to extract the features or hidden information.

7. Geology

- The seismic signals are used to determine the magnitude of earthquake and valconic eruptions.
- The seismic signals are also used to predict nuclear explosions.
- The seismic noise are also used to predict the movement of earth layers (tectonic plates).

8. Astronomy

- The analysis of light received from a star is used to determine the condition of the star.
- The analysis of images of various celestial bodies gives vital information about them.

1.6 Use of MATLAB in Signals and Systems

The ***MATLAB*** (**MAT**rix **LAB**oratory) is a software developed by The MathWork Inc, USA, which can run on any windows platform in a PC(Personal Computer). This software has number of tools for the study of various engineering subjects. It includes a tool for signal processing also. Using this tool a wide variety of studies can be made on signals and systems. Some of the analysis that is relevant to this particular text book are given below.

- Sketch or plot of signals as a function of independent variable.
- Spectrum analysis of signals.
- Solution of LTI systems.
- Perform convolution and deconvolution operations on signals.
- Perform various transforms on signals like Laplace transform, Fourier transform, $\mathcal{Z}$-transform, Fast Fourier Transform (FFT), etc.
- Determination of state model from transfer function and viceversa.
- Stability analysis of signals and systems in various domains.

CHAPTER 2

Continuous Time Signals and Systems

2.1 Introduction

Time is an important independent variable required to measure/monitor any activity. Hence, whatever phenomena we observe in nature are always measured as a function of time.

Time is a continuous independent variable represented by the letter 't'. Any physical observation or a measure is a continuous function of time represented by x(t) and called ***signal.*** The signal x(t) is called the continuous time (CT) signal which is a function of the independent variable, time. In x(t),the unit of time and the unit of the value of x(t) at any time is not considered but only the numerical values are considered.

The signal x(t) can be used to represent any physical quantity, and the start of any observation or a measure of the physical quantity is taken as time $t = 0$. The time after the start of observation is taken as positive time and the time before the start of observation is taken as negative time.

The physical devices which are all sources of continuous time signals are called ***continuous time systems***. The standard continuous time signals, mathematical operation on continuous time signals and classification of continuous time signals are discussed in this chapter. The mathematical representation of continuous time systems and their analyses are also presented in this chapter. Wherever required, the discussion on LTI systems are presented separately.

2.2 Standard Continuous Time Signals

1. Impulse signal

The impulse signal is a signal with infinite magnitude and zero duration, but with an area of A. Mathematically, impulse signal is defined as,

$$\text{Impulse Signal, } \delta(t) = \infty\ ;\ t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\, dt = A$$
$$= 0\ ;\ t \neq 0$$

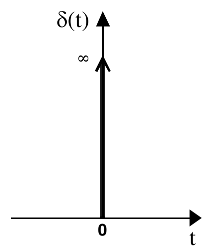


Fig 2.1 : Impulse signal (or Unit Impulse signal).

The unit impulse signal is a signal with infinite magnitude and zero duration, but with unit area. Mathematically, unit impulse signal is defined as,

$$\text{Unit Impulse Signal, } \delta(t) = \infty\ ;\ t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0\ ;\ t \neq 0$$

2. Step signal

The step signal is defined as,

$$x(t) = A\ ;\ t \geq 0$$
$$= 0\ ;\ t < 0$$

The unit step signal is defined as,

$$x(t) = u(t) = 1\ ;\ t \geq 0$$
$$= 0\ ;\ t < 0$$

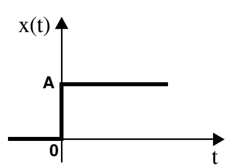


Fig 2.2 : Step signal.

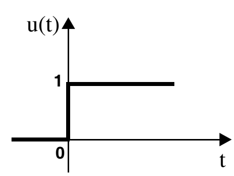


Fig 2.3 : Unit step signal.

3. Ramp signal

The ramp signal is defined as,

$$x(t) = At \; ; \; t \ge 0$$
$$= 0 \; ; \; t < 0$$

The unit ramp signal is defined as,

$$x(t) = t \; ; \; t \ge 0$$
$$= 0 \; ; \; t < 0$$

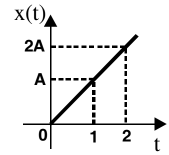


Fig 2.4 : *Ramp signal.*

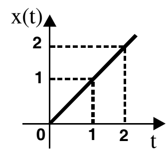


Fig 2.5 : *Unit ramp signal.*

4. Parabolic signal

The parabolic signal is defined as,

$$x(t) = \frac{At^2}{2} \; ; \; \text{for } t \ge 0$$
$$= 0 \; ; \; t < 0$$

The unit parabolic signal is defined as,

$$x(t) = \frac{t^2}{2} \; ; \; \text{for } t \ge 0$$
$$= 0 \; ; \; t < 0$$

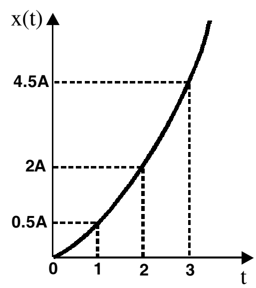


Fig 2.6 : *Parabolic signal.*

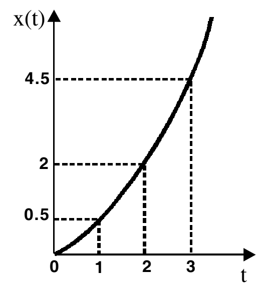


Fig 2.7 :*Unit parabolic signal.*

5. Unit pulse signal

The unit pulse signal is defined as,

$$x(t) = \Pi(t) = u\left(t + \frac{1}{2}\right) - u\left(t - \frac{1}{2}\right)$$

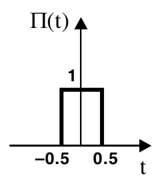


Fig 2.8 : *Unit pulse signal.*

6. Sinusoidal signal

Case i : ***Cosinusoidal signal***

The cosinusoidal signal is defined as,

$$x(t) = A\cos(\Omega_0 t + \phi)$$

where, $\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}$ = Angular frequency in rad/sec

F_0 = Frequency in cycles/sec or Hz

T = Time period in sec

When $\phi = 0$, $\quad x(t) = A\cos\Omega_0 t$

When ϕ = Positive, $\quad x(t) = A\cos(\Omega_0 t + \phi)$

When ϕ = Negative, $\quad x(t) = A\cos(\Omega_0 t - \phi)$

Case ii : Sinusoidal signal

The sinusoidal signal is defined as,

$$x(t) = A\sin(\Omega_0 t + \phi)$$

where, $\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}$ = Angular frequency in rad/sec

F_0 = Frequency in cycles/sec or Hz

T = Time period in sec

When $\phi = 0$, $x(t) = A\sin\Omega_0 t$

When ϕ = Positive, $x(t) = A\sin(\Omega_0 t + \phi)$

When ϕ = Negative, $x(t) = A\sin(\Omega_0 t - \phi)$

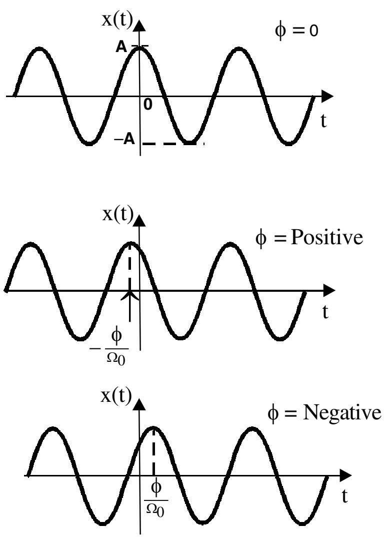


Fig 2.9 : Cosinusoidal signal.

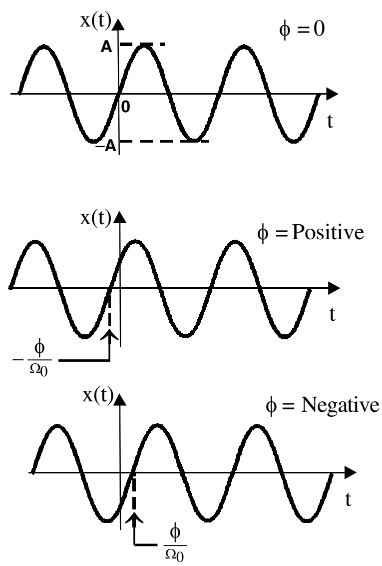


Fig 2.10 : Sinusoidal signal.

7. Exponential signal

Case i : Real exponential signal

The real exponential signal is defined as,

$$x(t) = A\,e^{bt}$$

where, A and b are real

Here, when b is positive, the signal x(t) will be an exponentially rising signal; and when b is negative the signal x(t) will be an exponentially decaying signal.

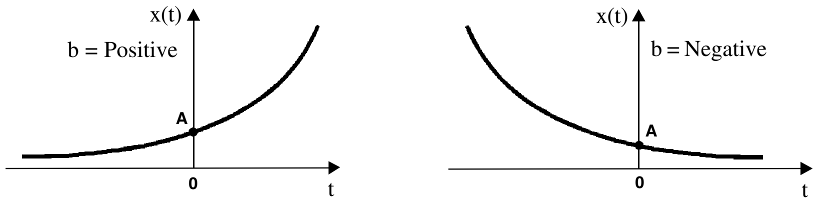


Fig 2.11 : *Real exponential signal.*

***Case ii :* Complex exponential signal**

The complex exponential signal is defined as,

$$x(t) = A\,e^{j\Omega_0 t}$$

$$\text{where, } \quad \Omega_0 = 2\pi F_0 = \frac{2\pi}{T} = \text{ Angular frequency in rad/sec}$$

$$F_0 = \text{Frequency in cycles/sec or Hz}$$

$$T = \text{Time period in sec}$$

The complex exponential signal can be represented in a complex plane by a rotating vector, which rotates with a constant angular velocity of Ω_0 rad/sec.

The complex exponential signal can be resolved into real and imaginary parts as shown below,

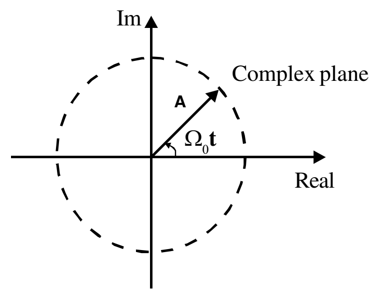


Fig 2.12 : *Complex exponential signal.*

$$x(t) = A\,e^{j\Omega_0 t} = A\,(\cos\Omega_0 t + j\sin\Omega_0 t)$$

$$= A\cos\Omega_0 t + jA\sin\Omega_0 t$$

$$\therefore \quad A\cos\Omega_0 t = \text{ Real part of } x(t)$$

$$A\sin\Omega_0 t = \text{ Imaginary part of } x(t)$$

From the above equation, we can say that a complex exponential signal is the vector sum of two sinusoidal signals of the form $\cos\Omega_0 t$ and $\sin\Omega_0 t$.

---

8. Exponentially rising/decaying sinusoidal signal

The exponential rising/decaying sinusoidal signal is defined as,

$$x(t) = A\,e^{bt}\sin\Omega_0 t$$

$$\text{where, } \quad \Omega_0 = 2\pi F_0 = \frac{2\pi}{T} = \text{ Angular frequency in rad/sec}$$

$$F_0 = \text{Frequency in cycles/sec or Hz}$$

$$T = \text{Time period in sec}$$

Here, A and b are real constants. When b is positive, the signal x(t) will be an exponentially rising sinusoidal signal; and when b is negative, the signal x(t) will be an exponentially decaying sinusoidal signal.

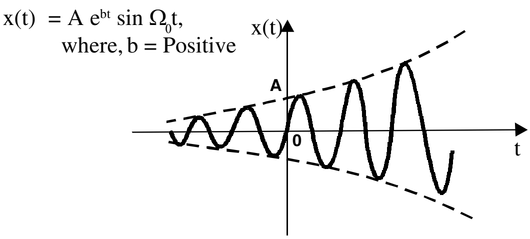


Fig 2.13 : Exponentially rising sinusoid.

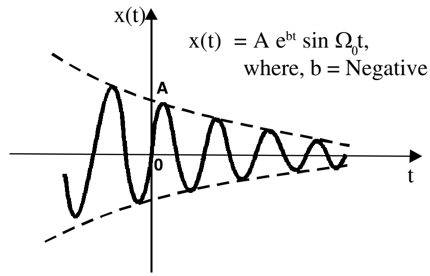


Fig 2.14 : Exponentially decaying sinusoid.

9. Triangular pulse signal

The Triangular pulse signal is defined as

$$x(t) = \Delta_a(t) = 1 - \frac{|t|}{a} \quad ; \quad |t| \le a$$
$$= 0 \quad ; \quad |t| > a$$

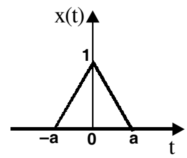


Fig 2.15 : *Triangular pulse signal.*

10. Signum signal

The Signum signal is defined as the sign of the independent variable t. Therefore, the Signum signal is expressed as,

$$x(t) = \mathrm{sgn}(t) = 1 \quad ; \quad t > 0$$
$$= 0 \quad ; \quad t = 0$$
$$= -1 \quad ; \quad t < 0$$

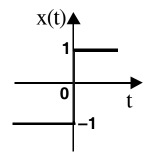


Fig 2.16 : *Signum signal.*

11. Sinc signal

The Sinc signal is defined as,

$$x(t) = \mathrm{sinc}(t) = \frac{\sin t}{t} \quad ; \quad -\infty < t < \infty$$

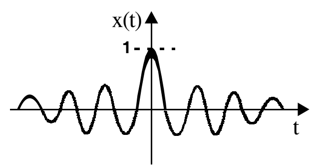


Fig 2.17 : *Sinc signal.*

12. Gaussian signal

The Gaussian signal is defined as,

$$x(t) = g_a(t) = e^{-a^2t^2} \quad ; \quad -\infty < t < \infty$$

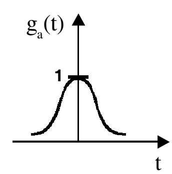


Fig 2.18 : *Gaussian signal.*

2.3 Classification of Continuous Time Signals

The continuous time signals are classified depending on their characteristics. Some ways of classifying continuous time signals are,

1. Deterministic and Nondeterministic signals
2. Periodic and Nonperiodic signals
3. Symmetric and Antisymmetric signals (Even and Odd signals)
4. Energy and Power signals
5. Causal and Noncausal signals

2.3.1 Deterministic and Nondeterministic Signals

The signal that can be completely specified by a mathematical equation is called a ***deterministic signal***. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

Examples of deterministic signals: $x_1(t) = At$

$x_2(t) = X_m \sin \Omega_0 t$

The signal whose characteristics are random in nature is called a ***nondeterministic signal.*** The noise signals from various sources like electronic amplifiers, oscillators, radio receivers, etc., are best examples of nondeterministic signals.

2.3.2 Periodic and Nonperiodic Signals

A periodic signal will have a definite pattern that repeats again and again over a certain period of time. Therefore the signal which satisfies the condition,

$\boxed{x(t + T) = x(t)}$ is called a ***periodic signal***.

A signal which does not satisfy the condition, $x(t + T) = x(t)$ is called an ***aperiodic or nonperiodic signal***. In periodic signals, the term T is called the ***fundamental time period*** of the signal. Hence, inverse of T is called the ***fundamental frequency,*** F_0 in cycles/sec or Hz, and $2\pi F_0 = \Omega_0$ is called the ***fundamental angular frequency*** in rad/sec.

The sinusoidal signals and complex exponential signals are always periodic with a periodicity of T, where, $T = \dfrac{1}{F_0} = \dfrac{2\pi}{\Omega_0}$. The proof of this concept is given below.

Proof :

a) Cosinusoidal signal

Let, $x(t) = A\cos\Omega_0 t$

$$\therefore x(t + T) = A\cos\Omega_0(t+T) = A\cos(\Omega_0 t + \Omega_0 T)$$

$$= A\cos\left(\Omega_0 t + \frac{2\pi}{T}T\right) \qquad \boxed{\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}}$$

$$= A\cos(\Omega_0 t + 2\pi) = A\cos\Omega_0 t = x(t) \qquad \boxed{\cos(\theta + 2\pi) = \cos\theta}$$

b) Sinusoidal signal

Let, $x(t) = A\sin\Omega_0 t$

$\therefore x(t+T) = A\sin\Omega_0(t+T) = A\sin(\Omega_0 t + \Omega_0 T)$

$= A\sin\left(\Omega_0 t + \dfrac{2\pi}{T}T\right)$

$= A\sin(\Omega_0 t + 2\pi) = A\sin\Omega_0 t = x(t)$

$\boxed{\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}}$

$\boxed{\sin(\theta + 2\pi) = \sin\theta}$

c) Complex exponential signal

Let, $x(t) = Ae^{j\Omega_0 t}$

$\therefore x(t+T) = Ae^{j\Omega_0(t+T)} = A\,e^{j\Omega_0 t}\,e^{j\Omega_0 T} = A\,e^{j\Omega_0 t}\,e^{j\frac{2\pi}{T}T} = A\,e^{j\Omega_0 t}\,e^{j2\pi}$

$= A\,e^{j\Omega_0 t}(\cos 2\pi + j\sin 2\pi) = A\,e^{j\Omega_0 t}\,(1 + j\,0) = x(t)$

$\boxed{\cos 2\pi = 1,\ \sin 2\pi = 0}$

When a continuous time signal is a mixture of two periodic signals with fundamental time periods T_1 and T_2, then the continuous time signal will be periodic, if the ratio of T_1 and T_2 (i.e., T_1/T_2) is a rational number. Now the periodicity of the continuous time signal will be the LCM (Least Common Multiple) of T_1 and T_2.

Note : *1. The ratio of two integers is called a rational number.*

Example of rational number : $\dfrac{5}{2}, \dfrac{7}{9}, \dfrac{8}{11}$.

Example of non-rational number : $\dfrac{\sqrt{2}}{5}, \dfrac{7}{2\pi}, \dfrac{4}{\sqrt{7}}$.

2. When T_1/T_2 is a rational number, then F_{01}/F_{02} and Ω_{01}/Ω_{02} are also rational numbers.

Example 2.1

Verify whether the following continuous time signals are periodic. If periodic, find the fundamental period.

a) $x(t) = 2\cos\dfrac{t}{4}$ b) $x(t) = e^{\alpha t}$; $\alpha > 1$ c) $x(t) = e^{\frac{-j2\pi t}{7}}$ d) $x(t) = 3\cos\left(5t + \dfrac{\pi}{6}\right)$ e) $x(t) = \cos^2\left(2t - \dfrac{\pi}{4}\right)$

Solution

a) Given that, $x(t) = 2\cos\dfrac{t}{4}$

The given signal is a cosinusoidal signal, which is always periodic.

On comparing x(t) with the standard form "$A\cos 2\pi F_0 t$" we get,

$$2\pi F_0 = \frac{1}{4} \quad\Rightarrow\quad F_0 = \frac{1}{8\pi}$$

Period, $T = \dfrac{1}{F_0} = 8\pi$

$\therefore$ x(t) is periodic with period, $T = 8\pi$.

b) Given that, $x(t) = e^{\alpha t}$; $\alpha > 1$

$\therefore x(t+T) = e^{\alpha(t+T)}$

$= e^{\alpha t}\,e^{\alpha T}$

For any value of α, $e^{\alpha T} \neq 1$ and so $x(t+T) \neq x(t)$

Since $x(t+T) \neq x(t)$, the signal x(t) is non-periodic.

c) Given that, $x(t) = e^{\frac{-j2\pi t}{7}}$

The given signal is a complex exponential signal, which is always periodic.

On comparing x(t) with the standard form "$A\,e^{-j2\pi F_0 t}$"

We get, $F_0 = \frac{1}{7}$

$$\therefore \text{ Peroid, } T = \frac{1}{F_0} = 7$$

$\therefore$ x(t) is periodic with period, T = 7.

---

d) Given that, $x(t) = 3\cos\left(5t + \frac{\pi}{6}\right)$

The given signal is a cosinusoidal signal, which is always periodic.

$$\therefore x(t+T) = 3\cos\left(5(t+T) + \frac{\pi}{6}\right) = 3\cos\left(5t + 5T + \frac{\pi}{6}\right) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 5T\right)$$

Let $5T = 2\pi$, $\quad \therefore T = \frac{2\pi}{5}$

$$\therefore x(t+T) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 5 \times \frac{2\pi}{5}\right) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 2\pi\right)$$

$$= 3\cos\left(5t + \frac{\pi}{6}\right) = x(t)$$

> For integer values of M,
> $\cos(\theta + 2\pi M) = \cos\theta$

Since $x(t + T) = x(t)$, the signal x(t) is periodic with period, $T = \frac{2\pi}{5}$

---

e) Given that, $x(t) = \cos^2\left(2t - \frac{\pi}{3}\right)$

$$x(t) = \cos^2\left(2t - \frac{\pi}{3}\right) = \frac{1 + \cos 2\left(2t - \frac{\pi}{3}\right)}{2} = \frac{1 + \cos\left(4t - \frac{2\pi}{3}\right)}{2}$$

> $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

$$\therefore x(t+T) = \frac{1 + \cos\left(4(t+T) - \frac{2\pi}{3}\right)}{2} = \frac{1 + \cos\left(4t + 4T - \frac{2\pi}{3}\right)}{2}$$

$$= \frac{1 + \cos\left(4t - \frac{2\pi}{3} + 4T\right)}{2}$$

Let $4T = 2\pi$, $\therefore T = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\therefore x(t+T) = \frac{1 + \cos\left(4t - \frac{2\pi}{3} + 4 \times \frac{\pi}{2}\right)}{2} = \frac{1 + \cos\left(\left(4t - \frac{2\pi}{3}\right) + 2\pi\right)}{2}$$

$$= \frac{1 + \cos\left(4t - \frac{2\pi}{3}\right)}{2} = \frac{1 + \cos 2\left(2t - \frac{\pi}{3}\right)}{2} = \cos^2\left(2t - \frac{\pi}{3}\right) = x(t)$$

Since $x(t + T) = x(t)$, the signal x(t) is periodic with period, $T = \frac{\pi}{2}$

> For integer values of M,
> $\cos(\theta + 2\pi M) = \cos\theta$

Since $x_1(t)$ and $x_2(t)$ are periodic and the ratio of T_1 and T_2 is a rational number, the signal x(t) is also periodic. Let T be the periodicity of x(t). Now the periodicity of x(t) is the LCM (Least Common Multiple) of T_1 and T_2, which is calculated as shown below.

$$T_1 = \frac{2\pi}{3} = \frac{2\pi}{3} \times \frac{21}{2\pi} = 7$$

$$T_2 = \frac{2\pi}{7} = \frac{2\pi}{7} \times \frac{21}{2\pi} = 3$$

Note : *To find LCM, first convert T_1 and T_2 to integers by multiplying by a common number. Find LCM of integer values of T_1 and T_2. Then divide this LCM by the common number.*

Now LCM of 7 and 3 is 21.

$$\therefore \text{ Period, } T = 21 \div \frac{21}{2\pi} = 21 \times \frac{2\pi}{21} = 2\pi$$

Proof : $x(t + T) = 2 \cos 3(t + T) + 3 \sin 7(t + T)$

$$= 2 \cos(3t + 3T) + 3 \sin(7t + 7T)$$

$$= 2 \cos(3t + 3 \times 2\pi) + 3 \sin(7t + 7 \times 2\pi)$$

$$= 2 \cos(3t + 6\pi) + 3 \sin(7t + 14\pi)$$

$$= 2 \cos 3t + 3 \sin 7t = x(t)$$

Put, $T = 2\pi$

For integer values of M,
$\cos(\theta + 2\pi M) = \cos\theta$
$\sin(\theta + 2\pi M) = \sin\theta$

(c) Given that, $x(t) = 5 \cos 4\pi t + 3 \sin 8\pi t$

Let, $x_1(t) = 5 \cos 4\pi t$

Let T_1 be the periodicity of $x_1(t)$. On comparing $x_1(t)$ with the standard form "A cos $2\pi F_{01}t$", we get,

$$F_{01} = 2 \; ; \qquad \therefore \text{ Period, } T_1 = \frac{1}{F_{01}} = \frac{1}{2}$$

Let, $x_2(t) = 3 \sin 8\pi t$

Let T_2 be the periodicity of $x_2(t)$. On comparing $x_2(t)$ with the standard form "A sin $2\pi F_{02}t$", we get,

$$F_{02} = 4 \; ; \qquad \therefore \text{ Period, } T_2 = \frac{1}{F_{02}} = \frac{1}{4}$$

$$\text{Now, } \frac{T_1}{T_2} = T_1 \times \frac{1}{T_2} = \frac{1}{2} \times \frac{4}{1} = 2$$

Since $x_1(t)$ and $x_2(t)$ are periodic and the ratio of T_1 and T_2 is a rational number, the signal x(t) is also periodic. Let T be the periodicity of x(t). Now, the periodicity of x(t) is the LCM (Least Common Multiple) of T_1 and T_2, which is calculated as shown below.

$$T_1 = \frac{1}{2} = \frac{1}{2} \times 4 = 2$$

$$T_2 = \frac{1}{4} = \frac{1}{4} \times 4 = 1$$

Note : *To find LCM, first convert T_1 and T_2 to integers by multiplying by a common number. Find LCM of integer values of T_1 and T_2. Then divide this LCM by the common number.*

Now LCM of 2 and 1 is 2.

$$\therefore \text{ Period, } T = 2 \div 4 = 2 \times \frac{1}{4} = \frac{1}{2}$$

Proof : $x(t + T) = 5 \cos 4\pi(t + T) + 3 \sin 8\pi(t + T)$

$$= 5 \cos(4\pi t + 4\pi T) + 3 \sin(8\pi t + 8\pi T)$$

$$= 5 \cos\left(4\pi t + 4\pi \times \frac{1}{2}\right) + 3 \sin\left(8\pi t + 8\pi \times \frac{1}{2}\right)$$

$$= 5 \cos(4\pi t + 2\pi) + 3 \sin(8\pi t + 2\pi)$$

$$= 5 \cos 4\pi t + 3 \sin 8\pi t = x(t)$$

Put, $T = \frac{1}{2}$

For integer values of M,
$\cos(\theta + 2\pi M) = \cos\theta$
$\sin(\theta + 2\pi M) = \sin\theta$

2.3.3 Symmetric (Even) and Antisymmetric (Odd) Signals

The signals may exhibit symmetry or antisymmetry with respect to t = 0.

When a signal exhibits symmetry with respect to t = 0 then it is called an ***even signal.*** Therefore, the even signal satisfies the condition, **x(–t) = x(t).**

When a signal exhibits antisymmetry with respect to t = 0, then it is called an ***odd signal***. Therefore, the odd signal satisfies the condition, **x(–t) = –x(t).**

Since $\cos(-\theta) = \cos\theta$, the cosinusoidal signals are even signals and since $\sin(-\theta) = -\sin\theta$, the sinusoidal signals are odd signals.

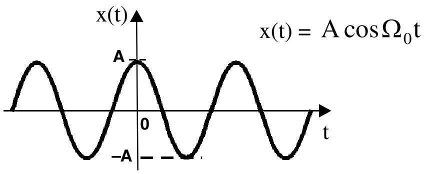


Fig 2.19a : *Symmetric or Even signal.*

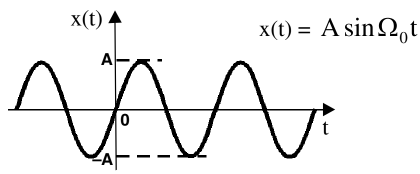


Fig 2.19b : *Antisymmetric or Odd signal.*

Fig 2.19 : *Symmetric and antisymmetric continuous time signals.*

A continuous time signal x(t) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $x(t) = x_e(t) + x_o(t)$

where, $x_e(t)$ = Even part of x(t) and $x_o(t)$ = Odd part of x(t)

Now, it can be proved that,

$$\boxed{\begin{aligned} x_e(t) &= \frac{1}{2}\left[x(t) + x(-t)\right] \\ x_o(t) &= \frac{1}{2}\left[x(t) - x(-t)\right] \end{aligned}}$$

Proof :

Let, $x(t) = x_e(t) + x_o(t)$ .....(2.1)

On replacing t by –t in equation (2.1) we get,

$$x(-t) = x_e(-t) + x_o(-t) \qquad .....(2.2)$$

Since $x_e(t)$ is even, $x_e(-t) = x_e(t)$

Since $x_o(t)$ is odd, $x_o(-t) = -x_o(t)$

Hence the equation (2.2) can be written as,

$$x(-t) = x_e(t) - x_o(t) \qquad .....(2.3)$$

On adding equations (2.1) & (2.3) we get,

$$x(t) + x(-t) = 2\,x_e(t)$$

$$\therefore\ x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$$

On subtracting equation (2.3) from equation (2.1) we get,

$$x(t) - x(-t) = 2\,x_o(t)$$

$$\therefore\ x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$$

The properties of signals with symmetry are given below without proof.

1. When a signal is even, then its odd part will be zero.
2. When a signal is odd, then its even part will be zero.
3. The product of two odd signals will be an even signal.
4. The product of two even signals will be an even signal.
5. The product of an even and odd signal will be an odd signal.

Example 2.3

Determine the even and odd part of the following continuous time signals.

a) $x(t) = e^t$ b) $x(t) = 3 + 2t + 5t^2$ c) $x(t) = \sin 2t + \cos t + \sin t \cos 2t$

Solution

a) Given that, $x(t) = e^t$

$$\therefore x(-t) = e^{-t}$$

$$\text{Even part, } x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[e^t + e^{-t}]$$

$$\text{Odd part, } x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[e^t - e^{-t}]$$

b) Given that, $x(t) = 3 + 2t + 5t^2$

$$\therefore x(-t) = 3 + 2(-t) + 5(-t)^2$$

$$= 3 - 2t + 5t^2$$

$$\text{Even part, } x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[3 + 2t + 5t^2 + 3 - 2t + 5t^2]$$

$$= \frac{1}{2}[6 + 10t^2] = 3 + 5t^2$$

$$\text{Odd part, } x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[3 + 2t + 5t^2 - 3 + 2t - 5t^2]$$

$$= \frac{1}{2}[4t] = 2t$$

c) Given that, $x(t) = \sin 2t + \cos t + \sin t \cos 2t$

$$\therefore x(-t) = \sin 2(-t) + \cos(-t) + \sin(-t) \cos 2(-t)$$

$$= -\sin 2t + \cos t - \sin t \cos 2t$$

$$\text{Even part, } x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[\sin 2t + \cos t + \sin t \cos 2t - \sin 2t + \cos t - \sin t \cos 2t]$$

$$= \frac{1}{2}[2 \cos t] = \cos t$$

$$\text{Odd part, } x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[\sin 2t + \cos t + \sin t \cos 2t + \sin 2t - \cos t + \sin t \cos 2t]$$

$$= \frac{1}{2}[2 \sin 2t + 2 \sin t \cos 2t] = \sin 2t + \sin t \cos 2t$$

2.3.4 Energy and Power Signals

The signals which have finite energy are called ***energy signals***. The nonperiodic signals like exponential signals will have constant energy and so nonperiodic signals are energy signals.

The signals which have finite average power are called ***power signals***. The periodic signals like sinusoidal and complex exponential signals will have constant power and so periodic signals are power signals.

The ***energy*** E of a continuous time signal x(t) is defined as,

$$\boxed{\text{Energy, } E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \text{ in joules}}$$

The average ***power*** of a continuous time signal x(t) is defined as,

$$\boxed{\text{Power, } P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \text{ in watts}}$$

For periodic signals, the average power over one period will be same as average power over an infinite interval.

$$\boxed{\therefore \text{ For periodic signals, power, } P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt}$$

For energy signals, the energy will be finite (or constant) and average power will be zero. For power signals the average power is finite (or constant) and energy will be infinite.

i.e., For energy signal, E is constant (i.e., $0 < E < \infty$) and $P = 0$.

For power signal, P is constant (i.e., $0 < P < \infty$) and $E = \infty$.

Proof :

The energy of a signal x(t) is defined as ,

$$E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \quad .....(2.4)$$

The power of a signal is defined as ,

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \underset{T \to \infty}{Lt} \frac{1}{2T} \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \quad .....(2.5)$$

Using equation (2.4), the equation (2.5) can be written as,

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \times E \quad .....(2.6)$$

In equation (2.6), When E = constant,

$$P = E \times \underset{T \to \infty}{Lt} \frac{1}{2T}$$

$$= E \times \frac{1}{2 \times \infty} = E \times 0 = 0$$

From the above analysis, we can say that when a signal has finite energy the power will be zero. Also, from the above analysis we can say that the power is finite only when energy is infinite.

Example 2.4

Determine the power and energy for the following continuous time signals.

a) $x(t) = e^{-2t}\,u(t)$

b) $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$

c) $x(t) = 3\cos 5\Omega_0 t$

Solution

a) Given that, $x(t) = e^{-2t}\,u(t)$

Here, $x(t) = e^{-2t}\,u(t)$; for all t

$\therefore\ x(t) = e^{-2t}$; for $t \ge 0$

$$\therefore \int_{-T}^{T} |x(t)|^2\,dt = \int_{0}^{T} \left(\left|e^{-2t}\right|\right)^2 dt = \int_{0}^{T} \left(e^{-2t}\right)^2 dt = \int_{0}^{T} e^{-4t}\,dt = \left[\frac{e^{-4t}}{-4}\right]_0^T$$

$$= \left[\frac{e^{-4T}}{-4} - \frac{e^{0}}{-4}\right] = \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$\text{Energy, } E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{\mathrm{Lt}} \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$= \frac{1}{4} - \frac{e^{-\infty}}{4} = \frac{1}{4} - \frac{0}{4} = \frac{1}{4} \text{ joules}$$

$$\text{Power, } P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2T}\left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$= \frac{1}{\infty}\left[\frac{1}{4} - \frac{e^{-\infty}}{4}\right] = 0 \times \left[\frac{1}{4} - 0\right] = 0$$

Since energy is constant and power is zero, the given signal is an energy signal.

---

b) Given that, $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$

Here, $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)} = 1\angle\left(2t+\frac{\pi}{4}\right)$

$\therefore\ |x(t)| = 1$

$$\int_{-T}^{T} |x(t)|^2\,dt = \int_{-T}^{T} 1 \times dt = [t]_{-T}^{T} = T + T = 2T$$

$$\text{Energy, } E = \underset{T\to\infty}{\mathrm{Lt}} \int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{\mathrm{Lt}}\ 2T = \infty$$

$$\text{Power, } P = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{2T} \times 2T = 1 \text{ watt}$$

Since power is constant and energy is infinite, the given signal is a power signal.

---

c) Given that, $x(t) = 3\cos 5\Omega_0 t$

$$\therefore \int_{-T}^{T} |x(t)|^2\, dt = \int_{-T}^{T} \left(|3\cos 5\Omega_0 t|\right)^2 dt = \int_{-T}^{T} \left|(3\cos 5\Omega_0 t)^2\right| dt = \int_{-T}^{T} (3\cos 5\Omega_0 t)^2\, dt$$

$$= \int_{-T}^{T} 9\cos^2 5\Omega_0 t\, dt = 9\int_{-T}^{T} \left(\frac{1+\cos 10\Omega_0 t}{2}\right) dt \qquad \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \frac{9}{2}\int_{-T}^{T} (1+\cos 10\Omega_0 t)\, dt = \frac{9}{2}\left[t + \frac{\sin 10\Omega_0 t}{10\Omega_0}\right]_{-T}^{T}$$

$$= \frac{9}{2}\left[T + \frac{\sin 10\Omega_0 T}{10\Omega_0} - \left(-T + \frac{\sin 10\Omega_0 (-T)}{10\Omega_0}\right)\right]$$

$$= \frac{9}{2}\left[2T + 2\frac{\sin 10\Omega_0 T}{10\Omega_0}\right] = \frac{9}{2}\left[2T + 2\frac{\sin 10\frac{2\pi}{T}T}{10\frac{2\pi}{T}}\right] \qquad \boxed{\sin(-\theta) = -\sin\theta} \quad \boxed{\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}}$$

$$= \frac{9}{2}\left[2T + \frac{T}{10\pi}\sin 20\pi\right] = \frac{9}{2}\left[2T + \frac{T}{10\pi}\times 0\right] = 9T \qquad \boxed{\text{For integer M, } \sin\pi M = 0}$$

$$\text{Energy, } E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2\, dt = \underset{T\to\infty}{Lt}\, 9T = \infty$$

$$\text{Power, } P = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2\, dt = \underset{T\to\infty}{Lt} \frac{1}{2T}\times 9T = \underset{T\to\infty}{Lt} \frac{9}{2} = \frac{9}{2} = 4.5 \text{ watts}$$

Since energy is infinite and power is constant, the given signal is a power signal.

2.3.5 Causal, Noncausal and Anticausal Signals

A signal is said to be ***causal***, if it is defined for $t \ge 0$.

Therefore if $x(t)$ is causal, then $x(t) = 0$, for $t < 0$.

A signal is said to be ***noncausal***, if it is defined for either $t \le 0$, or for both $t \le 0$ and $t > 0$.

Therefore if $x(t)$ is noncausal, then $x(t) \ne 0$, for $t < 0$.

When a noncausal signal is defined only for $t \le 0$, it is called ***anticausal signal***.

Examples of causal and noncausal signals

Step signal,	$x(t) = A \quad ; t \ge 0$	Causal signals
Unit step signal,	$x(t) = u(t) = 1 \quad ; t \ge 0$	
Exponential signal,	$x(t) = A\, e^{bt}\, u(t)$	
Complex exponential signal,	$x(t) = A\, e^{j\Omega_0 t}\, u(t)$	
Exponential signal,	$x(t) = A\, e^{bt} \quad ; \text{for all } t$	Noncausal signals
Complex exponential signal,	$x(t) = A\, e^{j\Omega_0 t} \quad ; \text{for all } t$	

Note : *On multiplying a noncausal signal by u(t), it becomes causal.*

2.4 Mathematical Operations on Continuous Time Signals

2.4.1 Scaling of Continuous Time Signals

The two types of scaling continuous time signals are,

1. Amplitude Scaling
2. Time Scaling

1. Amplitude Scaling

The ***amplitude scaling*** is performed by multiplying the amplitude of the signal by a constant.

Let x(t) be a continuous time signal. Now Ax(t) is the amplitude scaled version of x(t), where A is a constant.

When $|A| > 1$, then Ax(t) is the amplitude magnified version of x(t) and when $|A| < 1$, then Ax(t) is the amplitude attenuated version of x(t).

Example : 1

Let, $x(t) = at + be^{-ct}$

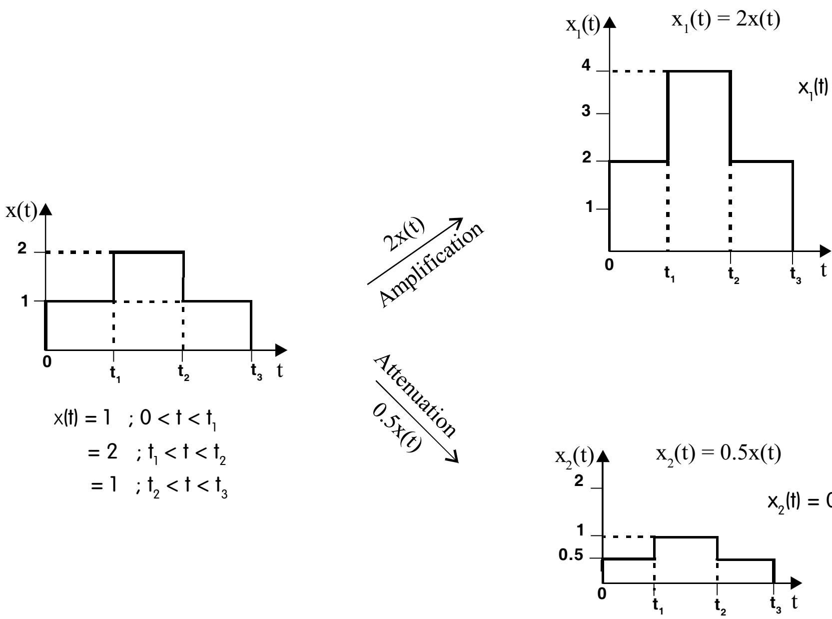
Let $x_1(t)$ and $x_2(t)$ be the amplitude scaled versions of x(t), scaled by constants 4 and $\frac{1}{4}\left(\frac{1}{4} = 0.25\right)$ respectively.

Now, $x_1(t) = 4x(t) = 4(at + be^{-ct}) = 4at + 4be^{-ct}$

$x_2(t) = 0.25x(t) = 0.25(at + be^{-ct}) = 0.25at + 0.25be^{-ct}$

Example : 2

A continuous time signal and its amplitude scaled version are shown in fig 2.20.



$x(t) = 1 \;;\; 0 < t < t_1$
$\quad\;\; = 2 \;;\; t_1 < t < t_2$
$\quad\;\; = 1 \;;\; t_2 < t < t_3$

$x_1(t) = 2x(t) = 2 \;;\; 0 < t < t_1$
$\quad\;\; = 4 \;;\; t_1 < t < t_2$
$\quad\;\; = 2 \;;\; t_2 < t < t_3$

$x_2(t) = 0.5x(t) = 0.5 \;;\; 0 < t < t_1$
$\quad\;\; = 1 \;;\; t_1 < t < t_2$
$\quad\;\; = 0.5 \;;\; t_2 < t < t_3$

Fig 2.20 : *A continuous time signal and its amplitude scaled version.*

2. *Time Scaling*

The ***time scaling*** is performed by multiplying the variable time by a constant.

If x(t) is a continuous time signal, then x(At) is the time scaled version of x(t), where A is a constant.

When |A| > 1, then x(At) is the time compressed version of x(t) and when |A| < 1, then x(At) is the time expanded version of x(t).

Example : 1

Let, $x(t) = at + be^{-ct}$

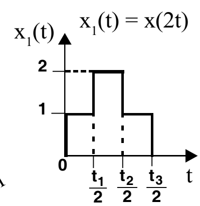
Let $x_1(t)$ and $x_2(t)$ be the time scaled versions of x(t), scaled by constants 4 and 1/4 (0.25)respectively.

Now, $x_1(t) = x(4t) = a \times 4t + be^{-c \times 4t} = 4at + be^{-4ct}$

$x_2(t) = x(0.25t) = a \times 0.25t + be^{-c \times 0.25t} = 0.25at + be^{-0.25ct}$

Example : 2

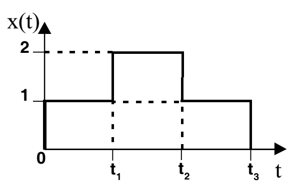
A continuous time signal and its time scaled version are shown in fig 2.21.



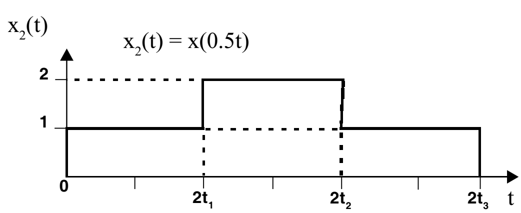
When $t = \frac{t_1}{2}$; $x_1\left(\frac{t_1}{2}\right) = x\left(2\frac{t_1}{2}\right) = x(t_1)$

When $t = \frac{t_2}{2}$; $x_1\left(\frac{t_2}{2}\right) = x\left(2\frac{t_2}{2}\right) = x(t_2)$

When $t = \frac{t_3}{2}$; $x_1\left(\frac{t_3}{2}\right) = x\left(2\frac{t_3}{2}\right) = x(t_3)$



$$x(t) = 1 \;;\; 0 < t < t_1$$
$$= 2 \;;\; t_1 < t < t_2$$
$$= 1 \;;\; t_2 < t < t_3$$



When $t = 2t_1$; $x_1(2t_1) = x(0.5 \times 2t_1) = x(t_1)$
When $t = 2t_2$; $x_1(2t_2) = x(0.5 \times 2t_2) = x(t_2)$
When $t = 2t_3$; $x_1(2t_3) = x(0.5 \times 2t_3) = x(t_3)$

Fig 2.21 : *A continuous time signal and its time scaled version.*

2.4.2 Folding (Reflection or Transpose) of Continuous Time Signals

The ***folding*** of a continuous time signal x(t) is performed by changing the sign of time base t in the signal x(t).

The folding operation produces a signal x(–t) which is a mirror image of the original signal x(t) with respect to the time origin t = 0.

Example : 1

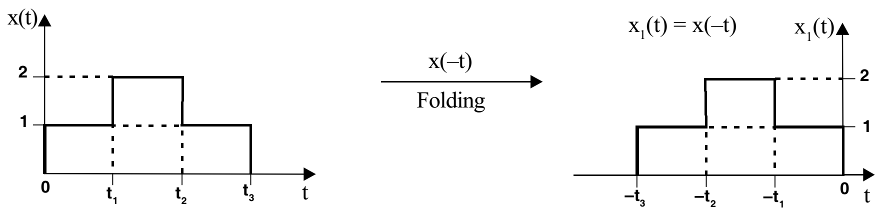
Let, $x(t) = at + be^{-ct}$

Let $x_1(t)$ be folded version of x(t).

Now, $x_1(t) = x(-t) = a(-t) + be^{-c(-t)} = -at + be^{ct}$

Example : 2

A continuous time signal and its folded version is shown in fig 2.22.



$x(t) = 1 \; ; \quad 0 < t < t_1$
$\quad\;\; = 2 \; ; \quad t_1 < t < t_2$
$\quad\;\; = 1 \; ; \quad t_2 < t < t_3$

When $t = -t_1$; $x_1(-t_1) = x(-(-t_1)) = x(t_1)$
When $t = -t_2$; $x_1(-t_2) = x(-(-t_2)) = x(t_2)$
When $t = -t_3$; $x_1(-t_3) = x(-(-t_3)) = x(t_3)$

***Fig 2.22 :** A continuous time signal and its folded version.*

2.4.3 Time Shifting of Continuous Time Signals

The ***time shifting*** of a continuous time signal x(t) is performed by replacing the independent variable t by t – m, to get the time shifted signal x(t – m), where m represents the time shift in seconds.

In x(t – m), if m is positive, then the time shift results in a delay by m seconds. The ***delay*** results in shifting the original signal x(t) to right, to generate the time shifted signal x(t – m).

In x(t – m), if m is negative, then the time shift results in an advance of the signal by |m| seconds. The ***advance*** results in shifting the original signal x(t) to left, to generate the time shifted signal x(t – m).

Example : 1

Let, $x(t) = at + be^{-ct}$

Let $x_1(t)$ and $x_2(t)$ be time shifted version of x(t), shifted by m units of time.

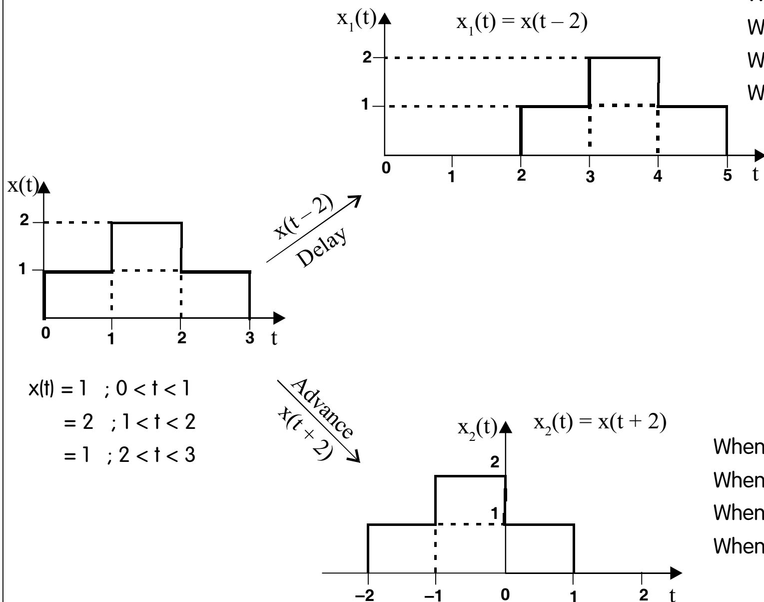
Let $x_1(t)$ be delayed version of x(t) and $x_2(t)$ be advanced version of x(t).

Now, $x_1(t) = a(t - m) + be^{-c(t - m)}$

$x_2(t) = a(t + m) + be^{-c(t + m)}$

Example : 2

A signal and its shifted version are shown in fig 2.23.



When t = 2; $x_1(2) = x(2-2) = x(0) = 1$
When t = 3; $x_1(3) = x(3-2) = x(1) = 1$
When t = 4; $x_1(4) = x(4-2) = x(2) = 2$
When t = 5; $x_1(5) = x(5-2) = x(3) = 1$

$$\begin{aligned} x(t) &= 1 \ ; 0 < t < 1 \\ &= 2 \ ; 1 < t < 2 \\ &= 1 \ ; 2 < t < 3 \end{aligned}$$

When t = –2 ; $x_1(-2) = x(-2+2) = x(0) = 1$
When t = –1 ; $x_1(-1) = x(-1+2) = x(1) = 1$
When t = 0 ; $x_1(0) = x(0+2) = x(2) = 2$
When t = 1 ; $x_1(1) = x(1+2) = x(3) = 1$

Fig 2.23 : A continuous time signal and its shifted version.

Delayed Unit Impulse Signal

The unit impulse signal is defined as,

$$\begin{aligned} \delta(t) &= \infty \ ; t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \\ &= 0 \ ; \ t \neq 0 \end{aligned}$$

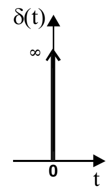


Fig 2.24a : Impulse.

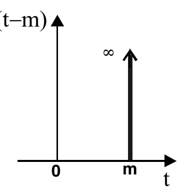


Fig 2.24b : Delayed impulse.

Fig 2.24 : Impulse and delayed impulse signal.

The unit impulse signal delayed by m units of time is denoted as $\delta(t-m)$, and it is defined as,

$$\begin{aligned} \delta(t-m) &= \infty ; t = m \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t-m)\, dt = 1 \\ &= 0 \ ; \ t \neq m \end{aligned}$$

Delayed Unit Step Signal

The unit step signal is defined as,

$$\begin{aligned} u(t) &= 1 \ ; \text{for } \ t \geq 0 \\ &= 0 \ ; \text{for } \ t < 0 \end{aligned}$$

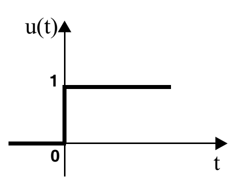


Fig 2.25a : Unit step signal.

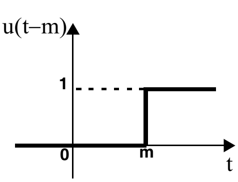


Fig 2.25b : Delayed unit step signal.

Fig 2.25 : Unit step and delayed unit step signal.

The unit step signal delayed by m units of time is denoted as $u(t-m)$, and it is defined as,

$$\begin{aligned} u(t-m) &= 1 \ ; t \geq m \\ &= 0 \ ; t < m \end{aligned}$$

2.4.4 Addition of Continuous Time Signals

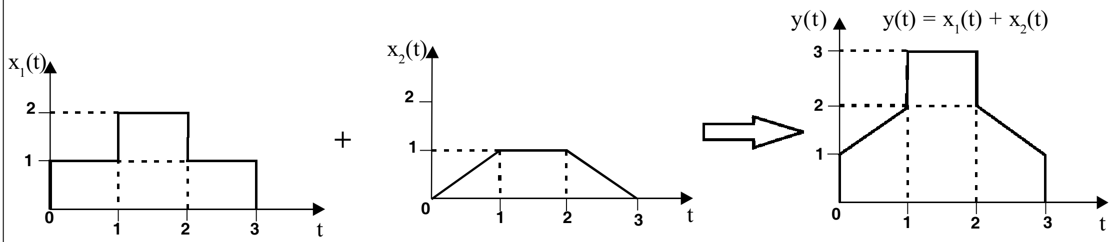
The ***addition*** of two continuous time signals is performed by adding the value of the two signals corresponding to the same instant of time.

The sum of two signals $x_1(t)$ and $x_2(t)$ is a signal $y(t)$, whose value at any instant is equal to the sum of the value of these two signals at that instant.

i.e., $y(t) = x_1(t) + x_2(t)$

Example :

Graphical addition of two continuous time signals is shown in fig 2.26.



$x_1(t) = 1 \quad ; 0 < t < 1$
$\quad = 2 \quad ; 1 < t < 2$
$\quad = 1 \quad ; 2 < t < 3$

$x_2(t) = t \quad ; 0 < t < 1$
$\quad = 1 \quad ; 1 < t < 2$
$\quad = 3 - t \; ; 2 < t < 3$

For t = 0 to 1 ; $y(t) = x_1(t) + x_2(t) = 1 + t$
For t = 1 to 2 ; $y(t) = x_1(t) + x_2(t) = 2 + 1 = 3$
For t = 2 to 3 ; $y(t) = x_1(t) + x_2(t) = 1 + 3 - t = 4 - t$

Fig 2.26 : *Addition of two continuous time signals.*

2.4.5 Multiplication of Continuous Time Signals

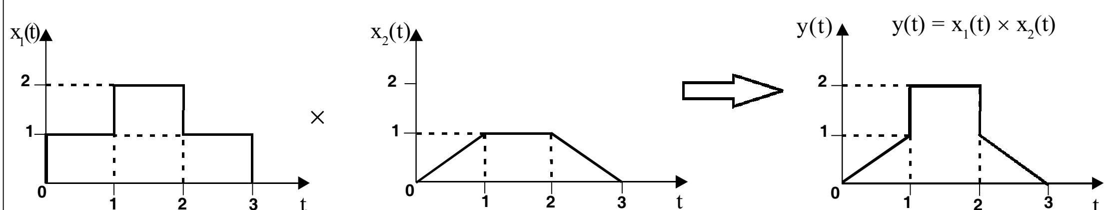
The ***multiplication*** of two continuous time signals is performed by multiplying the value of the two signals corresponding to the same instant of time.

The product of two signals $x_1(t)$ and $x_2(t)$ is a signal $y(t)$, whose value at any instant is equal to the product of the values of these two signals at that instant.

i.e., $y(t) = x_1(t) \times x_2(t)$

Example :

Graphical multiplication of two continuous time signals is shown in fig 2.27.



$x_1(t) = 1 ; 0 < t < 1$
$\quad = 2 ; 1 < t < 2$
$\quad = 1 ; 2 < t < 3$

$x_2(t) = t ; \; 0 < t < 1$
$\quad = 1 ; 1 < t < 2$
$\quad = 3 - t ; 2 < t < 3$

For t = 0 to 1; $y(t) = x_1(t) \times x_2(t) = 1 ´ t = t$
For t = 1 to 2; $y(t) = x_1(t) \times x_2(t) = 2 ´ 1 = 2$
For t = 2 to 3; $y(t) = x_1(t) \times x_2(t) = 1 ´ (3 - t) = 3 - t$

Fig 2.27 : *Multiplication of two continuous time signals.*

2.4.6 Differentiation and Integration of Continuous Time Signals

Differentiation is a mathematical operation used to estimate the rate of change of a continuous time signal at any instant of time.

Differentiation is denoted by the operator $\frac{d}{dt}$.

Therefore, the differentiation of a continuous time signal x(t) is denoted by $\frac{d}{dt}x(t)\left(\text{or } \frac{dx(t)}{dt}\right)$.

The differentiation of a continuous time signal x(t) is defined as,

$$\frac{dx(t)}{dt} = \underset{\Delta t \to 0}{Lt} \frac{x(t) - x(t - \Delta t)}{\Delta t}$$

Integration, is the inverse process of differentiation. More appropriately. Integration is the process of identifying the signal from its differentiation.

The integration is denoted by the operaor $\int .... dt$

Therefore, the integration of a continuous time signal x(t) is denoted by $\int x(t)\,dt$.

Differentiation and integration of standard continuous time signals are listed in table 2.1

Table 2.1 : Differentiation and Integration of Standard Continuous Time Signals

Signal, x(t)	**Differentiation of x(t),** $\frac{dx(t)}{dt}$	**Integration of x(t),** $\int x(t)\,dt$
$\delta(t)$	—	1
$u(t)$	$\delta(t)$	t
t	$u(t)$	$\frac{t^2}{2}$
t^2	$2t$	$\frac{t^3}{3}$
$\sin t$	$\cos t$	$-\cos t$
$\cos t$	$-\sin t$	$\sin t$
e^{-at}	$-ae^{-at}$	$\frac{e^{-at}}{-a}$
e^{at}	ae^{at}	$\frac{e^{at}}{a}$
$\sin\Omega_0 t$	$\Omega_0 \cos\Omega_0 t$	$\frac{-\cos\Omega_0 t}{\Omega_0}$
$\cos\Omega_0 t$	$-\Omega_0 \sin\Omega_0 t$	$\frac{\sin\Omega_0 t}{\Omega_0}$

The standard signals such as impulse, step, ramp and parabolic signals are related through integration and differentiation as shown below.

$$\underset{(\text{Impulse})}{\delta(t)} \xrightarrow{\text{integration}} \underset{(\text{Unit step})}{u(t)} \xrightarrow{\text{integration}} \underset{(\text{Unit ramp})}{t} \xrightarrow{\text{integration}} \underset{(\text{Unit parabolic})}{\frac{t^2}{2}}$$

$$\underset{(\text{Impulse})}{\delta(t)} \xleftarrow{\text{differentiation}} \underset{(\text{Unit step})}{u(t)} \xleftarrow{\text{differentiation}} \underset{(\text{Unit ramp})}{t} \xleftarrow{\text{differentiation}} \underset{(\text{Unit parabolic})}{\frac{t^2}{2}}$$

$$\boxed{\textbf{Note}: \left.\frac{d}{dt}u(t)\right|_{t=0} = \left.\frac{\Delta u}{\Delta t}\right|_{t=0} = \left.\frac{u(0^+) - u(0^-)}{\Delta t}\right|_{t=0} = \left.\frac{1-0}{0}\right|_{t=0} = \infty\Big|_{t=0} = Impulse}$$

Example 2.5

A continuous time signal is defined as,

$$x(t) = t \quad ; \; 0 \le t \le 3$$
$$= 0 \quad ; \; t > 3$$

Sketch the waveform of $x(-t)$ and $x(2-t)$.

Solution

The given signal is shown in fig 1.

The signal $x(-t)$ is the folded version of $x(t)$. The signal $x(-t)$ is shown in fig 2.

The signal $x(2-t) = x(-t+2)$ is the advanced version of the folded signal. The signal $x(-t+2)$ is shown in fig 3.

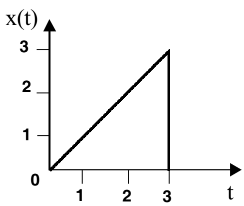


***Fig 1 :** x(t).*

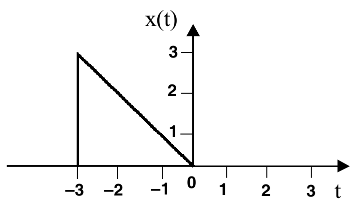


***Fig 2 :** x(–t).*

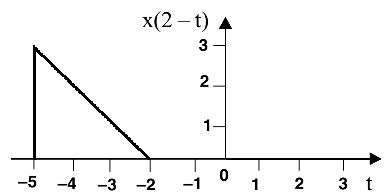
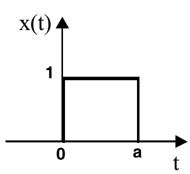


***Fig 3 :** x(2 – t).*

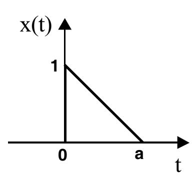
Example 2.6

Sketch the even and odd parts of the following signals.

a)



b)

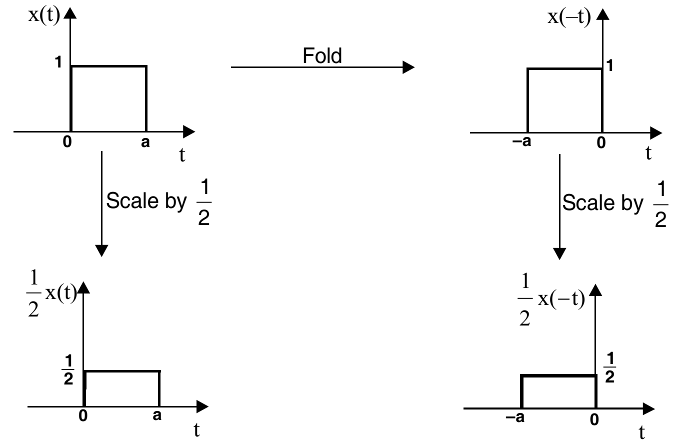


Solution

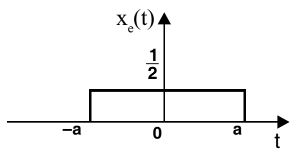
a) The even part of the signal is given by, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$ .....(1)

The odd part of the signal is given by, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$ .....(2)

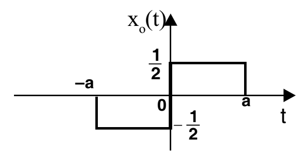
From equations (1) and (2), it is observed that the even and odd parts of the signal can be obtained from the folded and scaled versions of the signal. Hence the given signal is folded, scaled and then graphically added and subtracted to get the even and odd parts as shown below.



$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$



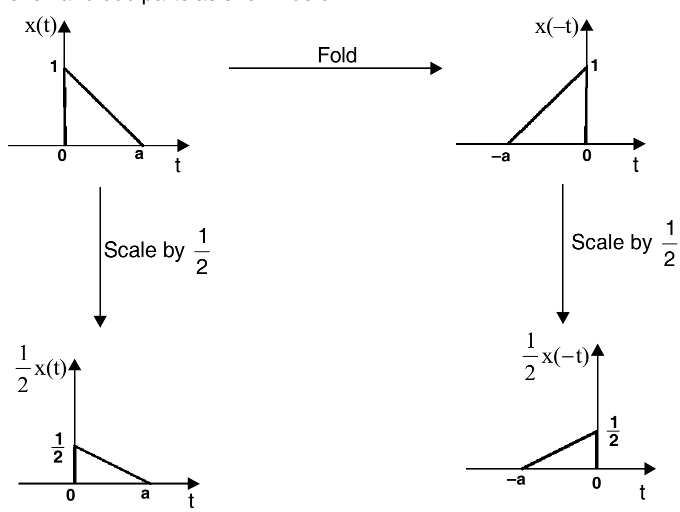
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$



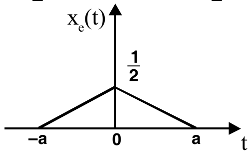
b) The even part of the signal is given by, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$ .....(1)

The odd part of the signal is given by, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$ .....(2)

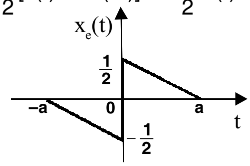
From equations (1) and (2), it is observed that the even and odd parts of the signal can be obtained from the folded and scaled versions of the signal. Hence the given signal is folded, scaled and then graphically added and subtracted to get the even and odd parts as shown below.



$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$



$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$



2.5 Impulse Signal

The impulse signal is a special signal which can be derived as follows.

Consider a pulse signal, $P_\Delta(t)$ with height A/Δ and width Δ as shown in fig 2.28. Now, the pulse signal, $P_\Delta(t)$ can be defined as,

$$P_\Delta(t) = \frac{A}{\Delta} \quad ; \; 0 \le t \le \Delta$$
$$= 0 \quad ; \; t > \Delta$$

The area of the pulse signal for any value of t is given by,

$$\text{Area} = \text{Height} \times \text{Width} = \frac{A}{\Delta} \times \Delta = A$$

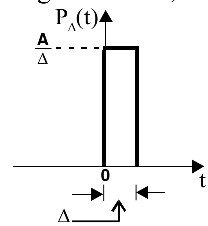


Fig 2.28 : *Pulse signal.*

In the signal, $P_\Delta(t)$ if the width Δ is reduced, then the height A/Δ increases, but the area of the pulse remains same as A. When the width Δ tends to zero, the height A/Δ tends to infinity. This limiting value of the pulse signal is called ***impulse signal***, δ(t). Even when the width Δ tends to zero, the area of the pulse remains as A.

$$\therefore \text{ Impulse Signal, } \delta(t) = \underset{\Delta\to 0}{\text{Lt}}\; P_\Delta(t) = \underset{\Delta\to 0}{\text{Lt}}\; \frac{A}{\Delta} \quad ; \; t = 0$$
$$= 0 \quad ; \; t \ne 0$$

$$\therefore \text{ Impulse Signal, } \delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0 \; ; \; t \ne 0$$

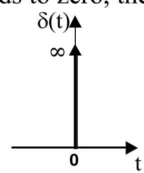


Fig 2.29 : *Impulse signal (or Unit impulse signal).*

In the above impulse signal if A=1, then the impulse signal is called a ***unit impulse signal***. The impulse signal or unit impulse signal can be represented graphically as shown in fig 2.29. An impulse with infinite magnitude and zero duration is a mathematical fiction and does not exist in reality. However a signal with large magnitude and short duration (when compared to time constant of a system) can be considered as an impulse signal. Practically, the magnitude of the impulse is measured by its area.

Definition of impulse signal : The impulse signal is a signal with infinite magnitude and zero duration, but with an area of A. Mathematically, an impulse signal is defined as,

$$\text{Impulse Signal, } \delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0 \; ; \; t \ne 0$$

Definition of unit impulse signal : The unit impulse signal is a signal with infinite magnitude and zero duration, but with unit area. Mathematically, a unit impulse signal is defined as,

$$\text{Unit Impulse Signal, } \delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$
$$= 0 \; ; \; t \ne 0$$

2.5.1 Properties of Impulse Signal

Property-1: $\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$

Proof :

Consider a narrow pulse signal, $P_\Delta(t)$ of width $\Delta\lambda$ and height $1/\Delta\lambda$ as shown in fig 2.30.

Now the pulse signal is defined as,

$$P_\Delta(t) = \frac{1}{\Delta\lambda} \quad ; \; 0 \le t \le \Delta\lambda$$

$$= 0 \quad ; \; t > \Delta\lambda$$

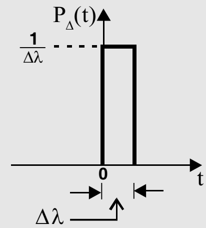


Fig 2.30.

Now the impulse signal can be represented as,

$$\delta(t) = \underset{\Delta\lambda \to 0}{\text{Lt}} P_\Delta(t)$$

On integrating the above equation we get,

$$\int_{-\infty}^{+\infty} \delta(t)\,dt = \int_{-\infty}^{+\infty} \underset{\Delta\lambda \to 0}{\text{Lt}} P_\Delta(t)\,dt = \underset{\Delta\lambda \to 0}{\text{Lt}} \int_{-\infty}^{+\infty} P_\Delta(t)\,dt = \underset{\Delta\lambda \to 0}{\text{Lt}} \int_{0}^{\Delta\lambda} \frac{1}{\Delta\lambda} dt = \underset{\Delta\lambda \to 0}{\text{Lt}} \frac{1}{\Delta\lambda} \int_{0}^{\Delta\lambda} dt$$

$$= \underset{\Delta\lambda \to 0}{\text{Lt}} \frac{1}{\Delta\lambda} \left[t\right]_0^{\Delta\lambda} = \underset{\Delta\lambda \to 0}{\text{Lt}} \frac{1}{\Delta\lambda} \left[\Delta\lambda - 0\right] = \underset{\Delta\lambda \to 0}{\text{Lt}} 1 = 1$$

Property-2: $\int_{-\infty}^{+\infty} x(t)\, \delta(t)\, dt = x(0)$

Proof :

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{+\infty} x(0)\,\delta(t)\,dt$$

Since δ(t) is nonzero only at t=0, x(t) is replaced by x(0).

$$= x(0) \int_{-\infty}^{+\infty} \delta(t)\,dt = x(0) \times 1 = x(0)$$

Since x(0) is constant, it is taken outside integration.

Using property-1

Property-3: $\int_{-\infty}^{+\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$

Proof :

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = \int_{-\infty}^{+\infty} x(t_0)\,\delta(t-t_0)\,dt$$

Since $\delta(t_0)$ is nonzero only at $t=t_0$, x(t) is replaced by $x(t_0)$.

$$= x(t_0) \int_{-\infty}^{+\infty} \delta(t-t_0)\,dt = x(t_0) \times 1 = x(t_0)$$

Since $x(t_0)$ is constant, it is taken outside integration.

Using property-1

Property-4: $\int_{-\infty}^{+\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$

Proof :

Consider the property-3 of impulse signal. $\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

On substituting $t = \lambda$ in the above equation we get, $\int_{-\infty}^{+\infty} x(\lambda)\,\delta(\lambda-t_0)\,d\lambda = x(t_0)$

On substituting $t_0 = t$ in the above equation we get, $\int_{-\infty}^{+\infty} x(\lambda)\, \delta(\lambda - t)\, d\lambda = x(t)$

Since impulse signal is even, $\delta(\lambda - t) = \delta(t - \lambda)$, Therefore the above equation is written as shown below.

$$\int_{-\infty}^{+\infty} x(\lambda)\, \delta(t - \lambda)\, d\lambda = x(t)$$

Property-5: $\delta(at) = \dfrac{1}{|a|}\delta(t)$

Proof :

Consider a narrow pulse signal, $P_\Delta(t)$ of width $\Delta\lambda$ and height $1/\Delta\lambda$ as shown in fig 2.31(a).

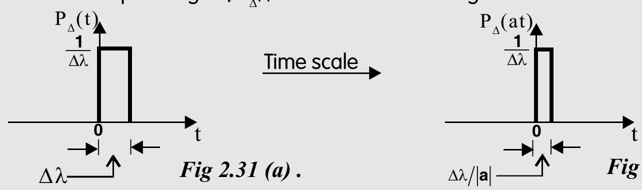


Fig 2.31 (a) . *Fig 2.31 (b).*

Fig 2.31 : A pulse signal and its time scaled version.

Now the signal, $P_\Delta(at)$ will be time scaled version of signal $P_\Delta(t)$ as shown in fig 2.31(b).Now the pulse signal and time scaled pulse signal can be mathematically defined as,

$$P_\Delta(t) = \frac{1}{\Delta\lambda} \quad ; \; 0 \le t \le \Delta\lambda \qquad\qquad P_\Delta(at) = \frac{1}{\Delta\lambda} \quad ; \; 0 \le t \le \frac{\Delta\lambda}{|a|}$$

$$= 0 \quad ; \; t > \Delta\lambda \qquad\qquad = 0 \quad ; \; t > \frac{\Delta\lambda}{|a|}$$

$$\therefore \int_{-\infty}^{+\infty} P_\Delta(t)\, dt = \int_{0}^{\Delta\lambda} \frac{1}{\Delta\lambda}\, dt \qquad ....(2.7)$$

$$\int_{-\infty}^{+\infty} P_\Delta(at)\, dt = \int_{0}^{\Delta\lambda/|a|} \frac{1}{\Delta\lambda}\, dt = \frac{1}{|a|}\int_{0}^{\Delta\lambda} \frac{1}{\Delta\lambda}\, dt = \frac{1}{|a|}\int_{-\infty}^{+\infty} P_\Delta(t)\, dt \qquad ....(2.8)$$

Using equation (2.7)

Now the impulse signal and time scaled impulse signal can be represented as,

$$\delta(t) = \underset{\Delta\lambda \to 0}{\text{Lt}}\; P_\Delta(t) \quad \text{and} \quad \delta(at) = \underset{\Delta\lambda \to 0}{\text{Lt}}\; P_\Delta(at)$$

On integrating the time scaled impulse signal we get,

$$\int_{-\infty}^{+\infty} \delta(at)\, dt = \int_{-\infty}^{+\infty} \underset{\Delta\lambda \to 0}{\text{Lt}}\; P_\Delta(at)\, dt = \underset{\Delta\lambda \to 0}{\text{Lt}} \int_{-\infty}^{+\infty} P_\Delta(at)\, dt = \underset{\Delta\lambda \to 0}{\text{Lt}}\; \frac{1}{|a|}\int_{-\infty}^{+\infty} P_\Delta(t)\, dt$$

Using equation (2.8)

$$= \frac{1}{|a|}\int_{-\infty}^{+\infty} \underset{\Delta\lambda \to 0}{\text{Lt}}\; P_\Delta(t)\, dt = \frac{1}{|a|}\int_{-\infty}^{+\infty} \delta(t)\, dt$$

Using definition of impulse signal

$$\therefore \int_{-\infty}^{+\infty} \delta(at)\, dt = \frac{1}{|a|}\int_{-\infty}^{+\infty} \delta(t)\, dt$$

On differentiating the above equation we get,

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

2.5.2 Representation of Continuous Time Signal as Integral of Impulses

Let x(t) be a continuous time signal as shown in fig 2.32.

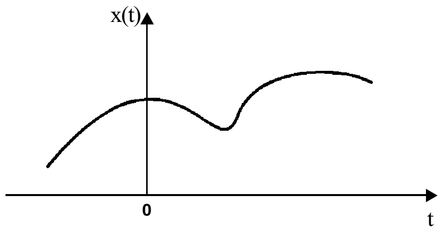


Fig 2.32.

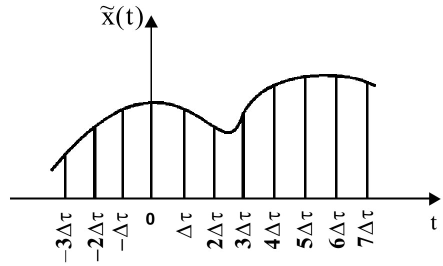


Fig 2.33.

Let us divide x(t) as narrow pulses of width Δτ as shown in fig 2.33.

Now the signal x(t) can be expressed as,

$$x(t) = \underset{\Delta\tau \to 0}{\text{Lt}} \; \tilde{x}(t)$$

Each narrow pulse of fig 2.33 can be interpretted as shown below,

$$\begin{aligned}
&\vdots \\
x(-2\Delta\tau) &= \tilde{x}(t) \;;\quad \text{for} \quad -2\Delta\tau < t < -\Delta\tau \\
x(-\Delta\tau) &= \tilde{x}(t) \;;\quad \text{for} \quad -\Delta\tau < t < 0 \\
x(0) &= \tilde{x}(t) \;;\quad \text{for} \quad 0 < t < \Delta\tau \\
x(\Delta\tau) &= \tilde{x}(t) \;;\quad \text{for} \quad \Delta\tau < t < 2\Delta\tau \\
x(2\Delta\tau) &= \tilde{x}(t) \;;\quad \text{for} \quad 2\Delta\tau < t < 3\Delta\tau \\
&\vdots
\end{aligned}$$

$$\therefore\; x(t) = \underset{\Delta\tau \to 0}{\text{Lt}} \; \tilde{x}(t)$$

$$= \underset{\Delta\tau \to 0}{\text{Lt}} \; [\ldots\ldots x(-2\Delta\tau) + x(-\Delta\tau) + x(0) + x(\Delta\tau) + x(2\Delta\tau) + \ldots\ldots] \qquad \ldots\ldots(2.9)$$

Consider the pulse signal of width Δτ and height 1/ Δτ as shown in fig 2.34. This pulse signal can be expressed as,

$$P_\Delta(t) = \frac{1}{\Delta\tau} \;;\quad 0 \le t \le \Delta\tau$$
$$= 0 \;;\quad \text{otherwise}$$

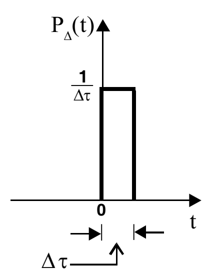


Fig 2.34.

Now, $P_\Delta(t) \times \Delta\tau$ = A pulse of unit amplitude.

∴ On multiplying $P_\Delta(t) \times \Delta\tau$ with the signal $\tilde{x}(t)$, the signal $\tilde{x}(0)$ is selected.

$$\therefore\; \tilde{x}(0) = \tilde{x}(t)\, P_\Delta(t)\, \Delta\tau$$

Consider the shifted version of the pulse signal of fig 2.34, as shown in fig 2.35.

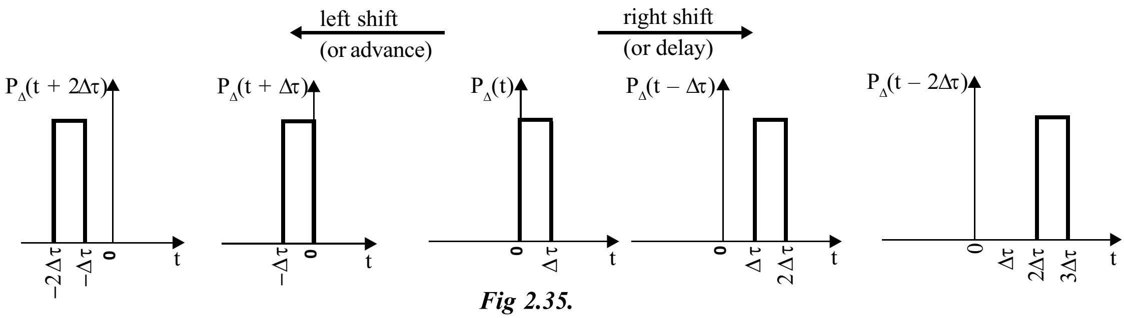


Fig 2.35.

If we multiply $\tilde{x}(t)$ with shifted pulse signals shown in fig 2.35, then each product will select one pulse of the signal $\tilde{x}(t)$ as shown below.

$$\begin{aligned}
&\vdots\\
x(-2\Delta\tau) &= \tilde{x}(t)\ P_\Delta(t+2\Delta\tau)\ \Delta\tau\\
x(-\Delta\tau) &= \tilde{x}(t)\ P_\Delta(t+\Delta\tau)\ \Delta\tau\\
x(0) &= \tilde{x}(t)\ P_\Delta(t)\ \Delta\tau\\
x(\Delta\tau) &= \tilde{x}(t)\ P_\Delta(t-\Delta\tau)\ \Delta\tau\\
x(2\Delta\tau) &= \tilde{x}(t)\ P_\Delta(t-2\Delta\tau)\ \Delta\tau\\
&\vdots
\end{aligned}$$

In the above equation $\tilde{x}(t)$ can be replaced by respective selected pulses itself as shown below,

$$\begin{aligned}
&\vdots\\
\therefore\ x(-2\Delta\tau) &= \tilde{x}(-2\Delta\tau)\ P_\Delta(t+2\Delta\tau)\ \Delta\tau\\
x(-\Delta\tau) &= \tilde{x}(-\Delta\tau)\ P_\Delta(t+\Delta\tau)\ \Delta\tau\\
x(0) &= \tilde{x}(0)\ P_\Delta(t)\ \Delta\tau\\
x(\Delta\tau) &= \tilde{x}(\Delta\tau)\ P_\Delta(t-\Delta\tau)\ \Delta\tau\\
x(2\Delta\tau) &= \tilde{x}(2\Delta\tau)\ P_\Delta(t-2\Delta\tau)\ \Delta\tau\\
&\vdots
\end{aligned}$$

On substituting the above equations in equation (2.9) we get,

$$\tilde{x}(t) = \underset{\Delta\tau\to 0}{\mathrm{Lt}} \left[\begin{array}{l} \ldots\ldots\ \tilde{x}(-2\Delta\tau)\ P_\Delta(t+2\Delta\tau)\,\Delta\tau + \tilde{x}(-\Delta\tau)\ P_\Delta(t+\Delta\tau)\,\Delta\tau + \tilde{x}(0)\ P_\Delta(t)\,\Delta\tau \\ \quad + \tilde{x}(\Delta\tau)\ P_\Delta(t-\Delta\tau)\,\Delta\tau + \tilde{x}(2\Delta\tau)\ P_\Delta(t-2\Delta\tau)\,\Delta\tau + \ldots\ldots\ldots\ldots \end{array}\right]$$

$$= \underset{\Delta\tau\to 0}{\mathrm{Lt}} \sum_{n=-\infty}^{+\infty} \tilde{x}(n\Delta\tau)\ P_\Delta(t-n\Delta\tau)\ \Delta\tau$$

On applying limit $\Delta\tau \to 0$ the signal $\tilde{x}(n\Delta\tau)$ becomes continuous, the signal $P_\Delta(t-n\Delta\tau)$ becomes an impulse and so the summation becomes integration.

Hence the above equation can be expressed as,

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\,\delta(t-\tau)\,d\tau \qquad \ldots(2.10)$$

The equation (2.10) is used to represent any continuous time signal x(t) as an integral of impulses.

2.6 Continuous Time System

A ***continuous time system*** (or Analog system) is a physical device that operates on a continuous time signal (or an analog signal) called input or excitation, according to some well defined rule, to produce another continuous time signal (or an analog signal) called output or response. We can say that the input signal x(t) is transformed by the system into a signal y(t), and the transformation can be expressed mathematically as shown in equation (2.11).The diagrammatic representation of continuous time system is shown in fig 2.36.

Response, $y(t) = \mathcal{H}\{x(t)\}$ .....(2.11)

where, $\mathcal{H}$ denotes the transformation (also called an operator).

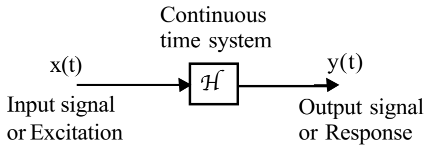


Fig 2.36 : *Representation of continuous time system.*

LTI System

A continuous time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship does not change with time. When a continuous time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a continuous time system is a unit impulse signal δ(t) then the output is called an ***impulse response*** of the system and it is denoted by h(t).

∴ Impulse Response, $h(t) = \mathcal{H}\{\delta(t)\}$ .....(2.12)

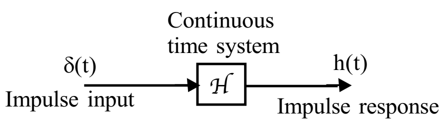


Fig 2.37 : *Continuous time system with impulse input.*

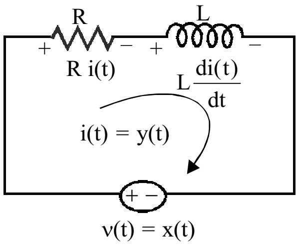
2.6.1 Mathematical Equation Governing LTI Continuous Time System

The electric heaters, motors, generators, etc., are examples of electrical continuous time systems. The continuous time systems that operate on electrical energy can be modelled by three basic elements Resistor(R), Inductor(L) and Capacitor(C). The models constructed using these fundamental elements are called ***electric circuits.***

In electric circuits the inputs and outputs are either voltage signals or current signals. The continuous time voltage signal is denoted by $v(t)$ and current signal by $i(t)$.

The basic RL, RC, and RLC circuits and their time domain KVL (Kirchoff's Voltage Law) equations are shown in fig 2.38, fig 2.39 and fig 2.40 respectively. From these circuits it can be observed that the equations governing the continuous time systems are differential equations.

Also, it can be shown that all continuous time systems like Mechanical systems, Thermal systems, Hydraulic systems, etc., are all governed by differential equations.



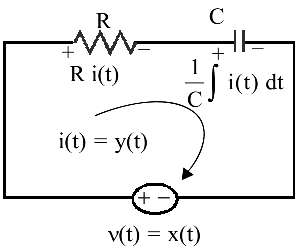
$$R\,i(t) + L\frac{di(t)}{dt} = \nu(t)$$

$\downarrow$ Replace i(t) by y(t) and ν(t) by x(t)

$$Ry(t) + L\frac{dy(t)}{dt} = x(t)$$

$$\therefore \frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{1}{L}x(t)$$

***Fig 2.38 :** RL circuit and the mathematical equation governing RL circuit.*



$$R\,i(t) + \frac{1}{C}\int i(t)\,dt = \nu(t)$$

$\downarrow$ differentiate

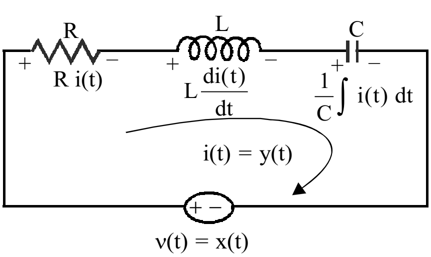
$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{d\nu(t)}{dt}$$

$\downarrow$ Replace i(t) by y(t) and ν(t) by x(t)

$$R\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

$$\therefore \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx(t)}{dt}$$

***Fig 2.39 :** RC circuit and the mathematical equation governing RC circuit.*



$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = \nu(t)$$

$\downarrow$ differentiate

$$R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}i(t) = \frac{d\nu(t)}{dt}$$

$\downarrow$ Replace i(t) by y(t) and ν(t) by x(t)

$$R\frac{dy(t)}{dt} + L\frac{d^2y(t)}{dt^2} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

$$\therefore \frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{L}\frac{dx(t)}{dt}$$

***Fig 2.40 :** RLC circuit and the mathematical equation governing RLC circuit.*

In general, the input-output relation of an LTI (Linear Time Invariant) continuous time system is represented by a constant coefficient differential equation shown below (equation (2.18)).

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + \,.......\, + a_{N-1} \frac{d}{dt} y(t) + a_N\, y(t) = b_0 \frac{d^M}{dt^M} x(t)$$

$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ...+ b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \qquad .....(2.13)$$

where, N = Order of the system, $M \le N$, and $a_0 = 1$.

The solution of the above differential equation is the response y(t) of the system, for the input x(t).

> ***Note :*** *A system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship do not change with time.*

2.6.2 Block Diagram and Signal Flow Graph Representation of LTI Continuous Time System

Block Diagram

A ***block diagram*** of a system is a pictorial representation of the functions performed by the system. The block diagram of a system is constructed using the mathematical equation governing the system.

The basic elements of a block diagram are Differentiator, Integrator, Constant Multiplier and Signal Adder. The symbols used for the basic elements and their input-ouput relation are listed in table 2.2.

Table 2.2 : Basic Elements of Block Diagram and Signal Flow Graph

Description	**Elements of block diagram**	**Elements of signal flow graph**
Differentiator	$x(t) \longrightarrow \boxed{\frac{d}{dt}} \longrightarrow \frac{d}{dt}x(t)$	$x(t) \circ \xrightarrow{\frac{d}{dt}} \circ\ \frac{d}{dt}x(t)$
Integrator (with zero initial condition)	$x(t) \longrightarrow \boxed{\int} \longrightarrow \int x(t)\,dt$	$x(t) \circ \xrightarrow{\int} \circ\ \int x(t)\,dt$
Constant Multiplier	$x(t) \longrightarrow \triangleright_{a} \longrightarrow a\,x(t)$	$x(t) \circ \xrightarrow{a} \circ\ a\,x(t)$
Signal Adder	$x_1(t)$, $x_2(t) \longrightarrow \oplus \longrightarrow x_1(t) + x_2(t)$	$x_1(t) \circ \xrightarrow{1}$, $x_2(t) \circ \xrightarrow{1}$ $\circ\ x_1(t) + x_2(t)$

Signal Flow Graph

A ***signal flow graph*** of a system is a graphical representation of the functions performed by the system. The signal flow graph shows the flow of signals from one point of a system to another and gives the relationship among the signals. The signal flow graph of a system is constructed using the mathematical equation governing the system.

The basic elements of a signal flow graph are nodes and directed branches. Each node represents a signal. The signal at a node is given by the sum of all incoming signals. Each branch has an input node and an output node. The direction of signal flow is marked by an arrow on the branch and the operation performed by the signal is indicated by an operator like integrator/differentiator. When the signal passes from the input node to the output node, it is operated by the operation specified by the branch. The basic operations performed by the branches of a signal flow graph are listed in table 2.2.

Example 2.7

Construct the block diagram and signal flow graph of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + 3\,y(t) = 4\,\frac{dx(t)}{dt} + 5\,x(t)$$

Solution

***Case i :* Block diagram and signal flow graph using differentiators**

Given that, $$\frac{d^2y(t)}{dt^2} + 2\,\frac{dy(t)}{dt} + 3\,y(t) = 4\,\frac{dx(t)}{dt} + 5\,x(t)$$

$$\therefore\ y(t) = -\frac{1}{3}\frac{d^2y(t)}{dt^2} - \frac{2}{3}\frac{dy(t)}{dt} + \frac{4}{3}\frac{dx(t)}{dt} + \frac{5}{3}x(t) \qquad .....(1)$$

The equation (1) is used to construct the block diagram and signal flow graph using differentiators as shown in fig 1 and fig 2 respectively.

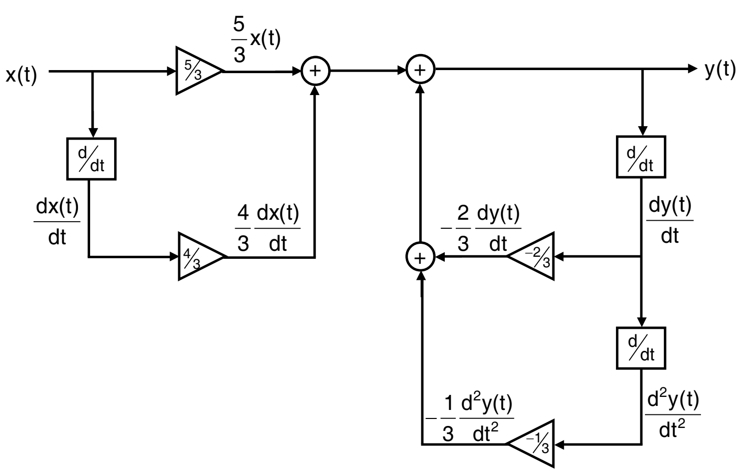


Fig 1 : *Block diagram using differentiators.*

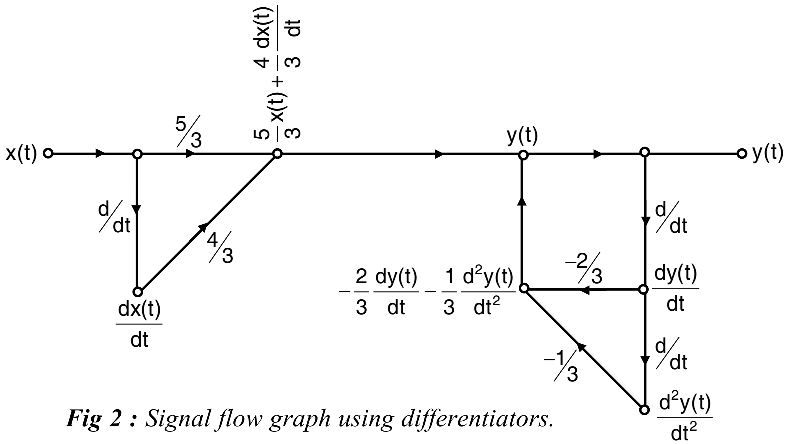


Fig 2 : Signal flow graph using differentiators.

Case ii : Block diagram and signal flow graph using integrators

Given that, $$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 3\,y(t) = 4\frac{dx(t)}{dt} + 5\,x(t)$$

Integrate with zero initial conditions

$$\frac{dy(t)}{dt} + 2\,y(t) + 3\int y(t)\,dt = 4\,x(t) + 5\int x(t)\,dt$$

Integrate with zero initial conditions

$$y(t) + 2\int y(t)\,dt + 3\iint y(t)\,dt\,dt = 4\int x(t)\,dt + 5\iint x(t)\,dt\,dt$$

$$\therefore\ y(t) = -2\int y(t)\,dt - 3\iint y(t)\,dt\,dt + 4\int x(t)\,dt + 5\iint x(t)\,dt\,dt \qquad .....(2)$$

The equation (2) is used to construct the block diagram and signal flow graph using integrators as shown in fig 3 and fig 4 respectively.

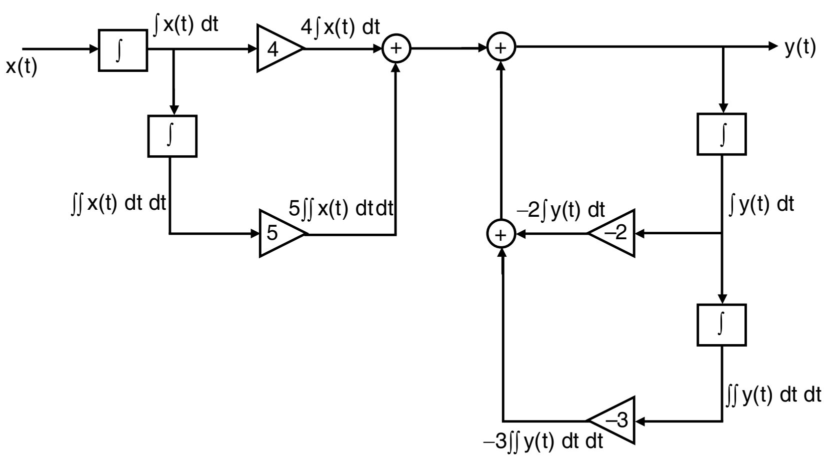


Fig 3 : Block diagram using integrators.

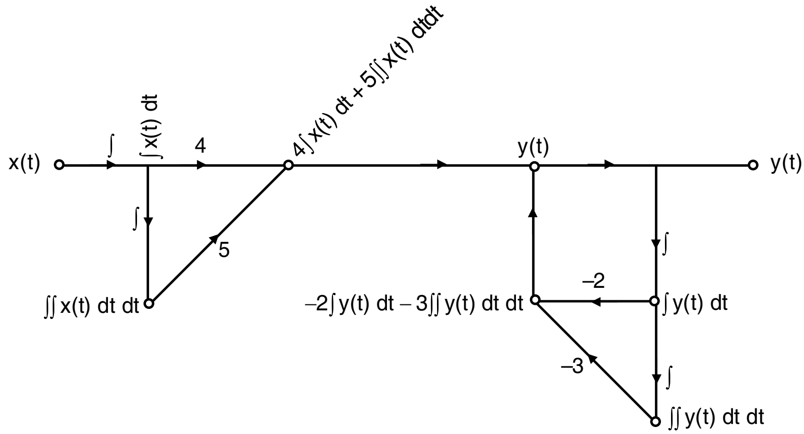


***Fig 4 :** Signal flow graph using integrators.*

2.7 Response of LTI Continuous Time System in Time Domain

The general equation governing an LTI continuous time system is,

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + ...+ a_{N-1} \frac{d}{dt} y(t) + a_N y(t) = b_0 \frac{d^M}{dt^M} x(t)$$

$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ...+ b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \qquad .....(2.14)$$

where, N = Order of the system, $M \le N$, and $a_0 = 1$.

The solution of the differential equation (2.14) is the ***response*** y(t) of the LTI system, which consists of two parts. In mathematics, the two parts of the solution y(t) are the homogeneous solution $y_h(t)$ and the particular solution $y_p(t)$.

$$\boxed{\therefore \text{ Response, } y(t) = y_h(t) + y_p(t)} \qquad .....(2.15)$$

The ***homogeneous solutio*n** is the response of the system when there is no input.The ***particular solution*** $y_p(t)$ is the solution of difference equation for specific input signal x(t) for $t \ge 0$.

In signals and systems the two parts of the solution y(t) are called zero-input response $y_{zi}(t)$ and zero-state response $y_{zs}(t)$.

$$\boxed{\therefore \text{ Response, } y(t) = y_{zi}(t) + y_{zs}(t)} \qquad .....(2.16)$$

The ***zero input response*** is given by the homogeneous solution with constants evaluated using initial values of output (or initial conditions). The zero-input response is mainly due to initial output conditions (or initial stored energy) in the system. Hence the zero-input response is also called ***free response or natural response***.

The ***zero-state response*** is given by the sum of homogeneous solution and particular solution with zero initial conditions.The zero-state response is the response of the system due to input signal and with zero initial output condition. Hence the zero-state response is also called ***forced response***.

2.7.1 Homogeneous Solution

The ***homogeneous solution*** is obtained when x(t) = 0. On substituting x(t) = 0 in the system equation (2.14) we get,

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + ... + a_{N-1}\frac{d}{dt}y(t) + a_N\, y(t) = 0 \qquad .....(2.17)$$

Now, the solution of equation (2.17) is the homogeneous solution.

Let us assume that the solution of equation (2.17) is in the form of an exponential.

i.e., $y(t) = Ce^{\lambda t}$

$$\therefore \frac{d}{dt}y(t) = C\lambda\, e^{\lambda t}$$

$$\frac{d^2}{dt^2}y(t) = C\lambda^2\, e^{\lambda t}$$

$$\vdots$$

$$\frac{d^N}{dt^N}y(t) = C\lambda^N\, e^{\lambda t}$$

On substituting the above assumed solution in equation (2.17) we get,

$$C\lambda^N e^{\lambda t} + a_1 C\lambda^{N-1} e^{\lambda t} + a_2 C\lambda^{N-2} e^{\lambda t} + ... + a_{N-1}C\lambda e^{\lambda t} + a_N\, Ce^{\lambda t} = 0$$

$$\therefore \left(\lambda^N + a_1\, \lambda^{N-1} + a_2\, \lambda^{N-2} + ... + a_{N-1}\lambda + a_N\right) Ce^{\lambda t} = 0$$

Now the ***characteristic polynomial*** of the system is given by,

$$\lambda^N + a_1\, \lambda^{N-1} + a_2\, \lambda^{N-2} + ... + a_{N-1}\lambda + a_N = 0$$

The characteristic polynomial has N roots, which are denoted as λ_1, λ_2,...λ_N.

The roots of the characteristic polynomial may be distinct real roots, repeated real roots or complex roots. The assumed solutions for various types of roots are given below.

Distinct Real Roots

Let the roots λ_1, λ_2, λ_3, ... λ_N be distinct real roots. Now the homogeneous solution will be in the form,

$$y_h(t) = C_1\, e^{\lambda_1 t} + C_2\, e^{\lambda_2 t} + ... + C_{N-1}\, e^{\lambda_{N-1} t} + C_N\, e^{\lambda_N t}$$

where, C_1, C_2, ...C_N are constants that can be evaluated using initial conditions.

Repeated Real Roots

Let one of the real roots λ_1 repeats p times and the remaining (N – p) roots are distinct real roots.

Now the homogeneous solution is in the form,

$$y_h(t) = (C_1 + C_2 t + C_3 t^2 + ... + C_p t^{p-1}) e^{\lambda_1 t} + C_{p+1} e^{\lambda_{p+1} t} + ... + C_N e^{\lambda_N t}$$

where, $C_1, C_2, C_3, ... C_N$ are constants that can be evaluated using initial conditions.

Complex Roots

Let the characteristic polynomial have a pair of complex roots λ and λ^* and the remaining (N – 2) roots be distinct real roots. Let, $\lambda = \alpha + j\beta$ and $\lambda^* = \alpha - j\beta$. Now, the homogeneous solution will be in the form,

$$y_h(t) = e^{\alpha t}(C_1 \cos\beta t + jC_2 \sin\beta t) + C_3 e^{\lambda_3 t} + ... + C_{N-1} e^{\lambda_{N-1} t} + C_N e^{\lambda_N t}$$

where, C_1, C_2, C_3 ... C_N are constants that can be evaluated using initial conditions.

2.7.2 Particular Solution

The ***particular solution***, $y_p(t)$ is the solution of N^{th} order differential equation governing the system for specific input signal x(t) for $t \geq 0$. Since the input signal may have a different form, the particular solution depends on the form or type of the input signal x(t).

If x(t) is constant, then $y_p(t)$ is also a constant.

> **Example :**
>
> Let $x(t) = u(t)$, now $y_p(t) = K\,u(t)$

If x(t) is exponential, then $y_p(t)$ is also an exponential.

> **Example :**
>
> Let $x(t) = e^{at}\,u(t)$, now $y_p(t) = K\,e^{at}\,u(t)$

If x(t) is sinusoid, then $y_p(t)$ is also a sinusoid.

> **Example :**
>
> Let $x(t) = A\cos\Omega_0 t$, now $y_p(t) = K_1 \cos\Omega_0 t + K_2 \sin\Omega_0 t$

The general form of a particular solution for various types of inputs are listed in table 2.3.

Table 2.3 : Particular Solution

Input signal, x(t)	Particular solution, $y_p(t)$
$x(t) = A$	$y_p(t) = K$
$x(t) = Au(t)$	$y_p(t) = K\,u(t)$
$x(t) = A\,e^{\alpha t}$ (if $\alpha \neq \lambda_i$)	$y_p(t) = K\,e^{\alpha t}$
$x(t) = A\,e^{\alpha t}$ (if $\alpha = \lambda_i$)	$y_p(t) = K\,t\,e^{\alpha t}$
$x(t) = A\cos\Omega_0 t$ $x(t) = A\sin\Omega_0 t$	$y_p(t) = K_1 \cos\Omega_0 t + K_2 \sin\Omega_0 t$

Note : *λ_i is one of the root of characteristic polynomial.*

2.7.3 Zero-Input and Zero-State Response

The ***zero-input response*** $y_{zi}(t)$ (or ***free response*** or ***natural response***) is obtained from the homogeneous solution $y_h(t)$ with constants evaluated using initial output (or initial conditions).

$$\therefore \textbf{ Zero-input response, } y_{zi}(t) = y_h(t)\Big|_{\text{with constants evaluated using initial output conditions}}$$

The zero-state response $y_{zs}(t)$ (or forced response) is obtained from the sum of homogeneous solution and particular solution and evaluating the constants with zero initial conditions.

$$\therefore \textbf{ Zero-state response, } y_{zs}(t) = \left[y_h(t) + y_p(t)\right]\Big|_{\text{with constants } C_1,\, C_2, \ldots C_N \text{ evaluated with zero initial output conditions}}$$

The sum of zero-input response and zero-state response will be the total response or complete response of a system.

2.7.4 Total Response

The total response y(t) of a continuous time system can be obtained by the following two methods.

Method -1

The ***total response*** y(t) is given by the sum of homogeneous solution and particular solution.

$\therefore$ Total response, $y(t) = y_h(t) + y_p(t)$

Procedure to Determine Total Response

1. Determine homogeneous solution $y_h(t)$ with constants C_1, C_2, ...C_N.
2. Determine particular solution $y_p(t)$ and estimate the constant K by evaluating the given system equation using $y_p(t)$ for any value of $t \geq 1$ so that no term of the system equation vanishes.
3. Now the total response y(t) is given by, the sum of $y_h(t)$ and $y_p(t)$

   $\therefore$ Total response, $y(t) = y_h(t) + y_p(t)$
4. The total response y(t) will have N number of constants C_1, C_2, ...C_N. Evaluate the constants C_1, C_2, ...C_N using initial outputs (or initial conditions).

Method -2

The ***total response*** y(t) is given by the sum of zero-input response and zero-state response.

$\therefore$ Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

Procedure to Determine Total Response

1. Determine the homogeneous solution $y_h(t)$ with constants C_1, C_2, ...C_N.
2. Determine the zero-input response $y_{zi}(t)$ which is obtained from the homogeneous solution $y_h(t)$ by evaluating the constants C_1, C_2, ...C_N using initial outputs (or initial conditions).
3. Determine particular solution $y_p(t)$ and estimate the constant K by evaluating the given system equation using $y_p(t)$ for any value of $t \geq 1$ so that no term of the system equation vanishes.
4. Determine the zero-state response, $y_{zs}(t)$ which is given by sum of homogeneous solution and particular solution and evaluating the constants C_1, C_2, ...C_N with zero initial conditions.
5. Now the total response y(t) is given by the sum of zero-input response and zero-state response.

   $\therefore$ Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

Example 2.8

Determine the natural response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t)\,; \quad y(0) = 1; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = -2$$

Solution

The natural response is response of the system due to initial conditions and so it is given by zero-input response.

$$\text{Zero - input response}\,, \;\; y_{zi}(t) = y_h(t)\,\big|_{\text{with constants evaluated using initial conditions}}$$

$$\text{where, } y_h(t) = \text{Homogeneous solution}$$

The given system equation is,

$$\mathbf{\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = \frac{dx(t)}{dt} + 4x(t)} \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 5y(t) = 0 \qquad .....(2)$$

$$\text{Let, } y(t) = C\,e^{\lambda t}\,; \quad \therefore \frac{d}{dt}y(t) = C\lambda\,e^{\lambda t}\,; \quad \frac{d^2}{dt^2}y(t) = C\lambda^2\,e^{\lambda t}$$

On substituting the above terms in equation (2) we get,

$$C\lambda^2 e^{\lambda t} + 6\,C\lambda\,e^{\lambda t} + 5\,Ce^{\lambda t} = 0$$

$$\therefore \left(\lambda^2 + 6\lambda + 5\right)Ce^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$\lambda^2 + 6\lambda + 5 = 0 \quad \Rightarrow \quad (\lambda + 1)\,(\lambda + 5) = 0 \quad \Rightarrow \quad \lambda = -1, -5$$

Now the homogeneous solution is given by,

$$\text{Homogeneous solution, } y_h(t) = C_1\,e^{\lambda_1 t} + C_2\,e^{\lambda_2 t} = C_1\,e^{-t} + C_2\,e^{-5t}$$

Natural Response (or Zero-input Response)

$$\text{Zero - input response}\,, \;\; y_{zi}(t) = y_h(t)\,\big|_{\text{with constants evaluated using initial conditions}}$$

$$= C_1\,e^{-t} + C_2\,e^{-5t}\Big|_{\text{with } C_1 \text{ and } C_2 \text{ evaluated using initial conditions}}$$

$$\therefore \frac{dy_{zi}(t)}{dt} = -C_1\,e^{-t} - 5C_2\,e^{-5t}$$

$$\text{At } t = 0, \;\; y_{zi}(0) = C_1\,e^0 + C_2\,e^0 = C_1 + C_2$$

$$\text{Given that, } y_{zi}(0) = 1, \quad \therefore C_1 + C_2 = 1 \qquad .....(3)$$

$$\text{At } t = 0, \;\; \frac{dy_{zi}(t)}{dt} = -C_1\,e^0 - 5C_2\,e^0 = -C_1 - 5C_2$$

$$\text{Given that, } \left.\frac{dy_{zi}(t)}{dt}\right|_{t=0} = -2, \quad \therefore -C_1 - 5C_2 = -2 \qquad .....(4)$$

On adding equations (3) and (4) we get,

$$-4C_2 = -1 \Rightarrow C_2 = \frac{1}{4}$$

From equation (3), $C_1 = 1 - C_2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Natural response, $y_{zi}(t) = \frac{3}{4}e^{-t} + \frac{1}{4}e^{-5t}\ ;\ t \geq 0 = \frac{1}{4}\left(3e^{-t} + e^{-5t}\right)u(t)$

Example 2.9

Determine the forced response of the system described by the equation,

$$5\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t), \quad \text{for the input, } x(t) = 2\,u(t).$$

Solution

The forced response is the response of the system due to input signal with zero initial conditions and so it is given by zero-state response.

$$\text{Zero state response}, \quad y_{zs}(t) = y_h(t) + y_p(t)\Big|_{\text{with constants evaluated with zero initial conditions}}$$

where, $y_h(t)$ = Homogeneous solution and $y_p(t)$ = Particular solution

The given system equation is,

$$\mathbf{5\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t)} \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$5\frac{dy(t)}{dt} + 10\,y(t) = 0 \qquad .....(2)$$

Let, $y(t) = C\,e^{\lambda t}$; $\quad \therefore \frac{d}{dt}y(t) = C\lambda\,e^{\lambda t}$

On substituting the above terms in equation (2) we get,

$$5\,C\lambda\,e^{\lambda t} + 10\,C e^{\lambda t} = 0$$

$$\therefore (5\,\lambda + 10)\,C e^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$5\lambda + 10 = 0 \quad \Rightarrow \quad \lambda + 2 = 0 \quad \Rightarrow \quad \lambda = -2$$

Now the homogeneous solution is given by,

$$\text{Homogeneous solution}, \quad y_h(t) = C e^{\lambda t} = C\,e^{-2t}$$

Particular Solution

The particular solution is the solution of the system equation (equation (1)) for specific input.

Here input, $x(t) = 2\,u(t)$

Let the particular solution, $y_p(t)$ is of the form,

$$y_p(t) = K\;x(t)$$

$$\therefore y_p(t) = 2K\,u(t)\,; \qquad \frac{dy_p(t)}{dt} = 2K\,\delta(t)$$

$$\boxed{\frac{d}{dt}u(t) = \delta(t)}$$

On substituting the above terms and the input in system equation (equation (1)) we get,

$$5\,\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t)$$

$$\Downarrow$$

$$10\,K\,\delta(t) + 20\,K\,u(t) = 4\,u(t)$$

$$\text{At } t = 1,\; 10\,K\,\delta(1) + 20\,K\,u(1) = 4\,u(1) \qquad \Rightarrow \qquad 20\,K = 4 \qquad \Rightarrow \qquad K = 1/5$$

$$\boxed{\delta(1) = 0,\;\; u(1) = 1}$$

$$\therefore \text{Particular solution,}\;\; y_p(t) = \frac{2}{5}\,u(t)$$

Forced Response (or Zero-State Response)

$$\text{Zero state response },\; y_{zs}(t) = y_h(t) + y_p(t)\Big|_{\text{with constants evaluated with zero initial conditions}}$$

$$= C\,e^{-2t} + \frac{2}{5}\,u(t)$$

$$\text{At } t = o, \quad y_{zs}(t) = C\,e^0 + \frac{2}{5}\,u(0) = C + \frac{2}{5}$$

$$\text{Since,} \quad y_{zs}(0) = 0, \quad C + \frac{2}{5} = 0 \quad \Rightarrow \quad C = -\frac{2}{5}$$

$$\therefore \text{ Forced Response,} \quad y_{zs}(t) = -\frac{2}{5}\,e^{-2t} + \frac{2}{5}\,u(t);\;\; \text{for } t \geq 0 = \frac{2}{5}\left(1 - e^{-2t}\right)u(t)$$

Example 2.10

Determine the complete response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt} \;; \quad y(0) = 0 \;; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 1, \quad \text{for the input, } x(t) = e^{-2t}\,u(t)$$

Solution

The given system equation is,

$$\mathbf{\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt}} \qquad .....(1)$$

Homogeneous Solution

The homogeneous solution is the solution of the system equation when $x(t) = 0$.

On substituting $x(t) = 0$ in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = 0 \qquad .....(2)$$

$$\text{Let, } y(t) = C\,e^{\lambda t}\,; \quad \therefore \frac{d}{dt}y(t) = C\lambda\,e^{\lambda t} \;\; \text{and} \;\; \frac{d^2}{dt}y(t) = C\lambda^2\,e^{\lambda t}$$

On substituting the above terms in equation (2) we get,

$$C\lambda^2\,e^{\lambda t} + 5\,C\lambda\,e^{\lambda t} + 4\,C\,e^{\lambda t} = 0$$

$$\therefore \left(\lambda^2 + 5\,\lambda + 4\right)C\,e^{\lambda t} = 0$$

The characteristic polynomial of the above equation is,

$$\lambda^2 + 5\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda + 4)\,(\lambda + 1) = 0 \quad \Rightarrow \quad \lambda = -4, -1$$

Now the homogeneous solution is given by,

$$\text{Homogeneous solution, } y_h(t) = C_1\,e^{\lambda_1 t} + C_2\,e^{\lambda_2 t} = C_1\,e^{-4t} + C_2\,e^{-t}$$

Particular Solution

The particular solution is the solution of the system equation (equation (1)) for specific input.

Here input, $x(t) = e^{-2t}\,u(t)$

$$\therefore x(t) = e^{-2t} \;;\; \text{for} \quad t \geq 0$$

$$\therefore \frac{dx(t)}{dt} = -2e^{-2t}$$

Let the particular solution, $y_p(t)$ is of the form,

$$y_p(t) = K\,x(t)$$

$$\therefore y_p(t) = K\,e^{-2t}\,; \qquad \frac{dy_p(t)}{dt} = -2K\,e^{-2t}\,; \qquad \frac{d^2y(t)}{dt^2} = 4K\,e^{-2t}$$

On substituting the above terms and the input in system equation (equation (1)) we get,

$$\frac{d^2y(t)}{dt^2} + 5\,\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt}$$

$$\Downarrow$$

$$4K\,e^{-2t} - 10K\,e^{-2t} + 4K\,e^{-2t} = -2\,e^{-2t}$$

On dividing throughout by e^{-2t} we get,

$$4K - 10K + 4K = -2 \quad \Rightarrow \quad -2K = -2 \quad \Rightarrow \quad K = 1$$

$$\therefore \text{ Particular solution, } y_p(t) = e^{-2t}$$

Total (or Complete) Response

Method - 1

Total response, $y(t) = y_h(t) + y_p(t)$

$$\therefore y(t) = C_1\,e^{-4t} + C_2\,e^{-t} + e^{-2t}$$

When $t = 0$, $y(t) = y(0) = C_1\,e^0 + C_2\,e^0 + e^0 = C_1 + C_2 + 1$

Given that $y(0) = 0$, $\therefore C_1 + C_2 + 1 = 0$ .....(3)

$$\text{Here, } \frac{dy(t)}{dt} = -4C_1\,e^{-4t} - C_2\,e^{-t} - 2\,e^{-2t}$$

$$\text{Now, } \left.\frac{dy(t)}{dt}\right|_{t=0} = -4C_1\,e^0 - C_2\,e^0 - 2\,e^0 = -4C_1 - C_2 - 2$$

Given that, $\left.\dfrac{dy(t)}{dt}\right|_{t=0} = 1$; $\quad \therefore -4C_1 - C_2 - 2 = 1$ .....(4)

On adding equation (3) and (4) we get,

$$-3C_1 - 1 = 1 \quad \Rightarrow \quad -3C_1 = 2 \quad \Rightarrow \quad C_1 = -\frac{2}{3}$$

From equation (3), $C_2 = -1 - C_1 = -1 + \frac{2}{3} = -\frac{1}{3}$

$$\therefore \text{ Total Response, } y(t) = -\frac{2}{3} e^{-4t} - \frac{1}{3} e^{-t} + e^{-2t} \quad ; \quad t \geq 0$$

$$\text{(or) } y(t) = \left(e^{-2t} - \frac{2}{3} e^{-4t} - \frac{1}{3} e^{-t}\right) u(t)$$

Method - 2

Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

Zero - input response, $y_{zi}(t) = y_h(t) \Big|_{\text{with constants evaluated using initial condition}}$

$$y_h(t) = C_1 e^{-4t} + C_2 e^{-t}$$

$$\therefore \frac{dy_h(t)}{dt} = -4C_1 e^{-4t} - C_2 e^{-t}$$

At $t = 0$, $y_h(0) = C_1 e^0 + C_2 e^0 = C_1 + C_2$

Given that, $y_h(0) = 0$, $\therefore C_1 + C_2 = 0$ .....(5)

At $t = 0$, $\frac{dy_h(t)}{dt} = -4C_1 e^0 - C_2 e^0 = -4C_1 - C_2$

Given that, $\left.\frac{dy_h(t)}{dt}\right|_{t=0} = 1$, $\therefore -4C_1 - C_2 = 1$ .....(6)

On adding equations (5) and (6) we get,

$$-3C_1 = 1 \Rightarrow C_1 = -\frac{1}{3}$$

From equation (5), $C_2 = -C_1 = \frac{1}{3}$

$$\therefore y_{zi}(t) = -\frac{1}{3} e^{-4t} + \frac{1}{3} e^{-t}$$

Zero - state response , $y_{zs}(t) = y_h(t) + y_p(t) \Big|_{\text{with constants evaluated with zero initial conditions}}$

$$\therefore y_{zs}(t) = C_1 e^{-4t} + C_2 e^{-t} + e^{-2t}$$

$$\therefore \frac{dy_{zs}(t)}{dt} = -4C_1 e^{-4t} - C_2 e^{-t} - 2 e^{-2t}$$

At $t = 0$, $y_{zs}(0) = C_1 e^0 + C_2 e^0 + e^0 = C_1 + C_2 + 1$

At $t = 0$, $\frac{dy_{zs}(t)}{dt} = -4C_1 e^0 - C_2 e^0 - 2 e^0 = -4C_1 - C_2 - 2$

Since initial conditions are zero,

$C_1 + C_2 + 1 = 0$ .....(7)

$-4C_1 - C_2 - 2 = 0$ .....(8)

On adding equations (7) and (8) we get,

$$-3C_1 - 1 = 0 \quad \Rightarrow \quad C_1 = -\frac{1}{3}$$

From the equation (7), $C_2 = -1 - C_1 = -1 + \frac{1}{3} = -\frac{2}{3}$

$$\therefore\ y_{zs}(t) = -\frac{1}{3}e^{-4t} - \frac{2}{3}e^{-t} + e^{-2t}$$

$\therefore$ Total Response, $y(t) = y_{zi}(t) + y_{zs}(t)$

$$= -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} - \frac{1}{3}e^{-4t} - \frac{2}{3}e^{-t} + e^{-2t}$$

$$= -\frac{2}{3}e^{-4t} - \frac{1}{3}e^{-t} + e^{-2t} \quad ; \quad t \geq 0$$

$$= \left(e^{-2t} - \frac{2}{3}e^{-4t} - \frac{1}{3}e^{-t}\right)u(t)$$

2.8 Classification of Continuous Time Systems

The continuous time systems are classified based on their characteristics. Some of the classifications of continuous time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. Feedback and nonfeedback systems

2.8.1 Static and Dynamic Systems

A continuous time system is called ***static*** or ***memoryless*** if its output at any instant of time t depends at most on the input signal at the same time but not on the past or future input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{aligned} y(t) &= a\,x(t) \\ y(t) &= t\,x(t) + 6\,x^3(t) \end{aligned}\right\}\ \text{Static systems}$$

$$\left.\begin{aligned} y(t) &= t\,x(t) + 3\,x(t^2) \\ y(t) &= x(t) + 3\,x(t-2) \end{aligned}\right\}\ \text{Dynamic systems}$$

2.8.2 Time Invariant and Time Variant Systems

A system is said to be ***time invariant*** if its input-output characteristics does not change with time.

Definition : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$x(t) \xrightarrow{\mathcal{H}} y(t) \text{ implies that, } x(t-m) \xrightarrow{\mathcal{H}} y(t-m)$$

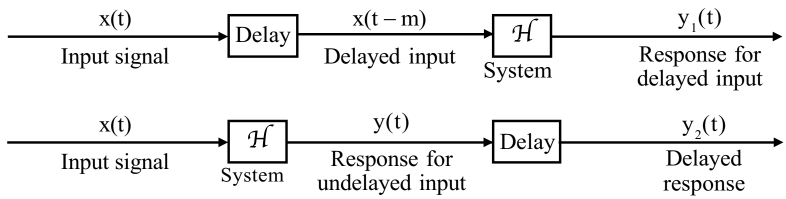
for every input signal x(t) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(t) = \mathcal{H}\{x(t)\}}$ then $\mathbf{y(t-m) = \mathcal{H}\{x(t-m)\}}$.

Alternative Definition for Time Invariance

A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

The diagrammatic explanation of the above definition of time invariance is shown in fig 2.41.



If $y_1(t) = y_2(t)$ then the system is time invariant

Fig 2.41 : *Diagrammatic explanation of time invariance.*

Procedure to test for time invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be $y_1(t)$.
2. Delay the response of the system for unshifted input by m units of time. Let this delayed response be $y_2(t)$.
3. Check whether $y_1(t) = y_2(t)$. If they are equal then the system is time invariant. Otherwise the system is time variant.

Example 2.11

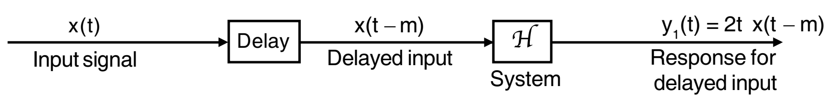
State whether the following systems are time invariant or not.

a) $y(t) = 2t\, x(t)$ b) $y(t) = x(t) \sin 20\pi t$ c) $y(t) = 3x(t^2)$ d) $y(t) = x(-t)$

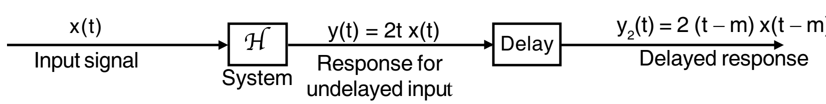
Solution

a) Given that, $y(t) = 2t\, x(t)$

<u>**Test 1** : Response for delayed input</u>



<u>**Test 2** : Delayed response</u>



<u>**Conclusion**</u> : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

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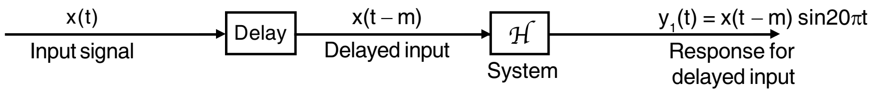
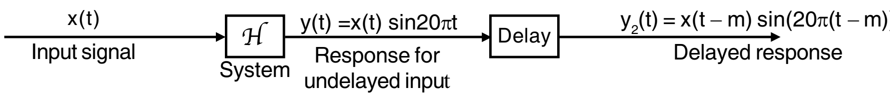
b) Given that, $y(t) = x(t) \sin 20\pi t$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

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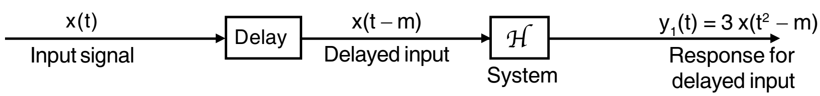
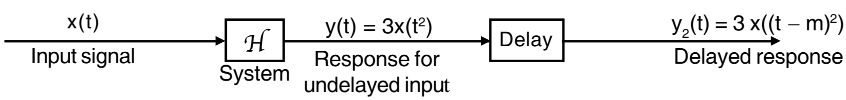
c) Given that, $y(t) = 3x(t^2)$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

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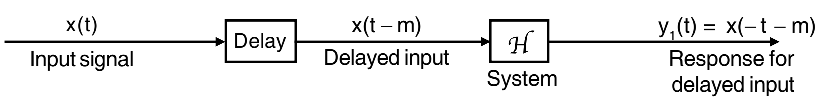
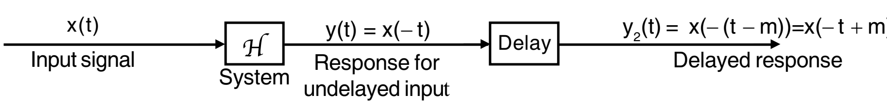
d) Given that, $y(t) = x(-t)$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

Example 2.12

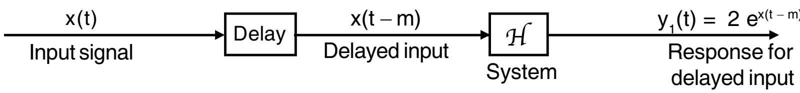
State whether the following systems are time invariant or not.

a) $y(t) = 2\,e^{x(t)}$ b) $y(t) = x(t) + C$ c) $y(t) = 3x^2(t)$ d) $y(t) = x(t) + \dfrac{dx(t)}{dt}$ e) $y(t) = x(t) + \int x(t)\,dt$

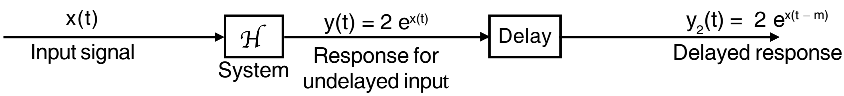
Solution

a) Given that, $y(t) = 2\,e^{x(t)}$

Test 1 : Response for delayed input



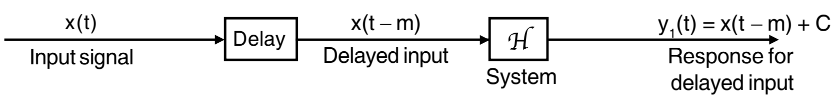
Test 2 : Delayed response



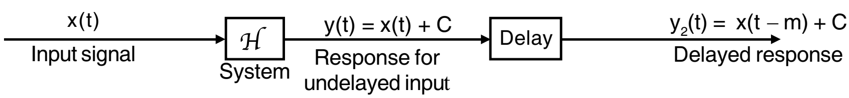
Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

b) Given that, $y(t) = x(t) + C$

Test 1 : Response for delayed input



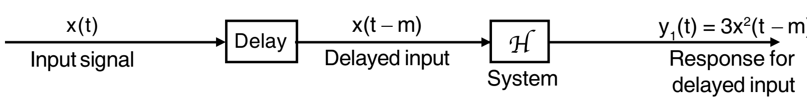
Test 2 : Delayed response



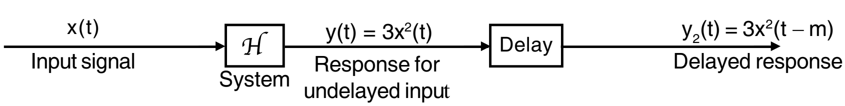
Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

c) Given that, $y(t) = 3x^2(t)$

Test 1 : Response for delayed input



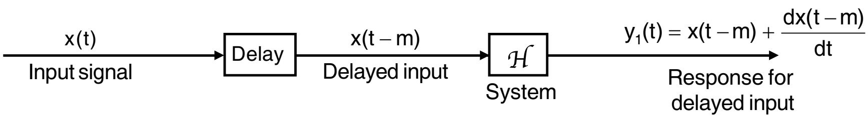
Test 2 : Delayed response



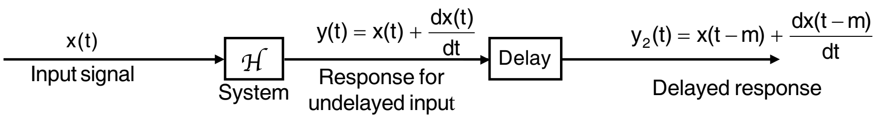
Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

d) Given that, $y(t) = x(t) + \frac{dx(t)}{dt}$

Test 1 : Response for delayed input



Test 2 : Delayed response

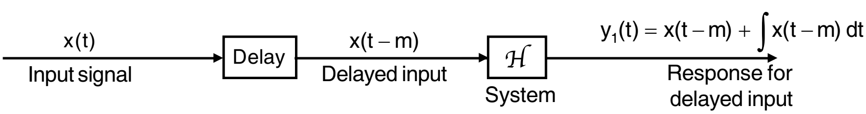


Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

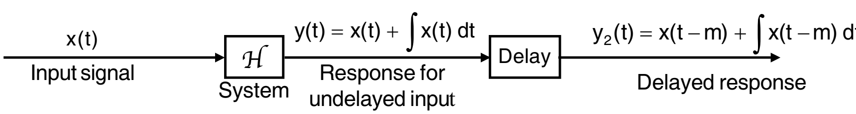
---

e) Given that, $y(t) = x(t) + \int x(t)\, dt$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

2.8.3 Linear and Nonlinear Systems

A ***linear system*** is the one that satisfies the superposition principle.

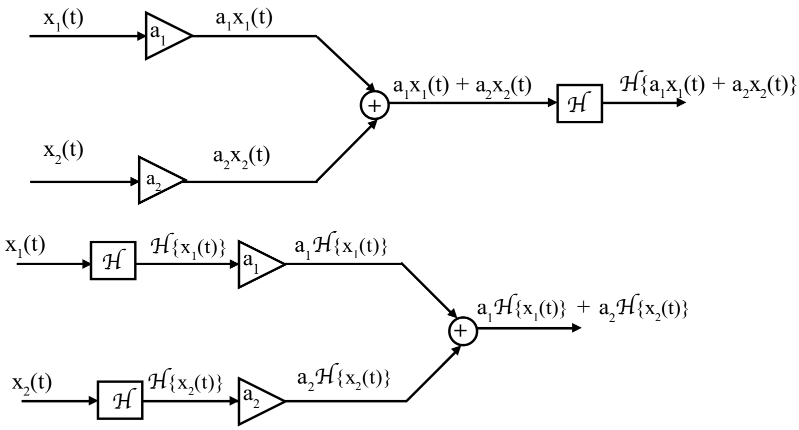
The ***principle of superposition*** requires that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

$$\mathcal{H}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, \mathcal{H}\{x_1(t)\} + a_2\, \mathcal{H}\{x_2(t)\}$$

for any arbitrary input signal $x_1(t)$ and $x_2(t)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, the system is ***nonlinear***. The diagrammatic explanation of linearity is shown in fig. 2.42.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, \mathcal{H}\{x_1(t)\} + a_2\, \mathcal{H}\{x_2(t)\}$

***Fig 2.42 :** Diagrammatic explanation of linearity.*

Procedure to test for linearity

1. Let $x_1(t)$ and $x_2(t)$ be two inputs to the system $\mathcal{H}$, and $y_1(t)$ and $y_2(t)$ be the corresponding responses.
2. Consider a signal, $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$ which is a weighed sum of $x_1(t)$ and $x_2(t)$.
3. Let $y_3(t)$ be the response for $x_3(t)$.
4. Check whether $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$. If equal then the system is linear, otherwise it is nonlinear.

Example 2.13

Test the following systems for linearity.

a) $y(t) = t\, x(t)$, b) $y(t) = x(t^2)$, c) $y(t) = x^2(t)$, d) $y(t) = A\, x(t) + B$, e) $y(t) = e^{x(t)}$.

Solution

a) Given that, $y(t) = t\, x(t)$

Let $\mathcal{H}$ be the system operating on $x(t)$ to produce, $y(t) = \mathcal{H}\{x(t)\} = t\, x(t)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = t\, x_1(t) \qquad ....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = t\, x_2(t) \qquad .....(2)$$

Let $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$. A linear combination of inputs $x_1(t)$ and $x_2(t)$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore\ y_3(t) = \mathcal{H}\{\, x_3(t)\} = \mathcal{H}\{\, a_1\, x_1(t) + a_2\, x_2(t)\}$$

$$= t(a_1\, x_1(t) + a_2\, x_2(t)) = a_1\, t\, x_1(t) + a_2\, t\, x_2(t)$$

$$= a_1\, y_1(t) + a_2\, y_2(t)) \qquad \text{Using equations (1) and (2)}$$

Since, $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$, the given system is linear.

b) Given that, $y(t) = x(t^2)$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = x(t^2)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = x_1(t^2) \qquad ....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = x_2(t^2) \qquad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore\ y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\}$$

$$= (a_1 x_1(t^2) + a_2 x_2(t^2))$$

$$= a_1 y_1(t) + a_2 y_2(t)) \qquad \boxed{\text{Using equations (1) and (2)}}$$

Since, $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, the given system is linear.

---

c) Given that, $y(t) = x^2(t)$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = x^2(t)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = x_1^2(t) \qquad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = x_2^2(t) \qquad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\} = (a_1 x_1(t) + a_2 x_2(t))^2$$

$$= a_1^2\ x_1^2(t) + a_2^2\ x_2^2(t) + 2\ a_1\ a_2\ x_1(t)\ x_1(t)$$

$$= a_1^2\ y_1(t) + a_2^2\ y_2(t) + 2\ a_1\ a_2\ x_1(t)\ x_1(t) \qquad \boxed{\text{Using equations (1) and (2)}}$$

Here, $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$. Hence the given system is nonlinear.

---

d) Given that, $y(t) = A\ x(t) + B$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = A\ x(t) + B$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = A\ x_1(t) + B \qquad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = A\ x_2(t) + B \qquad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\}$$

$$= A\ (a_1 x_1(t) + a_2 x_2(t)) + B = A\ a_1 x_1(t) + A\ a_2 x_2(t) + B$$

$$= a_1\ A\ x_1(t) + a_2\ A\ x_2(t) + B$$

$$= a_1\ (y_1(t) - B) + a_2\ (y_2(t) - B) + B \qquad \boxed{\text{Using equations (1) and (2)}}$$

Here, $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$. Hence the given system is nonlinear.

---

e) Given that, $y(t) = e^{x(t)}$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = e^{x(t)}$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = e^{x_1(t)} \quad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = e^{x_2(t)} \quad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\}$$

$$= e^{(a_1 x_1(t) + a_2 x_2(t))} = e^{a_1 x_1(t)}\ e^{a_2 x_2(t)}$$

$$= \left(e^{x_1(t)}\right)^{a_1} \left(e^{x_2(t)}\right)^{a_2} = (y_1(t))^{a_1}\ (y_2(t))^{a_2} \quad \boxed{\text{Using equations (1) and (2)}}$$

Here, $y_3(t) \neq a_1\, y_1(t) + a_2\, y_2(t)$. Hence the given system is nonlinear.

Example 2.14

Test the following systems for linearity.

a) $y(t) = 4\,x(t) + 2\,\dfrac{dx(t)}{dt}$ b) $\dfrac{d^2y(t)}{dt^2} + 2\,\dfrac{dy(t)}{dt} + 3\,y(t) = x(t)$

Solution

a) Given that, $y(t) = 4\,x(t) + 2\,\dfrac{dx(t)}{dt}$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = 4\,x(t) + 2\,\dfrac{dx(t)}{dt}$

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = 4\,x_1(t) + 2\,\frac{dx_1(t)}{dt} \quad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = 4\,x_2(t) + 2\,\frac{dx_2(t)}{dt} \quad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore\ y_3(t) = \mathcal{H}\{x_3(t)\}$$

$$= 4\,x_3(t) + 2\,\frac{dx_3(t)}{dt}$$

$$= 4(a_1\,x_1(t) + a_2\,x_2(t)) + 2\,\frac{d}{dt}(a_1\,x_1(t) + a_2\,x_2(t))$$

$$= 4a_1\,x_1(t) + 4a_2\,x_2(t) + 2a_1\,\frac{dx_1(t)}{dt} + 2a_2\,\frac{dx_2(t)}{dt}$$

$$= a_1\left(4\,x_1(t) + 2\,\frac{dx_1(t)}{dt}\right) + a_2\left(4\,x_2(t) + 2\,\frac{dx_2(t)}{dt}\right) \quad \boxed{\text{Using equations (1) and (2)}}$$

$$= a_1\,y_1(t) + a_2\,y_2(t)$$

Since, $y_3(t) = a_1 y_1(t) + a_2 y_2(t)$, the given system is linear.

b) Given that, $\dfrac{d^2y(t)}{dt^2} + 2\dfrac{dy(t)}{dt} + 3\,y(t) = x(t)$

Let $\mathcal{H}$ be the system operating on x(t) to produce, y(t).

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

When the input is $x_1(t)$, the response is $y_1(t)$. Hence the system equation for the input $x_1(t)$ can be written as,

$$\frac{d^2y_1(t)}{dt^2} + 2\frac{dy_1(t)}{dt} + 3\,y_1(t) = x_1(t) \quad .....(1)$$

When the input is $x_2(t)$, the response is $y_2(t)$. Hence the system equation for the input $x_2(t)$ can be written as,

$$\frac{d^2y_2(t)}{dt^2} + 2\frac{dy_2(t)}{dt} + 3\,y_2(t) = x_2(t) \quad .....(2)$$

Let, $x_3(t) = a_1\,x_1(t) + a_2\,x_2(t)$.

A linear combination of inputs $x_1(t)$ and $x_2(t)$

Let, $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

When the input is $x_3(t)$, the response is $y_3(t)$. Hence the system equation for the input $x_3(t)$ is given by,

$$\frac{d^2y_3(t)}{dt^2} + 2\frac{dy_3(t)}{dt} + 3\,y_3(t) = x_3(t) \quad .....(3)$$

Let us multiply equation (1) by a_1.

$$\therefore\ a_1\frac{d^2y_1(t)}{dt^2} + 2a_1\frac{dy_1(t)}{dt} + 3a_1\,y_1(t) = a_1\,x_1(t) \quad .....(4)$$

Let us multiply equation (2) by a_2.

$$\therefore\ a_2\frac{d^2y_2(t)}{dt^2} + 2a_2\frac{dy_2(t)}{dt} + 3a_2\,y_2(t) = a_2\,x_2(t) \quad .....(5)$$

On adding equation (4) and (5) we get,

$$a_1\frac{d^2y_1(t)}{dt^2} + 2a_1\frac{dy_1(t)}{dt} + 3a_1\,y_1(t) + a_2\frac{d^2y_2(t)}{dt^2} + 2a_2\frac{dy_2(t)}{dt} + 3a_2\,y_2(t) = a_1\,x_1(t) + a_2\,x_2(t)$$

$$\frac{d^2}{dt^2}[a_1\,y_1(t) + a_2\,y_2(t)] + 2\frac{d}{dt}[a_1\,y_1(t) + a_2\,y_2(t)] + 3[a_1\,y_1(t) + a_2\,y_2(t)] = a_1\,x_1(t) + a_2\,x_2(t) \quad .....(6)$$

On comparing equations (3) and (6) we can say that,

if, $x_3(t) = a_1\,x_1(t) + a_2\,x_2(t)$, then $y_3(t) = a_1\,y_1(t) + a_2\,y_2(t)$

Hence the sytem is linear.

2.8.4 Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time t depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time t depends on future inputs or outputs then the system is called a ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $t < 0$, (i.e., $h(t) = 0$ for $t < 0$).

Example 2.15

Test the casuality of the following systems.

a) $y(t) = x(t) - x(t-1)$ b) $y(t) = x(t) + 2\,x(3-t)$ c) $y(t) = t\,x(t)$

d) $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$ e) $y(t) = x(t) + \int_0^{3t} x(\lambda)\, d\lambda$ f) $y(t) = 2x(t) + \frac{dx(t)}{dt}$

Solution

a) Given that, $y(t) = x(t) - x(t-1)$

When $t = 0$, $y(0) = x(0) - x(-1)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-1)$.

When $t = 1$, $y(1) = x(1) - x(0)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(0)$.

From the above analysis we can say that for any value of t, the system output depends on present and past inputs. Hence the system is causal.

---

b) Given that, $y(t) = x(t) + 2\,x(3-t)$

When $t = -1$, $y(-1) = x(-1) + 2\,x(4)$ $\Rightarrow$ The response at $t = -1$, i.e., $y(-1)$ depends on the present input $x(-1)$ and future input $x(4)$.

When $t = 0$, $y(0) = x(0) + 2\,x(3)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(3)$.

When $t = 1$, $y(1) = x(1) + 2\,x(2)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and future input $x(2)$.

When $t = 2$, $y(2) = x(2) + 2\,x(1)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on the present input $x(2)$ and past input $x(1)$.

From the above analysis we can say that for $t< 2$, the system output depends on present and future inputs. Hence the system is noncausal.

---

c) Given that, $y(t) = t\,x(t)$

When $t = 0$, $y(0) = 0 \times x(0)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $t = 1$, $y(1) = 1 \times x(1)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $t = 2$, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on the present input $x(2)$.

From the above analysis we can say that the response for any value of t depends on the present input. Hence the system is causal.

---

d) Given that, $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$$y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda = x(t) + \left[z(\lambda)\right]_0^t = x(t) + z(t) - z(0), \qquad \text{where, } z(\lambda) = \int x(\lambda)\, d\lambda$$

When $t = 0$, $y(0) = x(0) + z(0) - z(0)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on present input.

When $t = 1$, $y(1) = x(1) + z(1) - z(0)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on present and past input.

When $t = 2$, $y(2) = x(2) + z(1) - z(0)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on present and past input.

From the above analysis we can say that the response for any value of t depends on the present and past input. Hence the system is causal.

---

e) Given that, $\mathbf{y(t) = x(t) + \int_0^{3t} x(\lambda)\, d\lambda}$

$$y(t) = x(t) + \int_0^{3t} x(\lambda)\, d\lambda = x(t) + \left[z(\lambda)\right]_0^{3t} = x(t) + z(3t) - z(0), \qquad \text{where } z(\lambda) = \int x(\lambda)\, d\lambda$$

When $t = 0$, $y(0) = x(0) + z(0) - z(0)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on present input.

When $t = 1$, $y(1) = x(1) + z(3) - z(0)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on present, past and future inputs.

When $t = 2$, $y(2) = x(2) + z(6) - z(0)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on present, past and future inputs.

From the above analysis we can say that the response for $t > 0$ depends on the present, past and future inputs. Hence the system is noncausal.

---

f) Given that, $\mathbf{y(t) = 2x(t) + \frac{dx(t)}{dt}}$

$$y(t) = 2\,x(t) + \frac{dx(t)}{dt}$$

$$= 2\,x(t) + \underset{\Delta t \to 0}{\text{Lt}} \frac{x(t) - x(t - \Delta t)}{\Delta t} \qquad \text{(Using definition of differentiation, refer section 2.4.6)}$$

In the above equation, for any value of t, the $x(t)$ is present input and $x(t-\Delta t)$ is the past input.

Therefore we can say that the response for any value of t depends on present and past input. Hence the system is causal.

Example 2.16

Test the causality of the following systems.

a) $y(t) = x(t) + 3\,x(t + 4)$ b) $y(t) = x(t^2)$

c) $y(t) = x(2t)$ d) $y(t) = x(-t)$

Solution

a) Given that, $\mathbf{y(t) = x(t) + 3\,x(t + 4)}$

When $t = 0$, $y(0) = x(0) + 3\,x(4)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(4)$.

When $t = 1$, $y(1) = x(1) + 3\,x(5)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and future input $x(5)$.

From the above analysis we can say that the response for any value of t depends on present and future inputs. Hence the system is noncausal.

---

b) Given that, $\mathbf{y(t) = x(t^2)}$

When $t = -1$; $y(-1) = x(1)$ $\Rightarrow$ The response at $t = -1$, depends on the future input $x(1)$.

When $t = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $t = 0$, depends on the present input $x(0)$.

When $t = 1$; $y(1) = x(1)$ $\Rightarrow$ The response at $t = 1$, depends on the present input $x(1)$.

When $t = 2$; $y(2) = x(4)$ $\Rightarrow$ The response at $t = 2$, depends on the future input $x(4)$.

From the above analysis we can say that the response for any value of t (except $t = 0$ & $t = 1$) depends on future input. Hence the system is noncausal.

---

c) Given that, $y(t) = x(2t)$

When $t = -1$; $y(-1) = x(-2)$ $\Rightarrow$ The response at $t = -1$, depends on the past input $x(-2)$.

When $t = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $t = 0$, depends on the present input $x(0)$.

When $t = 1$; $y(1) = x(2)$ $\Rightarrow$ The response at $t = 1$, depends on the future input $x(2)$.

From the above analysis we can say that the response of the system for $t > 0$, depends on future input. Hence the system is noncausal.

---

d) Given that, $y(t) = x(-t)$

When $t = -2$; $y(-2) = x(2)$ $\Rightarrow$ The response at $t = -2$, depends on the future input $x(2)$.

When $t = -1$; $y(-1) = x(1)$ $\Rightarrow$ The response at $t = -1$, depends on the future input $x(1)$.

When $t = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $t = 0$, depends on the present input $x(0)$.

When $t = 1$; $y(1) = x(-1)$ $\Rightarrow$ The response at $t = 1$, depends on the past input $x(-1)$.

From the above analysis we can say that the response of the system for $t < 0$ depends on future input. Hence the system is noncausal.

2.8.5 Stable and Unstable Systems

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(t) be the input of continuous time system and y(t) be the response or output for x(t).

The term ***bounded input*** refers to finite value of the input signal x(t) for any value of t. Hence if input x(t) is bounded then there exists a constant M_x such that $|x(t)| \le M_x$ and $M_x < \infty$, for all t.

> Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
>
> Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of t. Hence if output y(t) is bounded then there exists a constant M_y such that $|y(t)| \le M_y$ and $M_y < \infty$, for all t.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable.

Condition for Stability of an LTI System

For an LTI (Linear Time Invariant) system, the condition for BIBO stability can be transformed to a condition on impulse response, h(t). For BIBO stability of an LTI continuous time system, the integral of impulse response should be finite.

$$\boxed{\therefore \int_{-\infty}^{+\infty} |h(t)|\, dt < \infty \text{ , for stability of an LTI system.}}$$

Proof :

The response of a system y(t) for any input x(t) is given by convolution of the input and impulse response.

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad .....(2.18)$$

On taking the absolute value on both sides of equation (2.18), we get,

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)\ x(t-\tau)\ d\tau \right| \quad = \int_{-\infty}^{+\infty} |h(\tau)\ x(t-\tau)\ |d\tau$$

$$= \int_{-\infty}^{+\infty} |h(\tau)|\ |x(t-\tau)|\ d\tau \qquad .....(2.19)$$

If the input x(t) is bounded then there exists a constant M_x, such that $|x(t-\tau)| \leq M_x < \infty$. Hence equation(2.19) can be written as,

$$|y(t)| = M_x \int_{-\infty}^{+\infty} |h(\tau)|d\tau \qquad .....(2.20)$$

From equation (2.20) we can say that the output y(t) is bounded, if the impulse response satisfies the condition,

$$\int_{-\infty}^{+\infty} |h(\tau)|\ d\tau\ < \infty$$

Since τ is a dummy variable in the above condition we can replace τ by t.

$$\therefore \int_{-\infty}^{+\infty} |h(t)|dt\ < \infty$$

Example 2.17

Test the stability of the following systems.

a) y(t) = cos (x(t)) b) y(t) = x(–t – 2) c) y(t) = t x(t)

Solution

a) Given that, y(t) = cos (x(t))

The given system is a nonlinear system, and so the test for stability should be performed for specific inputs.

The value of cos θ lies between –1 to +1 for any value of θ. Therefore the output y(t) is bounded for any value of input x(t). Hence the given system is stable.

b) Given that, y(t) = x(–t – 2)

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, y(t) = t x(t)

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

Case i : Let x(t) tends to ∞ or constant, as **t** tends to infinity. In this case, y(t) = t x(t) will be infinity as **t** tends to inifnity and so the system is unstable.

Case ii : Let x(t) tends to 0, as **t** tends to infinity. In this case y(t) = t x(t) will be zero as **t** tends to infinity and so the system is stable.

Example 2.18

Test the stability of the LTI systems, whose impulse responses are given below.

a) $h(t) = e^{-5|t|}$ b) $h(t) = e^{4t}\,u(t)$ c) $h(t) = e^{-4t}\,u(t)$

d) $h(t) = t\,e^{-3t}\,u(t)$ e) $h(t) = t\cos t\,u(t)$ f) $h(t) = e^{-t}\sin t\,u(t)$

Solution

a) Given that, $h(t) = e^{-5|t|}$

For stability, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|e^{-5|t|}\right| dt = \int_{-\infty}^{+\infty} e^{-5|t|}\,dt$$

$$= \int_{-\infty}^{0} e^{5t}\,dt + \int_{0}^{+\infty} e^{-5t}\,dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$$

$$= \frac{e^{0}}{5} - \frac{e^{-\infty}}{5} + \frac{e^{-\infty}}{-5} - \frac{e^{0}}{-5} = \frac{1}{5} - 0 + 0 + \frac{1}{5} = \frac{2}{5}$$

Here, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt = \frac{2}{5}$ = constant. Hence the system is stable.

---

b) Given that, $h(t) = e^{4t}\,u(t)$

For stability, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|e^{4t}\,u(t)\right| dt = \int_{-\infty}^{+\infty} e^{4t}\,u(t)\,dt$$

$$= \int_{0}^{+\infty} e^{4t}\,dt = \left[\frac{e^{4t}}{4}\right]_{0}^{\infty} = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty$$

Here, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt = \infty$. Hence the system is unstable.

---

c) Given that, $h(t) = e^{-4t}\,u(t)$

For stability, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|e^{-4t}\,u(t)\right| dt = \int_{-\infty}^{+\infty} e^{-4t}\,u(t)\,dt$$

$$= \int_{0}^{+\infty} e^{-4t}\,dt = \left[\frac{e^{-4t}}{-4}\right]_{0}^{\infty} = \frac{e^{-\infty}}{-4} - \frac{e^{0}}{-4} = 0 + \frac{1}{4} = \frac{1}{4}$$

Here, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt = \frac{1}{4}$ = constant. Hence the system is stable.

---

d) Given that, $h(t) = t\,e^{-3t}\,u(t)$

For stability, $\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|t\,e^{-3t}\,u(t)\right|\,dt = \int_{0}^{+\infty} t\,e^{-3t}\,dt$$

$$= \left[t\,\frac{e^{-3t}}{-3} - \int 1 \times \frac{e^{-3t}}{-3}\,dt\right]_0^\infty = \left[-\frac{t\,e^{-3t}}{3} - \frac{e^{-3t}}{9}\right]_0^\infty$$

$$\boxed{\int u\,v = u\int v - \int\left[du \int v\right]}$$

$$= -\frac{\infty \times e^{-\infty}}{3} - \frac{e^{-\infty}}{9} + \frac{0 \times e^{0}}{3} + \frac{e^{0}}{9}$$

$$= -\frac{\infty \times 0}{3} - 0 + 0 + \frac{1}{9} = \frac{1}{9}$$

Since, $\int_{-\infty}^{+\infty} |h(t)|\,dt = \frac{1}{9}$ = constant, the system is stable.

e) Given that, $h(t) = t\cos t\,u(t)$

For stability, $\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|t\cos t\,u(t)\right|\,dt = \int_{0}^{+\infty} t\cos t\,dt$$

$$= \left[t\sin t - \int 1 \times \sin t\,dt\right]_0^\infty = \left[t\sin t + \cos t\right]_0^\infty$$

$$\boxed{\int u\,v = u\int v - \int\left[du \int v\right]}$$

$$= \infty \times \sin\infty + \cos\infty - 0 \times \sin 0 - \cos 0$$

$$= \infty + \cos\infty - 0 - 1 = \infty$$

Since, $\int_{-\infty}^{+\infty} |h(t)|\,dt = \infty$, the system is unstable.

f) Given that, $h(t) = e^{-t}\sin t\,u(t)$

For stability, $\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|e^{-t}\sin t\,u(t)\right|\,dt = \int_{0}^{+\infty} e^{-t}\sin t\,dt \qquad .....(1)$$

$$\int_0^\infty e^{-t}\sin t\,dt = \left[e^{-t}(-\cos t)\right]_0^\infty - \int_0^\infty \frac{e^{-t}}{-1}(-\cos t)\,dt$$

$$\boxed{\int u\,v = u\int v - \int\left[du \int v\right]}$$

$$= \left[-e^{-t}\cos t\right]_0^\infty - \int_0^\infty e^{-t}\cos t\,dt$$

$$= \left[-e^{-t}\cos t\right]_0^\infty - \left[\left[e^{-t}\sin t\right]_0^\infty - \int_0^\infty \frac{e^{-t}}{-1}\sin t\,dt\right]$$

$$\boxed{\int u\,v = u\int v - \int\left[du \int v\right]}$$

$$= \left[-e^{-t}\cos t\right]_0^\infty - \left[e^{-t}\sin t\right]_0^\infty - \int_0^\infty e^{-t}\sin t\,dt \qquad .....(2)$$

From equation (2) we can write,

$$2\int_0^\infty e^{-t}\,\text{sint}\;dt = \left[-e^{-t}\,\text{cost}\right]_0^\infty - \left[e^{-t}\,\text{sint}\right]_0^\infty$$

$$\therefore \int_0^\infty e^{-t}\,\text{sint}\;dt = \frac{1}{2}\left[-e^{-t}\,\text{cost}\right]_0^\infty - \frac{1}{2}\left[e^{-t}\,\text{sint}\right]_0^\infty \qquad .....(3)$$

Using equation (3), the equation (1) can be written as,

$$\int_{-\infty}^{+\infty}|h(t)|\;dt = \int_0^\infty e^{-t}\,\text{sint}\;dt = \frac{1}{2}\left[-e^{-t}\,\text{cost}\right]_0^\infty - \frac{1}{2}\left[e^{-t}\,\text{sint}\right]_0^\infty$$

$$= \frac{1}{2}\left[-e^{-\infty}\cos\infty + e^0\cos 0\right] - \frac{1}{2}\left[e^{-\infty}\sin\infty - e^0\sin 0\right]$$

$$= \frac{1}{2}\left[-0\times\cos\infty + 1\right] - \frac{1}{2}\left[0\times\sin\infty - 0\right]$$

$$= \frac{1}{2}[0+1] - \frac{1}{2}[0-0] = \frac{1}{2}$$

Since, $\displaystyle\int_{-\infty}^{+\infty}|h(t)|\;dt = \frac{1}{2}$ = constant, the system is stable.

Example 2.19

Determine the range of values of "a" and "b" for the stability of LTI system with impulse response.

$h(t) = e^{at}\,u(t) + e^{-bt}\,u(t)$

Solution

Given that, $h(t) = e^{at}\,u(t) + e^{-bt}\,u(t)$.

$$\int_{-\infty}^{+\infty}|h(t)|\;dt = \int_{-\infty}^{+\infty}\left|e^{at}u(t)+e^{-bt}u(t)\right|dt = \left|\int_{-\infty}^{+\infty}(e^{at}u(t)+e^{-bt}u(t))\;dt\right|$$

$$= \left|\int_0^\infty e^{at}\,dt + \int_0^\infty e^{-bt}\;dt\right| = \left|\left[\frac{e^{at}}{a}\right]_0^\infty + \left[\frac{e^{-bt}}{-b}\right]_0^\infty\right|$$

$$= \left|\frac{e^{a\times\infty}}{a} - \frac{e^{a\times 0}}{a} + \frac{e^{-b\times\infty}}{-b} - \frac{e^{-b\times 0}}{-b}\right| = \left|\frac{e^{a\times\infty}}{a} - \frac{1}{a} + \frac{e^{-b\times\infty}}{-b} + \frac{1}{b}\right|$$

In the above equation if "a" is negative and "b" is positive then it converges to finite value. Therefore, when "a" is negative and "b" is positive,

$$\int_0^\infty |h(t)|\;dt = \left|\frac{0}{a} - \frac{1}{a} + \frac{0}{-b} + \frac{1}{b}\right|$$

$$= \left|-\frac{1}{a} + \frac{1}{b}\right| = \left|\frac{a-b}{ab}\right| = \text{Constant}$$

Here the integral of impulse response is a constant when "a" is negative and "b" is positive.

Therefore the range of values of "a" and "b" for stability of LTI system are, **a < 0 and b > 0**.

2.8.6 Feedback and Nonfeedback Systems

The system in which the output y(t) at any time t depends on past output, past input and present input is called a ***feedback system.*** The integration and differentiation of a signal at any time depends on past value and so the equations governing feedback systems will have terms involving differentiations and integrations of output and input.

The equations governing feedback systems will be in the form,

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + ... + a_{N-1} \frac{d}{dt} y(t) + a_N\, y(t) = b_0 \frac{d^M}{dt^M} x(t)$$
$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... + b_{M-1} \frac{d}{dt} x(t) + b_M x(t)$$

The system in which the output depends only on the present and past input is called a ***nonfeedback system.*** The equations governing nonfeedback systems will not have terms involving differentiations and integrations of output.

The equations governing nonfeedback systems will be in the form,

$$y(t) = b_0 \frac{d^M}{dt^M} x(t) + b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... + b_{M-1} \frac{d}{dt} x(t) + b_M x(t)$$

2.9 Convolution of Continuous Time Signals

The ***convolution*** of two continuous time signals $x_1(t)$ and $x_2(t)$ is defined as,

$$\boxed{x_3(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda} \qquad ....(2.21)$$

where, $x_3(t)$ is the signal obtained by convolving $x_1(t)$ and $x_2(t)$,

and λ is a dummy variable used for integration.

The convolution relation of equation (2.21) can be symbolically expressed as,

$$x_3(t) = x_1(t) * x_2(t) \qquad ..... (2.22)$$

where the symbol $*$ indicates convolution operation.

2.9.1 Response of LTI Continuous Time System Using Convolution

In an LTI continuous time system, the response y(t) of the system for an arbitrary input x(t) is given by convolution of input x(t) with impulse response h(t) of the system. It is expressed as,

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda} \qquad ....(2.23)$$

where the symbol $*$ represents convolution operation.

In an LTI system, if the input x(t) is a unit step signal, then the response is called a ***unit step response***.

Proof :

Let y(t) be the response of system H for an input x(t)

$$\therefore\ y(t) = H[x(t)] \qquad .....(2.24)$$

From equation (2.10) we know that the signal x(t) can be expressed as an integral of impulses,

$$\text{i.e.,}\quad x(t) = \int_{-\infty}^{+\infty} x(\lambda)\ \delta(t-\lambda)\ d\lambda \qquad .....(2.25)$$

where, $\delta(t-\lambda)$ is the delayed unit impulse signal

From equation (2.24) and (2.25) we get,

$$y(t) = H\left[\int_{-\infty}^{+\infty} x(\lambda)\ \delta(t-\lambda)\ d\lambda\right]$$

$$= \int_{-\infty}^{+\infty} H\big[x(\lambda)\ \delta(t-\lambda)\big]\ d\lambda$$

In linear system, integration and system operation H can be interchanged

$$= \int_{-\infty}^{+\infty} x(\lambda)\ H\big[\delta(t-\lambda)\big]\ d\lambda \qquad .....(2.26)$$

The system H is a function of t and not a function of λ.

Let the response of the LTI system to the unit impulse input $\delta(t)$ be denoted by h(t).

$$\therefore\ h(t) = H[\delta(t)]$$

Then by time invariance property, the response of the system to delayed unit impulse input $\delta(t-\lambda)$ is given by,

$$H[\delta(t-\lambda)] = h(t-\lambda) \qquad .....(2.27)$$

Using equation (2.27), the equation (2.26) can be expressed as,

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)\ h(t-\lambda)\ d\lambda \qquad .....(2.28)$$

The equation (2.28) represents the convolution of input x(t) with the impulse response h(t) to yield the output y(t). Hence it is proved that the response y(t) of LTI continuous time system for an arbitrary input x(t) is given by convolution of input x(t) with impulse response h(t) of the system.

2.9.2 Properties of Convolution

The convolution of continuous time signals will satisfy the following properties.

Commutative property : $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

Associative property : $[\ x_1(t) * x_2(t)\ ] * x_3(t) = x_1(t) * [\ x_2(t) * x_3(t)\ ]$

Distributive property : $x_1(t) * [\ x_2(t) + x_3(t)\ ] = [\ x_1(t) * x_2(t)\ ] + [\ x_1(t) * x_3(t)\ ]$

Proof of Commutative Property :

Consider two continuous time signals, $x_1(t)$ and $x_2(t)$.

By Commutative property we can write,

$$\underset{\text{(LHS)}}{x_1(t) * x_2(t)} = \underset{\text{(RHS)}}{x_2(t) * x_1(t)}$$

$$LHS = x_1(t) * x_2(t)$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(t-m)\, dm \quad .....(2.29)$$

where m is a dummy variable used for convolution operation.

Let, $t - m = p$ | when $m = -\infty$, $p = t - m = t + \infty = +\infty$

$\therefore m = t - p$ | when $m = +\infty$, $p = t - m = t - \infty = -\infty$

$dm = -dp$

On replacing m by (t – p) and (t – m) by p in equation (2.29) we get,

$$LHS = -\int_{p=+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp = \int_{p=-\infty}^{+\infty} x_2(p)\, x_1(t-p)\, dp$$

Here p is a dummy variable used for convolution operation

$$= x_2(t) * x_1(t)$$

$$= RHS$$

Proof of Associative Property :

Consider three continuous time signals $x_1(t)$, $x_2(t)$ and $x_3(t)$. By Associative property we can write,

$$\underbrace{[\, x_1(t) * x_2(t)\,] * x_3(t)}_{LHS} = \underbrace{x_1(t) * [\, x_2(t) * x_3(t)\,]}_{RHS}$$

Let, $y_1(t) = x_1(t) * x_2(t)$ .....(2.30)

Let us replace t by p.

$$\therefore y_1(p) = x_1(p) * x_2(p)$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, dm \quad .....(2.31)$$

Let, $y_2(t) = x_2(t) * x_3(t)$ .....(2.32)

$$\therefore y_2(t) = \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t-q)\, dq$$

$$\therefore y_2(t-m) = \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t-q-m)\, dq \quad .....(2.33)$$

where p, m and q are dummy variables used for convolution operation.

$$LHS = [\, x_1(t) * x_2(t)\,] * x_3(t)$$

$$= y_1(t) * x_3(t) \quad \boxed{\text{Using equation (2.30)}}$$

$$= \int_{p=-\infty}^{+\infty} y_1(p)\, x_3(t-p)\, dp$$

$$= \int_{p=-\infty}^{+\infty} \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3(t-p)\, dm\, dp \quad \boxed{\text{Using equation (2.31)}}$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, dm \int_{p=-\infty}^{+\infty} x_2(p-m)\, x_3(t-p)\, dp \quad .....(2.34)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

$dp = dq$

On replacing (p – m) by q and p by (q + m) in the equation (2.34) we get,

$$\text{LHS} = \int_{m=-\infty}^{+\infty} x_1(m)\, dm \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t-q-m)\, dq$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, y_2(t-m)\, dm \qquad \boxed{\text{Using equation (2.33)}}$$

$$= x_1(t) * y_2(t)$$

$$= x_1(t) * [x_2(t) * x_3(t)] \qquad \boxed{\text{Using equation (2.32)}}$$

$$= \text{RHS}$$

Proof of Distributive Property :

Consider three continuous time signals $x_1(t)$, $x_2(t)$ and $x_3(t)$. By distributive property we can write,

$$\underbrace{x_1(t) * [\, x_2(t) + x_3(t)\,]}_{\text{LHS}} = \underbrace{[\, x_1(t) * x_2(t)\,] + [\, x_1(t) * x_3(t)\,]}_{\text{RHS}}$$

$$\text{LHS} = x_1(t) * [\, x_2(t) + x_3(t)\,]$$

$$= x_1(t) * x_4(t) \qquad \boxed{x_4(t) = x_2(t) + x_3(t)}$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\ x_4(t-m)\, dm \qquad \boxed{\text{m is dummy variable used for integration}}$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, [\, x_2(t-m) + x_3(t-m)\,]\, dm \qquad \boxed{\text{if, } x_4(t) = x_2(t) + x_3(t), \text{ then } x_4(t-m) = x_2(t-m) + x_3(t-m)}$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\ x_2(t-m)\, dm + \int_{m=-\infty}^{+\infty} x_1(m)\ x_3(t-m)\, dm$$

$$= [\, x_1(t) * x_2(t)\,] + [\, x_1(t) * x_3(t)\,]$$

$$= \text{RHS}$$

2.9.3 Interconnections of Continuous Time Systems

Smaller continuous time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two continuous time systems with impulse responses $h_1(t)$ and $h_2(t)$ are shown in fig 2.43.

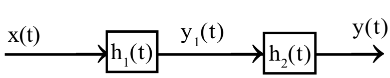


Fig 2.43a : *Cascade connection.*

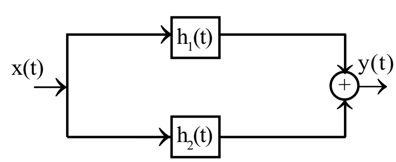


Fig 2.43b : *Parallel connection.*

Fig 2.43 : *Interconnection of continuous time systems.*

Cascade Connected Continuous Time Systems

Two cascade connected continuous time systems with impulse response $h_1(t)$ and $h_2(t)$ can be replaced by a single equivalent continuous time system whose impulse response is given by convolution of individual impulse responses.

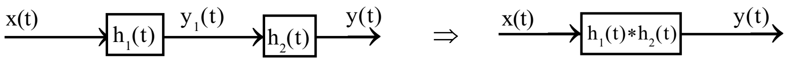


Fig 2.44 : *Cascade connected continuous time systems and their equivalent.*

Proof :

With reference to fig 2.44 we can write,

$$y_1(t) = x(t) * h_1(t) \qquad .....(2.35)$$

$$y(t) = y_1(t) * h_2(t) \qquad .....(2.36)$$

Using equation (2.35) the equation (2.36) can be written as,

$$y(t) = [\, x(t) * h_1(t) \,] * h_2(t)$$

$$= x(t) * [\, h_1(t) * h_2(t) \,]$$

Using associative property

$$= x(t) * h(t) \qquad .....(2.37)$$

where, $h(t) = h_1(t) * h_2(t)$

From equation (2.37) we can say that the overall impulse response of two cascaded continuous time systems is given by convolution of individual impulse responses.

Parallel Connected Continuous Time Systems

Two parallel connected continuous time systems with impulse responses $h_1(t)$ and $h_2(t)$ can be replaced by a single equivalent continuous time system whose impulse response is given by the sum of individual impulse responses.

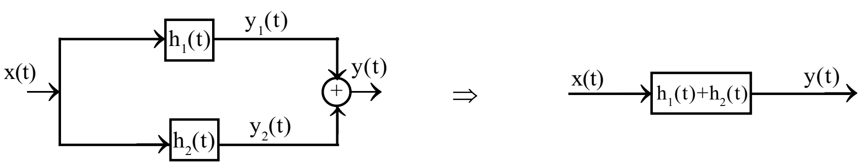


Fig 2.45 : *Parallel connected continuous time systems and their equivalent.*

Proof:

With reference to fig 2.45 we can write,

$$y_1(t) = x(t) * h_1(t) \qquad .....(2.38)$$

$$y_2(t) = x(t) * h_2(t) \qquad .....(2.39)$$

$$y(t) = y_1(t) + y_2(t) \qquad .....(2.40)$$

On substituting for $y_1(t)$ and $y_2(t)$ from equations (2.38) and (2.39) in equation (2.40) we get,

$$y(t) = [\, x(t) * h_1(t) \,] + [\, x(t) * h_2(t) \,] \qquad .....(2.41)$$

By using distributive property of convolution, the equation (2.41) can be written as shown below.

$$y(t) = x(t) * [\, h_1(t) + h_2(t) \,]$$

$$= x(t) * h(t) \qquad .....(2.42)$$

where, $h(t) = h_1(t) + h_2(t)$

From equation (2.42) we can say that the overall impulse response of two parallel connected continuous time systems is given by the sum of individual impulse responses.

2.9.4 Procedure to Perform Convolution

The ***convolution*** of two continuous time signals $x_1(t)$ and $x_2(t)$ is defined as,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

where, $x_3(t)$ is the signal obtained by convolving $x_1(t)$ and $x_2(t)$,
λ is a dummy variable used for integration,
$*$ indicates convolution operation.

The computation of $x_3(t)$ using the above convolution equation for any value of t involves the following operations,

1. ***Change of time index*** : The time index t in signals $x_1(t)$ and $x_2(t)$ is changed to λ to get $x_1(\lambda)$ and $x_2(\lambda)$.
2. ***Folding*** : The signal $x_2(\lambda)$ is folded to get $x_2(-\lambda)$.
3. ***Shifting*** : The signal $x_2(-\lambda)$ is shifted by t units of time to get $x_2(t-\lambda)$.
4. ***Multiplication*** : The signals $x_1(\lambda)$ and $x_2(t-\lambda)$ are multiplied to get a product signal.
5. ***Integration*** : The product signal is integrated to get $x_3(t)$. Let the product signal is nonzero in the interval $\lambda=\lambda_1$ to $\lambda=\lambda_2$, Now the signal $x_3(t)$ is given by,

$$x_3(t) = \int_{\lambda=\lambda_1}^{\lambda=\lambda_2} x_1(\lambda)\; x_2(t-\lambda)\; d\lambda$$

If both $x_1(t)$ and $x_2(t)$ are defined for $t > 0$, (i.e., both $x_1(t)$ and $x_2(t)$ are causal) then the product signal is nonzero in the interval $\lambda=0$ to $\lambda=t$, Now the signal $x_3(t)$ is given by,

$$x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\; x_2(t-\lambda)\; d\lambda$$

In order to determine the range of values of λ for product signal, graphical representation of signals will be very useful. The operations like folding, shifting and multiplication can be performed graphically to ascertain the range of values of λ over which the product signal is nonzero.

In the above convolution, if the signals $x_1(t)$ and $x_2(t)$ are defined by a single mathematical equation for $t=-\infty$ to $+\infty$, then time shift is valid for any value of t in the range $t=-\infty$ to $+\infty$. Therefore the time shift, multiplication and integration are performed only once by taking general time shift t.

In the above convolution, if the signals $x_1(t)$ and $x_2(t)$ are defined by different mathematical equations in various intervals of time, then the time shift, multiplication and integration are performed in each interval of time by considering a time shift t in each interval.

2.9.5 Unit Step Response Using Convolution

In general the response y(t) of a system is given by convolution of input x(t) and impulse response h(t) of the system.

$$y(t) = x(t) * h(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Let the input x(t) be unit step input u(t), and the corresponding response be s(t). Now the unit step response s(t) is given by,

$$\text{Unit Step Response, } s(t) = u(t) * h(t)$$

$$= h(t) * u(t) \qquad \boxed{\text{Using Commutative property}}$$

$$= \int_{\lambda=-\infty}^{\lambda=+\infty} h(\lambda)\, u(t-\lambda)\, d\lambda$$

In the above convolution operation, $u(\lambda) = 1$ for $\lambda > 0$,
$u(-\lambda) = 1$ for $\lambda < 0$,
$u(t-\lambda) = 1$ for $\lambda < t$, and $u(t-\lambda) = 0$ for $\lambda > t$.

Therefore the unit step response s(t) is given by,

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda$$

Example 2.20

Perform convolution of the following causal signals.

a) $x_1(t) = 2\,u(t)$, $x_2(t) = u(t)$ b) $x_1(t) = e^{-2t}\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$

c) $x_1(t) = t\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$ d) $x_1(t) = \cos t\,u(t)$, $x_2(t) = t\,u(t)$

Solution

a) Given that, $x_1(t) = 2\,u(t) = 2 \;;\; t \ge 0$

$x_2(t) = u(t) = 1 \;;\; t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$\boxed{\text{Since } x_1(t) \text{ and } x_2(t) \text{ are causal, the limits of integration is 0 to t.}}$

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} 2 \times 1 \;\; d\lambda = 2\int_{\lambda=0}^{\lambda=t} d\lambda$$

$$= 2\big[\lambda\big]_0^t = 2\,\big[t-0\big] = 2t \quad ; \text{ for } t \ge 0 = 2t\,u(t)$$

---

b) Given that, $x_1(t) = e^{-2t}\,u(t) = e^{-2t} \;;\; t \ge 0$

$x_2(t) = e^{-5t}\,u(t) = e^{-5t} \;;\; t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$\boxed{\text{Since } x_1(t) \text{ and } x_2(t) \text{ are causal, the limits of integration is 0 to t.}}$

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-2\lambda}\, e^{-5(t-\lambda)}\, d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-2\lambda}\, e^{-5t}\, e^{5\lambda} d\lambda$$

$$= e^{-5t}\int_{\lambda=0}^{\lambda=t} e^{-2\lambda+5\lambda}\,d\lambda = e^{-5t}\int_{\lambda=0}^{\lambda=t} e^{3\lambda}\,d\lambda = e^{-5t}\left[\frac{e^{3\lambda}}{3}\right]_0^t = e^{-5t}\left[\frac{e^{3t}}{3}-\frac{e^{0}}{3}\right]$$

$$= \frac{e^{-5t}}{3}\left(e^{3t}-1\right) = \frac{1}{3}\left(e^{-2t}-e^{-5t}\right);\ \text{for } t\ge 0 = \frac{1}{3}\left(e^{-2t}-e^{-5t}\right)u(t)$$

c) Given that, $x_1(t) = t\,u(t) = t \;;\; t \ge 0$

$x_2(t) = e^{-5t}\,u(t) = e^{-5t} \;;\; t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

Since $x_1(t)$ and $x_2(t)$ are causal, the limits of integration is 0 to t.

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda$$

$$= \int_{\lambda=0}^{\lambda=t} \lambda\,e^{-5(t-\lambda)}\,d\lambda = \int_{\lambda=0}^{\lambda=t} \lambda\,e^{-5t}\,e^{5\lambda}d\lambda$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = \lambda$	$v = e^{5\lambda}$

$$= e^{-5t}\int_{\lambda=0}^{\lambda=t} \lambda\,e^{5\lambda}\,d\lambda = e^{-5t}\left[\lambda\,\frac{e^{5\lambda}}{5} - \int\left[1\times\frac{e^{5\lambda}}{5}\right]d\lambda\right]_0^t$$

$$= e^{-5t}\left[\lambda\,\frac{e^{5\lambda}}{5} - \frac{e^{5\lambda}}{25}\right]_0^t = e^{-5t}\left[t\,\frac{e^{5t}}{5} - \frac{e^{5t}}{25} - 0\times\frac{e^{0}}{5} + \frac{e^{0}}{25}\right]$$

$$= \frac{e^{-5t}}{25}\left(5te^{5t} - e^{5t} + 1\right);\ \text{for } t\ge 0 = \frac{1}{25}\left(e^{-5t} + 5t - 1\right)u(t)$$

d) Given that, $x_1(t) = \cos t\,u(t) = \cos t \;;\; t \ge 0$

$x_2(t) = t\,u(t) = t \;;\; t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

Since $x_1(t)$ and $x_2(t)$ are causal, the limits of integration is 0 to t.	
$\int uv = u\int v - \int\left[du\int v\right]$	
$u = \lambda$	$v = \cos\lambda$

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda$$

$$= \int_{\lambda=0}^{\lambda=t} \cos\lambda \times (t-\lambda)\,d\lambda = t\int_{\lambda=0}^{\lambda=t} \cos\lambda\,d\lambda - \int_{\lambda=0}^{\lambda=t} \lambda\cos\lambda\,d\lambda$$

$$= t\left[\sin\lambda\right]_0^t - \left[\lambda\sin\lambda - \int\left[1\times\sin\lambda\right]d\lambda\right]_0^t = t\left[\sin\lambda\right]_0^t - \left[\lambda\sin\lambda + \cos\lambda\right]_0^t$$

$$= t\left[\sin t - \sin 0\right] - \left[t\sin t + \cos t - 0\times\sin 0 - \cos 0\right]$$

$$= t\sin t - 0 - t\sin t - \cos t + 0 + 1 = 1 - \cos t \quad ;\ \text{for } t\ge 0 = (1-\cos t)\,u(t)$$

Example 2.21

Determine the unit step response of the following systems whose impulse responses are given below.

a) $h(t) = 3t\,u(t)$ b) $h(t) = e^{-5t}\,u(t)$ c) $h(t) = u(t+2)$

d) $h(t) = u(t-2)$ e) $h(t) = u(t+2) + u(t-2)$

Solution

a) Given that, $h(t) = 3t\,u(t) = 3t \quad ; \; t \geq 0$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} 3\lambda\, d\lambda = 3\int_{\lambda=0}^{\lambda=t} \lambda\, d\lambda$$

$$= 3\left[\frac{\lambda^2}{2}\right]_0^t = 3\left[\frac{t^2}{2} - \frac{0}{2}\right] = \frac{3}{2}t^2\,; \;\text{for } t \geq 0 = \frac{3}{2}t^2\,u(t)$$

b) Given that, $h(t) = e^{-5t}\,u(t) = e^{-5t} \quad ; \; t \geq 0$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-5\lambda}\, d\lambda = \left[\frac{e^{-5\lambda}}{-5}\right]_0^t$$

$$= \left[\frac{e^{-5t}}{-5} - \frac{e^{0}}{-5}\right] = \frac{1}{5}(1 - e^{-5t});\;\text{for } t \geq 0 = \frac{1}{5}(1 - e^{-5t})\,u(t)$$

c) Given that, $h(t) = u(t+2) = 1 \quad ; \; t \geq -2$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda = \int_{\lambda=-2}^{\lambda=t} d\lambda = [\lambda]_{-2}^{t} = [t+2]$$

$$= t+2\,;\;\text{for } t \geq -2 = (t+2)\,u(t+2)$$

d) Given that, $h(t) = u(t-2) = 1 \quad ; \; t \geq 2$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda = \int_{\lambda=2}^{\lambda=t} d\lambda = [\lambda]_{2}^{t} = [t-2]$$

$$= t-2\,;\;\text{for } t \geq 2 = (t-2)\,u(t-2)$$

e) Given that, $h(t) = u(t+2) + u(t-2)$

Let, $h(t) = h_1(t) + h_2(t)$

where, $h_1(t) = u(t+2) = 1 \quad ; \; t \geq -2$

$h_2(t) = u(t-2) = 1 \quad ; \; t \geq 2$

Unit step response, $s(t) = h(t) * u(t) = [\,h_1(t) + h_2(t)\,] * u(t)$

$= [\,h_1(t) * u(t)\,] + [\,h_2(t) * u(t)\,] = s_1(t) + s_2(t)$

where, $s_1(t) = h_1(t) * u(t)$

$s_2(t) = h_2(t) * u(t)$

$$s_1(t) = \int_{\lambda=-\infty}^{\lambda=t} h_1(\lambda)\, d\lambda = \int_{\lambda=-2}^{\lambda=t} d\lambda = [\lambda]_{-2}^{t} = [t+2] = t+2 \quad ;\text{for } t > -2$$

$$s_2(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda = \int_{\lambda=2}^{\lambda=t} d\lambda = \int_{\lambda=2}^{\lambda=t} d\lambda = [\lambda]_{2}^{t} = [t-2] = t-2 \quad ;\text{for } t > 2$$

Now, Unit step response, $s(t) = s_1(t) \quad ; \;\text{for } t = -2 \text{ to } 2$

$= s_1(t) + s_2(t) \quad ; \;\text{for } t > 2$

$\therefore$ Unit step response, $s(t) = t+2 \quad ; \;\text{for } t = -2 \text{ to } 2$

$= t+2 + t-2 = 2t \quad ; \;\text{for } t > 2$

Example 2.22

Perform convolution of the following signals, by graphical method.

a) $x_1(t) = e^{-3t}\, u(t)$, $x_2(t) = t\, u(t)$ b) $x_1(t) = e^{-at}$; $0 \le t \le T$, $x_2(t) = 1$; $0 \le t \le 2T$

Solution

a) Given that, $\mathbf{x_1(t) = e^{-3t}\, u(t) = e^{-3t}\ ;\ t \ge 0}$

$\mathbf{x_2(t) = t\, u(t) \quad = t \quad ; t \ge 0}$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t, in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

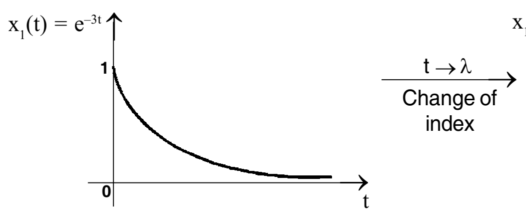


Fig 1 : $x_1(t)$.

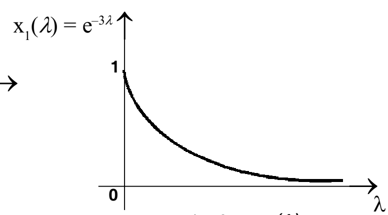


Fig 2 : $x_1(\lambda)$.

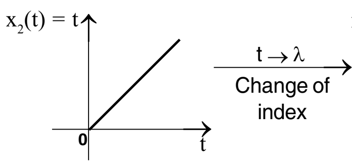


Fig 3 : $x_2(t)$.

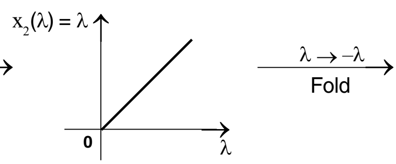


Fig 4 : $x_2(\lambda)$.

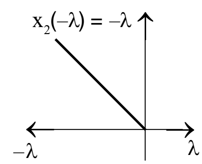


Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

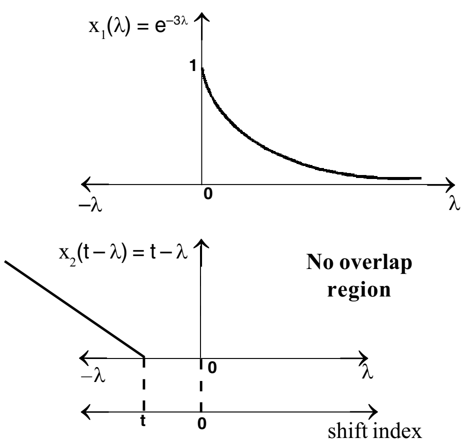


Fig 6 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.

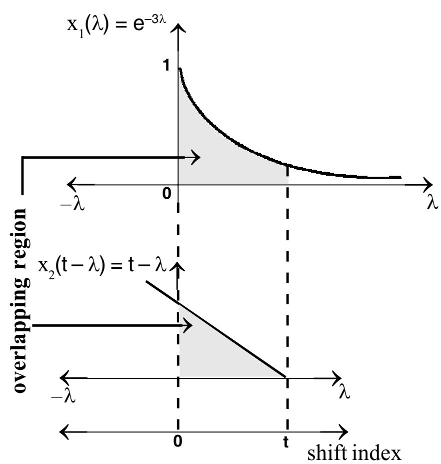


Fig 7 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 0$.

From fig 6 and fig 7, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift $t > 0$.

For any time shift $t > 0$, the non-zero product exists in the overlapping region shown in fig 7. Here the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence integration of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore\ x_3(\lambda) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \quad ; \quad t \geq 0$$

$$= \int_0^t e^{-3\lambda}(t-\lambda)\, d\lambda = \int_0^t t\, e^{-3\lambda}\, d\lambda - \int_0^t \lambda\, e^{-3\lambda}\, d\lambda$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = \lambda$	$v = e^{-3\lambda}$

$$= t\int_0^t e^{-3\lambda}\, d\lambda - \left[\lambda\frac{e^{-3\lambda}}{-3} - \int 1 \times \frac{e^{-3\lambda}}{-3}\, d\lambda\right]_0^t = t\left[\frac{e^{-3\lambda}}{-3}\right]_0^t - \left[\lambda\frac{e^{-3\lambda}}{-3} - \frac{e^{-3\lambda}}{9}\right]_0^t$$

$$= t\left[\frac{e^{-3t}}{-3} - \frac{e^0}{-3}\right] - \left[t\frac{e^{-3t}}{-3} - \frac{e^{-3t}}{9} - 0 + \frac{e^0}{9}\right] = -\frac{t\,e^{-3t}}{3} + \frac{t}{3} + \frac{t\,e^{-3t}}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}$$

$$= \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \quad ; \quad t \geq 0$$

$$= \left(\frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}\right)u(t) = \frac{1}{9}\left(e^{-3t} + 3t - 1\right)u(t)$$

b) Given that, $x_1(t) = e^{-at}$; $0 \leq t \leq T$

$x_2(t) = 1$; $0 \leq t \leq 2T$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

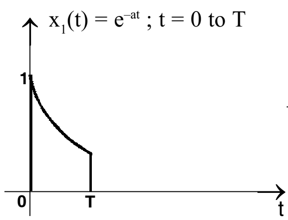


Fig 1 : $x_1(t)$.

$t \rightarrow \lambda$
Change of index

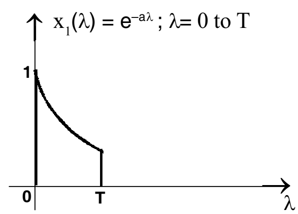


Fig 2 : $x_1(\lambda)$.

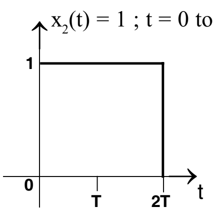


Fig 3 : $x_2(t)$.

$t \rightarrow \lambda$
Change of index

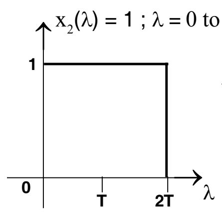


Fig 4 : $x_2(\lambda)$.

$\lambda \rightarrow -\lambda$
Fold

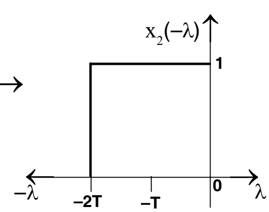


Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

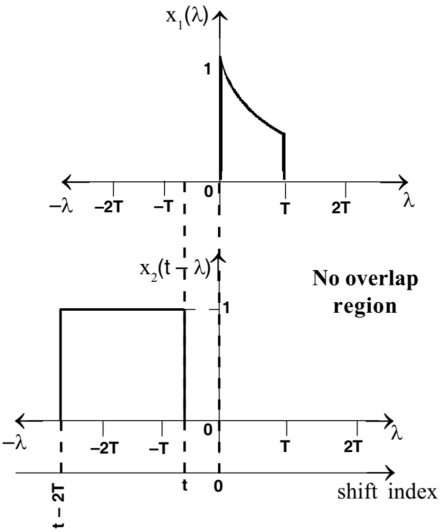


Fig 6 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.*

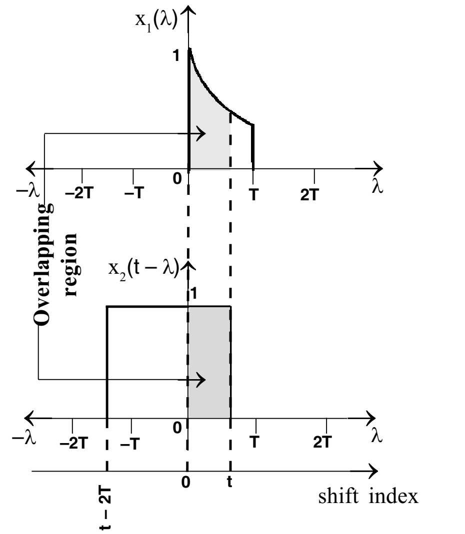


Fig 7 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 0$ to T.*

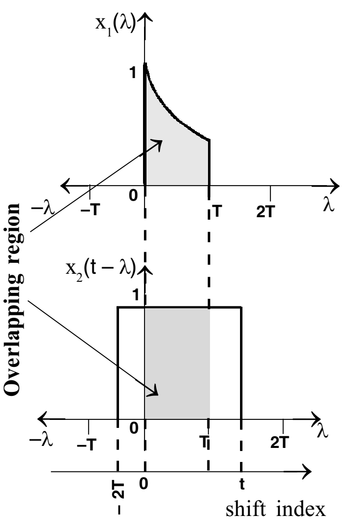


Fig 8 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = T$ to $2T$.*

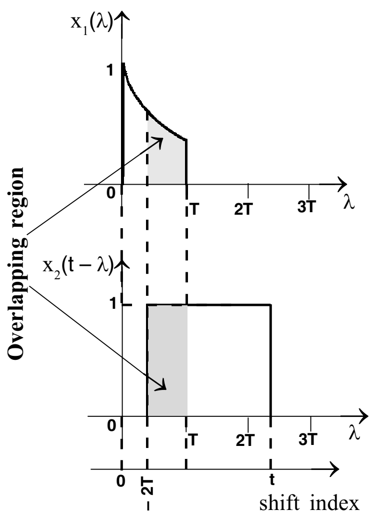


Fig 9 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 2T$ to $3T$.*

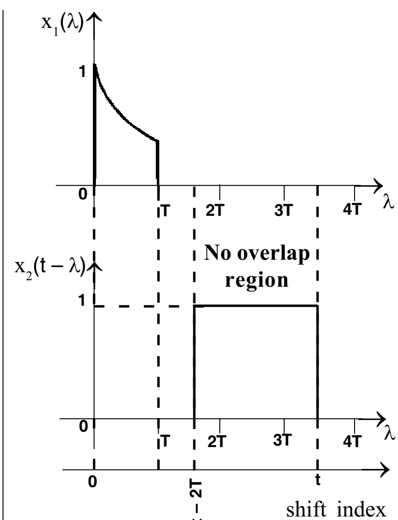


Fig 10 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 3T$.*

From fig 6 to fig 10, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift t = 0 to 3T. For any time shift in the range t = 0 to 3T, the non-zero product exists in the overlapping regions shown in fig 7, fig 8 and fig 9.

For t = 0 to T : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore\ x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^t e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_0^t$$

$$= \frac{e^{-at}}{-a} - \frac{e^0}{-a} = -\frac{1}{a}e^{-at} + \frac{1}{a} = \frac{1}{a}(1 - e^{-at})$$

For t = T to 2T : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = T$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = T$.

$$\therefore\ x_3(t) = \int_{\lambda=0}^{\lambda=T} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^T e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_0^T$$

$$= \frac{e^{-aT}}{-a} - \frac{e^0}{-a} = -\frac{1}{a}e^{-aT} + \frac{1}{a} = \frac{1}{a}(1 - e^{-aT})$$

For t = 2T to 3T : In this interval, the overlapping region is $\lambda = t - 2T$ to $\lambda = T$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t - 2T$ to $\lambda = T$.

$$\therefore\ x_3(t) = \int_{\lambda=t-2T}^{\lambda=T} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2T}^T e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_{t-2T}^T$$

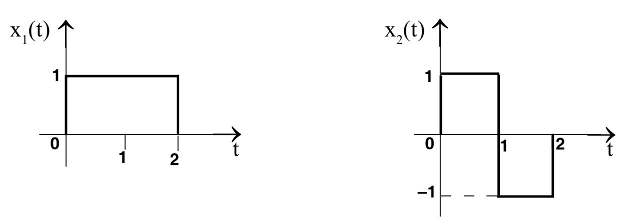
$$= \frac{e^{-aT}}{-a} - \frac{e^{-a(t-2T)}}{-a} = -\frac{1}{a}e^{-aT} + \frac{1}{a}e^{-a(t-2T)} = \frac{1}{a}(e^{-a(t-2T)} - e^{-aT})$$

$$\begin{aligned}
\therefore\ x_3(t) &= 0 && ;\quad t < 0 \\
&= \frac{1}{a}(1 - e^{-at}) && ;\quad 0 < t < T \\
&= \frac{1}{a}(1 - e^{-aT}) && ;\quad T < t < 2T \\
&= \frac{1}{a}(e^{-a(t-2T)} - e^{-aT}) && ;\quad 2T < t < 3T \\
&= 0 && ;\quad t > 3T
\end{aligned}$$

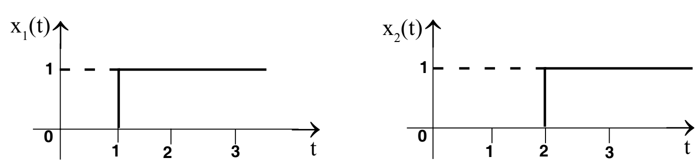
Example 2.23

Perform convolution of the following signals, by graphical method and sketch the resultant signal.

a)



b)



Solution

a) Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda = -\infty}^{\lambda = +\infty} x_1(\lambda)\, x_2(t - \lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

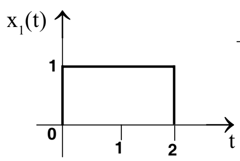


Fig 1 : *$x_1(t)$.*

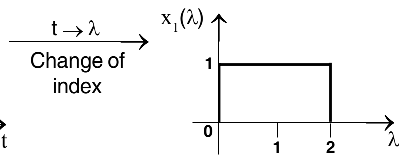


Fig 2 : *$x_1(\lambda)$.*

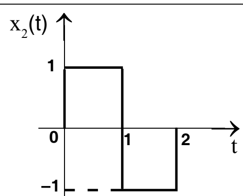


Fig 3 : *$x_2(t)$.*

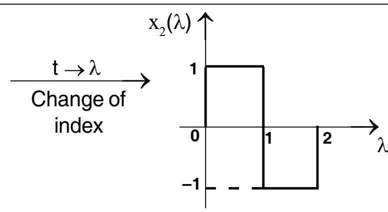


Fig 4 : *$x_2(\lambda)$.*

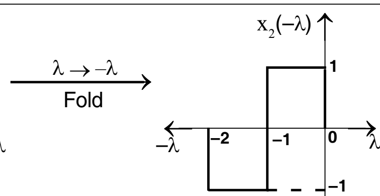


Fig 5 : *$x_2(-\lambda)$.*

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t - \lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

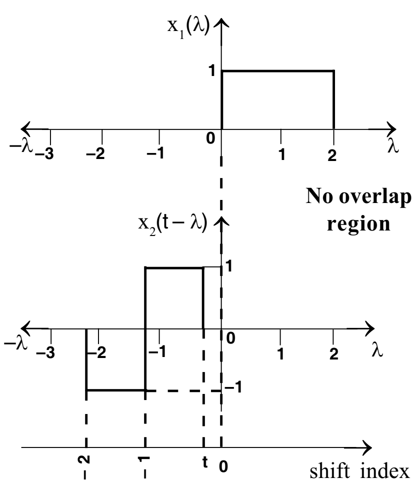


Fig 6 : *$x_1(\lambda)$ and $x_2(t - \lambda)$ when time shift, $t < 0$.*

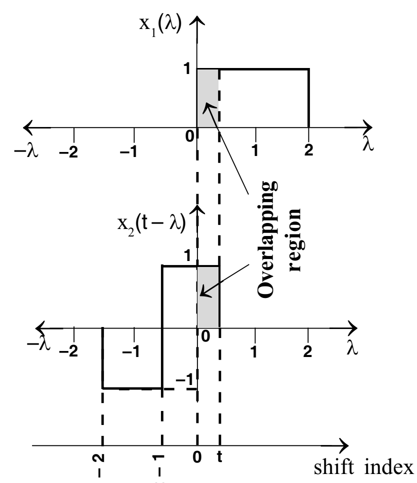


Fig 7 : *$x_1(\lambda)$ and $x_2(t - \lambda)$ when time shift, $t = 0$ to 1.*

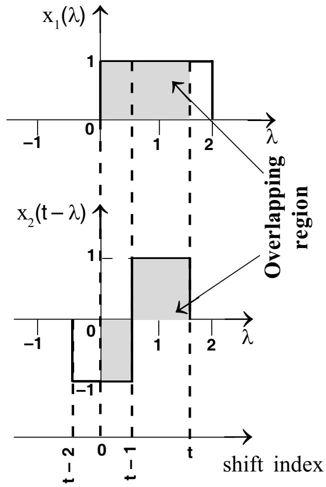


Fig 8 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t = 1 to 2.

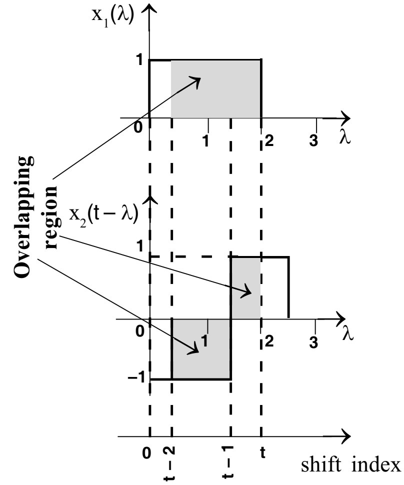


Fig 9 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t = 2 to 3.

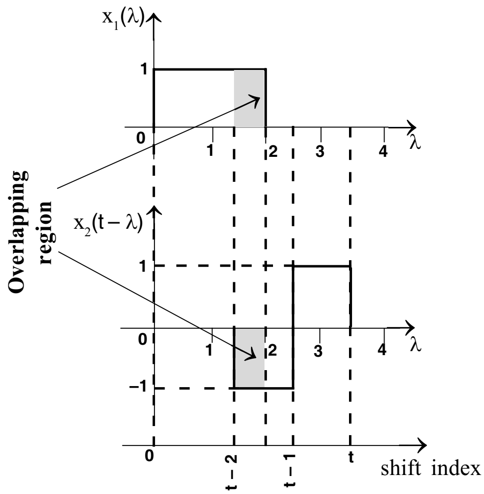


Fig 10 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t = 3 to 4.

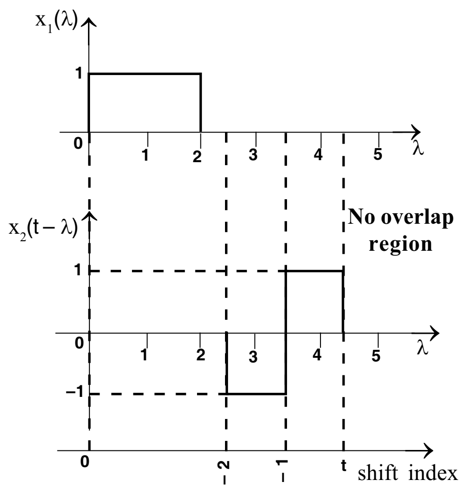


Fig 11 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t > 4.

From fig 6 to fig 11, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift t = 0 to 4. For any time shift in the range t = 0 to 4, the non-zero product exists in the overlapping regions shown in fig 7 to fig 10.

For t = 0 to 1 : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore\ x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^t 1 \times 1\, d\lambda = [\lambda]_0^t = t$$

For t = 1 to 2 : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore\ x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^{t-1} 1 \times (-1)\, d\lambda + \int_{t-1}^{t} 1 \times 1\, d\lambda = \left[-\lambda\right]_0^{t-1} + \left[\lambda\right]_{t-1}^{t}$$

$$= -(t-1) + 0 + t - (t-1) = -t + 1 + t - t + 1 = 2 - t$$

For t = 2 to 3 : In this interval, the overlapping region is $\lambda = t - 2$ to $\lambda = 2$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t-2$ to $\lambda = 2$.

$$\therefore\ x_3(t) = \int_{\lambda=t-2}^{\lambda=2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2}^{t-1} 1 \times (-1)\, d\lambda + \int_{t-1}^{2} 1 \times 1\, d\lambda = \left[-\lambda\right]_{t-2}^{t-1} + \left[\lambda\right]_{t-1}^{2}$$

$$= -(t-1) + (t-2) + 2 - (t-1) = -t + 1 + t - 2 + 2 - t + 1 = 2 - t$$

For t = 3 to 4 : In this interval, the overlapping region is $\lambda = t - 2$ to $\lambda = 2$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t-2$ to $\lambda = 2$.

$$\therefore\ x_3(t) = \int_{\lambda=t-2}^{\lambda=2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2}^{2} 1 \times (-1)\, d\lambda = \left[-\lambda\right]_{t-2}^{2}$$

$$= -2 + (t-2) = -2 + t - 2 = t - 4$$

The result of the convolution of $x_1(t)$ with $x_2(t)$ is given below and the resultant waveform is shown in fig 12.

$$x_1(t) * x_2(t) = x_3(t) = \begin{cases} 0 & ; t < 0 \\ t & ; 0 < t < 1 \\ 2 - t & ; 1 < t < 3 \\ t - 4 & ; 3 < t < 4 \\ 0 & ; t > 4 \end{cases}$$

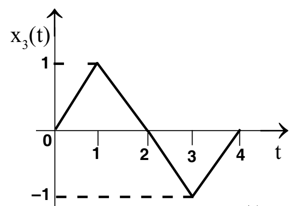


Fig 12 : $x_3(t)$.

At t = 0, $x_3(0) = 0$

At t = 1, $x_3(1) = t = 1$

At t = 2, $x_3(2) = 2 - t = 2 - 2 = 0$

At t = 3, $x_3(3) = 2 - t = 2 - 3 = -1$

At t = 4, $x_3(4) = t - 4 = 4 - 4 = 0$

b) Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

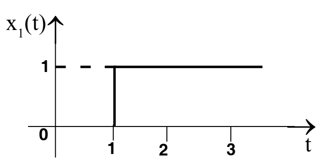


Fig 1 : $x_1(t)$.

$t \rightarrow \lambda$
Change of index

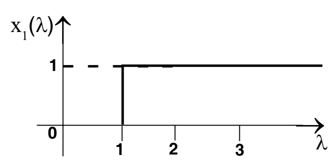


Fig 2 : $x_1(\lambda)$.

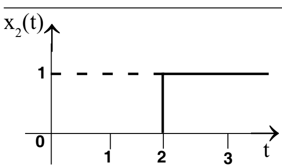


Fig 3 : $x_2(t)$.

$t \rightarrow \lambda$
Change of index

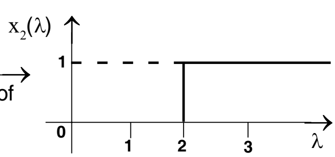


Fig 4 : $x_2(\lambda)$.

$\lambda \rightarrow -\lambda$
Fold

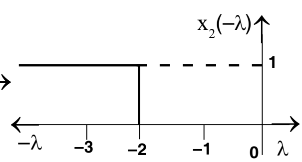


Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

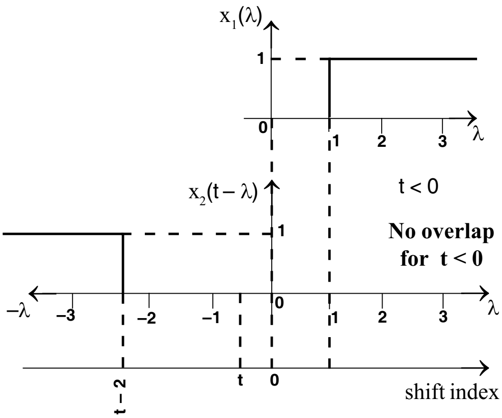


Fig 6 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t < 0.*

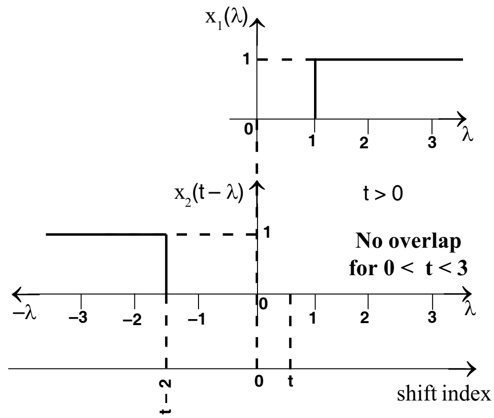


Fig 7 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t > 0.*

From fig 6, fig 7 and fig 8, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift $t \geq 3$. For any time shift t > 3, the non-zero product exists in the overlapping region shown in fig 8.

<u>For t > 3 :</u>

$$x_3(t) = \int_{\lambda=1}^{\lambda=t-2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_1^{t-2} 1 \times 1\, d\lambda = [\lambda]_1^{t-2}$$

$$= t - 2 - 1$$

$$= t - 3 \quad ; t \geq 3$$

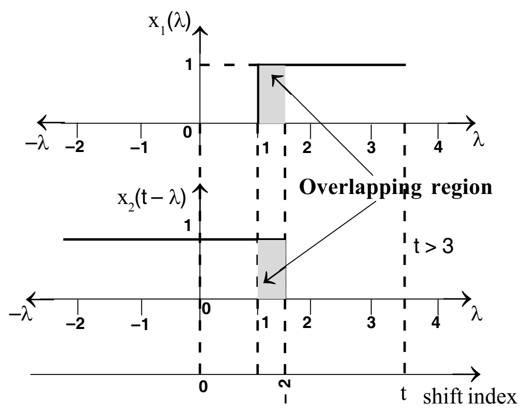


Fig 8 : *$x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, t >3.*

The result of the convolution of $x_1(t)$ with $x_2(t)$ is given below and the resultant waveform is shown in fig 9.

$$x_1(t) * x_2(t) = x_3(t) = t - 3 \quad ; t \geq 3$$

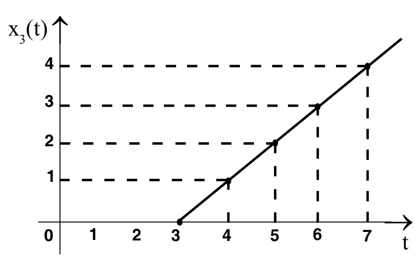


Fig 9 : *$x_3(t)$.*

At t = 3,	$x_3(t) = t - 3 = 3 - 3 = 0$
At t = 4,	$x_3(t) = t - 3 = 4 - 3 = 1$
At t = 5,	$x_3(t) = t - 3 = 5 - 3 = 2$
At t = 6,	$x_3(t) = t - 3 = 6 - 3 = 3$
At t = 7,	$x_3(t) = t - 3 = 7 - 3 = 4$

2.10 Inverse System and Deconvolution

Inverse System

The ***inverse system*** is used to recover the input from the response of a system. For a given system, the inverse system exists, if distinct inputs to a system leads to distinct outputs. The inverse systems exist for all LTI systems. Consider an amplifier with gain "A". When we pass the output of the amplifier through an attenuator with gain "1/A" then the output of the attenuator will be same as that of input of the amplifier. Therefore the attenuator with gain "1/A" is an inverse system for the amplifier with gain "A".

The inverse system is denoted by $\mathcal{H}^{-1}$. If x(t) is the input and y(t) is the output of a system, then y(t) is the input and x(t) is the output of its inverse system.

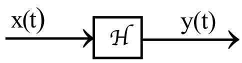


Fig 2.46a : *System.*

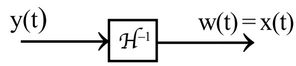


Fig 2.46b : *Inverse system.*

Fig 2.46 : *A system and its inverse system.*

Let h(t) be the impulse response of a system and h'(t) be the impulse response of inverse system. Let us connect the system and its inverse in cascade as shown in fig 2.47.

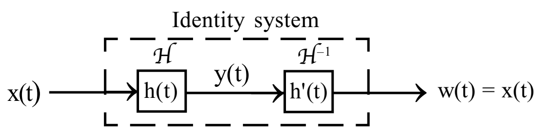


Fig 2.47 : *Cascade connection of a system and its inverse.*

Now it can be proved that,

$$h(t) * h'(t) = \delta(t) \qquad .....(2.43)$$

Therefore the cascade of a system and its inverse is the identity system.

Proof :

With reference to fig 2.47 we can write,

$$y(t) = x(t) * h(t) \qquad .....(2.44)$$

$$w(t) = y(t) * h'(t) \qquad .....(2.45)$$

On substituting for y(t) from equation (2.44) in equation (2.45) we get,

$$w(t) = x(t) * h(t) * h'(t) \qquad .....(2.46)$$

In equation (2.46), if $h(t) * h'(t) = \delta(t)$ then $x(t) * \delta(t) = x(t)$

In a inverse system, $w(t) = x(t)$, and so, $h(t) * h'(t) = \delta(t)$. Hence proved.

Deconvolution

In an LTI system the response y(t) is given by convolution of input x(t) and impulse response h(t).

i.e., $y(t) = x(t) * h(t)$

The process of recovering the input signal from the response of a system is called ***deconvolution***. Alternatively, the process of recovering x(t) from y(t), where $y(t) = x(t)*h(t)$, is called ***deconvolution***.

The deconvolution can be more conveniently handled in the Laplace domain (or s-domain), and so solved problems on deconvolution are presented in Chapter-3. [Refer example 3.29]

2.11 Summary of Important Concepts

1. An analog signal is a continuous function of an independent variable.
2. When the independent variable of an analog signal is time 't' then the analog signal is called continuous time (CT) signal.
3. In a continuous time signal the magnitude and the independent variable are continuous.
4. The sinusoidal and complex exponential signals are always periodic.
5. The sum of two periodic signals is also periodic if the ratio of their fundamental periods is a rational number.
6. A signal which is neither even nor odd can be expressed as a sum of even and odd signals.
7. Periodic signals are power signals and nonperiodic signals are energy signals.
8. The power of an energy signal is zero and the energy of a power signal is infinite.
9. The causal signals are defined only for $t \geq 0$.
10. The noncausal signals are defined for either $t \leq 0$ or all t.
11. Ideally, an impulse signal is a signal with infinite magnitude and zero duration.
12. Practically, an impulse signal is a signal with large magnitude and short duration.
13. The homogenous solution is the response of a system when there is no input, whereas the particular solution is the response for specific input.
14. The free or natural response is the response of the system due to initial stored energy, whereas the forced response is the response due to a particular input when there is no initial energy.
15. The response (or total response) of an LTI system is the sum of natural and forced response.
16. The response of a static system depends on present input whereas, the response of a dynamic system depends on present, past and future inputs.
17. The dynamic systems require memory whereas the static systems do not require memory.
18. In time invariant systems the input-output characteristics do not change with time.
19. In time invariant systems, if a delay is introduced either at input or at output, the response remains same.
20. Linear systems will satisfy the principle of superposition.
21. In linear systems, the response for weighted sum of inputs is equal to similar weighted sum of individual responses.
22. In causal systems, the response depends only on past and present values of inputs/outputs.
23. In noncausal systems, the response depends on future inputs/outputs.
24. In stable systems, for any bounded input, the output will be bounded.
25. The present output of a feedback system depends on past outputs.
26. The response of an LTI system is given by convolution of input and impulse response.
27. The unit step response of an LTI system is given by the integral of its impulse response.
28. The convolution operation satisfies the commutative, associative and distributive properties.
29. When LTI systems are connected in cascade, the overall impulse response is given by the convolution of individual impulse responses.
30. When LTI systems are connected in parallel, the overall impulse response is given by the sum of its individual impulse responses.
31. The inverse system is used to recover the input from the response of a system.
32. The cascade of a system and its inverse is the identity system.
33. The deconvolution is the process of recovering input from the response of a system.

2.12 Short Questions and Answers

Q2.1 ***Prove that sinc (0) = 1.***

Proof

$$\text{sinc}(0) = \underset{t\to 0}{\text{Lt}}\ \text{sinc}(t) = \underset{t\to 0}{\text{Lt}}\ \frac{\sin t}{t} = 1 \qquad \text{(Using L'Hospital's rule)}$$

Q2.2 ***Consider the complex valued exponential signal $x(t) = Ae^{\alpha t + j\Omega t}$, $a > 0$. Evaluate the real and imaginary components of x(t) for the following cases.***

i) α real, $\alpha = \alpha_1$ ii) α imaginary, $\alpha = j\Omega_1$ iii) α complex, $\alpha = \alpha_1 + j\Omega_1$

Solution

case i : $\alpha = \alpha_1$

$$x(t) = Ae^{\alpha t + j\Omega t} = Ae^{\alpha_1 t + j\Omega t} = Ae^{\alpha_1 t}\, e^{j\Omega t} = Ae^{\alpha_1 t}(\cos\Omega t + j\sin\Omega t)$$

$\therefore$ Real part of $x(t) = Ae^{\alpha_1 t}\cos\Omega t$; Imaginary part of $x(t) = Ae^{\alpha_1 t}\sin\Omega t$

case ii : $\alpha + j\Omega_1$

$$x(t) = Ae^{\alpha t + j\Omega t} = Ae^{j\Omega_1 t + j\Omega t} = Ae^{j(\Omega_1+\Omega)t} = A\big(\cos(\Omega_1 + \Omega)t + j\sin(\Omega_1+\Omega)t\big)$$

$\therefore$ Real part of $x(t) = A\cos(\Omega_1 + \Omega)t$; Imaginary part of $x(t) = A\sin(\Omega_1 + \Omega)t$

case iii : $\alpha = \alpha_1 + j\Omega_1$

$$x(t) = Ae^{\alpha t + j\Omega t} = Ae^{(\alpha_1 + j\Omega_1)t + j\Omega t} = Ae^{\alpha_1 t + j(\Omega_1+\Omega)t} = Ae^{\alpha_1 t}\, e^{j(\Omega_1+\Omega)t}$$

$$= Ae^{\alpha_1 t}\big(\cos(\Omega_1+\Omega)t + j\sin(\Omega_1+\Omega)t\big)$$

$\therefore$ Real part of $x(t) = Ae^{\alpha_1 t}\cos(\Omega_1+\Omega)t$; Imaginary part of $x(t) = Ae^{\alpha_1 t}\sin(\Omega_1+\Omega)t$

Q2.3 ***Determine whether the signal, $x(t) = 3\cos\sqrt{2}t + 7\cos 5\pi t$ is periodic.***

Solution

Given that $x(t) = 3\cos\sqrt{2}t + 7\cos\ 5\pi\, t$

Let, $x_1(t) = 3\cos\sqrt{2}t$. Let T_1 be period of $x_1(t)$. On comparing $x_1(t)$ with standard form $A\cos 2\pi F_{01}t$ we get,

$$2\pi F_{01} = \sqrt{2} \quad\Rightarrow\quad F_{01} = \frac{\sqrt{2}}{2\pi}, \qquad \therefore T_1 = \frac{1}{F_{01}} = \frac{2\pi}{\sqrt{2}}$$

Let, $x_2(t) = 7\cos 5\pi t$. Let T_2 be period of $x_2(t)$. On comparing $x_2(t)$ with standard form $A\cos 2\pi F_{02}t$ we get,

$$2\pi F_{02} = 5\pi \quad\Rightarrow\quad F_{02} = \frac{5\pi}{2\pi} = \frac{5}{2}, \qquad \therefore T_2 = \frac{1}{F_{02}} = \frac{2}{5}$$

Now, $\dfrac{T_1}{T_2} = T_1 \times \dfrac{1}{T_2} = \dfrac{2\pi}{\sqrt{2}} \times \dfrac{5}{2} = \dfrac{5\pi}{\sqrt{2}}$. Here T_1/T_2 is not a rational number and so $x(t)$ is nonperiodic.

Q2.4 ***Determine the period of the signal $x(t) = 0.1\, e^{-j\frac{2\pi}{3}t} + 0.3\sin\pi t$.***

Solution

Let, $x_1(t) = 0.1\, e^{-j\frac{2\pi}{3}t}$. Let T_1 be period of $x_1(t)$. On comparing $x_1(t)$ with standard form $A\, e^{-j2\pi F_{01}t}$ we get,

$$2\pi F_{01} = \frac{2\pi}{3} \quad\Rightarrow\quad F_{01} = \frac{1}{3} \qquad \therefore T_1 = \frac{1}{F_{01}} = 3$$

Let, $x_2(t) = 0.3\sin\pi t$. Let T_2 be period of $x_2(t)$. On comparing $x_2(t)$ with standard form $A\sin 2\pi F_{02}t$ we get,

$$2\pi F_{02} = \pi \quad\Rightarrow\quad F_{02} = \frac{1}{2} \qquad \therefore T_2 = \frac{1}{F_{02}} = 2$$

Here, $\frac{T_1}{T_2} = \frac{3}{2} = \text{Rational number.}$ $\therefore$ x(t) is periodic.

Let T be the period of x(t), which is given by LCM of T_1 and T_2. $\therefore T = \text{LCM of 2 and 3} = 2 \times 3 = 6.$

Q2.5 ***Determine the even and odd parts of a complex exponential signal.***

Solution

Complex exponential signal, $x(t) = Ae^{j\Omega_0 t} = A(\cos\Omega_0 t + j\sin\Omega_0 t) = A\cos\Omega_0 t + jA\sin\Omega_0 t$

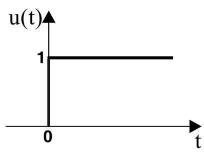
Now, $x(-t) = A\cos\Omega_0(-t) + jA\sin\Omega_0(-t) = A\cos\Omega_0 t - jA\sin\Omega_0 t$

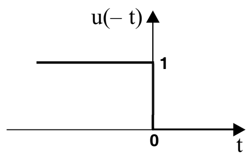
Even part, $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right] = A\cos\Omega_0 t$

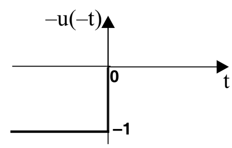
Odd part, $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right] = jA\sin\Omega_0 t$

Q2.6 ***Sketch the even parts and odd parts of a unit step signal.***

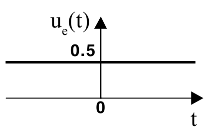
Solution



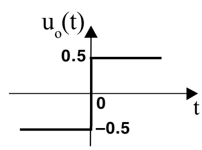




Even part, $u_e(t) = 0.5\left[u(t) + u(-t)\right] = 0.5u(t) + 0.5u(-t)$



Odd part, $u_o(t) = 0.5\left[u(t) - u(-t)\right] = 0.5u(t) - 0.5u(-t)$



Q2.7 ***Determine the energy and power of a unit step signal.***

Solution

Unit step signal, $u(t) = 1 \,;\, t \ge 0$
$\qquad = 0 \,;\, t < 0$

$$\therefore \int_{-T}^{T} |x(t)|^2 dt = \int_0^T |u(t)|^2 dt = \int_0^T 1^2 dt = \int_0^T dt = [t]_0^T = T - 0 = T$$

$$\text{Energy, } E = \underset{T\to\infty}{Lt} \int_{-T}^{T} |x(t)|^2 dt = \underset{T\to\infty}{Lt}\, T = \infty$$

$$\text{Power, } P = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \times T = \frac{1}{2} \text{ watts}$$

Q2.8 ***Compare energy and power signals.***

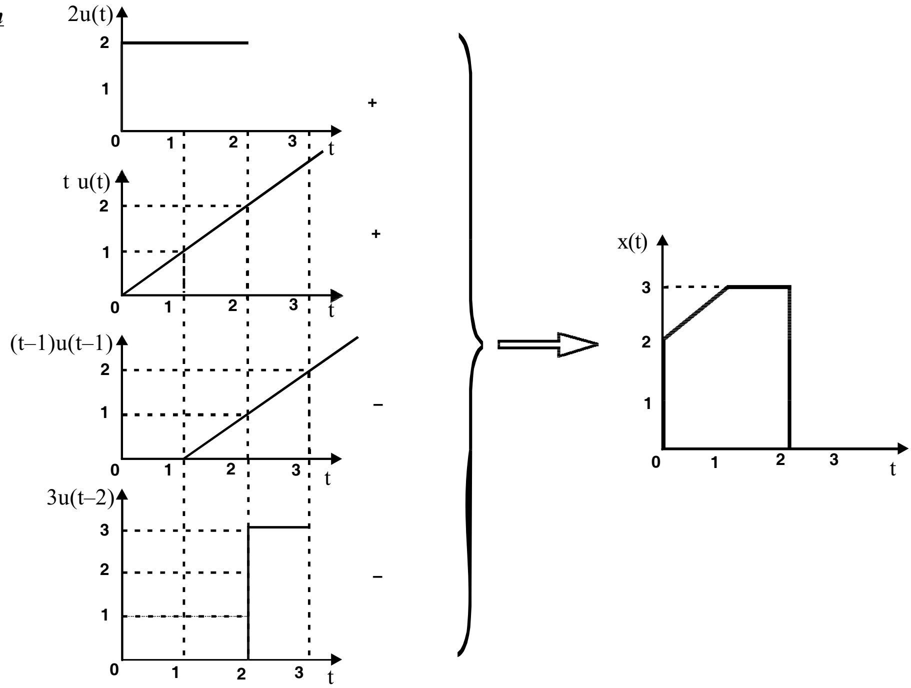
Energy Signal	Power Signal
1. Energy of the signal is constant.	1. Energy of the signal is infinite.
2. Power of the signal is zero.	2. Power of the signal is constant.
3. Nonperiodic signals.	3. Periodic signals.

Q2.9 ***Differentiate between causal and noncausal signals.***

The causal signals are defined only for $t \geq 0$, whereas the noncausal signals are defined for either $t \leq 0$ or for all t (i.e, for both $t \leq 0$ and $t > 0$).

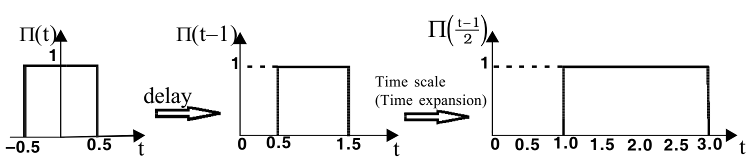
Q2.10 ***Sketch the signal, $x(t) = 2\,u(t) + t\,u(t) - (t-1)\,u(t-1) - 3\,u(t-2)$***

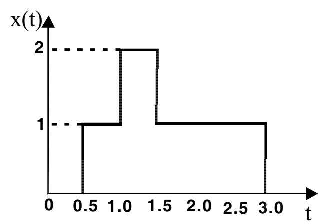
Solution



Q2.11 ***Sketch the signal,*** $x(t) = \Pi\left(\dfrac{t-1}{2}\right) + \Pi(t-1)$

Solution





Q2.12 ***A continuous time signal is shown in fig Q 2.12.***
Sketch the following versions of the signal.

$a)\; x(t-3) \qquad b)\; -2\,x(t) \qquad c)\; x(t-3) - 2\,x(t) \qquad d)\; \dfrac{d\,x(t)}{dt}$

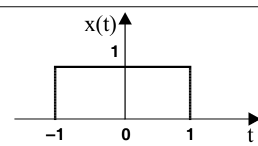
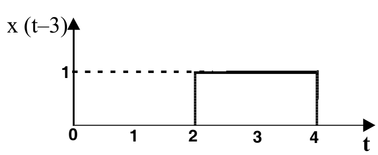


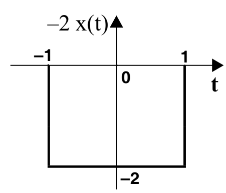
Fig Q : 2.12.

Solution

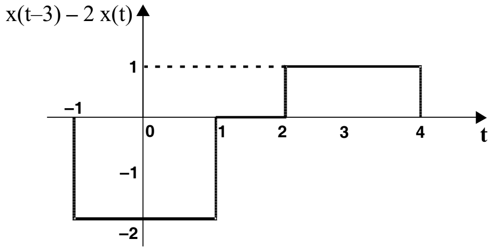
a)



b)



c)

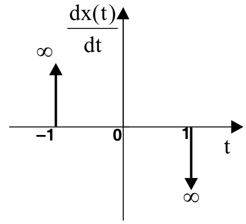


d) $x(t) = u(t+1) - u(t-1)$

$$\therefore \frac{d}{dt}x(t) = \frac{d}{dt}u(t+1) - \frac{d}{dt}u(t-1)$$

$$= \delta(t+1) - \delta(t-1)$$

$\frac{d}{dt}u(t) = \delta(t)$
Using time invariant property $\frac{d}{dt}u(t \pm k) = \delta(t \pm k)$



Q2.13 ***A continuous time signal is shown in fig Q 2.13. Find the following versions of the signal.***

a) x(– t) ***b) – x(t)***

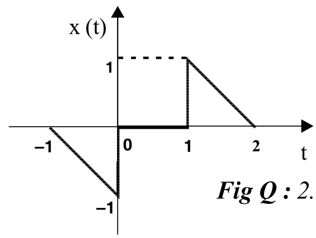
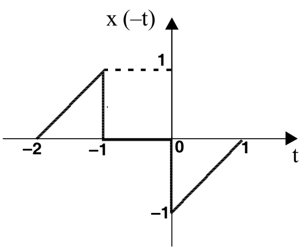


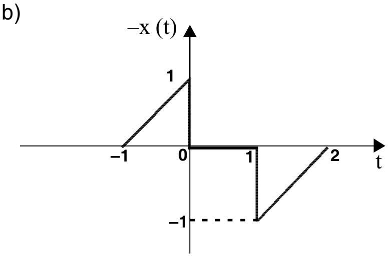
Fig Q : 2.13.

Solution

a)



b)



Q2.14 ***A continuous time signal is shown in fig Q 2.14 . Find the following versions of the signal. Comment on the result.***

a) x(t–2) ***b) x(–t)*** ***c) x(–t +2)***

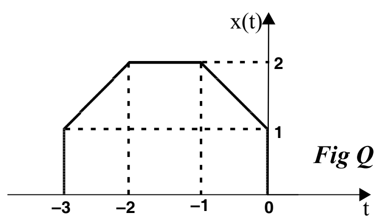
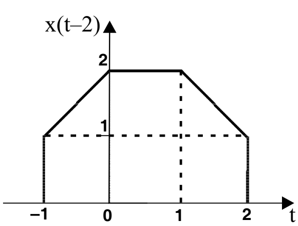


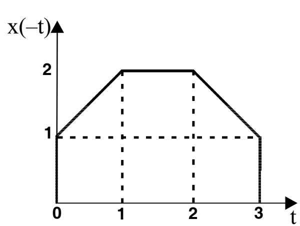
Fig Q : 2.14.

Solution

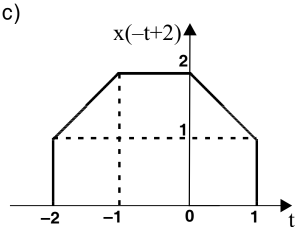
a)



b)



c)



Comment : The folded version of a delayed signal will be same as advance of its folded version (i.e., delay and fold is equal to fold and advance).

Q2.15 ***A continuous time signal is shown in fig Q 2.15. Find the following versions of the signal. Comment on the result.***

a) $x(t+2)$ b) $x(-t)$ c) $x(-t-2)$

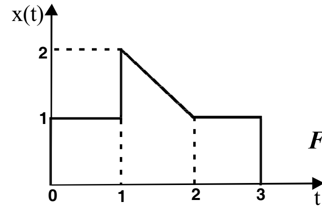
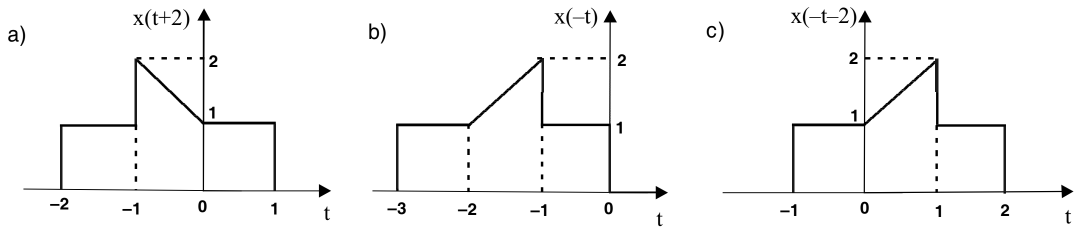


Fig Q : 2.15.

Solution



Comment : The folded version of an advanced signal will be same as delay of its folded version (i.e., advance and fold is equal to fold and delay).

Q2.16 ***Evaluate the following integrals,***

$$i)\ \int_{-\infty}^{\infty} \delta(t+3)\, e^{-t}\, dt \qquad ii)\ \int_{-\infty}^{\infty} \left[\delta(t)\cos t + \delta(t-1)\sin t\right] dt \qquad iii)\ \int_{0}^{5} \delta(t) \sin 2\pi t\, dt$$

Solution

$$\text{i)}\ \int_{-\infty}^{\infty} \delta(t+3)\, e^{-t}\, dt = \int_{-\infty}^{\infty} e^{-t}\delta(t-(-3))\, dt = e^{-t}\Big|_{t=-3} = e^{3} = 20.0855$$

Using Property of impulse

$$\int x(t)\,\delta(t-t_0)\, dt = x(t_0)$$

$$\text{ii)}\ \int_{-\infty}^{\infty} \left[\delta(t)\cos t + \delta(t-1)\sin t\right] dt = \int_{-\infty}^{\infty} \cos t\, \delta(t)\, dt + \int_{-\infty}^{\infty} \sin t\, \delta(t-1)\, dt = \cos t\Big|_{t=0} + \sin t\Big|_{t=1}$$

$$= \cos 0 + \sin 1 = 1 + 0.8415 = 1.8415$$

Using Property of impulse

$$\int x(t)\,\delta(t)\, dt = x(0)$$

$$\text{iii)}\ \int_{0}^{5} \delta(t) \sin 2\pi t\, dt = \sin 2\pi t\Big|_{t=0} = \sin 0 = 0$$

Q2.17 ***Determine natural response of the first order system governed by the equation,***

$$\frac{dy(t)}{dt} + 3y(t) = x(t)\ ;\quad y(0) = 2$$

Solution

Homogeneous Solution, $y_h(t)$

Put, $x(t) = 0$ in the given equation.

$$\therefore\ \frac{dy(t)}{dt} + 3y(t) = 0 \qquad \text{..... (1)}$$

Let, $y_h(t) = Ce^{\lambda t}$; $\quad \therefore\ \dfrac{dy_h(t)}{dt} = C\lambda e^{\lambda t}$

On substituting the above terms in equation (1) we get,

$$C\lambda e^{\lambda t} + 3Ce^{\lambda t} = 0 \qquad \Rightarrow \qquad C(\lambda+3)e^{\lambda t} = 0$$

The characteristic equation is, $\lambda + 3 = 0 \quad \Rightarrow \quad \lambda = -3$

$\therefore y_h(t) = Ce^{\lambda t} = Ce^{-3t}$

Natural Response

Natural response (or zero input response), $y_{zi}(t) = y_h(t)\Big|_{\text{with constants evaluated using initial conditions}} = Ce^{-3t}$

At $t = 0$, $y_{zi}(0) = Ce^0 = C$

Given that, $y(0) = 2$, $\therefore C = 2$

$\therefore$ Natural response, $y_{zi}(t) = 2e^{-3t}\ ;\ t \ge 0 \quad \Rightarrow \quad y_{zi}(t) = 2e^{-3t}u(t)$

Q2.18 ***Determine the unit step response of the first order system governed by the equation,***

$$\frac{dy(t)}{dt} + 0.5y(t) = x(t), \text{ with zero intial conditions.}$$

Solution

Homogeneous Solution, $y_h(t)$

Put, $x(t) = 0$ in the given equation.

$$\therefore \frac{dy(t)}{dt} + 0.5y(t) = 0 \qquad \text{..... (1)}$$

Let, $y_h(t) = Ce^{\lambda t}$; $\quad \therefore \dfrac{dy_h(t)}{dt} = C\lambda e^{\lambda t}$

On substituting the above terms in equation (1) we get,

$$C\lambda e^{\lambda t} + 0.5Ce^{\lambda t} = 0 \quad \Rightarrow \quad C(\lambda + 0.5)e^{\lambda t} = 0$$

The characteristic polynomial is, $\lambda + 0.5 = 0 \quad \therefore \lambda = -0.5$

$$\therefore y_h(t) = Ce^{-0.5t}\ ;\ t \ge 0 \quad \Rightarrow \quad y_h(t) = Ce^{-0.5t}\,u(t)$$

Particular Solution, $y_p(t)$

Here, $x(t) = u(t)$

Let, $y_p(t) = k\,u(t)$, $\quad \therefore \dfrac{dy_p(t)}{dt} = k\,\delta(t)$

On substituting the above terms in the given equation we get,

$$K\delta(t) + 0.5Ku(t) = u(t)$$

$$\text{At } t = 1, \quad K\delta(1) + 0.5Ku(1) = u(1) \quad \Rightarrow \quad K \times 0 + 0.5K \times 1 = 1 \quad \Rightarrow \quad \therefore K = \frac{1}{0.5} = 2$$

$$\therefore y_p(t) = 2\,u(t)$$

Unit Step Response (or Total Response)

Unit step response (or Total Response), $y(t) = y_h(t) + y_p(t) = Ce^{-0.5t}u(t) + 2u(t)$

At $t = 0$, $\quad y(0) = Ce^0 + 2 = C + 2$

Here, $y(0) = 0$, $\quad \therefore C + 2 = 0 \quad \Rightarrow \quad C = -2$

$\therefore$ Unit step response, $y(t) = Ce^{-0.5t}u(t) + 2u(t) = -2e^{-0.5t}u(t) + 2u(t) = 2(1 - e^{-0.5t})u(t)$

Q2.19 ***If two LTI systems with impulse responses, $h_1(t) = e^{-at}u(t)$ and $h_2(t) = e^{-bt}u(t)$ are connected in cascade, what will be the overall impulse response of cascaded system?***

Solution

When LTI systems are in cascade, the overall impulse response is given by convolution of individual impulse responses.

Let, $h_o(t)$ = Overall impulse response of cascaded system.

Now, $h_o(t) = h_1(t) * h_2(t) = \int_{\lambda=0}^{t} h_1(\lambda)h_2(t-\lambda)\,d\lambda = \int_{\lambda=0}^{t} e^{-a\lambda}e^{-b(t-\lambda)}d\lambda$

$$= \int_{\lambda=0}^{t} e^{-a\lambda}e^{-bt}e^{b\lambda}\,d\lambda = e^{-bt}\int_{\lambda=0}^{t} e^{-(a-b)\lambda}d\lambda = e^{-bt}\left[\frac{e^{-(a-b)\lambda}}{-(a-b)}\right]_0^t = e^{-bt}\left[\frac{e^{-(a-b)t}}{-(a-b)} - \frac{e^0}{-(a-b)}\right]$$

$$= \frac{1}{a-b}\left[-e^{-at} + e^{-bt}\right] \;;t \geq 0 \quad = \frac{1}{a-b}\left[-e^{-at} + e^{-bt}\right]u(t)$$

Q2.20 ***Find the overall impulse response of the system shown in fig Q2.20. Take, $h_1(t) = t\,u(t)$; $h_2(t) = 3\,u(t)$; $h_3(t) = 2\,u(t)$.***

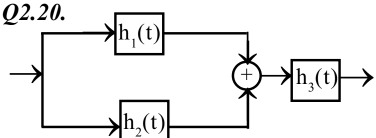


Fig Q : 2.20.

Solution

Overall impulse response, $h_o(t) = [h_1(t) + h_2(t)] * h_3(t) = [h_1(t) * h_3(t)] + [h_2(t) * h_3(t)]$

$$= \int_{\lambda=0}^{t} h_1(\lambda)h_3(t-\lambda)\,d\lambda + \int_{\lambda=0}^{t} h_2(\lambda)h_3(t-\lambda)\,d\lambda$$

$$= \int_{\lambda=0}^{t} \lambda \times 2\,d\lambda + \int_{\lambda=0}^{t} 3 \times 2\,d\lambda = 2\int_{\lambda=0}^{t} \lambda\,d\lambda + 6\int_{\lambda=0}^{t} d\lambda = 2\left[\frac{\lambda^2}{2}\right]_0^t + 6\,[\lambda]_0^t$$

$$= 2\left[\frac{t^2}{2} - 0\right] + 6\,[t-0] = t^2 + 6t \;\; ; \; t \geq 0 \;\; = \left[t^2 + 6t\right]u(t)$$

Q2.21 ***What is the importance of convolution?***

The convolution operation can be used to determine the response of an LTI system for any input from the knowledge of its impulse response. [The response of an LTI system is given by convolution of input and impulse response of the LTI system].

2.13 MATLAB Programs

Program 2.1

Write a MATLAB program to generate standard signals like unit impulse, unit step, unit ramp, parabolic, sinusoidal, triangular pulse, signum, sinc and Gaussian signals.

```
%****************** program to plot some standard signals

tmin=-5; dt=0.1; tmax=5;
t=tmin:dt:tmax;                %set a time vector

%****************** unit impulse signal
x1=1;
x2=0;
x=x1.*(t==0)+x2.*(t~=0);       %generate unit impulse signal
subplot(3,3,1);plot(t,x);      %plot the generated unit impulse signal
xlabel('t');ylabel('x(t)');title('unit impulse signal');

%****************** unit step signal
x1=1;
x2=0;
x=x1.*(t>=0)+x2.*(t<0);        %generate unit step signal
subplot(3,3,2);plot(t,x);      %plot the generated unit step signal
xlabel('t');ylabel('x(t)');title('unit step signal');

%****************** unit ramp signal
```

```
x1=t;
x2=0;
x=x1.*(t>=0)+x2.*(t<0);         %generate unit ramp signal
subplot(3,3,3);plot(t,x);       %plot the generated unit ramp signal
xlabel('t');ylabel('x(t)');title('unit ramp signal');

%****************** parabolic signal
A=0.4;
x1=(A*(t.^2))/2;
x2=0;
x=x1.*(t>=0)+x2.*(t<0);         %generate parabolic signal
subplot(3,3,4);plot(t,x);       %plot the generated parabolic signal
xlabel('t');ylabel('x(t)');title('parabolic signal');

%****************** sinusoidal signal
T=2;                            %declare time period
F=1/T;                          %compute frequency
x=sin(2*pi*F*t);                %generate sinusoidal signal
subplot(3,3,5);plot(t,x);       %plot the generated sinusoidal signal
xlabel('t');ylabel('x(t)');title('sinusoidal signal');

%****************** triangular pulse signal
a=2;
x1=1-abs(t)/a;
x2=0;
x=x1.*(abs(t)<=a)+x2.*(abs(t)>a); %generate triangular pulse signal
subplot(3,3,6);plot(t,x);         %plot the triangular pulse signal
xlabel('t');ylabel('x(t)');title('triangular pulse signal');

%****************** signum signal
```

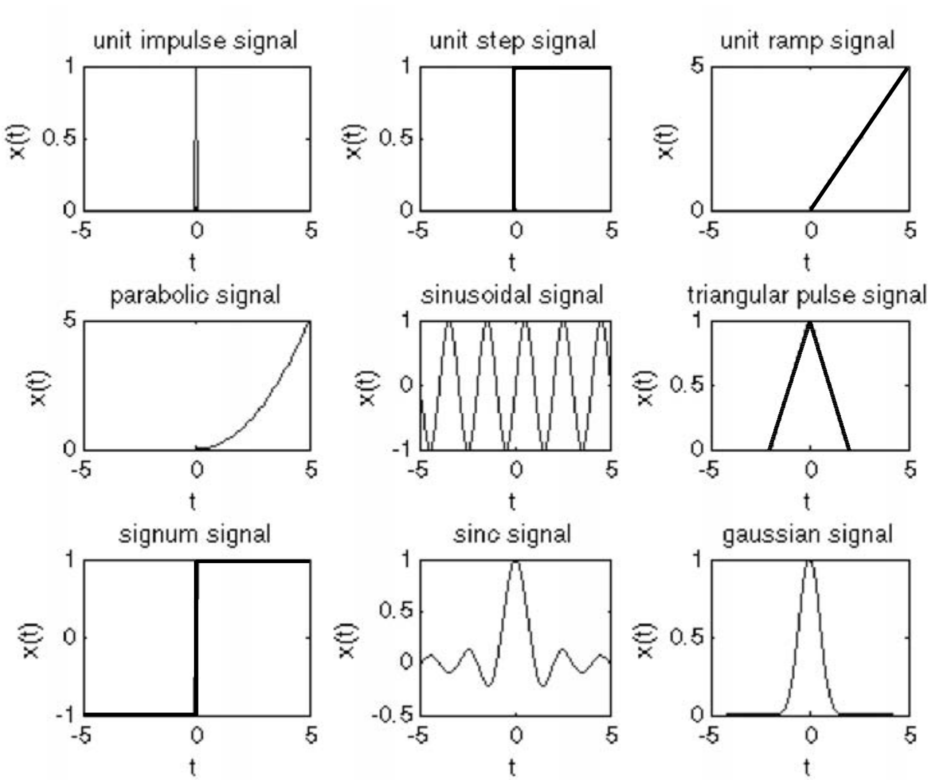


***Fig P2.1 :** Output waveforms of program 2.1.*

```
x1=1;
x2=0;
x3=-1;
x=x1.*(t>0)+x2.*(t==0)+x3.*(t<0); %generate signum signal
subplot(3,3,7);plot(t,x);         %plot the generated signum signal
xlabel('t');ylabel('x(t)');title('signum signal');

%******************* sinc Pulse
x=sinc(t);                        %generate sinc Pulse
subplot(3,3,8);plot(t,x);         %plot the generated sinc Pulse
xlabel('t');ylabel('x(t)');title('sinc signal');

%******************* gaussian signal
a=2;
x=exp(-a.*(t.^2));                %generate gaussian signal
subplot(3,3,9);plot(t,x);         %plot the generated gaussian signal
xlabel('t');ylabel('x(t)');title('gaussian signal');
```

OUTPUT

The output waveforms of program 2.1 are shown in fig P2.1.

Program 2.2

Write a MATLAB program to find the even and odd parts of the signal $x(t)=e^{2t}$.

```
%To find the even and odd parts of the signal, x(t)=exp(2*t)

tmin=-3; tmax=3; dt=.1;
t=tmin:dt:tmax;     %set a time vector

x1=exp(2*t);        %generate the given signal
x2=exp(-2*t);       %generate the time folded signal

if(x2==x1)
     disp('"The given signal is even signal"');
else if (x2==(-x1))
     disp('"The given signal is odd signal"');
else
     disp('"The given signal is neither even nor odd signal"');
end
end

xe=(x1+x2)/2;        %compute even part
xo=(x1-x2)/2;        %compute odd part

ymin=min([min(x1), min(x2), min(xe), min(xo)]);
ymax=max([max(x1), max(x2), max(xe), max(xo)]);

subplot(2,2,1);plot(t,x1);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x1(t)');title('signal x(t)');

subplot(2,2,2);plot(t,x2);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x2(t)');title('signal x(-t)');

subplot(2,2,3);plot(t,xe);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('xe(t)');title('even part of x(t)');

subplot(2,2,4);plot(t,xo);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('xo(t)');title('odd part of x(t)');
```

OUTPUT

"The given signal is neither even nor odd signal"

The input and output waveforms of the program 2.2 are shown in fig P2.2.

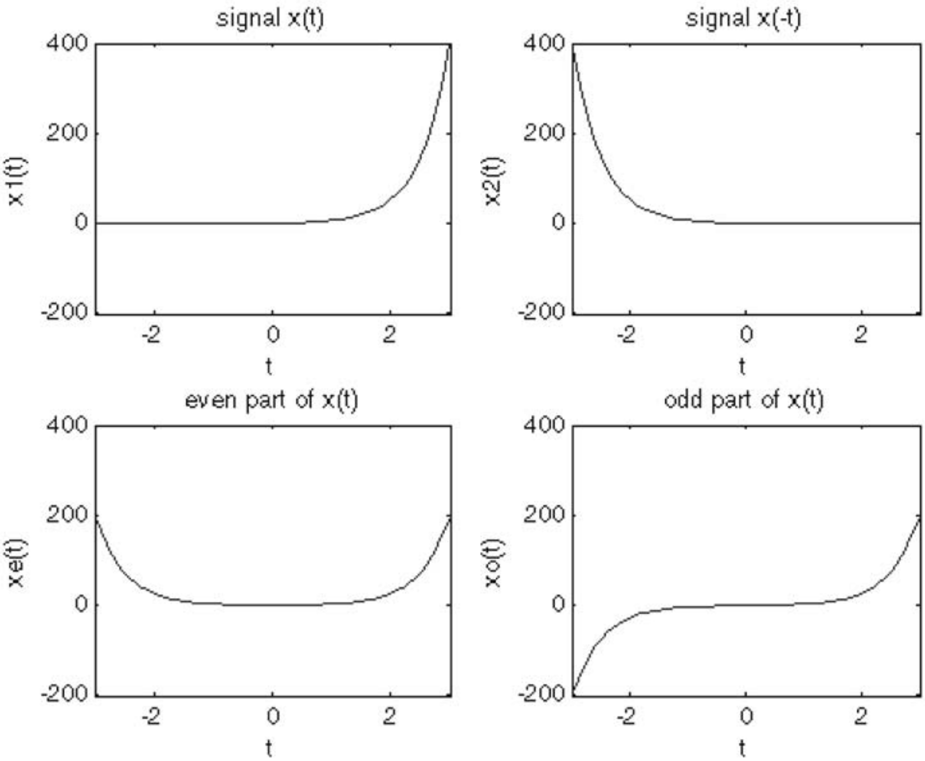


Fig P2.2 : *Input and Output waveforms of program 2.2.*

Program 2.3

Write a MATLAB program to find the energy and power of the signal x(t)=10sin(10πt).

```
%Program to compute the signal energy and power of the signal
%x(t)=10sin(10*pi*t)

tmax=10; dt=.01; T0=10;
t=-tmax:dt:tmax;   %set up a time vector

x=10*sin(10*pi*t); %generate the given signal x(t)
xsq=x.^2;          %compute the square of the given signal x(t)
E = trapz(t,xsq);  %use trapezoidal-rule numerical integration to
                   %find energy
P = E /(2*T0);     %divide the energy by period to compute power

disp([' Energy, E  =',num2str(E),' Joules']);
disp([' Power, P =',num2str(P),' Watts']);
```

OUTPUT

```
Energy, E  = 1000 Joules
Power,  P  = 50 Watts
```

Program 2.4

Write a MATLAB program to perform Amplitude scaling, Time scaling, and Time shift on the signal x(t) = 1+t; for t = 0 to 2.

Program to declare the given signal as function y(t)

```
% declare the given signal as function y(t)
function x = y(t)
x=(1.0 + t).*(t>=0 & t<=2);
```

> *Note: The above program should be stored as a separate file in the current working directory*

Program to perform amplitude & time scaling and time shift on y(t)

```
%To perform Amplitude scaling, Time scaling and Time shift
%on the signal x(t)=1.0+t; for t= 0 to 2
%include y.m file in work directory which declare the given signal
%as function y(t)

tmin=-3; tmax=5; dt=0.2;
t=tmin:dt:tmax;     %set a time vector

y0 =y(t);           %assign the given signal as y0
y1 =1.5*y(t);       %compute the amplified version of x(t)
y2 =0.5*y(t);       %compute the attenuated version of x(t)
y3=y(2*t);          %compute the time compressed version of x(t)
y4=y(0.5*t);        %compute the time expanded version of x(t)
y5=y(t-2);          %compute the delayed version of x(t)
y6=y(t+2);          %compute the advanced version of x(t)

%compute the min and max value for y-axis
ymin=min([min(y0), min(y1), min(y2), min(y3), min(y4),
min(y5),min(y6)]);
ymax=max([max(y0), max(y1), max(y2), max(y3), max(y4),
max(y5),max(y6)]);

%plot the given signal and amplitude scaled signal
subplot(3,3,1);plot(t,y0);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x(t)');title('Signal x(t)');
subplot(3,3,2);plot(t,y1);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x1(t)');title('Amplified signal 1.5x(t)');
subplot(3,3,3);plot(t,y2);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x2(t)');title('Attenuated signal 0.5x(t)');

%plot the given signal and time scaled signal
subplot(3,3,4);plot(t,y0);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x(t)');title('Signal x(t)');
subplot(3,3,5);plot(t,y3);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x3(t)');title('Time comp. signal x(2t)');
subplot(3,3,6);plot(t,y4);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x4(t)');title('Time expan. signal x(0.5t)');

%plot the given signal and time shifted signal
subplot(3,3,7);plot(t,y0);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x(t)');title('Signal x(t)');
subplot(3,3,8);plot(t,y5);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x5(t)');title('Delayed signal x(t-2)');
subplot(3,3,9);plot(t,y6);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x6(t)');title('Advanced signal x(t+2)');
```

OUTPUT

The input and output waveforms of program 2.4 are shown in fig P2.4.

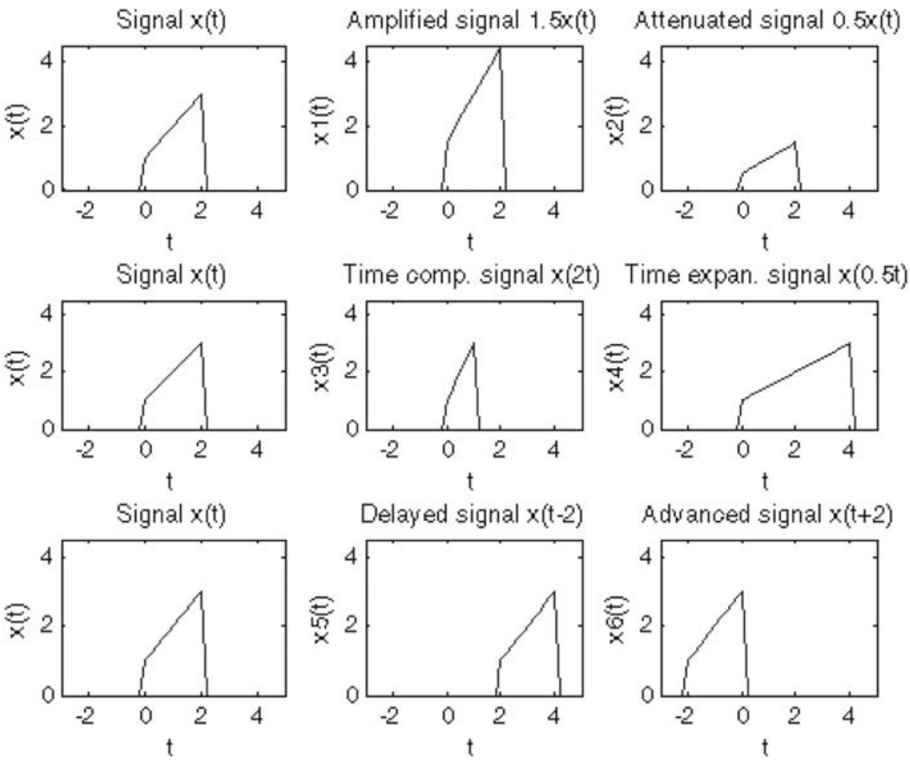


Fig P2.4 : Input and Output waveforms of program 2.4.

Program 2.5

Write a MATLAB program to perform addition and multiplication on the following two signals.

xa(t)=1; 0<t<1   xb(t)=t; 0<t<1
=2; 1<t<2   =1; 1<t<2
=1; 2<t<3   =3–t; 2<t<3

```
%To perform addition and multiplication of the following two signals
%1) xa(t)=1;0<t<1            2) xb(t)=t;0<t<1
%        =2;1<t<2                      =1;1<t<2
%        =1;2<t<3                      =3-t;2<t<3

tmin=-1; tmax=5; dt=0.1;
t=tmin:dt:tmax;    %Set a time vector

x1=1;
x2=2;
x3=3-t;
xa=x1.*(t>0&t<1)+x2.*(t>=1&t<=2)+x1.*(t>2&t<3);
xb=t.*(t>0&t<1)+x1.*(t>=1&t<=2)+x3.*(t>2&t<3);
xadd=xa+xb;          %Add the two signals
xmul=xa.*xb;         %Multiply two signals

xmin=min([min(xa), min(xb), min(xadd), min(xmul)]);
xmax=max([max(xa), max(xb), max(xadd), max(xmul)]);
subplot(2,3,1);plot(t,xa);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xa(t)');title('Signal xa(t)');
subplot(2,3,2);plot(t,xb);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xb(t)');title('Signal xb(t)');
```

```
subplot(2,3,3);plot(t,xadd);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xadd(t)');title('Sum of xa(t) and xb(t)');
subplot(2,3,4);plot(t,xa);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xa(t)');title('Signal xa(t)');

subplot(2,3,5);plot(t,xb);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xb(t)');title('Signal xb(t)');
subplot(2,3,6);plot(t,xmul);axis([tmin tmax xmin xmax]);
xlabel('t');ylabel('xmul(t)');title('Product of xa(t) and xb(t)');
```

OUTPUT

The input and output waveforms of program 2.5 are shown in fig P2.5.

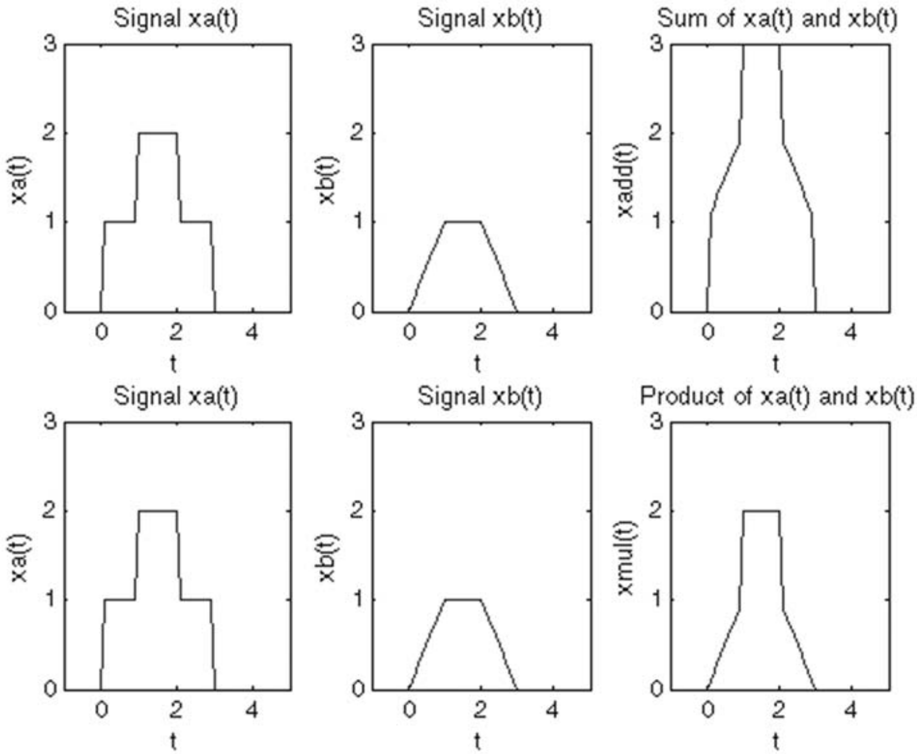


Fig P2.5 : *Input and Output waveforms of program 2.5.*

Program 2.6

Write a MATLAB program to perform convolution of the following two signals.

x1(t)=1; 1<t<10 x2(t)=1; 2<t<10

```
%*****************Program to perform convolution of two signals
%*****************x1(t)=1; t= 1 to 10 and x2(t)=1; t= 2 to 10

tmin=0; tmax=10; dt=0.01;
t=tmin:dt:tmax;                %set time vector for given signal

x1=1.*(t>=1 & t<=10);          %generate signal x1(t)
x2=1.*(t>=2 & t<=10);          %generate signal x2(t)
```

```
x3=conv(x1,x2);         %perform convolution of signals x1(t) and x2(t)

n3=length(x3);
t1=0:1:n3-1;             %set time vector for x3(t) signal

subplot(3,1,1);plot(t,x1);
xlabel('t');ylabel('x1(t)');
title('signal x1(t)');

subplot(3,1,2);plot(t,x2);
xlabel('t');ylabel('x2(t)');
title('signal x2(t)');

subplot(3,1,3);plot(t1,x3); xlim ([0 600]);
xlabel('t / dt');ylabel('x3(t) / dt');
title('signal, x3(t) = x1(t)* x2(t)');
```

OUTPUT

The input and output waveforms of program 2.6 are shown in fig P2.6.

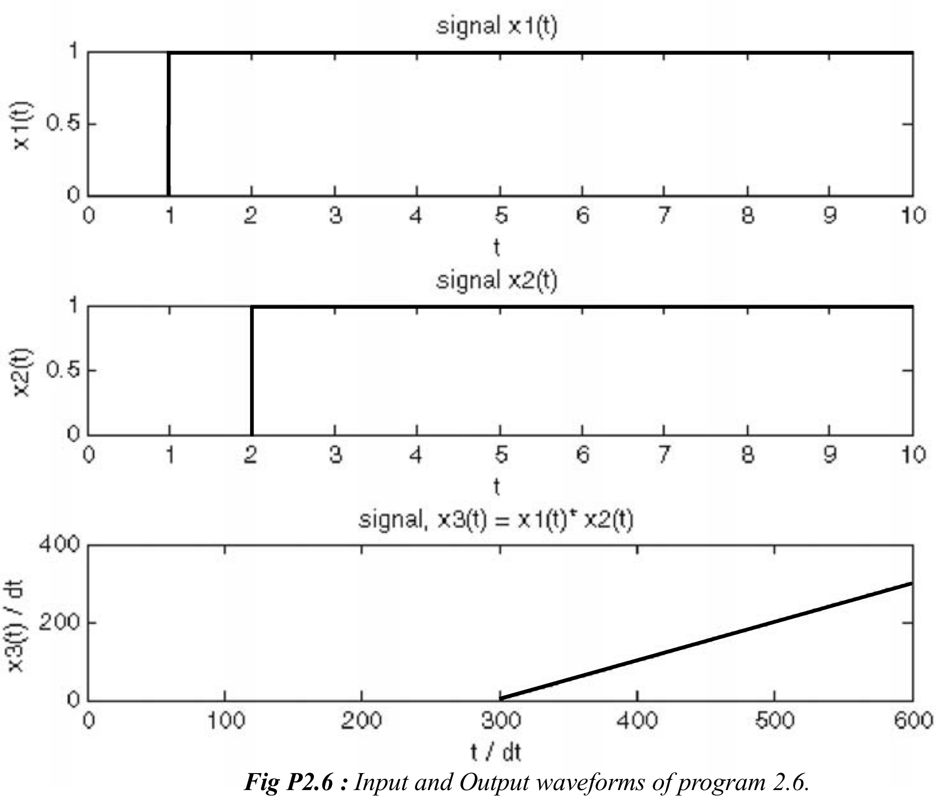


Fig P2.6 : *Input and Output waveforms of program 2.6.*

Program 2.7

Write a MATLAB program to perform convolution of the following two signals.

```
x1(t)=1; 0<t<2          x2(t)= 1;   0<t<1
                             =-1;   1<t<2
```

```
%*****************Program to perform convolution of two signals
%*****************x1(t)=1; t= 0 to 2 and x2(t)= 1; for t= 0 to 1
%                                            = -1; for  t= 1 to 2
tmin=0; tmax=4; dt=0.01;
t=tmin:dt:tmax;          %set time vector for given signal

x1=1.*(t>=0 & t<=2);     %generate signal x1(t)
xa=1;
xb=-1;
x2=xa.*(t>=0 & t<=1)+ xb.*(t>=1 & t<=2); % generate signal x2(t)
x3=conv(x1,x2);                          % perform convolution of
                                         % x1(t) & x2(t)

n3=length(x3);
t1=0:1:n3-1;             %set time vector for signal x3(t)

subplot(3,1,1);plot(t,x1);
xlabel('t');ylabel('x1(t)');title('signal x1(t)');

subplot(3,1,2);plot(t,x2);
xlabel('t');ylabel('x2(t)');title('signal x2(t)');
```

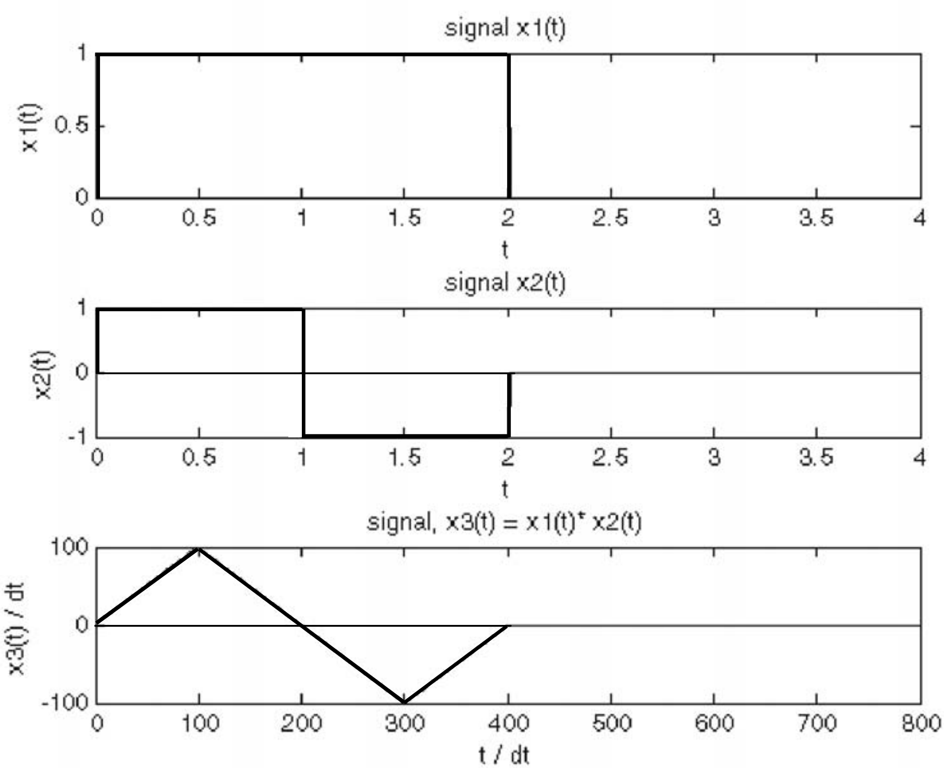


***Fig P2.7 :** Input and Output waveforms of program 2.7.*

```
subplot(3,1,3);plot(t1,x3);
xlabel('t / dt');ylabel('x3(t) / dt');
title('signal, x3(t) = x1(t)* x2(t)');
```

OUTPUT

The input and output waveforms of program 2.7 are shown in fig P2.7.

Program 2.8

Given that, $x1(t) = e^{-0.7t}$; 0<t<1 and x2(t)=1; 0<t<2
Write a MATLAB program to perform following operations.
1. Convolution of x1(t) and x2(t).
2. Deconvolution of output of convolution with x2(t) to extract x1(t).
3. Deconvolution of output of convolution with x1(t) to extract x2(t).

```
%***************** Program to perform convolution and deconvolution
%***************** x1(t)=exp(-0.7t);0<=t<=T and x2(t)=1;0<=t<=T
T=2;
tmin = 0; tmax=2*T; dt=0.01;
t=tmin:dt:tmax;                    %set time vector for given signals

x1=exp(-0.7*t).*(t>=0 & t<T);     %generate signal x1(t)
x2=1.*(t>=0 & t<T);               %generate signal x2(t)
x3=conv(x1,x2);                    %perform convolution of x1(t) and x2(t)

n3=length(x3);
```

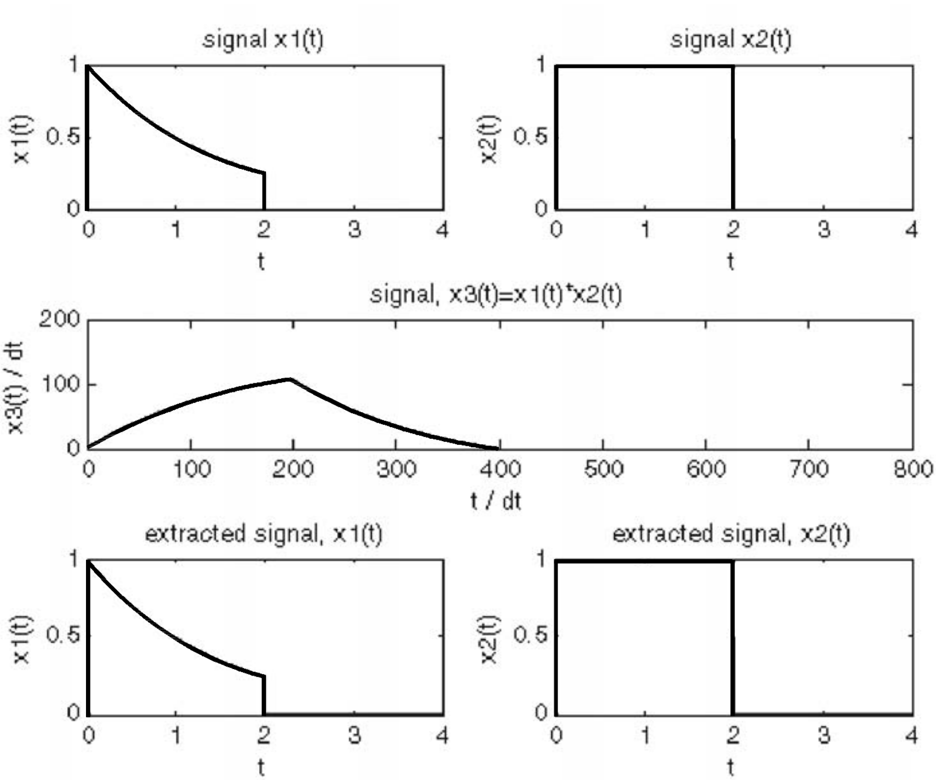


Fig P2.8 : *Input and Output waveforms of program 2.8.*

```
t1=0:1:n3-1;        %set time vector for x3(t) signal
%plot the given signals and the signal obtained by convolution
subplot(3,2,1);plot(t,x1);title('signal x1(t)');
xlabel('t');ylabel('x1(t)');
subplot(3,2,2);plot(t,x2);title('signal x2(t)');
xlabel('t');ylabel('x2(t)');
subplot(3,2,3:4);plot(t1,x3);title('signal, x3(t)=x1(t)*x2(t)');
xlabel('t / dt');ylabel('x3(t) / dt');

x1d=deconv(x3,x2); %perform deconvolution to extract x1(t)
x2d=deconv(x3,x1); %perform deconvolution to extract x2(t)

ymin=min([min(x1d), min(x2d)]);
ymax=max([max(x1d), max(x2d)]);

%plot the signals obtained by deconvolution
subplot(3,2,5);plot(t,x1d);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x1(t)');title('extracted signal, x1(t)');
subplot(3,2,6);plot(t,x2d);axis([tmin tmax ymin ymax]);
xlabel('t');ylabel('x2(t)');title('extracted signal, x2(t)');
```

OUTPUT

The input and output waveforms of program 2.8 are shown in fig P2.8.

2.14 Exercises

I. Fill in the blanks with appropriate words

1. The ———— signal is continuous function of an independent variable.
2. The ———— signal can be represented by a rotating vector in a complex plane.
3. The sum of two periodic signals will also be periodic if the ratio of their fundamental periods is a ———— number.
4. When a periodic signal is a sum of two or more periodic signals then the period of the signal is given by ———— of the period of its components.
5. When a signal exhibits ———— with respect to $t \geq 0$ it is called even signal.
6. When a signal exhibits antisymmetry with respect to $t = 0$ it is called ———— signal.
7. The periodic signals will have constant ————.
8. The nonperiodic signals will have constant ————.
9. The ———— signals are defined for $t \geq 0$.
10. When the sign of t is changed in x(t) the resultant signal is called ———— signal.
11. The ———— is the reverse process of differentiation.
12. A signal with large magnitude and short duration is called ————.
13 The response of a system for impulse input is called ————.
14. The ———— system is governed by constant coefficient differential equation.
15. The ———— response is the response of the system due to initial conditions alone.
16. The ———— response is the response of the system due to input alone.
17. The systems that do not require memory are called ———— systems.
18. A system is said to be ———— if its input-output characteristics do not change with time.
19. A ———— system is one that satisfies the superposition principle.
20. A ———— system is one in which the response does not depend on future inputs/outputs.
21. When the output of a system is finite for any finite input then the system is called ———— stable.
22. For ———— of LTI system the integral of impulse response should be finite.
23. In ———— system the present output depends on past outputs.

24. The response of an LTI system is given by ———————— of input and impulse response.
25. In —————————— connection the output of a system becomes input for another system.

Answers

1. analog	9. causal	17. static
2. complex exponential	10. folded	18. time invariant
3. rational	11. integration	19. linear
4. LCM (Least Common Multiple)	12. impulse	20. causal
5. symmetry	13. impulse response	21. BIBO
6. odd	14. LTI	22. stability
7. power	15. natural or free or zero input	23. feedback
8. energy	16. forced or zero state	24. convolution
		25. cascade

II. State whether the following statements are True/False

1. In an analog signal, both the magnitude of the signal and the independent variable are continuous.
2. A complex exponential signal is a three dimensional signal.
3. The sinc signal is a periodic signal.
4. The sinusoidal and complex exponential signals are always periodic.
5. The product of two odd signals will be an odd signal.
6. When the energy of a signal is finite then the power will be infinite.
7. When the power of a signal is finite then the energy will be zero.
8. The causal signals are defined for all t.
9. The time shift of a signal x(t) is obtained by replacing t by $t \pm m$.
10. The rate of change of a signal is called differentiation.
11. An impulse signal has infinite magnitude but constant area.
12. A system which satisfies the property of linearity and time invariance is called an LTI system.
13. For a constant coefficient differential equations, the homogeneous solution will be always exponential.
14. For a constant coefficient differential equation, the particular solution and input will be the same type of signal.
15. The systems that require memory are called dynamic systems.
16. In a time invariant system, if a delay is introduced either at input or output, then the overall response will remain same.
17. A nonlinear system is one which satisfies the principle of superposition.
18. The noncausal systems can be realised in real time.
19. The response of a noncausal system depends only on past inputs.
20. All LTI systems are BIBO stable.
21. The stability of non-LTI systems can be tested by using its impulse response.
22. The nonfeedback systems depend on past inputs.
23. Convolution is a linear operation.
24. The convolution can be used to determine the response of non-LTI systems.
25. The convolution operation satisfies commutative property.

Answers

1. True	6. False	11. True	16. True	21. False
2. True	7. False	12. True	17. False	22. True
3. False	8. False	13. True	18. False	23. True
4. True	9. True	14. True	19. False	24. False
5. False	10. True	15. True	20. False	25. True

III. Choose the right answer for the following questions

1. ***The unit impulse is defined as,***

a) $\delta(t) = \infty; t = 0$ **b)** $\delta(t) = \infty; t = 0$; $= 0; t \neq 0$ **c)** $\delta(t) = \infty;\ t = 0$ and $\int_{-\infty}^{+\infty} \delta(t)\, dt = A$ **d)** $\delta(t) = \infty;\ t = 0$; $= 0\ ;\ t \neq 0$ and $\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$

2. ***Which of the following is a periodic signal?***

a) $x(t) = A\,u(t)$ **b)** $x(t) = A\,e^{-jbt}$ **c)** $x(t) = A\,e^{bt}$ **d)** $x(t) = A\,t$

3. ***Which of the following is a nonperiodic signal?***

a) $x(t) = A\,e^{-j\sqrt{b}t}$ **b)** $x(t) = A\,e^{-j\frac{\pi t}{b}}$ **c)** $x(t) = A\,e^{bt}$ **d)** $x(t) = A\,e^{-jb\pi t}$

4. ***Which of the following statements is false?***

i) the product of two odd signals is an odd signal. ii) the product of two even signals is an even signal.
iii) the product of even and odd signals is an even signal. iv) the product of even and odd signals is an odd signal.

a) i and iii **b)** i only **c)** iii only **d)** iv only

5. ***Which of the following is a purely even signal?***

a) $x(t) = e^{at}$ **b)** $x(t) = \cos\Omega_0 t$ **c)** $x(t) = e^{j\Omega_0 t}$ **d)** $x(t) = u(t)$

6. ***Which of the following is a purely odd signal?***

a) $x(t) = e^{jbt}$ **b)** $x(t) = Ae^{bt}$ **c)** $x(t) = A\,\sin n\Omega_0 t$ **d)** $x(t) = t^2$

7. ***Which of the following is an energy signal?***

a) $x(t) = Ae^{j\Omega_0 t}$ **b)** $x(t) = A\sin\Omega_0 t$ **c)** $x(t) = B\,\cos\Omega_0 t$ **d)** $x(t) = e^{-at}\,u(t)$

8. ***Which of the following statements are true?***

i) the periodic signals are power signals. ii) the nonperiodic signals are energy signals.
iii) for energy signals, the power is zero. iv) for power signals, the energy is zero.

a) i only **b)** i and ii only **c)** i, ii and iii only **d)** all of the above

9. ***Which of the following signal is a causal signal?***

a) $x(t) = A$ **b)** $x(t) = t$ **c)** $x(t) = u(t)$ **d)** $x(t) = e^{-at}$

10. ***Which of the following represents the pulse signal shown in fig 10.?***

a) $u(t + 0.5) - u(t - 0.5)$
b) $u(t - 0.5) + u(t + 0.5)$
c) $u(t - 0.5) - u(t + 0.5)$
d) $u(t + 0.5) + u(t - 0.5)$

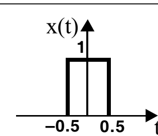


Fig 10.

11. ***The differentiation of a unit step signal is,***

a) ramp signal **b)** impulse signal **c)** exponential signal **d)** parabolic signal

12. ***The integral of a unit impulse signal is,***

a) infinity **b)** zero **c)** one **d)** constant

13. ***If x(t) is a continuous time signal and $\delta(t)$ is a unit impulse signal then,*** $\int_{-\infty}^{\infty} \mathbf{x(t)\,\delta(t - t_0)}$ ***is equal to,***

a) $x(t)$ **b)** $\delta(t)$ **c)** $x(t_0)$ **d)** $\delta(t_0)$

14. If x(t) is a continuous time signal and δ(t) is unit impulse signal then, $\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau - T)\,d\tau$ is equal to,

a) $x(t)$ **b)** $\delta(t)$ **c)** $x(t)\,\delta(t)$ **d)** $\delta(t-\tau)$

15. Which of the following responses of an LTI system does not depend on initial conditions?

a) natural response **b)** free response **c)** forced response **d)** total response

16. The homogeneous solution of the differential equation $\frac{dy(t)}{dt} + a\,y(t) = x(t)\ ;\ y(0) = b$, will be in the form,

a) Ce^{At} **b)** $C_1 + C_2e^{At}$ **c)** e^{At} **d)** $C_1 + C_2$

17. The total solution of the differential equation $\frac{dy(t)}{dt} + a\,y(t) = x(t)\ ;\ y(0) = b$, will be in the form,

a) $e^{At}\,x(t)$ **b)** $Ce^{At} + K\,x(t)$ **c)** $K_1 + K_2\,x(t)$ **d)** $e^{At} + x(t)$

18. Which of the following is not a static system?

a) $y(t) = x(t^2)$ **b)** $y(t) = x^2(t)$ **c)** $y(t) = t\,x(t)$ **d)** $y(t) = e^{x(t)}$

19. Which of the following is a time invariant system?

a) $y(t) = t\,x(t)$ **b)** $y(t) = e^{x(t)}$ **c)** $y(t) = x(-t)$ **d)** $y(t) = x(t^2)$

20. Which of the following is a linear system?

a) $y(t) = x(t^2)$ **b)** $y(t) = x^2(t)$ **c)** $y(t) = x(t) + C$ **d)** $y(t) = e^{x(t)}$

21. Which of the following is a nonlinear system?

a) $y(t) = t\,x(t)$ **b)** $y(t) = \int x(t)\,dt$ **c)** $y(t) = \frac{dx(t)}{dt}$ **d)** $y(t) = \sqrt{x(t)}$

22. Which of the following is a causal system?

a) $y(t) = x(t^2)$ **b)** $y(t) = x^2(t)$ **c)** $y(t) = x(-t)$ **d)** $y(t) = x(2t)$

23. Which of the following is a noncausal system?

a) $y(t) = \frac{dx(t)}{dt}$ **b)** $y(t) = \int_0^t x(\lambda)\,d\lambda$ **c)** $y(t) = e^{x(t)}$ **d)** $y(t) = x(t^2)$

24. Which of the following statements are true ?

i) an LTI system is always stable.

ii) an LTI system is stable only if the integral of its impulse response is finite.

iii) in a system, if the input is bounded then the output is always bounded.

iv) in a system, even if the input is unbounded the output can be bounded

a) ii only **b)** ii and iv only **c)** iii only **d)** i and iv only

25. Which of the following is a stable system?

a) $y(t) = t\,x(t)$ **b)** $y(t) = t^2x(t)$ **c)** $y(t) = e^t x(t)$ **d)** $y(t) = e^{-t}x(t)$

26. Which of the following impulse responses of LTI systems represents an unstable system?

a) $h(t) = e^{at}\,u(t)$ **b)** $h(t) = e^{-at}\,u(t)$ **c)** $h(t) = t\,e^{-t}\,u(t)$ **d)** $h(t) = e^{-t}\sin t\,u(t)$

27. Which of the following impulse responses of LTI systems represents a stable system?

a) $h(t) = e^t\cos t\,u(t)$ **b)** $h(t) = e^t\sin t\,u(t)$ **c)** $h(t) = e^{-t}\cos t\,u(t)$ **d)** $h(t) = t\sin t\,u(t)$

28. Which of the following represents the convolution of two causal signals $x_1(t)$ and $x_2(t)$?

i) $\int_{\lambda=0}^{t} x_1(\lambda)x_2(t-\lambda)\,d\lambda$ ii) $\int_{\lambda=0}^{t} x_2(\lambda)x_1(t-\lambda)\,d\lambda$ iii) $\int_{t=0}^{\lambda} x_1(t)x_2(\lambda-t)\,dt$ iv) $\int_{t=0}^{\lambda} x_2(t)x_1(\lambda-t)\,dt$

a) i only **b)** ii only **c)** i and ii only **d)** all of the above

29. If $h_1(t)$ and $h_2(t)$ are impulse responses of two stable LTI systems in cascade, then overall impulse response of the cascaded system is,

a) $h_1(t) * h_2(t)$ **b)** $h_1(t)\ h_2(t)$ **c)** $h_1(t) + h_2(t)$ **d)** $h_1(t) - h_2(t)$

30. If the impulse response of an LTI causal system is in the form e^{-at} then its response for step input of value A will be in the form,

a) $A\left(1-e^{-at}\right)u(t)$ **b)** $Ae^{-at}u(t)$ **c)** $\dfrac{A}{a}\left(1-e^{-at}\right)u(t)$ **d)** $\left(A-e^{-at}\right)u(t)$

Answers

1. d	5. b	9. c	13. c	17. b	21. d	25. d	29. a
2. b	6. c	10. a	14. a	18. a	22. b	26. a	30. c
3. c	7. d	11. b	15. c	19. b	23. d	27. c	
4. a	8. c	12. c	16. a	20. a	24. b	28. d	

IV. Answer the following questions

1. Define and sketch impulse and unit impulse signal.
2. Define and sketch step and unit step signal.
3. Define and sketch ramp signal.
4. Define and sketch sinc signal.
5. Define and sketch signum signal.
6. What are the various ways of classifying the continuous time signals?
7. Define deterministic and nondeterministic signals. Give examples.
8. Define periodic and nonperiodic signals. Give examples.
9. Define even and odd signals. Give examples.
10. Prove that the even part of a signal is given by $x_e(t)=\frac{1}{2}\left[x(t)+x(-t)\right]$ and the odd part of a signal is given by $x_o(t)=\frac{1}{2}\left[x(t)-x(-t)\right]$.
11. Define energy of a continuous time signal. Give examples.
12. Define power of a continuous time signal. Give examples.
13. Prove that power of an energy signal is zero.
14. Prove that energy of a power signal is infinite.
15. Define causal and noncausal signals. Give examples.
16. List the properties of impulse signal.
17. Write any two properties of impulse signal and prove.
18. Show that any continuous time signal x(t) can be expressed as an integral of impulses.
19. What is homogenous and particular solution?
20. What is free and forced response?
21. What is zero-input and zero-state response?
22. Define static and dynamic systems. Give examples.
23. Explain the time invariant property of a system.
24. Define linear systems.
25. Define causal and noncausal systems. Give examples.
26. Define stable and unstable systems.

27. What is the condition for stability of an LTI system?
28. Define feedback and nonfeedback systems.
29. Define convolution of two continuous time signals.
30. Show that the response of an LTI system can be obtained by convolution of input and impluse response?
31. List the properties of convolution.
32. State and prove the associative and commutative properties of convolution.
33. State and prove the distributive property of convolution.
34. What are the two ways of interconnection of LTI systems?
35. What is the relation between impulse response and unit step response of an LTI system?

V. Solve the following problems

E2.1 Verify whether the following signals are periodic. If periodic find the fundamental period.

a) $x(t)=4\sin 7t$ **b)** $x(t)=2e^{0.7t}$ c) $x(t)=3e^{-j0.1\pi t}$ **d)** $x(t)=9\sin\left(6t+\frac{\pi}{3}\right)$ e) $x(t)=\sin^2\left(3t-\frac{\pi}{5}\right)$

E2.2 Determine the periodicity of the following continuous time signals.

a) $x(t)=4\sin\frac{2\pi t}{7}+5\sin\frac{2\pi t}{9}$ **b)** $x(t)=0.5\,e^{-j\frac{2\pi t}{5}}+0.7\cos 8\pi t$ c) $x(t)=6\sin 5t+3\cos 4\pi t$

E2.3 *Determine the even part and odd part of the following continuous time signals.*

a) $x(t)=e^{j2t}$ **b)** $x(t)=4+e^{3t}$ **c)** $x(t)=t+3t^2+\cos^2 t$ **d)** $x(t)=\sin^2 t+e^{j5\pi t}$

E2.4 Determine the energy and power of the following continuous time signals.

a) $x(t)=0.9\,e^{-3t}\,u(t)$ **b)** $x(t)=3\,e^{-j0.5\pi t}$ **c)** $x(t)=1.2\sin 7\Omega_{\circ}t$ **d)** $x(t)=t\,u(t)$

E2.5 Sketch the even and odd parts of the signals shown in fig E2.5.1, E2.5.2, E2.5.3, E2.5.4.

a)

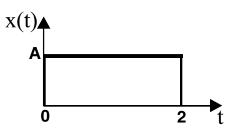


Fig E2.5.1.

b)

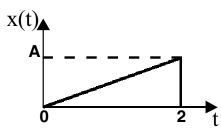


Fig E2.5.2.

c)

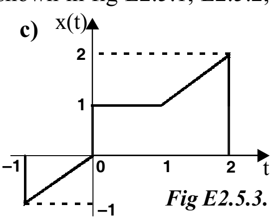


Fig E2.5.3.

d)

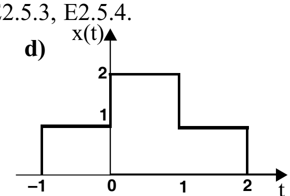


Fig E2.5.4.

E2.6 A continuous time signal is shown in fig E2.6
Determine the following versions of the signal.

a) $x(-t)$ **b)** $-x(t)$ **c)** $x(t+3)$

d) $x(-t+3)$ **e)** $x(t-3)$ **f)** $x(-t-3)$

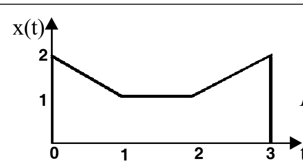


Fig E2.6.

E2.7 Evaluate the following integrals.

a) $\int_{-\infty}^{+\infty} e^{-0.5t}\delta(t-3)dt$ **b)** $\int_{-\infty}^{+\infty} e^{j0.1t}\delta(t-10)dt$ **c)** $\int_{-\infty}^{+\infty} t^3\delta(t+1)dt$

E2.8 Determine the natural response of the following systems.

a) $\frac{d^3y(t)}{dt^3}+8\frac{d^2y(t)}{dt^2}+17\frac{dy(t)}{dt}+10y(t)=\frac{dx(t)}{dt}+0.5x(t)\ ;\ y(0)=2,\left.\frac{dy(t)}{dt}\right|_{t=0}=-4\ ;\ \left.\frac{d^2y(t)}{dt^2}\right|_{t=0}=10$

b) $\frac{d^2y(t)}{dt^2}+1.6\frac{dy(t)}{dt}+0.63y(t)=0\ ;\ \left.\frac{dy(t)}{dt}\right|_{t=0}=-0.05,\ y(0)=0.02$

c) $\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+0.75y(t)=0\ ;\ y(0)=0.25,\ \left.\frac{dy(t)}{dt}\right|_{t=0}=-1$

E2.9 Determine the forced response of the following systems.

a) $\frac{d^3y(t)}{dt^3}+7\frac{d^2y(t)}{dt^2}+14\frac{dy(t)}{dt}+8y(t)=\frac{dx(t)}{dt}+0.5x(t)\ ;\ x(t)=e^{-3t}u(t)$

b) $\frac{d^2y(t)}{dt^2}+1.8\frac{dy(t)}{dt}+0.45y(t)=0.5\frac{dx(t)}{dt}+0.18x(t)\ ;\quad x(t)=u(t)$

c) $\frac{d^2y(t)}{dt^2}+0.7\frac{dy(t)}{dt}+0.1y(t)=0.4x(t)\ ;\ x(t)=0.2e^{-0.3t}u(t)$

E2.10 Determine the total response of the following systems.

a) $\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+0.21y(t)=\frac{dx(t)}{dt}+x(t)\ \ ;x(t)=0.2e^{-0.5t}u(t)\ ;\left.\frac{dy(t)}{dt}\right|_{t=0}=-1\ ;\ y(0)=0.3$

b) $\frac{d^2y(t)}{dt^2}+9\frac{dy(t)}{dt}+14y(t)=2x(t)\ \ ;x(t)=0.7u(t)\ ;\left.\frac{dy(t)}{dt}\right|_{t=0}=-0.5\ ;\ y(0)=0.2$

c) $\frac{d^2y(t)}{dt^2}+3.1\frac{dy(t)}{dt}+2.2y(t)=0.1x(t)\ \ ;x(t)=0.7e^{-2.5t}u(t)\ ;\left.\frac{dy(t)}{dt}\right|_{t=0}=-0.5\ ;\ y(0)=0$

E2.11 Verify whether the following systems are time invariant or time variant.

a) $y(t)=e^t\,x(t)$ **b)** $y(t)=\cos t\,x(t)$ **c)** $y(t)=x(-t+2)$

d) $y(t)=3\,x(t)+5$ **e)** $y(t)=3\frac{dx(t)}{dt}+7x(t)$ **f)** $y(t)=e^{x(t)}+\int x(t)\,dt$

E2.12 Verify whether the following systems are linear or nonlinear.

a) $y(t)=t^2x(t)$ **b)** $y(t)=\sqrt{x(t)}$ **c)** $y(t)=e^tx(t)$ **d)** $y(t)=\int x(t)\,dt$ **e)** $y(t)=t+x(t)$ **f)** $y(t)=\cos t\,x(t)$

E2.13 Determine the linearity of the LTI systems governed by the following differential equations.

a) $\frac{d^2y(t)}{dt^2}+0.3\frac{dy(t)}{dt}+0.5y(t)=2\,x(t)$ **b)** $y(t)=4\frac{d^2x(t)}{dt^2}+2\frac{dx(t)}{dt}+5\,x(t)$

E2.14 Verify whether the following systems are causal or noncausal.

a) $y(t)=e^t\,x(t)$ **b)** $y(t)=(t-2)\,u(t+2)$ **c)** $y(t)=(t+2)\,x(t)$

d) $y(t)=e^{x(t)}+\frac{dx(t)}{dt}$ **e)** $y(t)=\cos t\,x(t)$ **f)** $y(t)=\sin t\,x(t+2)$

E2.15 Verify whether the following systems are stable or unstable.

a) $y(t)=e^t\,x(t)$ **b)** $y(t)=e^{-t}\,x(t)$ **c)** $y(t)=t^2\,x(t)$

d) $y(t)=\sin t\,x(t)$ **e)** $y(t)=e^{-t}\sin t\,x(t)$ **f)** $y(t)=x(t+2)$

E2.16 Verify the stability of LTI systems whose impulse responses are given below.

a) $h(t)=t^2\,u(t)$ **b)** $h(t)=e^{-7t}\,u(t)$ **c)** $h(t)=e^{5t}\,u(t)$

d) $h(t)=t\sin t\,u(t)$ **e)** $h(t)=\left(A+Be^{-Ct}\right)u(t)$ **f)** $h(t)=2e^{-3t}\cos t\,u(t)$

E2.17 Determine the overall impulse response of the following system.

a)

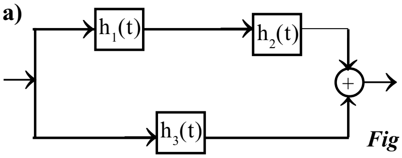


Fig E2.17a

$h_1(t)=e^{-6t}u(t);\quad h_2(t)=e^{2t}u(t);\quad h_3(t)=t\,u(t)$

b)

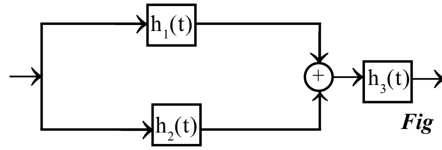


Fig E2.17b

$h_1(t)=\cos t\,u(t);\quad h_2(t)=\sin t\,u(t);\quad h_3(t)=u(t)$

E2.18 Perform the convolution of the following signals.

a) $x_1(t)=t^2u(t);\quad x_2(t)=u(t-1)$ **b)** $x_1(t)=te^{-4t}u(t);\quad x_2(t)=u(t)$

c) $x_1(t)=t\,u(t);\quad x_2(t)=\sin t\,u(t)$

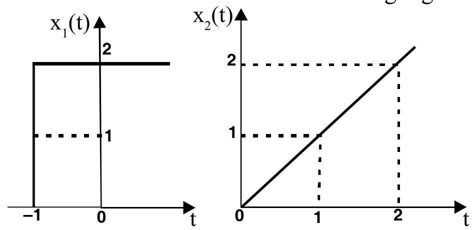
E2.19 Determine the unit step response of the following systems whose impulse responses are given below.

a) $h(t) = te^{-9t}u(t)$ **b)** $h(t) = e^{-t}\cos t\, u(t)$

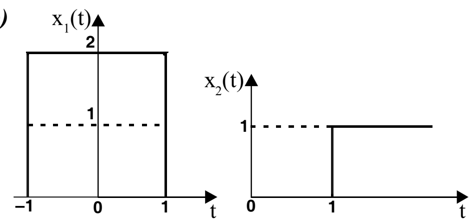
c) $h(t) = e^{-5t}u(t-2)$ **d)** $h(t) = u(t-3) + u(t+2)$

E2.20 Perform convolution of the following signals by graphical method and sketch the resultant waveform.

a)



b)



Answers

E2.1 **a)** $T = \frac{2\pi}{7}$ **b)** nonperiodic **c)** $T = 20$ **d)** $T = \frac{\pi}{3}$ **e)** $T = \frac{\pi}{3}$

E2.2) **a)** $T = 63$ **b)** $T = 5$ **c)** nonperiodic

E2.3 **a)** $x_e(t) = \cos 2t$; $x_o(t) = j\sin 2t$ **b)** $x_e(t) = 4 + \frac{1}{2}(e^{3t} + e^{-3t})$; $x_o(t) = \frac{1}{2}(e^{3t} - e^{-3t})$

c) $x_e(t) = 3t^2 + \cos^2 t$; $x_o(t) = t$ **d)** $x_e(t) = \sin^2 t + \cos 5\pi t$; $x_o(t) = j\sin 5\pi t$

E2.4 **a)** $E = 0.135$ joules, $P = 0$ **b)** $E = \infty$, $P = 9$ watts **c)** $E = \infty$, $P = 0.72$ watts **d)** $E = \infty$, $P = \infty$

E2.5 **a)**

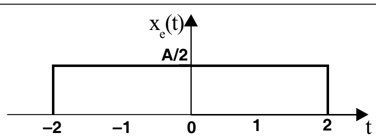


Fig E2.5a.1

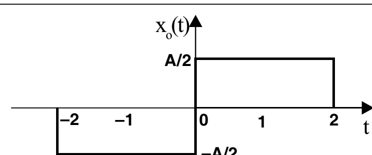


Fig E2.5a.2

b)

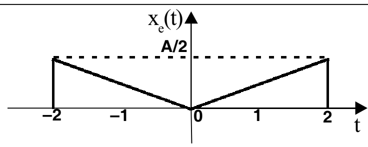


Fig E2.5b.1

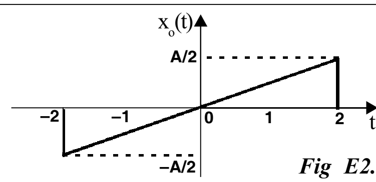


Fig E2.5b.2

c)

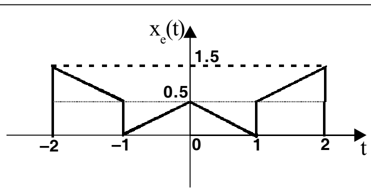


Fig E2.5c.1

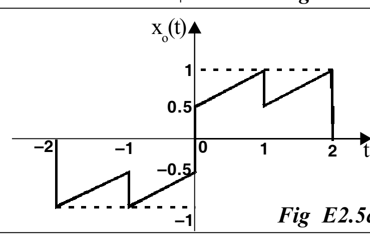


Fig E2.5c.2

d)

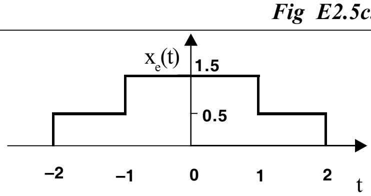


Fig E2.5d.1

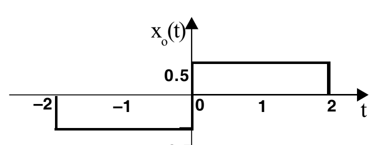


Fig E2.5d.2

E2.6 a)

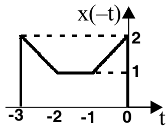


Fig E2.6a

b)

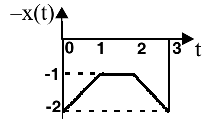


Fig E2.6b

c)

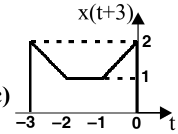


Fig E2.6c

d)

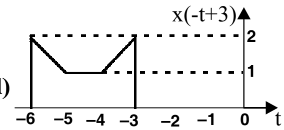


Fig E2.6d

e)

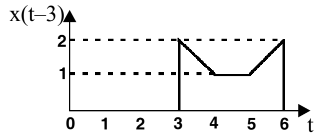


Fig E2.6e

f)

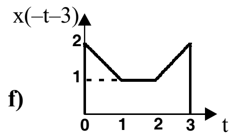


Fig E2.6f

E2.7 **a)** $e^{-1.5} = 0.2231$ **b)** $e^{j} = \cos 1 + j\sin 1 = 0.5403 + j0.8415$ **c)** $t^3\Big|_{t=-1} = -1$

(Note : Calculate in radians mode)

E2.8 **a)** $y_{zi}(t) = \left(\frac{1}{2}e^{-t} + \frac{4}{3}e^{-2t} + \frac{1}{6}e^{-5t}\right)u(t)$ **b)** $y_{zi}(t) = (-0.16e^{-0.7t} + 0.18e^{-0.9t})\,u(t)$

c) $y_{zi}(t) = (-0.625e^{-0.5t} + 0.875e^{-1.5t})\,u(t)$

E2.9 **a)** $y_{zs}(t) = \left(\frac{-5}{12}e^{-t} + \frac{5}{4}e^{-2t} + \frac{5}{12}e^{-4t} - \frac{5}{4}e^{-3t}\right)u(t)$ **b)** $y_{zs}(t) = (-0.5e^{-0.3t} + 0.1e^{-1.5t} + 0.4)\,u(t)$

c) $y_{zs}(t) = \left(\frac{8}{3}e^{-0.2t} + \frac{4}{3}e^{-0.5t} - 4e^{-0.3t}\right)u(t)$

E2.10 **a)** $y(t) = \left(-0.725e^{-0.2t} + 3.525e^{-0.7t} - 2.5e^{-0.5t}\right)u(t)$ **b)** $y(t) = \left(0.04e^{-2t} + 0.06e^{-7t} + 0.1\right)u(t)$

c) $y_{zs}(t) = (-0.5e^{-1.1t} + 0.4e^{-2t} + 0.1e^{-2.5t})\,u(t)$

E2.11 **a)** **b)** **c)** – Time variant **d)** **e)** **f)** – Time invariant	**E2.12)** **a)** **c)** **d)** **f)** – Linear **b)** **e)** – NonLinear
E2.13 **a)** **b)** – Linear	**E2.14)** **a)** **c)** **d)** **e)** – Causal **b)** **f)** – Noncausal
E2.15 **b)** **d)** **e)** **f)** – Stable **a)** **c)** – Unstable	**E2.16)** **b)** **f)** – stable **a)** **c)** **d)** **e)** – Unstable

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E2.17 **a)** $h_o(t) = \left[h_1(t) * h_2(t)\right] + h_3(t) = \left(\frac{1}{8}e^{2t} - \frac{1}{8}e^{-6t} + t\right)u(t)$

b) $h_o(t) = \left[h_1(t) + h_2(t)\right] * h_3(t) = (\sin t - \cos t + 1)\,u(t)$

E2.18 **a)** $\left(\frac{t^3 - 1}{3}\right)u(t-1)$ **b)** $\frac{1}{16}\left(1 - e^{-4t} - 4te^{-4t}\right)u(t)$

c) $(t + \sin t)\,u(t)$

E2.19 **a)** $s(t) = \frac{1}{81}\left(1 - e^{-9t} - 9te^{-9t}\right)u(t)$ **b)** $s(t) = \frac{1}{2}\left(1 + e^{-t}\sin t - e^{-t}\cos t\right)u(t)$

c) $s(t) = \frac{e^{-10}}{5}\left(1 - e^{-5(t-2)}\right)u(t-2)$ **d)** $s(t) = (t-3)\,u(t-3) + (t+2)\,u(t+2)$

E2.20 **a)** $x_1(t) * x_2(t) = (t+1)^2\,u(t+1)$

(or) $x_1(t) * x_2(t) = (t+1)^2 \;;\; t \ge -1$

b) $x_1(t) * x_2(t) = 2t - 2(t-2)\,u(t-2)$

(or) $x_1(t) * x_2(t) = 2t \;;\; t = 0$ to 2

$= 4 \;;\; t \ge 2$

CHAPTER 3

Laplace Transform

3.1 Introduction

The Laplace transform is used to transform a time signal to complex frequency domain. (The complex frequency domain is also known as Laplace domain or s-domain). This transformation was first proposed by Laplace (in the year 1780) and later adopted for various engineering applications for solving differential equations. Hence this transformation is called ***Laplace transform.***

In signals and systems the Laplace transform is used to transform a time domain system to s-domain. In time domain the equations governing a system will be in the form of differential equations. While transforming the system to s-domain, the differential equations are transformed to simple algebraic equations and so the analysis of systems will be much easier in s-domain.

In this chapter a brief discussion about Laplace transform and its applications for analysis of signals and systems are presented.

Complex Frequency

The ***complex frequency*** is defined as,

Complex frequency, $s = \sigma + j\Omega$

where, σ = ***Neper frequency*** in neper per second
Ω = ***Radian (or Real) frequency*** in radian per second

The complex frequency is involved in the time domain signal of the form Ke^{st}. The signal Ke^{st} can be thought of as an universal signal which represents all types of signals and takes a particular form for various choices of σ and Ω as shown below.

Let, $x(t) = A\,e^{st} = A\,e^{(\sigma + j\Omega)t}$ .....(3.1)

Let us analyse the signal of equation (3.1) for various choice of σ and Ω.

Case i : When $\sigma = 0,\ \Omega = \Omega_0$

On substituting $\sigma = 0$ in equation (3.1) we get,

$$\begin{aligned} x(t) &= A\,e^{j\Omega_0 t} \\ &= A\,(\cos\Omega_0 t + j\sin\Omega_0 t) \\ &= A\cos\Omega_0 t + j\,A\sin\Omega_0 t \end{aligned} \quad .....(3.2)$$

The real part of equation (3.2) represents a cosinusoidal signal and the imaginary part represents a sinusoidal signal.

i.e., $\text{Re}[x(t)] = A\cos\Omega_0 t$ ..... Cosinusoidal signal

$\text{Im}[x(t)] = A\sin\Omega_0 t$ ..... Sinusoidal signal

Note : *Re - stands for 'real part of'*
Im - stands for 'imaginary part of'

Case ii : When Ω = 0

On substituting $\Omega = 0$ in equation (3.1) we get,

$$x(t) = A\,e^{\sigma t} \qquad \text{....(3.3)}$$

In equation (3.3) if σ is positive then the signal will be an exponentially increasing signal.

In equation (3.3) if σ is negative then the signal will be an exponentially decreasing signal.

i.e., $x(t) = A\,e^{\sigma t}$ ..... Exponentially increasing signal

$x(t) = A\,e^{-\sigma t}$ ..... Exponentially decreasing signal

---

Case iii : When σ = 0 and Ω = 0

On substituting $\sigma = 0$ and $\Omega = 0$ in equation (3.1) we get,

$$x(t) = A\,e^{0} = A \qquad \text{.....(3.4)}$$

The equation (3.4) represents a step signal.

Case iv : When σ ≠ 0 , Ω ≠ 0 and Ω = Ω₀ .

When both σ and Ω are non-zero and when $\Omega = \Omega_0$, the equation (3.1) can be expressed as shown below.

$$\begin{aligned} x(t) &= A\,e^{(\sigma + j\Omega_0)t} = A\,e^{\sigma t}\;A\,e^{j\Omega_0 t} \\ &= A\,e^{\sigma t}(\cos\Omega_0 t + j\sin\Omega_0 t) \\ &= A\,e^{\sigma t}\cos\Omega_0 t + j\,A\,e^{\sigma t}\sin\Omega_0 t) \end{aligned} \qquad \text{.....(3.5)}$$

The real part of equation (3.5) represents an exponentially increasing/decreasing cosinusoidal signal.

The imaginary part of equation (3.5) represents an exponentially increasing/decreasing sinusoidal signal.

i.e., $\text{Re}[x(t)] = A\,e^{\sigma t}\cos\Omega_0 t$

$\longrightarrow A\,e^{\sigma t}\cos\Omega_0 t$ ..... Exponentially increasing cosinusoidal signal

$\longrightarrow A\,e^{-\sigma t}\cos\Omega_0 t$ ..... Exponentially decreasing cosinusoidal signal

$\text{Im}[x(t)] = A\,e^{\sigma t}\sin\Omega_0 t$

$\longrightarrow A\,e^{\sigma t}\sin\Omega_0 t$ ..... Exponentially increasing sinusoidal signal

$\longrightarrow A\,e^{-\sigma t}\sin\Omega_0 t$ ..... Exponentially decreasing sinusoidal signal

---

Complex Frequency Plane or s-Plane

The complex frequency is defined as,

Complex frequency, $s = \sigma + j\Omega$

where, σ = Real part of s

Ω = Imaginary part of s

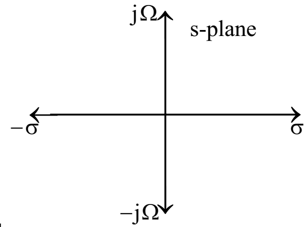


Fig 3.1: *Complex frequency plane or s-plane.*

The σ and Ω can take values from $-\infty$ to $+\infty$. A two dimensional complex plane with values of σ on horizontal axis and values Ω on vertical axis as shown in fig 3.1 is called ***complex frequency plane*** or ***s-plane.***

The ***s-plane*** is used to represent various critical frequencies (poles and zeros) of signals which are functions of s and to study the path taken by these critical frequencies when some parameters of the signals are varied. This study will be useful to design systems for a desired response.

Definition of Laplace Transform

In order to transform a time domain signal x(t) to s-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞. The transformed signal is represented as X(s) and the transformation is denoted by the script letter $\mathcal{L}$.

Symbolically the ***Laplace transform*** of x(t) is denoted as,

$$X(s) = \mathcal{L}\{x(t)\}$$

Let x(t) be a continuous time signal defined for all values of t. Let X(s) be Laplace transform of x(t). Now the ***Laplace transform*** of x(t) is defined as,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.6)$$

If x(t) is defined for $t \geq 0$, (i.e., if x(t) is causal) then,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.7)$$

The definition of Laplace transform as given by equation (3.6) is called ***Two sided Laplace transform*** or ***Bilateral Laplace Transform*** and the definition of Laplace transform as given by equation (3.7) is called ***One sided Laplace transform*** or ***Unilateral Laplace transform***.

Definition of Inverse Laplace Transform

The s-domain signal X(s) can be transformed to time domain signal x(t) by using inverse Laplace transform.

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds}$$

The signal x(t) and X(s) are called ***Laplace transform pair*** and can be expressed as,

$$\boxed{x(t) \underset{\mathcal{L}^{-1}}{\overset{\mathcal{L}}{\Longleftrightarrow}} X(s)}$$

Existence of Laplace Transform

The computation of Laplace transform involves integral of x(t) from $t = -\infty$ to $+\infty$. Therefore Laplace transform of a signal exists if the integral, $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges (i.e., finite). The integral will converge if the signal x(t) is sectionally continuous in every finite interval of t and if it is of exponential order as t approaches infinity.

A causal signal x(t) is said to be ***exponential order*** if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}|x(t)|$ approaches zero as t approaches infinity.

i.e., if, $\underset{t \to \infty}{\text{Lt}}\; e^{-\sigma t}\, |x(t)| = 0$, then x(t) is of exponential order.

For a causal signal, if $\underset{t \to \infty}{\text{Lt}}\; e^{-\sigma t}\, |x(t)| = 0$ for $\sigma > \sigma_c$, and if $\underset{t \to \infty}{\text{Lt}}\; e^{-\sigma t}|x(t)| = \infty$ for $\sigma < \sigma_c$, then σ_c is called ***abscissa of convergence***, (where σ_c is a point on real axis in s-plane).

The integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges only if the real part of s is greater than the abscissa of convergence σ_c. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence.

3.2 Region of Convergence

The Laplace transform of a signal is given by $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. The ROC for the following three types of signals are discussed here.

Case i : Right sided (causal) signal

Case ii : Left sided (anticausal) signal

Case iii : Two sided signal.

Case i : Right sided (causal) signal

Let, $x(t) = e^{-at} u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-at}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{e^{-(\sigma + j\Omega + a)\infty}}{-(s+a)} - \frac{e^0}{-(s+a)} = -\frac{e^{-(\sigma+a)\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

$$\boxed{\text{Put, } s = \sigma + j\Omega}$$

$$\therefore\ \mathcal{L}\{x(t)\} = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

where, $k = \sigma + a = \sigma - (-a)$

When $\sigma > -a$, $k = \sigma - (-a) = \text{Positive}$, $\therefore\ e^{-k\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) = \text{Negative}$, $\therefore\ e^{-k\infty} = e^{+\infty} = \infty$

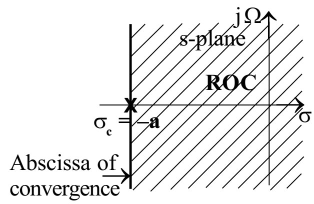


Fig 3.2 : ROC of $x(t) = e^{-at}\,u(t)$.

Hence we can say that, X(s) converges, when $\sigma > -a$, and does not converge for $\sigma < -a$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a$.

When $\sigma > -a$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = -\frac{0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence, $\sigma_c = -a$, as shown in fig 3.2.

<u>Case ii :</u> Left sided (anticausal) signal

Let, $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \le 0$, where $b > 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{+\infty} e^{-bt}\,u(-t)\,e^{-st}\,dt = \int_{-\infty}^{0} e^{-bt}\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-(s+b)t}\,dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma+j\Omega+b)\infty}}{-(s+b)}$$

$$= -\frac{1}{s+b} + \frac{e^{(\sigma+b)\times\infty}e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+b}$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

where, $k = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $k = \sigma - (-b) = \text{Positive}$, $\therefore\ e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b) = \text{Negative}$, $\therefore\ e^{k\infty} = e^{-\infty} = 0$

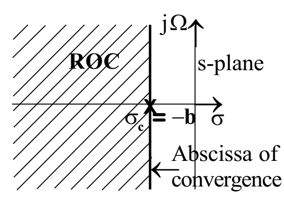


Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

When $\sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{1}{s+b} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{0\times e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal the ROC includes all points on the s-plane to the left of abscissa of convergence, $\sigma_c = -b$, as shown in fig 3.3.

Case iii: ***Two sided signal***

Let, $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$, where $a > 0$, $b > 0$, and $a > b$ (i.e., $-a < -b$)

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \left[e^{-at}\, u(t) + e^{-bt}\, u(-t)\right] e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt + \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt + \int_{-\infty}^{0} e^{-(s+b)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \left[\frac{e^{-(\sigma+j\Omega+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(\sigma+j\Omega+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \frac{e^{-(\sigma+j\Omega+a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)} + \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma+j\Omega+b)\infty}}{-(s+b)}$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

$$= -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

where, $p = \sigma + a = \sigma - (-a)$ and $q = \sigma + b = \sigma - (-b)$

When $\sigma > -a$, $p = \sigma - (-a) =$ Positive, $\therefore e^{-p\infty} = e^{-\infty} = 0$
When $\sigma < -a$, $p = \sigma - (-a) =$ Negative, $\therefore e^{-p\infty} = e^{+\infty} = \infty$

When $\sigma > -b$, $q = \sigma - (-b) =$ Positive, $\therefore e^{q\infty} = e^{\infty} = \infty$
When $\sigma < -b$, $q = \sigma - (-b) =$ Negative, $\therefore e^{q\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when σ lies between $-a$ and $-b$ (i.e., $-a < \sigma < -b$), and does not converge for $\sigma < -a$ and $\sigma > -b$.

$\therefore$ Abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$.

When $-a < \sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

$$= -\frac{0 \times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{0 \times e^{j\Omega\times\infty}}{s+b}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

Therefore for a two sided signal the ROC includes all points on the s-plane in the region in between two abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$, as shown in fig 3.4.

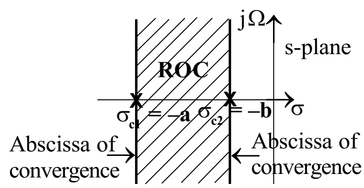


Fig 3.4 : *ROC of* $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$.

Example 3.1

Determine the Laplace transform of the following continuous time signals and their ROC.

a) $x(t) = A\,u(t)$ b) $x(t) = t\,u(t)$ c) $x(t) = e^{-3t}\,u(t)$ d) $x(t) = e^{-3t}\,u(-t)$ e) $x(t) = e^{-4|t|}$

Solution

a) Given that, $x(t) = A\,u(t) = A$; $t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_0^{\infty} A\,e^{-st}\,dt = A\int_0^{\infty} e^{-st}\,dt$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

$$= A\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)\infty}}{-s} - \frac{e^0}{-s}\right] = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > 0$.

When $\sigma > 0$, the X(s) is given by,

$$X(s) = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = A\left[\frac{0\times e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

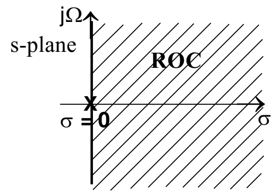


Fig 1 : ROC of $x(t) = A\,u(t)$.

$\therefore\ \mathcal{L}\{A\,u(t)\} = \dfrac{A}{s}$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.
(or ROC is right half of s-plane).

---

b) Given that, $x(t) = t\,u(t) = t$; $t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt = \int_0^{\infty} t\,e^{-st}\,dt$$

$\boxed{\int u\,v = u\int v - \int\left[du\int v\right]}$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1\times\frac{e^{-st}}{-s}\,dt = \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \left[t\,\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} - \left[\frac{e^{-(\sigma+j\Omega)t}}{s^2}\right]_0^{\infty}$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

$$= \left[\infty\times\frac{e^{-(\sigma+j\Omega)\infty}}{-s} - 0\times\frac{e^0}{-s} - \frac{e^{-(\sigma+j\Omega)\infty}}{s^2} + \frac{e^0}{s^2}\right]$$

$$= \left[\infty\times\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} - 0 - \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > 0$.

When $\sigma > 0$, the X(s) is given by,

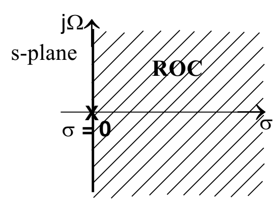


Fig 2 : ROC of $x(t) = t\,u(t)$.

$$X(s) = \left[\infty\times\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

$$= \left[\infty\times\frac{0\times e^{-j\Omega\times\infty}}{-s} - \frac{0\times e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

$$\therefore\ \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.

(or ROC is right half of s-plane).

c) Given that, $x(t) = e^{-3t}\,u(t) = e^{-3t}$; $t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{0}^{\infty} e^{-3t}\,e^{-st}\,dt = \int_{0}^{\infty} e^{-(s+3)t}\,dt$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^{\infty} = \frac{e^{-(s+3)\infty}}{-(s+3)} - \frac{e^0}{-(s+3)} = -\frac{e^{-(\sigma + j\Omega + 3)\infty}}{s+3} + \frac{1}{s+3}$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$= -\frac{e^{-(\sigma+3)\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3)$ = Positive. $\therefore e^{-k\infty} = e^{-\infty} = 0$
When, $\sigma < -3$, $k = \sigma - (-3)$ = Negative. $\therefore e^{-k\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > -3$.

When $\sigma > -3$, the X(s) is given by,

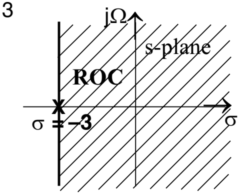


***Fig 3 :** ROC of $x(t) = e^{-3t}u(t)$.*

$$X(s) = -\frac{e^{-k\times\infty}\,e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{0\times e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = \frac{1}{s+3}$$

$$\therefore\ \mathcal{L}\left\{e^{-3t}\,u(t)\right\} = \frac{1}{s+3}$$; with ROC as all points in s-plane to the right of line passing through $\sigma = -3$.

d) Given that, $x(t) = e^{-3t}\,u(-t) = e^{-3t}$; $t \leq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{0} e^{-3t}\,e^{-st}\,dt = \int_{-\infty}^{0} e^{-(s+3)t}\,dt$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_{-\infty}^{0} = \frac{e^0}{-(s+3)} - \frac{e^{(s+3)\infty}}{-(s+3)} = -\frac{1}{s+3} + \frac{e^{(\sigma + j\Omega + 3)\infty}}{s+3}$$

$$= -\frac{1}{s+3} + \frac{e^{(\sigma+3)\times\infty}\,e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3)$ = Positive. $\therefore e^{k\infty} = e^{\infty} = \infty$
When, $\sigma < -3$, $k = \sigma - (-3)$ = Negative. $\therefore e^{k\infty} = e^{-\infty} = 0$

Therefore we can say that, X(s) converges when $\sigma < -3$.

When $\sigma < -3$, the X(s) is given by,

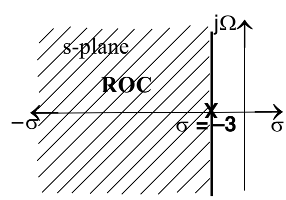


***Fig 4 :** ROC of $x(t) = e^{-3t}u(-t)$.*

$$X(s) = -\frac{1}{s+3} + \frac{e^{k\times\infty}\,e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3} + \frac{0\times e^{j\Omega\times\infty}}{s+3} = -\frac{1}{s+3}$$

$$\therefore\ \mathcal{L}\left\{e^{-3t}\,u(-t)\right\} = -\frac{1}{s+3}$$; with ROC as all points in s-plane to the left of line passing through $\sigma = -3$.

e) Given that, $x(t) = e^{-4|t|} = e^{4t}$; $t \le 0$
$= e^{-4t}$; $t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{4t}\, e^{-st}\, dt + \int_{0}^{\infty} e^{-4t} e^{-st}\, dt = \int_{-\infty}^{0} e^{-(s-4)t}\, dt + \int_{0}^{\infty} e^{-(s+4)t}\, dt$$

$$= \left[\frac{e^{-(s-4)t}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_{0}^{\infty} = \left[\frac{e^{0}}{-(s-4)} - \frac{e^{(s-4)\infty}}{-(s-4)}\right] + \left[\frac{e^{-(s+4)\infty}}{-(s+4)} - \frac{e^{0}}{-(s+4)}\right]$$

$$= -\frac{1}{s-4} + \frac{e^{(\sigma + j\Omega - 4)\infty}}{s-4} - \frac{e^{-(\sigma + j\Omega + 4)\infty}}{s+4} + \frac{1}{s+4}$$

$$\boxed{\text{Put, } s = \sigma + j\Omega}$$

$$= -\frac{1}{s-4} + \frac{e^{(\sigma - 4)\times\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-(\sigma + 4)\times\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

When, $\sigma < 4$, $\sigma - 4$ = Negative. $\quad \therefore e^{(\sigma-4)\infty} = e^{-\infty} = 0$
When, $\sigma > 4$, $\sigma - 4$ = Positive. $\quad \therefore e^{(\sigma-4)\infty} = e^{\infty} = \infty$

When, $\sigma < -4$, $\sigma + 4$ = Negative. $\quad \therefore e^{-(\sigma+4)\infty} = e^{\infty} = \infty$
When, $\sigma > -4$, $\sigma + 4$ = Positive. $\quad \therefore e^{-(\sigma+4)\infty} = e^{-\infty} = 0$

Therefore we can say that, X(s) converges when σ lies between -4 and $+4$.

When σ lies between -4 and $+4$, the X(s) is given by,

$$X(s) = -\frac{1}{s-4} + \frac{e^{(\sigma - 4)\times\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-(\sigma + 4)\times\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + \frac{e^{-\infty}\, e^{j\Omega\times\infty}}{s-4} - \frac{e^{-\infty}\, e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + \frac{0 \times e^{j\Omega\times\infty}}{s-4} - \frac{0 \times e^{-j\Omega\times\infty}}{s+4} + \frac{1}{s+4}$$

$$= -\frac{1}{s-4} + 0 - 0 + \frac{1}{s+4}$$

$$= \frac{1}{s+4} - \frac{1}{s-4} = \frac{s - 4 - (s+4)}{(s+4)(s-4)} = -\frac{8}{s^2 - 16}$$

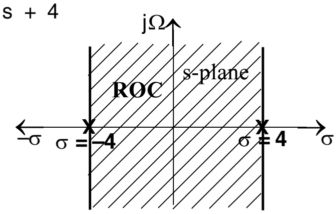


***Fig 5 :** ROC of $x(t) = e^{-4|t|}$.*

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

$$\therefore \mathcal{L}\left\{e^{-4|t|}\right\} = -\frac{8}{s^2 - 16}$$; with ROC as all points in s-plane in between the lines passing through $\sigma = -4$ and $\sigma = 4$.

Example 3.2

Determine the Laplace transform of the following signals.

a) $x(t) = \sin\Omega_0 t\, u(t)$ b) $x(t) = \cos\Omega_0 t\, u(t)$ c) $x(t) = \cosh\Omega_0 t\, u(t)$ d) $x(t) = e^{-a} \sin\Omega_0 t\, u(t)$ e) $x(t) = e^{-at} \cos\Omega_0 t\, u(t)$

Solution

a) Given that, $x(t) = \sin\Omega_0 t\, u(t) = \sin\Omega_0 t$; $t \ge 0$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} \sin\Omega_0 t\, e^{-st}\, dt = \int_{0}^{\infty} \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\, e^{-st}\, dt$$

$$= \frac{1}{2j}\int_{0}^{\infty} \left(e^{-(s - j\Omega_0)t} - e^{-(s + j\Omega_0)t}\right) dt = \frac{1}{2j}\left[\frac{e^{-(s - j\Omega_0)t}}{-(s - j\Omega_0)} - \frac{e^{-(s + j\Omega_0)t}}{-(s + j\Omega_0)}\right]_{0}^{\infty}$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s-j\Omega_0)} - \frac{e^{-\infty}}{-(s+j\Omega_0)} - \frac{e^{0}}{-(s-j\Omega_0)} + \frac{e^{0}}{-(s+j\Omega_0)}\right]$$

$$= \frac{1}{2j}\left[0 - 0 + \frac{1}{s-j\Omega_0} - \frac{1}{s+j\Omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{s+j\Omega_0 - s + j\Omega_0}{(s-j\Omega_0)\,(s+j\Omega_0)}\right] = \frac{1}{2j}\left[\frac{2j\Omega_0}{s^2+\Omega_0^{\,2}}\right] = \frac{\Omega_0}{s^2+\Omega_0^{\,2}} \qquad \boxed{(a+b)(a-b) = a^2-b^2} \quad \boxed{j^2 = -1}$$

$$\therefore\ \mathcal{L}\{\sin\Omega_0 t\ u(t)\} = \frac{\Omega_0}{s^2+\Omega_0^{\,2}}$$

---

b) Given that, $x(t) = \cos\Omega_0 t\ u(t) = \cos\Omega_0 t \quad ; \quad t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cos\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\, e^{-st}\, dt \qquad \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-j\Omega_0)t} + e^{-(s+j\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s-j\Omega_0)t}}{-(s-j\Omega_0)} + \frac{e^{-(s+j\Omega_0)t}}{-(s+j\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-j\Omega_0)} + \frac{e^{-\infty}}{-(s+j\Omega_0)} - \frac{e^{0}}{-(s-j\Omega_0)} - \frac{e^{0}}{-(s+j\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s-j\Omega_0} + \frac{1}{s+j\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+j\Omega_0 + s - j\Omega_0}{(s-j\Omega_0)\,(s+j\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2+\Omega_0^{\,2}}\right] = \frac{s}{s^2+\Omega_0^{\,2}} \qquad \boxed{(a+b)(a-b) = a^2-b^2} \quad \boxed{j^2 = -1}$$

$$\therefore\ \mathcal{L}\{\cos\Omega_0 t\ u(t)\} = \frac{s}{s^2+\Omega_0^{\,2}}$$

---

c) Given that, $x(t) = \cosh\Omega_0 t\ u(t) = \cosh\Omega_0 t \quad ; \quad t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cosh\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{\Omega_0 t} + e^{-\Omega_0 t}}{2}\, e^{-st}\, dt \qquad \boxed{\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-\Omega_0)t} + e^{-(s+\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s-\Omega_0)t}}{-(s-\Omega_0)} + \frac{e^{-(s+\Omega_0)t}}{-(s+\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-\Omega_0)} + \frac{e^{-\infty}}{-(s+\Omega_0)} - \frac{e^{0}}{-(s-\Omega_0)} - \frac{e^{0}}{-(s+\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s-\Omega_0} + \frac{1}{s+\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\Omega_0 + s - \Omega_0}{(s-\Omega_0)\,(s+\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-\Omega_0^{\,2}}\right] = \frac{s}{s^2-\Omega_0^{\,2}} \qquad \boxed{(a+b)(a-b) = a^2-b^2}$$

$$\therefore\ \mathcal{L}\{\cosh\Omega_0 t\ u(t)\} = \frac{s}{s^2-\Omega_0^{\,2}}$$

---

d) Given that, $x(t) = e^{-at} \sin\Omega_0 t\ u(t) = e^{-at} \sin\Omega_0 t \quad ; \quad t \geq 0$

By definition of Laplace transform,

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty e^{-at} \sin\Omega_0 t\, e^{-st}\, dt = \int_0^\infty e^{-at}\, \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\, e^{-st}\, dt$$

$$= \frac{1}{2j}\int_0^\infty \left(e^{-(s+a-j\Omega_0)t} - e^{-(s+a+j\Omega_0)t}\right) dt = \frac{1}{2j}\left[\frac{e^{-(s+a-j\Omega_0)t}}{-(s+a-j\Omega_0)} - \frac{e^{-(s+a+j\Omega_0)t}}{-(s+a+j\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2j}\left[\frac{e^{-\infty}}{-(s+a-j\Omega_0)} - \frac{e^{-\infty}}{-(s+a+j\Omega_0)} - \frac{e^{0}}{-(s+a-j\Omega_0)} + \frac{e^{0}}{-(s+a+j\Omega_0)}\right]$$

$$= \frac{1}{2j}\left[0 - 0 + \frac{1}{s+a-j\Omega_0} - \frac{1}{s+a+j\Omega_0}\right] = \frac{1}{2j}\left[\frac{(s+a+j\Omega_0) - (s+a-j\Omega_0)}{(s+a-j\Omega_0)\,(s+a+j\Omega_0)}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\Omega_0}{(s+a)^2 + \Omega_0^2}\right] = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$$

$\boxed{(a+b)(a-b) = a^2 - b^2}$ $\boxed{j^2 = -1}$

$$\therefore\ \mathcal{L}\{e^{-at} \sin\Omega_0 t\ u(t)\} = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$$

---

e) Given that, $x(t) = e^{-at} \cos\Omega_0 t\ u(t) = e^{-at} \cos\Omega_0 t \quad ; \quad t \geq 0$

By definition of Laplace transform,

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty e^{-at} \cos\Omega_0 t\, e^{-st}\, dt = \int_0^\infty e^{-at}\, \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\, e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s+a-j\Omega_0)t} + e^{-(s+a+j\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s+a-j\Omega_0)t}}{-(s+a-j\Omega_0)} + \frac{e^{-(s+a+j\Omega_0)t}}{-(s+a+j\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s+a-j\Omega_0)} + \frac{e^{-\infty}}{-(s+a+j\Omega_0)} - \frac{e^{0}}{-(s+a-j\Omega_0)} - \frac{e^{0}}{-(s+a+j\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s+a-j\Omega_0} + \frac{1}{s+a+j\Omega_0}\right] = \frac{1}{2}\left[\frac{s+a+j\Omega_0 + s+a-j\Omega_0}{(s+a-j\Omega_0)\,(s+a+j\Omega_0)}\right]$$

$$= \frac{1}{2}\left[\frac{2(s+a)}{(s+a)^2 + \Omega_0^2}\right] = \frac{s+a}{(s+a)^2 + \Omega_0^2}$$

$\boxed{(a+b)(a-b) = a^2 - b^2}$ $\boxed{j^2 = -1}$

$$\therefore\ \mathcal{L}\{e^{-at} \cos\Omega_0 t\ u(t)\} = \frac{s+a}{(s+a)^2 + \Omega_0^2}$$

Example 3.3

Determine the Laplace transform of the signals shown below.

a)

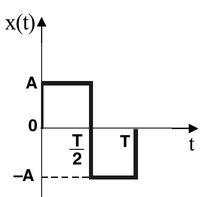


Fig 3.3.1.

b)

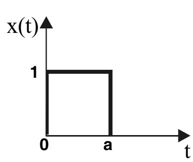


Fig 3.3.2.

c)

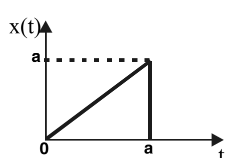


Fig 3.3.3.

d)

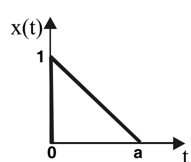


Fig 3.3.4.

Solution

a)

The mathematical equation of the signal shown in fig 3.3.1 is,

$x(t) = A \quad ;\text{for} \quad 0 < t < T/2$

$\quad = -A \quad ;\text{for} \quad T/2 < t < T$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{T} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{T/2} A\, e^{-st}\, dt + \int_{T/2}^{T} (-A)\, e^{-st}\, dt = \left[\frac{A\, e^{-st}}{-s}\right]_0^{T/2} + \left[\frac{-A\, e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= \left[\frac{A\, e^{\frac{-sT}{2}}}{-s} - \frac{A\, e^{0}}{-s}\right] + \left[\frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}\right]$$

$$= -\frac{A\, e^{\frac{-sT}{2}}}{s} + \frac{A}{s} + \frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2\, e^{\frac{-sT}{2}}\right] = \frac{A}{s}\left[1 - e^{\frac{-sT}{2}}\right]^2 \qquad \boxed{(a-b)^2 = a^2 + b^2 - 2ab}$$

---

b)

The mathematical equation of the signal shown in fig 3.3.2 is,

$x(t) = 1 \quad ;\text{for} \quad 0 \le t \le a$

$\quad = 0 \quad ;\text{for} \quad t > a$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{a} 1 \times e^{-st}\, dt = \int_{0}^{a} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{a}$$

$$= \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = -\frac{e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}\left(1 - e^{-as}\right)$$

---

c)

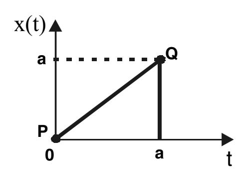


Fig 1.

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, a]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

$$\frac{x(t) - 0}{0 - a} = \frac{t - 0}{0 - a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t$$

$$\therefore\ x(t) = t \quad ;\text{for } t = 0 \text{ to } a$$
$$= 0 \quad ;\text{for } t > a$$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_0^a t\, e^{-st}\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}\, dt\right]_0^a = \left[-\frac{t\, e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a$$

$\int uv = u\int v - \int[du \int v]$	
$u = t$	$v = e^{-st}$

$$= \left[-\frac{a\, e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2}\right] = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a\, e^{-as}}{s}$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1 + as)\right]$$

d)

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 1]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, 0]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

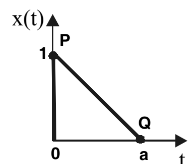


Fig 1.

$$\frac{x(t) - 1}{1 - 0} = \frac{t - 0}{0 - a} \Rightarrow x(t) - 1 = -\frac{t}{a} \Rightarrow x(t) = 1 - \frac{t}{a}$$

$$\therefore\ x(t) = 1 - \frac{t}{a} \quad ;\text{for } t = 0 \text{ to } a$$
$$= 0 \quad ;\text{for } t > a$$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$= \int_0^a \left(1 - \frac{t}{a}\right) e^{-st}\, dt = \int_0^a e^{-st}\, dt - \frac{1}{a}\int_0^a t\, e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^a - \frac{1}{a}\left[t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}\, dt\right]_0^a$$

$\int uv = u\int v - \int[du \int v]$	
$u = t$	$v = e^{-st}$

$$= \left[-\frac{e^{-st}}{s} \right]_0^a - \frac{1}{a}\left[-\frac{t\,e^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_0^a$$

$$= \left[-\frac{e^{-as}}{s} + \frac{e^0}{s} \right] - \frac{1}{a}\left[-\frac{a\,e^{-as}}{s} - \frac{e^{-as}}{s^2} + 0 + \frac{e^0}{s^2} \right]$$

$$= -\frac{e^{-as}}{s} + \frac{1}{s} + \frac{e^{-as}}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2}$$

$$= \frac{1}{s} + \frac{e^{-as}}{as^2} - \frac{1}{as^2} = \frac{1}{as^2}\left[e^{-as} + as - 1 \right]$$

Example 3.4

Determine the Laplace transform of the sine pulse shown in fig 3.4.1.

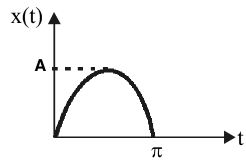


Fig 3.4.1.

Solution

The mathematical equation of the sine pulse shown in fig 3.4.1. is,

$$x(t) = A \sin t \quad ;\text{for } 0 < t < \pi$$

$$= 0 \quad ;\text{for } t > \pi$$

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\pi A \sin t\, e^{-st}\, dt$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= A\int_0^\pi \sin t\, e^{-st}\, dt = A\int_0^\pi \frac{e^{jt} - e^{-jt}}{2j}\, e^{-st}\, dt$$

$$= \frac{A}{2j}\int_0^\pi \left[e^{-(s-j)t} - e^{-(s+j)t} \right] dt = \frac{A}{2j}\left[\frac{e^{-(s-j)t}}{-(s-j)} - \frac{e^{-(s+j)t}}{-(s+j)} \right]_0^\pi$$

$$= \frac{A}{2j}\left[\frac{e^{-(s+j)t}}{(s+j)} - \frac{e^{-(s-j)t}}{(s-j)} \right]_0^\pi = \frac{A}{2j}\left[\frac{(s-j)e^{-(s+j)t} - (s+j)\,e^{-(s-j)t}}{(s+j)(s-j)} \right]_0^\pi$$

$$= \frac{A}{2j}\left[\frac{(s-j)\,e^{-st}\,e^{-jt} - (s+j)\,e^{-st}\,e^{jt}}{s^2 - j^2} \right]_0^\pi$$

$$= \frac{A}{2j(s^2+1)}\left[(s-j)\,e^{-st}\,e^{-jt} - (s+j)\,e^{-st}\,e^{jt} \right]_0^\pi$$

$$= \frac{A}{2j(s^2+1)}\left[(s-j)\,e^{-s\pi}\,e^{-j\pi} - (s+j)\,e^{-s\pi}\,e^{j\pi} - (s-j)\,e^0 + (s+j)\,e^0 \right]$$

$$= \frac{A}{2j(s^2+1)}\left[-(s-j)\,e^{-s\pi} + (s+j)\,e^{-s\pi} - (s-j) + (s+j) \right]$$

$$\boxed{\begin{aligned} e^{\pm j\pi} &= \cos\pi \pm j\sin\pi \\ &= -1 \pm j0 = -1 \end{aligned}}$$

$$= \frac{A}{2j(s^2+1)}\left[e^{-\pi s}(-s + j + s + j) - s + j + s + j \right]$$

$$= \frac{A}{2j(s^2+1)}\left(2j\,e^{-\pi s} + 2j \right) = \frac{A}{s^2+1}\left(e^{-\pi s} + 1 \right)$$

Table 3.1 : Laplace Transform of Some Standard Signals

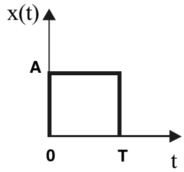
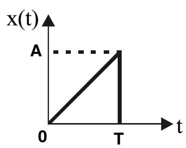
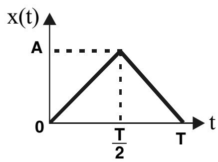
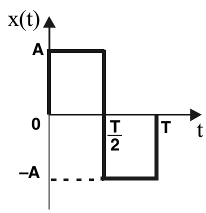
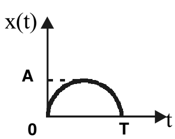
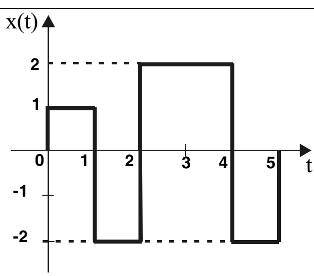
Waveform	x(t)	X(s) = $\mathcal{L}\{x(t)\}$
	$x(t) = A \; ; \; 0 < t < T$ $= 0 \; ; \; t > T$	$X(s) = \frac{A}{s}(1 - e^{-sT})$
	$x(t) = \frac{At}{T} \; ; \; 0 < t < T$ $= 0 \; ; \; t > T$	$X(s) = \frac{A}{Ts^2}\left[1 - e^{-sT}(1 + sT)\right]$
	$x(t) = \frac{2At}{T} \; ; \; 0 < t < \frac{T}{2}$ $= 2A - \frac{2At}{T} \; ; \; \frac{T}{2} < t < T$	$X(s) = \frac{2A}{Ts^2}\left(1 - e^{\frac{-sT}{2}}\right)^2$
	$x(t) = A \; ; \; 0 < t < \frac{T}{2}$ $= -A \; ; \; \frac{T}{2} < t < T$	$X(s) = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$
	$x(t) = A \sin t \; ; \; 0 < t < T$ $= 0 \; ; \; t > T$	$X(s) = \frac{A}{s^2 + 1}\left(e^{-sT} + 1\right)$
	$x(t) = 1 \; ; \; 0 < t < 1$ $= -2 \; ; \; 1 < t < 2$ $= 2 \; ; \; 2 < t < 4$ $= -2 \; ; \; 4 < t < 5$ $= 0 \; ; \; t > 5$	$X(s) = \frac{1}{s}(1 - 3e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s})$

Table 3.1 Continued.....

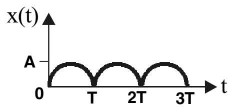
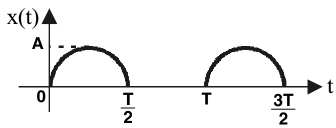
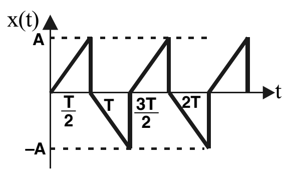
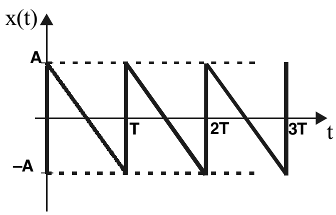
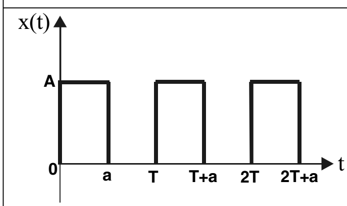
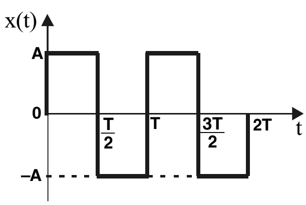
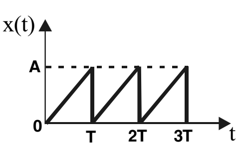
Waveform	x(t)	$X(s) = \mathcal{L}\{x(t)\}$
	$x(t) = A \sin t \; ; 0 < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{A}{s^2 + 1}\left(\dfrac{1 + e^{-sT}}{1 - e^{-sT}}\right)$
	$x(t) = A \sin t \; ; 0 < t < \dfrac{T}{2}$ $= 0 \; ; \dfrac{T}{2} < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{A}{\left(s^2 + 1\right)\left(1 - e^{-\frac{sT}{2}}\right)}$
	$x(t) = \dfrac{2At}{T} \; ; 0 < t < \dfrac{T}{2}$ $= A - \dfrac{2At}{T} \; ; \dfrac{T}{2} < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{2A\left[1 - \left(1 + \dfrac{Ts}{2}\right)e^{-\frac{sT}{2}}\right]}{Ts^2\left(1 + e^{-\frac{sT}{2}}\right)}$
	$x(t) = A - \dfrac{2At}{T} \; ; 0 < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{2A}{Ts}\left(\dfrac{T}{2}\,\dfrac{1 + e^{-sT}}{1 - e^{-sT}} - \dfrac{1}{s}\right)$
	$x(t) = A \; ; \; 0 < t < a$ $= 0 \; ; \; a < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{A}{s}\,\dfrac{1 - e^{-as}}{1 - e^{-sT}}$
	$x(t) = A \; ; \; 0 < t < \dfrac{T}{2}$ $= -A \; ; \; \dfrac{T}{2} < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{A}{s}\left(\dfrac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$
	$x(t) = \dfrac{At}{T} \; ; \; 0 < t < T$ and $\; x(t + nT) = x(t)$	$X(s) = \dfrac{A}{Ts^2}\left[\dfrac{1 - e^{-sT}(1 + sT)}{1 - e^{-sT}}\right]$

Table 3.2 : Some Standard Laplace Transform Pairs

Note : σ = Real part of s

x(t)	**X(s)**	**ROC**
$\delta(t)$	1	Entire s-plane
$u(t)$	$\frac{1}{s}$	$\sigma > 0$
$t\,u(t)$	$\frac{1}{s^2}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}\,u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-at}\,u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
$-e^{-at}\,u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
$t^n\,u(t)$ where, $n = 1, 2, 3 \ldots$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$t\,e^{-at}\,u(t)$	$\frac{1}{(s+a)^2}$	$\sigma > -a$
$\frac{1}{(n-1)!}\,t^{n-1}\,e^{-at}\,u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$	$\sigma > -a$
$t^n\,e^{-at}\,u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$\sigma > 0$
$\cos\Omega_0 t\,u(t)$	$\frac{s}{s^2+\Omega_0^2}$	$\sigma > 0$
$\sinh\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2-\Omega_0^2}$	$\sigma > \Omega_0$
$\cosh\Omega_0 t\,u(t)$	$\frac{s}{s^2-\Omega_0^2}$	$\sigma > \Omega_0$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$\sigma > -a$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$\sigma > -a$

3.3 Properties and Theorems of Laplace Transform

The properties and theorems of Laplace transform are listed in table 3.3. The proof of properties and theorems are presented in this section.

1. Amplitude Scaling

In amplitude scaling, if the amplitude (or magnitude) of a time domain signal is multiplied by a constant A, then its Laplace transform is also multiplied by the same constant.

i.e., if $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{A\,x(t)\} = A\,X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt \qquad .....(3.8)$$

$$\mathcal{L}\{A\,x(t)\} = \int_{-\infty}^{+\infty} A\,x(t)\,e^{-st}\,dt$$

$$= A\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

$$= A\,X(s)$$

Using equation (3.8)

2. Linearity

The linearity property states that, Laplace transform of weighted sum of the two or more signals is equal to similar weighted sum of Laplace transforms of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$, then

$$\mathcal{L}\{a_1\,x_1(t) + a_2\,x_2(t)\} = a_1\,X_1(s) + a_2\,X_2(s)$$

Proof :

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\,e^{-st}\,dt \qquad .....(3.9)$$

$$X_2(s) = \mathcal{L}\{x_2(t\} = \int_{-\infty}^{+\infty} x_2(t)\,e^{-st}\,dt \qquad .....(3.10)$$

$$\mathcal{L}\{a_1\,x_1(t) + a_2\,x_2(t)\} = \int_{-\infty}^{+\infty} \left[a_1\,x_1(t) + a_2\,x_2(t)\right] e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{+\infty} x_1(t)\,e^{-st}\,dt + a_2\int_{-\infty}^{+\infty} x_2(t)\,e^{-st}\,dt$$

$$= a_1\,X_1(s) + a_2\,X_2(s)$$

Using equations (3.9) and (3.10)

3. Time Differentiation

The time differentiation property states that if a causal signal x(t) is piecewise continuous, and Laplace transform of x(t) is X(s) then, Laplace transform of $\frac{d}{dt}x(t)$ is given by sX(s) − x(0).

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0) \quad ; \quad \text{where, } x(0) \text{ is value of } x(t) \text{ at } t = 0.$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\, e^{-st}\, dt \qquad .....(3.11)$$

$$\therefore \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \int_0^\infty e^{-st}\frac{dx(t)}{dt}\, dt$$

$\int u\, v = u \int v - \int \left[du \int v\right]$	
$u = e^{-st}$	$v = \frac{dx(t)}{dt}$

$$= \left[e^{-st}\, x(t)\right]_0^\infty - \int_0^\infty -s\, e^{-st}\, x(t)\, dt$$

$$= e^{-\infty}\, x(\infty) - e^0\, x(0) + s\int_0^\infty x(t)\, e^{-st}\, dt \qquad \boxed{e^{-\infty} = 0 \text{ and } e^0 = 1}$$

$$= s\int_0^\infty x(t)\, e^{-st}\, dt - x(0) = s\,X(s) - x(0) \qquad \boxed{\text{Using equation (3.11)}}$$

4. Time Integration

The time integration property states that, if a causal signal x(t) is continuous and Laplace transform of x(t) is X(s), then the Laplace transform of $\int x(t)\, dt$ is given by, $\frac{X(s)}{s} + \frac{\left[\int x(t)\, dt\right]_{t=0}}{s}$

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\int x(t)\, dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\, dt\right]_{t=0}}{s}$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\, e^{-st}\, dt \qquad .....(3.12)$$

$$\therefore \mathcal{L}\left\{\int x(t)\, dt\right\} = \int_0^\infty \left[\int x(t)\, dt\right] e^{-st}\, dt$$

$\int uv = u \int v - \int \left[du \int v\right]$	
$u = \int x(t)\, dt$	$v = e^{-st}$

$$= \left[\left[\int x(t)\, dt\right]\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty x(t)\,\frac{e^{-st}}{-s}\, dt$$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty} \frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0} \frac{e^{0}}{-s} + \frac{1}{s}\int_0^{\infty} x(t)\,e^{-st}\,dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^{\infty} x(t)\,e^{-st}\,dt$$

$e^{-\infty} = 0$ and $e^{0} = 1$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

Using equation (3.12)

5. Frequency shifting

The frequency shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{e^{\pm at}\,x(t)\right\} = X(s \mp a) \qquad \left[\text{i.e., } \mathcal{L}\left\{e^{at}\,x(t)\right\} = X(s-a) \text{ and } \mathcal{L}\left\{e^{-at}\,x(t)\right\} = X(s+a)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt \qquad .....(3.13)$$

$$\therefore \mathcal{L}\left\{e^{\pm at}\,x(t)\right\} = \int_{-\infty}^{+\infty} e^{\pm at}\,x(t)\,e^{-st}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\,e^{-(s \mp a)t}\,dt$$

$$= X(s \mp a)$$

The term $\int_{-\infty}^{+\infty} x(t)\,e^{-(s\mp a)t}\,dt$ is similar to the form of definition of Laplace transform (equation(3.13)) except that s is replaced by $(s \mp a)$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\,e^{-(s\mp a)t}dt = X(s \mp a)$$

6. Time shifting

The time shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(t \pm a)\} = e^{\pm as}\,X(s) \qquad \left[\text{i.e., } \mathcal{L}\{x(t+a)\} = e^{as}\,X(s) \text{ and } \mathcal{L}\{x(t-a)\} = e^{-as}\,X(s)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt \qquad .....(3.14)$$

$$\therefore \mathcal{L}\{x(t \pm a)\} = \int_{-\infty}^{+\infty} x(t \pm a)\,e^{-st}\,dt = \int_{-\infty}^{+\infty} x(\tau)\,e^{-s(\tau \mp a)}\,d\tau$$

Let, $t \pm a = \tau$
$\therefore t = \tau \mp a$
On differentiating
$dt = d\tau$

$$= \int_{-\infty}^{+\infty} x(\tau)\,e^{-s\tau} \times e^{\pm as}\,d\tau = e^{\pm as}\int_{-\infty}^{+\infty} x(\tau)\,e^{-s\tau}\,d\tau$$

Since τ is a dummy variable for integration we can change τ to t.

$$= e^{\pm as}\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = e^{\pm as}\,X(s)$$

Using equation (3.14)

7. Frequency Differentiation

The frequency differentiation property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On differentiating the above equation with respect to s we get,

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left(\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{+\infty} x(t)\left(-t\,e^{-st}\right)dt$$

Interchanging the order of integration and differentiation

$$= \int_{-\infty}^{+\infty} (-t\,x(t))\,e^{-st}\,dt = \mathcal{L}\{-t\,x(t)\} = -\mathcal{L}\{t\,x(t)\}$$

$$\therefore\ \mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

8. Frequency Integration

The frequency integration property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{1}{t}\,x(t)\right\} = \int_{s}^{\infty} X(s)\,ds$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On integrating the above equation with respect to s between limits s to ∞ we get,

$$\int_{s}^{\infty} X(s)\,ds = \int_{s}^{\infty}\left[\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right]ds$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\int_{s}^{\infty} e^{-st}\,ds\right]dt$$

Interchanging the order of integrations.

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[0 + \frac{e^{-st}}{t}\right]dt = \int_{-\infty}^{+\infty}\left[\frac{1}{t}\,x(t)\right]e^{-st}\,dt = \mathcal{L}\left\{\frac{1}{t}\,x(t)\right\}$$

9. Time scaling

The time scaling property of Laplace transform says that,

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{s}{a}\right)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.15)$$

$$\therefore\ \mathcal{L}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = \frac{1}{a}\, X\left(\frac{s}{a}\right)$$

Put, $at = \tau$

$\therefore\ t = \frac{\tau}{a}$

On differentiating

$dt = \frac{d\tau}{a}$

The above transform is applicable for positive values of "a".

If "a" happens to be negative it can be proved that,

$$\mathcal{L}\{x(at)\} = -\frac{1}{a}\, X\left(\frac{s}{a}\right)$$

Hence in general,

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{s}{a}\right)$$ for both positive and negative values of "a"

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau$ is similar to the form of definition of Laplace transform(equation(3.15)) except that s is replaced by $\left(\frac{s}{a}\right)$.

$$\therefore\ \int_{-\infty}^{+\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = X\left(\frac{s}{a}\right)$$

10. Periodicity

The periodicity property of Laplace transform says that,

If $x(t) = x(t+nT)$, and $x_1(t)$ be one period of $x(t)$, and $\mathcal{L}\{x_1(t)\} = \int_0^T x_1(t)\, e^{-st}\, dt$, then

$$\mathcal{L}\{x(t + nT)\} = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Proof :

By definition of Laplace transform,

$$\mathcal{L}\{x(t + nT)\} = \int_0^{\infty} x(t + nT)\, e^{-st}\, dt$$

$$= \int_0^T x_1(t)\, e^{-st}\, dt + \int_T^{2T} x_1(t - T)\, e^{-s(t + T)}\, dt + \int_{2T}^{3T} x_1(t - 2T)\, e^{-s(t + 2T)}\, dt + \ldots\ldots\ldots\ldots.$$

$$\ldots\ldots\ldots\ldots + \int_{pT}^{(p + 1)T} x_1(t - pT)\, e^{-s(t + pT)}\, dt + \ldots\ldots\ldots$$

$$= \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t - pT)\, e^{-s(t+pT)}\, dt$$

$$= \sum_{p=0}^{\infty} \int_{0}^{T} x_1(t)\, e^{-st}\, e^{-psT}\, dt$$

The periodic signal will be identical in every period and so, $x_1(t+pT) = x_1(t)$.

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-psT} \right) dt$$

Interchanging the order of integration and summation

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-sT} \right)^p dt$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\frac{1}{1 - e^{-sT}} \right) dt$$

$$= \frac{1}{1 - e^{-sT}} \int_{0}^{T} x_1(t)\, e^{-st}\, dt$$

The term $\frac{1}{1 - e^{-sT}}$ is independent of **t**

11. Initial Value Theorem

The initial value theorem states that, if x(t) and its derivative are Laplace transformable then,

$$\underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

i.e., Initial value of signal, $x(0) = \underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to \infty$ on both sides of the above equation we get,

$$\underset{s \to \infty}{\mathrm{Lt}}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to \infty}{\mathrm{Lt}}\ \left[s\,X(s) - x(0)\right]$$

By definition of Laplace transform, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt$

$$\underset{s \to \infty}{\mathrm{Lt}} \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to \infty}{\mathrm{Lt}}\ \left[s\,X(s) - x(0)\right]$$

Here $\frac{dx(t)}{dt}$ and x(0) are not functions of s

$$\int_0^{\infty} \frac{dx(t)}{dt} \left(\underset{s \to \infty}{\mathrm{Lt}}\ e^{-st} \right) dt = \left(\underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s) \right) - x(0)$$

$\underset{s \to \infty}{\mathrm{Lt}}\ e^{-st} = 0$

$$0 = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s) - x(0)$$

$$\therefore\ x(0) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

$$\therefore\ \underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

$x(0) = \underset{t \to 0}{\mathrm{Lt}}\ x(t)$

12. Final Value Theorem

The final value theorem states that if x(t) and its derivative are Laplace transformable then $\underset{t \to \infty}{\text{Lt}}\ x(t) = \underset{s \to 0}{\text{Lt}}\ s\,X(s)$

i.e., Final value of signal, $x(\infty) = \underset{t \to \infty}{\text{Lt}}\ x(t) = \underset{s \to 0}{\text{Lt}}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\dfrac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to 0$ on both sides of the above equation we get,

$$\underset{s \to 0}{\text{Lt}}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to 0}{\text{Lt}}\left[s\,X(s) - x(0)\right]$$

By definition of Laplace transform
$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt$$

$$\underset{s \to 0}{\text{Lt}} \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to 0}{\text{Lt}}\left[s\,X(s) - x(0)\right]$$

Here $\dfrac{dx(t)}{dt}$ and x(0) are not functions of s

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s \to 0}{\text{Lt}}\ e^{-st}\right) dt = \left(\underset{s \to 0}{\text{Lt}}\ s\,X(s)\right) - x(0)$$

$$\underset{s \to 0}{\text{Lt}}\ e^{-st} = 1$$

$$\int_0^\infty \frac{dx(t)}{dt}\, dt = \underset{s \to 0}{\text{Lt}}\ s\,X(s) - x(0)$$

$$\left[x(t)\right]_0^\infty = \underset{s \to 0}{\text{Lt}}\ s\,X(s) - x(0)$$

$$x(\infty) - x(0) = \underset{s \to 0}{\text{Lt}}\ s\,X(s) - x(0)$$

$$\therefore\ x(\infty) = \underset{s \to 0}{\text{Lt}}\ s\,X(s)$$

$$\therefore\ \underset{t \to \infty}{\text{Lt}}\ x(t) = \underset{s \to 0}{\text{Lt}}\ s\,X(s)$$

$$x(\infty) = \underset{t \to \infty}{\text{Lt}}\ x(t)$$

13. Convolution Theorem

The convolution theorem of Laplace transform says that, Laplace transform of convolution of two signals is given by the product of the Laplace transform of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$ then,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s) \qquad .....(3.16)$$

The equation (3.16) is also known as convolution property of Laplace transform.

With reference to chapter-2, section 2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t - \lambda)\, d\lambda \qquad ....(3.17)$$

where, λ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals.

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \qquad .....(3.18)$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt \qquad .....(3.19)$$

Let $x_3(t)$ be the signal obtained by convolution of $x_1(t)$ and $x_2(t)$.Now from equation (3.17) we get,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\, d\lambda \qquad .....(3.20)$$

Let, $\mathcal{L}\{x_3(t)\} = X_3(s)$. Now by definition of Laplace transform we can write,

$$X_3(s) = \mathcal{L}\{x_3(t)\} = \int_{-\infty}^{+\infty} x_3(t)\, e^{-st}\, dt \qquad ..... (3.21)$$

On substituing for $x_3(t)$ from equation (3.20) in equation (3.21) we get,

$$X_3(s) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\; d\lambda \right] e^{-st}\, dt \qquad ..... (3.22)$$

Let, $e^{-st} = e^{s\lambda} \times e^{-s\lambda} \times e^{-st} = e^{-s\lambda} \times e^{-s(t-\lambda)} = e^{-s\lambda} \times e^{-sM}$ ..... (3.23)

where, $M = t - \lambda$ and so, $dM = dt$ .....(3.24)

Using equations (3.23) and (3.24), the equation (3.22) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(M)\, e^{-s\lambda}\; e^{-sM}\; d\lambda\, dM$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda)\, e^{-s\lambda}\, d\lambda \times \int_{-\infty}^{+\infty} x_2(M)\, e^{-sM}\, dM \qquad .....(3.25)$$

In equation (3.25), λ and M are dummy variables used for integration, and so they can be changed to t.

Therefore equation (3.25) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \times \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt$$

$$= X_1(s)\, X_2(s)$$

Using equations (3.18) and (3.19)

$$\therefore\ \mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s)$$

Table 3.3 : Properties of Laplace Transform

Note : $\mathcal{L}\{x(t)\} = X(s)$; $\mathcal{L}\{x_1(t)\} = X_1(s)$; $\mathcal{L}\{x_2(t)\} = X_2(s)$

Property	Time domain signal	s-domain signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1 x_1(t) \pm a_2 x_2(t)$	$a_1 X_1(s) \pm a_2 X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$s\,X(s) - x(0)$
	$\frac{d^n}{dt^n}x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$s^n X(s) - \sum\limits_{K=1}^{n} s^{n-K} \left.\frac{d^{(K-1)}x(t)}{dt^{K-1}}\right\|_{t=0}$
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$
	$\int \ldots\ldots \int x(t)\,(dt)^n$ where $n = 1, 2, 3 \ldots\ldots$	$\frac{X(s)}{s^n} + \sum\limits_{K=1}^{n} \frac{1}{s^{n-K+1}} \left[\int \ldots \int x(t)\,(dt)^k\right]_{t=0}$
Frequency shifting	$e^{\pm at}\,x(t)$	$X(s \mp a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm \alpha s}\,X(s)$
Frequency differentiation	$t\,x(t)$	$-\frac{dX(s)}{ds}$
	$t^n\,x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$(-1)^n \frac{d^n}{ds^n} X(s)$
Frequency integration	$\frac{1}{t}\,x(t)$	$\int\limits_{s}^{\infty} X(s)\,ds$
Time scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{s}{a}\right)$
Periodicity	$x(t + nT)$	$\frac{1}{1 - e^{-sT}} \int\limits_{0}^{T} x_1(t)\,e^{-st}\,dt$ where, $x_1(t)$ is one period of $x(t)$.
Initial value theorem	$\underset{t \to 0}{Lt}\; x(t) = x(0)$	$\underset{s \to \infty}{Lt}\; s\,X(s)$
Final value theorem	$\underset{t \to \infty}{Lt}\; x(t) = x(\infty)$	$\underset{s \to 0}{Lt}\; s\,X(s)$
Convolution theorem	$x_1(t) * x_2(t) = \int\limits_{-\infty}^{+\infty} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda$	$X_1(s)\,X_2(s)$

Example 3.5

Determine Laplace transform of periodic square wave shown in fig 3.5.1.

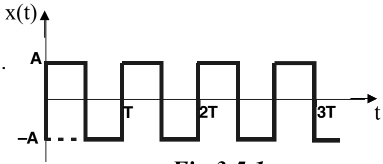


Fig 3.5.1.

Solution

The given waveform satisfy the condition, $x(t + nT) = x(t)$, and so it is periodic.

Let $x_1(t)$ be one period of x(t). The equation for one period of the periodic waveform of fig 3.5.1 is,

$$x_1(t) = A \quad ; \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A \quad ; \text{for } t = \frac{T}{2} \text{ to } T$$

From periodicity property of Laplace transform,

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of x(t).

$$\therefore \mathcal{L}\{x(t)\} = X(s) = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Using the result of example 3.3(a), the above equation can be written as,

From example 3.3(a) we get,
$$\int_0^T x_1(t)\, e^{-st}\, dt = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$$

$$X(s) = \frac{1}{1 - e^{-sT}} \left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$= \frac{1}{\left(1 + e^{-\frac{sT}{2}}\right)\left(1 - e^{-\frac{sT}{2}}\right)} \left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$\boxed{a^2 - b^2 = (a + b)\,(a - b)}$$

$$= \frac{A}{s}\left(\frac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$$

Example 3.6

Determine the Laplace transform of the periodic signal whose waveform is shown in fig 3.6.1.

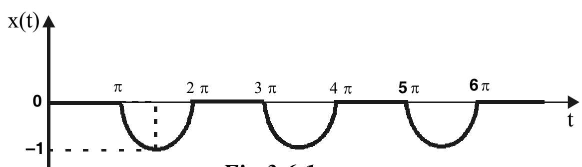


Fig 3.6.1.

Solution

The given waveform satisfies the condition, $x(t + nT) = x(t)$, and so it is periodic. Here, $T = 2\pi$.

Let $x_1(t)$ be one period of x(t). The equation for one period of the periodic waveform of fig 3.6.1 is,

$$x_1(t) = 0 \quad ; \text{for } t = 0 \text{ to } \pi$$

$$= \sin t \quad ; \text{for } t = \pi \text{ to } 2\pi$$

From periodicity property of Laplace transform,

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of x(t).

$$\therefore\ X(s) = \mathcal{L}\{x(t)\} = \frac{1}{1-e^{-sT}}\int_0^T x_1(t)\, e^{-st}\, dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} x_1(t)\, e^{-st}\, dt \qquad \boxed{\text{Put, } T = 2\pi}$$

$$= \frac{1}{1-e^{-2\pi s}}\left[\int_0^{\pi} 0 \times e^{-st}\, dt + \int_{\pi}^{2\pi} \sin\, t \times e^{-st}\, dt\right] = \frac{1}{1-e^{-2\pi s}}\int_{\pi}^{2\pi} \sin\, t\, e^{-st}\, dt \qquad \boxed{\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}}$$

$$= \frac{1}{1-e^{-2\pi s}}\int_{\pi}^{2\pi} \frac{e^{jt} - e^{-jt}}{2j}\, e^{-st}\, dt = \frac{1}{2j\left(1-e^{-2\pi s}\right)}\int_{\pi}^{2\pi}\left[e^{-(s-j)t} - e^{-(s+j)t}\right] dt$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[\frac{e^{-(s-j)t}}{-(s-j)} - \frac{e^{-(s+j)t}}{-(s+j)}\right]_{\pi}^{2\pi} = \frac{1}{2j\left(1 - e^{-2\pi s}\right)}\left[-\frac{e^{jt}\, e^{-st}}{s-j} + \frac{e^{-jt}\, e^{-st}}{s+j}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[-\frac{e^{j2\pi}e^{-2\pi s}}{s-j} + \frac{e^{-j2\pi}\, e^{-2\pi s}}{s+j} + \frac{e^{j\pi}\, e^{-\pi s}}{s-j} - \frac{e^{-j\pi}e^{-\pi s}}{s+j}\right]$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[-\frac{e^{-2\pi s}}{s-j} + \frac{e^{-2\pi s}}{s+j} - \frac{e^{-\pi s}}{s-j} + \frac{e^{-\pi s}}{s+j}\right] \qquad \boxed{e^{\pm j\pi} = -1 \text{ and } e^{\pm j2\pi} = 1} \quad \boxed{a^2 - b^2 = (a+b)\,(a-b)}$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[\frac{-(s+j)\, e^{-2\pi s} + (s-j)\, e^{-2\pi s} - (s+j)\, e^{-\pi s} + (s-j)\, e^{-\pi s}}{(s-j)\,(s+j)}\right] \qquad \boxed{j^2 = -1}$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[\frac{-se^{-2\pi s} - je^{-2\pi s} + se^{-2\pi s} - j\, e^{-2\pi s} - se^{-\pi s} - je^{-\pi s} + se^{-\pi s} - je^{-\pi s}}{s^2+1}\right]$$

$$= \frac{1}{2j\left(1-e^{-2\pi s}\right)}\left[\frac{-2je^{-\pi s} - 2je^{-2\pi s}}{s^2+1}\right] = \frac{-2je^{-\pi s}\left(1+e^{-\pi s}\right)}{2j\left(s^2+1\right)\left(1+e^{-\pi s}\right)\left(1-e^{-\pi s}\right)} = \frac{e^{-\pi s}}{\left(s^2+1\right)\left(e^{-\pi s}-1\right)}$$

Example 3.7

Determine the Laplace transform of the following signals using properties of Laplace transform.

a) $x(t) = (t^2 - 2t)\, u(t-1)$ b) Unit ramp signal starting at $t = a$. c)

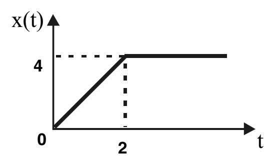


Fig 3.7.1.

Solution

a) Given that, $x(t) = (t^2 - 2t)\, u(t-1) = t^2\, u(t-1) - 2t\, u(t-1)$

From table 3.2 we get,

$$\mathcal{L}\left\{t^2\, u(t)\right\} = \frac{2}{s^3} \qquad .....(1)$$

$$\mathcal{L}\{t\, u(t)\} = \frac{1}{s^2}\quad ; \quad \therefore\ \mathcal{L}\{2t\, u(t)\} = \frac{2}{s^2} \qquad .....(2)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\, u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\, u(t-a)\} = e^{-as}\, X(s)$

$$\therefore\ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\left(t^2 - 2t\right) u(t-1)\right\} = \mathcal{L}\left\{t^2\, u(t-1)\right\} - \mathcal{L}\{2t\, u(t-1)\}$$

$$= e^{-s}\, \mathcal{L}\left\{t^2\, u(t)\right\} - e^{-s}\, \mathcal{L}\{2t\, u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-s}\, \frac{2}{s^3} - e^{-s}\, \frac{2}{s^2} = 2e^{-s}\left(\frac{1-s}{s^3}\right) = \frac{2e^{-s}(1-s)}{s^3} \qquad \boxed{\text{Using equations (1) and (2)}}$$

b) Given that, Unit ramp signal starting at t = a.

The unit ramp starting at t = a, is unit ramp delayed by "a" units of time. The unit ramp waveform and the ramp waveform starting at t = a are shown in fig 1 and fig 2 respectively. The equation of unit ramp and delayed ramp are given below.

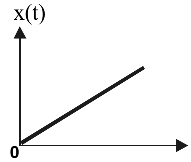


Fig 1 : Ramp.

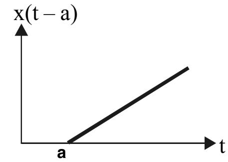


Fig 2 : Ramp starting at t = a.

Unit ramp, $x(t) = t\,u(t)$

Delayed unit ramp, $x(t-a) = (t-a)\,u(t-a)$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \qquad .....(3)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\,u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\,u(t-a)\} = e^{-as}\,X(s)$

Therefore Laplace transform of delayed ramp signal is,

$$\mathcal{L}\{x(t-a)\} = e^{-as}\,\mathcal{L}\{x(t)\}$$

$$= e^{-as}\,\mathcal{L}\{t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-as}\,\frac{1}{s^2} = \frac{e^{-as}}{s^2} \qquad \boxed{\text{Using equation (3)}}$$

---

c)

The given signal can be decomposed into two signals as shown in fig 4 and fig 5.

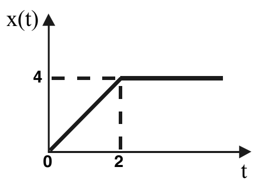


Fig 3.

⇒

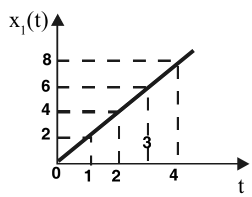


Fig 4.

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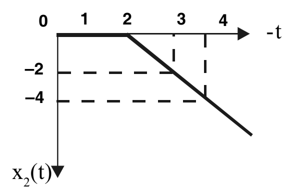


Fig 5.

The mathematical equations of signals, $x_1(t)$ and $x_2(t)$ are given below.

$x_1(t) = 2t\,u(t)$

$x_2(t) = -2(t-2)\,u(t-2)$

$\therefore \quad x(t) = x_1(t) + x_2(t) = 2t\,u(t) - 2(t-2)\,u(t-2)$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \quad ; \quad \therefore \mathcal{L}\{2t\,u(t)\} = \frac{2}{s^2} \qquad .....(4)$$

$$\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{x_1(t) + x_2(t)\} = \mathcal{L}\{2t\,u(t) - 2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - \mathcal{L}\{2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - e^{-2s}\,\mathcal{L}\{2t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= \frac{2}{s^2} - e^{-2s}\,\frac{2}{s^2} = \frac{2\left(1 - e^{-2s}\right)}{s^2} \qquad \boxed{\text{Using equation (4)}}$$

Example 3.8

Let, $X(s) = \mathcal{L}\{x(t)\}$. Determine the initial value, $x(0)$ and the final value, $x(\infty)$, for the following signals using initial value and final value theorems.

a) $X(s) = \dfrac{1}{s(s-1)}$ b) $X(s) = \dfrac{s+1}{s^2+2s+2}$ c) $X(s) = \dfrac{7s+6}{s(3s+5)}$

d) $X(s) = \dfrac{s^2+1}{s^2+6s+5}$ e) $X(s) = \dfrac{s+5}{s^2(s+9)}$

Solution

a) Given that, $X(s) = \dfrac{1}{s(s-1)}$

$$\text{Initial value, } x(0) = \mathop{Lt}_{s\to\infty} s\,X(s) = \mathop{Lt}_{s\to\infty} s\,\frac{1}{s(s-1)} = \mathop{Lt}_{s\to\infty} \frac{1}{(s-1)} = \mathop{Lt}_{s\to\infty} \frac{1}{s\left(1-\frac{1}{s}\right)}$$

$$= \mathop{Lt}_{s\to\infty} \frac{1}{s}\,\frac{1}{\left(1-\frac{1}{s}\right)} = \frac{1}{\infty}\,\frac{1}{\left(1-\frac{1}{\infty}\right)} = 0 \times \frac{1}{1-0} = 0$$

$$\text{Final value, } x(\infty) = \mathop{Lt}_{s\to 0} sX(s) = \mathop{Lt}_{s\to 0} s\,\frac{1}{s(s-1)} = \mathop{Lt}_{s\to 0} \frac{1}{(s-1)} = \frac{1}{0-1} = -1$$

b) Given that, $X(s) = \dfrac{s+1}{s^2+2s+2}$

$$\text{Initial value, } x(0) = \mathop{Lt}_{s\to\infty} sX(s) = \mathop{Lt}_{s\to\infty} s\frac{s+1}{s^2+2s+2} = \mathop{Lt}_{s\to\infty} s\,\frac{s\left(1+\frac{1}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

$$= \mathop{Lt}_{s\to\infty} \frac{1+\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} = \frac{1+\frac{1}{\infty}}{1+\frac{2}{\infty}+\frac{2}{\infty}} = \frac{1+0}{1+0+0} = 1$$

$$\text{Final value, } x(\infty) = \mathop{Lt}_{s\to 0} sX(s) = \mathop{Lt}_{s\to 0} s\,\frac{s+1}{s^2+2s+2}$$

$$= 0 \times \frac{0+1}{0+0+2} = 0$$

c) Given that, $X(s) = \dfrac{7s+6}{s(3s+5)}$

$$\text{Initial value, } x(0) = \mathop{Lt}_{s\to\infty} sX(s) = \mathop{Lt}_{s\to\infty} s\frac{7s+6}{s(3s+5)} = \mathop{Lt}_{s\to\infty} s\,\frac{s\left(7+\frac{6}{s}\right)}{s^2\left(3+\frac{5}{s}\right)}$$

$$= \mathop{Lt}_{s\to\infty} \frac{7+\frac{6}{s}}{3+\frac{5}{s}} = \frac{7+\frac{6}{\infty}}{3+\frac{5}{\infty}} = \frac{7+0}{3+0} = \frac{7}{3}$$

$$\text{Final value, } x(\infty) = \mathop{Lt}_{s\to 0} sX(s) = \mathop{Lt}_{s\to 0} s\,\frac{7s+6}{s(3s+5)}$$

$$= \mathop{Lt}_{s\to 0} \frac{7s+6}{3s+5} = \frac{0+6}{0+5} = \frac{6}{5}$$

d) Given that, $X(s) = \dfrac{s^2+1}{s^2+6s+5}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{\text{Lt}}\ sX(s) = \underset{s\to\infty}{\text{Lt}}\ s\,\frac{s^2+1}{s^2+6s+5} = \underset{s\to\infty}{\text{Lt}}\ s\,\frac{s^2\left(1+\frac{1}{s^2}\right)}{s^2\left(1+\frac{6}{s}+\frac{1}{s^2}\right)}$$

$$= \underset{s\to\infty}{\text{Lt}}\ s\,\frac{1+\frac{1}{s^2}}{1+\frac{6}{s}+\frac{1}{s^2}} = \infty \times \frac{1+\frac{1}{\infty}}{1+\frac{6}{\infty}+\frac{1}{\infty}} = \infty \times \frac{1+0}{1+0+0} = \infty$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{\text{Lt}}\ sX(s) = \underset{s\to 0}{\text{Lt}}\ s\,\frac{s^2+1}{s^2+6s+5} = 0 \times \frac{0+1}{0+0+5} = 0$$

---

e) Given that, $X(s) = \dfrac{s+5}{s^2(s+9)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{\text{Lt}}\ sX(s) = \underset{s\to\infty}{\text{Lt}}\ s\,\frac{s+5}{s^2(s+9)} = \underset{s\to\infty}{\text{Lt}}\ s\,\frac{s\left(1+\frac{5}{s}\right)}{s^3\left(1+\frac{9}{s}\right)}$$

$$= \underset{s\to\infty}{\text{Lt}}\ \frac{1}{s}\,\frac{1+\frac{5}{s}}{1+\frac{9}{s}} = \frac{1}{\infty} \times \frac{1+\frac{1}{\infty}}{1+\frac{9}{\infty}} = 0 \times \frac{1+0}{1+0} = 0$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{\text{Lt}}\ sX(s) = \underset{s\to 0}{\text{Lt}}\ s\,\frac{s+5}{s^2(s+9)} = \underset{s\to 0}{\text{Lt}}\ \frac{s+5}{s^2+9s} = \frac{0+5}{0+0} = \infty$$

Example 3.9

Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem of Laplace transform.

a) $x_1(t) = u(t+5),\ x_2(t) = \delta(t-7)$

b) $x_1(t) = u(t-2),\ x_2(t) = \delta(t+6)$

c) $x_1(t) = u(t+1),\ x_2(t) = r(t-2)$; where $r(t) = t\,u(t)$

Solution

$\mathcal{L}\{u(t)\} = \dfrac{1}{s}$ and $\mathcal{L}\{\delta(t)\} = 1$

if $\mathcal{L}\{x(t)\} = X(s)$ then $\mathcal{L}\{x(t\pm a)\} = e^{\pm as}X(s)$

a) Given that, $x_1(t) = u(t+5)$, $x_2(t) = \delta(t-7)$

$$x_1(t) = u(t+5)$$

$$\therefore X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t+5)\} = e^{5s}\,\mathcal{L}\{u(t)\} = e^{5s} \times \frac{1}{s} = \frac{e^{5s}}{s} \quad .....(1)$$

$$x_2(t) = \delta(t-7)$$

$$\therefore X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t-7)\} = e^{-7s}\,\mathcal{L}\{\delta(t)\} = e^{-7s} \times 1 = e^{-7s} \quad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{5s}}{s} \times e^{-7s} = \frac{e^{5s-7s}}{s} = \frac{e^{-2s}}{s}$$

Using equations (1) and (2)

$$\therefore x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t-2} = u(t)\big|_{t=t-2} = u(t-2)$$

b) Given that, $x_1(t) = u(t-2)$, $x_2(t) = \delta(t+6)$

$$\boxed{\begin{array}{l}\mathcal{L}\{u(t)\} = \dfrac{1}{s} \text{ and } \mathcal{L}\{\delta(t)\} = 1 \\ \text{if } \mathcal{L}\{x(t)\} = X(s) \text{ then } \mathcal{L}\{x(t \pm a)\} = e^{\pm as}X(s)\end{array}}$$

$$x_1(t) = u(t-2)$$

$$\therefore\ X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t-2)\} = e^{-2s}\,\mathcal{L}\{u(t)\} = e^{-2s} \times \frac{1}{s} = \frac{e^{-2s}}{s} \qquad .....(1)$$

$$x_2(t) = \delta(t+6)$$

$$\therefore\ X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t+6)\} = e^{6s}\,\mathcal{L}\{\delta(t)\} = e^{6s} \times 1 = e^{6s} \qquad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{-2s}}{s} \times e^{6s} = \frac{e^{-2s+6s}}{s} = \frac{e^{4s}}{s} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{4s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t+4} = u(t)\Big|_{t=t+4} = u(t+4)$$

---

c) Given that, $x_1(t) = u(t+1)$, $x_2(t) = r(t-2)$; where $r(t) = t\,u(t)$

$$\boxed{\begin{array}{l}\mathcal{L}\{u(t)\} = \dfrac{1}{s} \text{ and } \mathcal{L}\{r(t)\} = \mathcal{L}\{t\,u(t)\} = \dfrac{1}{s^2} \\ \text{if } \mathcal{L}\{x(t)\} = X(s) \text{ then } \mathcal{L}\{x(t \pm a)\} = e^{\pm as}X(s)\end{array}}$$

$$x_1(t) = u(t+1)$$

$$\therefore\ X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t+1)\} = \frac{e^{s}}{s} \qquad .....(1)$$

$$x_2(t) = r(t-2)$$

$$\therefore\ X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{r(t-2)\} = e^{-2s}\,\mathcal{L}\{r(t)\} = \frac{e^{-2s}}{s^2} \qquad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{s}}{s} \times \frac{e^{-2s}}{s^2} = \frac{e^{-s}}{s^3} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}\Bigg|_{t=t-1} = \left[\frac{t^2}{2}\,u(t)\right]\Bigg|_{t=t-1} \qquad \boxed{\mathcal{L}\left\{\frac{t^{n-1}}{(n-1)!}\,u(t)\right\} = \frac{1}{s^n}}$$

$$= \frac{(t-1)^2}{2}\,u(t-1)$$

Example 3.10

Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem of Laplace transform and sketch the resultant waveform, where, $x_1(t) = u(t) - u(t-1)$ and $x_2(t) = u(t) - u(t-2)$.

Solution

$$x_1(t) = u(t) - u(t-1)$$

$$\therefore\ X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t) - u(t-1)\} = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-1)\} = \mathcal{L}\{u(t)\} - e^{-s}\mathcal{L}\{u(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} \qquad .....(1)$$

$$x_2(t) = u(t) - u(t-2)$$

$$\therefore\ X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{u(t) - u(t-2)\} = \mathcal{L}\{u(t)\} - \mathcal{L}\{u(t-2)\} = \mathcal{L}\{u(t)\} - e^{-2s}\mathcal{L}\{u(t)\} = \frac{1}{s} - \frac{e^{-2s}}{s} \qquad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s) = \left(\frac{1}{s} - \frac{e^{-s}}{s}\right)\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right) \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-3s}}{s^2}\right\}$$

$$\boxed{\begin{aligned}\mathcal{L}\{t\,u(t)\} &= \frac{1}{s^2}\\ \mathcal{L}\{(t-a)\,u(t-a)\} &= \frac{e^{-as}}{s^2}\end{aligned}}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$$

$$= t\,u(t) - (t-1)\,u(t-1) - (t-2)\,u(t-2) + (t-3)\,u(t-3)$$

<u>To sketch the resultant waveform</u>

<u>For t = 0 to 1</u>

When t = 0 to1, $u(t) = 1$, $u(t-1) = 0$, $u(t-2) = 0$, $u(t-3) = 0$

$$\therefore\ x_1(t) * x_2(t) = t \times 1 - (t-1)\times 0 - (t-2)\times 0 + (t-3)\times 0 = t$$

<u>For t = 1 to 2</u>

When t = 1 to 2, $u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 0$, $u(t-3) = 0$

$$\therefore\ x_1(t) * x_2(t) = t \times 1 - (t-1)\times 1 - (t-2)\times 0 + (t-3)\times 0 = 1$$

<u>For t = 2 to 3</u>

When t = 2 to 3, $u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 1$, $u(t-3) = 0$

$$\therefore\ x_1(t) * x_2(t) = t \times 1 - (t-1)\times 1 - (t-2)\times 1 + (t-3)\times 0 = 3 - t$$

<u>For t > 3</u>

When t > 3, $u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 1$, $u(t-3) = 1$

$$\therefore\ x_1(t) * x_2(t) = t \times 1 - (t-1)\times 1 - (t-2)\times 1 + (t-3)\times 1 = 0$$

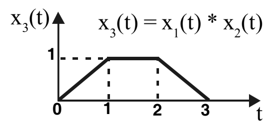


Fig 1.

Using the calculated values, the resultant waveform is sketched as shown in fig 1.

3.4 Poles and Zeros of Rational Function of s

Let X(s) be Laplace transform of x(t). When X(s) is expressed as a ratio of two polynomials in s, then the s-domain signal X(s) is called a ***rational function*** of s.

The zeros and poles are two critical complex frequencies at which a rational function of s takes two extreme values, such as zero and infinity respectively.

Let X(s) is expressed as a ratio of two polynomials in s as shown in equation (3.26).

$$X(s) = \frac{P(s)}{Q(s)}$$

$$= \frac{b_0\,s^M + b_1\,s^{M-1} + b_2\,s^{M-2} + \ldots + b_{M-1}\,s + b_M}{a_0\,s^N + a_1\,s^{N-1} + a_2\,s^{N-2} + \ldots + a_{N-1}\,s + a_N} \qquad \ldots(3.26)$$

where, P(s) = Numerator polynomial of X(s)

Q(s) = Denominator polynomial of X(s)

In equation (3.26) let us scale the coefficients of numerator polynomial by b_0 and the coefficients of denominator polynomial by a_0, and the equation (3.26) can be expressed in factorized form as shown in equation (3.27).

$$X(S) = \frac{b_0\left(s^M + \frac{b_1}{b_0}s^{M-1} + \frac{b_2}{b_0}s^{M-2} + ... + \frac{b_{M-1}}{b_0}s + \frac{b_M}{b_0}\right)}{a_0\left(s^N + \frac{a_1}{a_0}s^{N-1} + \frac{a_2}{a_0}s^{N-2} + ... + \frac{a_{N-1}}{a_0}s + \frac{a_N}{a_0}\right)}$$

$$= G\,\frac{(s - z_1)(s - z_2)\,.....\,(s - z_M)}{(s - p_1)(s - p_2)\,.....\,(s - p_N)} \qquad .....(3.27)$$

where, $G = \frac{b_0}{a_0}$ = Scaling factor

$z_1, z_2, ..., z_M$ = Roots of numerator polynomial, P(s)

$p_1, p_2, ..., p_N$ = Roots of denominator polynomial, Q(s)

In equation (3.27) if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial $z_1, z_2, ..., z_M$ are called ***zeros*** of X(s). Since s is complex frequency, the ***zeros*** can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation (3.27), if the value of s is equal to any one of the roots of the denominator polynomial then the signal X(s) will become infinite. Therefore the roots of denominator polynomial $p_1, p_2, ..., p_N$ are called ***poles*** of X(s). Since **s** is complex frequency, the ***poles*** can be defined as values of complex frequencies at which the signal X(s) become infinite. *Since the signal X(s) attains infinte value at poles, the ROC of X(s) does not include poles.*

In a realizable system, *the number of zeros will be less than or equal to number of poles.* Also for every zero, we can associate one pole. (When number of finite zeros are less than poles, the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate |X(s)| for various values of s, then |X(s)| will be zero for $s = z_i$ and infnite for $s = p_i$. Hence the plot of |X(s)| in a three dimensional plane will look like a pole (or pillar like structure) and so the point $s = p_i$ is called a pole.

3.4.1 Representation of Poles and Zeros in s-Plane

We know that, Complex frequency, $s = \sigma + j\Omega$

where, σ = Real part of s, Ω = Imaginary part of s

Hence the s-plane is a complex plane, with σ on real axis and Ω on imaginary axis as shown in fig 3.5. In the s-plane, the zeros are marked by small circle " **o** " and the poles are marked by letter "**X**".

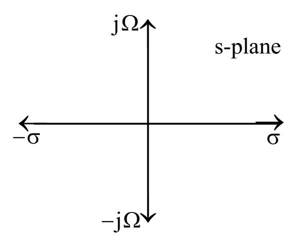


Fig 3.5: *s-plane.*

For example consider the rational function of s shown below.

$$X(s) = \frac{(s+2)(s+5)}{s(s^2 + 6s + 13)}$$

The roots of quadratic $s^2 + 6s + 13 = 0$ are,

$$s = \frac{-6 \pm \sqrt{36 - 4 \times 13}}{2} = \frac{-6 \pm j4}{2} = -3 + j\,2, \;\; -3 - j\,2$$

$$\therefore s^2 + 6s + 13 = (s + 3 - j2)(s + 3 + j2)$$

$$\therefore X(s) = \frac{(s+2)\,(s+5)}{s(s^2+6s+13)} = \frac{(s+2)\,(s+5)}{s(s+3-j2)\,(s+3+j2)} \qquad .....(3.28)$$

The zeros of the above function are,

$z_1 = -2$

$z_2 = -5$

The poles of the above function are,

$p_1 = 0$

$p_2 = -3 + j2$

$p_3 = -3 - j2$

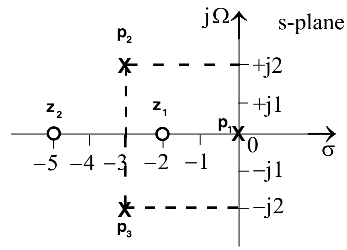


***Fig 3.6 :** Pole-zero plot of the function of equation (3.28).*

The pole-zero plot of rational function of s of equation (3.28) is shown in fig 3.6.

3.4.2 ROC of Rational Function of s

<u>Case i :</u> Right sided (causal) signal

Let x(t) be a right sided (causal) signal defined as,

$$x(t) = e^{-a_1 t}\,u(t) + e^{-a_2 t}\,u(t) + e^{-a_3 t}\,u(t) \;\; ; \;\; \text{where } -a_1 < -a_2 < -a_3$$

Now, the Laplace transform of x(t) is,

$$X(s) = \frac{1}{s + a_1} + \frac{1}{s + a_2} + \frac{1}{s + a_3} = \frac{N(s)}{(s + a_1)\,(s + a_2)\,(s + a_3)}$$

where, $N(s) = (s + a_2)\,(s + a_3) + (s + a_1)\,(s + a_3) + (s + a_1)\,(s + a_2)$

The poles of X(s) are,

$p_1 = -a_1 \;\; ; \;\; p_2 = -a_2 \;\; ; \;\; p_3 = -a_3$

Let, σ = Real part of s.

Now, the convergence criteria for X(s) are,

$\sigma > -a_1 \;\; ; \;\; \sigma > -a_2 \;\; ; \;\; \sigma > -a_3$

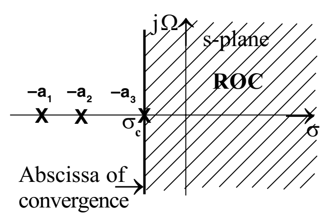


***Fig 3.7 :** ROC of a causal signal.*

Since $-a_1 < -a_2 < -a_3$, the X(s) converges, when $\sigma > -a_3$, and does not converge for $\sigma < -a_3$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a_3$.

Therefore ROC of X(s) is all points to the right of abscissa of convergence (or right of the line passing through $-a_3$) in s-plane as shown in fig.3.7. *In terms of poles of X(s) we can say that the ROC is right of right most pole of X(s), (i.e., right of the pole with largest real part).*

Case ii : Left sided (anticausal) signal

Let x(t) be a left sided (anticausal) signal defined as,

$$x(t) = e^{-a_1 t}\, u(-t) + e^{-a_2 t}\, u(-t) + e^{-a_3 t}\, u(-t) \quad ; \quad \text{where } -a_1 < -a_2 < -a_3$$

Now, the Laplace transform of x(t) is,

$$X(s) = -\frac{1}{s+a_1} - \frac{1}{s+a_2} - \frac{1}{s+a_3} = \frac{N(s)}{(s+a_1)(s+a_2)(s+a_3)}$$

where, $N(s) = -(s + a_2)(s + a_3) - (s + a_1)(s + a_3) - (s + a_1)(s + a_2)$

The poles of X(s) are,

$p_1 = -a_1 \quad ; \quad p_2 = -a_2 \quad ; \quad p_3 = -a_3$

Let, σ = Real part of s.

Now the convergence criteria for X(s) are,

$\sigma < -a_1 \quad ; \quad \sigma < -a_2 \quad ; \quad \sigma < -a_3$

Since $-a_1 < -a_2 < -a_3$, the X(s) converges, when $\sigma < -a_1$, and does not converge for $\sigma > -a_1$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a_1$.

Therefore ROC of X(s) is all points to the left of abscissa of convergence (or left of the line passing through $-a_1$) in s-plane as shown in fig.3.8. *In terms of poles of X(s) we can say that the ROC is left of left most pole of X(s), (i.e., left of the pole with smallest real part).*

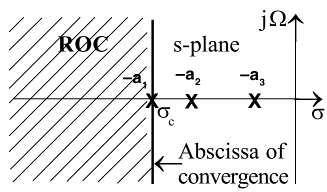


Fig 3.8 : *ROC of an anticausal signal.*

Case iii : Two sided signal

Let x(t) be a two sided signal defined as,

$$x(t) = e^{-a_1 t}\, u(t) + e^{-a_2 t}\, u(t) + e^{-a_3 t}\, u(-t) + e^{-a_4 t}\, u(-t) \quad ; \quad \text{where } -a_1 < -a_2 < -a_3 < -a_4$$

Now, the Laplace transform of x(t) is,

$$X(s) = \frac{1}{s+a_1} + \frac{1}{s+a_2} - \frac{1}{s+a_3} - \frac{1}{s+a_4}$$

$$= \frac{N(s)}{(s+a_1)(s+a_2)(s+a_3)(s+a_4)}$$

where, $N(s) = (s + a_2)(s + a_3)(s + a_4) + (s + a_1)(s + a_3)(s + a_4)$
$\qquad - (s + a_1)(s + a_2)(s + a_4) - (s + a_1)(s + a_2)(s + a_3)$

The poles of X(s) are,

$p_1 = -a_1 \quad ; \quad p_2 = -a_2 \quad ; \quad p_3 = -a_3 \quad ; \quad p_4 = -a_4$

Let, σ = Real part of s.

Now, the convergence criteria for X(s) are,

$\sigma > -a_1 \quad ; \quad \sigma > -a_2 \quad ; \quad \sigma < -a_3 \quad ; \quad \sigma < -a_4$

Since $-a_1 < -a_2 < -a_3 < -a_4$, the function X(s) converges, when σ lies between $-a_2$ and $-a_3$ (i.e., $-a_2 < \sigma < -a_3$), and does not converge for $\sigma < -a_2$ and $\sigma > -a_3$.

$\therefore$ Abscissa of convergences, $\sigma_{c1} = -a_2$ and $\sigma_{c2} = -a_3$.

Therefore ROC of X(s) is all points in the region in between two abscissa of convergences (or region inbetween the two lines passing through $-a_2$ and $-a_3$) in s-plane as shown in fig.3.9.

Let X(s) be s-domain representation of a signal with causal and noncausal part. Let a_x be the magnitude of largest pole of causal part of the signal and let a_y be the magnitude of smallest pole of anticausal part of the signal and let $a_x < a_y$. *Now in term of poles of X(s) we can say that the ROC is the region inbetween two lines passing through a_x and a_y, where $a_x < a_y$.*

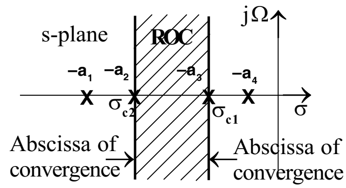


***Fig 3.9 :** ROC of a two sided signal.*

3.4.3 Properties of ROC

The various concepts of ROC that has been discussed in section 3.2 and 3.4.2 are summarized as properties of ROC and given below.

Property-1 : The ROC of X(s) consists of strips parallel to the jΩ - axis in the s-plane.

Property-2 : If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s- plane.

Property-3 : If x(t) is right sided, and if the line passing through Re(s) = σ_0 is in ROC, then all values of s for which Re(s) > σ_0 will also be in ROC.

Property-4 : If x(t) is left sided, and if the line passing through Re(s) = σ_0 is in ROC, then all values of s for which Re(s) < σ_0 will also be in ROC.

Property-5 : If x(t) is two sided, and if the line passing through Re(s) = σ_0 is in ROC, then the ROC will consists of a strip in the s-plane that includes the line passing through Re(s) = σ_0 .

Property-6 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then its ROC is bounded by poles or extends to infinity.

Property-7 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then ROC does not include any poles of X(s).

Property-8 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is right sided, then ROC is the region in s-plane to the right of the rightmost pole.

Property-9 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is left sided, then ROC is the region in s-plane to the left of the leftmost pole.

Example 3.11

Determine the poles and zeros of the rational function of s given below. Also sketch the pole-zero plot.

$$X(s) = \frac{(s + 1)\left(s^2 + 10s + 41\right)}{(s + 4)\left(s^2 + 4s + 13\right)}$$

Solution

In the given function, both the numerator and denominator polynomials are third order polynomials. Hence the given function has three zeros and three ploes.

Let z_1, z_2, z_3, be zeros and p_1, p_2, p_3, be poles of X(s).

To Determine Zeros

Consider the numerator polynomial.

$$(s + 1)(s^2 + 10s + 41) = 0$$

The roots of quadratic $s^2 + 10s + 41 = 0$ are,

$$s = \frac{-10 \pm \sqrt{10^2 - 4 \times 41}}{2} = \frac{-10 \pm \sqrt{-64}}{2} = \frac{-10}{2} \pm \frac{j\sqrt{64}}{2} = -5 \pm j4 = -5 + j4,\ -5 - j4$$

$$\therefore (s + 1)(s^2 + 10s + 41) = (s + 1)(s + 5 - j4)(s + 5 + j4) = 0$$

$\therefore$ The roots of numerator polynomial are,

$$s = -1\ ;\ s = -5 + j4\ ;\ s = -5 - j4$$

$\therefore$ Zeros of X(s) are,

$$z_1 = -1\ ;\ z_2 = -5 + j4\ ;\ z_3 = -5 - j4$$

To Determine Poles

Consider the denominator polynomial.

$$(s + 4)(s^2 + 4s + 13) = 0$$

The roots of quadratic $s^2 + 4s + 13 = 0$ are,

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = \frac{-4 \pm \sqrt{-36}}{2}$$

$$= \frac{-4}{2} \pm \frac{j\sqrt{36}}{2} = -2 \pm j3 = -2 + j3,\ -2 - j3$$

$$\therefore\ (s + 4)(s^2 + 4s + 13) = (s + 4)(s + 2 - j3)(s + 2 + j3) = 0$$

$\therefore$ The roots of denominator polynomial are,

$$s = -4\ ;\ s = -2 + j3\ ;\ s = -2 - j3$$

$\therefore$ Poles of X(s) are,

$$p_1 = -4\ ;\ p_2 = -2 + j3\ ;\ p_3 = -2 - j3$$

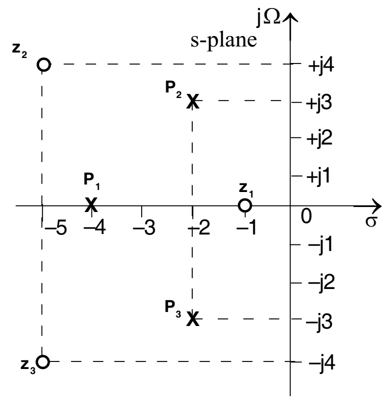


***Fig 1 :** Pole-zero plot.*

Pole-Zero Plot

The pole-zero plot of function X(s) is shown in fig 1. The zeros are denoted by "o" and poles are denoted by "**X**".

3.5 Inverse Laplace Transform

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j} \int_{s = \sigma - j\Omega}^{s = \sigma + j\Omega} X(s)\, e^{st}\, ds}$$

Performing inverse Laplace transform by using the above fundamental definition is tedious. But the inverse Laplace transform by partial fraction expansion method will be much easier. In this method the s-domain signal is expressed as a sum of first order and second order sections. Then the inverse Laplace transform is obtained by comparing each section with standard transform pair, (listed in table 3.2).

In the following section the inverse Laplace transform by partial fraction exapansion method is explained with example.

3.5.1 Inverse Laplace Transform by Partial Fraction Expansion Method

Let Laplace transform of x(t) be X(s). The s-domain signal X(s) will be a ratio of two polynomials in s (i.e., rational function of s). The roots of the denominator polynomial are called poles. The roots of numerator polynomials are called zeros. (For definition of poles and zeros please refer section 3.4). In signals and systems, three different types of s-domain signals are encountered. They are,

Case i : Signals with separate poles.

Case ii : Signals with multiple poles.

Case iii : Signals with complex conjugate poles.

The inverse Laplace transform by partial fraction expansion method of all the three cases are explained with an example.

Case - i : When s-Domain Signal X(s) has Distinct Poles

$$\text{Let,}\quad X(s) = \frac{K}{s(s+p_1)(s+p_2)} \qquad .....(3.29)$$

By partial fraction expansion technique, the equation (3.29) can be expressed as,

$$X(s) = \frac{K}{s(s+p_1)(s+p_2)} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2} \qquad .....(3.30)$$

The residues K_1, K_2 and K_3 are given by,

$$K_1 = X(s) \times s \big|_{s=0}$$

$$K_2 = X(s) \times (s+p_1) \big|_{s=-p_1}$$

$$K_3 = X(s) \times (s+p_2) \big|_{s=-p_2}$$

$$\text{We Know that, } \mathcal{L}\{u(t)\} = \frac{1}{s},\quad \mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}$$

By using the above standard Laplace transform pairs the inverse Laplace transform of equation (3.30) can be obtained as shown below.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{s+p_2}\right\}$$

$$\therefore\ x(t) = K_1\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + K_2\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_1}\right\} + K_3\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_2}\right\}$$

$$= K_1\,u(t) + K_2\,e^{-p_1 t}\,u(t) + K_3\,e^{-p_2 t}\,u(t)$$

Example 3.12

Determine the inverse Laplace transform of $X(s) = \dfrac{2}{s(s+1)(s+2)}$

Solution

Given that, $X(s) = \dfrac{2}{s(s+1)(s+2)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

The residue K_1 is obtained by multiplying X(s) by s and letting s = 0.

$$K_1 = X(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s\Bigg|_{s=0} = \frac{2}{(s+1)(s+2)}\Bigg|_{s=0} = \frac{2}{1 \times 2} = 1$$

The residue K_2 is obtained by multiplying X(s) by (s + 1) and letting s = –1.

$$K_2 = X(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)} \times (s+1)\Bigg|_{s=-1} = \frac{2}{s(s+2)}\Bigg|_{s=-1} = \frac{2}{-1(-1+2)} = -2$$

The residue K_3 is obtained by multiplying X(s) by (s + 2) and letting s = –2.

$$K_3 = X(s) \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)} \times (s+2)\Bigg|_{s=-2} = \frac{2}{s(s+1)}\Bigg|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

$$\therefore\ X(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

Now, $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right\}$

$$= u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$$

$$= (1 - 2e^{-t} + e^{-2t})\,u(t) = (1 - e^{-t})^2\,u(t)$$

$\boxed{(x-y)^2 = x^2 - 2xy + y^2}$

Case - ii : When s-Domain Signal X(s) has Multiple Poles

Let, $$X(s) = \frac{K}{s(s+p_1)(s+p_2)^2} \qquad .....(3.31)$$

By partial fraction expansion technique, the equation (3.31) can be expressed as,

$$X(s) = \frac{K}{s(s+p_1)(s+p_2)^2} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{(s+p_2)} \qquad .....(3.32)$$

The residues K_1, K_2, K_3, and K_4 are given by,

$$K_1 = X(s) \times s\Big|_{s=0} \quad ; \quad K_2 = X(s) \times (s+p_1)\Big|_{s=-p_1}$$

$$K_3 = X(s) \times (s+p_2)^2\Big|_{s=-p_2} \quad ; \quad K_4 = \frac{d}{ds}\Big[X(s) \times (s+p_2)^2\Big]\Bigg|_{s=-p_2}$$

We know that, $\mathcal{L}\{u(t)\} = \frac{1}{s}$, $\mathcal{L}\{e^{-at}\,u(t)\} = \frac{1}{s+a}$, $\mathcal{L}\{t\,e^{-at}\,u(t)\} = \frac{1}{(s+a)^2}$

By using the above standard Laplace transform pairs the inverse Laplace transform of equation (3.31) can be obtained as shown below.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{s+p_2}\right\}$$

$$\therefore\ x(t) = K_1\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + K_2\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_1}\right\} + K_3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+p_2)^2}\right\} + K_4\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_2}\right\}$$

$$= K_1\,u(t) + K_2\,e^{-p_1 t}\,u(t) + K_3\,t\,e^{-p_2 t}\,u(t) + K_4\,e^{-p_2 t}\,u(t)$$

In general if the pole p_2 in equation (3.31) has multiplicity of q, as shown below,

$$X(s) = \frac{K}{s(s + p_1)(s + p_2)^q}$$

then the partial fraction of the above signal can be expressed as,

$$X(s) = \frac{K}{s(s + p_1)(s + p_2)^q} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^q} + \frac{K_{12}}{(s+p_2)^{q-1}} + \frac{K_{22}}{(s+p_2)^{q-2}} + \ ...... \ + \frac{K_{(q-1)2}}{s+p_2}$$

The residue K_1, K_2, K_3 are evaluated as shown above. The q – 1 residues $K_{12}, K_{22}, ... K_{(q-1)2}$ can be evaluated using the equation,

$$K_{r2} = \frac{1}{r!}\frac{d^r}{dq^r}\left[X(s) \times (s+2)^q\right] \quad ; \quad r = 1, 2, ... q-1$$

Example 3.13

Determine the inverse Laplace transform of $X(s) = \dfrac{2}{s(s + 1)(s + 2)^2}$

Solution

Given that, $X(s) = \dfrac{2}{s(s+1)(s+2)^2}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{K}{s(s + 1)(s + 2)^2} = \frac{K_1}{s} + \frac{K_2}{(s + 1)} + \frac{K_3}{(s + 2)^2} + \frac{K_4}{(s + 2)}$$

The residue K_1 is obtained by multiplying X(s) by s and letting s = 0.

$$K_1 = X(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s\Bigg|_{s=0} = \frac{2}{(s+1)(s+2)^2}\Bigg|_{s=0} = \frac{2}{1 \times 2^2} = 0.5$$

The residue K_2 is obtained by multiplying X(s) by (s + 1) and letting s = –1.

$$K_2 = X(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)^2} \times (s+1)\Bigg|_{s=-1} = \frac{2}{-1(-1 + 2)^2} = -2$$

The residue K_3 is obtained by multiplying X(s) by $(s + 2)^2$ and letting s = –2.

$$K_3 = X(s) \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\Bigg|_{s=-2} = \frac{2}{-2(-2 + 1)} = 1$$

The residue K_4 is obtained by differentiating the product $X(s) \times (s + 2)^2$ with respect to s and then letting s = –2.

$$K_4 = \frac{d}{ds}\left[X(s) \times (s+2)^2\right]\Bigg|_{s=-2} = \frac{d}{ds}\left[\frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\right]\Bigg|_{s=-2}$$

$$= \frac{d}{ds}\left[\frac{2}{s(s+1)}\right]\Bigg|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Bigg|_{s=-2} = \frac{-2(2(-2) + 1)}{(-2)^2(-2 + 1)^2} = 1.5$$

$\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$	
$u = 2$	$v = s(s+1)$ $= s^2 + s$

$$\therefore X(s) = \frac{2}{s(s + 1)(s + 2)^2} = \frac{0.5}{s} - \frac{2}{s + 1} + \frac{1}{(s + 2)^2} + \frac{1.5}{s + 2}$$

Now, $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\dfrac{0.5}{s} - \dfrac{2}{s + 1} + \dfrac{1}{(s + 2)^2} + \dfrac{1.5}{s + 2}\right\}$

$$= 0.5\,u(t) - 2\,e^{-t}\,u(t) + t\,e^{-2t}\,u(t) + 1.5\,e^{-2t}\,u(t)$$

$$= \left(0.5 - 2\,e^{-t} + t\,e^{-2t} + 1.5\,e^{-2t}\right) u(t)$$

Case iii : ***When s-Domain Signal X(s) has Complex Conjugate Poles***

Let, $X(s) = \dfrac{K}{(s + p_1)(s^2 + bs + c)}$ .....(3.33)

By partial fraction expansion technique, the equation (3.33) can be expressed as,

$$X(s) = \frac{K}{(s + p_1)(s^2 + bs + c)} = \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + bs + c} \quad ..... (3.34)$$

The residue K_1 is given by, $K_1 = X(s) \times (s + p_1)\big|_{s = -p_1}$

The residues K_2 and K_3 are solved by cross multiplying the equation (3.34) and then equating the coefficients of like power of s.

Finally express X(s) as shown below,

$$\boxed{\text{Arranging, } s^2 + bs, \text{ in the form of } (x+y)^2}$$

$$\boxed{(x+y)^2 = x^2 + 2xy + y^2}$$

$$X(s) = \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + bs + c}$$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + 2 \times \frac{b}{2} s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{\left(s + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

$$\boxed{\text{Put, } \frac{b}{2} = a \text{ and } c - \frac{b^2}{4} = \Omega_0^2}$$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{(s + a)^2 + \Omega_0^2}$$

$$X(s) = \frac{K_1}{s + p_1} + K_2 \frac{s + \frac{K_3}{K_2}}{(s + a)^2 + \Omega_0^2} = \frac{K_1}{s + p_1} + K_2 \frac{s + a + \frac{K_3}{K_2} - a}{(s + a)^2 + \Omega_0^2}$$

$$= \frac{K_1}{s + p_1} + K_2 \frac{s + a + K_4}{(s + a)^2 + \Omega_0^2}$$

$$\boxed{\text{Put, } \frac{K_3}{K_2} - a = K_4}$$

$$= \frac{K_1}{s + p_1} + K_2 \frac{s + a}{(s + a)^2 + \Omega_0^2} + \frac{K_2 K_4}{\Omega_0} \frac{\Omega_0}{(s + a)^2 + \Omega_0^2}$$

$$\boxed{\text{Put, } \frac{K_2 K_4}{\Omega_0} = K_5}$$

$$\therefore\ X(s) = \frac{K_1}{s + p_1} + K_2 \frac{s + a}{(s + a)^2 + \Omega_0^2} + K_5 \frac{\Omega_0}{(s + a)^2 + \Omega_0^2} \quad .....(3.35)$$

We know that, $\mathcal{L}\left\{e^{-at}u(t)\right\} = \dfrac{1}{s+a}$; $\mathcal{L}\left\{e^{-at}\cos\Omega_0 t\, u(t)\right\} = \dfrac{s+a}{(s+a)^2 + \Omega_0^2}$;

$$\mathcal{L}\left\{e^{-at}\sin\Omega_0 t\, u(t)\right\} = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$$

By using the above standard Laplace transform pairs the inverse Laplace transform of equation (3.35) can be obtained as shown below.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s + p_1} + K_2 \frac{s + a}{(s + a)^2 + \Omega_0^{\,2}} + K_5 \frac{\Omega_0}{(s + a)^2 + \Omega_0^{\,2}}\right\}$$

$$\therefore x(t) = K_1\, \mathcal{L}^{-1}\left\{\frac{1}{s + p_1}\right\} + K_2\, \mathcal{L}^{-1}\left\{\frac{s + a}{(s + a)^2 + \Omega_0^{\,2}}\right\} + K_5\, \mathcal{L}^{-1}\left\{\frac{\Omega_0}{(s + a)^2 + \Omega_0^{\,2}}\right\}$$

$$= K_1\, e^{-p_1 t}\, u(t) + K_2\, e^{-at}\, \cos\Omega_0 t\, u(t) + K_5\, e^{-at}\, \sin\Omega_0 t\, u(t)$$

Example 3.14

Determine the inverse Laplace transform of $X(s) = \dfrac{1}{(s + 2)(s^2 + s + 1)}$

Solution

Given that, $X(s) = \dfrac{1}{(s + 2)(s^2 + s + 1)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{1}{(s + 2)(s^2 + s + 1)} = \frac{K_1}{s + 2} + \frac{K_2 s + K_3}{s^2 + s + 1}$$

The residue K_1 is obtained by multiplying X(s) by (s + 2) and letting s = –2.

$$\therefore K_1 = X(s) \times (s+2)\Big|_{s=-2} = \frac{1}{(s+2)(s^2+s+1)} \times (s+2)\Bigg|_{s=-2} = \frac{1}{(-2)^2 - 2 + 1} = \frac{1}{3}$$

To solve K_2 and K_3, cross multiply the following equation and substitute the value of K_1. Then equate the coefficients of like power of s.

$$\frac{1}{(s + 2)(s^2 + s + 1)} = \frac{K_1}{s + 2} + \frac{K_2 s + K_3}{s^2 + s + 1}$$

$$1 = K_1 \left(s^2 + s + 1\right) + (K_2 s + K_3)(s + 2)$$

$$1 = \frac{1}{3}\left(s^2 + s + 1\right) + K_2 s^2 + 2K_2 s + K_3 s + 2K_3$$

$$1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + K_2 s^2 + 2K_2 s + K_3 s + 2K_3$$

$$1 = \left(\frac{1}{3} + K_2\right)s^2 + \left(\frac{1}{3} + 2K_2 + K_3\right)s + \frac{1}{3} + 2K_3$$

On equating the coefficients of s^2 terms,

$$0 = \frac{1}{3} + K_2 \quad \Rightarrow \quad K_2 = -\frac{1}{3}$$

On equating the coefficients of s terms,

$$0 = \frac{1}{3} + 2K_2 + K_3 \quad \Rightarrow \quad K_3 = -\frac{1}{3} - 2K_2 = -\frac{1}{3} - 2\times\left(-\frac{1}{3}\right) = \frac{1}{3}$$

$$\therefore X(s) = \frac{1}{(s + 2)(s^2 + s + 1)} = \frac{\frac{1}{3}}{s+2} + \frac{\frac{-1}{3}s + \frac{1}{3}}{s^2 + s + 1} = \frac{\frac{1}{3}}{s+2} + \frac{-\frac{1}{3}(s-1)}{s^2 + s + 1}$$

Arranging, s^2+s, in the form of $(x+y)^2$

$\boxed{(x+y)^2 = x^2+2xy+x^2}$

$$= \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s-1}{(s^2 + (2 \times 0.5s) + 0.5^2) + (1 - 0.5^2)} = \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s - 1}{(s + 0.5)^2 + 0.75}$$

$$= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5-1-0.5}{(s+0.5)^2+\left(\sqrt{0.75}\right)^2} = \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5-1.5}{(s+0.5)^2+0.866^2}$$

$$= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5}{(s+0.5)^2+0.866^2} + \frac{1}{3}\times\frac{1.5}{0.866}\frac{0.866}{(s+0.5)^2+0.866^2}$$

$$\therefore\ x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s+0.5}{(s+0.5)^2+0.866^2} + 0.577\frac{0.866}{(s+0.5)^2+0.866^2}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{s+0.5}{(s+0.5)^2+(0.866)^2}\right\} + 0.577\,\mathcal{L}^{-1}\left\{\frac{0.866}{(s+0.5)^2+0.866^2}\right\}$$

$$= \frac{1}{3}e^{-2t}\,u(t) - \frac{1}{3}e^{-0.5t}\cos 0.866t\,u(t) + 0.577\,e^{-0.5t}\sin 0.866t\,u(t)$$

3.5.2 Inverse Laplace Transform Using Convolution Theorem

The convolution theorem of equation (3.16) is useful to evaluate the inverse Laplace transform of complicated s-domain signals.

Let x(t) be inverse Laplace transform of X(s). Let, the s-domain signal X(s) be expressed as a product of two s-domain signals $X_1(s)$ and $X_2(s)$. Let, $x_1(t)$ and $x_2(t)$ be inverse Laplace transform of $X_1(s)$ and $X_2(s)$ respectively.

Now, the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ are computed separately to get $x_1(t)$ and $x_2(t)$, and then the inverse Laplace transform of X(s) is obtained by convolution of $x_1(t)$ and $x_2(t)$, as shown below.

Let, $X(s) = X_1(s)\,X_2(s)$, $\mathcal{L}^{-1}\{X(s)\} = x(t)$, $\mathcal{L}^{-1}\{X_1(s)\} = x_1(t)$, $\mathcal{L}^{-1}\{X_2(s)\} = x_2(t)$.

First determine inverse Laplace transform of $X_1(s)$ and $X_2(s)$, to get $x_1(t)$ and $x_2(t)$.

Now, by convolution property,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

Here, $X(s) = X_1(s)\,X_2(s)$.

$$\therefore X(s) = \mathcal{L}\{x_1(t) * x_2(t)\}$$

On taking inverse Laplace transform of the above equation we get,

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda \qquad \boxed{\text{Using equation (3.17)}}$$

Example 3.15

Find the inverse Laplace transform of the s-domain signal, $X(s) = \dfrac{4}{s^2(s^2+16)}$ using convolution theorem.

Solution

$$\text{Let,}\quad X(s) = \frac{4}{s^2(s^2+16)} = \frac{4}{s^2(s^2+4^2)} = \frac{4}{s^2+4^2}\times\frac{1}{s^2} = X_1(s)\,X_2(s)$$

$$\text{where,}\quad X_1(s) = \frac{4}{s^2+4^2} \quad\text{and}\quad X_2(s) = \frac{1}{s^2}$$

$$x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \sin 4t\,u(t)$$

$$x_2(t) = \mathcal{L}^{-1}\{X_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\,u(t)$$

$$\therefore x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\}$$

By convolution theorem, the inverse Laplace transform of $X_1(s)\,X_2(s)$ is given by convolution of $x_1(t)$ and $x_2(t)$.

$$\therefore x(t) = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\} = x_1(t) * x_2(t)$$

Since $x_1(t)$ and $x_2(t)$ are causal, limits of integration are changed to 0 to t.

$$= \int_{-\infty}^{+\infty} x_1(\lambda)\,x_2(t-\lambda)\,d\lambda = \int_0^t \sin 4\lambda\,(t-\lambda)\,d\lambda = \int_0^t (t-\lambda)\sin 4\lambda\,d\lambda$$

$$= \left[(t-\lambda)\left(\frac{-\cos 4\lambda}{4}\right) - \int(-1)\left(\frac{-\cos 4\lambda}{4}\right)d\lambda\right]_0^t$$

$\int u\,v = u\int v - \int\left[du\int v\right]$	
$u = t-\lambda$	$v = \sin 4\lambda$

$$= \left[-(t-\lambda)\left(\frac{\cos 4\lambda}{4}\right) - \frac{\sin 4\lambda}{16}\right]_0^t$$

$$= \left[-(t-t)\left(\frac{\cos 4t}{4}\right) - \frac{\sin 4t}{16} + (t-0)\,\frac{\cos 0}{4} + \frac{\sin 0}{16}\right]$$

$\cos 0 = 1,\quad \sin 0 = 0$

$$= 0 - \frac{\sin 4t}{16} + \frac{t}{4} + 0 = \frac{1}{4}\left(t - \frac{\sin 4t}{4}\right);\ t \ge 0 = \frac{1}{4}\left(t - \frac{\sin 4t}{4}\right)u(t)$$

Example 3.16

Find the inverse Laplace transform of the following s-domain signals.

a) $X(s) = \dfrac{3s^2+8s+23}{(s+3)\,(s^2+2s+10)}$

b) $X(s) = \dfrac{8s^2+11s}{(s+2)\,(s+1)^3}$

Solution

a) Given that, $\mathbf{X(s) = \dfrac{3s^2+8s+23}{(s+3)\,(s^2+2s+10)}}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{3s^2+8s+23}{(s+3)\,(s^2+2s+10)} = \frac{K_1}{s+3} + \frac{K_2\,s + K_3}{s^2+2s+10}$$

The residue K_1 is obtained by multiplying X(s) by (s + 3) and letting s = –3.

$$\therefore K_1 = X(s)\times(s+3)\Big|_{s=-3} = \frac{3s^2+8s+23}{(s+3)\,(s^2+2s+10)}\times(s+3)\Bigg|_{s=-3}$$

$$= \frac{3\times(-3)^2+8\times(-3)+23}{(-3)^2+2\times(-3)+10} = \frac{27-24+23}{9-6+10} = \frac{26}{13} = 2$$

To solve K_2 and K_3 cross multiply the following equation and substitute the value of K_1. Then equate the coefficients of like power of s.

$$\frac{3s^2+8s+23}{(s+3)\,(s^2+2s+10)} = \frac{K_1}{s+3} + \frac{K_2\,s + K_3}{s^2+2s+10}$$

$$3s^2+8s+23 = K_1\left(s^2+2s+10\right) + (K_2\,s + K_3)\,(s+3)$$

$$3s^2+8s+23 = K_1s^2+2K_1s+10K_1+K_2\,s^2+3\,K_2\,s+K_3\,s+3\,K_3$$

$$3s^2+8s+23 = (K_1+K_2)s^2+(2K_1+3K_2+K_3)s+10K_1+3\,K_3$$

On equating the coefficients of s^2 terms, we get,

$$3 = K_1+K_2 \qquad \Rightarrow \qquad K_2 = 3-K_1 = 3-2 = 1$$

On equating the coefficients of s terms, we get,

$$8 = 2K_1 + 3K_2 + K_3 \quad \Rightarrow \quad K_3 = 8 - 2K_1 - 3K_2 = 8 - 2\times 2 - 3\times 1 = 1$$

$$\therefore X(s) = \frac{3s^2+8s+23}{(s+3)(s^2+2s+10)} = \frac{2}{s+3} + \frac{s+1}{s^2+2s+10}$$

$$= \frac{2}{s+3} + \frac{s+1}{s^2+2s+1+9} = \frac{2}{s+3} + \frac{s+1}{(s+1)^2+3^2}$$

$$\therefore x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+3} + \frac{s+1}{(s+1)^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = 2e^{-3t}u(t) + e^{-t}\cos 3t\, u(t)$$

---

b) Given that, $X(s) = \dfrac{8s^2+11s}{(s+2)(s+1)^3}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{8s^2+11s}{(s+2)(s+1)^3} = \frac{K_1}{s+2} + \frac{K_2}{(s+1)^3} + \frac{K_3}{(s+1)^2} + \frac{K_4}{s+1}$$

The residue K_1 is obtained by multiplying X(s) by (s + 2) and letting s = –2.

$$\therefore K_1 = X(s) \times (s+2)\Big|_{s=-2} = \frac{8s^2+11s}{(s+2)(s+1)^3} \times (s+2)\Bigg|_{s=-2} = \frac{8\times(-2)^2+11\times(-2)}{(-2+1)^3} = -10$$

The residue K_2 is obtained by multiplying X(s) by $(s+1)^3$ and letting s = –1.

$$\therefore K_2 = X(s) \times (s+1)^3\Big|_{s=-1} = \frac{8s^2+11s}{(s+2)(s+1)^3} \times (s+1)^3\Bigg|_{s=-1} = \frac{8\times(-1)^2+11\times(-1)}{(-1+2)^3} = -3$$

The residue K_3 is obtained by differentiating the product $X(s)\times(s+1)^3$ with respect to s and then letting s = –1.

$$\therefore K_3 = \frac{d}{ds}\left[X(s) \times (s+1)^3\right]\Bigg|_{s=-1} = \frac{d}{ds}\left[\frac{8s^2+11s}{(s+2)(s+1)^3} \times (s+1)^3\right]\Bigg|_{s=-1}$$

$$= \frac{d}{ds}\left[\frac{8s^2+11s}{s+2}\right]\Bigg|_{s=-1} = \frac{(16s+11)(s+2)-(8s^2+11s)\times 1}{(s+2)^2}\Bigg|_{s=-1}$$

$$\boxed{\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}}$$

$$= \frac{16s^2+32s+11s+22-8s^2-11s}{(s+2)^2}\Bigg|_{s=-1}$$

$$= \frac{8s^2+32s+22}{(s+2)^2}\Bigg|_{s=-1} = \frac{8\times(-1)^2-32+22}{(-1+2)^2} = -2$$

The residue K_4 is obtained by differentiating the product $X(s)\times(s+1)^3$ twice with respect to s and then dividing by 2! and letting s = –1.

$$\therefore K_4 = \frac{1}{2!}\frac{d^2}{ds^2}\left[X(s) \times (s+1)^3\right]\Bigg|_{s=-1} = \frac{1}{2}\frac{d}{ds}\left[\frac{d}{ds}\left[X(s) \times (s+1)^3\right]\right]\Bigg|_{s=-1}$$

$$= \frac{1}{2}\frac{d}{ds}\left[\frac{8s^2+32s+22}{(s+2)^2}\right]\Bigg|_{s=-1} = \frac{1}{2}\,\frac{(16s+32)(s+2)^2-(8s^2+32s+22)\,2(s+2)}{(s+2)^4}\Bigg|_{s=-1}$$

$$= \frac{1}{2}\,\frac{(16\times(-1)+32)(-1+2)^2-(8\times(-1)^2+32\times(-1)+22)\,2(-1+2)}{(-1+2)^4}$$

$$= \frac{1}{2}\times\frac{-16+32-(8-32+22)\times 2}{1} = 10$$

$$\therefore X(s) = \frac{8s^2 + 11s}{(s+2)\,(s+1)^3} = \frac{-10}{s+2} - \frac{3}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{10}{s+1}$$

$$\therefore\ x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{-10}{s+2} - \frac{3}{(s+1)^3} - \frac{2}{(s+1)^2} + \frac{10}{s+1}\right\}$$

$$= -10\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + 10\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -10\,e^{-2t}\,u(t) - 3\times\frac{1}{2!}\,t^2\,e^{-t}\,u(t) - 2\,t\,e^{-t}\,u(t) + 10\,e^{-t}\,u(t)$$

$$= (-10\,e^{-2t} - 1.5\,t^2\,e^{-t} - 2\,t\,e^{-t} + 10\,e^{-t})\,u(t)$$

$$= \left[(10 - 2\,t - 1.5\,t^2)e^{-t} - 10\,e^{-2t}\right]u(t)$$

Example 3.17

Find the inverse Laplace transform of $X(s) = \dfrac{4}{(s+2)\,(s+4)}$ if the ROC is,

i) $-2 > \text{Re}\{s\} > -4$ ii) $\text{Re}\{s\} < -4$ iii) $\text{Re}\{s\} > -2$

Solution

Given that , $\mathbf{X(s) = \dfrac{4}{(s+2)\,(s+4)}}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{4}{(s+2)\,(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

The residue K_1 is obtained by multiplying X(s) by (s + 2) and letting s = –2.

$$\therefore\ K_1 = X(s) \times (s+2)\Big|_{s=-2} = \frac{4}{(s+2)\,(s+4)} \times (s+2)\Bigg|_{s=-2} = \frac{4}{-2+4} = 2$$

The residue K_2 is obtained by multiplying X(s) by (s + 4) and letting s = –4.

$$\therefore\ K_2 = X(s) \times (s+4)\Big|_{s=-4} = \frac{4}{(s+2)\,(s+4)} \times (s+4)\Bigg|_{s=-4} = \frac{4}{-4+2} = -2$$

$$\therefore X(s) = \frac{4}{(s+2)\,(s+4)} = \frac{2}{s+2} - \frac{2}{s+4}$$

Case i :

Given that ROC lies between lines passing through s = –2 to s = –4. Hence x(t) will be two sided signal.

The term corresponding to the pole, p = –2 will be anticausal signal and the term corresponding to the pole, p = –4 will be causal signal.

$$\therefore\ x(t) = -2\,e^{-2t}\,u(-t) - 2\,e^{-4t}\,u(t)$$

Case ii :

Given that ROC is left of the line passing through s = –4. Hence x(t) will be anticausal signal.

$$\therefore\ x(t) = -2\,e^{-2t}\,u(-t) + 2\,e^{-4t}\,u(-t) = 2\left[e^{-4t} - e^{-2t}\right]u(-t)$$

Case iii :

Given that ROC is right of the line passing through s = – 2. Hence x(t) will be causal signal.

$$\therefore\ x(t) = 2\,e^{-t}\,u(t) - 2\,e^{-4t}\,u(t) = 2\left[e^{-t} - e^{-4t}\right]u(t)$$

3.6 Analysis of LTI Continuous Time System Using Laplace Transform

3.6.1 Transfer Function of LTI Continuous Time System

In general, the input-output relation of a LTI (Linear Time Invariant) continuous time system is represented by the constant coefficient differential equation shown below, (equation (3.36)).

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + ... + a_{N-1}\frac{d}{dt}y(t) + a_N\,y(t)$$

$$= b_0\frac{d^M}{dt^M}x(t) + b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + ... + b_{M-1}\frac{d}{dt}x(t) + b_M\,x(t) \quad .....(3.36)$$

where, N = Order of the system and $M \leq N$.

On taking Laplace transform of equation (3.36) with zero initial conditions we get,

$$s^N\,Y(s) + a_1\,s^{N-1}\,Y(s) + a_2\,s^{N-2}\,Y(s) + ... + a_{N-1}\,s\,Y(s) + a_N\,Y(s)$$

$$= b_0\,s^M\,X(s) + b_1\,s^{M-1}\,X(s) + b_2\,s^{M-2}\,X(s) + ... + b_{M-1}\,s\,X(s) + b_M\,X(s)$$

$$Y(s)\left(s^N + a_1\,s^{N-1} + a_2\,s^{N-2} + ... + a_{N-1}\,s + a_N\right)$$

$$= X(s)\left(b_0\,s^M + b_1\,s^{M-1} + b_2\,s^{M-2} + ... + b_{M-1}\,s + b_M\right)$$

$$\therefore\ \frac{Y(s)}{X(s)} = \frac{b_0\,s^M + b_1\,s^{M-1} + b_2\,s^{M-2} + ... + b_{M-1}\,s + b_M}{s^N + a_1\,s^{N-1} + a_2\,s^{N-2} + ... + a_{N-1}\,s + a_N} \quad .....(3.37)$$

The ***transfer function*** of a continuous time system is defined as the ratio of Laplace transform of output and Laplace transform of input. Hence the equation (3.37) is the transfer function of an LTI continuous time system.

The equation (3.37) is a rational function of s (i.e., ratio of two polynomials in s). The numerator and denominator polynomials of equation (3.36) can be expressed in the factorized form as shown in equation (3.38).

$$\frac{Y(s)}{X(s)} = G\,\frac{(s-z_1)\,(s-z_2)\,(s-z_3)\ .....\ (s-z_M)}{(s-p_1)\,(s-p_2)\,(s-p_3)\ .....\ (s-p_N)} \quad .....(3.38)$$

where, $z_1, z_2, z_3, ...\ z_M$ are roots of numerator polynomial
(or zeros of continuous time system)

$p_1, p_2, p_3, ...\ p_N$ are roots of denominator polynomial
(or poles of continuous time system)

3.6.2 Impulse Response and Transfer Function

Let, x(t) = Input of a LTI continuous time system

y(t) = Output / Response of the LTI continuous time system for the input x(t)

h(t) = Impulse response (i.e., response for impulse input)

Now, the response y(t) of the continuous time system is given by convolution of input and impulse response. (Refer chapter - 2, section 2.9.1, equation (2.23)).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)\,h(t-\lambda)\,d\lambda \quad ....(3.39)$$

On taking Laplace transform of equation(3.39) we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

If $\mathcal{L}\{x(t)\} = X(s)$
and $\mathcal{L}\{h(t)\} = H(s)$
then by convolution property
$\mathcal{L}\{x(t) * h(t)\} = X(s)\,H(s)$

Using convolution property of Laplace transform the above equation can be written as,

$$Y(s) = X(s)\,H(s)$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} \qquad .....(3.40)$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} = G\,\frac{(s-z_1)\,(s-z_2)\,(s-z_3)\,.....\,(s-z_M)}{(s-p_1)\,(s-p_2)\,(s-p_3)\,.....\,(s-p_N)}$$

Using equation (3.38)

From equation (3.40) we can conclude that *the transfer function of LTI continuous time system is also given by Laplace transform of the impulse response.*

Alternatively we can say that *the inverse Laplace transform of transfer function is the impulse response of the system.*

$$\boxed{\therefore\ \text{Impulse response,}\quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{Y(s)}{X(s)}\right\}}$$

Using equation (3.40)

3.6.3 Response of LTI Continuous Time System Using Laplace Transform

In general, the input-output relation of an LTI (Linear Time Invariant) continuous time system is represented by the constant coefficient differential equation shown below, (equation (3.41)).

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + ... + a_{N-1}\frac{d}{dt}y(t) + a_N\,y(t)$$

$$= b_0\frac{d^M}{dt^M}x(t) + b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + ... + b_{M-1}\frac{d}{dt}x(t) + b_M\,x(t) \quad .....(3.41)$$

The solution of the above differential equation (equation (3.41)) is the ***(total) response*** y(t) of LTI system, which consists of two parts.

In signals and systems the two parts of the solution y(t) are called zero-input response $y_{zi}(t)$ and zero-state response $y_{zs}(t)$.

$$\therefore\ \text{Response,}\ \ y(t) = y_{zi}(t) + y_{zs}(t)$$

Zero-Input Response (or Free Response or Natural Response) Using Laplace Transform

The ***zero-input response*** $y_{zi}(t)$ is mainly due to initial output (or initial stored energy) in the system. The zero-input response is obtained from system equation (equation (3.41)) when input x(t) = 0.

On substituting x(t) = 0 and $y(t) = y_{zi}(t)$ in equation (3.41) we get,

$$\frac{d^N}{dt^N}y_{zi}(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y_{zi}(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y_{zi}(t) + ... + a_{N-1}\frac{d}{dt}y_{zi}(t) + a_N\,y_{zi}(t) = 0$$

On taking Laplace transform of the above equation with non-zero initial conditions for output we can form an equation for $Y_{zi}(s)$.

The zero-input response $y_{zi}(t)$ is given by inverse Laplace transform of $Y_{zi}(s)$.

Zero-State Response (or Forced Response) Using Laplace Transform

The ***zero-state response*** $y_{zs}(t)$ is the response of the system due to input signal and with zero initial output. The zero-state response is obtained from the differential equation governing the system (equation(3.41)) for specific input signal x(t) for $t \geq 0$ and with zero initial output.

On substituting $y(t) = y_{zs}(t)$ in equation (3.41) we get,

$$\frac{d^N}{dt^N} y_{zs}(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y_{zs}(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y_{zs}(t) + ... + a_{N-1} \frac{d}{dt} y_{zs}(t) + a_N\, y_{zs}(t)$$

$$= b_0 \frac{d^M}{dt^M} x(t) + b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... + b_{M-1} \frac{d}{dt} x(t) + b_M\, x(t)$$

On taking Laplace transform of the above equation with zero initial conditions for output (i.e., $y_{zs}(t)$) and non-zero initial values for input (i.e., x(t)) we can form an equation for $Y_{zs}(s)$.

The zero-state response $y_{zs}(t)$ is given by inverse Laplace transform of $Y_{zs}(s)$.

Total Response

The ***total response*** y(t) is the response of the system due to input signal and initial output (or intial stored energy). The total response is obtained from the differential equation governing the system (equation(3.41)) for specific input signal x(t) for $t \geq 0$ and with non-zero initial conditions.

On taking Laplace transform of equation (3.41) with non-zero initial conditions for both input and output, and then substituting for X(s) we can form an equation for Y(s).

The total response y(t) is given by inverse Laplace transform of Y(s).

Alternatively the total response y(t) is given by sum of zero-input response $y_{zi}(t)$ and zero-state response $y_{zs}(t)$.

$\therefore$ Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

3.6.4 Convolution and Deconvolution Using Laplace Transform

Convolution

The ***convolution*** operation is performed to find the response y(t) of LTI continuous time system from the input x(t) and impulse response h(t).

$\therefore$ Response, $y(t) = x(t) * h(t)$

If $\mathcal{L}\{x(t)\} = X(s)$
and $\mathcal{L}\{h(t)\} = H(s)$
then by convolution property
$\mathcal{L}\{x(t) * h(t)\} = X(s)\,H(s)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$\therefore \quad Y(s) = X(s)\,H(s) \qquad .....(3.42)$$

$\therefore$ Response, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$= \mathcal{L}^{-1}\{X(s)\,H(s)\}$

Procedure :
1. Take Laplace transform of x(t) to get X(s).
2. Take Laplace transform of h(t) to get H(s).
3. Compute the product of X(s)H(s). Let, X(s)H(s) = Y(s).
4. Take inverse Laplace transform of Y(s) to get y(t).

Deconvolution

The ***deconvolution*** operation is performed to extract the input x(t) of an LTI continuous time system from the response y(t) of the system.

From equation (3.42) get,

$$X(s) = \frac{Y(s)}{H(s)}$$

On taking inverse Laplace transform of the above equation we get,

$$\text{Input, } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{Y(s)}{H(s)}\right\}$$

Procedure : 1. Take Laplace transform of y(t) to get Y(s).

2. Take Laplace transform of h(t) to get H(s).
3. Divide Y(s) by H(s) to get X(s) (i.e., X(s) = Y(s) / H(s)).
4. Take inverse Laplace transform of X(s) to get x(t).

3.6.5 Stability in s-Domain

ROC of a Stable LTI System

Let H(s) be Laplace transform of h(t). Now by definition of Laplace transform we get,

$$H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} h(t)\, e^{-(\sigma+j\Omega)t}\, dt$$

$$\boxed{\text{Put, } s = \sigma + j\Omega}$$

Let us evaluate H(s) for $\sigma = 0$.

$$\therefore H(s) = \int_{-\infty}^{+\infty} h(t) e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} e^{-j\Omega t}\, h(t)\, dt = \left[e^{-j\Omega t}\left(\int h(t)\, dt\right) - \int -j\Omega e^{-j\Omega t}\left(\int h(t)\, dt\right) dt\right]_{-\infty}^{+\infty} \quad .....(3.43)$$

For a stable LTI system, $\int h(t)\,dt$ is constant.

$$\boxed{\int uv = u\int v - \int\left[du\int v\right]}$$

Therefore, in equation (3.43), put, $\int h(t)\,dt = A$, where A is constant.

$$\therefore H(s) = \left[e^{-j\Omega t}A + j\Omega\int e^{-j\Omega t}A\,dt\right]_{-\infty}^{+\infty} = A\left[e^{-j\Omega t} + j\Omega\int e^{-j\Omega t}\,dt\right]_{-\infty}^{+\infty} \quad .....(3.44)$$

The evaluation of equation (3.44), is evaluation of Laplace transform for $s = j\Omega$, (i.e., evaluation of H(s) along imaginary axis) and so we can say that H(s) exists if the ROC includes the imaginary axis. Hence *for a stable LTI continuous time system the ROC should include the imaginary axis of s-plane.*

Location of Poles for Stability of Causal Systems

Let h(t) be impulse response of an LTI causal system. Now if h(t) satisfies the condition,

$$\boxed{\int_0^{\infty} h(t)\, dt < \infty} \quad ....(3.45)$$

then the LTI continuous time causal system is stable.

Let, $h(t) = e^{at}\, u(t)$

$$\text{Now, } \int_0^{\infty} h(t)\, dt = \int_0^{\infty} e^{at}\, u(t)\, dt = \int_0^{\infty} e^{at}\, dt = \left[\frac{e^{at}}{a}\right]_0^{\infty} = \frac{e^{a\times\infty}}{a} - \frac{e^{a\times 0}}{a} = \frac{e^{a\times\infty}}{a} - \frac{1}{a}$$

Let a be negative, and let k = – a, so that k is always positive.

$$\text{Now,} \int_0^{\infty} h(t)\,dt = \frac{e^{a\times\infty}}{a} - \frac{1}{a} = \frac{e^{-k\infty}}{-k} + \frac{1}{k} = \frac{e^{-\infty}}{-k} + \frac{1}{k} \qquad \boxed{e^{-\infty} = 0}$$

$$= \frac{0}{-k} + \frac{1}{k} = \frac{1}{k} = \text{Constant, and so system is stable.} \qquad ....(3.46)$$

Let a be positive.

$$\text{Now,} \int_0^{\infty} h(t)\,dt = \frac{e^{a\times\infty}}{a} - \frac{1}{a} \qquad \boxed{e^{\infty} = \infty}$$

$$= \frac{e^{a\infty}}{a} - \frac{1}{a} = \frac{\infty}{a} - \frac{1}{a} = \infty, \text{ and so system is unstable.} \qquad .....(3.47)$$

From the above discussion, the stability condition of equation (3.45) can be transformed as a condition on location of poles of transfer function of the LTI continuous time causal system in s-plane.

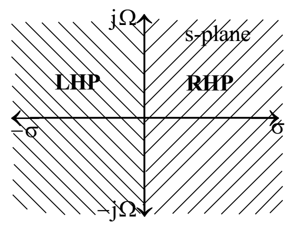


Fig 3.10 : *s-plane.*

Consider the s-plane shown in fig 3.10. The area to the right of vertical axis is called right half plane (RHP) and the area to the left of vertical axis is called left half plane (LHP).

The transfer function of a continuous time system is given by Laplace transform of its impulse response.

Let, $h(t) = e^{at}\,u(t)$

$$\therefore \text{Transfer function, } H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{e^{at}\,u(t)\} = \frac{1}{s-a}$$

Here, the transfer function H(s) has pole at s = a.

If, a < 0, (i.e., if a is negative), then the pole will lie on the left half of s-plane, and from equation (3.46) we can say that the causal system is stable.

If, a > 0, (i.e., if a is positive), then the pole will lie on the right half of s-plane. and from equation (3.47) we can say that the causal system is unstable.

Therefore we can say that, *for a stable LTI continuous time causal system the poles should lie on the left half of s-plane.*

General Condition for Stability in s-Plane

On combining the condition for location of poles and the ROC we can say that,

1. *For a stable LTI continuous time causal system, the poles should lie on the left half of s-plane and the imaginary axis should be included in the ROC.*
2. *For a stable LTI continuous time noncausal system, the imaginary axis should be included in the ROC.*

The various types of impulse response of LTI continuous time system and their transfer functions and the locations of poles are summarized in table 3.4.

Table-3.4 : Impulse Response and Location of Poles of Transfer Function in s-Plane

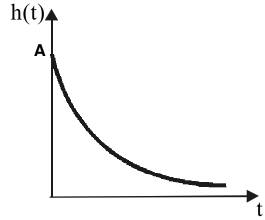
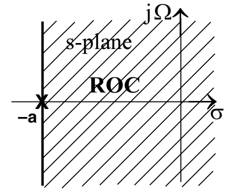
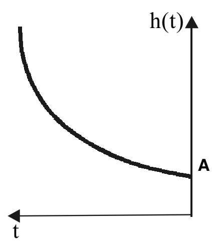
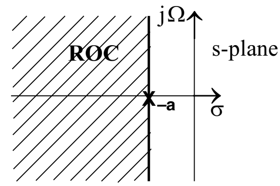
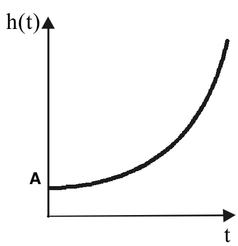
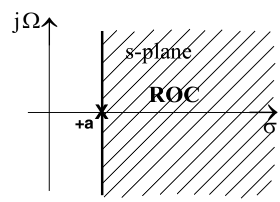
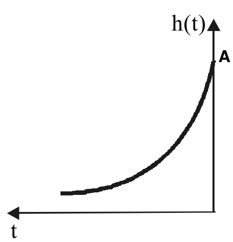
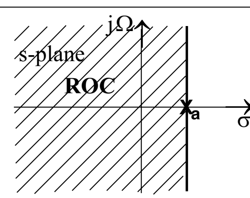
Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = A\,e^{-at}\,u(t)\ ;\ a > 0$ 	$H(s) = \dfrac{A}{s+a}$ Pole at $s = -a$. ROC is $\sigma > -a$, where σ is real part of s.	 The pole at $s = -a$, lies on left half of s-plane. ROC includes imaginary axis. Causal system. Since pole lies on LHP and the imaginary axis is included in ROC, the system is stable.
$h(t) = A\,e^{-at}\,u(-t)\ ;\ a > 0$ 	$H(s) = -\dfrac{A}{s+a}$ Pole at $s = -a$. ROC is $\sigma < -a$, where σ is real part of s.	 The pole at $s = -a$, lies on left half of s-plane. The ROC does not include imaginary axis. Noncausal system. Since imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,e^{at}\,u(t)\ ;\ a > 0$ 	$H(s) = \dfrac{A}{s-a}$ Pole at $s = +a$. ROC is $\sigma > +a$, where σ is real part of s.	 The pole at $s = +a$, lies on right half of s-plane. ROC does not include imaginary axis. Causal system. Since pole lies on RHP and imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,e^{at}\,u(-t)\ ;\ a > 0$ 	$H(s) = -\dfrac{A}{s-a}$ Pole at $s = +a$. ROC is $\sigma < +a$, where σ is real part of s.	 The pole at $s = +a$, lies on right half of s-plane. The ROC includes imaginary axis. Noncausal system. Since imaginary axis is included in ROC, the system is stable.

Table-3.4 : Continued....

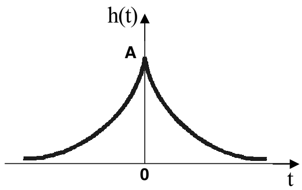
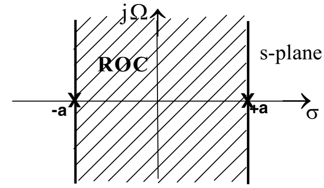
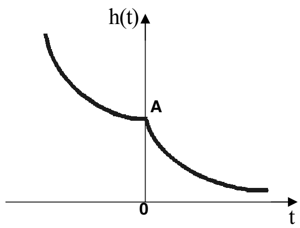
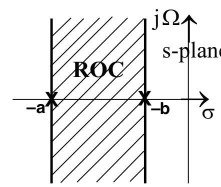
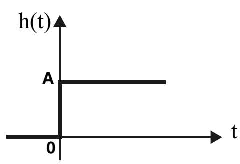
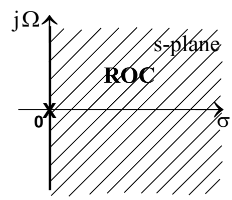
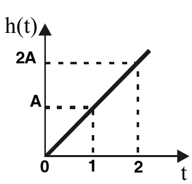
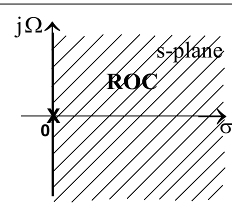
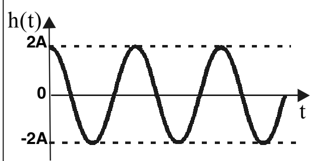
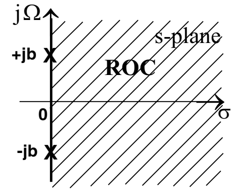
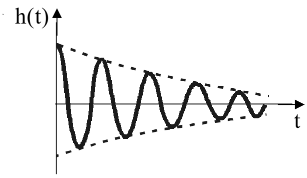
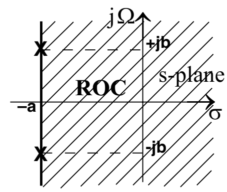
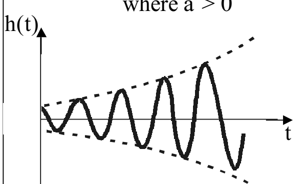
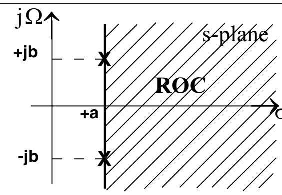
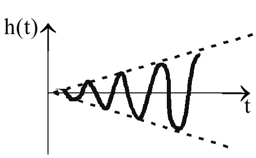
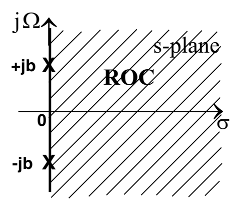
Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = A\,e^{-a\lvert t \rvert}$; $a > 0$ 	$H(s) = \dfrac{A}{s+a} - \dfrac{A}{s-a}$ Poles at $s = -a, +a$. ROC is $-a < \sigma < +a$, where σ is real part of s.	 The pole at $s = +a$, lies on RHP and pole at $s = -a$, lies on LHP. ROC includes imaginary axis. Noncausal system. Since the imaginary axis is included in ROC, the system is stable.
$h(t) = A\,e^{-at}u(t) + A\,e^{-bt}u(-t)$ where $a > 0, b > 0, a > b$ 	$H(s) = \dfrac{A}{s+a} - \dfrac{A}{s+b}$ Poles at $s = -a, -b$. ROC is $-a < \sigma < -b$, where σ is real part of s.	 The poles at $s = -a, -b$, lie on LHP. The ROC does not include imaginary axis. Noncausal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,u(t)$ 	$H(s) = \dfrac{A}{s}$ Pole at $s = 0$. ROC is $\sigma > 0$, where σ is real part of s.	 The pole at $s = 0$ lies on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\,t\,u(t)$ 	$H(s) = \dfrac{A}{s^2}$ Double pole at $s = 0$. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at $s = 0$ lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

Table-3.4 : Continued....

Impulse response h(t)	Transfer function $H(s) = \mathcal{L}\{h(t)\}$	Location of poles in s-plane and ROC
$h(t) = 2A \cos bt\, u(t)$ 	$H(s) = \dfrac{A}{s + jb} + \dfrac{A}{s - jb}$ Poles at s = –jb, +jb. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at s = –jb, +jb, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\, e^{-at} \cos bt\, u(t)$, where $a > 0$ 	$H(s) = \dfrac{A}{s + a + jb} + \dfrac{A}{s + a - jb}$ Poles at s = –a – jb, –a + jb. ROC is $\sigma > -a$, where σ is real part of s.	 The poles at s = –a–jb, –a+jb, lie on left half of s-plane. The ROC includes the imaginary axis. Causal system. Since poles lie on LHP and the imaginary axis is included in ROC, the system is stable.
$h(t) = 2A\, e^{at} \cos bt\, u(t)$, where $a > 0$ 	$H(s) = \dfrac{A}{s - a + jb} + \dfrac{A}{s - a - jb}$ Poles at s = a – jb, a + jb. ROC is $\sigma > a$, where σ is real part of s.	 The poles at s = a–jb, a+jb, lie on right half of s-plane. The ROC does not include imaginary axis. Causal system. Since poles lie on RHP and the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\, t \cos bt\, u(t)$ 	$H(s) = \dfrac{A}{(s + jb)^2} + \dfrac{A}{(s - jb)^2}$ Double poles at s = – jb, + jb. ROC is $\sigma > 0$, where σ is real part of s.	 The poles at s = –jb, +jb, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

Example 3.18

Using Laplace transform, determine the natural response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 6\,\frac{dy(t)}{dt} + 5\,y(t) = \frac{dx(t)}{dt} + 4\,x(t) \quad ; \quad y(0) = 1 \quad ; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = -2$$

Solution

The natural response is the response of the system due to initial values of output alone. Hence for natural response the input x(t) is considered as zero. Therefore the natural response is also called zero-input response, $y_{zi}(t)$.

On substituting input x(t) = 0 and $y(t) = y_{zi}(t)$ in the given system equation we get,

$$\frac{d^2y_{zi}(t)}{dt^2} + 6\,\frac{dy_{zi}(t)}{dt} + 5\,y_{zi}(t) = 0$$

On taking Laplace transform of the above equation we get,

$$s^2\,Y_{zi}(s) - s\,y(0) - y'(0) + 6\,[sY_{zi}(s) - y(0)] + 5\,Y_{zi}(s) = 0$$

On substituting the given initial conditions of output in the above equation we get,

$$s^2\,Y_{zi}(s) - s \times 1 - (-2) + 6\,[s\,Y_{zi}(s) - 1] + 5\,Y_{zi}(s) = 0$$

$$s^2\,Y_{zi}(s) - s + 2 + 6s\,Y_{zi}(s) - 6 + 5\,Y_{zi}(s) = 0$$

$$(s^2 + 6s + 5)\;Y_{zi}(s) - s - 4 = 0$$

$$(s^2 + 6s + 5)\;Y_{zi}(s) = s + 4$$

$$\therefore\; Y_{zi}(s) = \frac{s+4}{s^2 + 6s + 5} = \frac{s+4}{(s+1)\,(s+5)}$$

The roots of quadratic $s^2 + 6s + 5 = 0$ are,

$$s = \frac{-6 \pm \sqrt{6^2 - 4 \times 5}}{2} = \frac{-6 \pm 4}{2} = -1, -5$$

By partial fraction expansion technique, $Y_{zi}(s)$ can be expressed as shown below.

$$Y_{zi}(s) = \frac{s+4}{(s+1)\,(s+5)} = \frac{A}{s+1} + \frac{B}{s+5}$$

$$A = \left.\frac{s+4}{(s+1)\,(s+5)} \times (s+1)\right|_{s=-1} = \frac{-1+4}{-1+5} = \frac{3}{4}$$

$$B = \left.\frac{s+4}{(s+1)\,(s+5)} \times (s+5)\right|_{s=-5} = \frac{-5+4}{-5+1} = \frac{1}{4}$$

$$\therefore\; Y_{zi}(s) = \frac{3}{4}\,\frac{1}{s+1} + \frac{1}{4}\,\frac{1}{s+5}$$

On taking inverse Laplace transform of the above equation we get natural response.

$$\therefore \left.\begin{array}{l}\text{Natural response} \\ \text{(or zero-input response)}\end{array}\right\}\; y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{3}{4}\,\frac{1}{s+1} + \frac{1}{4}\,\frac{1}{s+5}\right\}$$

$$= \frac{3}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}\,\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$= \frac{3}{4}\,e^{-t}\,u(t) + \frac{1}{4}\,e^{-5t}\,u(t)$$

$$= \frac{1}{4}\left(3\,e^{-t} + e^{-5t}\right)u(t)$$

Note : *Compare the above result with example 2.8 of chapter - 2.*

Example 3.19

Using Laplace transform determine the forced response of the system described by the equation,

$$5\,\frac{dy(t)}{dt} + 10\,y(t) = 2\,x(t) \quad ; \quad \text{for the input, } x(t) = 2\,u(t)$$

Solution

The forced response is the response of the system due to input alone. For forced response, the system equation is solved for the given input with zero initial output. (But initial values of input should be considered).

Input, $x(t) = 2\,u(t)$

$$\therefore\ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{2\,u(t)\} = \frac{2}{s} \qquad .....(1)$$

On substituting $y(t) = y_{zs}(t)$ in the given system equation we get,

$$5\,\frac{dy_{zs}(t)}{dt} + 10\,y_{zs}(t) = 2\,x(t)$$

On taking Laplace transform of the above equation with zero initial output conditions we get,

$$5s\,Y_{zs}(s) + 10\,Y_{zs}(s) = 2\,X(s)$$

$$5(s+2)\;Y_{zs}(s) = 2\,X(s)$$

$$\therefore\ Y_{zs}(s) = \frac{2}{5(s+2)}\,X(s)$$

$$= \frac{2}{5(s+2)} \times \frac{2}{s} = \frac{4}{5}\,\frac{1}{s(s+2)}$$

Using equation (1)

By partial fraction expansion technique, $Y_{zs}(s)$ can be expressed as shown below.

$$Y_{zs}(s) = \frac{4}{5}\,\frac{1}{s(s+2)} = \frac{4}{5}\left[\frac{A}{s} + \frac{B}{s+2}\right]$$

$$A = \frac{1}{s(s+2)} \times s\,\bigg|_{s=0} = \frac{1}{0+2} = \frac{1}{2}$$

$$B = \frac{1}{s(s+2)} \times (s+2)\,\bigg|_{s=-2} = \frac{1}{-2} = -\frac{1}{2}$$

$$\therefore\ Y_{zs}(s) = \frac{4}{5}\left[\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{s+2}\right] = \frac{2}{5}\left[\frac{1}{s} - \frac{1}{s+2}\right]$$

On taking inverse Laplace transform of the above equation we get forced response.

$$\left.\begin{array}{l}\therefore\ \text{Forced response} \\ \text{(or zero - state response)}\end{array}\right\}\ y_{zs}(t) = \mathcal{L}^{-1}\{Y_{zs}(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{5}\left[\frac{1}{s} - \frac{1}{s+2}\right]\right\}$$

$$= \frac{2}{5}\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}\right]$$

$$= \frac{2}{5}\left[u(t) - e^{-2t}\,u(t)\right]$$

$$= \frac{2}{5}\left(1 - e^{-2t}\right)u(t)$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{e^{-at}\,u(t)\} = \frac{1}{s+a}$$

***Note :** Compare the above result with example 2.9 of chapter - 2.*

Example 3.20

Using Laplace transform determine the complete response of the system described by the equation,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt} \quad ; \quad y(0) = 0 \quad ; \quad \left.\frac{dy(t)}{dt}\right|_{t=0} = 1, \text{ for the input, } x(t) = e^{-2t}\,u(t)$$

Solution

Method - I

Input, $x(t) = e^{-2t}\,u(t)$

$$\therefore\ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{e^{-2t}\,u(t)\right\} = \frac{1}{s+2} \qquad .....(1)$$

Initial value of input, $x(0) = x(t)\big|_{t=0} = e^0\,u(0) = 1$

The given sytem equation is,

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4\,y(t) = \frac{dx(t)}{dt}$$

On taking Laplace transform of the above equation we get,

$$s^2\,Y(s) - s\,y(0) - y'(0) + 5\,[s\,Y(s) - y(0)] + 4\,Y(s) = s\,X(s) - x(0)$$

On substituting the initial values of output and input in the above equation we get,

$$s^2\,Y(s) - s \times 0 - 1 + 5\,[s\,Y(s) - 0] + 4\,Y(s) = s\,X(s) - 1$$

$$s^2\,Y(s) + 5s\,Y(s) + 4\,Y(s) = s\,X(s)$$

$$\left(s^2 + 5s + 4\right)Y(s) = s \times \frac{1}{s+2} \qquad \boxed{\text{Using equation (1)}}$$

$$\therefore\ Y(s) = \frac{s}{(s+2)\left(s^2 + 5s + 4\right)}$$

$$= \frac{s}{(s+2)(s+1)(s+4)}$$

$$= \frac{s}{(s+1)(s+2)(s+4)}$$

> The roots of quadratic $s^2 + 5s + 4 = 0$ are,
> $$s = \frac{-5 \pm \sqrt{5^2 - 4 \times 4}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

By partial fraction expansion technique, Y(s) can be expressed as shown below.

$$Y(s) = \frac{s}{(s+1)(s+2)(s+4)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$A = \left.\frac{s}{(s+1)(s+2)(s+4)} \times (s+1)\right|_{s=-1} = \frac{-1}{(-1+2)(-1+4)} = \frac{-1}{1 \times 3} = -\frac{1}{3}$$

$$B = \left.\frac{s}{(s+1)(s+2)(s+4)} \times (s+2)\right|_{s=-2} = \frac{-2}{(-2+1)(-2+4)} = \frac{-2}{-1 \times 2} = 1$$

$$C = \left.\frac{s}{(s+1)(s+2)(s+4)} \times (s+4)\right|_{s=-4} = \frac{-4}{(-4+1)(-4+2)} = \frac{-4}{-3 \times (-2)} = -\frac{2}{3}$$

$$\therefore\ Y(s) = -\frac{1}{3}\,\frac{1}{s+1} + \frac{1}{s+2} - \frac{2}{3}\,\frac{1}{s+4}$$

On taking inverse Laplace transform of the above equation we get complete / total response of the system.

$$\therefore \text{ Total response, } y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{-\frac{1}{3}\frac{1}{s+1} + \frac{1}{s+2} - \frac{2}{3}\frac{1}{s+4}\right\}$$

$$= -\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{1}{3}e^{-t}u(t) + e^{-2t}u(t) - \frac{2}{3}e^{-4t}u(t)$$

$$\boxed{\mathcal{L}\left\{e^{-at}u(t)\right\} = \frac{1}{s+a}}$$

$$= \left(-\frac{1}{3}e^{-t} + e^{-2t} - \frac{2}{3}e^{-4t}\right)u(t)$$

Method - II

Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

where, $y_{zi}(t)$ = Zero-input response

$y_{zs}(t)$ = Zero-state response

Zero-input Response

On substituting $x(t) = 0$ and $y(t) = y_{zi}(t)$ in the system equation we get,

$$\frac{d^2y_{zi}(t)}{dt^2} + 5\frac{dy_{zi}(t)}{dt} + 4y_{zi}(t) = 0$$

On taking Laplace transform of the above equation we get,

$$s^2 Y_{zi}(s) - s\,y(0) - y'(0) + 5[s\,Y_{zi}(s) - y(0)] + 4Y_{zi}(s) = 0$$

On substituting the initial values of output in the above equation we get,

$$s^2 Y_{zi}(s) - s \times 0 - 1 + 5[s\,Y_{zi}(s) - 0] + 4Y_{zi}(s) = 0$$

$$\therefore (s^2 + 5s + 4)\ Y_{zi}(s) = 1$$

$$Y_{zi}(s) = \frac{1}{s^2 + 5s + 4} = \frac{1}{(s+1)(s+4)}$$

The roots of quadratic $s^2 + 5s + 4 = 0$ are,

$$s = \frac{-5 \pm \sqrt{5^2 - 4 \times 4}}{2} = \frac{-5 \pm 3}{2} = -1, -4$$

By partial fraction expansion technique, $Y_{zi}(s)$ can be expressed as shown below.

$$Y_{zi}(s) = \frac{1}{(s+1)(s+4)} = \frac{D}{s+1} + \frac{E}{s+4}$$

$$D = \frac{1}{(s+1)(s+4)} \times (s+1)\Big|_{s=-1} = \frac{1}{-1+4} = \frac{1}{3}$$

$$E = \frac{1}{(s+1)(s+4)} \times (s+4)\Big|_{s=-4} = \frac{1}{-4+1} = -\frac{1}{3}$$

$$\therefore Y_{zi}(s) = \frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}$$

On taking inverse Laplace transform of $Y_{zi}(s)$ we get zero-input response, $y_{zi}(t)$

$$\therefore \text{ Zero-input response, } y_{zi}(t) = \mathcal{L}^{-1}\{Y_{zi}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s+1} - \frac{1}{3}\frac{1}{s+4}\right\}$$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{\mathcal{L}\left\{e^{-at}u(t)\right\} = \frac{1}{s+a}}$$

$$= \frac{1}{3}e^{-t}u(t) - \frac{1}{3}e^{-4t}u(t)$$

Zero-state Response

Given that, $x(t) = e^{-2t}\,u(t)$

$$\therefore\ x(0) = x(t)\big|_{t=0} = e^{0}\,u(0) = 1 \qquad .....(2)$$

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{e^{-2t}\,u(t)\right\} = \frac{1}{s+2} \qquad .....(3)$$

On substituting $y(t) = y_{zs}(t)$ in the system equation we get,

$$\frac{d^2y_{zs}(t)}{dt^2} + 5\,\frac{dy_{zs}(t)}{dt} + 4\,y_{zs}(t) = \frac{dx(t)}{dt}$$

On taking Laplace transform of the above equation with zero initial output we get,

$$s^2\,Y_{zs}(s) + 5s\,Y_{zs}(s) + 4\,Y_{zs}(s) = s\,X(s) - x(0)$$

$$\therefore \left(s^2 + 5s + 4\right)\,Y_{zs}(s) = s \times \frac{1}{s+2} - 1$$

Using equations (2) and (3)

$$(s+1)\,(s+4)\,Y_{zs}(s) = \frac{s - s - 2}{s+2}$$

$$\therefore\ Y_{zs}(s) = \frac{-2}{(s+1)\,(s+2)\,(s+4)}$$

By partial fraction expansion technique, $Y_{zs}(s)$ can be expressed as shown below.

$$Y_{zs}(s) = \frac{-2}{(s+1)\,(s+2)\,(s+4)} = \frac{F}{s+1} + \frac{G}{s+2} + \frac{H}{s+4}$$

$$F = \frac{-2}{(s+1)\,(s+2)\,(s+4)} \times (s+1)\bigg|_{s=-1} = \frac{-2}{(-1+2)\,(-1+4)} = \frac{-2}{1 \times 3} = -\frac{2}{3}$$

$$G = \frac{-2}{(s+1)\,(s+2)\,(s+4)} \times (s+2)\bigg|_{s=-2} = \frac{-2}{(-2+1)\,(-2+4)} = \frac{-2}{-1 \times 2} = 1$$

$$H = \frac{-2}{(s+1)\,(s+2)\,(s+4)} \times (s+4)\bigg|_{s=-4} = \frac{-2}{(-4+1)\,(-4+2)} = \frac{-2}{-3 \times (-2)} = -\frac{1}{3}$$

$$\therefore\ Y_{zs}(s) = -\frac{2}{3}\,\frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{3}\,\frac{1}{s+4}$$

On taking inverse Laplace transform of $Y_{zs}(s)$ we get zero-state response, $y_{zs}(t)$

$$\therefore\ \text{Zero-state response, } y_{zs}(t) = \mathcal{L}^{-1}\left\{Y_{zs}(s)\right\} = \mathcal{L}^{-1}\left\{-\frac{2}{3}\,\frac{1}{s+1} + \frac{1}{s+2} - \frac{1}{3}\,\frac{1}{s+4}\right\}$$

$$= -\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$= -\frac{2}{3}\,e^{-t}\,u(t) + e^{-2t}\,u(t) - \frac{1}{3}\,e^{-4t}\,u(t)$$

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

Total / Complete Response

Total response, $y(t) = y_{zi}(t) + y_{zs}(t)$

$$= \left[\frac{1}{3}\,e^{-t}\,u(t) - \frac{1}{3}\,e^{-4t}\,u(t)\right] + \left[-\frac{2}{3}\,e^{-t}\,u(t) + e^{-2t}\,u(t) - \frac{1}{3}\,e^{-4t}\,u(t)\right]$$

$$= -\frac{1}{3}\,e^{-t}\,u(t) + e^{-2t}\,u(t) - \frac{2}{3}\,e^{-4t}\,u(t) = \left(-\frac{1}{3}\,e^{-t} + e^{-2t} - \frac{2}{3}\,e^{-4t}\right)u(t)$$

Note : *Compare the above result with example 2.10 of chapter - 2.*

Example 3.21

Determine the impulse response h(t) of the following system. Assume zero initial conditions.

a) $y(t) = x(t - t_0)$

b) $T_0 \dfrac{d^2y(t)}{dt^2} + y(t) = x(t)$

c) $\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3\,y(t) = \dfrac{dx(t)}{dt} + 2\,x(t)$

d) $\dfrac{d^2y(t)}{dt^2} + 3\dfrac{dy(t)}{dt} + 2\,y(t) = x(t)$

Solution

a) Given that, $y(t) = x(t - t_0)$

On taking Laplace transform with zero initial condition we get,

$$Y(s) = e^{-st_0}\,X(s)$$

Input, $x(t) = \delta(t)$; $\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1$

When the input is impulse, the output is denoted by h(t). Let $\mathcal{L}\{h(t)\} = H(s)$.

$$\therefore\ Y(s) = e^{-st_0}\,X(s) \quad \Rightarrow \quad H(s) = e^{-st_0}$$

$$\boxed{Y(s) = H(s) \quad ; \quad X(s) = 1}$$

$$\text{Impulse response, } h(t) = \mathcal{L}^{-1}\{H(s)\}$$
$$= \mathcal{L}^{-1}\left\{e^{-st_0}\right\}$$
$$= \delta(t - t_0)$$

$$\boxed{\begin{aligned} \mathcal{L}\{\delta(t)\} &= 1 \\ \mathcal{L}\{\delta(t - t_0)\} &= e^{-st_0} \end{aligned}}$$

b) Given that, $T_0 \dfrac{d^2y(t)}{dt^2} + y(t) = x(t)$

On taking Laplace transform with zero initial condition we get,

$$T_0\, s^2\, Y(s) + Y(s) = X(s)$$

$$T_0\left(s^2 + \frac{1}{T_0}\right)Y(s) = X(s)$$

Input, $x(t) = \delta(t)$; $\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1$

When the input is impulse, the output is denoted by h(t). Let $\mathcal{L}\{h(t)\} = H(s)$.

$$\therefore\ T_0\left(s^2 + \frac{1}{T_0}\right)Y(s) = X(s) \quad \Rightarrow \quad T_0\left(s^2 + \frac{1}{T_0}\right)H(s) = 1$$

$$\boxed{Y(s) = H(s) \quad ; \quad X(s) = 1}$$

$$\therefore\ H(s) = \frac{1}{T_0\left(s^2 + \dfrac{1}{T_0}\right)}$$

$$\text{Impulse response, } h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{T_0\left(s^2 + \dfrac{1}{T_0}\right)}\right\}$$

$$\boxed{\mathcal{L}\{\sin \Omega_0 t\, u(t)\} = \frac{\Omega_0}{s^2 + \Omega_0^2}}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{T_0}}\ \frac{\dfrac{1}{\sqrt{T_0}}}{s^2 + \left(\dfrac{1}{\sqrt{T_0}}\right)^2}\right\} = \frac{1}{\sqrt{T_0}}\ \sin\left(\frac{1}{\sqrt{T_0}}\,t\right)\ u(t)$$

c) Given that, $\dfrac{d^2y(t)}{dt^2} + 4\dfrac{dy(t)}{dt} + 3\,y(t) = \dfrac{dx(t)}{dt} + 2\,x(t)$

On taking Laplace transform with zero initial condition we get,

$$s^2\,Y(s) + 4s\,Y(s) + 3\,Y(s) = s\,X(s) + 2\,X(s)$$

$$(s^2 + 4s + 3)\,Y(s) = [s + 2]\,X(s)$$

Input, $x(t) = \delta(t)$

$$\therefore\; X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1$$

When the input is impulse, the output is denoted by h(t). Let $\mathcal{L}\{h(t)\} = H(s)$.

$$\therefore\; (s^2 + 4s + 3)\,Y(s) = (s + 2)\,X(s) \;\Rightarrow\; (s^2 + 4s + 3)\,H(s) = s + 2$$

$\boxed{Y(s) = H(s)\;;\; X(s) = 1}$

$$\therefore\; H(s) = \frac{s+2}{s^2 + 4s + 3} = \frac{s+2}{(s+1)(s+3)}$$

The roots of quadratic $s^2 + 4s + 3 = 0$ are,

$$s = \frac{-4 \pm \sqrt{4^2 - 4 \times 3}}{2} = \frac{-4 \pm 2}{2} = -1, -3$$

By partial fraction expansion technique, H(s) can be expressed as,

$$H(s) = \frac{s+2}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

$$A = \frac{s+2}{(s+1)(s+3)} \times (s+1)\bigg|_{s=-1} = \frac{-1+2}{-1+3} = \frac{1}{2} = 0.5$$

$$B = \frac{s+2}{(s+1)(s+3)} \times (s+3)\bigg|_{s=-3} = \frac{-3+2}{-3+1} = \frac{1}{2} = 0.5$$

$$\therefore\; H(s) = 0.5\,\frac{1}{s+1} + 0.5\,\frac{1}{s+3} = 0.5\left(\frac{1}{s+1} + \frac{1}{s+3}\right)$$

$$\therefore\; \text{Impulse response,}\quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{0.5\left(\frac{1}{s+1} + \frac{1}{s+3}\right)\right\}$$

$$= 0.5\left[\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}\right]$$

$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{s+a}}$

$$= 0.5\left[e^{-t}\,u(t) + e^{-3t}\,u(t)\right] = 0.5\left(e^{-t} + e^{-3t}\right)u(t)$$

---

d) Given that, $\dfrac{d^2y(t)}{dt^2} + 3\dfrac{dy(t)}{dt} + 2\,y(t) = x(t)$

On taking Laplace transform with zero initial conditions we get,

$$s^2\,Y(s) + 3s\,Y(s) + 2\,Y(s) = X(s)$$

$$(s^2 + 3s + 2)\,Y(s) = X(s)$$

Input, $x(t) = \delta(t)\;;\;\therefore\; X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1$

When the input is impulse, the output is denoted by h(t). Let $\mathcal{L}\{h(t)\} = H(s)$.

$$\therefore\; (s^2 + 3s + 2)\,Y(s) = X(s) \;\Rightarrow\; (s^2 + 3s + 2)\,H(s) = 1$$

$\boxed{Y(s) = H(s)\;;\; X(s) = 1}$

$$\therefore\; H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

The roots of quadratic $s^2 + 3s + 2 = 0$ are,

$$s = \frac{-3 \pm \sqrt{3^2 - 4 \times 2}}{2} = \frac{-3 \pm 1}{2} = -1, -2$$

By partial fraction expansion technique, H(s) can be expressed as,

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = \frac{1}{(s+1)(s+2)} \times (s+1)\Bigg|_{s=-1} = \frac{1}{-1+2} = 1$$

$$B = \frac{1}{(s+1)(s+2)} \times (s+2)\Bigg|_{s=-2} = \frac{1}{-2+1} = -1$$

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$\therefore\ H(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\therefore\ \text{Impulse response,}\quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= e^{-t}\,u(t) - e^{-2t}\,u(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

Example 3.22

Perform convolution of $x_1(t) = e^{-2t}\cos 3t\,u(t)$ and $x_2(t) = 4\sin 3t\,u(t)$ using Laplace transform.

Solution

Given that, $x_1(t) = e^{-2t}\cos 3t\,u(t)$ and $x_2(t) = 4\sin 3t\,u(t)$

$$\therefore\ X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\left\{e^{-2t}\cos 3t\,u(t)\right\} = \frac{s+2}{(s+2)^2+3^2} = \frac{s+2}{s^2+4s+13}$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\left\{4\sin 3t\,u(t)\right\} = 4 \times \frac{3}{s^2+3^2} = \frac{12}{s^2+9}$$

Now by convolution theorem, $x_1(t) * x_2(t) = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\}$

Let, $X(s) = X_1(s)\,X_2(s)$

$$\therefore\ X(s) = \frac{s+2}{s^2+4s+13} \times \frac{12}{s^2+9} = \frac{12(s+2)}{(s^2+4s+13)(s^2+9)} \qquad .....(1)$$

By partial fraction expansion, X(s) can be expressed as,

$$X(s) = \frac{12(s+2)}{(s^2+4s+13)(s^2+9)} = \frac{A\,s+B}{s^2+4s+13} + \frac{C\,s+D}{s^2+9}$$

On cross - multiplying the equation (1) we get,

$$12(s+2) = (A\,s+B)(s^2+9) + (C\,s+D)(s^2+4s+13)$$

$$12s+24 = A\,s^3 + 9A\,s + B\,s^2 + 9B + C\,s^3 + 4C\,s^2 + 13C\,s + D\,s^2 + 4D\,s + 13D$$

$$12s+24 = (A+C)s^3 + (B+4C+D)s^2 + (9A+13C+4D)s + (9B+13D) \qquad .....(2)$$

On equating the coefficients of s^3 terms of equation (2) we get,

$$A+C=0 \quad\Rightarrow\quad A=-C \qquad .....(3)$$

On equating the coefficients of s^2 terms of equation (2) we get,

$$B+4C+D=0 \qquad .....(4)$$

On equating the coefficients of s terms of equation (2) we get

$$9A+13C+4D=12 \qquad .....(5)$$

On equating constants of equation (2) we get,

$$9B+13D=24 \qquad .....(6)$$

On substituting A = –C in equation (5) we get,

$$9(-C)+13C+4D=12$$

$$\therefore\ 4C+4D=12 \quad\Rightarrow\quad C+D=3 \quad\Rightarrow\quad C=3-D \qquad .....(7)$$

On substituting C = 3 – D in equation (4) we get

$$B + 4(3 - D) + D = 0$$

$$\therefore B - 3D = -12 \qquad .....(8)$$

$$\text{Equation (8)} \times -9 \quad \Rightarrow \quad -9B + 27D = 108$$
$$\text{Equation (6)} \times 1 \quad \Rightarrow \quad 9B + 13D = 24$$
$$\overline{\qquad 40D = 132 \qquad} \qquad \therefore \; D = \frac{132}{40} = 3.3$$

From equation (7), $C = 3 - D = 3 - 3.3 = -0.3$

From equation (3), $A = -C = 0.3$

From equation (6), $B = \dfrac{24 - 13D}{9} = \dfrac{24 - 13 \times 3.3}{9} = -2.1$

$$\therefore X(s) = \frac{As + B}{s^2 + 4s + 13} + \frac{Cs + D}{s^2 + 9} = \frac{0.3s - 2.1}{s^2 + 4s + 13} + \frac{-0.3s + 3.3}{s^2 + 9}$$

$$= \frac{0.3\left(s - \dfrac{2.1}{0.3}\right)}{(s^2 + (2 \times 2s) + 2^2) + 3^2} + \frac{-0.3s + 3.3}{s^2 + 9}$$

Arranging, s^2+4s, in the form of $(x+y)^2$

$(x+y)^2 = x^2+2xy+x^2$

$$= \frac{0.3(s - 7)}{(s + 2)^2 + 3^2} + \frac{-0.3s + 3.3}{s^2 + 9} = \frac{0.3(s + 2 - 9)}{(s + 2)^2 + 3^2} + \frac{-0.3s + 3.3}{s^2 + 3^2}$$

$$= \frac{0.3(s + 2) - 0.3 \times 9}{(s + 2)^2 + 3^2} + \frac{-0.3s + 3.3}{s^2 + 3^2}$$

$$= 0.3\,\frac{s + 2}{(s + 2)^2 + 3^2} - 0.9\,\frac{3}{(s + 2)^2 + 3^2}$$
$$- 0.3\,\frac{s}{s^2 + 3^2} + 1.1\,\frac{3}{s^2 + 3^2}$$

$$\mathcal{L}\left\{e^{-at}\cos\Omega_0 t\,u(t)\right\} = \frac{s+a}{(s+a)^2 + \Omega_0^2}$$
$$\mathcal{L}\left\{e^{-at}\sin\Omega_0 t\,u(t)\right\} = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$$
$$\mathcal{L}\{\cos\Omega_0 t\,u(t)\} = \frac{s}{s^2 + \Omega_0^2}$$
$$\mathcal{L}\{\sin\Omega_0 t\,u(t)\} = \frac{\Omega_0}{s^2 + \Omega_0^2}$$

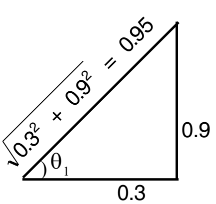
On taking inverse Laplace transform of the above equation we get,

$$x(t) = 0.3\,e^{-2t}\cos 3t - 0.9\,e^{-2t}\sin 3t - 0.3\cos 3t + 1.1\sin 3t \;; \quad t \ge 0$$
$$= e^{-2t}(0.3\cos 3t - 0.9\sin 3t) + (1.1\sin 3t - 0.3\cos 3t)\,u(t)$$

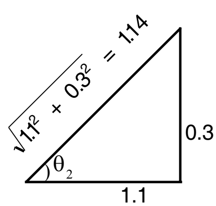
***Note :** The result of Example 3.22 can be further simplified as shown below.*

$$x(t) = e^{-2t}(\cos 3t \times 0.3 - \sin 3t \times 0.9) + (\sin 3t \times 1.1 - \cos 3t \times 0.3) \;; \quad t \ge 0$$

Let us construct two right angled triangles as shown below.



$$\cos\theta_1 = \frac{0.3}{0.95} \qquad \therefore 0.3 = 0.95\cos\theta_1$$
$$\sin\theta_1 = \frac{0.9}{0.95} \qquad \therefore 0.9 = 0.95\sin\theta_1$$
$$\tan\theta = \frac{0.9}{0.3} \;; \quad \therefore \theta_1 = \tan^{-1}\frac{0.9}{0.3} = 71.6^\circ$$



$$\cos\theta_2 = \frac{1.1}{1.14} \qquad \therefore 1.1 = 1.14\cos\theta_2$$
$$\sin\theta_2 = \frac{0.3}{1.14} \qquad \therefore 0.3 = 1.14\sin\theta_2$$
$$\tan\theta = \frac{0.3}{1.1} \;; \quad \therefore \theta_2 = \tan^{-1}\frac{0.3}{1.1} = 15.3^\circ$$

Using the relations obtained from right angled triangle, the x(t) can be written as,

$$x(t) = e^{-2t}(\cos 3t \times 0.95\cos\theta_1 - \sin 3t \times 0.95\sin\theta_1) + (\sin 3t \times 1.14\cos\theta_2 - \cos 3t \times 1.14\sin\theta_2)$$
$$= 0.95\,e^{-2t}(\cos 3t\cos 71.6^\circ - \sin 3t\sin 71.6^\circ) + 1.14(\sin 3t\cos 15.3^\circ - \cos 3t\sin 15.3^\circ)$$
$$= 0.95\,e^{-2t}\cos(3t + 71.6^\circ) + 1.14\sin(3t - 15.3^\circ) \;; \quad t \ge 0$$

Example 3.23

Find the transfer function of the systems governed by the following differential equations.

a) $\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 5y(t) = 2\frac{dx(t)}{dt} + x(t)$

b) $\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 11y(t) = x(t-2)$

Solution

a) Given that, $\mathbf{\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 5y(t) = 2\frac{dx(t)}{dt} + x(t)}$

On taking Laplace transform of the above equation with zero initial conditions we get,

$$s^3Y(s) + 4s^2Y(s) + 3sY(s) + 5Y(s) = 2sX(s) + X(s)$$

$$\left(s^3 + 4s^2 + 3s + 5\right) Y(s) = (2s+1)\, X(s)$$

$$\therefore \text{Transfer Function,} \quad \frac{Y(s)}{X(s)} = \frac{2s+1}{s^3 + 4s^2 + 3s + 5}$$

if, $\mathcal{L}\{x(t)\} = X(s)$,
then, $\mathcal{L}\left\{\frac{d^n x(t)}{dt^n}\right\} = s^n X(s)$
with zero initial conditions

b) Given that, $\mathbf{\frac{d^3y(t)}{dt^3} + 8\frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 11y(t) = x(t-2)}$

On taking Laplace transform of the above equation with zero initial conditions we get,

$$s^3Y(s) + 8s^2Y(s) + 6sY(s) + 11Y(s) = e^{-2s}\, X(s)$$

$$\left(s^3 + 8s^2 + 6s + 11\right) Y(s) = e^{-2s}\, X(s)$$

$$\therefore \text{Transfer Function,} \quad \frac{Y(s)}{X(s)} = \frac{e^{-2s}}{s^3 + 8s^2 + 6s + 11}$$

if, $\mathcal{L}\{x(t)\} = X(s)$,
then, $\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$

Example 3.24

Find the transfer function of the systems governed by the following impulse responses.

a) $h(t) = (2 + t)\, e^{-3t}\, u(t)$ b) $h(t) = t^2\, u(t) - e^{-4t}\, u(t) + e^{-7t}\, u(t)$ c) $h(t) = u(t) + 0.5\, e^{-6t}\, u(t) + 0.2\, e^{-3t} \cos t\, u(t)$

Solution

a) Impulse response, $\mathbf{h(t) = (2 + t)\, e^{-3t}\, u(t)}$

The transfer function is given by Laplace transform of Impulse response.

$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}, \quad \mathcal{L}\{t\, u(t)\} = \frac{1}{s^2}$

$$\therefore \text{Transfer function,} \quad H(s) = \mathcal{L}\{h(t)\}$$

$$= \mathcal{L}\{(2 + t)\, e^{-3t}\, u(t)\} = \mathcal{L}\{2\, e^{-3t}\, u(t)\} + \mathcal{L}\{t\, e^{-3t} u(t)\}$$

$$= 2\mathcal{L}\{e^{-3t}\, u(t)\} + \mathcal{L}\{t\, u(t)\}\Big|_{s=s+3} = 2 \times \frac{1}{s+3} + \frac{1}{s^2}\Big|_{s=s+3}$$

$$= \frac{2}{s+3} + \frac{1}{(s+3)^2} = \frac{2(s+3)+1}{(s+3)^2} = \frac{2s+7}{s^2+6s+9}$$

if, $\mathcal{L}\{x(t)\} = X(s)$,
then, $\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$

b) Impulse response, $\mathbf{h(t) = t^2\, u(t) - e^{-4t}\, u(t) + e^{-7t}\, u(t)}$

The transfer function is given by Laplace transform of Impulse response.

$$\therefore \text{Transfer function,} \quad H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{t^2 u(t) - e^{-4t} u(t) + e^{-7t} u(t)\}$$

$$= \mathcal{L}\{t^2 u(t)\} - \mathcal{L}\{e^{-4t} u(t)\} + \mathcal{L}\{e^{-7t} u(t)\}$$

$\mathcal{L}\{t^n u(t)\} = \frac{n!}{s^{n+1}}; \quad \mathcal{L}\{e^{-at} u(t)\} = \frac{1}{s+a}$

$$= \frac{2}{s^3} - \frac{1}{s+4} + \frac{1}{s+7}$$

$$= \frac{2(s+4)(s+7) - s^3(s+7) + s^3(s+4)}{s^3(s+4)(s+7)}$$

$$= \frac{2(s^2+7s+4s+28) - s^4 - 7s^3 + s^4 + 4s^3}{s^3(s^2+11s+28)}$$

$$= \frac{-3s^3 + 2s^2 + 22s + 56}{s^5 + 11s^4 + 28s^3}$$

c) Impulse response, $h(t) = u(t) + 0.5\,e^{-6t}\,u(t) + 0.2\,e^{-3t}\cos t\,u(t)$

The transfer function is given by Laplace transform of Impulse response.

$\therefore$ Transfer function, $H(s) = \mathcal{L}\{h(t)\}$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$
$$\mathcal{L}\left\{e^{-at}u(t)\right\} = \frac{1}{s+a}$$
$$\mathcal{L}\{\cos t\, u(t)\} = \frac{s}{s^2+1}$$

$$= \mathcal{L}\left\{u(t) + 0.5\,e^{-6t}\,u(t) + 0.2\,e^{-3t}\cos t\,u(t)\right\}$$

$$= \mathcal{L}\{u(t)\} + 0.5 \times \mathcal{L}\left\{e^{-6t}u(t)\right\} + 0.2 \times \mathcal{L}\{\cos t\,u(t)\}\Big|_{s=s+3}$$

if, $\mathcal{L}\{x(t)\} = X(s)$,
then, $\mathcal{L}\left\{e^{-at}x(t)\right\} = X(s+a)$

$$= \frac{1}{s} + 0.5 \times \frac{1}{s+6} + 0.2 \times \frac{s}{s^2+1}\Bigg|_{s=s+3}$$

$$= \frac{1}{s} + \frac{0.5}{s+6} + \frac{0.2(s+3)}{(s+3)^2+1}$$

$$= \frac{1}{s} + \frac{0.5}{s+6} + \frac{0.2(s+3)}{s^2+6s+9+1}$$

$$= \frac{1}{s} + \frac{0.5}{s+6} + \frac{0.2(s+3)}{s^2+6s+10}$$

$$= \frac{(s+6)(s^2+6s+10) + 0.5s\,(s^2+6s+10) + 0.2s(s+3)(s+6)}{s(s+6)(s^2+6s+10)}$$

$$= \frac{s^3+6s^2+10s+6s^2+36s+60 + 0.5s^3+3s^2+5s + 0.2s(s^2+9s+18)}{s(s^3+6s^2+10s+6s^2+36s+60)}$$

$$= \frac{s^3+6s^2+10s+6s^2+36s+60 + 0.5s^3+3s^2+5s+0.2s^3+1.8s^2+3.6s}{s(s^3+12s^2+46s+60)}$$

$$= \frac{1.7s^3+16.8s^2+54.6s+60}{s^4+12s^3+46s^2+60s}$$

Example 3.25

The unit step response of continuous time systems are given below. Determine the transfer function of the systems.

a) $s(t) = u(t) + e^{-2t}\,u(t)$ b) $s(t) = t^2\,u(t) + t\,e^{-4t}\,u(t)$ c) $s(t) = t\,u(t) + \sin t\,u(t)$

Solution

a) Unit step response, $s(t) = u(t) + e^{-2t}\,u(t)$

On taking Laplace transform of unit step response we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\left\{u(t) + e^{-2t}u(t)\right\}$$

Let, $\mathcal{L}\{s(t)\} = S(s)$

$$\mathcal{L}\{u(t)\} = \frac{1}{s};\quad \mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}$$

$$\therefore\ S(s) = \mathcal{L}\left\{u(t) + e^{-2t}u(t)\right\} = \mathcal{L}\{u(t)\} + \mathcal{L}\left\{e^{-2t}u(t)\right\}$$

$$= \frac{1}{s} + \frac{1}{s+2} = \frac{s+2+s}{s(s+2)} = \frac{2s+2}{s(s+2)} \quad ....(1)$$

Let, $U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$; where u(t) is unit step signal. .....(2)

Transfer function, $H(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{S(s)}{U(s)}$

$$= S(s) \times \frac{1}{U(s)} = \frac{2s+2}{s(s+2)} \times s = \frac{2s+2}{s+2}$$ Using equations (1) and (2)

b) Unit step response, $s(t) = t^2\, u(t) + t\, e^{-4t}\, u(t)$

On taking Laplace transform of unit step response we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{t^2\, u(t) + t\, e^{-4t} u(t)\}$$

Let, $\mathcal{L}\{s(t)\} = S(s)$

$\mathcal{L}\{t^n\, u(t)\} = \frac{n!}{s^{n+1}}$

$$\therefore S(s) = \mathcal{L}\{t^2\, u(t) + t\, e^{-4t} u(t)\} = \mathcal{L}\{t^2\, u(t)\} + \mathcal{L}\{t\, e^{-4t} u(t)\}$$

$$= \mathcal{L}\{t^2\, u(t)\} + \mathcal{L}\{t\, u(t)\}\Big|_{s=s+4} = \frac{2}{s^3} + \frac{1}{s^2}\Big|_{s=s+4} = \frac{2}{s^3} + \frac{1}{(s+4)^2}$$

if, $\mathcal{L}\{x(t)\} = X(s)$, then, $\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$

$$= \frac{2(s+4)^2 + s^3}{s^3(s+4)^2} = \frac{2(s^2+8s+16)+s^3}{s^3(s^2+8s+16)} = \frac{s^3+2s^2+16s+32}{s^3(s^2+8s+16)} \quad ....(1)$$

Let, $U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$; where u(t) is unit step signal .....(2)

Transfer function, $H(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{S(s)}{U(s)}$

$$= S(s) \times \frac{1}{U(s)} = \frac{s^3+2s^2+16s+32}{s^3(s^2+8s+16)} \times s = \frac{s^3+2s^2+16s+32}{s^4+8s^3+16s^2}$$ Using equations (1) and (2)

c) Unit step response, $s(t) = t\, u(t) + \sin t\, u(t)$

On taking Laplace transform of unit step response we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{t\, u(t) + \sin t\, u(t)\}$$

Let, $\mathcal{L}\{s(t)\} = S(s)$

$\mathcal{L}\{t u(t)\} = \frac{1}{s^2}$

$\mathcal{L}\{\sin t\, u(t)\} = \frac{1}{s^2+1}$

$$\therefore S(s) = \mathcal{L}\{t\, u(t) + \sin t\, u(t)\} = \mathcal{L}\{t\, u(t)\} + \mathcal{L}\{\sin t\, u(t)\}$$

$$= \frac{1}{s^2} + \frac{1}{s^2+1} = \frac{s^2+1+s^2}{s^2(s^2+1)} = \frac{2s^2+1}{s^2(s^2+1)} \quad ....(1)$$

Let, $U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$; where u(t) is unit step signal .....(2)

Transfer function, $H(s) = \frac{\text{Laplace transform of output}}{\text{Laplace transform of input}} = \frac{S(s)}{U(s)}$

$$= S(s) \times \frac{1}{U(s)} = \frac{2s^2+1}{s^2(s^2+1)} \times s = \frac{2s^2+1}{s^3+s}$$ Using equations (1) and (2)

Example 3.26

Find the impulse response of continuous time systems governed by the following transfer functions.

a) $H(s) = \frac{1}{s^2(s-2)}$ b) $H(s) = \frac{1}{s(s+1)\,(s-2)}$ c) $H(s) = \frac{1}{s^2+s+1}$

Solution

a) Transfer function, $H(s) = \frac{1}{s^2(s-2)}$

The impulse response is obtained by taking inverse Laplace transform of the transfer function.

$$\therefore \text{Impulse response, } h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-2}\right\}$$

Using partial fraction expansion technique

$$A = \frac{1}{s^2(s-2)} \times s^2 \bigg|_{s=0} = \frac{1}{0-2} = -\frac{1}{2}$$

$$B = \frac{d}{ds}\left[\frac{1}{s^2(s-2)} \times s^2\right]\bigg|_{s=0} = \frac{d}{ds}\left[\frac{1}{s-2}\right]\bigg|_{s=0} = \frac{-1}{(s-2)^2}\bigg|_{s=0} = \frac{-1}{(0-2)^2} = -\frac{1}{4}$$

$$C = \frac{1}{s^2(s-2)} \times (s-2)\bigg|_{s=2} = \frac{1}{2^2} = \frac{1}{4}$$

$$\therefore \text{Impulse response, } h(t) = \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s} + \frac{1}{4}\frac{1}{s-2}\right\}$$

$$= -\frac{1}{2}t\,u(t) - \frac{1}{4}u(t) + \frac{1}{4}e^{2t}u(t) = \frac{1}{4}\left(e^{2t} - 2t - 1\right)u(t)$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$
$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$$
$$\mathcal{L}\{e^{at}u(t)\} = \frac{1}{s-a}$$

---

b) Transfer function, $\mathbf{H(s) = \dfrac{1}{s(s+1)(s-2)}}$

The impulse response is obtained by taking inverse Laplace transform of the transfer function.

$$\therefore \text{Impulse response, } h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s-2}\right\}$$

$$A = \frac{1}{s(s+1)(s-2)} \times s\bigg|_{s=0} = \frac{1}{(0+1)(0-2)} = -\frac{1}{2}$$

Using partial fraction expansion technique

$$B = \frac{1}{s(s+1)(s-2)} \times (s+1)\bigg|_{s=-1} = \frac{1}{-1\times(-1-2)} = \frac{1}{3}$$

$$C = \frac{1}{s(s+1)(s-2)} \times (s-2)\bigg|_{s=2} = \frac{1}{2\times(2+1)} = \frac{1}{6}$$

$$\therefore \text{Impulse response, } h(t) = \mathcal{L}^{-1}\left\{-\frac{1}{2}\frac{1}{s} + \frac{1}{3}\frac{1}{s+1} + \frac{1}{6}\frac{1}{s-2}\right\}$$

$$= -\frac{1}{2}u(t) + \frac{1}{3}e^{-t}u(t) + \frac{1}{6}e^{2t}u(t) = \frac{1}{6}\left(e^{2t} + 2e^{-t} - 3\right)u(t)$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$
$$\mathcal{L}\{e^{\pm at}u(t)\} = \frac{1}{s\mp a}$$

---

c) Transfer function, $\mathbf{H(s) = \dfrac{1}{s^2+s+1}}$

The impulse response is obtained by taking inverse Laplace transform of the transfer function.

$$\therefore \text{Impulse response, } h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\}$$

Arranging, s^2+s, in the form of $(x+y)^2$

$(x+y)^2 = x^2+2xy+y^2$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 2\times 0.5\times s + 0.5^2) + (1-0.5^2)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+0.5)^2 + 0.75}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+0.5)^2 + \left(\sqrt{0.75}\right)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+0.5)^2 + 0.866^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{0.866} \times \frac{0.866}{(s+0.5)^2 + 0.866^2}\right\}$$

$$\therefore \text{ Impulse response, } h(t) = \frac{1}{0.866} \times \mathcal{L}^{-1}\left\{\frac{0.866}{(s+0.5)^2 + 0.866^2}\right\}$$

$$= \frac{1}{0.866}\ e^{-0.5t}\ \sin(0.866t)\ u(t)$$

$$\boxed{\mathcal{L}\left\{e^{-at}\sin\Omega_0 t\ u(t)\right\} = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}}$$

Example 3.27

The input and impulse response of continuous time systems are given below. Find the output of the continuous time systems.

a) $x(t) = \delta(t)$, $h(t) = e^{-at}\,u(t)$

b) $x(t) = e^{-2t}\,u(t)$, $h(t) = u(t)$

c) $x(t) = e^{-3t}\,u(t)$, $h(t) = u(t-1)$

d) $x(t) = \cos 4t\ u(t) + \cos 7t\ u(t)$, $h(t) = \delta(t-3)$

Solution

a) Given that, $x(t) = \delta(t)$ and $h(t) = e^{-at}\,u(t)$

$$\boxed{\mathcal{L}\{\delta(t)\} = 1;\quad \mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$\text{Let, } X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\delta(t)\} = 1 \qquad .....(1)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a} \qquad .....(2)$$

Response / Output, $y(t) = x(t) * h(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$= X(s)\,H(s)$$

Using convolution theorem of Laplace transform

$$= 1 \times \frac{1}{s+a} = \frac{1}{s+a}$$

Using equations (1) and (2)

Response / Output is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t)$$

b) Given that, $x(t) = e^{-2t}\,u(t)$ and $h(t) = u(t)$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s};\quad \mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$\text{Let, } X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{e^{-2t}\,u(t)\right\} = \frac{1}{s+2} \qquad .....(1)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s} \qquad .....(2)$$

Response / Output, $y(t) = x(t) * h(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$= X(s)\,H(s)$$

Using convolution theorem of Laplace transform

$$= \frac{1}{s+2} \times \frac{1}{s} = \frac{1}{s(s+2)}$$

Using equations (1) and (2)

Response / Output is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+2)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+2}\right\}$$

Using partial fraction expansion technique

$$A = \frac{1}{s\,(s+2)} \times s\,\bigg|_{s=0} = \frac{1}{0+2} = \frac{1}{2}$$

$$B = \frac{1}{s\,(s+2)} \times (s+2)\,\bigg|_{s=-2} = \frac{1}{-2} = -\frac{1}{2}$$

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \frac{1}{2}u(t) - \frac{1}{2}e^{-2t}u(t) = \frac{1}{2}\left(1 - e^{-2t}\right)u(t)$$

c) Given that, $x(t) = e^{-3t}\,u(t)$ and $h(t) = u(t-1)$

$$\text{Let, } X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{e^{-3t}\,u(t)\} = \frac{1}{s+3} \quad .....(1)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-1)\} = e^{-s} \times \mathcal{L}\{u(t)\} = e^{-s} \times \frac{1}{s} = \frac{e^{-s}}{s} \quad .....(2)$$

Response / Output, $y(t) = x(t) * h(t)$

if $\mathcal{L}\{u(t)\} = \frac{1}{s}$, then $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$= X(s)\,H(s)$$

Using convolution theorem of Laplace transform

$$= \frac{1}{s+3} \times \frac{e^{-s}}{s} = \frac{e^{-s}}{s(s+3)}$$

Using equations (1) and (2)

Response / Output is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s+3)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s+3)}\right\}\Bigg|_{t=t-1} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+3}\right\}\Bigg|_{t=t-1}$$

Using partial fraction expansion technique

$$A = \frac{1}{s\,(s+3)} \times s\Bigg|_{s=0} = \frac{1}{0+3} = \frac{1}{3}$$

$$B = \frac{1}{s\,(s+3)} \times (s+3)\Bigg|_{s=-3} = \frac{1}{-3} = -\frac{1}{3}$$

if, $\mathcal{L}\{x(t)\} = X(s)$, then, $\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\frac{1}{s} - \frac{1}{3}\frac{1}{s+3}\right\}\Bigg|_{t=t-1} = \left[\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}\right]\Bigg|_{t=t-1}$$

$$= \left[\frac{1}{3}u(t) - \frac{1}{3}e^{-3t}u(t)\right]\Bigg|_{t=t-1} = \left[\frac{1}{3}u(t-1) - \frac{1}{3}e^{-3(t-1)}u(t-1)\right] = \frac{1}{3}\left(1 - e^{-3(t-1)}\right)u(t-1)$$

d) Given that, $x(t) = \cos 4t\ u(t) + \cos 7t\ u(t)$ and $h(t) = \delta(t-3)$

$\mathcal{L}\{\cos\Omega_0 t\ u(t)\} = \frac{s}{s^2 + \Omega_0^2}$

$$\text{Let, } X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\cos 4t\ u(t) + \cos 7t\ u(t)\} = \frac{s}{s^2+4^2} + \frac{s}{s^2+7^2} \quad .....(1)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{\delta(t-3)\} = e^{-3s} \times \mathcal{L}\{\delta(t)\} = e^{-3s} \times 1 = e^{-3s} \quad .....(2)$$

Response / Output, $y(t) = x(t) * h(t)$

if $\mathcal{L}\{\delta(t)\} = 1$, then $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$= X(s)\,H(s)$$

Using convolution theorem of Laplace transform

$$= \left[\frac{s}{s^2+4^2} + \frac{s}{s^2+7^2}\right] \times e^{-3s} = e^{-3s} \times \left[\frac{s}{s^2+4^2} + \frac{s}{s^2+7^2}\right]$$

Using equations (1) and (2)

Response / Output is given by inverse Laplace transform of the above equation.

if, $\mathcal{L}\{x(t)\} = X(s)$, then, $\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$

$$\therefore \text{ Response, } y(t) = \mathcal{L}^{-1}\left\{e^{-3s} \times \left[\frac{s}{s^2+4^2} + \frac{s}{s^2+7^2}\right]\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2} + \frac{s}{s^2+7^2}\right\}\Bigg|_{t=t-3}$$

$$= \left[\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+7^2}\right\}\right]\Bigg|_{t=t-3} = \left[\cos 4t\ u(t) + \cos 7t\ u(t)\right]\Big|_{t=t-3}$$

$$= \cos 4(t-3)\ u(t-3) + \cos 7(t-3)\ u(t-3) = \left(\cos 4(t-3) + \cos 7(t-3)\right)u(t-3)$$

Example 3.28

Perform convolution of the following causal signals, using Laplace transform.

a) $x_1(t) = 2\,u(t)$, $x_2(t) = u(t)$ b) $x_1(t) = e^{-2t}\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$

c) $x_1(t) = t\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$ d) $x_1(t) = \cos t\,u(t)$, $x_2(t) = t\,u(t)$

Solution

$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s} \quad \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}}$

a) Given that, $x_1(t) = 2\,u(t)$

$x_2(t) = u(t)$

Let, $X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{2\,u(t)\} = 2\,\mathcal{L}\{u(t)\} = 2 \times \frac{1}{s} = \frac{2}{s}$ .....(1)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$ .....(2)

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{2}{s} \times \frac{1}{s} = \frac{2}{s^2}$$

Using equations (1) and (2)

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$= 2 \times t\,u(t) = 2\,t\,u(t)$$

---

b) Given that, $x_1(t) = e^{-2t}\,u(t)$

$x_2(t) = e^{-5t}\,u(t)$

$\boxed{\mathcal{L}\{e^{-at}\,u(t)\} = \frac{1}{s+a}}$

Let, $X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{e^{-2t}\,u(t)\} = \frac{1}{s+2}$ .....(1)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{e^{-5t}\,u(t)\} = \frac{1}{s+5}$ .....(2)

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{1}{s+2} \times \frac{1}{s+5} = \frac{1}{(s+2)\,(s+5)}$$

Using equations (1) and (2)

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)\,(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s+2} + \frac{B}{s+5}\right\}$$

Using partial fraction expansion technique

$$A = \frac{1}{(s+2)\,(s+5)} \times (s+2)\Bigg|_{s=-2} = \frac{1}{-2+5} = \frac{1}{3}$$

$$B = \frac{1}{(s+2)\,(s+5)} \times (s+5)\Bigg|_{s=-5} = \frac{1}{-5+2} = -\frac{1}{3}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{1}{s+5}\right\} = \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{3}\,e^{-2t}\,u(t) - \frac{1}{3}\,e^{-5t}\,u(t) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)u(t)$$

---

c) Given that, $x_1(t) = t\,u(t)$

$x_2(t) = e^{-5t}\,u(t)$

$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s};\ \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2};\ \mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}}$

Let, $X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$ .....(1)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{e^{-5t}\,u(t)\} = \frac{1}{s+5}$ .....(2)

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{1}{s^2} \times \frac{1}{s+5} = \frac{1}{s^2(s+5)}$$

Using equations (1) and (2)

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2\,(s+5)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+5}\right\}$$

Using partial fraction expansion technique

$$A = \frac{1}{s^2(s+5)} \times s^2 \bigg|_{s=0} = \frac{1}{0+5} = \frac{1}{5}$$

$$B = \frac{d}{ds}\left[\frac{1}{s^2(s+5)} \times s^2\right]\bigg|_{s=0} = \frac{d}{ds}\left[\frac{1}{s+5}\right]\bigg|_{s=0} = \frac{-1}{(s+5)^2}\bigg|_{s=0} = \frac{-1}{(0+5)^2} = -\frac{1}{25}$$

$$C = \frac{1}{s^2(s+5)} \times (s+5)\bigg|_{s=-5} = \frac{1}{(-5)^2} = \frac{1}{25}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{5}\,\frac{1}{s^2} - \frac{1}{25}\,\frac{1}{s} + \frac{1}{25}\,\frac{1}{s+5}\right\}$$

$$= \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{25}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{25}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\}$$

$$= \frac{1}{5}\,t\,u(t) - \frac{1}{25}\,u(t) + \frac{1}{25}\,e^{-5t}\,u(t)$$

$$= \frac{1}{25}\left(e^{-5t} + 5t - 1\right)u(t)$$

---

d) Given that, $x_1(t) = \cos t\ u(t)$

$x_2(t) = t\ u(t)$

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2};\quad \mathcal{L}\{\cos \Omega_0 t\ u(t)\} = \frac{s}{s^2 + \Omega_0^2}$$

Let, $$X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{\cos t\ u(t)\} = \frac{s}{s^2+1} \quad .....(1)$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{t\ u(t)\} = \frac{1}{s^2} \quad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s) = \frac{1}{s^2} \times \frac{s}{s^2+1}$$

Using equations (1) and (2)

$$= \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Cs+D}{s^2+1} \quad .....(3)$$

Using partial fraction expansion technique

On cross multiplying equation (3) we get,

$$1 = A\,(s^2+1) + (Cs+D)s$$

$$\therefore\ 1 = As^2 + A + Cs^2 + Ds \quad .....(4)$$

On equating constants of equation (4) we get, $A = 1$

On equating coefficients of s of equation (4) we get, $D = 0$

On equating coefficients of s^2 of equation (4) we get, $A + C = 0 \ \Rightarrow\ C = -A = -1$

$$\therefore \mathcal{L}\{x_1(t) * x_2(t)\} = \frac{A}{s} + \frac{Cs+D}{s^2+1}$$

$$= \frac{1}{s} - \frac{s}{s^2+1}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= u(t) - \cos t\, u(t)$$

$\mathcal{L}\{u(t)\} = \frac{1}{s}$

$\mathcal{L}\{\cos \Omega_0 t\, u(t)\} = \frac{s}{s^2 + \Omega_0^2}$

Note : *Compare the result of Example 3.28 with Example 2.20 of chapter - 2.*

Example 3.29

Perform deconvolution operation to extract the signal $x_1(t)$.

a) $x_1(t) * x_2(t) = 2t\, u(t)$; $x_2(t) = u(t)$

b) $x_1(t) * x_2(t) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)u(t)$; $x_2(t) = e^{-5t}\, u(t)$

c) $x_1(t) * x_2(t) = \frac{1}{25}\left(e^{-5t} + 5t - 1\right)u(t)$; $x_2(t) = e^{-5t}\, u(t)$

d) $x_1(t) * x_2(t) = u(t) - \cos t\, u(t)$; $x_2(t) = t\, u(t)$

Solution

a) Given that, $x_1(t) * x_2(t) = 2t\, u(t)$; $x_2(t) = u(t)$

Let, $x_1(t) * x_2(t) = x_3(t)$ .....(1)

$\therefore\ x_3(t) = 2t\, u(t)$

Let, $X_3(s) = \mathcal{L}\{x_3(t)\} = \mathcal{L}\{2t\, u(t)\} = \frac{2}{s^2}$ .....(2)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$ .....(3)

On taking Laplace transform of equation (1) we get,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \mathcal{L}\{x_3(t)\}$$

$$X_1(s)\, X_2(s) = X_3(s)$$

Using convolution property of Laplace transform

$$\therefore X_1(s) = \frac{X_3(s)}{X_2(s)} = X_3(s) \times \frac{1}{X_2(s)} = \frac{2}{s^2} \times s = \frac{2}{s}$$

Using equations (2) and (3)

$$\therefore\ x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s}\right\} = 2\, u(t)$$

b) Given that, $x_1(t) * x_2(t) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)u(t)$; $x_2(t) = e^{-5t}\, u(t)$

Let, $x_1(t) * x_2(t) = x_3(t)$ .....(1)

$$\therefore x_3(t) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right)u(t) = \frac{e^{-2t}}{3}u(t) - \frac{e^{-5t}}{3}u(t)$$

Let, $X_3(s) = \mathcal{L}\{x_3(t)\} = \mathcal{L}\left\{\frac{e^{-2t}}{3}u(t) - \frac{e^{-5t}}{3}u(t)\right\} = \frac{1}{3(s+2)} - \frac{1}{3(s+5)}$ .....(2)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{e^{-5t}\, u(t)\} = \frac{1}{s+5}$ .....(3)

On taking Laplace transform of equation (1) we get,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \mathcal{L}\{x_3(t)\}$$

$$X_1(s)\, X_2(s) = X_3(s)$$

Using convolution property of Laplace transform

$$\therefore\ X_1(s) = \frac{X_3(s)}{X_2(s)} = X_3(s) \times \frac{1}{X_2(s)}$$

$$= \left[\frac{1}{3(s+2)} - \frac{1}{3(s+5)}\right] \times (s+5)$$

Using equations (2) and (3)

$$= \frac{s+5}{3(s+2)} - \frac{1}{3} = \frac{s+5-(s+2)}{3(s+2)}$$

$$\therefore\ X_1(s) = \frac{s+5-s-2}{3(s+2)} = \frac{3}{3(s+2)} = \frac{1}{s+2}$$

$$\therefore\ x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}\,u(t)$$

---

c) Given that, $\mathbf{x_1(t) * x_2(t) = \frac{1}{25}\left(e^{-5t} + 5t - 1\right)u(t)\ ;\ \ x_2(t) = e^{-5t}\,u(t)}$

Let, $x_1(t) * x_2(t) = x_3(t)$ .....(1)

$$\therefore x_3(t) = \frac{1}{25}\left(e^{-5t} + 5t - 1\right)u(t) = \frac{e^{-5t}}{25}u(t) + \frac{t}{5}u(t) - \frac{1}{25}u(t)$$

Let, $X_3(s) = \mathcal{L}\{x_3(t)\} = \mathcal{L}\left\{\frac{e^{-5t}}{25}u(t) + \frac{t}{5}u(t) - \frac{1}{25}u(t)\right\} = \frac{1}{25(s+5)} + \frac{1}{5s^2} - \frac{1}{25s}$ .....(2)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\left\{e^{-5t}\,u(t)\right\} = \frac{1}{s+5}$ .....(3)

On taking Laplace transform of equation (1) we get,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \mathcal{L}\{x_3(t)\}$$

$$X_1(s)\ X_2(s) = X_3(s)$$

Using convolution property of Laplace transform

$$\therefore\ X_1(s) = \frac{X_3(s)}{X_2(s)} = X_3(s)\ \times \frac{1}{X_2(s)}$$

$$= \left[\frac{1}{25(s+5)} + \frac{1}{5s^2} - \frac{1}{25s}\right] \times (s+5) = \frac{1}{25} + \frac{s+5}{5s^2} - \frac{s+5}{25s}$$

Using equations (2) and (3)

$$= \frac{s^2 + 5(s+5) - s(s+5)}{25s^2} = \frac{s^2 + 5s + 25 - s^2 - 5s}{25s^2} = \frac{25}{25s^2} = \frac{1}{s^2}$$

$$\therefore x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\,u(t)$$

---

d) Given that, $\mathbf{x_1(t) * x_2(t) = u(t) - \cos t\ u(t)\ ;\ \ x_2(t) = t\,u(t)}$

Let, $x_1(t) * x_2(t) = x_3(t)$ .....(1)

$$\therefore\ x_3(t) = u(t) - \cos t\ u(t)$$

Let, $X_3(s) = \mathcal{L}\{x_3(t)\} = \mathcal{L}\{u(t) - \cos t\ u(t)\} = \frac{1}{s} - \frac{s}{s^2+1}$ .....(2)

$X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{t\ u(t)\} = \frac{1}{s^2}$ .....(3)

On taking Laplace transform of equation (1) we get,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = \mathcal{L}\{x_3(t)\}$$

$$X_1(s)\ X_2(s) = X_3(s)$$

Using convolution property of Laplace transform

$$\therefore\ X_1(s) = \frac{X_3(s)}{X_2(s)} = X_3(s)\ \times \frac{1}{X_2(s)} = \left[\frac{1}{s} - \frac{s}{s^2+1}\right] \times s^2$$

Using equations (2) and (3)

$$= s - \frac{s^3}{s^2+1} = \frac{s(s^2+1) - s^3}{s^2+1} = \frac{s^3 + s - s^3}{s^2+1} = \frac{s}{s^2+1}$$

$$\therefore x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t\ u(t)$$

Note : *Compare the results of Example 3.29 with Example 3.28.*

Example 3.30

The impulse response of continuous time systems are given below. Determine the unit step response of the systems using convolution theorem of Laplace transform, and verify the result with Example 2.21 of chapter-2.

a) $h(t) = 3t\, u(t)$ b) $h(t) = e^{-5t}\, u(t)$ c) $h(t) = u(t + 2)$

d) $h(t) = u(t - 2)$ e) $h(t) = u(t + 2) + u(t - 2)$

Solution

a) Given that, h(t) = 3t u(t)

Let, $$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{3t\, u(t)\} = 3\,\mathcal{L}\{t\, u(t)\} = 3 \times \frac{1}{s^2} = \frac{3}{s^2} \qquad .....(1)$$

$$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \text{ ; where u(t) is unit step signal} \qquad .....(2)$$

Unit step response, $s(t) = h(t) * u(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{h(t) * u(t)\}$$

$$= H(s)\, U(s)$$

Using convolution theorem of Laplace transform

$$= \frac{3}{s^2} \times \frac{1}{s} = \frac{3}{s^3}$$

Using equations (1) and (2)

Unit step response is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Unit step response, } \quad s(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{3}{2} \times \frac{2}{s^{2+1}}\right\}$$

$$= \frac{3}{2}\,\mathcal{L}^{-1}\left\{\frac{2}{s^{2+1}}\right\} = \frac{3}{2}\, t^2\, u(t)$$

$\boxed{\mathcal{L}\{t^n\, u(t)\} = \frac{n!}{s^{n+1}}}$

---

b) Given that, $h(t) = e^{-5t}\, u(t)$

Let, $$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{e^{-5t}\, u(t)\} = \frac{1}{s+5} \qquad .....(1)$$

$$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s} \text{ ; where u(t) is unit step signal} \qquad .....(2)$$

Unit step response, $s(t) = h(t) * u(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{h(t) * u(t)\} = H(s)\, U(s)$$

Using convolution theorem of Laplace transform

$$= \frac{1}{s+5} \times \frac{1}{s} = \frac{1}{s(s+5)}$$

Using equations (1) and (2)

Unit step response is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Unit step response, } \quad s(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s+5)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s+5}\right\}$$

Using partial fraction expansion technique

$$A = \frac{1}{s\,(s+5)} \times s\,\bigg|_{s=0} = \frac{1}{0+5} = \frac{1}{5}$$

$$B = \frac{1}{s\,(s+5)} \times (s+5)\,\bigg|_{s=-5} = \frac{1}{-5} = -\frac{1}{5}$$

$$\therefore \text{ Unit step response, } \quad s(t) = \mathcal{L}^{-1}\left\{\frac{1}{5}\,\frac{1}{s} - \frac{1}{5}\,\frac{1}{s+5}\right\}$$

$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s} \; ; \; \mathcal{L}\{e^{-at}\, u(t)\} = \frac{1}{s+a}}$

$$= \frac{1}{5}\, u(t) - \frac{1}{5}\, e^{-5t}\, u(t) = \frac{1}{5}\left(1 - e^{-5t}\right) u(t)$$

c) Given that, h(t) = u(t + 2)

Let, $H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{u(t+2)\} = e^{2s}\,\mathcal{L}\{u(t)\} = e^{2s} \times \frac{1}{s} = \frac{e^{2s}}{s}$ .....(1)

$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$; where u(t) is unit step signal .....(2)

Unit step response, $s(t) = h(t) * (t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{h(t) * u(t)\}$$

Using convolution theorem of Laplace transform

$$= H(s)\,U(s)$$

Using equations (1) and (2)

$$= \frac{e^{2s}}{s} \times \frac{1}{s} = \frac{e^{2s}}{s^2}$$

if $\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$, then $\mathcal{L}\{(t+a)\,u(t+a)\} = \frac{e^{as}}{s^2}$

Unit step response is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Unit step response, } s(t) = \mathcal{L}^{-1}\left\{\frac{e^{2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Bigg|_{t=t+2} = t\,u(t)\big|_{t=t+2} = (t+2)\,u(t+2)$$

To verify the above result with Example 2.21(c) of chapter-2

When $t > -2$, $u(t+2) = 1$, $\therefore\ s(t) = t + 2\ ;\ t > -2$

---

d) Given that, h(t) = u(t – 2)

Let, $H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{u(t-2)\} = e^{-2s}\mathcal{L}\{u(t)\} = e^{-2s} \times \frac{1}{s} = \frac{e^{-2s}}{s}$ .....(1)

$U(s) = \mathcal{L}\{u(t)\} = \frac{1}{s}$; where u(t) is unit step signal .....(2)

Unit step response, $s(t) = h(t) * u(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{s(t)\} = \mathcal{L}\{h(t) * u(t)\}$$

Using convolution theorem of Laplace transform

$$= H(s)\,U(s)$$

Using equations (1) and (2)

$$= \frac{e^{-2s}}{s} \times \frac{1}{s} = \frac{e^{-2s}}{s^2}$$

if $\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$, then $\mathcal{L}\{(t-a)\,u(t-a)\} = \frac{e^{-as}}{s^2}$

Unit step response is given by inverse Laplace transform of the above equation.

$$\therefore \text{ Unit step response, } s(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\Bigg|_{t=t-2} = t\,u(t)\big|_{t=t-2} = (t-2)\,u(t-2)$$

To verify the above result with Example 2.21(d) of chapter-2

When $t > 2$, $u(t-2) = 1$, $\therefore\ s(t) = t - 2\ ;\ t > 2$

---

e) Given that, h(t) = u(t + 2) + u(t – 2)

$$\begin{aligned}\text{Unit step response, } s(t) &= h(t) * u(t)\\ &= [u(t+2) + u(t-2)] * u(t)\\ &= [u(t+2) * u(t)] + [u(t-2) * u(t)]\\ &= (t+2)\,u(t+2) + (t-2)\,u(t-2)\end{aligned}$$

Using the results of (c) and (d)

To verify the above result with Example 2.21(e) of chapter-2

When $t = -2$ to 2, $u(t+2) = 1$, $u(t-2) = 0$

$\therefore\ s(t) = (t+2) \times 1 + (t-2) \times 0 = t + 2\ ;\ -2 < t < 2$

When $t > 2$, $u(t+2) = 1$, $u(t-2) = 1$

$\therefore\ s(t) = (t+2) \times 1 + (t-2) \times 1 = t + 2 + t - 2 = 2t\ ;\ t < 2$

Example 3.31

Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem of Laplace transform and verify the result with Example 2.23 of chapter-2.

a)

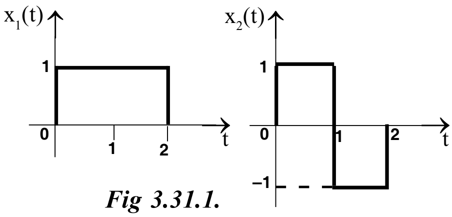


Fig 3.31.1.

b)

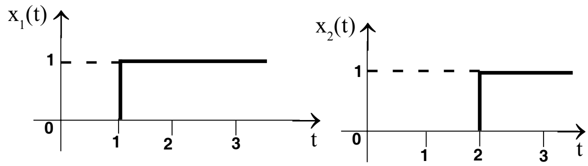


Fig 3.31.2.

Solution

a)

The mathematical equation of the signal shown in fig 1 is,

$$x_1(t) = u(t) - u(t-2)$$

On taking Laplace transform of above equation we get,

$$\text{if } \mathcal{L}\{u(t)\} = \frac{1}{s}, \text{then} \quad \mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

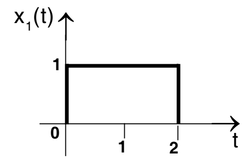


Fig 1.

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{\, u(t) - u(t-2)\,\}$$

$$= \mathcal{L}\{u(t)\} - e^{-2s}\mathcal{L}\{u(t)\} = \frac{1}{s} - \frac{e^{-2s}}{s} \qquad ......(1)$$

The mathematical equation of the signal shown in fig 2 is,

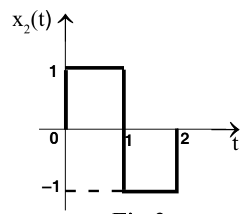


Fig 2.

$$x_2(t) = 1 \quad ; \quad 0 < t < 1$$
$$= -1 \quad ; \quad 1 < t < 2$$

Let, $X_2(s) = \mathcal{L}\{x_2(t)\}$

By the definition of Laplace transform,

$$X_2(s) = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt = \int_0^1 1 \times e^{-st}\, dt + \int_1^2 -1 \times e^{-st}\, dt = \int_0^1 e^{-st}\, dt - \int_1^2 e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^1 - \left[\frac{e^{-st}}{-s}\right]_1^2 = \left[\frac{e^{-s}}{-s} - \frac{e^{0}}{-s}\right] - \left[\frac{e^{-2s}}{-s} - \frac{e^{-s}}{-s}\right]$$

$$= -\frac{e^{-s}}{s} + \frac{1}{s} + \frac{e^{-2s}}{s} - \frac{e^{-s}}{s}$$

$$= \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \qquad .....(2)$$

By using convolution property of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s)$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\{X_1(s)\, X_1(s)\}$$

$$= \mathcal{L}^{-1}\left\{\left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)\left(\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s}\right)\right\} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s^2} + \frac{2e^{-3s}}{s^2} - \frac{e^{-4s}}{s^2}\right\}$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{2e^{-3s}}{s^2} - \frac{e^{-4s}}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{2e^{-3s}}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{e^{-4s}}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\bigg|_{t=t-1} + 2\,\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\bigg|_{t=t-3} - \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\bigg|_{t=t-4}$$

$$= t\,u(t) - 2[t\,u(t)]\big|_{t=t-1} + 2[t\,u(t)]\big|_{t=t-3} - [t\,u(t)]\big|_{t=t-4}$$

$$= t\,u(t) - 2(t-1)\,u(t-1) + 2(t-3)\,u(t-3) - (t-4)\,u(t-4)$$

$$\boxed{\text{if } \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2},\ \text{then } \mathcal{L}\{(t-a)\,u(t-a)\} = \frac{e^{-as}}{s^2}}$$

<u>To verify the above result with Example 2.23(a) of chapter-2</u>

<u>For t = 0 to 1</u>

When t = 0 to1, $u(t) = 1$, $u(t-1) = 0$, $u(t-3) = 0$, $u(t-4) = 0$

$$\therefore\ x_1(t) * x_2(t) = t\times 1 - 2(t-1)\times 0 + 2(t-3)\times 0 - (t-4)\times 0 = t$$

<u>For t = 1 to 3</u>

When t = 1 to 2, $u(t) = 1$, $u(t-1) = 1$, $u(t-3) = 0$, $u(t-4) = 0$

$$\therefore\ x_1(t) * x_2(t) = t\times 1 - 2(t-1)\times 1 + 2(t-3)\times 0 - (t-4)\times 0 = t - 2t + 2 = 2 - t$$

<u>For t = 3 to 4</u>

When t = 3 to 4, $u(t) = 1$, $u(t-1) = 1$, $u(t-3) = 1$, $u(t-4) = 0$

$$\therefore\ x_1(t) * x_2(t) = t\times 1 - 2(t-1)\times 1 + 2(t-3)\times 1 - (t-4)\times 0 = t - 2t + 2 + 2t - 6 = t - 4$$

<u>For t > 4</u>

When t > 4, $u(t) = 1$, $u(t-1) = 1$, $u(t-3) = 1$, $u(t-4) = 1$

$$\therefore\ x_1(t) * x_2(t) = t\times 1 - 2(t-1)\times 1 + 2(t-3)\times 1 - (t-4)\times 1 = t - 2t + 2 + 2t - 6 - t + 4 = 0$$

---

b)

The mathematical equation of the signal shown in fig 1 is,

$$x_1(t) = u(t-1)$$

On taking Laplace transform of above equation we get,

$$X_1(s) = \mathcal{L}\{u(t-1)\} = e^{-s}\mathcal{L}\{u(t)\} = \frac{e^{-s}}{s} \qquad .....(1)$$

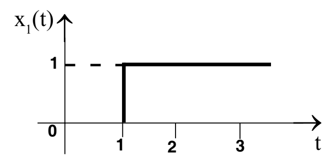


Fig 1.

The mathematical equation of the signal shown in fig 2 is,

$$x_2(t) = u(t-2)$$

On taking Laplace transform of above equation we get,

$$X_1(s) = \mathcal{L}\{u(t-2)\} = e^{-2s}\mathcal{L}\{u(t)\} = \frac{e^{-2s}}{s} \qquad .....(2)$$

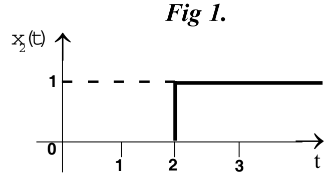


Fig 2.

By using convolution property of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$\therefore\ x_1(t) * x_2(t) = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-s}}{s} \times \frac{e^{-2s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}\bigg|_{t=t-3} = t\,u(t)\big|_{t=t-3} = (t-3)\,u(t-3)$$

$\boxed{\text{Using equations (1) and (2)}}$

$$\boxed{\text{if, } \mathcal{L}\{x(t)\} = X(s),\ \text{then, } \mathcal{L}\{x(t-a)\} = e^{-as}X(s)}$$

<u>To verify the above result with Example 2.23(b) of chapter-2</u>

<u>For t < 3</u>

When t < 3, $u(t-3) = 0$, $\quad \therefore\ x_1(t) * x_2(t) = (t-3)\times 0 = 0$

<u>For t ≥ 3</u>

When t > 3, $u(t-3) = 1$ $\quad \therefore\ x_1(t) * x_2(t) = (t-3)\times 1 = t-3$

3.7 Structures for Realization of LTI Continuous Time Systems in s-Domain

In time domain, the input-output relation of a LTI (Linear Time Invariant) continuous time system is represented by constant coefficient differential equation shown in equation (3.48).

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + ... + a_{N-1} \frac{d}{dt} y(t) + a_N y(t) = b_0 \frac{d^M}{dt^M} x(t)$$
$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + ... \ b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \quad .....(3.48)$$

In s-domain, the input-output relation of a LTI (Linear Time Invariant) continuous time system is represented by the transfer function H(s), which is a rational function of s, as shown in equation (3.49).

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + ... + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + ... + a_{N-1} s + a_N} \quad .....(3.49)$$

where, N = Order of the system, $M \le N$ and $a_0 = 1$

The above two representations of continuous time system can be viewed as a computational procedure (or algorithm) to determine the output signal y(t) from the input signal x(t).

The computations in the above equation can be arranged into various equivalent sets of differential equations, with each set of equations defining a computational procedure or algorithm for implementing the system.

Table 3.5 : Basic Elements of Block Diagram in Time Domain and s-Domain

Elements of block diagram	Time domain representation	s-domain representation
Differentiator	$x(t) \longrightarrow \boxed{d/dt} \longrightarrow \frac{d}{dt}x(t)$	$X(s) \longrightarrow \boxed{s} \longrightarrow s\,X(s)$
Integrator (with zero initial condition)	$x(t) \longrightarrow \boxed{\int} \longrightarrow \int x(t)\,dt$	$X(s) \longrightarrow \boxed{1/s} \longrightarrow \frac{X(s)}{s}$
Constant Multiplier	$x(t) \longrightarrow \triangleright_a \longrightarrow a\,x(t)$	$X(s) \longrightarrow \triangleright_a \longrightarrow a\,X(s)$
Signal Adder	$x_1(t),\ x_2(t) \longrightarrow \oplus \longrightarrow x_1(t) + x_2(t)$	$X_1(s),\ X_2(s) \longrightarrow \oplus \longrightarrow X_1(s) + X_2(s)$

For each set of equations, we can construct a block diagram consisting of integrators, adders and multipliers. Such block diagrams are referred to as realization of system or equivalently as ***structure*** for realizing system.The basic elements used to construct block diagrams are listed in table 3.5. (For block diagram representation of continuous time system refer chapter - 2, section 2.6.2).

Some of the block diagram representations of the system gives a direct relation between time domain equation and s-domain equation.

The main advantage of rearranging the sets of differential equations is to reduce the computational complexity and memory requirements.

The different types of structures for realizing continuous time systems are,

1. Direct form-I structure
2. Direct form-II structure
3. Cascade structure
4. Parallel structure

3.7.1 Direct Form-I Structure

Consider the differential equation governing the continuous time system.

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + \;.....\; + a_{N-1}\frac{d}{dt}y(t) + a_N y(t) = b_0\frac{d^M}{dt^M}x(t)$$
$$+ b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \;.....\; + b_{M-1}\frac{d}{dt}x(t) + b_M x(t)$$

$$\therefore \frac{d^N}{dt^N}y(t) = -a_1\frac{d^{N-1}}{dt^{N-1}}y(t) - a_2\frac{d^{N-2}}{dt^{N-2}}y(t) - \;.....\; - a_{N-1}\frac{d}{dt}y(t) - a_N\, y(t)$$
$$+ b_0\frac{d^M}{dt^M}x(t) + b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \;.....\; + b_{M-1}\frac{d}{dt}x(t) + b_M\, x(t)$$

On taking Laplace transform of the above equation with zero initial conditions we get,

$$s^N\,Y(s) = -a_1\,s^{N-1}Y(s) - a_2\,s^{N-2}Y(s) - \;.....\; - a_{N-1}s\,Y(s) - a_N Y(s)$$
$$+ b_0 s^M\,X(s) + b_1 s^{M-1}\,X(s) + b_2 s^{M-2}\,X(s) + \;.....\; + b_{M-1}s\,X(s) + b_M X(s)$$

On dividing throughout by s^N and letting $M = N$ we get,

$$Y(s) = -a_1\frac{Y(s)}{s} - a_2\frac{Y(s)}{s^2} - \;.....\; - a_{N-1}\frac{Y(s)}{s^{N-1}} - a_N\frac{Y(s)}{s^N}$$
$$+ b_0\,X(s) + b_1\frac{X(s)}{s} + b_2\frac{X(s)}{s^2} + \;.....\; + b_{N-1}\frac{X(s)}{s^{N-1}} + b_N\frac{X(s)}{s^N}$$

The above equation of Y(s) can be directly represented by a block diagram as shown in fig 3.11 and this structure is called direct form-I structure. This structure uses separate integrators for input and output signals. Hence for realizing this structure more memory is required. The direct form structure provides a direct relation between time domain and s-domain equations.

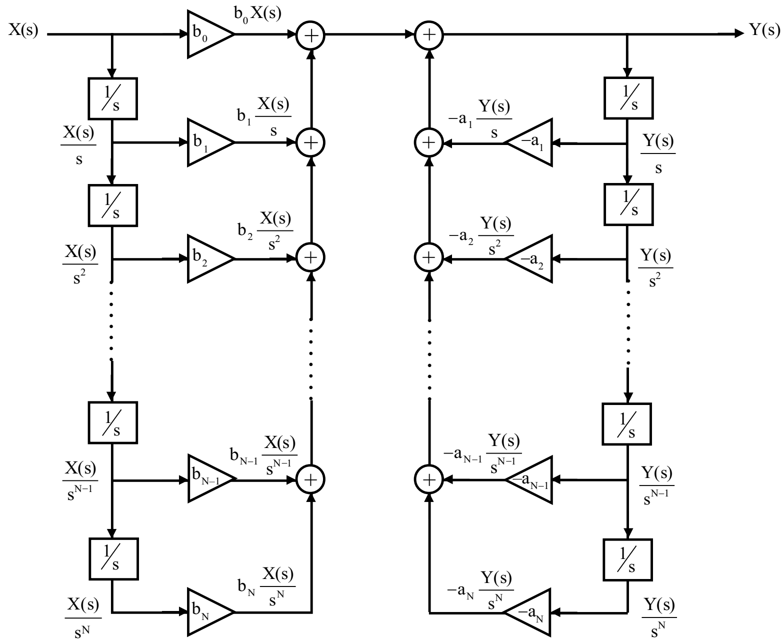


***Fig 3.11 :** Direct form-I structure of continuous time system.*

3.7.2 Direct Form-II Structure

An alternative structure called direct form-II structure can be realized which uses less number of integrators than the direct form-I structure.

Consider the differential equation governing the continuous time system.

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + \;.....\; + a_{N-1}\frac{d}{dt}y(t) + a_N y(t) = b_0\frac{d^M}{dt^M}x(t)$$
$$+ b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \;.....\; + b_{M-1}\frac{d}{dt}x(t) + b_M x(t)$$

On taking Laplace transform of the above equation with zero initial conditions we get,

$$s^N Y(s) + a_1 s^{N-1}Y(s) + a_2 s^{N-2}Y(s) + \;.....\; + a_{N-1}s\,Y(s) + a_N Y(s) = b_0 s^M X(s)$$
$$+ b_1 s^{M-1}X(s) + b_2 s^{M-2}X(s) + \;.....\; + b_{M-1}s\,X(s) + b_M X(s)$$

On dividing throughout by s^N and letting M = N we get,

$$Y(s) + a_1\frac{Y(s)}{s} + a_2\frac{Y(s)}{s^2} + \;.....\; + a_{N-1}\frac{Y(s)}{s^{N-1}} + a_N\frac{Y(s)}{s^N} = b_0X(s)$$
$$+ b_1\frac{X(s)}{s} + b_2\frac{X(s)}{s^2} + \;.....\; + b_{N-1}\frac{X(s)}{s^{N-1}} + b_N\frac{X(s)}{s^N}$$

$$Y(s)\left[1 + a_1\frac{1}{s} + a_2\frac{1}{s^2} + ... + a_{N-1}\frac{1}{s^{N-1}} + a_N\frac{1}{s^N}\right]$$
$$= X(s)\left[b_0 + b_1\frac{1}{s} + b_2\frac{1}{s^2} + ... + b_{N-1}\frac{1}{s^{N-1}} + b_N\frac{1}{s^N}\right]$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{b_0 + b_1\frac{1}{s} + b_2\frac{1}{s^2} + ... + b_{N-1}\frac{1}{s^{N-1}} + b_N\frac{1}{s^N}}{1 + a_1\frac{1}{s} + a_2\frac{1}{s^2} + ... + a_{N-1}\frac{1}{s^{N-1}} + a_N\frac{1}{s^N}}$$

Let, $\dfrac{Y(s)}{X(s)} = \dfrac{W(s)}{X(s)} \times \dfrac{Y(s)}{W(s)}$

where, $$\frac{W(s)}{X(s)} = \frac{1}{1 + a_1\frac{1}{s} + a_2\frac{1}{s^2} + ... + a_{N-1}\frac{1}{s^{N-1}} + a_N\frac{1}{s^N}} \quad .....(3.50)$$

and $$\frac{Y(s)}{W(s)} = b_0 + b_1\frac{1}{s} + b_2\frac{1}{s^2} + ... + b_{N-1}\frac{1}{s^{N-1}} + b_N\frac{1}{s^N} \quad .....(3.51)$$

On cross multiplying equation (3.50) we get,

$$W(s) + a_1\frac{W(s)}{s} + a_2\frac{W(s)}{s^2} + ... + a_{N-1}\frac{W(s)}{s^{N-1}} + a_N\frac{W(s)}{s^N} = X(s)$$

$$\therefore W(s) = -a_1\frac{W(s)}{s} - a_2\frac{W(s)}{s^2} - ..... - a_{N-1}\frac{W(s)}{s^{N-1}} - a_N\frac{W(s)}{s^N} + X(s) \quad .....(3.52)$$

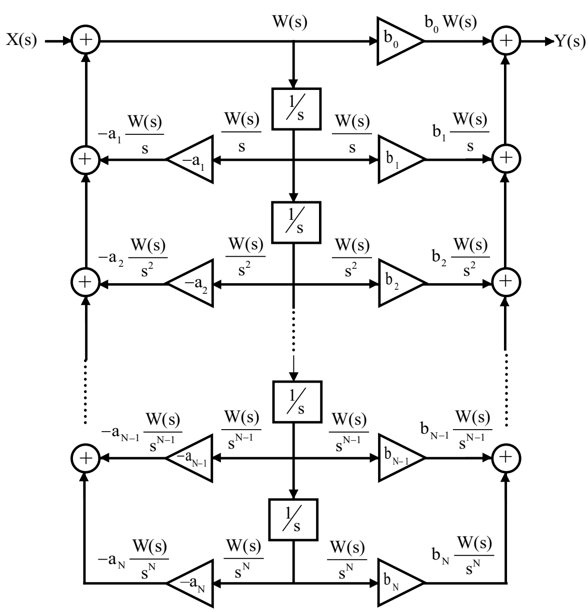


***Fig 3.12 :** Direct form-II structure of continuous time system for N = M.*

On cross multiplying equation (3.51) we get,

$$Y(s) = b_0 W(s) + b_1 \frac{W(s)}{s} + b_2 \frac{W(s)}{s^2} + ... + b_{N-1} \frac{W(s)}{s^{N-1}} + b_N \frac{W(s)}{s^N} \quad .....(3.53)$$

The equations (3.52) and (3.53) represent the continuous time system in s-domain and can be realized by a direct structure called direct form-II structure as shown in fig 3.12.

Conversion of Direct Form-I Structure to Direct Form-II Structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig 3.13. By linearity property the order of cascading can be interchanged as shown in fig 3.14 and fig 3.15.

In fig 3.15 we can observe that the input to the integrators in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of integrators in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same. Therefore, instead of having separate integrators for $\mathcal{H}_1$ and $\mathcal{H}_2$, a single set of integrators can be used. Hence the integrators can be merged to combine the cascaded systems to a single system and the resultant structure will be direct form-II structure as that of fig 3.12.

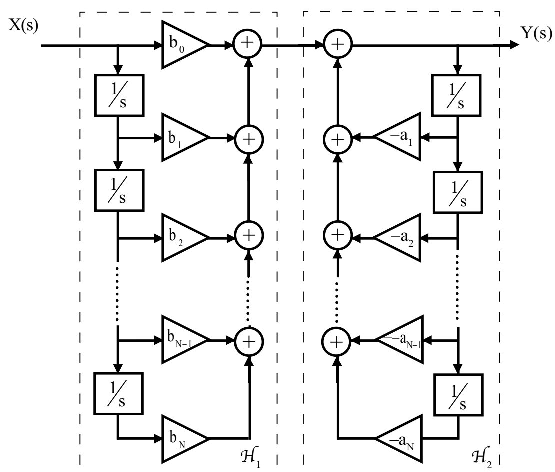


***Fig 3.13 :** Direct form-I structure as cascade of two systems.*

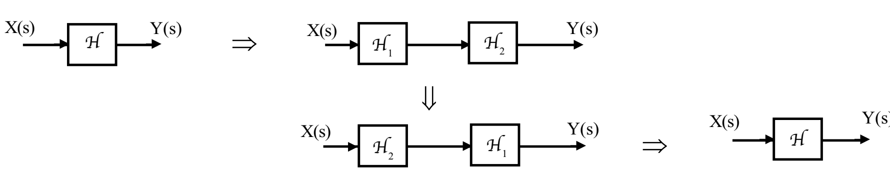


***Fig 3.14 :** Conversion of Direct form-I structure to Direct form-II structure.*

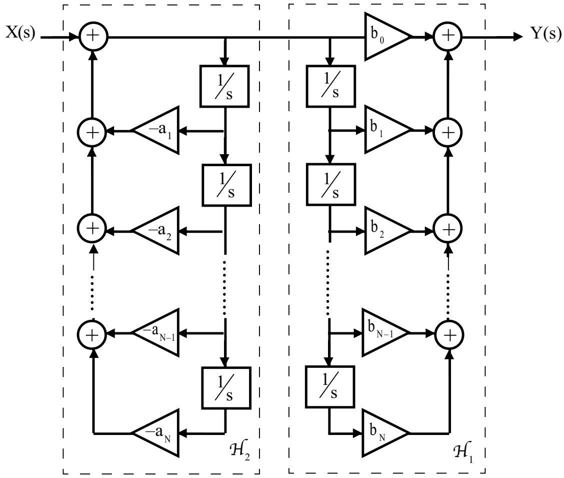


***Fig 3.15 :** Direct form-I structure after interchanging the order of cascading.*

3.7.3 Cascade Structure

The transfer function H(s) of a continuous time system can be expressed as a product of a number of second order or first order sections, as shown in equation (3.54).

$$H(s) = \frac{Y(s)}{X(s)} = H_1(s) \times H_2(s) \times H_3(s) \,..... \, H_m(s) \qquad .....(3.54)$$

$$= \prod_{i=1}^{m} H_i(s)$$

$$\text{where,} \quad H_i(s) = \frac{c_{0i} + c_{1i}\frac{1}{s} + c_{2i}\frac{1}{s^2}}{d_{0i} + d_{1i}\frac{1}{s} + d_{2i}\frac{1}{s^2}} \qquad \boxed{\text{Second order section}}$$

$$\text{or} \quad H_i(s) = \frac{c_{0i} + c_{1i}\frac{1}{s}}{d_{0i} + d_{1i}\frac{1}{s}} \qquad \boxed{\text{First order section}}$$

The individual second order or first order sections can be realized either in direct form-I or direct form-II structure. The overall system is obtained by cascading the individual sections as shown in fig 3.16.

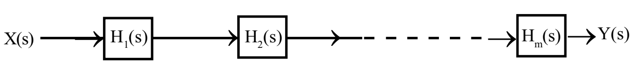


***Fig 3.16 :** Cascade structure of continuous time system.*

The difficulty in cascade structure are,

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second order sections.
3. Scaling multipliers should be provided between individual sections to prevent the system variables from becoming too large or too small.

3.7.4 Parallel Structure

The transfer function H(s) of a continuous time system can be expressed as a sum of first and second order sections, using partial fraction expansion technique as shown in equation (3.55).

$$H(s) = \frac{Y(s)}{X(s)} = C + H_1(s) + H_2(s) + H_3(s) + \ldots.. + H_m(s) \qquad \ldots..(3.55)$$

$$= C + \sum_{i=1}^{m} H_i(s)$$

where, $H_i(s) = \dfrac{c_{0i} + c_{1i}\dfrac{1}{s}}{d_{0i} + d_{1i}\dfrac{1}{s} + d_{2i}\dfrac{1}{s^2}}$ $\boxed{\text{Second order section}}$

or $H_i(s) = \dfrac{c_{0i}}{d_{0i} + d_{1i}\dfrac{1}{s}}$ $\boxed{\text{First order section}}$

The individual first and second order sections can be realized either in direct form-I or direct form-II structure. The overall system is obtained by connecting individual sections in parallel as shown in fig 3.17.

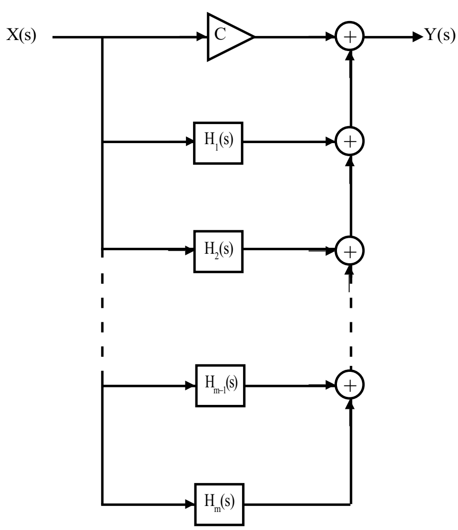


Fig 3.17 : *Parallel structure of continuous time system.*

Example 3.32

Find the direct form-I and direct form-II structures of the continuous time system represented by the equation,

$$\frac{d^2y(t)}{dt^2} + 0.6\,\frac{dy(t)}{dt} + 0.7\,y(t) = \frac{d^2x(t)}{dt^2} + 0.5\,\frac{dx(t)}{dt} + 0.4\,x(t)$$

Solution

Direct Form-I Structure

Given that, $$\frac{d^2y(t)}{dt^2} + 0.6\,\frac{dy(t)}{dt} + 0.7\,y(t) = \frac{d^2x(t)}{dt^2} + 0.5\,\frac{dx(t)}{dt} + 0.4\,x(t)$$

On taking Laplace transform of the above equation we get,

$$s^2\,Y(s) + 0.6s\,Y(s) + 0.7\,Y(s) = s^2\,X(s) + 0.5s\,X(s) + 0.4\,X(s)$$

On dividiing throughout by s^2 we get,

$$Y(s) + 0.6\,\frac{Y(s)}{s} + 0.7\,\frac{Y(s)}{s^2} = X(s) + 0.5\,\frac{X(s)}{s} + 0.4\,\frac{X(s)}{s^2} \quad .....(1)$$

$$\therefore\; Y(s) = -0.6\,\frac{Y(s)}{s} - 0.7\,\frac{Y(s)}{s^2} + X(s) + 0.5\,\frac{X(s)}{s} + 0.4\,\frac{X(s)}{s^2} \quad .....(2)$$

The direct form-I structure of the given continuous time system is realized using equation (2) as shown in fig 1.

Direct Form-II Structure

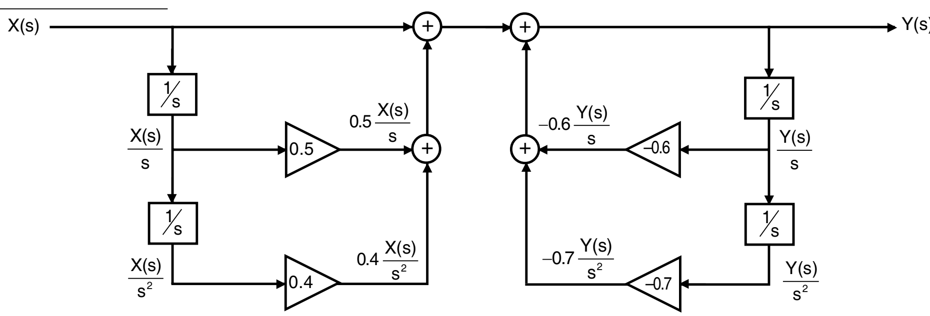


Fig 1 : *Direct form-I structure.*

On rearranging equation (1) we get,

$$Y(s)\left(1 + 0.6\,\frac{1}{s} + 0.7\,\frac{1}{s^2}\right) = X(s)\left(1 + 0.5\,\frac{1}{s} + 0.4\,\frac{1}{s^2}\right)$$

$$\frac{Y(s)}{X(s)} = \frac{1 + 0.5\,\frac{1}{s} + 0.4\,\frac{1}{s^2}}{1 + 0.6\,\frac{1}{s} + 0.7\,\frac{1}{s^2}}$$

Let, $$\frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)}\,\frac{Y(s)}{W(s)}$$

where, $$\frac{W(s)}{X(s)} = \frac{1}{1 + 0.6\,\frac{1}{s} + 0.7\,\frac{1}{s^2}} \quad .....(3)$$

$$\frac{Y(s)}{W(s)} = 1 + 0.5\,\frac{1}{s} + 0.4\,\frac{1}{s^2} \quad .....(4)$$

On cross multiplying equation (3) we get,

$$W(s) + 0.6\frac{W(s)}{s} + 0.7\frac{W(s)}{s^2} = X(s)$$

$$\therefore\ W(s) = -0.6\frac{W(s)}{s} - 0.7\frac{W(s)}{s^2} + X(s) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(s) = W(s) + 0.5\frac{W(s)}{s} + 0.4\frac{W(s)}{s^2} \qquad .....(6)$$

The direct form-II structure of the given system is realized using equations (5) and (6) as shown in fig 2.

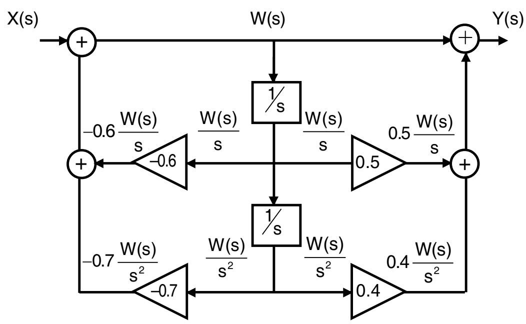


Fig 2 : *Direct form -II structure.*

Example 3.33

Find the direct form-I and direct form-II structures of the continuous time system represented by transfer function,

$$H(s) = \frac{8s^3 - 4s^2 + 11s - 2}{\left(s - \frac{1}{4}\right)\left(s^2 - s + \frac{1}{2}\right)}.$$

Solution

Direct Form-I Structure

Given that, $$\mathbf{H(s) = \frac{8s^3 - 4s^2 + 11s - 2}{\left(s - \frac{1}{4}\right)\left(s^2 - s + \frac{1}{2}\right)}}$$

Let, $H(s) = \dfrac{Y(s)}{X(s)}$; where Y(s) = Output in s-domain and X(s) = Input in s-domain.

$$\therefore\ \frac{Y(s)}{X(s)} = \frac{8s^3 - 4s^2 + 11s - 2}{\left(s - \frac{1}{4}\right)\left(s^2 - s + \frac{1}{2}\right)} = \frac{8s^3 - 4s^2 + 11s - 2}{s^3 - s^2 + \frac{1}{2}s - \frac{1}{4}s^2 + \frac{1}{4}s - \frac{1}{8}}$$

$$= \frac{8s^3 - 4s^2 + 11s - 2}{s^3 - \frac{5}{4}s^2 + \frac{3}{4}s - \frac{1}{8}} = \frac{s^3\left(8 - 4\frac{1}{s} + 11\frac{1}{s^2} - 2\frac{1}{s^3}\right)}{s^3\left(1 - \frac{5}{4}\frac{1}{s} + \frac{3}{4}\frac{1}{s^2} - \frac{1}{8}\frac{1}{s^3}\right)}$$

$$\therefore\ \frac{Y(s)}{X(s)} = \frac{8 - 4\dfrac{1}{s} + 11\dfrac{1}{s^2} - 2\dfrac{1}{s^3}}{1 - \dfrac{5}{4}\dfrac{1}{s} + \dfrac{3}{4}\dfrac{1}{s^2} - \dfrac{1}{8}\dfrac{1}{s^3}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y(s) - \frac{5}{4}\frac{Y(s)}{s} + \frac{3}{4}\frac{Y(s)}{s^2} - \frac{1}{8}\frac{Y(s)}{s^3} = 8X(s) - 4\frac{X(s)}{s} + 11\frac{X(s)}{s^2} - 2\frac{X(s)}{s^3}$$

$$\therefore\ Y(s) = 8X(s) - 4\frac{X(s)}{s} + 11\frac{X(s)}{s^2} - 2\frac{X(s)}{s^3} + \frac{5}{4}\frac{Y(s)}{s} - \frac{3}{4}\frac{Y(s)}{s^3} + \frac{1}{8}\frac{Y(s)}{s^3} \qquad .....(2)$$

The direct form-I structure can be obtained from equation (2) as shown in fig 1.

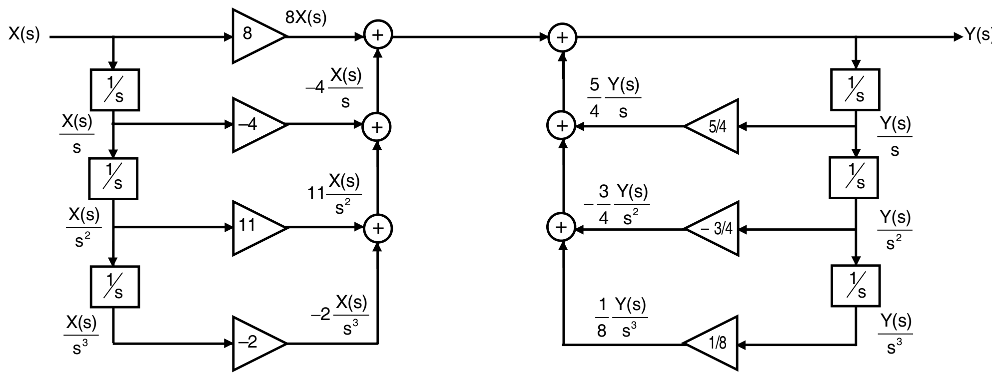


***Fig 1 :** Direct form-I structure.*

Direct Form-II Structure

From equation (1) we get,

$$\frac{Y(s)}{X(s)} = \frac{8 - 4\dfrac{1}{s} + 11\dfrac{1}{s^2} - 2\dfrac{1}{s^3}}{1 - \dfrac{5}{4}\dfrac{1}{s} + \dfrac{3}{4}\dfrac{1}{s^2} - \dfrac{1}{8}\dfrac{1}{s^3}}$$

Let, $\dfrac{Y(s)}{X(s)} = \dfrac{W(s)}{X(s)}\dfrac{Y(s)}{W(s)}$

where, $$\frac{W(s)}{X(s)} = \frac{1}{1 - \dfrac{5}{4}\dfrac{1}{s} + \dfrac{3}{4}\dfrac{1}{s^2} - \dfrac{1}{8}\dfrac{1}{s^3}} \qquad .....(3)$$

$$\frac{Y(s)}{W(s)} = 8 - 4\frac{1}{s} + 11\frac{1}{s^2} - 2\frac{1}{s^3} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W(s) - \frac{5}{4}\frac{W(s)}{s} + \frac{3}{4}\frac{W(s)}{s^2} - \frac{1}{8}\frac{W(s)}{s^3} = X(s)$$

$$\therefore\ W(s) = X(s) + \frac{5}{4}\frac{W(s)}{s} - \frac{3}{4}\frac{W(s)}{s^2} + \frac{1}{8}\frac{W(s)}{s^3} \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(s) = 8\,W(s) - 4\,\frac{W(s)}{s} + 11\,\frac{W(s)}{s^2} - 2\,\frac{W(s)}{s^3} \qquad .....(6)$$

The equations (5) and (6) can be realized by a direct form-II structure as shown in fig 2.

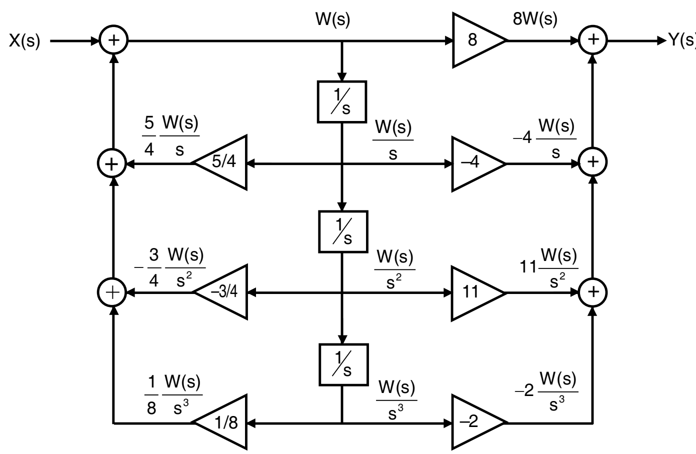


Fig 2 : *Direct form-II structure.*

Example 3.34

Obtain the direct form-I, direct form-II, cascade and parallel structure of the continuous time LTI system governed by the equation,

$$\frac{d^3y(t)}{dt^3} + \frac{3}{8}\frac{d^2y(t)}{dt^2} - \frac{3}{32}\frac{dy(t)}{dt} - \frac{1}{64}y(t) = 3\frac{d^3x(t)}{dt^2} + 2\frac{d^2x(t)}{dt}.$$

Solution

Direct Form-I Structure

Given that, $$\boldsymbol{\frac{d^3y(t)}{dt^3} + \frac{3}{8}\frac{d^2y(t)}{dt^2} - \frac{3}{32}\frac{dy(t)}{dt} - \frac{1}{64}y(t) = 3\frac{d^3x(t)}{dt^2} + 2\frac{d^2x(t)}{dt}}$$

On taking Laplace transform of the above equation we get,

$$s^3\,Y(s) + \frac{3}{8}s^2\,Y(s) - \frac{3}{32}s\,Y(s) - \frac{1}{64}Y(s) = 3s^3\,X(s) + 2s^2\,X(s) \qquad .....(1)$$

On dividing equation (1) throughout by s^3 we get,

$$Y(s) + \frac{3}{8}\frac{Y(s)}{s} - \frac{3}{32}\frac{Y(s)}{s^2} - \frac{1}{64}\frac{Y(s)}{s^3} = 3\,X(s) + 2\,\frac{X(s)}{s} \qquad .....(2)$$

$$\therefore\ Y(s) = -\frac{3}{8}\frac{Y(s)}{s} + \frac{3}{32}\frac{Y(s)}{s^2} + \frac{1}{64}\frac{Y(s)}{s^3} + 3\,X(s) + 2\,\frac{X(s)}{s} \qquad .....(3)$$

The direct form-I structure of the given continuous time system is realized using equation (3) as shown in fig 1.

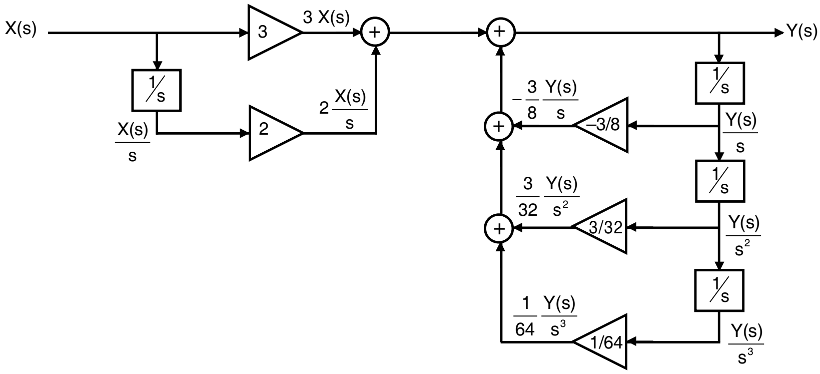


***Fig 1 :** Direct form-I structure.*

Direct Form-II Structure

On rearranging equation (2) we get,

$$Y(s)\left(1 + \frac{3}{8}\,\frac{1}{s} - \frac{3}{32}\,\frac{1}{s^2} - \frac{1}{64}\,\frac{1}{s^3}\right) = X(s)\left(3 + 2\,\frac{1}{s}\right)$$

$$\frac{Y(s)}{X(s)} = \frac{3 + 2\,\frac{1}{s}}{1 + \frac{3}{8}\,\frac{1}{s} - \frac{3}{32}\,\frac{1}{s^2} - \frac{1}{64}\,\frac{1}{s^3}}$$

Let, $\dfrac{Y(s)}{X(s)} = \dfrac{W(s)}{X(s)}\,\dfrac{Y(s)}{W(s)}$

where, $$\frac{W(s)}{X(s)} = \frac{1}{1 + \frac{3}{8}\,\frac{1}{s} - \frac{3}{32}\,\frac{1}{s^2} - \frac{1}{64}\,\frac{1}{s^3}} \qquad .....(4)$$

$$\frac{Y(s)}{W(s)} = 3 + 2\,\frac{1}{s} \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$W(s) + \frac{3}{8}\,\frac{W(s)}{s} - \frac{3}{32}\,\frac{W(s)}{s^2} - \frac{1}{64}\,\frac{W(s)}{s^3} = X(s)$$

$$\therefore\ W(s) = X(s) - \frac{3}{8}\,\frac{W(s)}{s} + \frac{3}{32}\,\frac{W(s)}{s^2} + \frac{1}{64}\,\frac{W(s)}{s^3} \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(s) = 3\,W(s) + 2\,\frac{W(s)}{s} \qquad .....(7)$$

The equations (6) and (7) can be realized by a direct form-II structure as shown in fig 2.

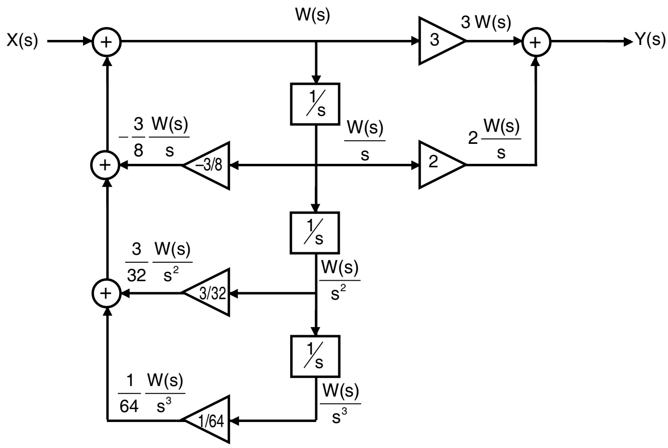


***Fig 2 :** Direct form-II structure.*

Cascade Structure

On rearranging equation (1) we get,

$$Y(s)\left(s^3 + \frac{3}{8}s^2 - \frac{3}{32}s - \frac{1}{64}\right) = X(s)\left(3s^3 + 2s^2\right)$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{3s^3 + 2s^2}{s^3 + \frac{3}{8}s^2 - \frac{3}{32}s - \frac{1}{64}} \qquad .....(8)$$

For cascade realization, the numerator and denominator polynomials of equation(8) should be expressed in the factorized form, as shown below.

s = –1/8 is one of the root of denominator polynomial of equation (8).

–1/8	1	3/8	–3/32	–1/64
	↓	–1/8	–2/64	+1/64
	1	2/8	–8/64	0

$$\therefore \frac{Y(s)}{X(s)} = \frac{3s^2\left(s + \frac{2}{3}\right)}{\left(s + \frac{1}{8}\right)\left(s^2 + \frac{2}{8}s - \frac{8}{64}\right)}$$

$$= \frac{3s^2\left(s + \frac{2}{3}\right)}{\left(s + \frac{1}{8}\right)\left(s^2 + \frac{1}{4}s - \frac{1}{8}\right)}$$

$$= \frac{3s^2\left(s + \frac{2}{3}\right)}{\left(s + \frac{1}{8}\right)\left(s + \frac{1}{2}\right)\left(s - \frac{1}{4}\right)} \qquad .....(9)$$

Since there are three first order factors in the denominator of equation (9), H(s) can be expressed as a product of three sections as shown in equation (10).

$$\text{Let,} \quad \frac{Y(s)}{X(s)} = H(s) = \frac{s}{s + \frac{1}{8}} \times \frac{s}{s + \frac{1}{2}} \times \frac{3s + 2}{s - \frac{1}{4}} = H_1(s) \times H_2(s) \times H_3(s) \qquad .....(10)$$

$$\text{where,} \quad H_1(s) = \frac{s}{s + \frac{1}{8}} \quad ; \quad H_2(s) = \frac{s}{s + \frac{1}{2}} \quad \text{and} \quad H_3(s) = \frac{3s + 2}{s - \frac{1}{4}}$$

The transfer function $H_1(s)$ can be realized in direct form-II structure as shown below.

$$\text{Let,} \quad H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{s}{s + \frac{1}{8}} = \frac{1}{1 + \frac{1}{8}\frac{1}{s}}$$

On cross multiplying the above function we get,

$$Y_1(s) + \frac{1}{8}\frac{Y_1(s)}{s} = X(s)$$

$$\therefore \; Y_1(s) = -\frac{1}{8}\frac{Y_1(s)}{s} + X(s)$$

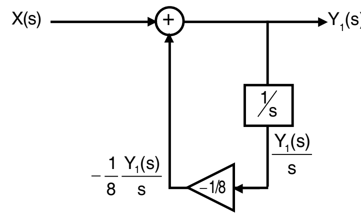


***Fig 3 :** Direct form-II structure of $H_1(s)$.*

The direct form-II structure of $H_1(s)$ is obtained using the above equation as shown in fig 3.

The transfer function $H_2(s)$ can be realized in direct form-II structure as shown below.

$$\text{Let,} \quad H_2(s) = \frac{Y_2(s)}{Y_1(s)} = \frac{s}{s + \frac{1}{2}} = \frac{1}{1 + \frac{1}{2}\frac{1}{s}}$$

On cross multiplying the above function we get,

$$Y_2(s) + \frac{1}{2}\frac{Y_2(s)}{s} = Y_1(s)$$

$$\therefore \; Y_2(s) = -\frac{1}{2}\frac{Y_2(s)}{s} + Y_1(s)$$

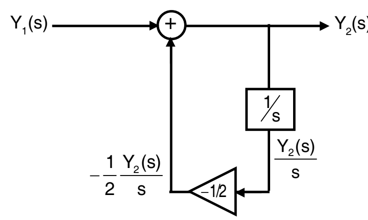


***Fig 4 :** Direct form-II structure of $H_2(s)$.*

The direct form-II structure of $H_2(s)$ is obtained using the above equation as shown in fig 4.

The transfer function $H_3(s)$ can be realized in direct form-II structure as shown below.

$$H_3(s) = \frac{3s + 2}{s - \frac{1}{4}} = \frac{3 + 2\,\frac{1}{s}}{1 - \frac{1}{4}\frac{1}{s}}$$

$$\text{Let,} \quad H_3(s) = \frac{Y(s)}{Y_2(s)} = \frac{W(s)}{Y_2(s)}\,\frac{Y(s)}{W(s)} = \frac{3 + 2\,\frac{1}{s}}{1 - \frac{1}{4}\frac{1}{s}}$$

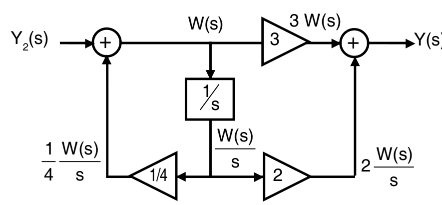


***Fig 5 :** Direct form-II structure of $H_3(s)$.*

$$\text{where,} \quad \frac{W(s)}{Y_2(s)} = \frac{1}{1 - \frac{1}{4}\frac{1}{s}} \quad \text{and} \quad \frac{Y(s)}{W(s)} = 3 + 2\,\frac{1}{s}$$

On cross multiplying the above functions we get,

$$W(s) = Y_2(s) + \frac{1}{4}\frac{W(s)}{s} \quad \text{and} \quad Y(s) = 3\,W(s) + 2\,\frac{W(s)}{s}$$

The direct form-II structure of $H_3(s)$ is obtained using the above equations as shown in fig 5.

The cascade structure of the given system is obtained by connecting the individual sections shown in fig 3, fig 4 and fig 5 in cascade as shown in fig 6.

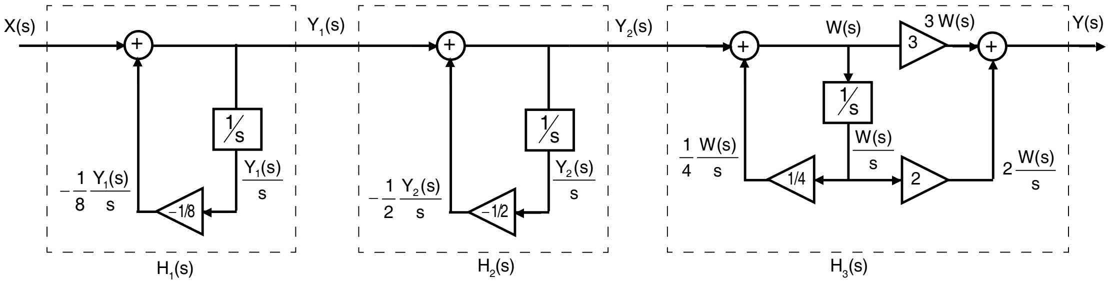


***Fig 6 :** Cascade structure of the system.*

Parallel Structure

From equation (8) we get,

$$\frac{Y(s)}{X(s)} = \frac{3s^3 + 2s^2}{s^3 + \frac{3}{8}s^2 - \frac{3}{32}s - \frac{1}{64}}$$

$$= 3 + \frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{s^3 + \frac{3}{8}s^2 - \frac{3}{32}s - \frac{1}{64}}$$

$$= 3 + \frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{\left(s + \frac{1}{8}\right)\left(s + \frac{1}{2}\right)\left(s - \frac{1}{4}\right)}$$

Dividing numerator by denominator

$$\begin{array}{r|l} & 3 \\ \hline s^3 + \frac{3}{8}s^2 - \frac{3}{32}s - \frac{1}{64} & 3s^3 + 2s^2 \\ & 3s^3 + \frac{9}{8}s^2 - \frac{9}{32}s - \frac{3}{64} \\ & (-) \quad (-) \quad (+) \quad (+) \\ \hline & \frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64} \end{array}$$

Using equation (9)

By partial fraction expansion technique the above function can be expressed as shown below.

$$\frac{Y(s)}{X(s)} = H(s) = 3 + \frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{\left(s + \frac{1}{8}\right)\left(s + \frac{1}{2}\right)\left(s - \frac{1}{4}\right)} = 3 + \frac{A}{s + \frac{1}{8}} + \frac{B}{s + \frac{1}{2}} + \frac{C}{s - \frac{1}{4}}$$

$$A = \left.\frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{\left(s + \frac{1}{8}\right)\left(s + \frac{1}{2}\right)\left(s - \frac{1}{4}\right)} \times \left(s + \frac{1}{8}\right)\right|_{s=-\frac{1}{8}} = \frac{\frac{7}{8}\left(-\frac{1}{8}\right)^2 + \frac{9}{32}\left(-\frac{1}{8}\right) + \frac{3}{64}}{\left(-\frac{1}{8} + \frac{1}{2}\right)\left(-\frac{1}{8} - \frac{1}{4}\right)}$$

$$= \frac{\frac{7}{512} - \frac{9}{256} + \frac{3}{64}}{\left(\frac{3}{8}\right)\left(-\frac{3}{8}\right)} = \frac{\frac{7 - 18 + 24}{512}}{-\frac{9}{64}} = \frac{\frac{13}{512}}{-\frac{9}{64}} = \frac{13}{512} \times \left(-\frac{64}{9}\right) = -\frac{13}{72}$$

$$B = \left.\frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{\left(s + \frac{1}{8}\right)\left(s + \frac{1}{2}\right)\left(s - \frac{1}{4}\right)} \times \left(s + \frac{1}{2}\right)\right|_{s=-\frac{1}{2}} = \frac{\frac{7}{8}\left(-\frac{1}{2}\right)^2 + \frac{9}{32}\left(-\frac{1}{2}\right) + \frac{3}{64}}{\left(-\frac{1}{2} + \frac{1}{8}\right)\left(-\frac{1}{2} - \frac{1}{4}\right)}$$

$$= \frac{\frac{7}{32} - \frac{9}{64} + \frac{3}{64}}{\left(-\frac{3}{8}\right)\left(-\frac{3}{4}\right)} = \frac{\frac{14 - 9 + 3}{64}}{\frac{9}{32}} = \frac{\frac{8}{64}}{\frac{9}{32}} = \frac{8}{64} \times \frac{32}{9} = \frac{4}{9}$$

$$C = \left.\frac{\frac{7}{8}s^2 + \frac{9}{32}s + \frac{3}{64}}{\left(s+\frac{1}{8}\right)\left(s+\frac{1}{2}\right)\left(s-\frac{1}{4}\right)} \times \left(s - \frac{1}{4}\right)\right|_{s=\frac{1}{4}} = \frac{\frac{7}{8}\left(\frac{1}{4}\right)^2 + \frac{9}{32}\left(\frac{1}{4}\right) + \frac{3}{64}}{\left(\frac{1}{4}+\frac{1}{8}\right)\left(\frac{1}{4}+\frac{1}{2}\right)}$$

$$= \frac{\frac{7}{128} + \frac{9}{128} + \frac{3}{64}}{\left(\frac{3}{8}\right)\left(\frac{3}{4}\right)} = \frac{\frac{7+9+6}{128}}{\frac{9}{32}} = \frac{\frac{22}{128}}{\frac{9}{32}} = \frac{22}{128} \times \frac{32}{9} = \frac{11}{18}$$

$$\therefore\ H(s) = 3 + \frac{A}{s+\frac{1}{8}} + \frac{B}{s+\frac{1}{2}} + \frac{C}{s-\frac{1}{4}} = 3 + \frac{-\frac{13}{72}}{s+\frac{1}{8}} + \frac{\frac{4}{9}}{s+\frac{1}{2}} + \frac{\frac{11}{18}}{s-\frac{1}{4}}$$

Let, $H(s) = 3 + H_1(s) + H_2(s) + H_3(s)$

$$\text{where,}\quad H(s) = \frac{Y(s)}{X(s)}\ ;\quad H_1(s) = \frac{-\frac{13}{72}}{s+\frac{1}{8}};\quad H_2(s) = \frac{\frac{4}{9}}{s+\frac{1}{2}};\quad H_3(s) = \frac{\frac{11}{18}}{s-\frac{1}{4}}$$

Now, $H(s) = \dfrac{Y(s)}{X(s)} = 3 + H_1(s) + H_2(s) + H_3(s)$

$$\therefore\ Y(s) = 3X(s) + H_1(s)\,X(s) + H_2(s)\,X(s) + H_3(s)\,X(s)$$

Let, $Y(s) = 3X(s) + Y_1(s) + Y_2(s) + Y_3(s)$ .....(11)

$$\text{where,}\quad Y_1(s) = H_1(s)\,X(s) = \frac{-\frac{13}{72}}{s+\frac{1}{8}}X(s) \Rightarrow \frac{Y_1(s)}{X(s)} = \frac{-\frac{13}{72}\frac{1}{s}}{1+\frac{1}{8}\frac{1}{s}}$$

$$Y_2(s) = H_2(s)\,X(s) = \frac{\frac{4}{9}}{s+\frac{1}{2}}X(s) \Rightarrow \frac{Y_2(s)}{X(s)} = \frac{\frac{4}{9}\frac{1}{s}}{1+\frac{1}{2}\frac{1}{s}}$$

$$Y_3(s) = H_2(s)\,X(s) = \frac{\frac{11}{18}}{s-\frac{1}{4}}X(s) \Rightarrow \frac{Y_3(s)}{X(s)} = \frac{\frac{11}{18}\frac{1}{s}}{1-\frac{1}{4}\frac{1}{s}}$$

In equation (11), the output Y(s) is expressed as sum of four components. The first component is simply the input multiplied by a constant, and so it is realised as a constant multiplier in the parallel structure shown in fig 10. The other three components involve first order section of transfer function, and so they are realized by direct form-II structures as shown below.

The transfer function $H_1(s)$ can be realized in direct form-II structure as shown below.

$$\text{Let,}\quad \frac{Y_1(s)}{X(s)} = \frac{W_1(s)}{X(s)}\,\frac{Y_1(s)}{W_1(s)} = \frac{-\frac{13}{72}\frac{1}{s}}{1+\frac{1}{8}\frac{1}{s}}$$

$$\text{where,}\quad \frac{W_1(s)}{X(s)} = \frac{1}{1+\frac{1}{8}\frac{1}{s}} \quad\text{and}\quad \frac{Y_1(s)}{W_1(s)} = -\frac{13}{72}\,\frac{1}{s} \qquad .....(12)$$

On cross multiplying and rearranging the transfer functions of equation(12) we get,

$$W_1(s) = X(s) - \frac{1}{8}\frac{W_1(s)}{s} \quad \text{and} \quad Y_1(s) = -\frac{13}{72}\frac{W_1(s)}{s}$$

The direct form-II structure of $H_1(s)$ is obtained using the above equations as shown in fig 7.

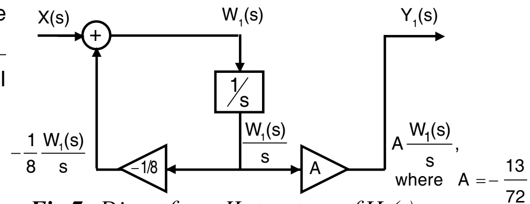


***Fig 7 :** Direct form-II structure of $H_1(s)$.*

The transfer function $H_2(s)$ can be realized in direct form-II structure as shown below.

$$\text{Let,}\quad \frac{Y_2(s)}{X(s)} = \frac{W_2(s)}{X(s)}\frac{Y_2(s)}{W_2(s)} = \frac{\frac{4}{9}\frac{1}{s}}{1+\frac{1}{2}\frac{1}{s}}$$

$$\text{where,}\quad \frac{W_2(s)}{X(s)} = \frac{1}{1+\frac{1}{2}\frac{1}{s}} \quad \text{and} \quad \frac{Y_2(s)}{W_2(s)} = \frac{4}{9}\frac{1}{s}$$

On cross multiplying and rearranging the above functions we get,

$$W_2(s) = X(s) - \frac{1}{2}\frac{W_2(s)}{s} \quad \text{and} \quad Y_2(s) = \frac{4}{9}\frac{W_2(s)}{s}$$

The direct form-II structure of $H_2(s)$ is obtained using the above equations as shown in fig 8.

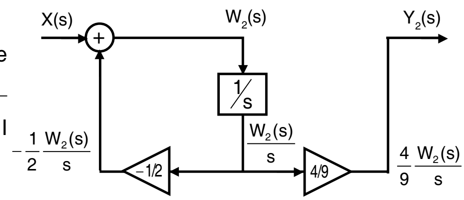


***Fig 8 :** Direct form-II structure of $H_2(s)$.*

The transfer function $H_3(s)$ can be realized in direct form-II structure as shown below.

$$\text{Let,}\quad \frac{Y_3(s)}{X(s)} = \frac{W_3(s)}{X(s)}\frac{Y_3(s)}{W_3(s)} = \frac{\frac{11}{18}\frac{1}{s}}{1-\frac{1}{4}\frac{1}{s}}$$

$$\text{where,}\quad \frac{W_3(s)}{X(s)} = \frac{1}{1-\frac{1}{4}\frac{1}{s}} \quad \text{and} \quad \frac{Y_3(s)}{W_3(s)} = \frac{11}{18}\frac{1}{s}$$

On cross multiplying and rearranging the above functions we get,

$$W_3(s) = X(s) + \frac{1}{4}\frac{W_3(s)}{s} \quad \text{and} \quad Y_3(s) = \frac{11}{18}\frac{W_3(s)}{s} \qquad .....(13)$$

The direct form-II structure of $H_3(s)$ is obtained using equation (13) as shown in fig 9.

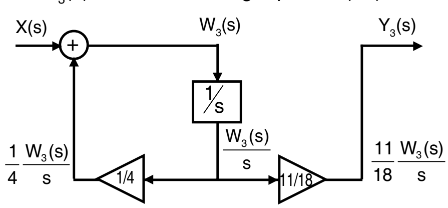


***Fig 9 :** Direct form-II structure of $H_3(s)$.*

The parallel structure of the given system is obtained by connecting the individual sections shown in fig 7, fig 8 and fig 9 in parallel as shown in fig 10.

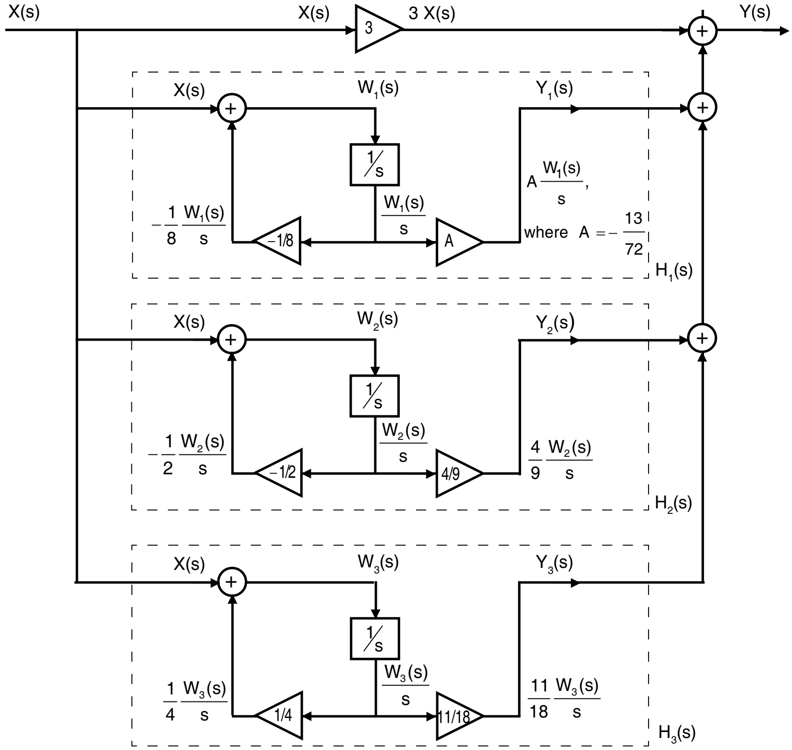


***Fig 10 :** Parallel structure of the system.*

3.8 Summary of Important Concepts

1. The Laplace transform is used to transform a time domain signal to complex frequency domain.
2. The signal Ke^{st} can be thought of as an universal signal which represents all types of signals.
3. Complex frequency, s is defined as, $s = \sigma + j\Omega$.
4. The Complex frequency plane or s-plane is a two dimensional complex plane with values of σ on horizontal axis and values of Ω on vertical axis.
5. In the definition of Laplace transform if the limits of integral is, 0 to $+\infty$, then the Laplace transform is called one sided Laplace transform or unilateral Laplace transform.
6. In the definition of Laplace transform if the limits of integral is, $-\infty$ to $+\infty$, then the Laplace transform is called two sided Laplace transform or bilateral Laplace transform.
7. A causal signal x(t) is said to be of exponential order if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}\left|x(t)\right|$ approaches zero as t approaches infinity.
8. For a causal signal, if $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-\sigma t}\left|x(t)\right| = 0$ for $\sigma > \sigma_c$, and if $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-\sigma t}\left|x(t)\right| = \infty$ for $\sigma < \sigma_c$, then σ_c is called abscissa of convergence, (where σ_c is a point on real axis in s-plane).
9. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\ e^{-st}\ dt$ converges is called Region Of Convergence (ROC).
10. For a causal signal, the ROC includes all points on the s-plane to the right of abscissa of convergence.
11. For an anticausal signal, the ROC includes all points on the s-plane to the left of abscissa of convergence.
12. For a two sided signal the ROC includes all points on the s-plane in the region in between two abscissa of convergences.
13. The convolution theorem of Laplace transform says that, Laplace transform of convolution of two time domain signals is given by the product of the Laplace transform of the individual signals.
14. When X(s) is expressed as a ratio of two polynomials in s, then the s-domain signal X(s) is called a rational function of s.

15. The zeros and poles are two critical complex frequencies at which a rational function of s takes two extreme values zero and infinity respectively.
16. Since the signal X(s) attains infinte value at poles, the ROC of X(s) does not include poles.
17. In a realizable system, the number of zeros will be less than or equal to number of poles.
18. For a causal signal x(t), in terms of poles of X(s), the ROC is the region to the right of right most pole of X(s), (i.e., right of the pole with largest real part).
19. For an anticausal signal x(t), in terms of poles of X(s), the ROC is the region to the left of left most pole of X(s), (i.e., left of the pole with smallest real part).
20. For a signal with causal and anticausal part, in terms of poles of X(s), the ROC is the region in between largest pole of causal part and smallest pole of anticausal part.
21. The transfer function of a continuous time system is defined as the ratio of Laplace transform of output and Laplace transform of input.
22. The transfer function of an LTI continuous time system is also given by Laplace transform of the impulse response.(Alternatively, the inverse Laplace transform of transfer function is the impulse response of the system).
23. For a stable LTI continuous time system the ROC should include the imaginary axis of s-plane.
24. For a stable LTI continuous time causal system, the poles should lie on the left half of s-plane and the imaginary axis should be included in the ROC.
25. For a stable LTI continuous time noncausal system, the imaginary axis should be included in the ROC.

3.9 Short Questions and Answers

Q3.1 ***Find the Laplace transform of $x(t) = \sin^2 t$.***

Solution

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{\sin^2 t\} = \mathcal{L}\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}\Big[\mathcal{L}\{1\} - \mathcal{L}\{\cos 2t\}\Big] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right] = \frac{2}{s(s^2+4)}$$

Q3.2 ***Let x(t) and X(s) be Laplace transform pair. Given that, $X(s) = \dfrac{s}{s^2+1}$. Find Laplace transform of $y(t) = x\left(\dfrac{t}{5}\right) - x(2t)$.***

Solution

If $\mathcal{L}\{x(t)\} = X(s)$ then by time scaling property $\mathcal{L}\{x(at)\} = \dfrac{1}{|a|}X\left(\dfrac{s}{a}\right)$

$$\mathcal{L}\left\{x\left(\frac{t}{5}\right)\right\} = 5\,X(5s) = 5\,\frac{5s}{(5s)^2+1} = \frac{25s}{25s^2+1} = \frac{s}{s^2+\dfrac{1}{25}} = \frac{s}{s^2+0.04}$$

$$\mathcal{L}\{x(2t)\} = \frac{1}{2}X\left(\frac{s}{2}\right) = \frac{1}{2}\,\frac{\dfrac{s}{2}}{\left(\dfrac{s}{2}\right)^2+1} = \frac{1}{4}\,\frac{s}{\dfrac{s^2}{4}+1} = \frac{s}{s^2+4}$$

$$\text{Now, } \mathcal{L}\{y(t)\} = \mathcal{L}\left\{x\left(\frac{t}{5}\right) - x(2t)\right\} = \mathcal{L}\left\{x\left(\frac{t}{5}\right)\right\} - \mathcal{L}\{x(2t)\}$$

$$= \frac{s}{s^2+0.04} - \frac{s}{s^2+4} = \frac{s(s^2+4)-s(s^2+0.04)}{(s^2+0.04)\,(s^2+4)} = \frac{3.96\,s}{(s^2+0.04)\,(s^2+4)}$$

Q3.3 ***Find the inverse Laplace transform of $X(s) = \dfrac{1}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+4}\right]$.***

Solution

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]\right\} = \frac{1}{2}\left[\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}\right]$$

$$= \frac{1}{2}[1+\cos 2t] = \frac{1+\cos 2t}{2} = \cos^2 t$$

Q3.4 ***Let x(t) and X(s) be Laplace transform pair. Given that, $x(t) = e^{-t}\,u(t)$. Find inverse Laplace transform of $e^{-3s}\,X(2s)$.***

Solution

$$X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{e^{-t}\,u(t)\} = \frac{1}{s+1} \quad \Rightarrow \quad X(2s) = \frac{1}{2s+1} = \frac{1}{2}\;\frac{1}{s+\frac{1}{2}}$$

Now, $\mathcal{L}^{-1}\{X(2s)\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\;\frac{1}{s+\frac{1}{2}}\right\} = \frac{1}{2}\,e^{-\frac{t}{2}}\,u(t)$

$$\therefore \mathcal{L}^{-1}\{e^{-3s}\,X(2s\} = \mathcal{L}^{-1}\{X(2s)\}\Big|_{t=t-3} = \frac{1}{2}\,e^{-\frac{t}{2}}\,u(t)\Big|_{t=t-3} = \frac{1}{2}\,e^{-\frac{(t-3)}{2}}\,u(t-3)$$

Q3.5 ***Find inverse Laplace transform of,***

i) $X(s) = \dfrac{s}{(s^2+9)}$ $\quad Re\{s\} > 0$ $\qquad$ ***ii)*** $X(s) = \dfrac{s}{(s^2+9)}$ $\quad Re\{s\} < 0$

Solution

i) When Re{s} > 0, the x(t) will be causal or right sided signal.

$$\therefore\; x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t\;u(t)$$

ii) When Re{s} < 0, the x(t) will be anticausal and left sided signal.

$$\therefore\; x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} = \cos 3t\;u(-t)$$

Q3.6 ***Given that, $\mathcal{L}\{x(t)\} = X(s) = \dfrac{4s+1}{s^2+6s+3}$. Determine the initial value, x(0).***

Solution

By initial value theorem,

Initial value, $x(0) = \underset{t\to 0}{\mathrm{Lt}}\; x(t) = \underset{s\to\infty}{\mathrm{Lt}}\; sX(s) = \underset{s\to\infty}{\mathrm{Lt}}\; s \times \dfrac{4s+1}{s^2+6s+3}$

$$= \underset{s\to\infty}{\mathrm{Lt}}\; s \times \frac{s\left(4+\frac{1}{s}\right)}{s^2\left(1+\frac{6}{s}+\frac{3}{s^2}\right)} = \underset{s\to\infty}{\mathrm{Lt}}\; \frac{4+\frac{1}{s}}{1+\frac{6}{s}+\frac{3}{s^2}} = \frac{4+0}{1+0+0} = 4$$

Q3.7 ***Given that, $\mathcal{L}\{x(t)\} = X(s) = \dfrac{s+4}{s\left(s^2+7s+10\right)}$. Determine the final value, $x(\infty)$.***

Solution

By final value theorem,

Final value, $x(\infty) = \underset{t\to\infty}{\mathrm{Lt}}\; x(\infty) = \underset{s\to 0}{\mathrm{Lt}}\; sX(s)$

$$= \underset{s\to 0}{\mathrm{Lt}}\; s \times \frac{s+4}{s\left(s^2+7s+10\right)} = \underset{s\to 0}{\mathrm{Lt}}\; \frac{s+4}{s^2+7s+10} = \frac{0+4}{0+0+10} = \frac{4}{10} = 0.4$$

Q3.8 ***Given that, $X(s) = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \ldots.. + b_1 s + b_0}{s(s^n + a_{n-1} s^{n-1} + \ldots.. + a_1 s + a_0}$. Find $\underset{t\to\infty}{\mathrm{Lt}}\; x(t)$.***

Solution

Using final value theorem,

$$\underset{t\to\infty}{\mathrm{Lt}}\; x(t) = \underset{s\to 0}{\mathrm{Lt}}\; s\,X(s) = \underset{s\to 0}{\mathrm{Lt}}\; s\;\frac{b_m\,s^m + b_{m-1}\,s^{m-1} + \ldots\ldots + b_1 s + b_0}{s(s^n + a_{n-1}s^{n-1} + \ldots.. + a_1 s + a_0)} = \frac{0+0+\ldots\ldots+0+b_0}{0+0+\ldots..+0+a_0} = \frac{b_0}{a_0}$$

Q3.9 ***If*** $X(s) = \dfrac{2}{s+3}$ ***. Find Laplace transform of*** $\dfrac{d}{dt}x(t)$ ***.***

Solution

Using initial value theorem,

$$x(0) = \underset{s\to\infty}{\text{Lt}}\ s\,X(s) = \underset{s\to\infty}{\text{Lt}}\ s \times \frac{2}{s+3} = \underset{s\to\infty}{\text{Lt}}\ s \times \frac{2}{s\left(1+\dfrac{3}{s}\right)} = \frac{2}{1+\dfrac{3}{\infty}} = \frac{2}{1+0} = 2$$

Using time differentiation property,

$$\mathcal{L}\left\{\frac{d}{dt}\,x(t)\right\} = s\,X(s) - x(0) = s \times \frac{2}{s+3} - 2 = \frac{2s - 2(s+3)}{s+3} = \frac{-6}{s+3}$$

Q3.10 ***If*** $X(s) = \dfrac{0.4}{s+0.2}$ ***. Find Laplace transform of t x(t).***

Solution

Using frequency differentiation property,

$$\mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s) = -\frac{d}{ds}\left[\frac{0.4}{s+0.2}\right] = -\frac{d}{ds}\left[0.4\,(s+0.2)^{-1}\right] = -\left[0.4 \times (-1) \times (s+0.2)^{-2}\right] = \frac{0.4}{(s+0.2)^2}$$

Q3.11 ***The impulse response of a system is*** e^{-4t} ***u(t), and for the input e(t), the response is*** $(1 - e^{-4t})$ ***u(t). Find the input x(t).***

Solution

Given that, $h(t) = e^{-4t}\,u(t)$

$$\therefore H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\left\{e^{-4t}\,u(t)\right\} = \frac{1}{s+4}$$

Given that, $y(t) = \left(1 - e^{-4t}\right)u(t) = u(t) - e^{-4t}\,u(t)$

$$\therefore Y(s) = \mathcal{L}\{y(t)\} = \mathcal{L}\left\{u(t) - e^{-4t}\,u(t)\right\} = \frac{1}{s} - \frac{1}{s+4}$$

Using convolution theorem of Laplace transform

We know that, $y(t) = x(t) * h(t)$

On taking Laplace transform of above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\} \quad \Rightarrow \quad Y(s) = X(s)\,H(s)$$

$$\therefore X(s) = \frac{Y(s)}{H(s)} = Y(s) \times \frac{1}{H(s)} = \left(\frac{1}{s} - \frac{1}{s+4}\right) \times (s+4) = \frac{s+4}{s} - 1 = \frac{s+4-s}{s} = \frac{4}{s}$$

Now input, $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\dfrac{4}{s}\right\} = 4\,u(t)$

Q3.12 ***The impulse response of a system is*** $(4e^{-t} - 2e^{-3t})$ ***u(t). Determine the stability of the system.***

Solution

Given that, $h(t) = \left(4e^{-t} - 2e^{-3t}\right)u(t) = 4e^{-t}\,u(t) - 2e^{-3t}\,u(t)$

$$\therefore H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\left\{4e^{-t}\,u(t) - 2e^{-3t}\,u(t)\right\}$$

$$= \frac{4}{s+1} - \frac{2}{s+3} = \frac{4(s+3) - 2(s+1)}{(s+1)\,(s+3)} = \frac{2s+10}{(s+1)\,(s+3)} = \frac{2(s+5)}{(s+1)\,(s+3)}$$

The poles of H(s) are lying at s = –1 and s = –3, and so they are lying on left half s-plane.

The ROC is the region to the right of right most pole and so imaginary axis is included in ROC.

Since the poles are lying on left half s-plane and the imaginary axis is included in ROC, the given causal system is stable.

Q3.13 ***The impulse response of a system is $2e^{-5|t|}$. Determine the stability of the system.***

Solution

Given that, $h(t) = 2e^{-5|t|} = 2e^{5t}$; $t \le 0$

$= 2e^{-5t}$; $t \ge 0$

$$\therefore h(t) = 2e^{5t}\, u(-t) + 2e^{-5t}\, u(t)$$

$$H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\left\{2e^{5t}\, u(-t) + 2e^{-5t}\, u(t)\right\}$$

$$= -\frac{2}{s-5} + \frac{2}{s+5} = \frac{-2(s+5)+2(s-5)}{(s-5)\,(s+5)} = \frac{-20}{(s-5)\,(s+5)}$$

The poles are lying at s = 5 and s = –5, and so one pole is lying on right half s-plane and another pole lying on left half s-plane.

The ROC is the region in between the lines passing through s = 5 and s = –5 and so imaginary axis is included in ROC.

Since imaginary axis is included in ROC, the given noncausal system is stable.

Q3.14 ***Draw the direct form-I structure for the system represented by the transfer function, $H(s) = \dfrac{2s^2 + 3s}{s^2 + 6s + 5}$.***

Solution

$$\text{Let, } H(s) = \frac{Y(s)}{X(s)} = \frac{2s^2 + 3s}{s^2 + 6s + 5}$$

On cross multiplying the above equation we get,

$$s^2Y(s) + 6\, sY(s) + 5Y(s) = 2s^2\, X(s) + 3s\, X(s)$$

On dividing the above equation by s^2 we get,

$$Y(s) + \frac{6}{s}Y(s) + \frac{5}{s^2}Y(s) = 2X(s) + \frac{3}{s}X(s)$$

$$\therefore Y(s) = 2X(s) + \frac{3}{s}X(s) - \frac{6}{s}Y(s) - \frac{5}{s^2}Y(s)$$

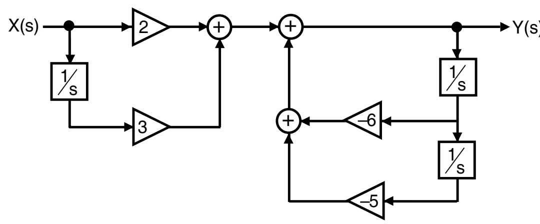


Fig Q3.14 : *Direct form-I structure.*

The above equation can be used to construct the direct form-I structure as shown in fig Q3.14.

Q3.15 ***Draw the direct form-II structure for the system represented by the transfer function, $H(s) = \dfrac{0.4s^2 + 0.7s}{s^2 + 0.5s + 0.9}$.***

Solution

$$\text{Let, } H(s) = \frac{Y(s)}{X(s)} = \frac{0.4s^2 + 0.7s}{s^2 + 0.5s + 0.9} = \frac{s^2\left(0.4 + \frac{0.7}{s}\right)}{s^2\left(1 + \frac{0.5}{s} + \frac{0.9}{s^2}\right)} = \frac{0.4 + \frac{0.7}{s}}{1 + \frac{0.5}{s} + \frac{0.9}{s^2}}$$

$$\text{Let, } \frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)} \times \frac{Y(s)}{W(s)}$$

$$\text{where, } \frac{W(s)}{X(s)} = \frac{1}{1 + \frac{0.5}{s} + \frac{0.9}{s^2}} \qquad \text{..... (1)}$$

$$\frac{Y(s)}{W(s)} = 0.4 + \frac{0.7}{s} \qquad \text{..... (2)}$$

On cross multiplying and rearranging equation (1) we get,

$$W(s) = X(s) - \frac{0.5}{s}W(s) - \frac{0.9}{s^2}W(s) \qquad \text{..... (3)}$$

On cross multiplying equation (2) we get,

$$Y(s) = 0.4\, W(s) + \frac{0.7}{s}W(s) \qquad \text{..... (4)}$$

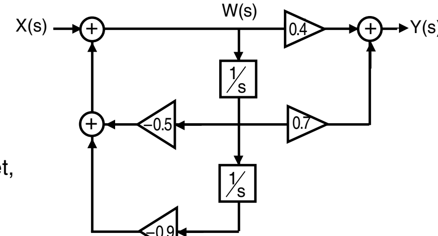


Fig Q3.15 : *Direct form -II structure.*

The direct form-II structure is constructed using equations (3) and (4) as shown in fig Q3.15

3.10 MATLAB Programs

Program 3.1

Write a MATLAB program to find Laplace transform of the following standard causal signals.

a) t b) e^{-at} c) te^{-at} d) $\cos\Omega t$ e) $\sinh\Omega t$ f) $e^{-at}\sin\Omega t$

```
%****Program to find Laplace transform of some standard signals

  clear all
  syms t real;     %Let a, t be any real variable
  syms a real;
  syms s complex; %Let s be complex variable

 %(a)
  x=t;
  disp('(a) Laplace transform of "t" is');
  laplace(x)

 %(b)
  x=exp(-a*t);
  disp('(b) Laplace transform of  "exp(-a*t)" is');
  laplace(x)

 %(c)
  x=t*exp(-a*t);
  disp('(c) Laplace transform of  "t*exp(-a*t)" is');
  laplace(x)

 %(d)
  sg=real(s);      %s - complex frequency ; sg - sigma ; o-omega
  o=imag(s);
  s=sg+(i*o);
  x=cos(o*t);
  disp('(d) Laplace transform of "cos(o*t)" is');
  laplace(x)

 %(e)
  x=sinh(o*t);
  disp('(e) Laplace transform of " sinh(o*t)" is');
  laplace(x)

 %(f)
  x=exp(-a*t)*sin(o*t);
  disp('(f) Laplace transform of  "exp(-a*t)*sin(o*t)" is');
  laplace(x)
```

OUTPUT

```
(a) Laplace transform of "t" is
      ans =
            1/s^2
(b) Laplace transform of  "exp(-a*t)" is
      ans =
            1/(s+a)
(c) Laplace transform of  "t*exp(-a*t)" is
      ans =
             1/(s+a)^2
```

```
(d) Laplace transform of "cos(o*t)" is
      ans =
            s/(s^2-(1/2*s-1/2*conj(s))^2)
(e) Laplace transform of " sinh(o*t)" is
      ans =
            -i*(1/2*s-1/2*conj(s))/(s^2+(1/2*s-1/2*conj(s))^2)
(f) Laplace transform of  "exp(-a*t)*sin(o*t)" is
      ans =
            -i*(1/2*s-1/2*conj(s))/((s+a)^2-(1/2*s-1/2*conj(s))^2)
```

Program 3.2

Write a MATLAB program to find Laplace transform of the following standard causal signals.

a) t^n b) $t^{n-1}/(n-1)!$ c) $t^n e^{-at}$ d) $t^{n-1} e^{-at}/(n-1)!$

```
%****Program to find Laplace transform of some standard signals

  clear all
  syms t real;                        %Let t,a,n be any real variable
  syms a real;
  syms n real;

  n=input('Enter the value of n'); %get value of n from keyboard
%(a)
  x=t^n;
  disp('(a) Laplace transform of "t^n" is');
  laplace(x)

%(b)
  x=t^(n-1)/factorial(n-1);
  disp('(b) Laplace transform of "t^(n-1)/(n-1)!" is');
  laplace(x)

%(c)
  x=(t^n)*exp(-a*t);
  disp('(c) Laplace transform of "(t^n)*exp(-a*t)" is');
  laplace(x)

%(d)
  x=(t^(n-1)*exp(-a*t))/factorial(n-1);
  disp('(d) Laplace transform of "(t^(n-1)*exp(-a*t))/(n-1)!" is');
  laplace(x)
```

OUTPUT

```
Enter the value of n4
  (a) Laplace transform of "t^n" is
        ans =
              24/s^5
  (b) Laplace transform of "t^(n-1)/(n-1)!" is
        ans =
              1/s^4
  (c) Laplace transform of "(t^n)*exp(-a*t)" is
        ans =
             24/(s+a)^5
  (d) Laplace transform of "(t^(n-1)*exp(-a*t))/(n-1)!" is
        ans =
             1/(s+a)^4
```

Program 3.3

Write a MATLAB program to find Laplace transform of the following causal signals.

a) t^2-3t b) 1+0.4e^{-2t}sin3t c) 3sin2t+3cos2t

```
%****Program to find Laplace transform of given signals

 clear all
 syms t real; %Let t be real variable

%(a)
 x=t^2-(3*t);
 disp('(a) Laplace transform of "t^2-(3*t)" is');
 X=laplace(x);
 simplify(X)

%(b)
 x=1+0.4*exp(-(2*t))*sin(3*t);
 disp('(b) Laplace transform of "1+(0.4*exp(-(2*t)))*sin(3*t)" is');
 X=laplace(x);
 simplify(X)

%(c)
 x=3*sin(2*t)+3*cos(2*t);
 disp('(c) Laplace transform of "3*sin(2*t)+3*cos(2*t)" is');
 X=laplace(x);
 simplify(X)
```

OUTPUT

```
(a) Laplace transform of "t^2-(3*t)" is
      ans =
            -(-2+3*s)/s^3
(b) Laplace transform of "1+(0.4*exp(-(2*t)))*sin(3*t)" is
      ans =
            1/5*(5*s^2+26*s+65)/s/(s^2+4*s+13)
(c) Laplace transform of "3*sin(2*t)+3*cos(2*t)" is
      ans =
            3*(s+2)/(s^2+4)
```

Program 3.4

Write a MATLAB program to find inverse Laplace transform of the following s-domain signals.

a) 2/s(s+1)(s+2) b) 2/s(s+1)$(s+2)^2$ c) 1/(s^2+s+1)(s+2)

```
%****Program to determine inverse Laplace transform

clear all;
syms s complex;

X=2/(s*(s+1)*(s+2));
disp('Inverse Laplace transform of 2/(s(s+1)(s+2)) is');
x=ilaplace(X);
simplify(x)

X=2/(s*(s+1)*(s+2)^2);
disp('Inverse Laplace transform of 2/(s(s+1)(s+2)^2) is');
x=ilaplace(X);
simplify(x)
```

```
X=1/((s^2+s+1)*(s+2));
disp('Inverse Laplace transform of 1/((s^2+s+1)*(s+2)) is');
x=ilaplace(X);
simplify(x)
```

OUTPUT

```
Inverse Laplace transform of 2/(s(s+1)(s+2)) is
      ans =
      1+exp(-2*t)-2*exp(-t)
Inverse Laplace transform of 2/(s(s+1)(s+2)^2) is
      ans =
      t*exp(-2*t)-2*exp(-t)+3/2*exp(-2*t)+1/2
Inverse Laplace transform of 1/((s^2+s+1)*(s+2)) is
      ans =
      -1/3*exp(-1/2*t)*cos(1/2*3^(1/2)*t)
      +1/3*3^(1/2)*exp(-1/2*t)*sin(1/2*3^(1/2)*t)+1/3*exp(-2*t)
```

Program 3.5

Write a MATLAB program to perform convolution of signals, $x_1(t)= t^2-3t$ and $x_2(t)=t$, using Laplace transform, and then to perform deconvolution using the result of convolution to extract $x_1(t)$ and $x_2(t)$.

```
%****Program for convolution & deconvolution using Laplace transform

clear all;
syms t real;
x1t=(t^2-(2*t));
x2t=t;

X1s=laplace(x1t);
X2s=laplace(x2t);
X3s = X1s*X2s;              %product of laplace transform of inputs
con12=ilaplace(X3s);
disp('Convolution of x1(t) and x2(t) is');
simplify(con12)             % convolution output

decon_X1s=X3s/X1s;
decon_x1t=ilaplace(decon_X1s);
disp('The signal x1(t) obtained by deconvolution is');
simplify(decon_x1t)

decon_X2s=X3s/X2s;
decon_x2t=ilaplace(decon_X2s);
disp('The signal x2(t) obtained by deconvolution is');
simplify(decon_x2t)
```

OUTPUT

```
Convolution of x1(t) and x2(t) is
      ans =
           1/12*t^4-1/3*t^3
The signal x1(t) obtained by deconvolution is
      ans =
          t
The signal x2(t) obtained by deconvolution is
      ans =
          t^2-2*t
```

Program 3.6

Write a MATLAB program to find residues and poles of s-domain signal, $(1.5s^3+4.45s^2+4.25s+0.2)/(s^3+3.5s^2+3.5s+1)$

```
%*****Program to find residues and poles of s-domain signal

clear all
s=tf('s');

b=[1.5 4.45 4.25 0.2];          % Numerator coefficients
a=[1 3.5 3.5 1];                % Denominator coefficients

disp('The given transfer function is,');
f=tf([b], [a])

disp('The residues, poles and direct terms of given transfer
function are,');
disp('(r - residue ; p - poles ; k - direct terms)');
[r,p,k]=residue(b,a)

disp('The numerator and denominator coefficients extracted from
r,p,k are,');
[b,a]=residue(r,p,k)
```

OUTPUT

```
The given transfer function is,
        Transfer function:
              1.5 s^3 + 4.45 s^2 + 4.25 s + 0.2
               ------------------
              s^3 + 3.5 s^2 + 3.5 s + 1
The residues, poles and direct terms of given transfer function are,
(r - residue ; p - poles ; k - direct terms)
        r =
            -1.6667
             2.2000
            -1.3333
        p =
            -2.0000
            -1.0000
            -0.5000
        k =
             1.5000
The numerator and denominator coefficients extracted from r,p,k are,
        b =
            1.5000     4.4500     4.2500     0.2000
        a =
            1.0000     3.5000     3.5000     1.0000
```

Program 3.7

Write a MATLAB program to find the impulse response of the LTI system governed by the transfer function, $H(s)=1/(s^2+s+1)$.

```
%****Program to find impulse response of continuous time LTI system

syms s complex;
H=1/(s^2+s+1);
disp('Impulse response h(t) is');
h=ilaplace(H);
simplify(h)

s=tf('s');
H=1/(s^2+s+1);
t=0:0.01:20;                % set a time vector

h=impulse(H,t);             % impulse response of system H
```

```
plot(t,h);
xlabel('time in seconds');
ylabel('h(t)');
```

OUTPUT

```
Impulse response h(t) is
      ans =
           2/3*3^(1/2)*exp(-1/2*t)*sin(1/2*3^(1/2)*t)
```

The sketch of impulse response is shown in fig P3.7.

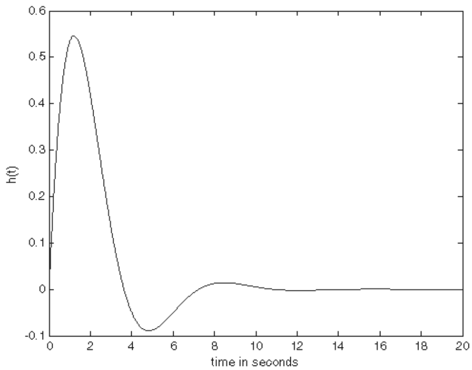


Fig P3.7 : *Impulse response of continuous time LTI system.*

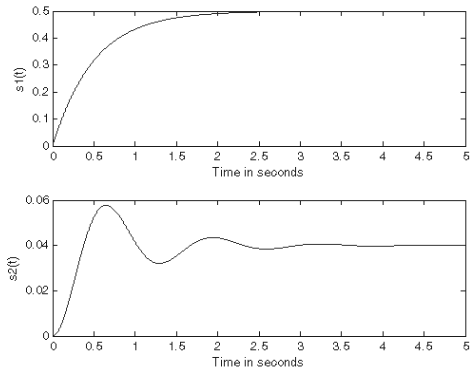


Fig P3.8 : *Step response of first and second order system.*

Program 3.8

Write a MATLAB program to find the step response of the first and second order LTI systems governed by the transfer functions,
$H(s)=1/(s+2)$ and $H(s)=1/(s^2+2.5s+25)$.

```
%********Program to find the step response of I and II order systems
syms s complex;
H1=1/(s+2);
disp('Step response of first order system is');
h1=ilaplace(H1);
simplify(h1)

H2=1/(s^2+2.5*s+25);
disp('Step response of second order system is');
h2=ilaplace(H2);
simplify(h2)

s=tf('s');
H1=1/(s+2);
H2=1/(s^2+2.5*s+25);
t1=0:0.0005:5;          % set a time vector
s1=step(H1,t1);         % step response of first order system
s2=step(H2,t1);         % step response of second order system
subplot(2,1,1);plot(t1,s1);
xlabel('Time in seconds'); ylabel('s1(t)');

subplot(2,1,2);plot(t1,s2);
xlabel('Time in seconds');
ylabel('s2(t)');
```

OUTPUT

```
Step response of first order system is
        ans =
               exp(-2*t)
Step response of second order system is
        ans =
                4/75*15^(1/2)*exp(-5/4*t)*sin(5/4*15^(1/2)*t)
```

The sketch of step response of first and second order system are shown in fig P3.8.

3.11 Exercises

I. Fill in the blanks with appropriate words

1. The Laplace transform will transform a time domain signal to ———— signal.
2. The imaginary part of complex frequency is called ———— frequency.
3. The real part of complex frequency is called ———— frequency.
4. The ROC of Laplace transform of a causal signal includes all points on the s-plane to the ———— of abscissa of convergence.
5. A signal is said to be ———— if $e^{-\sigma t}|x(t)|$ approaches zero as t approaches infinity.
6. If Laplace transform of x(t) is X(s) then Laplace transform of Kx(t) is ———— .
7. If Laplace transform of x(t) is X(s) then Laplace transform of x(t – a) is ———— .
8. When X(s) is a ratio of two polynomials in s, then X(s) is called ———— of s.
9. The ———— are critical frequencies at which the signal X(s) become infinite.
10. The ———— are critical frequencies at which the signal X(s) become zero.
11. The ROC of X(s) consists of stripes parallel to the ———— in s-plane.
12. The ———— is the ratio of Laplace transform of output and input.
13. The Laplace transform of ———— gives the transfer function of the system.
14. For a stable LTI system the ROC should include ———— of s-plane.
15. For a stable causal system ———— should lie on left half of s-plane.

Answers

1.	s-domain	6.	K X(s)	11.	$j\Omega$ – axis
2.	radian / real	7.	$e^{-as}X(s)$	12.	transfer function
3.	neper	8.	rational function	13.	impulse response
4.	right	9.	poles	14.	imaginary axis
5.	exponential order	10.	zeros	15.	poles

II. State whether the following statements are True/False

1. The signal Ke^{st} is an universal signal which represents all types of signals.
2. The Laplace transform is used to transform a frequency domain signal to complex frequency domain.
3. The Laplace transform exists only if the signal is of exponential order.
4. The Laplace transform of the sum of two or more signals is equal to sum of transforms of individual signals.
5. Laplace transform of convolution of two signals is given by the product of the Laplace transform of the individual signals.
6. For a rational signal X(s), the ROC includes poles of X(s).
7. For a realizable system the number of finite poles and zeros should be equal.
8. For a causal rational signal X(s) the ROC is the region to the right of rightmost pole.

9. For an anticausal rational signal X(s) the ROC is the region to the left of leftmost pole.
10. If x(t) is finite and integrable then the ROC is entire s-plane.
11. While forming transfer functions the initial conditions should not be neglected.
12. Zeros and poles of transfer function are complex frequencies.
13. In a transfer function, if s takes the value of a pole, then the transfer function becomes zero.
14. The inverse Laplace transform of the transfer function gives the impulse response.
15. The direct form-I structure uses less number of integrators than direct form-II structure.

Answers

1. True	6. False	11. False
2. False	7. False	12. True
3. True	8. True	13. False
4. True	9. True	14. True
5. True	10. True	15. False

III. Choose the right answer for the following questions

1. When $\Omega = 0$ in the signal $x(t) = Ae^{st}$, where $s = \sigma + j\Omega$, the signal x(t) represents,

a) exponential signal **b)** step signal **c)** ramp signal **d)** sinusoidal signal

2. The mathematical operation $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ is called,

a) Laplace transform **b)** two sided Laplace transform
c) one sided Laplace transform **d)** generalized frequency domain transform

3. The term Ω in $s = \sigma + j\Omega$ is called,

a) frequency **b)** real frequency **c)** complex frequency **d)** critical frequency

4. A causal signal x(t) is said to be exponential order for any positive value of σ if,

a) $\underset{t\to\infty}{\mathrm{Lt}}\ e^{+\sigma t}|x(t)| = \infty$ **b)** $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-\sigma t}|x(t)| = \infty$ **c)** $\underset{t\to\infty}{\mathrm{Lt}}\ e^{+\sigma t}|x(t)| = 0$ **d)** $\underset{t\to\infty}{\mathrm{Lt}}\ e^{-\sigma t}|x(t)| = 0$

5. The ROC of a causal signal x(t) is,

a) entire s-plane **b)** region in between two abscissa of convergence
c) right of abscissa of convergence **d)** left of abscissa of convergence

6. The Laplace transform of the causal signal $t^n u(t)$ is,

a) $\frac{n!}{s^{n+1}}$ **b)** $\frac{n!}{s^n}$ **c)** $\frac{n}{s^{n+1}}$ **d)** $\frac{n}{s^n}$

7. If x(t) and X(s) are Laplace transform pairs, then Laplace transform of $e^{-at}\, x(t)$ is,

a) $e^{as}X(s)$ **b)** $e^{-as}X(s)$ **c)** $X(s - a)$ **d)** $X(s + a)$

8. If x(t) and X(s) are Laplace transform pairs, then Laplace transform of $\frac{x(t)}{t}$ is,

a) $\int_0^\infty X(s)\, ds$ **b)** $\int_s^\infty X(s)\, ds$ **c)** $\frac{1}{s}\int_0^\infty X(s)\, ds$ **d)** $\frac{1}{s}\int_s^\infty X(s)\, ds$

9. If x(t) is periodic with period T, then Laplace transform of x(t) is defined as,

a) $\frac{1}{1-e^{-sT}}\int_0^T x(t)\, e^{-st}\, dt$ **b)** $\frac{1}{1+e^{-sT}}\int_0^T x(t)\, e^{-st}\, dt$ **c)** $\frac{1}{1-e^{sT}}\int_0^T x(t)\, e^{-st}\, dt$ **d)** $\frac{1}{1+e^{sT}}\int_0^T x(t)\, e^{-st}\, dt$

10. *If x(t) and X(s) are Laplace transform pair, and X(s) is rational, then which of the following statements are true.*

i) poles and zeros are critical frequencies.

ii) the ROC of X(s) does not include poles.

iii) the number of finite zeros will be less than or equal to number of poles.

a) i and iii only **b)** i only **c)** ii only **d)** all of the above

---

11. *If x(t) and X(s) are Laplace transform pair, and X(s) is rational, then which of the following statements are true regarding the ROC of X(s).*

i) The ROC is bounded by poles or extends to infinity.

ii) If x(t) is right sided, then ROC is right of right most pole.

iii) If x(t) is left sided, then ROC is left of left most pole.

a) i only **b)** ii only **c)** ii and iii only **d)** all of the above

---

12. *The inverse Laplace transform of $X(s) = \dfrac{2}{s^2 + 2s + 5}$ is,*

a) $x(t) = e^{-t}\cos 2t$ **b)** $x(t) = e^{-t}\sin 2t$ **c)** $x(t) = e^{-2t}\cos 5t$ **d)** $x(t) = e^{-2t}\sin 5t$

---

13. *The inverse Laplace transform of $X(s) = \dfrac{4}{s+5}$ for ROC Re{s} > –4 and Re{s} < –4 are respectively.*

a) $4\,e^{-5t}\,u(t)$ and $4\,e^{-5t}\,u(-t)$ **b)** $4\,e^{5t}\,u(t)$ and $4\,e^{5t}\,u(-t)$

c) $4\,e^{-5t}\,u(t)$ and $-4\,e^{-5t}\,u(-t)$ **d)** $-4\,e^{-5t}\,u(t)$ and $-4\,e^{-5t}\,u(t)$

---

14. *For a continuous time LTI system which of the following statements are true.*

i) The transfer function is ratio of Laplace transform of output and input.

ii) The transfer function is ratio of Laplace transform of input and output.

iii) The transfer function is Laplace transform of impulse response.

a) i only **b)** i and iii **c)** ii only **d)** iii only

---

15. *If x(t), y(t) and h(t) are input, output and impulse response of LTI continuous time system respectively then,*

a) $h(t) = x(t) * y(t)$ **b)** $x(t) = y(t) * h(t)$ **c)** $y(t) = x(t) * h(t)$ **d)** $y(t) = h(t) * h(t)$

---

16. *If X(s), Y(s) and H(s) are Laplace transform of input, output and impulse response of LTI continuous time system respectively then,*

a) $x(t) = \mathcal{L}^{-1}\left\{\dfrac{H(s)}{Y(s)}\right\}$ **b)** $x(t) = \mathcal{L}^{-1}\left\{\dfrac{Y(s)}{H(s)}\right\}$ **c)** $x(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{Y(s)\,H(s)}\right\}$ **d)** $x(t) = \mathcal{L}^{-1}\{Y(s)\,H(s)\}$

---

17. *The convolution of u(t) with u(t) will be equal to,*

a) $\delta(t)$ **b)** $u(t)$ **c)** $t\,u(t)$ **d)** $t^2\,u(t)$

---

18. *The convolution of $e^{-at}\,u(t)$ with $e^{-at}\,u(t)$ will be equal to,*

a) $t\,u(t)$ **b)** $e^{-at}\,u(t)$ **c)** $e^{-2at}\,u(t)$ **d)** $te^{-at}\,u(t)$

---

19. *Given that $H(s) = e^{-4s}$. What is the impulse response of the system?*

a) $\delta(t-4)$ **b)** $u(t-4)$ **c)** $e^{-4t}\,u(t)$ **d)** $e^{4t}\,u(t)$

---

20. *Which of the following statements are true regarding the stability of continuous time system?*

i) For stability of LTI system, the ROC should include imaginary axis.

ii) For stability of causal system the poles should lie on left half s-plane.

iii) For stability of noncausal system there is no restriction on location of poles in s-plane.

a) i and ii only **b)** i and iii only **c)** ii and iii only **d)** all of the above

---

Answers

1. a	6. a	11. d	16. b
2. b	7. d	12. b	17. c
3. b	8. b	13. c	18. d
4. d	9. a	14. b	19. a
5. c	10. d	15. c	20. d

IV. Answer the following questions

1. Define complex frequency.
2. Define Laplace transform of a signal
3. What is the condition to be satisfied for the existence of Laplace transform?
4. What is abscissa of convergence?
5. What is region of convergence(ROC)?
6. Define inverse Laplace transform.
7. Write any two properties of Laplace transform.
8. State and prove initial value theorem.
9. State and prove final value theorem.
10. Define the convolution theorem of Laplace transform.
11. Write any two properties of ROC of Laplace transform.
12. Write a procedure to determine inverse Laplace transform using convolution theorem.
13. Define transfer function of a continuous time system.
14. Define poles and zeros.
15. Write the procedure to perform convolution using Laplace transform.
16. Write the procedure to perform deconvolution using Laplace transform.
17. What is the relation between impulse response and transfer function of the system?
18. What is the condition for stability of a continuous time LTI system?
19. What is the condition for stability of LTI causal system?
20. What are the basic elements of block diagram?

V. Solve the following problems

E3.1 Find the Laplace transform and the ROC of the following signals using the fundamental definition of Laplace transform.

a) $x(t) = \left(t^2 + 2t + 4\right) u(t)$ **b)** $x(t) = \cos^2 6t\ u(t)$ **c)** $x(t) = e^{-5t} \sin 7t\ u(t)$ **d)** $x(t) = e^{at+b}\ u(t)$

E3.2 Find the Laplace transform of the following signals.

a) $x(t) = (4e^{-2t} \cos 5t - 3e^{-2t} \sin 5t)\ u(t)$ **b)** $x(t) = (t-2)^3\ u(t-2)$

c) $x(t) = (2 - 4t + t^2)\ e^{-t}\ u(t)$ **d)** $x(t) = \delta(t) + (t\,e^{-3t} + 2\cos 5t)\ u(t)$

E3.3 Find the Laplace transform of the following signals.

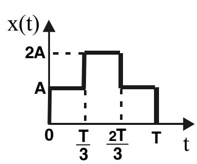
a)


Fig E3.3.1.

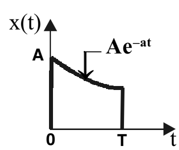
b)


Fig E3.3.2.

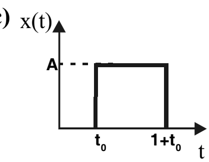
c)


Fig E3.3.3.

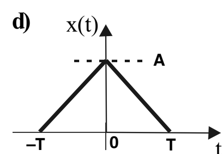
d)


Fig E3.3.4.

E3.4 Find the Laplace transform of the signal shown in fig E3.4.

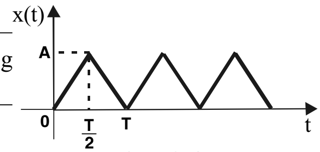


Fig E3.4.

E3.5 Find the initial value, x(0) in time domain for the following s-domain signals.

a) $X(s)=\dfrac{10(s+1)}{(s+2)\,(s+6)}$ **b)** $X(s)=\dfrac{2s^2+2}{8s^3+4s^2+3s+5}$

c) $X(s)=\dfrac{4}{s^3+2s^2+7s+1}$ **d)** $X(s)=\dfrac{1}{s^3}+\dfrac{7}{s^2}+\dfrac{4}{s}$

E3.6 Find the final value, $x(\infty)$ in time domain for the following s-domain signals.

a) $X(s)=\dfrac{10(s+1)}{s(s+2)\,(s+4)}$ **b)** $X(s)=\dfrac{s}{s^2+4}$ **c)** $X(s)=\dfrac{(s+2)\,(s+3)}{s^2(s+5)}$ **d)** $X(s)=\dfrac{s(s+1)}{s^2+2s+1}$

E3.7 Perform the convolution of $x_1(t)$ and $x_2(t)$ using Laplace transform.

a) $x_1(t)=t\,e^{-2t}\,u(t)$; $x_2(t)=3\sin 4t\,u(t)$ **b)** $x_1(t)=(t-2)\,u(t-2)$; $x_2(t)=e^{-3t}\,u(t)$

c) $x_1(t)=4t\,e^{-5t}\,u(t)$; $x_2(t)=u(t)$ **d)** $x_1(t)=t^2\,u(t)$; $x_2(t)=\delta(t-7)$

e) $x_1(t)=\sin t\,u(t)$; $x_2(t)=t\,e^{-t}\,u(t)$

E3.8 Determine the poles and zeros of the following s-domain signal and sketch the pole-zero plot.

$$X(s)=\frac{(s+2)\,(s^2+8s+20)}{(s+3)\,(s^2+2s+2)}$$

E3.9 Find the inverse Laplace transform of the following signals.

a) $X(s)=\dfrac{s}{s^2+4s+3}$ **b)** $X(s)=\dfrac{2s+1}{s^3+7s^2+10s}$

c) $X(s)=\dfrac{1}{s^3+2s^2+2s+1}$ **d)** $X(s)=\dfrac{(s+2)}{s^2+4s+5}$

E3.10 Find the inverse Laplace transform of the following signals.

a) $X(s)=\dfrac{s^2+2s-2}{s(s+2)\,(s-3)}$ **b)** $X(s)=\dfrac{4s^2+15s+62}{(s+1)\,(s^2+4s+20)}$

c) $X(s)=\dfrac{2s^2+4s+34}{(s+2)\,(s^2+6s+25)}$ **d)** $X(s)=\dfrac{s^3+8s^2+23s+28}{(s^2+4s+13)\,(s^2+2s+5)}$

E3.11 Find the inverse Laplace transform of $X(s)=\dfrac{1}{s^2(s^2-a^2)}$ using convolution theorem.

E3.12 Find the impulse response of the systems represented by following differential equations.

a) $\dfrac{dy(t)}{dt}=x(t-t_0)$ **b)** $3\dfrac{d^2y(t)}{dt}+y(t)=0.5\,x(t)$

c) $\dfrac{d^2y(t)}{dt^2}+0.8\dfrac{dy}{dt}+0.15\,y(t)=0.2\dfrac{dx(t)}{dt}+x(t)$

E3.13 Using Laplace transform, determine the natural response of the system represented by following equations.

a) $\dfrac{d^2y(t)}{dt^2}+1.5\,\dfrac{dy(t)}{dt}+0.36\,y(t)=0.1\,\dfrac{dx(t)}{dt}+0.7\,x(t)$; $y(0)=0.3$; $\left.\dfrac{dy(t)}{dt}\right|_{t=0}=-0.2$

b) $\dfrac{d^2y(t)}{dt^2}+10\,\dfrac{dy(t)}{dt}+21\,y(t)=8\,x(t)$; $y(0)=2$; $\left.\dfrac{dy(t)}{dt}\right|_{t=0}=-3$

E3.14 Using Laplace transform, determine the forced response of the system represented by following equations.

a) $\dfrac{d^2y(t)}{dt^2}+9\dfrac{dy(t)}{dt}+20\,y(t)=0.2\dfrac{dx(t)}{dt}+2\,x(t)$; Input $x(t)=6\,u(t)$

b) $\dfrac{d^3y(t)}{dt^3}+10\dfrac{d^2y(t)}{dt^2}+27\dfrac{dy(t)}{dt}+18\,y(t)=12\,x(t)$; Input $x(t)=e^{-5t}\,u(t)$

E3.15 Using Laplace transform, determine the complete response of the system represented by following equations.

a) $\dfrac{d^2y(t)}{dt^2}+1.4\dfrac{dy(t)}{dt}+0.4\,y(t)=0.2\dfrac{dx(t)}{dt}+1.25\,x(t)$; $y(0)=1$; $\left.\dfrac{dy(t)}{dt}\right|_{t=0}=-2$; for input $x(t)=2e^{-3t}u(t)$

b) $\dfrac{d^2y(t)}{dt^2}+11\dfrac{dy(t)}{dt}+24\,y(t)=3\,x(t)$; $y(0)=2$; $\left.\dfrac{dy(t)}{dt}\right|_{t=0}=-0.5$; for input $x(t)=4\,u(t)$

E3.16 Find the transfer function of the system governed by the following differential equations.

a) $\dfrac{d^3y(t)}{dt^3}+0.4\dfrac{d^2y(t)}{dt^2}+0.3\dfrac{dy(t)}{dt}+0.1\,y(t)=0.3\dfrac{dx(t)}{dt}+x(t-1)$

b) $\dfrac{d^3y(t)}{dt^3}+7\dfrac{d^2y(t)}{dt^2}+5\dfrac{dy(t)}{dt}+14\,y(t)=3\dfrac{d^2x(t)}{dt^2}+2\dfrac{dx(t)}{dt}+4x(t)$

c) $4\dfrac{d^2y(t)}{dt^2}+2\dfrac{dy(t)}{dt}+8\,y(t)=x(t-3)$

E3.17 Find the transfer function of the system governed by the following impulse responses.

a) $h(t)=\delta(t-2)+e^{-4t}\,u(t)$ **b)** $h(t)=\delta(t)+t\,e^{-3t}\,u(t)+e^{-9t}u(t)$

c) $h(t)=2\,u(t)+3e^{-6t}\,u(t)+4e^{-7t}\,\sin 2t\,u(t)$

E3.18 Determine the transfer function of the system, whose unit step responses are given below.

a) $s(t)=[t+\cosh 2t]\,u(t)$ **b)** $s(t)=[t^2+2\,e^{-5t}\,\sin 3t]\,u(t)$

c) $s(t)=[3t^2-4e^{-7t}+4e^{-t}]\,u(t)$

E3.19 Find the impulse response of the continuous time systems governed by the following transfer functions.

a) $H(s)=\dfrac{1}{s^2(s+\sqrt{2})}$ **b)** $H(s)=\dfrac{4}{s(s^2-16)}$ **c)** $H(s)=\dfrac{3}{s^2+18s+90}$

E3.20 Find the response of LTI systems whose input and impulse responses are given below.

a) $x(t)=0.5e^{-2t}\,u(t)$; $h(t)=u(t)$ **b)** $x(t)=t\,e^{-t}\,u(t)$; $h(t)=\delta(t-2)$

c) $x(t)=3e^{-2t}\cos 4t\,u(t)$; $h(t)=e^{-6t}\,u(t)$

E3.21 Perform deconvolution operation to extract the signal $x_1(t)$.

a) $x_1(t)*x_2(t)=(1+t)\,u(t)$; $x_2(t)=\delta(t)$

b) $x_1(t)*x_2(t)=(t\,e^{-t}+3t^2)\,u(t)$; $x_2(t)=2e^{-3t}u(t)$

c) $x_1(t)*x_2(t)=t^2\,u(t)$; $x_2(t)=2\,\cosh t\,u(t)$

E3.22 Find the unit step response of the systems whose impulse responses are given below.

a) $h(t)=(t+\sin t)\,u(t)$ **b)** $h(t)=\delta(t-3)+u(t-3)$ **c)** $h(t)=\left(t\,e^{-2t}+e^{-4t}\right)u(t)$

E3.23 Draw the direct form-I and direct form-II structure of the continuous time systems respresented by the following equations.

a) $\dfrac{d^2y(t)}{dt^2}+5\,\dfrac{dy(t)}{dt}+2\,y(t)=3\,\dfrac{d^2x(t)}{dt^2}+4\dfrac{dx(t)}{dt}+6\,x(t)$

b) $\dfrac{d^3y(t)}{dt^3}+0.9\,\dfrac{d^2y(t)}{dt^2}+1.2\,\dfrac{dy(t)}{dt}\;+2.1\,y(t)=3.1\dfrac{d^2x(t)}{dt^2}+0.6\dfrac{dx(t)}{dt}+0.9\,x(t)$

E3.24 Draw the direct form-I and direct form-II structure of the continuous time systems governed by the following transfer functions.

a) $H(s)=\dfrac{4s^2+2s-0.9}{s^3+2s^2-13s+0.2}$ **b)** $H(s)=\dfrac{2s^3+3s^2+7s+10}{s^3-9s^2+12s+13}$

E3.25 Draw the cascade and parallel structure for the LTI system governed by the following transfer functions.

a) $H(s)=\dfrac{(s+2)\,(s-5)}{(s+4)\,(s^2+s+3)}$ **b)** $H(s)=\dfrac{s^2+2s}{(s+6)\,(s^2-7s+10)}$

Answers

E3.1 **a)** $X(s)=\dfrac{4}{s^3}(s^2+0.5s+0.5)$, ROC : $\sigma>0$ **b)** $X(s)=\dfrac{s^2+72}{s(s^2+144)}$, ROC : $\sigma>0$

c) $X(s)=\dfrac{7}{(s+5)^2+49}$, ROC : $\sigma>-5$ **d)** $X(s)=\dfrac{e^b}{s+a}$, ROC $\sigma>-a$

E3.2 **a)** $X(s)=\dfrac{4s-7}{s^2+2s+29}$ **b)** $X(s)=\dfrac{6e^{-2s}}{s^4}$

c) $X(s)=\dfrac{2s^2}{(s+1)^3}$ **d)** $X(s)=\dfrac{s^4+8s^3+47s^2+168s+250}{(s+3)^2\;(s^2+25)}$

E3.3 **a)** $X(s)=\dfrac{A}{s}\left[1+e^{-\frac{sT}{3}}-e^{-\frac{2sT}{3}}-e^{-sT}\right]$ **b)** $X(s)=\dfrac{A}{(s+a)}\left[1-e^{-(s+a)T}\right]$

c) $X(s)=\dfrac{A}{s}\left[e^{-st_0}(1-e^{-s})\right]$ **d)** $X(s)=\dfrac{2A}{Ts^2}(\cosh sT-1)$

E3.4 $X(s)=\dfrac{2A}{Ts^2}\left(\dfrac{1-e^{-\frac{sT}{2}}}{1+e^{-\frac{sT}{2}}}\right)$

E3.5 **a)** $x(0)=10$ **b)** $x(0)=0.25$ **c)** $x(0)=0$ **d)** $x(0)=4$

E3.6 **a)** $x(\infty)=1.25$ **b)** $x(\infty)=0$ **c)** $x(\infty)=\infty$ **d)** $x(\infty)=0$

E3.7 **a)** $x_1(t)*x_2(t)=(0.6\,te^{-2t}+0.12e^{-2t}-0.12\,\cos 4t-0.09\,\sin 4t)\,u(t)$

b) $x_1(t)\,*\,x_2(t)=\dfrac{1}{3}\left[(t-2)-\dfrac{1}{3}+\dfrac{1}{3}e^{-3(t-2)}\right]u(t-2)$ **c)** $x_1(t)\,*\,x_2(t)=0.16\left[1-5\,t\,e^{-5t}-e^{-5t}\right]u(t)$

d) $x_1(t)\,*\,x_2(t)=(t-7)^2\;u(t-7)$ **e)** $x_1(t)\,*\,x_2(t)=0.5\left[(t+1)e^{-t}-\;\cos t\right]u(t)$

E3.8 $p_1=-3,\quad p_2=-1+j1,\quad p_3=-1-j$

$z_1=-2,\quad z_2=-4+j2,\quad z_3=-4-j2$

E3.9 **a)** $x(t) = [1.5e^{-3t} - 0.5e^{-t}]\, u(t)$

b) $x(t) = [0.1 + 0.5e^{-2t} - 0.6\,e^{-5t}]\, u(t)$

c) $x(t) = \left[e^{-t} + e^{-0.5t}\left(0.577\,\sin 0.866t - \cos 0.866t\right)\right] u(t)$

(or) $x(t) = \left[e^{-t} + 1.155e^{-0.5t}\ \sin\,(0.866t\ - 60^{o})\right] u(t)$

d) $x(t) = e^{-2t}\cos t\, u(t)$

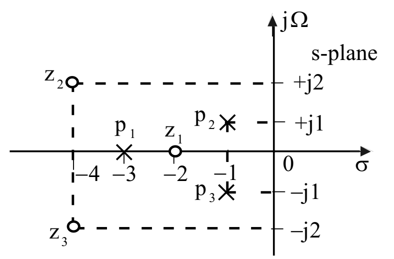


***Fig 3.8**: Pole-zero plot for exercise problem E3.8.*

E3.10 **a)** $x(t) = \left(\dfrac{1}{3} - \dfrac{1}{5}e^{-2t} + \dfrac{13}{15}e^{3t}\right) u(t)$

b) $x(t) = [3e^{-t} + e^{-2t}\cos 4t]\, u(t)$

c) $x(t) = 2[e^{-2t} - e^{-3t}\sin 4t]\, u(t)$

d) $x(t) = [e^{-2t}\sin 3t + e^{-t}\cos 2t]\, u(t)$

E3.11 $x(t) = \dfrac{1}{a^2}\left(\dfrac{\sinh at}{a} - t\right) u(t)$

E3.12 **a)** $h(t) = u(t - t_0)$

b) $h(t) = \dfrac{0.5}{\sqrt{3}}\,\sin\!\left(\dfrac{1}{\sqrt{3}}t\right) u(t)$

c) $h(t) = [4.7e^{-0.3t} - 4.5e^{-0.5t}]\, u(t)$

E3.15 **a)** $y(t) = \left(0.25e^{-3t} + 1.5833\,e^{-t} - 0.8333\,e^{-0.4t}\right) u(t)$

b) $y(t) = \left(0.5 - 0.8e^{-8t} + 2.3e^{-3t}\right) u(t)$

E3.13 **a)** $y_{zi}(t) = \left[0.1778e^{-0.3t} + 0.1222\,e^{-1.2t}\right] u(t)$

b) $y_{zi}(t) = \left(\dfrac{-3}{4}e^{-7t} + \dfrac{11}{4}e^{-3t}\right) u(t)$

E3.16 **a)** $\dfrac{Y(s)}{X(s)} = \dfrac{0.3s + e^{-s}}{s^3 + 0.4s^2 + 0.3s + 0.1}$

b) $\dfrac{Y(s)}{X(s)} = \dfrac{3s^2 + 2s + 4}{s^3 + 7s^2 + 5s + 14}$

c) $\dfrac{Y(s)}{X(s)} = \dfrac{e^{-3s}}{4s^2 + 2s + 8}$

E3.14 **a)** $y_{zs}(t) = \left(0.6 - 1.8e^{-4t} + 1.2\,e^{-5t}\right) u(t)$

b) $y_{zs}(t) = \left(0.3e^{-t} - e^{-3t} + 1.5e^{-5t} - 0.8e^{-6t}\right) u(t)$

E3.17 **a)** $H(s) = \dfrac{e^{-2s}(s+4) + 1}{(s+4)}$

b) $H(s) = \dfrac{s^3 + 16s^2 + 70s + 99}{s^3 + 15s^2 + 63s + 81}$

c) $H(s) = \dfrac{5s^3 + 90s^2 + 481s + 636}{s(s^3 + 20s^2 + 137s + 318)}$

E3.18 **a)** $H(s) = \dfrac{2s^2 - 4}{s^2 - 4}$

b) $H(s) = \dfrac{7s^2 + 10s + 34}{s(s^2 + 10s + 34)}$

c) $H(s) = \dfrac{24s^3 + 6s^2 + 48s + 42}{s^4 + 8s^3 + 7s^2}$

E3.19 **a)** $h(t) = \dfrac{1}{2}\left[\sqrt{2}t - 1 + e^{-\sqrt{2}t}\right] u(t)$

b) $h(t) = \dfrac{1}{4}\ [\cosh 4t - 1]\, u(t)$

c) $h(t) = e^{-9t}\sin 3t\, u(t)$

E3.20 **a)** $y(t) = 0.25\left[1 - e^{-2t}\right] u(t)$

b) $y(t) = (t-2)\; e^{-(t-2)}\; u(t-2)$

c) $y(t) = 0.375\left[e^{-2t}(\sin 4t + \cos 4t) - e^{-6t}\right] u(t)$

E3.21 **a)** $x_1(t) = (t + 1)\; u(t)$

b) $x_1(t) = \left[(t + 0.5)\; e^{-t} + 3t + 4.5t^2\right] u(t)$

c) $x_1(t) = \left(t - \dfrac{t^3}{6}\right) u(t)$

E3.22 **a)** $s(t) = [1 + t - \cos t]\, u(t)$

b) $s(t) = u(t-3) + (t-3)\; u(t-3)$

c) $s(t) = 0.25\left[2 - 2\;te^{-2t} - e^{-2t} - e^{-4t}\right] u(t)$

E3.23 a)

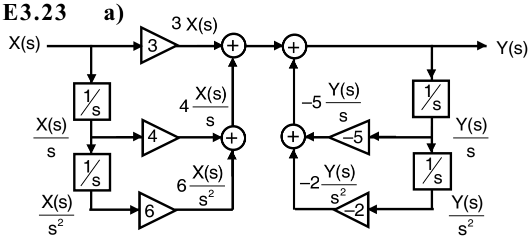


Fig : Direct form-I structure.

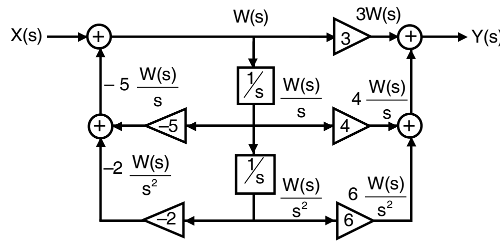


Fig : Direct form-II structure.

E3.23 b)

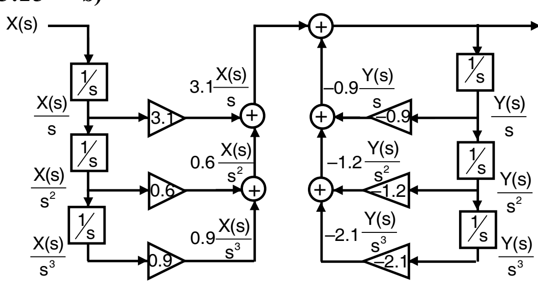


Fig : Direct form-I structure.

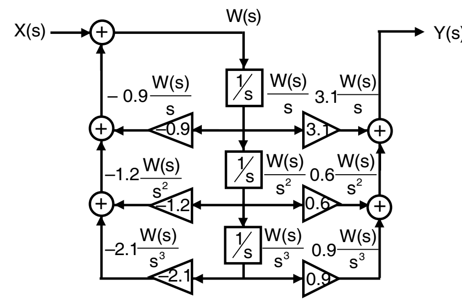


Fig : Direct form-II structure.

E3.24 a)

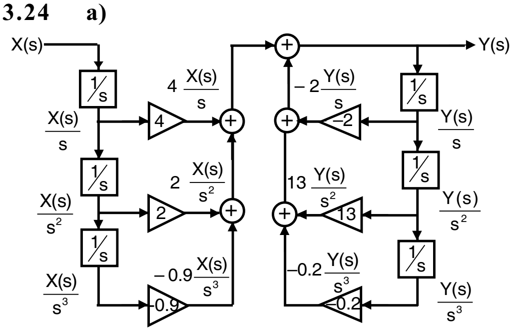


Fig : Direct form-I structure.

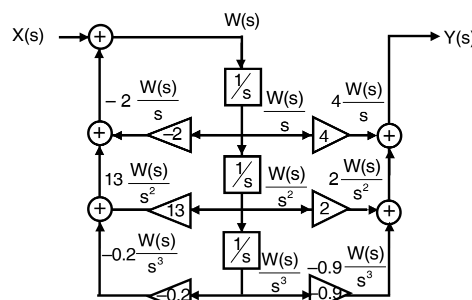


Fig : Direct form-II structure.

E3.24 b)

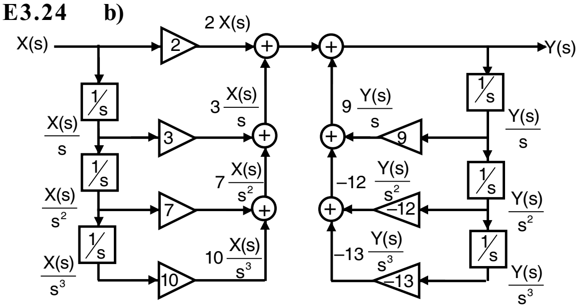


Fig : Direct form-I structure.

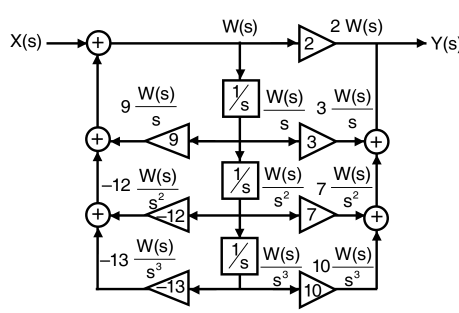


Fig : Direct form-II structure.

E3.25 **a)**

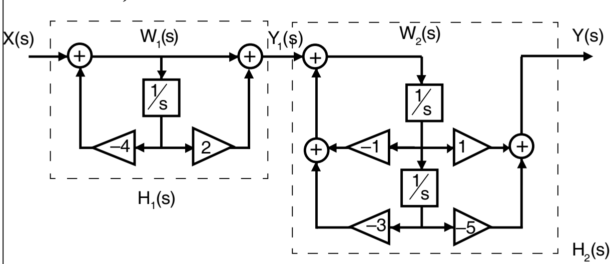


Fig *: Cascade structure.*

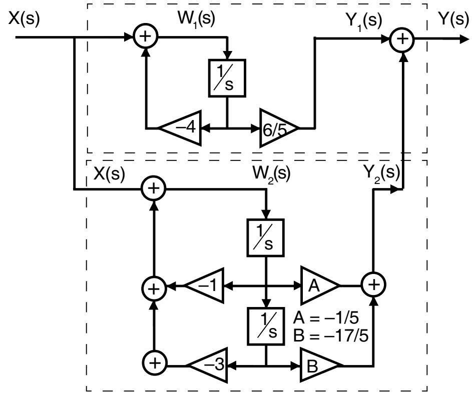


Fig *: Parallel structure.*

b)

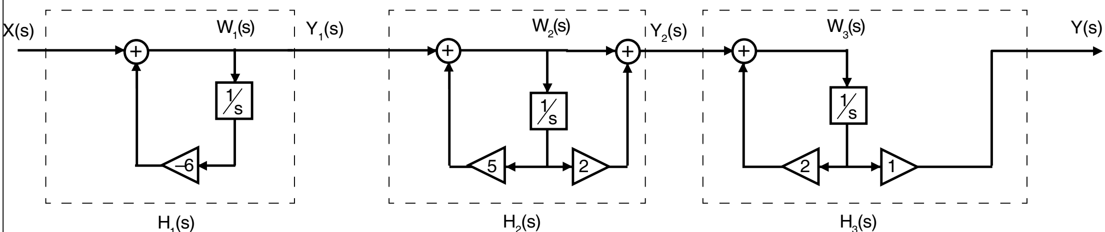


Fig *: Cascade structure .*

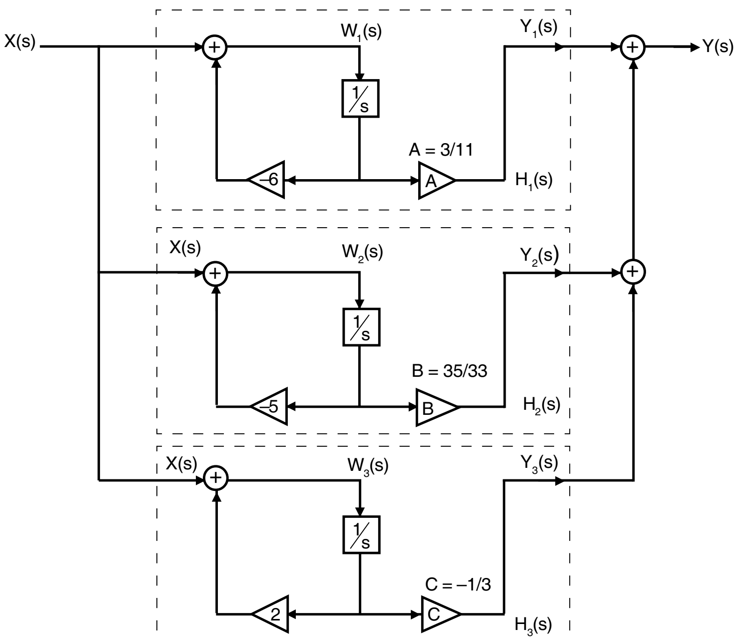


Fig *: Parallel structure.*

CHAPTER 4

Fourier Series and Fourier Transform of Continuous Time Signals

4.1 Introduction

The French mathematician Jean Baptiste Joseph Fourier (J.B.J. Fourier) has shown that any periodic non-sinusoidal signal can be expressed as a linear weighted sum of harmonically related sinusoidal signals. This leads to a method called ***Fourier series*** in which a periodic signal is represented as a function of frequency.

The Fourier representation of periodic signals has been extended to non-periodic signals by letting the fundamental period T tend to infinity, and this Fourier method of representing non-periodic signals as a function of frequency is called ***Fourier transform***. The Fourier represention of signals is also known as frequency domain representation. In general, the Fourier series representation can be obtained only for periodic signals, but the Fourier transform technique can be applied to both periodic and non-periodic signals to obtain the frequency domain representation of the signals.

The Fourier representation of signals can be used to perform frequency domain analysis of signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general, these plots are called ***frequency spectrum***.

4.2 Trigonometric Form of Fourier Series

4.2.1 Definition of Trigonometric Form of Fourier Series

The ***trigonometric form of Fourier series*** of a periodic signal, x(t), with period T is defined as,

$$\boxed{x(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t} \qquad .....(4.1)$$

$$\therefore\ x(t) = \frac{1}{2}a_0 + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + a_3 \cos 3\Omega_0 t + ...$$
$$+ b_1 \sin\Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + ...$$

where, $\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$n\Omega_0$ = Harmonic frequencies

a_0, a_n, b_n = Fourier coefficients of trigonometric form of Fourier series

> ***Note :*** *1. Here $a_0/2$ is the value of constant component of the signal x(t).*
>
> *2. The Fourier coefficient a_n and b_n are maximum amplitudes of n^{th} harmonic component.*

The ***Fourier coefficients*** can be evaluated using the following formulae.

$$a_0 = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{2}{T}\int_0^T x(t)\,dt \qquad .....(4.2)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \qquad .....(4.3)$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin n\Omega_0 t\,dt \quad \text{(or)} \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt \qquad .....(4.4)$$

In the above formulae, the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from $\mathbf{t_0}$ **to** $\mathbf{t_0 + T}$, where t_0 is any time instant.

4.2.2 Conditions for Existence of Fourier Series

The Fourier series exists only if the following Dirichlet's conditions are satisfied.

1. The signal x(t) is well defined and single valued, except possibly at a finite number of points.
2. The signal x(t) must possess only a finite number of discontinuities in the period T.
3. The signal must have a finite number of positive and negative maxima in the period T.

> ***Note :*** *1. The value of signal x(t) at $t = t_0$ is $x(t_0)$ if $t = t_0$ is a point of continuity.*
>
> *2. The value of signal x(t) at $t = t_0$ is $\dfrac{x(t_0^+) + x(t_0^-)}{2}$ if $t = t_0$ is a point of discontinuity.*

4.2.3 Derivation of Equations for a_0, a_n and b_n

Evaluation of a_0

The Fourier coefficient a_0 is given by,

$$a_0 = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\,dt \quad \text{(or)} \quad a_0 = \frac{2}{T}\int_0^T x(t)\,dt$$

Proof :

Consider the trigonometric Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\,dt = \int_0^T \frac{a_0}{2}\,dt + \int_0^T \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t\,dt + \int_0^T \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t\,dt$$

$$= \frac{a_0}{2}\int_0^T dt + \sum_{n=1}^{\infty} a_n \int_0^T \cos n\Omega_0 t\,dt + \sum_{n=1}^{\infty} b_n \int_0^T \sin n\Omega_0 t\,dt$$

$$\therefore \int_0^T x(t)\,dt = \frac{a_0}{2}\left[t\right]_0^T + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^T + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_0^T$$

$$= \frac{a_0}{2}\left[T-0\right] + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\Omega_0 T}{n\Omega_0} - \frac{\sin 0}{n\Omega_0}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\Omega_0 T}{n\Omega_0} + \frac{\cos 0}{n\Omega_0}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{1}{n\frac{2\pi}{T}}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n T\left[\frac{\sin n2\pi}{n2\pi}\right] + \sum_{n=1}^{\infty} b_n T\left[\frac{-\cos n2\pi}{n2\pi} + \frac{1}{n2\pi}\right]$$

$$= \frac{T}{2}a_0 + \sum_{n=1}^{\infty} a_n T \times 0 + \sum_{n=1}^{\infty} b_n T\left[-\frac{1}{n2\pi} + \frac{1}{n2\pi}\right] = \frac{T}{2}a_0$$

$$\therefore a_0 = \frac{2}{T}\int_0^T x(t)\,dt$$

$\Omega_0 = \frac{2\pi}{T}$
$\sin 0 = 0$
$\cos 0 = 1$

$\sin n2\pi = 0$;
$\cos n2\pi = 1$
for integer n

Evaluation of a_n

The Fourier coefficient a_n is given by,

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt$$

Proof :

Consider the trigonometric form of Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + \ldots\ldots + a_k\cos k\Omega_0 t + \ldots\ldots$$

$$+ b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + \ldots\ldots + b_k\sin k\Omega_0 t + \ldots\ldots$$

Let us multiply the above equation by $\cos k\Omega_0 t$

$$\therefore x(t)\cos k\Omega_0 t = \frac{a_0}{2}\cos k\Omega_0 t + a_1\cos\Omega_0 t\cos k\Omega_0 t + a_2\cos 2\Omega_0 t\cos k\Omega_0 t + \ldots\ldots$$

$$\ldots\ldots + a_k\cos^2 k\Omega_0 t + \ldots\ldots + b_1\sin\Omega_0 t\cos k\,\Omega_0\,t + b_2\sin 2\Omega_0 t\cos k\Omega_0 t + \ldots\ldots$$

$$\ldots\ldots + b_k\sin k\Omega_0 t\cos k\,\Omega_0 t + \ldots\ldots$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\cos k\Omega_0 t\,dt = \int_0^T \frac{a_0}{2}\cos k\Omega_0 t\,dt + \int_0^T a_1\cos\Omega_0 t\cos k\Omega_0 t\,dt$$

$$+ \int_0^T a_2\cos 2\Omega_0 t\cos k\Omega_0 t\,dt + \ldots\ldots + \int_0^T a_k\cos^2 k\Omega_0 t\,dt + \ldots\ldots + \int_0^T b_1\sin\Omega_0 t\cos k\Omega_0 t\,dt$$

$$+ \int_0^T b_2\sin 2\Omega_0 t\cos k\,\Omega_0 t\,dt + \ldots\ldots + \int_0^T b_k\sin k\Omega_0 t\cos k\Omega_0 t\,dt + \ldots\ldots \qquad \ldots\ldots(4.5)$$

$$\therefore \int_0^T x(t)\cos k\Omega_0 t\, dt = \int_0^T a_k \cos^2 k\Omega_0 t\, dt = a_k \int_0^T \frac{1+\cos 2k\Omega_0 t}{2}\, dt$$

$$= \frac{a_k}{2}\int_0^T (1+\cos 2k\Omega_0 t)\, dt = \frac{a_k}{2}\left[t + \frac{\sin 2k\Omega_0 t}{2k\Omega_0}\right]_0^T$$

$$= \frac{a_k}{2}\left[T + \frac{\sin 2k\Omega_0 T}{2k\Omega_0} - 0 - \frac{\sin 0}{2k\Omega_0}\right] = \frac{a_k}{2}\left[T + \frac{\sin 2k\frac{2\pi}{T}T}{2k\frac{2\pi}{T}}\right] = \frac{T}{2}a_k$$

In equation (4.5) all definite integrals will be zero, except $\int_0^T a_k \cos^2 k\Omega_0 t\, dt$.

$\Omega_0 = \frac{2\pi}{T}$

$\sin 0 = 0$

$\sin 2k2\pi = 0$ for integer k

$$\therefore a_k = \frac{2}{T}\int_0^T x(t)\cos k\Omega_0 t\, dt \qquad .....(4.6)$$

The equation (4.6) gives the k^{th} coefficient a_k. Hence the n^{th} coefficient a_n is given by,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\, dt$$

Evaluation of b_n

The Fourier coefficient b_n is given by,

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} x(t)\sin n\Omega_0 t\, dt \quad \text{(or)} \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

Proof :

Consider the trigonometric form of Fourier series of x(t) (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_2\cos 2\Omega_0 t + ... + a_k\cos k\Omega_0 t + ...$$

$$+ b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + ... + b_k\sin k\Omega_0 t + ...$$

Let us multiply the above equation by $\sin k\Omega_0 t$

$$\therefore x(t)\sin k\Omega_0 t = \frac{a_0}{2}\sin k\Omega_0 t + a_1\cos\Omega_0 t\sin k\Omega_0 t + a_2\cos 2\Omega_0 t\sin k\Omega_0 t + ...$$

$$..... + a_k\cos k\Omega_0 t\sin k\Omega_0 t + ... + b_1\sin\Omega_0 t\sin k\Omega_0 t$$

$$+ b_2\sin 2\Omega_0 t\sin k\Omega_0 t + ... + b_k\sin^2 k\Omega_0 t + ...$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\sin k\Omega_0 t\, dt = \int_0^T \frac{a_0}{2}\sin k\Omega_0 t\, dt + \int_0^T a_1\cos\Omega_0 t\sin k\Omega_0 t\, dt$$

$$+ \int_0^T a_2\cos 2\Omega_0 t\sin k\Omega_0 t\, dt + ... + \int_0^T a_k\cos k\Omega_0 t\sin k\Omega_0 t\, dt + ...$$

$$..... + \int_0^T b_1\sin\Omega_0 t\sin k\Omega_0 t\, dt + \int_0^T b_2\sin 2\Omega_0 t\sin k\Omega_0 t\, dt + ...$$

$$+ \int_0^T b_k\sin^2 k\Omega_0 t\, dt + ... \qquad .....(4.7)$$

$$\therefore \int_0^T x(t)\sin k\Omega_0 t\, dt = \int_0^T b_k \sin^2 k\Omega_0 t\, dt = b_k \int_0^T \frac{1-\cos 2k\Omega_0 t}{2}\, dt$$

$$= \frac{b_k}{2}\int_0^T (1-\cos 2k\Omega_0 t)\, dt = \frac{b_k}{2}\left[t - \frac{\sin 2k\Omega_0 t}{2k\Omega_0}\right]_0^T$$

$$= \frac{b_k}{2}\left[T - \frac{\sin 2k\Omega_0 T}{2k\Omega_0} - 0 + \frac{\sin 0}{2k\Omega_0}\right]$$

$$= \frac{b_k}{2}\left[T - \frac{\sin 2k\frac{2\pi}{T}T}{2k\frac{2\pi}{T}}\right] = \frac{T}{2}\, b_k$$

In equation (4.7) all definite integrals will be zero, except $\int_0^T b_k \sin^2 k\Omega_0 t\, dt$.

$\Omega_0 = \frac{2\pi}{T}$

$\sin 0 = 0$

$\sin 2k2\pi = 0$ for integer k

$$\therefore b_k = \frac{2}{T}\int_0^T x(t)\sin k\Omega_0 t\, dt \qquad .....(4.8)$$

The equation (4.8) gives k^{th} coefficient b_k. Hence the n^{th} coefficient b_n is given by,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

4.3 Exponential Form of Fourier Series

4.3.1 Definition of Exponential Form of Fourier Series

The ***exponential form of Fourier series*** of a periodic signal x(t) with period T is defined as,

$$\boxed{x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}} \qquad .....(4.9)$$

where, $\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}$ = Fundamental frequency in rad/sec

F_0 = Fundamental frequency in cycles/sec or Hz

$\pm n\Omega_0$ = Harmonic frequencies

c_n = Fourier coefficients of exponential form of Fourier series.

The ***Fourier coefficient c_n*** can be evaluated using the following equation.

$$\boxed{c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \quad \text{(or)} \quad c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt} \qquad .....(4.10)$$

In equation (4.10), the limits of integration are either **–T/2 to +T/2** or **0 to T**. In general, the limit of integration is one period of the signal and so the limits can be from $\mathbf{t_0}$ **to** $\mathbf{t_0 + T}$, where t_0 is any time instant.

4.3.2 Negative Frequency

The exponential form of Fourier series representation of a signal x(t) has complex exponential harmonic components for both positive and negative frequencies. When the positive and negative complex exponential components of same harmonic are added, it gives rise to real sine or cosine signals.

Alternatively, when the real sine or cosine signal has to be represented in terms of complex exponential then a signal with negative frequency is required.

Here it should be understood that the signal with negative frequency is not a physically realizable signal, but it is required for mathematical representation of real signals in terms of complex exponential signals.

4.3.3 Derivation of Equation for c_n

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \quad \text{(or)} \quad c_n = \frac{1}{T}\int_{0}^{T} x(t)\, e^{-jn\Omega_0 t}\, dt$$

Proof :

Consider the exponential form of Fourier series of x(t), (equation (4.9)).

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} = \ldots + c_{-k} e^{-jk\Omega_0 t} + \ldots + c_{-2} e^{-j2\Omega_0 t} + c_{-1} e^{-j\Omega_0 t}$$
$$+ c_0 + c_1 e^{j\Omega_0 t} + c_2 e^{j2\Omega_0 t} + \ldots + c_k e^{jk\Omega_0 t} + \ldots$$

Let us multiply the above equation by $e^{-jk\Omega_0 t}$.

$$\therefore\ x(t)\, e^{-jk\Omega_0 t} = \ldots + c_{-k} e^{-j2k\Omega_0 t} + \ldots + c_{-2} e^{-j2\Omega_0 t} e^{-jk\Omega_0 t} + c_{-1} e^{-j\Omega_0 t} e^{-jk\Omega_0 t}$$
$$+ c_0 e^{-jk\Omega_0 t} + c_1 e^{j\Omega_0 t} e^{-jk\Omega_0 t} + c_2 e^{j2\Omega_0 t} e^{-jk\Omega_0 t} + \ldots + c_k + \ldots$$

Let us integrate the above equation between limits 0 to T.

$$\therefore \int_0^T x(t)\, e^{-jk\Omega_0 t}\, dt = \ldots + \int_0^T c_{-k} e^{-j2k\Omega_0 t}\, dt + \ldots + \int_0^T c_{-2} e^{-j2\Omega_0 t} e^{-jk\Omega_0 t}\, dt$$
$$+ \int_0^T c_{-1} e^{-j\Omega_0 t} e^{-jk\Omega_0 t}\, dt + \int_0^T c_0 e^{-jk\Omega_0 t}\, dt + \int_0^T c_1 e^{j\Omega_0 t} e^{-jk\Omega_0 t}\, dt$$
$$+ \int_0^T c_2 e^{j2\Omega_0 t} e^{-jk\Omega_0 t}\, dt + \ldots + \int_0^T c_k\, dt + \ldots \qquad \ldots(4.11)$$

In equation (4.11) all definite integrals will be zero, except $\int_0^T c_k\, dt$

$$= \int_0^T c_k\, dt = c_k \left[t\right]_0^T = c_k \left[T - 0\right] = T c_k$$

$$\therefore\ c_k = \frac{1}{T}\int_0^T x(t)\, e^{-jk\Omega_0 t}\, dt \qquad \ldots(4.12)$$

The equation (4.12) gives the K^{th} coefficient, c_k. Hence the n^{th} coefficient, c_n is given by,

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt$$

4.3.4 Relation Between Fourier Coefficients of Trigonometric and Exponential Form

The relation between Fourier coefficients of trigonometric form and exponential form are given below.

$$c_0 = \frac{a_0}{2} \qquad \ldots(4.13)$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad \text{for } n = 1, 2, 3, 4, \ldots \qquad \ldots(4.14)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) \quad \text{for} \quad -n = -1, -2, -3, -4, ... \qquad .....(4.15)$$

$$\therefore \; |c_n| = \frac{1}{2}\sqrt{a_n^2 + b_n^2} \; \text{for all values of n, except when n = 0.} \qquad .....(4.16)$$

Proof :

Consider the trigonometric form of Fourier series of x(t), (equation (4.1)).

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \left[\frac{e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}}{2}\right] + \sum_{n=1}^{\infty} b_n \left[\frac{e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}}{2j}\right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} e^{jn\Omega_0 t} + \frac{a_n}{2} e^{-jn\Omega_0 t} - j\frac{b_n}{2} e^{jn\Omega_0 t} + j\frac{b_n}{2} e^{-jn\Omega_0 t}\right]$$

$$\boxed{\frac{1}{j} = \frac{j}{j^2} = -j}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} - j\frac{b_n}{2}\right] e^{jn\Omega_0 t} + \sum_{n=1}^{\infty}\left[\frac{a_n}{2} + j\frac{b_n}{2}\right] e^{-jn\Omega_0 t}$$

$$\text{Let, } c_0 = \frac{a_0}{2} \;\; ; \;\; c_n = \frac{a_n - jb_n}{2} \;\; ; \;\; c_n^* = \frac{a_n + jb_n}{2}$$

$$\therefore \; x(t) = c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t} + \sum_{n=1}^{\infty} c_n^* e^{-jn\Omega_0 t}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t} + \sum_{n=-\infty}^{-1} c_{-n} e^{jn\Omega_0 t}$$

$$\boxed{C_{-n} = C_n^*}$$

$$= \sum_{n=-\infty}^{-1} c_{-n} e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n e^{jn\Omega_0 t}$$

$$= \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

$$\boxed{\begin{array}{l} e^{jn\Omega_0 t} = 1 \\ \text{for } n = 0 \end{array}}$$

$$\therefore \; c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n) \quad \text{for} \quad n = 1, 2, 3, ..., \infty$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) \quad \text{for} \; -n = -1, -2, -3, ..., -\infty$$

4.3.5 Frequency Spectrum(or Line Spectrum) of Periodic Continuous Time Signals

Let x(t) be a periodic continuous time signal. Now, exponential form of Fourier series of x(t) is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t}$$

where, c_n is the Fourier coefficient of n^{th} harmonic component.

The Fourier coefficient, c_n is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_n = |c_n| \angle c_n$$

where, $|c_n|$ = Magnitude of c_n ; $\angle c_n$ = Phase of c_n

The term, $|c_n|$ represents the magnitude of n^{th} harmonic component and the term $\angle c_n$ represents the phase of the n^{th} harmonic component.

The plot of harmonic magnitude / phase of a signal versus "n" (or harmonic frequency $n\Omega_0$) is called ***Frequency spectrum (or Line spectrum)***. The plot of harmonic magnitude versus "n" (or $n\Omega_0$) is called ***magnitude (line) spectrum*** and the plot of harmonic phase versus "n" (or $n\Omega_0$) is called ***phase (line) spectrum***.

Consider the ramp waveform shown in fig 4.1. The Fourier coefficient c_n for this ramp waveform is given by,

$$c_0 = \frac{A}{2}, \qquad c_n = \frac{jA}{2n\pi}$$

(Please refer example 4.12 for derivation of c_n)

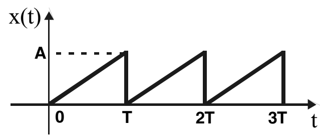


Fig 4.1 : *Ramp waveform.*

$$\text{Let, } A = 20, \qquad \therefore\ c_n = \frac{j20}{2n\pi} = \frac{j10}{n\pi}$$

$$\vdots$$

$$\text{When } n = -3,\ c_{-3} = -j\frac{10}{3\pi} = -j1.061 = 1.061\angle -90^o = 1.061\angle -\pi/2$$

$$\text{When } n = -2,\ c_{-2} = -j\frac{10}{2\pi} = -j1.592 = 1.592\angle -90^o = 1.592\angle -\pi/2$$

$$\text{When } n = -1,\ c_{-1} = -j\frac{10}{\pi} = -j3.183 = 3.183\angle -90^o = 3.183\angle -\pi/2$$

$$\text{When } n = 0,\ c_0 = \frac{20}{2} = 10 = 10\angle 0$$

$$\text{When } n = 1,\ c_1 = j\frac{10}{\pi} = j3.183 = 3.183\angle +90^o = 3.183\angle \pi/2$$

$$\text{When } n = 2,\ c_2 = j\frac{10}{2\pi} = j1.592 = 1.592\angle +90^o = 1.592\angle \pi/2$$

$$\text{When } n = 3,\ c_3 = j\frac{10}{3\pi} = j1.061 = 1.061\angle +90^o = 1.061\angle \pi/2$$

$$\vdots$$

Using the above calculated values the magnitude and phase spectrums are sketched as shown in fig 4.2 and fig 4.3 respectively.

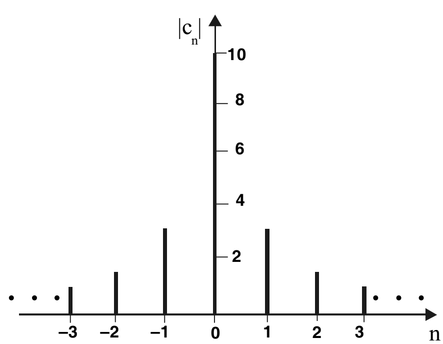


Fig 4.2 : *Magnitude spectrum of ramp waveform.*

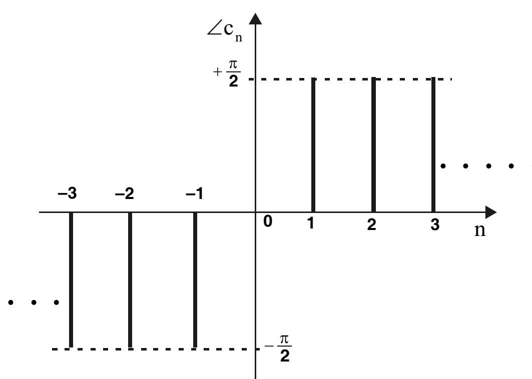


Fig 4.3 : *Phase spectrum of ramp waveform.*

4.4 Fourier Coefficients of Signals With Symmetry

4.4.1 Even Symmetry

A signal, x(t) is called ***even signal***, if the signal satisfies the condition ***x(–t) = x(t)***.

The waveform of an even periodic signal exhibits symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the symmetry of a waveform with respect to t = 0 or vertical axis is called ***even symmetry***.

Examples of even signals are,

$x(t) = 1 + t^2 + t^4 + t^6$

$x(t) = A \cos \Omega_0 t$

In order to determine the even symmetry of a waveform, fold the waveform with respect to vertical axis. After folding, if the waveshape remains same then it is said to have even symmetry.

For even signals the Fourier coefficient a_0 is optional, a_n exists and b_n are zero. The Fourier coefficient a_0 is zero if the average value of one period is equal to zero. For an even signal the Fourier coefficients are given by,

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \qquad \text{(or)} \qquad a_0 = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt \quad \text{(or)} \quad a_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\cos n\Omega_0 t\,dt\ ; \qquad b_n = 0$$

Proof :

Consider the equation for a_0, (equation (4.2)).

Dividing the integral into two parts.

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

Change of integral index,
Let, $t = -\tau\ ;\ \therefore dt = -d\tau$
When $t = 0,\ \tau = -t = 0$
When $t = -T/2,\ \tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

Since τ is dummy variable, Let $\tau = t$.

Since x(t) is even, x(–t) = x(t)

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{T}\int_0^{T/2} x(-t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

$$= \frac{2}{T}\int_0^{T/2} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} x(t)\,dt$$

Consider the equation for a_n, (equation (4.3)).

Dividing the integral into two parts.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Change of integral index,
Let, $t = -\tau\ ;\ \therefore dt = -d\tau$
When $t = 0,\ \tau = -t = 0$
When $t = -T/2,\ \tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\cos n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$\cos(-\theta) = \cos\theta$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\cos n\Omega_0\tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Since τ is dummy variable, Let $\tau = t$.

$$= \frac{2}{T}\int_0^{T/2} x(-t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Since x(t) is even, x(–t) = x(t)

$$= \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Consider the equation for b_n, (equation (4.4)).

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,\sin n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\,\sin n\Omega_0 t\,dt + \frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt$$

Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\,\sin n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt$$

Change of integral index,
Let, $t = -\tau$; $\therefore dt = -d\tau$
When $t = 0$, $\tau = -t = 0$
When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= -\frac{2}{T}\int_{0}^{T/2} x(-\tau)\,\sin n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt$$

$\sin(-\theta) = -\sin\theta$

$$= -\frac{2}{T}\int_{0}^{T/2} x(-t)\,\sin n\Omega_0 t\,dt + \frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt$$

Since τ is dummy variable, Let $\tau = t$.

$$= -\frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt + \frac{2}{T}\int_{0}^{T/2} x(t)\,\sin n\Omega_0 t\,dt = 0$$

Since $x(t)$ is even, $x(-t) = x(t)$

The waveform of some even periodic signals and their Fourier series are given below.

The waveform shown in fig 4.4, has even symmetry, half wave symmetry and quarter wave symmetry.Hence for this waveform, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.4 is given by equation (4.17). [Please refer example 4.1 for the derivation of Fourier series]

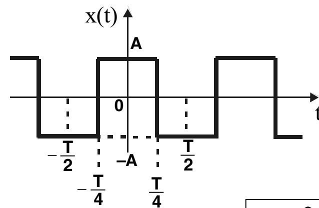


Fig 4.4.

$$\Omega_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{4A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - .....\right] \quad .....(4.17)$$

The waveform shown in fig 4.5, has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is substracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The trigonometric Fourier series representation of the waveform of fig 4.5 is given by equation (4.18). [Please refer example 4.3 for the derivation of Fourier series]

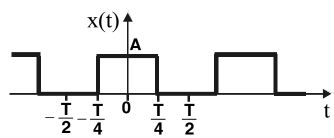


Fig 4.5.

$$\Omega_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left[\frac{\cos\Omega_0 t}{1} - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \frac{\cos 9\Omega_0 t}{9} - .....\right] .....(4.18)$$

The waveform shown in fig 4.6 has even symmetry and so $b_n = 0$. The trigonometric Fourier series representation of the waveform of fig 4.6 is given by equation (4.19).

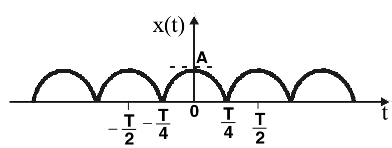


Fig 4.6.

$$\Omega_0 = \frac{2\pi}{T}$$

$$x(t) = \frac{2A}{\pi} + \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{(2^2 - 1)} - \frac{\cos 4\Omega_0 t}{(4^2 - 1)} + \frac{\cos 6\Omega_0 t}{(6^2 - 1)} - \frac{\cos 8\Omega_0 t}{(8^2 - 1)} + ...\right] \quad .....(4.19)$$

The waveform shown in fig 4.7 has even symmetry and so $b_n = 0$. The trigonometric Fourier series representation of the waveform of fig 4.7 is given by equation (4.20). [Please refer example 4.4 for the derivation of Fourier series].

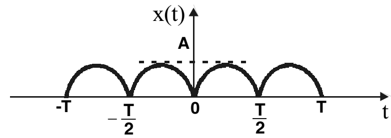


Fig 4.7.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{(2^2 - 1)} + \frac{\cos 4\Omega_0 t}{(4^2 - 1)} + \frac{\cos 6\Omega_0 t}{(6^2 - 1)} + \frac{\cos 8\Omega_0 t}{(8^2 - 1)} + ...\right] \qquad .....(4.20)$$

The waveform shown in fig. 4.8 has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig 4.8 is given by equation (4.21). [Please refer example 4.2 for the derivation of Fourier series].

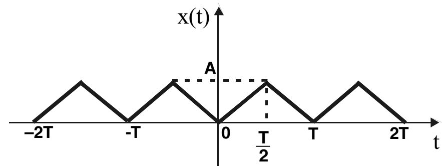


Fig 4.8.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\left[\frac{\cos \Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + ...\right] \qquad .....(4.21)$$

The waveform shown in fig. 4.9 has even symmetry and so $b_n = 0$. If the dc component $(a_0/2)$ is subtracted from this waveform then it will have half wave and quarter wave symmetry, and so the Fourier series has odd harmonics of cosine terms. The Fourier series representation of the waveform of fig 4.9 is given by equation (4.22). [Please refer example 4.11 for the derivation of Fourier series].

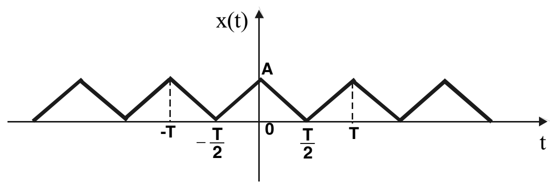


Fig 4.9.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos \Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \frac{\cos 7\Omega_0 t}{7^2} + ...\right] \qquad .....(4.22)$$

4.4.2 Odd Symmetry

A signal, x(t) is called ***odd signal*** if it satisfies the condition ***x(–t) = –x(t)***.

The waveform of odd periodic signal will exhibit anti-symmetry with respect to t = 0 (i.e., with respect to vertical axis) and so the anti-symmetry of a waveform with respect to t = 0 or vertical axis is called ***odd symmetry***.

Examples of odd signals are,
$x(t) = t + t^3 + t^5 + t^7$
$x(t) = A \sin \Omega_0 t$

In order to determine the odd symmetry of a waveform, invert either the right side (or the left side) of the waveform with respect to horizontal axis and then fold the waveform with respect to vertical axis. After inverting one half and folding, if the waveshape remains same then it is said to have odd symmetry.

For odd signals a_0 and a_n are zero and b_n exists. For odd signal the Fourier coefficients are given by,

$$a_0 = 0 \quad ; \quad a_n = 0$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt \quad \text{or} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\sin n\Omega_0 t\,dt$$

Proof :

Consider the equation for a_0, (equation (4.2)).

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

Change of integral index,
Let, $t = -\tau$; $\therefore dt = -d\tau$
When $t = 0$, $\tau = -t = 0$
When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{T}\int_0^{T/2} x(-t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt$$

Since τ is dummy variable, Let $\tau = t$.

Since $x(t)$ is odd, $x(-t) = -x(t)$

$$= -\frac{2}{T}\int_0^{T/2} x(t)\,dt + \frac{2}{T}\int_0^{T/2} x(t)\,dt = 0$$

Consider the equation for a_n, (equation (4.3)).

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\cos n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Change of integral index,
Let, $t = -\tau$; $\therefore dt = -d\tau$
When $t = 0$, $\tau = -t = 0$
When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= \frac{2}{T}\int_0^{T/2} x(-\tau)\cos n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

$\cos(-\theta) = \cos\theta$

$$= \frac{2}{T}\int_0^{T/2} x(-t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

Since τ is dummy variable, Let $\tau = t$.

Since $x(t)$ is odd, $x(-t) = -x(t)$

$$= -\frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = 0$$

Consider the equation for b_n, (equation (4.4)).

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\Omega_0 t\,dt = \frac{2}{T}\int_{-T/2}^{0} x(t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

Dividing the integral into two parts.

$$= \frac{2}{T}\int_{T/2}^{0} x(-\tau)\sin n\Omega_0(-\tau)\,(-d\tau) + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

Change of integral index,
Let, $t = -\tau$; $\therefore dt = -d\tau$
When $t = 0$, $\tau = -t = 0$
When $t = -T/2$, $\tau = -t = -(-T/2) = T/2$

$$= -\frac{2}{T}\int_0^{T/2} x(-\tau)\sin n\Omega_0 \tau\,d\tau + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

$\sin(-\theta) = -\sin\theta$

$$= -\frac{2}{T}\int_0^{T/2} x(-t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

Since τ is dummy variable, Let $\tau = t$.

Since $x(t)$ is odd, $x(-t) = -x(t)$

$$= \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt + \frac{2}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\,dt$$

The waveform of some odd periodic signals and their Fourier series are given below. Certain signals will become odd after subtraction of the dc component ($a_0/2$), such a signal waveform is shown in fig 4.13.

The waveform shown in fig 4.10 has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.10 is given by equation (4.23). [Please refer example 4.5 for derivation of Fourier series].

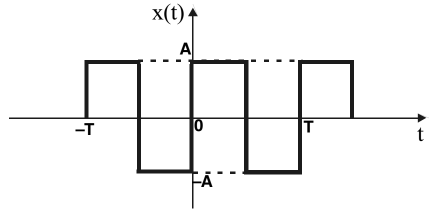


Fig 4.10.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{4A}{\pi}\left[\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \frac{\sin 7\Omega_0 t}{7} + ...\right] \qquad .....(4.23)$$

The waveform shown in fig 4.11 has odd symmetry, half wave symmetry and quarter wave symmetry. Hence for this waveform, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n. Therefore the Fourier series consists of odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.11 is given by equation (4.24). [Please refer example 4.6 for derivation of Fourier series].

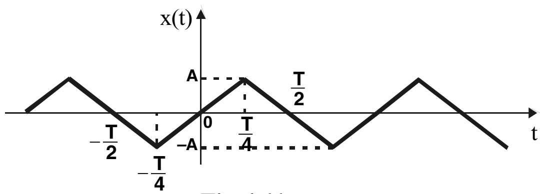


Fig 4.11.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{8A}{\pi^2}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + ...\right] \qquad .....(4.24)$$

The waveform shown in fig 4.12 has odd symmetry and so $a_0 = 0$, $a_n = 0$, and b_n exists for all values of n. Hence the Fourier series has both even and odd harmonics of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.12 is given by equation (4.25). [Please refer example 4.7 for derivation of Fourier series].

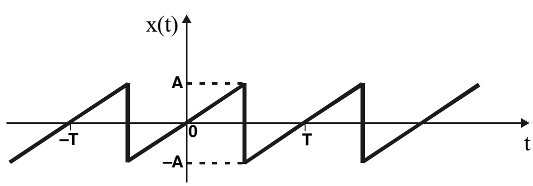


Fig 4.12.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{2A}{\pi}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} - \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} - .....\right] \qquad .....(4.25)$$

The waveform shown in fig 4.13 is neither even nor odd. But it can be shown that if the dc component ($a_0/2$) is subtracted from this waveform it becomes odd signal. Hence the Fourier coefficients $a_n = 0$ and b_n exists for all values of n. Therefore the Fourier series has a dc component and all harmonics (both even and odd harmonics) of sine terms. The trigonometric Fourier series representation of the waveform of fig 4.13 is given by equation (4.26). [Please refer example 4.8 for derivation of Fourier series].

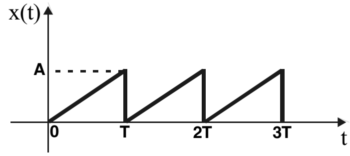


Fig 4.13.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} + \;.....\right] \quad .....(4.26)$$

4.4.3 Half Wave Symmetry (or Alternation Symmetry)

The periodic waveforms in which each period/cycle consists of two equal and opposite half period/cycle are called alternating waveforms, because this type of waveform will have alternate positive and negative half cycles. Such waveforms are said to have ***half wave symmetry*** or ***alternation symmetry***.

The waveforms with half wave symmetry will satisfy the condition,

$$\boxed{x\left(t \pm \frac{T}{2}\right) = -\,x(t)}$$

When a waveform has half wave symmetry, the Fourier series will consist of odd harmonic terms alone. The waveforms shown in fig 4.4, 4.10 and 4.11 exhibit half wave symmetry. Certain waveform will exhibit half wave symmetry after subtraction of the dc component ($a_0/2$), such waveforms are shown in fig 4.5, 4.8 and 4.9.

Some of the waveforms with only half wave symmetry and their Fourier series are given below.

The waveform shown in fig 4.14 has half wave symmetry. Hence the Fourier series consists of odd harmonic terms alone. The trigonometric Fourier series representation of the waveform of fig 4.14 is given by equation (4.27). [Please refer example 4.10 for the derivation of Fourier series].

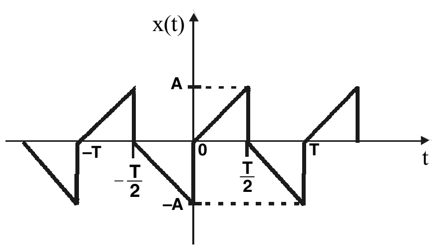


Fig 4.14.

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$x(t) = -\frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \;.....\right)$$
$$+ \frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \;.....\right) \quad .....(4.27)$$

The waveform shown in fig 4.15 has half wave symmemtry. Hence the Fourier series consists of odd harmonic terms alone. The trigonometric Fourier series representation of the waveform of fig 4.15 is given by equation (4.28).

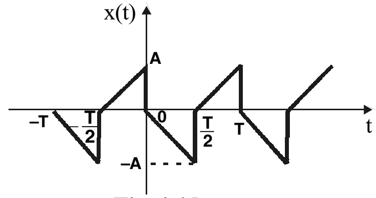


Fig 4.15.

$$x(t) = \frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right) \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$- \frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + ...\right) \qquad .....(4.28)$$

4.4.4 Quarter Wave Symmetry

A waveform with half wave symmetry if in addition has even/odd symmetry then it is said to have ***quarter wave symmetry***. In a waveform with quarter wave symmetry, each quarter period will have identical shape, but may have opposite sign. The existence of the type of Fourier coefficients for waveform with quarter wave symmetry is shown below.

$$x\left(t \pm \frac{T}{2}\right) = -x(t)$$

x(t) has half wave symmerty.
Fourier series has odd harmonic terms.

$$x(-t) = x(t) \text{ and } x\left(t \pm \frac{T}{2}\right) = -x(t)$$

x(t) has even and half wave symmetries.
(i.e., x(t) has quarter wave symmetry).
Fourier series will have odd harmonics of cosine terms.

$$x(-t) = -x(t) \text{ and } x\left(t \pm \frac{T}{2}\right) = -x(t)$$

x(t) has odd and half wave symmetries.
(i.e., x(t) has quarter wave symmetry).
Fourier series will have odd harmonics of sine terms.

The waveforms shown in fig 4.4, 4.10 and 4.11 has quarter wave symmetry. Certain waveforms will exhibit quarter wave symmetry after subtraction of dc component ($a_0/2$), such waveforms are shown in fig 4.5, 4.8 and 4.9.

4.5 Properties of Fourier Series

The properties of exponential form of Fourier series coefficients are listed in table 4.1. The proof of these properties are left as exercise to the readers.

Table 4.1 : Properties of Exponential Form of Fourier Series Coefficients

Note : *c_n and d_n are exponential form of Fourier series coefficients of x(t) and y(t) respectively.*

Property	Continuous time periodic signal	Fourier series coefficients
Linearity	$A\,x(t) + B\,y(t)$	$A\,c_n + B\,d_n$
Time shifting	$x(t - t_0)$	$c_n\, e^{-jn\Omega_0 t_0}$
Frequency shifting	$e^{-jm\Omega_0 t}\, x(t)$	c_{n-m}
Conjugation	$x^*(t)$	c^*_{-n}
Time reversal	$x(-t)$	c_{-n}
Time scaling	$x(\alpha t)$; $\alpha > 0$ (x(t) is period with period T/α)	c_n (No change in Fourier coefficient)
Multiplication	$x(t)\, y(t)$	$\sum\limits_{m=-\infty}^{+\infty} c_m\, d_{n-m}$
Differentiation	$\dfrac{d}{dt} x(t)$	$jn\Omega_0\, c_n$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (Finite valued and periodic only if $a_0 = 0$)	$\dfrac{1}{jn\Omega_0}\, c_n$
Periodic convolution	$\int_T x(\tau)\, y(t-\tau)\, d\tau$	$T\, c_n\, d_n$
Symmetry of real signals	x(t) is real	$c_n = c^*_{-n}$ $\lvert c_n \rvert = \lvert c_{-n} \rvert$; $\angle c_n = -\angle c_{-n}$ $Re\{c_n\} = Re\{c_{-n}\}$ $Im\{c_n\} = -Im\{c_{-n}\}$
Real and even	x(t) is real and even	c_n are real and even
Real and odd	x(t) is real and odd	c_n are imaginary and odd
Parseval's relation	Average power, P of x(t) is defined as, $P = \dfrac{1}{T}\int_T \lvert x(t) \rvert^2\, dt$	The average power, P in terms of Fourier series coefficients is, $P = \sum\limits_{n=-\infty}^{+\infty} \lvert c_n \rvert^2$
	$\dfrac{1}{T}\int_T \lvert x(t) \rvert^2\, dt = \sum\limits_{n=-\infty}^{+\infty} \lvert c_n \rvert^2$	

Note : 1. *The term $\lvert c_n \rvert^2$ respresent the power in n^{th} harmonic component of x(t). The total average power in a periodic signal is equal to the sum of power in all of its harmonics.*

2. *The term $\lvert c_n \rvert^2$ for n = 0, 1, 2, ... is the distribution of power as a function of frequency and so it is called* ***power density spectrum*** *or* ***power spectral density*** *of the periodic signal*

4.6 Diminishing of Fourier Coefficients

The basic four waveforms (parabolic, ramp, square and impulse) related through integration and differentiation are shown in figure 4.16.

The basic waveform shown in figure 4.16, are related through integration on going from bottom to top, and they are related through differentiation on going from top to bottom.

It can be stated that the Fourier coefficients of the signals will be proportional to $\frac{1}{n^k}$ where k is the number of times, the signal has to be differentiated to produce impulse.

If a waveform has square like structure with jump continuity (or discontinuity) then it has to be differentiated one time to produce impulse. In this case the Fourier coefficients a_n and b_n will be proportional to $\frac{1}{n}$. (Refer Fourier series of waveform shown in fig 4.4, 4.5 and 4.10).

If a waveform has ramp like structure (and without jump continuity) then it has to be differentiated twice to produce impulse. In this case the Fourier coefficient a_n and b_n will be proportional to $\frac{1}{n^2}$.(Refer Fourier series of waveform shown in fig 4.8, 4.9 and 4.11).

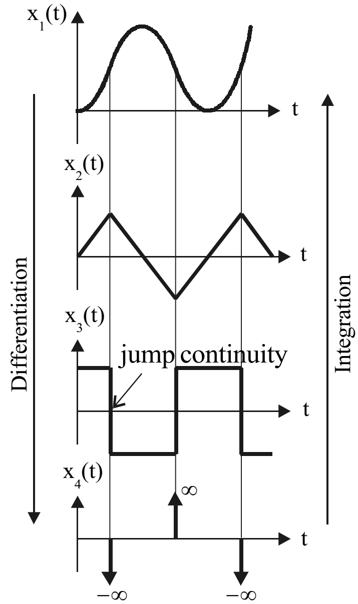


Fig 4.16 : Waveform related through integration and differentiation.

If a waveform has parabolic wiggles (and without jump continuity) then it has to be differentiated thrice to produce impulse. In this case the Fourier coefficient a_n and b_n will be proportional to $\frac{1}{n^3}$ .The above concepts are summarized in table 4.2.

Table : 4.2

Condition		Example	Proportionality for a_n and b_n
Jump in	**Impulse in**		
$x(t)$	$\frac{d}{dt}x(t)$	Square wave	$\frac{1}{n}$
$\frac{d}{dt}x(t)$	$\frac{d^2}{dt^2}x(t)$	Triangular wave	$\frac{1}{n^2}$
$\frac{d^2}{dt^2}x(t)$	$\frac{d^3}{dt^3}x(t)$	Parabolic wave	$\frac{1}{n^3}$
$\frac{d^{k-1}}{dt^{k-1}}x(t)$	$\frac{d^k}{dt^k}x(t)$	-	$\frac{1}{n^k}$

From the above discussion *we can say that the Fourier coefficients will decrease more rapidly with increasing n for a waveform with parabolic wiggles when compared to ramp or square wave* (i.e., the magnitudes of higher harmonics decreases more rapidly when the waveform has parabolic wiggles).

Similarly when we compare the waveform with ramp like structure and waveform with jumps like square wave, we can say that the magnitudes of higher harmonics (or Fourier coefficients) decreases more rapidly with increasing n in waveform with ramp like structure than the waveform with jumps.

4.7 Gibbs Phenomenon

The exponential form of Fourier series of a continuous time periodic signal x(t) is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t}$$

The above equation is frequency domain representation of the signal x(t) as a sum of infinite series with each term in the series representing a harmonic frequency component. When the signal x(t) is reconstructed or synthesised with only N number of terms of the infinite seires, the reconstructed signal exhibits oscillations (or overshoot or ripples), especially in signals with discontinuities.

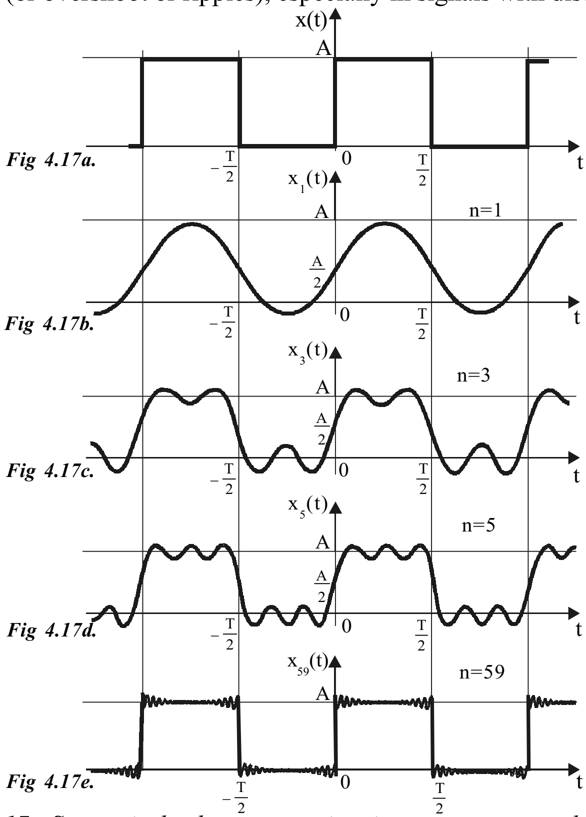


Fig 4.17 : *Successively closer approximations to a square pulse signal.*

Consider a periodic square pulse signal shown in fig 4.17a. The reconstructed signal using N-terms of Fourier series are shown in fig 4.17b, c, d and e. (Refer MATLAB program 4.5 in section 4.18). It can be observed that the reconstructed signal exhibits oscillations and the oscillations are compressed towards points of discontinuities with increasing value of N. Also it can be observed that, at the points of discontinuity, the Fourier series converges to average value of the signal on either side of discontinuity. This phenomenon was named after a famous mathematician, Josiah Gibbs, as ***Gibbs phenomenon*** and the oscillations are called ***Gibbs oscillations***. (Josiah Gibbs is a famous mathematician who first provided mathematical explanation for this phenomenon).

Also in reconstruction of signal with finite terms of Fourier series it can be observed that the peak overshoot of the oscillations remains constant irrespective of the value of N, but with increasing N the peak overshoot shifts towards the point of discontinuity.

4.8 Solved Problems in Fourier Series

Example 4.1

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.1.1.

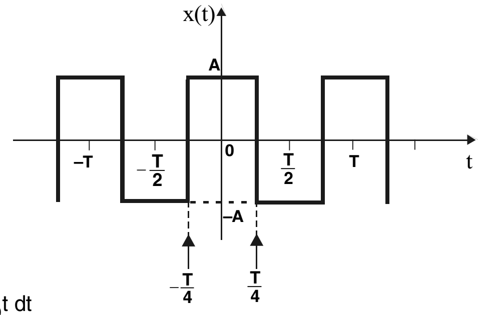


Fig 4.1.1.

Solution

The waveform shown in fig 4.1.1 has even symmetry, half wave symmetry and quarter wave symmetry.

$$\therefore\ a_0 = 0, \quad b_n = 0 \quad \text{and} \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of the square wave is,

$$x(t) = A \quad ; \quad \text{for} \quad t = 0 \text{ to } \frac{T}{4}$$
$$= -A \quad ; \quad \text{for} \quad t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/4} A\cos n\Omega_0 t\,dt + \frac{4}{T}\int_{T/4}^{T/2} (-A)\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/4}^{T/2} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} - \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_{T/4}^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right] - \frac{4A}{T}\left[\frac{\sin\left(n\frac{2\pi}{T}\frac{T}{2}\right)}{n\frac{2\pi}{T}} - \frac{\sin\left(n\frac{2\pi}{T}\frac{T}{4}\right)}{n\frac{2\pi}{T}}\right]$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi}\sin\frac{n\pi}{2} - 0\right] - \frac{4A}{T}\left[\frac{T}{2n\pi}\sin n\pi - \frac{T}{2n\pi}\sin\frac{n\pi}{2}\right] \qquad \boxed{\sin 0 = 0}$$

$$= \frac{2A}{n\pi}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\sin\frac{n\pi}{2} = \frac{4A}{n\pi}\sin\frac{n\pi}{2} \qquad \boxed{\begin{array}{l}\sin n\pi = 0 \\ \text{for integer } n\end{array}}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$$\therefore\ a_n = 0 \quad ; \quad \text{for even values of n}$$

$$a_n = \frac{4A}{n\pi}\sin\frac{n\pi}{2} \quad ; \quad \text{for odd values of n}$$

$$\therefore\ a_1 = \frac{4A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{4A}{\pi}$$

$$a_3 = \frac{4A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{4A}{3\pi}$$

$$a_5 = \frac{4A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{4A}{5\pi}$$

$$a_7 = \frac{4A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{4A}{7\pi} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore\ x(t) = \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= a_1 \cos\Omega_0 t + a_3 \cos 3\Omega_0 t + a_5 \cos 5\Omega_0 t + a_7 \cos 7\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\cos\Omega_0 t - \frac{4A}{3\pi}\cos 3\Omega_0 t + \frac{4A}{5\pi}\cos 5\Omega_0 t - \frac{4A}{7\pi}\cos 7\Omega_0 t + ...$$

$$= \frac{4A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + ...\right]$$

Example 4.2

Find the Fourier series of the waveform shown in fig 4.2.1.

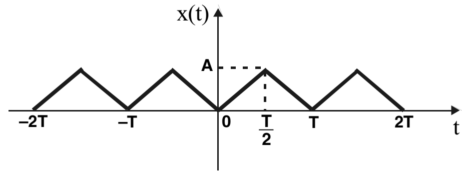


Fig 4.2.1.

Solution

The given waveform has even symmetry and so $b_n = 0$

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\, dt \quad ; \quad a_n = \frac{4}{T}\int_0^{T/2} x(t) \cos n\Omega_0 t\, dt \quad ; \quad b_n = 0$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

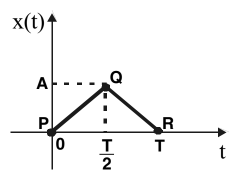


Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\dfrac{T}{2}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} \Rightarrow x(t) = \frac{2A}{T}t$$

$$\therefore\ x(t) = \frac{2A}{T}t \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\, dt = \frac{4}{T}\int_0^{T/2} \frac{2A}{T} t\, dt = \frac{8A}{T^2}\int_0^{T/2} t\, dt$$

$$= \frac{8A}{T^2}\left[\frac{t^2}{2}\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{T^2}{8} - 0\right] = A$$

Evaluation a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2}\frac{2A}{T}t\cos n\Omega_0 t\,dt = \frac{8A}{T^2}\int_0^{T/2} t\cos n\Omega_0 t\,dt$$

$$\boxed{\int uv = u\int v - \int\left[du\int v\right] \quad u = t \quad v = \cos\Omega_0 t}$$

$$= \frac{8A}{T^2}\left[t\frac{\sin n\Omega_0 t}{n\Omega_0} - \int 1\times\left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^{T/2}$$

$$= \frac{8A}{T^2}\left[\frac{t\sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} = \frac{8A}{T^2}\left[\frac{t\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{8A}{T^2}\left[\frac{T}{2}\frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} - \frac{0\times\sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$\boxed{\sin 0 = 0 \quad \cos 0 = 1}$$

$$= \frac{8A}{T^2}\left[\frac{T^2}{4n\pi}\sin n\pi + \frac{T^2}{4n^2\pi^2}\cos n\pi - \frac{T^2}{4n^2\pi^2}\right] = \frac{2A}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$\boxed{\sin n\pi = 0 \text{ for integer values of } n}$$

For even integer values of n, $\cos n\pi = +1$

For odd integer values of n, $\cos n\pi = -1$

$\therefore$ $a_n = 0$; for even values of n, and

$$a_n = \frac{2A}{n^2\pi^2}\left[\cos n\pi - 1\right] = -\frac{4A}{n^2\pi^2} \quad ; \text{ for odd values of n.}$$

$$\therefore a_1 = -\frac{4A}{1^2\pi^2} \quad ; \quad a_3 = -\frac{4A}{3^2\pi^2} \quad ; \quad a_5 = -\frac{4A}{5^2\pi^2} \quad ; \text{ and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for odd values of n.

$$\therefore x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n\cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + ...$$

$$= \frac{A}{2} - \frac{4A}{1^2\pi^2}\cos\Omega_0 t - \frac{4A}{3^2\pi^2}\cos 3\Omega_0 t - \frac{4A}{5^2\pi^2}\cos 5\Omega_0 t - ......$$

$$= \frac{A}{2} - \frac{4A}{\pi^2}\left[\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right]$$

Example 4.3

Determine the trigonometric form of Fourier series of the waveform shown in fig 4.3.1.

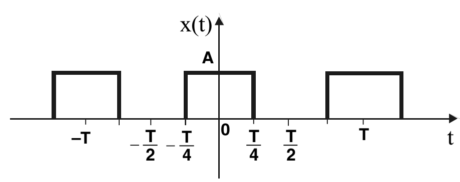


Fig 4.3.1.

Solution

The waveform of fig 4.3.1 has even symmetry.

$$\therefore b_n = 0, \quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \quad ; \quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of the given periodic rectangular pulse is,

$$x(t) = A \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{4}$$

$$= 0 \quad ; \quad \text{for } t = \frac{T}{4} \text{ to } \frac{T}{2}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/4} A\,dt = \frac{4}{T}\left[At\right]_0^{T/4}$$

$$= \frac{4}{T}\left[A\frac{T}{4} - 0\right] = A$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/4} A\cos\, n\Omega_0 t\,dt = \frac{4A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_0^{T/4}$$

$$= \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/4} = \frac{4A}{T}\left[\frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} - \frac{\sin 0}{n\frac{2\pi}{T}}\right]$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}} \quad \boxed{\sin 0 = 0}$$

$$= \frac{4A}{T} \times \frac{T}{2n\pi}\sin\frac{n\pi}{2} = \frac{2A}{n\pi}\sin\frac{n\pi}{2}$$

For even values of n, $\sin\frac{n\pi}{2} = 0$

For odd values of n, $\sin\frac{n\pi}{2} = \pm 1$

$\therefore\ a_n = 0\ ;$ for even values of n, and

$a_n = \frac{2A}{n\pi}\sin\frac{n\pi}{2}\ ;$ for odd values of n.

$$\therefore\ a_1 = \frac{2A}{1\times\pi}\sin\frac{\pi}{2} = +\frac{2A}{\pi}$$

$$a_3 = \frac{2A}{3\times\pi}\sin\frac{3\pi}{2} = -\frac{2A}{3\pi}$$

$$a_5 = \frac{2A}{5\times\pi}\sin\frac{5\pi}{2} = +\frac{2A}{5\pi}$$

$$a_7 = \frac{2A}{7\times\pi}\sin\frac{7\pi}{2} = -\frac{2A}{7\pi} \quad \text{and so on.}$$

Fourier Series

The trigonomteric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$ and a_n exists only for odd values of n.

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=\text{odd}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_1\cos\Omega_0 t + a_3\cos 3\Omega_0 t + a_5\cos 5\Omega_0 t + a_7\cos 7\Omega_0 t + \ldots..$$

$$= \frac{A}{2} + \frac{2A}{\pi}\cos\Omega_0 t - \frac{2A}{3\pi}\cos 3\Omega_0 t + \frac{2A}{5\pi}\cos 5\Omega_0 t - \frac{2A}{7\pi}\cos 7\Omega_0 t + \ldots..$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\cos\Omega_0 t - \frac{\cos 3\Omega_0 t}{3} + \frac{\cos 5\Omega_0 t}{5} - \frac{\cos 7\Omega_0 t}{7} + \ldots..\right]$$

Example 4.4

Determine the trigonometric form of Fourier series of the full wave rectified sine wave shown in fig 4.4.1.

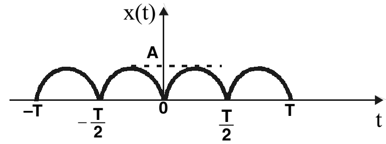


Fig 4.4.1.

Solution

The waveform shown in fig 4.4.1 is the output of full wave rectifier and it has even symmetry.

$$\therefore\ b_n = 0,\quad a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt \ ;\quad a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt$$

The mathematical equation of full wave rectified output is,

$$x(t) = A\sin\Omega_0 t \ ;\quad \text{for } t = 0 \text{ to } \frac{T}{2} \quad \text{and} \quad \Omega_0 = \frac{2\pi}{T}$$

Evaluation of a_0

$$a_0 = \frac{4}{T}\int_0^{T/2} x(t)\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\,dt = \frac{4A}{T}\left[-\frac{\cos\Omega_0 t}{\Omega_0}\right]_0^{T/2}$$

$$= \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}t}{\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[-\frac{\cos\frac{2\pi}{T}\frac{T}{2}}{\frac{2\pi}{T}} + \frac{\cos 0}{\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{\pi}\left[-\cos\pi + \cos 0\right] = \frac{2A}{\pi}\left[1+1\right] = \frac{4A}{\pi} \qquad \boxed{\cos\pi = -1 \quad \cos 0 = 1}$$

Evaluation of a_n

$$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\Omega_0 t\,dt = \frac{4}{T}\int_0^{T/2} A\sin\Omega_0 t\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T}\int_0^{T/2}\frac{\sin(\Omega_0 t + n\Omega_0 t) + \sin(\Omega_0 t - n\Omega_0 t)}{2}\,dt \qquad \boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{2A}{T}\int_0^{T/2}\sin(1+n)\Omega_0 t\,dt + \frac{2A}{T}\int_0^{T/2}\sin(1-n)\Omega_0 t\,dt$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\Omega_0 t}{(1+n)\Omega_0}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\Omega_0 t}{(1-n)\Omega_0}\right]_0^{T/2}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}}\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}}\right]_0^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} + \frac{\cos 0}{(1+n)\frac{2\pi}{T}}\right] + \frac{2A}{T}\left[\frac{-\cos(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} + \frac{\cos 0}{(1-n)\frac{2\pi}{T}}\right] \qquad \boxed{\cos 0 = 1}$$

$$= -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi} \qquad .....(1)$$

The equation (1) for a_n can be evaluated for all values of n except n = 1. For n = 1, a_n has to be estimated seperately as shown below.

$$a_1 = \frac{4}{T}\int_0^{T/2} x(t)\ \cos\Omega_0 t\ dt = \frac{4}{T}\int_0^{T/2} A\ \sin\Omega_0 t\ \cos\Omega_0 t\ dt$$

$$= \frac{4}{T}\int_0^{T/2} A\frac{\sin 2\Omega_0 t}{2}\ dt = \frac{2A}{T}\int_0^{T/2} \sin 2\Omega_0 t\ dt \qquad \boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$= \frac{2A}{T}\left[\frac{-\cos 2\Omega_0 t}{2\Omega_0}\right]_0^{T/2} = \frac{2A}{T}\left[\frac{-\cos\left(2\times\frac{2\pi}{T}\times\frac{T}{2}\right)}{2\times\frac{2\pi}{T}} + \frac{\cos 0}{2\times\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[-\frac{T}{4\pi}\cos 2\pi + \frac{T}{4\pi}\right] = \frac{2A}{T}\left[-\frac{T}{4\pi} + \frac{T}{4\pi}\right] = 0 \qquad \boxed{\cos 2\pi = \cos 0 = 1}$$

For values of n > 1, the a_n are calculated using equation (1) as shown below.

$$\therefore\ a_n = -\frac{A\cos(1+n)\pi}{(1+n)\pi} + \frac{A}{(1+n)\pi} - \frac{A\cos(1-n)\pi}{(1-n)\pi} + \frac{A}{(1-n)\pi}$$

When n is even integer, (1 + n) and (1 – n) will be odd, $\therefore \cos(1+n)\pi = -1$; $\cos(1-n)\pi = -1$

When n is odd integer, (1 + n) and (1 – n) will be even, $\therefore \cos(1+n)\pi = 1$; $\cos(1-n)\pi = 1$

$$\therefore\ a_n = 0\ ; \text{ for odd values of n}$$

$$a_n = \frac{A}{(1+n)\pi} + \frac{A}{(1+n)\pi} + \frac{A}{(1-n)\pi} + \frac{A}{(1-n)\pi}\ ; \text{ for even values of n}$$

$$= \frac{2A}{(1+n)\pi} + \frac{2A}{(1-n)\pi} = \frac{2A(1-n) + 2A(1+n)}{(1+n)(1-n)\pi} = \frac{4A}{(1-n^2)\pi}$$

$$\therefore\ a_2 = \frac{4A}{\left(1-2^2\right)\pi} = -\frac{4A}{3\pi}$$

$$a_4 = \frac{4A}{\left(1-4^2\right)\pi} = -\frac{4A}{15\pi}$$

$$a_6 = \frac{4A}{\left(1-6^2\right)\pi} = -\frac{4A}{35\pi}$$

$$a_8 = \frac{4A}{\left(1-8^2\right)\pi} = -\frac{4A}{63\pi} \quad \text{and so on}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $b_n = 0$, and a_n exists only for even values of n.

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=\text{even}} a_n \cos n\Omega_0 t$$

$$= \frac{a_0}{2} + a_2\cos 2\Omega_0 t + a_4\cos 4\Omega_0 t + a_6\cos 6\Omega_0 t + a_8\cos 8\Omega_0 t + ...$$

$$= \frac{2A}{\pi} - \frac{4A}{3\pi}\cos 2\Omega_0 t - \frac{4A}{15\pi}\cos 4\Omega_0 t - \frac{4A}{35\pi}\cos 6\Omega_0 t - \frac{4A}{63\pi}\cos 8\Omega_0 t - .....$$

$$= \frac{2A}{\pi} - \frac{4A}{\pi}\left[\frac{\cos 2\Omega_0 t}{3} + \frac{\cos 4\Omega_0 t}{15} + \frac{\cos 6\Omega_0 t}{35} + \frac{\cos 8\Omega_0 t}{63} + ...\right]$$

Example 4.5

Determine the Fourier series of the square wave shown in fig 4.5.1.

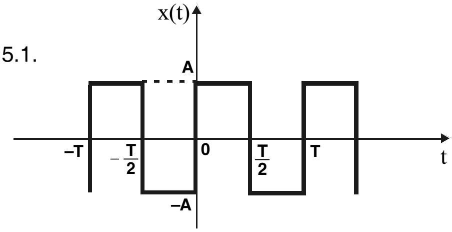


Fig 4.5.1.

Solution

The given waveform has odd symmetry, half-wave symmetry and quarter wave symmetry.

$$\therefore\ a_0 = 0, \quad a_n = 0, \quad b_n = \frac{4}{T}\int_0^{T/2} x(t)\ \sin n\Omega_0 t\ dt$$

The mathematical equation of the given waveform is,

$$x(t) = A \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A \quad ; \quad \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of b_n

$$\boxed{\Omega_0 = \frac{2\pi}{T}} \quad \boxed{\cos 0 = 1}$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\ \sin n\Omega_0 t\ dt = \frac{4}{T}\int_0^{T/2} A\ \sin n\Omega_0 t\ dt = \frac{4A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_0^{T/2}$$

$$= \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}}\right]_0^{T/2} = \frac{4A}{T}\left[\frac{-\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos 0}{n\frac{2\pi}{T}}\right] = \frac{4A}{T}\left[-\frac{T}{2n\pi}\cos n\pi + \frac{T}{2n\pi}\right]$$

$\cos n\pi = -1, \qquad$ for $n = \text{odd}$

$\cos n\pi = +1, \qquad$ for $n = \text{even}$

$$\therefore\ b_n = 0 \qquad ; \text{ for even values of n}$$

$$= \frac{4A}{T}\left[\frac{T}{2n\pi} + \frac{T}{2n\pi}\right] = \frac{4A}{n\pi}; \text{ for odd values of n}$$

$$\therefore\ b_1 = \frac{4A}{\pi} \quad ; \quad b_3 = \frac{4A}{3\pi} \quad ; \quad b_5 = \frac{4A}{5\pi} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore\ x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + \ldots..$$

$$= \frac{4A}{\pi}\sin\Omega_0 t + \frac{4A}{3\pi}\sin 3\Omega_0 t + \frac{4A}{5\pi}\sin 5\Omega_0 t + \ldots..$$

$$= \frac{4A}{\pi}\left[\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \ldots..\right]$$

Example 4.6

Determine the trigonometric form of Fourier series of the signal shown in fig 4.6.1.

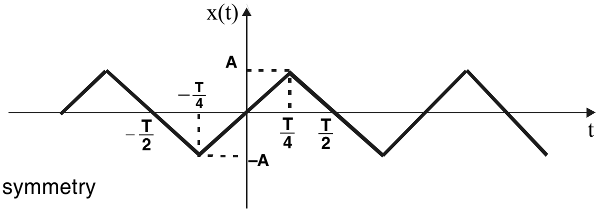


Fig 4.6.1.

Solution

The given signal has odd symmetry, half wave symmetry and quarter wave symmetry, and so $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\,\sin n\Omega_0 t\,dt \quad \text{(or)} \quad b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,\sin n\Omega_0 t\,dt$$

***Note :** Here x(t) is governed by single mathematical equation in the range $-\frac{T}{4}$ to $+\frac{T}{4}$. And so the calculations will be simple, if the integral limit is $-\frac{T}{4}$ to $+\frac{T}{4}$*

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

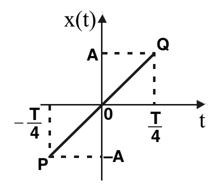


Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = \left[-\frac{T}{4}, -A\right]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\frac{T}{4}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - (-A)}{-A - A} = \frac{t - \left(-\frac{T}{4}\right)}{-\frac{T}{4} - \frac{T}{4}} \;\Rightarrow\; \frac{x(t) + A}{-2A} = \frac{t + \frac{T}{4}}{-\frac{T}{2}}$$

$$\Downarrow$$

$$-\frac{x(t)}{2A} - \frac{1}{2} = -\frac{2t}{T} - \frac{1}{2} \;\Rightarrow\; -\frac{x(t)}{2A} = -\frac{2t}{T} \;\Rightarrow\; x(t) = \frac{4A}{T}t$$

$$\therefore\; x(t) = \frac{4A}{T}t \quad ; \quad \text{for } t = -\frac{T}{4} \text{ to } +\frac{T}{4}$$

Evaluation of b_n

$$b_n = \frac{4}{T}\int_{-T/4}^{+T/4} x(t)\,\sin n\Omega_0 t\,dt = \frac{4}{T}\int_{-T/4}^{+T/4} \frac{4A}{T}t\,\sin n\Omega_0 t\,dt = \frac{16A}{T^2}\int_{-T/4}^{+T/4} t\,\sin n\Omega_0 t\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{16A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_{-T/4}^{+T/4}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{16A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_{-T/4}^{T/4} = \frac{16A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_{-T/4}^{+T/4}$$

$$= \frac{16A}{T^2}\left[-\frac{T}{4}\frac{\cos n\frac{2\pi}{T}\frac{T}{4}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{4}}{n^2\frac{4\pi^2}{T^2}} + \frac{T}{4}\frac{\cos n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\left(-\frac{T}{4}\right)}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$b_n = \frac{16A}{T^2}\left[-\frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2} + \frac{T^2}{8n\pi}\cos\frac{n\pi}{2} + \frac{T^2}{4n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= -\frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2A}{n\pi}\cos\frac{n\pi}{2} + \frac{4A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}$$

For odd integer values of n, $\sin\frac{n\pi}{2} = \pm 1$

For even integer values of n, $\sin\frac{n\pi}{2} = 0$

$$\therefore b_n = 0 \quad ; \text{ for even values of n}$$

$$= \frac{8A}{n^2\pi^2}\sin\frac{n\pi}{2}; \text{ for odd values of n}$$

$$\therefore\ b_1 = \frac{8A}{1^2\pi^2}\sin\frac{\pi}{2} = +\frac{8A}{\pi^2}$$

$$b_3 = \frac{8A}{3^2\pi^2}\sin\frac{3\pi}{2} = -\frac{8A}{3^2\pi^2}$$

$$b_5 = \frac{8A}{5^2\pi^2}\sin\frac{5\pi}{2} = +\frac{8A}{5^2\pi^2}$$

$$b_7 = \frac{8A}{7^2\pi^2}\sin\frac{7\pi}{2} = -\frac{8A}{7^2\pi^2} \quad \text{and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$ and b_n exists only for odd values of n.

$$\therefore\ x(t) = \sum_{n=\text{odd}} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + b_7 \sin 7\Omega_0 t + ...$$

$$= \frac{8A}{\pi^2}\sin\Omega_0 t - \frac{8A}{3^2\pi^2}\sin 3\Omega_0 t + \frac{8A}{5^2\pi^2}\sin 5\Omega_0 t - \frac{8A}{7^2\pi^2}\sin 7\Omega_0 t + ...$$

$$= \frac{8A}{\pi^2}\left[\sin\Omega_0 t - \frac{\sin 3\Omega_0 t}{3^2} + \frac{\sin 5\Omega_0 t}{5^2} - \frac{\sin 7\Omega_0 t}{7^2} + ...\right]$$

Example 4.7

Determine the trigonometric form of Fourier series for the signal shown in fig 4.7.1.

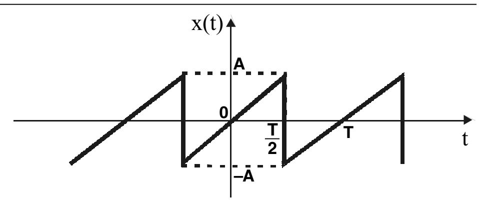


Fig 4.7.1.

Solution

The given signal has odd symmetry and so $a_0 = 0$, $a_n = 0$,

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\ \sin n\Omega_0 t\ dt$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\dfrac{T}{2}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-\frac{T}{2}} \Rightarrow x(t) = \frac{2At}{T}$$

$$\therefore x(t) = \frac{2At}{T} \quad ; \quad \text{for} \quad t = 0 \text{ to } \frac{T}{2}$$

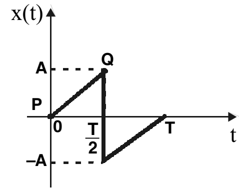


Fig 1.

Evaluation of b_n

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\Omega_0 t\, dt = \frac{4}{T}\int_0^{T/2} \frac{2At}{T}\sin n\Omega_0 t\, dt = \frac{8A}{T^2}\int_0^{T/2} t\sin n\Omega_0 t\, dt$$

$$= \frac{8A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) dt\right]_0^{T/2}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{8A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_0^{T/2} = \frac{8A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^{T/2}$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$= \frac{8A}{T^2}\left[-\frac{\frac{T}{2}\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} + \frac{0 \times \cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\boxed{\sin 0 = 0}$

$$= \frac{8A}{T^2}\left[-\frac{T^2}{4n\pi}\cos n\pi + \frac{T^2}{4n^2\pi^2}\sin n\pi\right] = -\frac{2A}{n\pi}\cos n\pi$$

$\boxed{\sin n\pi = 0 \text{ for integer } n}$

For even integer values of n, $\cos n\pi = +1$

For odd integer values of n, $\cos n\pi = -1$

$$\therefore b_n = -\frac{2A}{n\pi} \quad \text{for } n = \text{even}$$

$$= +\frac{2A}{n\pi} \quad \text{for } n = \text{odd}$$

$$\therefore b_1 = +\frac{2A}{\pi}\ ;\ b_2 = -\frac{2A}{2\pi}\ ;\ b_3 = +\frac{2A}{3\pi}\ ;\ b_4 = -\frac{2A}{4\pi}\ ;\ b_5 = +\frac{2A}{5\pi} \text{ and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, $a_0 = 0$, $a_n = 0$

$$\therefore\ x(t) = \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

$$= b_1 \sin\Omega_0 t + b_2 \sin 2\Omega_0 t + b_3 \sin 3\Omega_0 t + b_4 \sin 4\Omega_0 t + b_5 \sin 5\Omega_0 t + ...$$

$$= \frac{2A}{\pi}\sin\Omega_0 t - \frac{2A}{2\pi}\sin 2\Omega_0 t + \frac{2A}{3\pi}\sin 3\Omega_0 t - \frac{2A}{4\pi}\sin 4\Omega_0 t + \frac{2A}{5\pi}\sin 5\Omega_0 t - .....$$

$$= \frac{2A}{\pi}\left[\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} - \frac{\sin 4\Omega_0 t}{4} + \frac{\sin 5\Omega_0 t}{5} - .....\right]$$

Example 4.8

Determine the trigonometric form of the Fourier series of the ramp signal shown in fig 4.8.1.

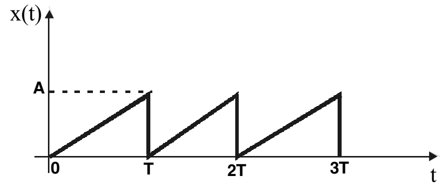


Fig 4.8.1.

Solution

The given signal is neither even nor odd.

$$\therefore\ a_0 = \frac{2}{T}\int_0^T x(t)\,dt \quad ; \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \quad ; \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

Note : *It can be shown that after subtracting $a_0/2$ from the signal, it becomes odd. Hence a_n will be equal to zero.*

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = [T, A]$

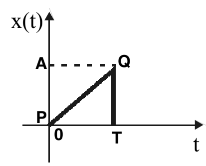


Fig 1.

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - T} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-T} \Rightarrow x(t) = \frac{At}{T}$$

$$\therefore\ x(t) = \frac{At}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

Evaluation of a_0

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^T \frac{A}{T}t\,dt = \frac{2A}{T^2}\int_0^T t\,dt$$

$$= \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{2A}{T^2}\left[\frac{T^2}{2} - 0\right] = A$$

Evaluation of a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_0^T \frac{At}{T}\cos n\Omega_0 t\,dt = \frac{2A}{T^2}\int_0^T t\cos n\Omega_0 t\,dt$$

$$= \frac{2A}{T^2}\left[t\left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^T$$

$\int uv = u\int v - \int \left[du \int v\right]$	
$u = t$	$v = \cos n\Omega_0 t$

$$a_n = \frac{2A}{T^2}\left[t\frac{\sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^T = \frac{2A}{T^2}\left[\frac{t\,\sin n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^T \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T^2}\left[\frac{T\,\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} - \frac{0\times\sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right] \qquad \boxed{\begin{array}{l}\sin 0 = 0\\ \cos 0 = 1\end{array}}$$

$$= \frac{2A}{T^2}\left[\frac{T^2}{2n\pi}\sin n2\pi + \frac{T^2}{4n^2\pi^2}\cos n2\pi - 0 - \frac{T^2}{4n^2\pi^2}\right] \qquad \boxed{\begin{array}{l}\sin n2\pi = 0\\ \cos n2\pi = 1\\ \text{for integer n}\end{array}}$$

$$= \frac{2A}{T^2}\left[0 + \frac{T^2}{4n^2\pi^2} - \frac{T^2}{4n^2\pi^2}\right] = 0$$

Evaluation of b_n

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt = \frac{2}{T}\int_0^T \frac{At}{T}\sin n\Omega_0 t\,dt = \frac{2A}{T^2}\int_0^T t\sin n\Omega_0 t\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{2A}{T^2}\left[t\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1\times\left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right)dt\right]_0^T$$

$$= \frac{2A}{T^2}\left[-t\frac{\cos n\Omega_0 t}{n\Omega_0} + \frac{\sin n\Omega_0 t}{n^2\Omega_0^2}\right]_0^T = \frac{2A}{T^2}\left[-t\frac{\cos n\frac{2\pi}{T}t}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}t}{n^2\frac{4\pi^2}{T^2}}\right]_0^T \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T^2}\left[-\frac{T\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} + \frac{0\times\cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right] \qquad \boxed{\sin 0 = 0}$$

$$= \frac{2A}{T^2}\left[-\frac{T^2}{2n\pi}\cos n2\pi + \frac{T^2}{4n^2\pi^2}\sin n2\pi\right] = -\frac{A}{n\pi} \qquad \boxed{\begin{array}{l}\sin n2\pi = 0\\ \cos n2\pi = 1\\ \text{for integer n}\end{array}}$$

$$\therefore\ b_1 = -\frac{A}{\pi}\ ;\ b_2 = -\frac{A}{2\pi}\ ;\ b_3 = -\frac{A}{3\pi}\ ;\ b_4 = -\frac{A}{4\pi}\ \text{ and so on.}$$

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n\sin n\Omega_0 t$$

Here, $a_n = 0$

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n\sin n\Omega_0 t$$

$$= \frac{a_0}{2} + b_1\sin\Omega_0 t + b_2\sin 2\Omega_0 t + b_3\sin 3\Omega_0 t + b_4\sin 4\Omega_0 t + ...$$

$$= \frac{A}{2} - \frac{A}{\pi}\sin\Omega_0 t - \frac{A}{2\pi}\sin 2\Omega_0 t - \frac{A}{3\pi}\sin 3\Omega_0 t - \frac{A}{4\pi}\sin 4\Omega_0 t - .....$$

$$= \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 4\Omega_0 t}{4} + ...\right]$$

Example 4.9

Determine the Fourier series representation of the half-wave rectifier output shown in fig 4.9.1.

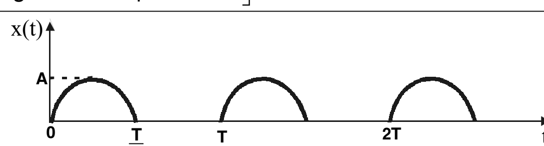


Fig 4.9.1.

Solution

The signal shown in fig 4.9.1 is neither even nor odd.

$$\therefore\ a_0 = \frac{2}{T}\int_0^T x(t)\,dt \quad ; \quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt \quad ; \quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\,dt$$

The mathematical equation representing half-wave rectified output is,

$$x(t) = A\sin\Omega_0 t \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= 0 \quad ; \quad \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of a_0

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\,dt = \frac{2A}{T}\left[\frac{-\cos\Omega_0 t}{\Omega_0}\right]_0^{T/2}$$

$$= \frac{2A}{T}\left[\frac{-\cos\frac{2\pi}{T}t}{\frac{2\pi}{T}}\right]_0^{T/2} = \frac{2A}{T}\left[\frac{-\cos\frac{2\pi}{T}\frac{T}{2}}{\frac{2\pi}{T}} + \frac{\cos 0}{\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{2A}{T}\left[-\frac{T}{2\pi}\cos\pi + \frac{T}{2\pi}\right] = \frac{2A}{T}\left[-\frac{T}{2\pi}\times(-1) + \frac{T}{2\pi}\right] \qquad \boxed{\begin{array}{l}\cos 0 = 1\\ \cos\pi = -1\end{array}}$$

$$= \frac{2A}{T}\times\frac{T}{\pi} = \frac{2A}{\pi}$$

Evaluation of a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\cos n\Omega_0 t\,dt \qquad \boxed{2\sin A\cos B = \sin(A+B) + \sin(A-B)}$$

$$= \frac{2A}{T}\int_0^{T/2} \frac{\sin(\Omega_0 t + n\Omega_0 t) + \sin(\Omega_0 t - n\Omega_0 t)}{2}\,dt = \frac{A}{T}\int_0^{T/2}\left[\sin(1+n)\Omega_0 t + \sin(1-n)\Omega_0 t\right]dt$$

$$= \frac{A}{T}\left[-\frac{\cos(1+n)\Omega_0 t}{(1+n)\Omega_0} - \frac{\cos(1-n)\Omega_0 t}{(1-n)\Omega_0}\right]_0^{T/2} = \frac{A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}} - \frac{\cos(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}}\right]_0^{T/2}$$

$$= \frac{A}{T}\left[\frac{-\cos(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} - \frac{\cos(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} + \frac{\cos 0}{(1+n)\frac{2\pi}{T}} + \frac{\cos 0}{(1-n)\frac{2\pi}{T}}\right] \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[-\frac{T\cos(1+n)\pi}{(1+n)2\pi} - \frac{T\cos(1-n)\pi}{(1-n)2\pi} + \frac{T}{(1+n)2\pi} + \frac{T}{(1-n)2\pi}\right] \qquad \boxed{\cos 0 = 1}$$

The above expression for a_n can be evaluated for all values of n except for n = 1. For n = 1, a_n has to be evaluated separately as shown below.

$$a_1 = \frac{2}{T}\int_0^T x(t)\cos\Omega_0 t\,dt = \frac{2}{T}\int_0^T A\sin\Omega_0 t\cos\Omega_0 t\,dt \qquad \boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$= \frac{2A}{T}\int_0^T \frac{\sin 2\Omega_0 t}{2}\,dt = \frac{A}{T}\left[\frac{-\cos 2\Omega_0 t}{2\Omega_0}\right]_0^T = \frac{A}{T}\left[\frac{-\cos\frac{4\pi}{T}t}{\frac{4\pi}{T}}\right]_0^T \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[-\frac{T}{4\pi}\cos\frac{4\pi T}{T} + \frac{T}{4\pi}\cos 0\right] = \frac{A}{T}\left[-\frac{T}{4\pi} + \frac{T}{4\pi}\right] = 0 \qquad \boxed{\begin{array}{l}\cos 4\pi = 1\\ \cos 0 = 1\end{array}}$$

$\therefore\ a_1 = 0$

$$a_n = -\frac{A\cos(1+n)\pi}{(1+n)2\pi} - \frac{A\cos(1-n)\pi}{(1-n)2\pi} + \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} \quad ; \text{ for all n except } n = 1$$

When n is even, the terms (n + 1) and (n – 1) are odd, $\therefore \cos(1+n)\pi = -1,\ \cos(1-n)\pi = -1$

When n is odd, the terms (n + 1) and (n – 1) are even, $\therefore \cos(1+n)\pi = 1,\ \cos(1-n)\pi = 1$

$\therefore\ a_n = 0$; for odd values of n

$$a_n = \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} + \frac{A}{(1+n)2\pi} + \frac{A}{(1-n)2\pi} \quad ; \text{ for even values of n}$$

$$= \frac{A}{(1+n)\pi} + \frac{A}{(1-n)\pi} = \frac{A(1-n) + A(1+n)}{(1+n)(1-n)\pi} = \frac{2A}{(1-n^2)\pi}$$

$$\therefore\ a_2 = \frac{2A}{(1-2^2)\pi} = -\frac{2A}{3\pi}$$

$$a_4 = \frac{2A}{(1-4^2)\pi} = -\frac{2A}{15\pi}$$

$$a_6 = \frac{2A}{(1-6^2)\pi} = -\frac{2A}{35\pi}$$

$$a_8 = \frac{2A}{(1-8^2)\pi} = -\frac{2A}{63\pi} \text{ and so on.}$$

Evaluation of b_n

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\ \sin n\Omega_0 t\, dt \qquad \boxed{2\sin A\sin B = \cos(A-B) - \cos(A+B)}$$

$$= \frac{2A}{T}\int_0^{T/2} \frac{\cos(\Omega_0 t - n\Omega_0 t) - \cos(\Omega_0 t + n\Omega_0 t)}{2}\, dt = \frac{A}{T}\int_0^{T/2}\left[\cos(1-n)\Omega_0 t - \cos(1+n)\Omega_0 t\right] dt$$

$$= \frac{A}{T}\left[\frac{\sin(1-n)\Omega_0 t}{(1-n)\Omega_0} - \frac{\sin(1+n)\Omega_0 t}{(1+n)\Omega_0}\right]_0^{T/2} = \frac{A}{T}\left[\frac{\sin(1-n)\frac{2\pi}{T}t}{(1-n)\frac{2\pi}{T}} - \frac{\sin(1+n)\frac{2\pi}{T}t}{(1+n)\frac{2\pi}{T}}\right]_0^{T/2} \qquad \boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[\frac{\sin(1-n)\frac{2\pi}{T}\frac{T}{2}}{(1-n)\frac{2\pi}{T}} - \frac{\sin(1+n)\frac{2\pi}{T}\frac{T}{2}}{(1+n)\frac{2\pi}{T}} - \frac{\sin 0}{(1-n)\frac{2\pi}{T}} + \frac{\sin 0}{(1+n)\frac{2\pi}{T}}\right] \qquad \boxed{\sin 0 = 0}$$

$$= \frac{A\sin(1-n)\pi}{(1-n)2\pi} - \frac{A\sin(1+n)\pi}{(1+n)2\pi}$$

The above expression for b_n can be evaluated for all values of n except for n = 1. For n = 1, b_n has to be evaluated separately as shown below.

$$b_1 = \frac{2}{T}\int_0^T x(t)\sin\Omega_0 t\, dt = \frac{2}{T}\int_0^{T/2} A\sin\Omega_0 t\sin\Omega_0 t\, dt = \frac{2A}{T}\int_0^{T/2}\sin^2\Omega_0 t\, dt \qquad \boxed{\sin^2\theta = \frac{1-\cos 2\theta}{2}}$$

$$= \frac{2A}{T}\int_0^{T/2}\frac{1-\cos 2\Omega_0 t}{2}\, dt = \frac{A}{T}\int_0^{T/2}(1-\cos 2\Omega_0 t)\, dt = \frac{A}{T}\left[t - \frac{\sin 2\Omega_0 t}{2\Omega_0}\right]_0^{T/2}$$

$$\therefore\ b_1 = \frac{A}{T}\left[t - \frac{\sin\frac{4\pi}{T}t}{\frac{4\pi}{T}}\right]_0^{T/2} = \frac{A}{T}\left[\frac{T}{2} - \frac{\sin\frac{4\pi}{T}\frac{T}{2}}{\frac{4\pi}{T}} - 0 + \frac{\sin 0}{\frac{4\pi}{T}}\right]$$

$$\boxed{\begin{array}{c}\Omega_0 = \dfrac{2\pi}{T}\\ \hline \sin 0 = 0\end{array}}$$

$$= \frac{A}{2} - \frac{A}{4\pi}\sin 2\pi = \frac{A}{2}$$

$$\boxed{\sin 2\pi = 0}$$

$$\therefore\ b_1 = \frac{A}{2}$$

$$b_n = \frac{A\sin(1-n)\pi}{(1-n)2\pi} - \frac{A\sin(1+n)\pi}{(1+n)2\pi}$$

For integer values of n, except when n = 1, $\sin(1-n)\pi = 0$.

For integer values of n, $\sin(1+n)\pi = 0$.

$\therefore$ $b_n = 0$ for all values of n except n = 1.

Fourier Series

The trigonometric form of Fourier series of x(t) is given by,

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here, a_n exists only for even values of n and $b_n = 0$ for all values of n except when n = 1.

$$\therefore\ x(t) = \frac{a_0}{2} + \sum_{n=\text{even}} a_n \cos n\Omega_0 t + b_1 \sin n\Omega_0 t$$

$$= \frac{a_0}{2} + a_2\cos 2\Omega_0 t + a_4 \cos 4\Omega_0 t + a_6\cos 6\Omega_0 t + a_8 \cos 8\Omega_0 t + \ldots\ldots + b_1 \sin\Omega_0 t$$

$$= \frac{A}{\pi} - \frac{2A}{3\pi}\cos 2\Omega_0 t - \frac{2A}{15\pi}\cos 4\Omega_0 t - \frac{2A}{35\pi}\cos 6\Omega_0 t - \frac{2A}{63\pi}\cos 8\Omega_0 t - \ldots\ldots + \frac{A}{2}\sin\Omega_0 t$$

$$= \frac{A}{\pi} + \frac{2A}{\pi}\left[\frac{\pi}{4}\sin\Omega_0 t - \frac{\cos 2\Omega_0 t}{3} - \frac{\cos 4\Omega_0 t}{15} - \frac{\cos 6\Omega_0 t}{35} - \frac{\cos 8\Omega_0 t}{63} - \ldots\ldots\right]$$

Example 4.10

Find the Fourier series of the signal shown in fig 4.10.1.

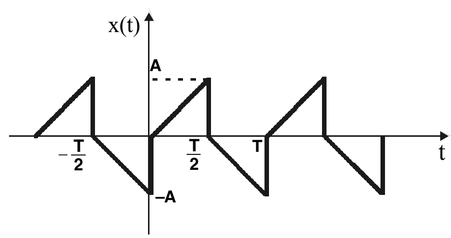


Fig 4.10.1.

Solution

The given signal has half-wave symmetry and so a_0 will be zero. The Fourier coefficients a_n and b_n will exist only for odd integer values of n.

$$\therefore\ a_0 = 0\ ;\quad a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\, dt\ ;\quad b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, y = x(t), x = t.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

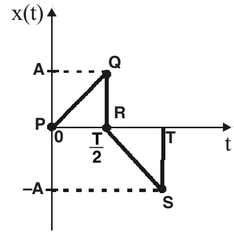


Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = \left[\dfrac{T}{2}, A\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{-A} = \frac{t}{-\frac{T}{2}} \Rightarrow x(t) = \frac{2At}{T}$$

Consider points R and S, as shown in fig 1.

Coordinates of point-R = $[t_3, x(t_3)] = \left[\dfrac{T}{2}, 0\right]$

Coordinates of point-S = $[t_4, x(t_4)] = [T, -A]$

On substituting the coordinates of points R and S in equation (1) we get,

$$\frac{x(t) - 0}{0 - (-A)} = \frac{t - \frac{T}{2}}{\frac{T}{2} - T} \Rightarrow \frac{x(t)}{A} = \frac{t - \frac{T}{2}}{-\frac{T}{2}} \Rightarrow \frac{x(t)}{A} = -\frac{2t}{T} + 1 \Rightarrow x(t) = A - \frac{2At}{T}$$

Now the mathematical equation of the waveform is given by,

$$x(t) = \frac{2At}{T} \quad ; \quad \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= A - \frac{2At}{T} \quad ; \quad \text{for } t = \frac{T}{2} \text{ to } T$$

Evaluation of a_0

The given signal has half wave symmetry and so $a_0 = 0$.

Proof :

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\left[\int_0^{T/2} \frac{2At}{T}\,dt + \int_{T/2}^{T}\left(A - \frac{2At}{T}\right)dt\right] = \frac{2}{T}\left[\left[\frac{2At^2}{2T}\right]_0^{T/2} + \left[At - \frac{2At^2}{2T}\right]_{T/2}^{T}\right]$$

$$= \frac{2}{T}\left[\frac{2AT^2}{8T} - 0 + AT - \frac{2AT^2}{2T} - \frac{AT}{2} + \frac{2AT^2}{8T}\right] = \frac{2}{T}\left[\frac{AT}{4} + AT - AT - \frac{AT}{2} + \frac{AT}{4}\right] = 0$$

Evaluation of a_n

The coefficient a_n for a signal with half-wave symmetry is,

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\,dt$$

$$\therefore a_n = \frac{2}{T}\int_0^{T/2} \frac{2At}{T}\cos n\Omega_0 t\,dt + \frac{2}{T}\int_{T/2}^{T}\left(A - \frac{2At}{T}\right)\cos n\Omega_0 t\,dt$$

$$= \frac{4A}{T^2}\int_0^{T/2} t\cos n\Omega_0 t\,dt + \frac{2A}{T}\int_{T/2}^{T}\cos n\Omega_0 t\,dt - \frac{4A}{T^2}\int_{T/2}^{T} t\cos n\Omega_0 t\,dt$$

$$\therefore\ a_n = \frac{4A}{T^2}\left[t \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) dt\right]_0^{T/2} + \frac{2A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[t \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{\sin n\Omega_0 t}{n\Omega_0}\right) dt\right]_{T/2}^{T}$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = \cos\Omega_0 t$

$$= \frac{4A}{T^2}\left[\frac{t \sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} + \frac{2A}{T}\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[\frac{t \sin n\Omega_0 t}{n\Omega_0} - \left(\frac{-\cos n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_{T/2}^{T}$$

$$= \frac{4A}{T^2}\left[\frac{\frac{T}{2}\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} - \frac{0 \times \sin 0}{n\frac{2\pi}{T}} - \frac{\cos 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\boxed{\Omega_0 = \frac{2\pi}{T}}$

$$+ \frac{2A}{T}\left[\frac{\sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}}\right]$$

$\sin 0 = 0$
$\cos 0 = 1$

$$- \frac{4A}{T^2}\left[\frac{T \sin n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} - \frac{\frac{T}{2} \times \sin\frac{n2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} - \frac{\cos\frac{n2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= \frac{A}{n\pi}\sin n\pi + \frac{A}{n^2\pi^2}\cos n\pi - \frac{A}{n^2\pi^2} + \frac{A}{n\pi}\sin n2\pi - \frac{A}{n\pi}\sin n\pi$$

$$- \frac{2A}{n\pi}\sin n2\pi - \frac{A}{n^2\pi^2}\cos n2\pi + \frac{A}{n\pi}\sin n\pi + \frac{A}{n^2\pi^2}\cos n\pi$$

For integer n
$\sin n\pi = 0$
$\sin n2\pi = 0$
$\cos n2\pi = 1$

$$= 0 + \frac{A}{n^2\pi^2}\cos n\pi - \frac{A}{n^2\pi^2} + 0 - 0 - 0 - \frac{A}{n^2\pi^2} + 0 + \frac{A}{n^2\pi^2}\cos n\pi$$

$$= \frac{2A}{n^2\pi^2}\cos n\pi - \frac{2A}{n^2\pi^2} = \frac{2A}{n^2\pi^2}(\cos n\pi - 1)$$

When n is even integer, $\cos n\pi = +1$
When n is odd integer, $\cos n\pi = -1$

$$\therefore\ a_n = 0 \quad ; \text{ for even integer values of n}$$

$$= -\frac{4A}{\pi^2 n^2} \quad ; \text{ for odd integer values of n}$$

$$\therefore\ a_1 = -\frac{4A}{1^2\pi^2} \quad ; \quad a_3 = -\frac{4A}{3^2\pi^2} \quad ; \quad a_5 = -\frac{4A}{5^2\pi^2} \quad \text{and so on.}$$

Evaluation of b_n

The coefficient b_n for a signal with half wave symmetry is,

$$b_n = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\, dt$$

$$\therefore\ b_n = \frac{2}{T}\int_0^{T/2}\frac{2At}{T}\sin n\Omega_0 t\, dt + \frac{2}{T}\int_{T/2}^{T}\left(A - \frac{2At}{T}\right)\sin n\Omega_0 t\, dt$$

$$= \frac{4A}{T^2}\int_0^{T/2} t\sin n\Omega_0 t\, dt + \frac{2A}{T}\int_{T/2}^{T}\sin n\Omega_0 t\, dt - \frac{4A}{T^2}\int_{T/2}^{T} t\sin n\Omega_0 t\, dt$$

$$\therefore\ b_n = \frac{4A}{T^2}\left[t \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) dt\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[t \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) - \int 1 \times \left(\frac{-\cos n\Omega_0 t}{n\Omega_0}\right) dt\right]_{T/2}^{T}$$

$\int uv = u\int v - \int \left[du \int v\right]$	
$u = t$	$v = \sin n\Omega_0 t$

$$= \frac{4A}{T^2}\left[\frac{-t\cos n\Omega_0 t}{n\Omega_0} - \left(\frac{-\sin n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_0^{T/2} + \frac{2A}{T}\left[\frac{-\cos n\Omega_0 t}{n\Omega_0}\right]_{T/2}^{T}$$

$$- \frac{4A}{T^2}\left[\frac{-t\cos n\Omega_0 t}{n\Omega_0} - \left(\frac{-\sin n\Omega_0 t}{n^2\Omega_0^2}\right)\right]_{T/2}^{T}$$

$$= \frac{4A}{T^2}\left[\frac{-\frac{T}{2}\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}} + \frac{0 \times \cos 0}{n\frac{2\pi}{T}} - \frac{\sin 0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$\Omega_0 = \frac{2\pi}{T}$
$\sin 0 = 0$

$$+ \frac{2A}{T}\left[-\frac{\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}}\right]$$

$$- \frac{4A}{T^2}\left[\frac{-T\cos n\frac{2\pi}{T}T}{n\frac{2\pi}{T}} + \frac{\sin n\frac{2\pi}{T}T}{n^2\frac{4\pi^2}{T^2}} + \frac{\frac{T}{2} \times \cos n\frac{2\pi}{T}\frac{T}{2}}{n\frac{2\pi}{T}} - \frac{\sin n\frac{2\pi}{T}\frac{T}{2}}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$= -\frac{A}{n\pi}\cos n\pi + \frac{A}{n^2\pi^2}\sin n\pi - \frac{A}{n\pi}\cos n2\pi + \frac{A}{n\pi}\cos n\pi + \frac{2A}{n\pi}\cos n2\pi$$

$$+ \frac{A}{n^2\pi^2}\sin n2\pi - \frac{A}{n\pi}\cos n\pi + \frac{A}{n^2\pi^2}\sin n\pi$$

For integer n
$\sin n\pi = 0$
$\sin n2\pi = 0$
$\cos n2\pi = 1$

$$= -\frac{A}{n\pi}\cos n\pi + 0 - \frac{A}{n\pi} + \frac{A}{n\pi}\cos n\pi + \frac{2A}{n\pi} + 0 - \frac{A}{n\pi}\cos n\pi + 0$$

$$= \frac{A}{n\pi} - \frac{A}{n\pi}\cos n\pi = \frac{A}{n\pi}(1 - \cos n\pi)$$

When n is even integer, $\cos n\pi = +1$

When n is odd integer, $\cos n\pi = -1$

$$\therefore\ b_n = 0 \quad ; \quad \text{for even integer values of n.}$$

$$= \frac{2A}{n\pi} \quad ; \quad \text{for odd integer values of n.}$$

$$\therefore\ b_1 = \frac{2A}{\pi} \quad ; \quad b_3 = \frac{2A}{3\pi} \quad ; \quad b_5 = \frac{2A}{5\pi} \quad \text{and so on.}$$

Fourier Series of x(t)

The Fourier series of x(t) is

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\Omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t$$

Here $a_0 = 0$ and the Fourier coefficients a_n and b_n exist only for odd values of n.

$$\therefore\ x(t) = \sum_{n = odd} a_n \cos n\Omega_0 t + \sum_{n = odd} b_n \sin n\Omega_0 t$$

$$= a_1 \cos\Omega_0 t + a_3 \cos 3\Omega_0 t + a_5 \cos 5\Omega_0 t + ...$$

$$+ b_1 \sin\Omega_0 t + b_3 \sin 3\Omega_0 t + b_5 \sin 5\Omega_0 t + ...$$

$$\therefore\ x(t) = -\frac{4A}{\pi^2}\cos\Omega_0 t - \frac{4A}{3^2\pi^2}\cos 3\Omega_0 t - \frac{4A}{5^2\pi^2}\cos 5\Omega_0 t - \ldots..$$
$$+\frac{2A}{\pi}\sin\Omega_0 t + \frac{2A}{3\pi}\sin 3\Omega_0 t + \frac{2A}{5\pi}\sin 5\Omega_0 t + \ldots..$$
$$= -\frac{4A}{\pi^2}\left(\cos\Omega_0 t + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + \ldots..\right)$$
$$+\frac{2A}{\pi}\left(\sin\Omega_0 t + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + \ldots..\right)$$

Example 4.11

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.11.1. Hence determine the trigonometric form of Fourier series.

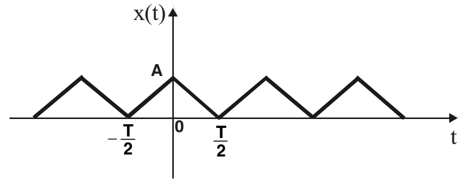


Fig 4.11.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P, Q and R as shown in fig 1.

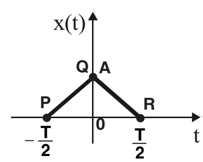


Fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = \left[\dfrac{-T}{2}, 0\right]$

Coordinates of point-Q = $[t_2, x(t_2)] = [0, A]$

Coordinates of point-R = $[t_3, x(t_3)] = \left[\dfrac{T}{2}, 0\right]$

On substituting the coordinates of points P and Q in equation (1) we get,

$$\frac{x(t) - 0}{0 - A} = \frac{t + \frac{T}{2}}{\frac{-T}{2} - 0} \Rightarrow \frac{x(t)}{-A} = \frac{-2t}{T} - 1 \Rightarrow x(t) = A + \frac{2At}{T}$$

On substituting the coordinates of points Q and R in equation (1) we get,

$$\frac{x(t) - A}{A - 0} = \frac{t - 0}{0 - \frac{T}{2}} \Rightarrow \frac{x(t)}{A} - 1 = \frac{-2t}{T} \Rightarrow x(t) = A - \frac{2At}{T}$$

$$\therefore\ x(t) = A + \frac{2At}{T}\ ;\ \text{for } t = -\frac{T}{2} \text{ to } 0$$
$$= A - \frac{2At}{T}\ ;\ \text{for } t = 0 \text{ to } \frac{T}{2}$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt$$

When $n = 0$, $c_0 = \dfrac{1}{T}\displaystyle\int_{-T/2}^{+T/2} x(t)\, e^0\, dt = \dfrac{1}{T}\int_{-T/2}^{+T/2} x(t)\, dt$

$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right)dt + \frac{1}{T}\int_{0}^{+T/2}\left(A - \frac{2At}{T}\right)dt$$

$$\therefore\ c_0 = \frac{A}{T}\int_{-T/2}^{0} dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\,dt + \frac{A}{T}\int_{0}^{T/2} dt - \frac{2A}{T^2}\int_{0}^{T/2} t\,dt$$

$$= \frac{A}{T}\left[t\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_{-T/2}^{0} + \frac{A}{T}\left[t\right]_{0}^{T/2} - \frac{2A}{T^2}\left[\frac{t^2}{2}\right]_{0}^{T/2}$$

$$= \frac{A}{T}\left[0 + \frac{T}{2}\right] + \frac{2A}{T^2}\left[0 - \frac{T^2}{8}\right] + \frac{A}{T}\left[\frac{T}{2} - 0\right] - \frac{2A}{T^2}\left[\frac{T^2}{8} - 0\right]$$

$$= \frac{A}{2} - \frac{A}{4} + \frac{A}{2} - \frac{A}{4} = \frac{2A}{2} - \frac{2A}{4} = A - \frac{A}{2} = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,e^{-jn\Omega_0 t}\,dt$$

$$= \frac{1}{T}\int_{-T/2}^{0}\left(A + \frac{2At}{T}\right)e^{-jn\Omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}\left(A - \frac{2At}{T}\right)e^{-jn\Omega_0 t}\,dt$$

$$= \frac{A}{T}\int_{-T/2}^{0} e^{-jn\Omega_0 t}\,dt + \frac{2A}{T^2}\int_{-T/2}^{0} t\,e^{-jn\Omega_0 t}\,dt + \frac{A}{T}\int_{0}^{T/2} e^{-jn\Omega_0 t}\,dt - \frac{2A}{T^2}\int_{0}^{T/2} t\,e^{-jn\Omega_0 t}\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = t$	$v = e^{-jn\Omega_0 t}$

$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1\times\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\,dt\right]_{-T/2}^{0}$$

$$+ \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{T/2} - \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1\times\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\,dt\right]_{0}^{T/2}$$

$$= \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-T/2}^{0} + \frac{2A}{T^2}\left[t\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{-T/2}^{0} + \frac{A}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{0}^{+T/2}$$

$$- \frac{2A}{T^2}\left[\frac{t\,e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_{0}^{T/2}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{A}{T}\left[\frac{e^0}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}}\right] + \frac{2A}{T^2}\left[\frac{0\times e^0}{-jn\frac{2\pi}{T}} - \frac{e^0}{-n^2\frac{4\pi^2}{T^2}} + \frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}\left(-\frac{T}{2}\right)}}{-n^2\frac{4\pi^2}{T^2}}\right]$$

$$+ \frac{A}{T}\left[\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^0}{-jn\frac{2\pi}{T}}\right] - \frac{2A}{T^2}\left[\frac{T}{2}\frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-jn\frac{2\pi}{T}} - \frac{e^{-jn\frac{2\pi}{T}\frac{T}{2}}}{-n^2\frac{4\pi^2}{T^2}} - \frac{0\times e^0}{-jn\frac{2\pi}{T}} + \frac{e^0}{-n^2\frac{4\pi^2}{T^2}}\right]$$

$$= -\frac{A}{j2n\pi} + \frac{A\,e^{jn\pi}}{j2n\pi} - 0 + \frac{A}{2n^2\pi^2} - \frac{A\,e^{jn\pi}}{j2n\pi} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{j2n\pi}$$

$$+ \frac{A}{j2n\pi} + \frac{A\,e^{-jn\pi}}{j2n\pi} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2} - 0 + \frac{A}{2n^2\pi^2}$$

$$= \frac{A}{n^2\pi^2} - \frac{A\,e^{jn\pi}}{2n^2\pi^2} - \frac{A\,e^{-jn\pi}}{2n^2\pi^2}$$

We know that,

$$e^{\pm jn\pi} = \cos n\pi \pm j\sin n\pi$$
$$= +1 \pm j0 = 1 \quad ; \quad \text{for even n}$$
$$= -1 \pm j0 = -1 \quad ; \quad \text{for odd n.}$$

$\therefore$ When n is even,

$$c_n = \frac{A}{n^2\pi^2} - \frac{A}{2n^2\pi^2} - \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} - \frac{A}{n^2\pi^2} = 0$$

$\therefore$ When n is odd,

$$c_n = \frac{A}{n^2\pi^2} + \frac{A}{2n^2\pi^2} + \frac{A}{2n^2\pi^2} = \frac{A}{n^2\pi^2} + \frac{A}{n^2\pi^2} = \frac{2A}{n^2\pi^2}$$

$$\therefore \; c_{-1} = \frac{2A}{(-1)^2\pi^2} = \frac{2A}{1^2\pi^2}$$
$$c_{-3} = \frac{2A}{(-3)^2\pi^2} = \frac{2A}{3^2\pi^2}$$
$$c_{-5} = \frac{2A}{(-5)^2\pi^2} = \frac{2A}{5^2\pi^2}$$

and so on

$$c_1 = \frac{2A}{1^2\pi^2}$$
$$c_3 = \frac{2A}{3^2\pi^2}$$
$$c_5 = \frac{2A}{5^2\pi^2}$$

and so on

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n\, e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n\, e^{jn\Omega_0 t}$$

Here c_n exist only for odd values of n.

$$\therefore \; x(t) = \sum_{\substack{n\,=\,\text{negative}\\ \text{odd integer}}} c_n\, e^{jn\Omega_0 t} + c_0 + \sum_{\substack{n\,=\,\text{positive}\\ \text{odd integer}}} c_n\, e^{jn\Omega_0 t}$$
$$= \;.....\; + c_{-5}\, e^{-j5\Omega_0 t} + c_{-3}\, e^{-j3\Omega_0 t} + c_{-1}\, e^{-j\Omega_0 t} + c_0 + c_1\, e^{j\Omega_0 t} + c_3\, e^{j3\Omega_0 t} + c_5\, e^{j5\Omega_0 t} + \;.....$$

$$x(t) = \;.....\; + \frac{2A}{5^2\pi^2} e^{-j5\Omega_0 t} + \frac{2A}{3^2\pi^2} e^{-j3\Omega_0 t} + \frac{2A}{1^2\pi^2} e^{-j\Omega_0 t} + \frac{A}{2} + \frac{2A}{1^2\pi^2} e^{j\Omega_0 t}$$
$$+ \frac{2A}{3^2\pi^2} e^{j3\Omega_0 t} + \frac{2A}{5^2\pi^2} e^{j5\Omega_0 t} + \;.....$$
$$= \frac{2A}{\pi^2}\left(..... + \frac{1}{5^2} e^{-j5\Omega_0 t} + \frac{1}{3^2} e^{-j3\Omega_0 t} + \frac{1}{1^2} e^{-j\Omega_0 t}\right) + \frac{A}{2}$$
$$+ \frac{2A}{\pi^2}\left(\frac{1}{1^2} e^{j\Omega_0 t} + \frac{1}{3^2} e^{j3\Omega_0 t} + \frac{1}{5^2} e^{j5\Omega_0 t} + ...\right)$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right) + \frac{1}{3^2}\left(e^{j3\Omega_0 t} + e^{-j3\Omega_0 t}\right) + \frac{1}{5^2}\left(e^{j5\Omega_0 t} + e^{-j5\Omega_0 t}\right) + ...\right]$$
$$= \frac{A}{2} + \frac{2A}{\pi^2}\left[\frac{1}{1^2}\, 2\cos\Omega_0 t + \frac{1}{3^2}\, 2\cos 3\Omega_0 t + \frac{1}{5^2}\, 2\cos 5\Omega_0 t + ...\right]$$
$$= \frac{A}{2} + \frac{4A}{\pi^2}\left[\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right]$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Example 4.12

Determine the exponential form of the Fourier series representation of the signal shown in fig 4.12.1. Hence determine the trigonometric form of Fourier series.

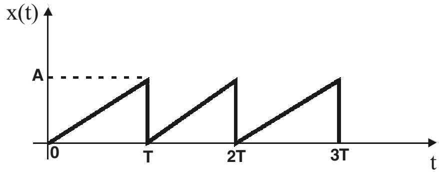


Fig 4.12.1.

Solution

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point-P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point-Q = $[t_2, x(t_2)] = [T, A]$

On substituting the coordinates of points P and Q in equation (1) we get,

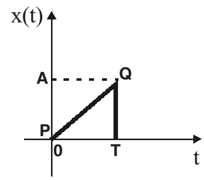


Fig 1.

$$\frac{x(t) - 0}{0 - A} = \frac{t - 0}{0 - T} \quad \Rightarrow \quad \frac{x(t)}{-A} = \frac{t}{-T} \quad \Rightarrow \quad x(t) = \frac{At}{T}$$

$$\therefore\ x(t) = \frac{At}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

Evaluation of c_0

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt$$

$$\text{When } n = 0,\ c_0 = \frac{1}{T}\int_0^T x(t)\, e^0\, dt = \frac{1}{T}\int_0^T x(t)\, dt$$

$$= \frac{1}{T}\int_0^T \frac{At}{T}\, dt = \frac{A}{T^2}\int_0^T t\, dt = \frac{A}{T^2}\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{T^2}{2} - 0\right] = \frac{A}{2}$$

Evaluation of c_n

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_0^T \frac{At}{T}\, e^{-jn\Omega_0 t}\, dt = \frac{A}{T^2}\int_0^T t\, e^{-jn\Omega_0 t}\, dt$$

$\int uv = u\int v - \int \left[du \int v\right]$	
$u = t$	$v = e^{-jn\Omega_0 t}$

$$= \frac{A}{T^2}\left[t\,\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0} - \int 1 \times \frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\, dt\right]_0^T = \frac{A}{T^2}\left[\frac{t\, e^{-jn\Omega_0 t}}{-jn\Omega_0} - \frac{e^{-jn\Omega_0 t}}{(-jn\Omega_0)^2}\right]_0^T$$

$$= \frac{A}{T^2}\left[\frac{t\, e^{-jn\frac{2\pi}{T}t}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}t}}{n^2\frac{4\pi^2}{T^2}}\right]_0^T = \frac{A}{T^2}\left[\frac{T\, e^{-jn\frac{2\pi}{T}T}}{-jn\frac{2\pi}{T}} + \frac{e^{-jn\frac{2\pi}{T}T}}{n^2\frac{4\pi^2}{T^2}} - 0 - \frac{e^0}{n^2\frac{4\pi^2}{T^2}}\right]$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= -\frac{A}{jn2\pi}\, e^{-jn2\pi} + \frac{A}{n^2 4\pi^2}\, e^{-jn2\pi} - \frac{A}{n^2 4\pi^2}$$

$$= -\frac{A}{jn2\pi} + \frac{A}{n^2 4\pi^2} - \frac{A}{n^2 4\pi^2} = -\frac{A}{jn2\pi}$$

$$\boxed{\begin{aligned} e^{-jn2\pi} &= \cos n2\pi - j\sin n2\pi \\ &= 1 - j0 = 1;\ \text{for integer } n \end{aligned}}$$

$$\therefore\ c_{-1} = \frac{A}{j2\pi} \qquad c_1 = -\frac{A}{j2\pi}$$

$$c_{-2} = \frac{A}{j4\pi} \qquad c_2 = -\frac{A}{j4\pi}$$

$$c_{-3} = \frac{A}{j6\pi} \qquad c_3 = -\frac{A}{j6\pi}$$

and so on. and so on.

Exponential Form of Fourier Series

The exponential form of Fourier series is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t} = \sum_{n=-\infty}^{-1} c_n\, e^{jn\Omega_0 t} + c_0 + \sum_{n=1}^{\infty} c_n\, e^{jn\Omega_0 t}$$

$$= \ldots\ldots c_{-3}\, e^{-j3\Omega_0 t} + c_{-2}\, e^{-j2\Omega_0 t} + c_{-1}\, e^{-j\Omega_0 t} + c_0 + c_1\, e^{j\Omega_0 t} + c_2\, e^{j2\Omega_0 t} + c_3\, e^{j3\Omega_0 t} + \ldots\ldots$$

$$= \ldots\ldots + \frac{A}{j6\pi} e^{-j3\Omega_0 t} + \frac{A}{j4\pi} e^{-j2\Omega_0 t} + \frac{A}{j2\pi} e^{-j\Omega_0 t} + \frac{A}{2} - \frac{A}{j2\pi} e^{j\Omega_0 t} - \frac{A}{j4\pi} e^{j2\Omega_0 t} - \frac{A}{j6\pi} e^{j3\Omega_0 t} \ldots\ldots$$

$$= \frac{A}{j2\pi}\left[\ldots\ldots + \frac{e^{-j3\Omega_0 t}}{3} + \frac{e^{-j2\Omega_0 t}}{2} + \frac{e^{-j\Omega_0 t}}{1}\right] + \frac{A}{2} - \frac{A}{j2\pi}\left[\frac{e^{j\Omega_0 t}}{1} + \frac{e^{j2\Omega_0 t}}{2} + \frac{e^{j3\Omega_0 t}}{3} + \ldots\ldots\right]$$

Trigonometric Form of Fourier Series

The trigonometric form of Fourier series can be obtained as shown below.

$$x(t) = \frac{A}{2} - \frac{A}{\pi}\left[\frac{1}{1}\left(\frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right) + \frac{1}{2}\left(\frac{e^{j2\Omega_0 t} - e^{-j2\Omega_0 t}}{2j}\right) + \frac{1}{3}\left(\frac{e^{j3\Omega_0 t} - e^{-j3\Omega_0 t}}{2j}\right) + \ldots\ldots\right]$$

$$= \frac{A}{2} - \frac{A}{\pi}\left[\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + \ldots\ldots\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

4.9 Fourier Transform

4.9.1 Development of Fourier Transform From Fourier Series

The exponential form of Fourier series representation of a periodic signal is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} \qquad \ldots..(4.29)$$

$$\text{where, } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \qquad \ldots..(4.30)$$

In the Fourier representation using equation (4.29), the c_n for various values of n are the spectral components of the signal x(t), located at intervals of fundamental frequency Ω_0. Therefore the frequency spectrum is discrete in nature.

The Fourier representation of a signal using equation (4.29) is applicable for periodic signals. For Fourier representation of non-periodic signals, let us consider that the fundamental period tends to infinity. When the fundamental period tends to infinity, the fundamental frequency Ω_0 tends to zero or becomes very small. Since fundamental frequency Ω_0 is very small, the spectral components will lie very close to each other and so the frequency spectrum becomes continuous.

In order to obtain the Fourier representation of a non-periodic signal let us consider that the fundamental frequency Ω_0 is very small.

Let, $\Omega_0 \rightarrow \Delta\Omega$

On replacing Ω_0 by $\Delta\Omega$ in equation (4.29) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Delta\Omega t}$$

On substituting for c_n in the above equation from equation (4.30) (by taking τ as dummy variable for integration) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} \qquad .....(4.31)$$

We know that, $\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}$; $\quad \therefore \dfrac{1}{T} = \dfrac{\Omega_0}{2\pi}$

Since $\Omega_0 \rightarrow \Delta\Omega$, $\quad \dfrac{1}{T} = \dfrac{\Delta\Omega}{2\pi}$ $\qquad .....(4.32)$

On substituting for $\dfrac{1}{T}$ from equation (4.32) in equation (4.31) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\Omega}{2\pi} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

For non-periodic signals, the fundamental period T tends to infinity. On letting limit T tends to infinity in the above equation we get,

$$x(t) = \underset{T\rightarrow\infty}{Lt}\ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

When $T \rightarrow \infty$; $\quad \sum \rightarrow \int$; $\quad \Delta\Omega \rightarrow \Omega$

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)\, e^{-jn\Omega\tau}\, d\tau \right] e^{jn\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{jn\Omega t}\, d\Omega \qquad .....(4.33)$$

$\boxed{\text{Since } \tau \text{ is a dummy variable, Let } \tau = t.}$

$$\text{where, } X(j\Omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-jn\Omega\tau}\, d\tau = \int_{-\infty}^{+\infty} x(t)\, e^{-jn\Omega t}\, dt \qquad .....(4.34)$$

The equation (4.34) is Fourier transform of x(t) and equation (4.33) is inverse Fourier transform of x(t).

Since the equation (4.34) extracts the frequency components of the signal, transformation using equation (4.34) is also called ***analysis*** of the signal x(t). Since the equation (4.33) combines the frequency components of the signal, the inverse transformation using equation (4.33) is also called ***synthesis*** of the signal x(t).

Definition of Fourier Transform

Let, x(t) = Continuous time signal

X(jΩ) = Fourier transform of x(t)

The Fourier transform of continuous time signal, x(t) is defined as,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

Also, X(jΩ) is denoted as $\mathcal{F}\{x(t)\}$ where "$\mathcal{F}$" is the symbol used to denote the Fourier transform operation.

$$\boxed{\therefore\ \mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt} \qquad .....(4.35)$$

> **Note :** *Sometimes the Fourier transform is expressed as a function of cyclic frequency F, rather than radian frequency Ω. The Fourier transform as a function of cyclic frequency F, is defined as,*
>
> $$X(jF) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

Condition for Existence of Fourier Transform

The Fourier transform of x(t) exists if it satisfies the following Dirichlet condition.

1. The x(t) be absolutely integrable.

   i.e., $\int_{-\infty}^{+\infty} x(t)\, dt < \infty$

2. The x(t) should have a finite number of maxima and minima within any finite interval.
3. The x(t) can have a finite number of discontinuities within any interval.

Definition of Inverse Fourier Transform

The ***inverse Fourier transform*** of X(jΩ) is defined as,

$$\boxed{x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega} \qquad .....(4.36)$$

The signals x(t) and X(jΩ) are called ***Fourier transform pair*** and can be expressed as shown below,

$$\boxed{x(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\rightleftarrows}} X(j\Omega)}$$

> **Note :** *When Fourier transform is expressed as a function of cyclic frequency F, the inverse Fourier transform is defined as,*
>
> $$x(t) = F^{-1}\{X(jF)\} = \int_{-\infty}^{+\infty} X(jF)\, e^{j2\pi Ft}\, dF$$

4.9.2 Frequency Spectrum Using Fourier Transform

The X(jΩ) is a complex function of Ω. Hence it can be expressed as a sum of real part and imaginary part as shown below.

$\therefore\ X(j\Omega) = X_r(j\Omega) + jX_i(j\Omega)$

where, $X_r(j\Omega)$ = Real part of $X(j\Omega)$

$X_i(j\Omega)$ = Imaginary part of $X(j\Omega)$

The magnitude of $X(j\Omega)$ is called ***Magnitude spectrum.***

$$\boxed{\therefore\ \text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X_r^2(j\Omega) + X_i^2(j\Omega)}} \quad .....(4.37)$$

(or)

$$\text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X(j\Omega)\ X^*(j\Omega)} \quad .....(4.38)$$

where, $X^*(j\Omega)$ = Conjugate of $X(j\Omega)$

The phase of $X(j\Omega)$ is called ***Phase spectrum***.

$$\boxed{\therefore\ \text{Phase spectrum, } \angle X(j\Omega) = \tan^{-1}\frac{X_i(j\Omega)}{X_r(j\Omega)}} \quad .....(4.39)$$

The magnitude spectrum will always have even symmetry and phase spectrum will have odd symmetry. The magnitude and phase spectrum together called ***frequency spectrum***.

4.10 Properties of Fourier Transform

1. Linearity

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

The linearity property of Fourier transform says that,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X_1(j\Omega) = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \quad \text{and} \quad X_2(j\Omega) = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt \quad .....(4.40)$$

Consider the linear combination $a_1\, x_1(t) + a_2\, x_2(t)$. On taking Fourier transform of this signal we get,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = \int_{-\infty}^{+\infty} [a_1\, x_1(t) + a_2\, x_2(t)]\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} a_1\, x_1(t)\, e^{-j\Omega t}\, dt + \int_{-\infty}^{+\infty} a_2\, x_2(t)\, e^{-j\Omega t}\, dt$$

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt$$

Using equation (4.40)

$$= a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

2. Time shifting

The time shifting property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(t - t_0)\} = e^{-j\Omega_0 t}\, X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.41)$$

$$\therefore \mathcal{F}\{x(t - t_0)\} = \int_{-\infty}^{+\infty} x(t - t_0)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega(\tau + t_0)}\, d\tau$$

Let, $t - t_0 = \tau$
$\therefore t = \tau + t_0$
On differentiating
$dt = d\tau$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau} \times e^{-j\Omega t_0}\, d\tau = e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau$$

Since τ is a dummy variable for ingetration we can change τ to t.

$$= e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = e^{-j\Omega t_0}\, X(j\Omega)$$

Using equation (4.41)

3. Time scaling

The time scaling property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{j\Omega}{a}\right)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore \mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\left(\frac{\tau}{a}\right)}\, \frac{d\tau}{a}$$

Put, $at = \tau$; $\therefore t = \frac{\tau}{a}$; $dt = \frac{d\tau}{a}$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = \frac{1}{a}\, X\left(\frac{j\Omega}{a}\right)$$

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau$ is similar to the form of Fourier transform except that Ω is replaced by $\left(\frac{\Omega}{a}\right)$.

$$\therefore \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = X\left(\frac{j\Omega}{a}\right)$$

The above transform is applicable for positive values of "a".

If "a" happens to be negative then it can be proved that,

$$\mathcal{F}\{x(at)\} = -\frac{1}{a}\, X\left(\frac{j\Omega}{a}\right)$$

Hence in general,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{j\Omega}{a}\right) \quad \text{for both positive and negative values of "a"}$$

4. Time reversal

The time reversal property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(-t)\} = X(-j\Omega)$$

Proof :

From time scaling property we know that,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{j\Omega}{a}\right)$$

Let, a = –1.

$$\therefore \mathcal{F}\{x(-t)\} = X(-j\Omega)$$

5. Conjugation

The conjugation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x^*(t)\} = X^*(-j\Omega)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\{x^*(t)\} = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\Omega t}\, dt$$

$$= \left[\int_{-\infty}^{+\infty} x(t)\, e^{j\Omega t}\, dt\right]^* = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt\right]^*$$

$$= [X(-j\Omega)]^* = X^*(-j\Omega)$$

The term, $\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $-\Omega$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t} dt = X(-j\Omega)$$

6. Frequency shifting

The frequency shifting property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{e^{j\Omega_0 t}\, x(t)\} = X(j(\Omega - \Omega_0))$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\{e^{j\Omega_0 t}\, x(t)\} = \int_{-\infty}^{+\infty} \left[e^{j\Omega_0 t}\, x(t)\right] e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(t) e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega - \Omega_0)t}\, dt = X\big(j(\Omega - \Omega_0)\big)$$

The term $\int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega-\Omega_0)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega - \Omega_0$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-j(\Omega - \Omega_0)t} dt = X\big(j(\Omega - \Omega_0)\big)$$

7. Time differentiation

The differentiation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = j\Omega\, X(j\Omega)$$

Proof :

Consider the definition of Fourier transform of x(t).

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.42)$$

$$\therefore\ \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \int_{-\infty}^{+\infty} \left(\frac{d}{dt}x(t)\right) e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} e^{-j\Omega t}\left(\frac{d}{dt}x(t)\right) dt$$

$$\therefore\ \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \left[e^{-j\Omega t}\ x(t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty}(-j\Omega)\,e^{-j\Omega t}\,x(t)\,dt$$

$\boxed{\int uv = u\int v - \int\left[du\int v\right]}$

$$= e^{-\infty}\,x(\infty) - e^{+\infty}\,x(-\infty) + j\Omega\int_{-\infty}^{+\infty}x(t)\,e^{-j\Omega t}\,dt$$

$\boxed{\begin{array}{c} x(-\infty) = 0 \\ e^{-\infty} = 0 \end{array}}$

$$= j\Omega\int_{-\infty}^{+\infty}x(t)\,e^{-j\Omega t}\,dt = j\Omega\,X(j\Omega)$$

Using equation (4.42)

8. Time integration

The integration property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ and $X(0) = 0$ then

$$\mathcal{F}\left\{\int_{-\infty}^{t}x(\tau)\,d\tau\right\} = \frac{1}{j\Omega}X(j\Omega)$$

Proof :

Consider a continuous time signal x(t). Let X(jΩ) be Fourier transform of x(t). Since integration and differentiation are inverse operations, x(t) can be expressed as shown below.

$$\frac{d}{dt}\left[\int_{-\infty}^{t}x(\tau)\,d\tau\right] = x(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\left\{\frac{d}{dt}\left[\int_{-\infty}^{t}x(\tau)\,d\tau\right]\right\} = \mathcal{F}\{x(t)\}$$

$$j\Omega\ \mathcal{F}\left\{\int_{-\infty}^{t}x(\tau)\,d\tau\right\} = \mathcal{F}\{x(t)\}$$

Using time differentiation property of Fourier transform.

$$\therefore\ \mathcal{F}\left\{\int_{-\infty}^{t}x(\tau)\,d\tau\right\} = \frac{1}{j\Omega}X(j\Omega)$$

$\boxed{\mathcal{F}\{x(t)\} = X(j\Omega)}$

9. Frequency differentiation

The frequency differentiation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$, then

$$\mathcal{F}\{t\,x(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty}x(t)\,e^{-j\Omega t}\,dt$$

On differentiating the above equation with respect to Ω we get,

$$\frac{d}{d\Omega}X(j\Omega) = \frac{d}{d\Omega}\left(\int_{-\infty}^{+\infty}x(t)\,e^{-j\Omega t}\,dt\right)$$

$$= \int_{-\infty}^{+\infty}x(t)\left(\frac{d}{d\Omega}e^{-j\Omega t}\right)dt$$

Interchanging the order of integration and differentiation

$$\therefore \frac{d}{d\Omega}X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\left(-jt\, e^{-j\Omega t}\right) dt = \frac{1}{j}\int_{-\infty}^{+\infty} (t\, x(t))\, e^{-j\Omega t}\, dt$$

$$-j = -j \times \frac{j}{j} = \frac{1}{j}$$

$$= \frac{1}{j}\mathcal{F}\{t\, x(t)\}$$

Using definition of Fourier transform.

$$\therefore \mathcal{F}\{t\, x(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

10. Convolution theorem

The convolution theorem of Fourier transform says that, Fourier transform of convolution of two signals is given by the product of the Fourier transform of the individual signals.

i.e., if $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$ then,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\, X_2(j\Omega) \qquad .....(4.43)$$

The equation (4.43) is also known as convolution property of Fourier transform.

With reference to chapter-2, section -2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \qquad .....(4.44)$$

where τ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals. Now, by definition of Fourier transform,

$$X_1(j\Omega) = \mathcal{F}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \qquad .....(4.45)$$

$$X_2(j\Omega) = \mathcal{F}\{x_2(t\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt \qquad .....(4.46)$$

Using definition of Fourier transform we can write,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty}\left[x_1(t) * x_2(t\right] e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-j\Omega t}\, dt \qquad .....(4.47)$$

Let, $e^{-j\Omega t} = e^{j\Omega\tau} \times e^{-j\Omega\tau} \times e^{-j\Omega t} = e^{-j\Omega\tau} \times e^{-j\Omega(t-\tau)} = e^{-j\Omega\tau} \times e^{-j\Omega M}$ .....(4.48)

where, $M = t - \tau$ and so, $dM = dt$ .....(4.49)

Using equations (4.48) and (4.49), the equation (4.47) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(\tau)\, x_2(M)\, e^{-j\Omega\tau}\, e^{-j\Omega M}\, d\tau\, dM$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-j\Omega\tau}\, d\tau \times \int_{-\infty}^{+\infty} x_2(M)\, e^{-j\Omega M}\, dM \qquad .....(4.50)$$

In equation (4.50), τ and M are dummy variables used for integration, and so they can be changed to t.

Therefore equation (4.50) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \times \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt$$

$$= X_1(j\Omega)\, X_2(j\Omega)$$

Using equations (4.45) and (4.46)

11. Frequency convolution

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$.

The frequency convolution property of Fourier transform says that,

$$\mathcal{F}\{x_1(t)\,x_2(t)\} = \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\,X_2(j(\Omega-\lambda))\,d\lambda$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt$$

$$\therefore\ \mathcal{F}\{x_1(t)x_2(t)\} = \int_{t=-\infty}^{t=+\infty} x_1(t)x_2(t)\,e^{-j\Omega t}\,dt \qquad .....(4.51)$$

By the definition of inverse Fourier transform we get,

$$x_1(t) = \mathcal{F}^{-1}\{X_1(j\Omega)\} = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X_1(j\Omega)\,e^{j\Omega t}\,d\Omega = \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\,e^{j\lambda t}\,d\lambda \qquad .....(4.52)$$

Here Ω is the vairable used for ingetration. Let us change Ω to λ.

On substituting for $X_1(t)$ from equation (4.52) in equation (4.51) we get,

$$\mathcal{F}\{x_1(t)x_2(t)\} = \int_{t=-\infty}^{t=+\infty}\left[\frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\,e^{j\lambda t}\,d\lambda\right] x_2(t)\,e^{-j\Omega t}\,dt$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\,e^{-j\Omega t}\,e^{j\lambda t}\,dt\right]d\lambda$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\,e^{-j(\Omega-\lambda)t}\,dt\right]d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\,X_2(j(\Omega-\lambda))\,d\lambda$$

The term, $\int_{t=-\infty}^{t=+\infty} x_2(t)\,e^{-j(\Omega-\lambda)t}\,dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega-\lambda$.

$$\therefore\ \int_{t=-\infty}^{t=+\infty} x_2(t)\,e^{-j(\Omega-\lambda)t}dt = X_2(j(\Omega-\lambda))$$

12. Parseval's relation

The Parseval's relation says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\Omega)|^2\,d\Omega$$

Proof :

Let $x(t)$ be a continuous time signal and $x^*(t)$ be conjugate of $x(t)$.

Now, $|x(t)|^2 = x(t)\,x^*(t)$

On integrating the above equation with respect to t we get,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2\,dt = \int_{t=-\infty}^{t=+\infty} x(t)\,x^*(t)\,dt \qquad .....(4.53)$$

By definition of inverse Fourier transform, we can write,

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X(j\Omega)\,e^{j\Omega t}\,d\Omega$$

On taking conjugate of the above equation we get,

$$x^*(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega \qquad .....(4.54)$$

Using equation (4.54) the equation (4.53) can be written as,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2\, dt = \int_{t=-\infty}^{t=+\infty} x(t)\left[\frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega\right] dt$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\left[\int_{t=-\infty}^{t=+\infty} x(t)\, e^{-j\Omega t}\, dt\right] d\Omega$$

Using definition of Fourier transform.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, X(j\Omega)\, d\Omega$$

$X(j\Omega)\, X^*(j\Omega) = |X(j\Omega)|^2$

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} |X(j\Omega)|^2\, d\Omega$$

Note : *The term $|X(j\Omega)|^2$ represents the distribution of energy as function of Ω and so it is called* ***energy density spectrum*** *or* ***energy spectral density*** *of the signal x(t).*

13. Duality

If $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

and if $x_2(t) \equiv X_1(j\Omega)$, i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions

then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$, i.e., $X_2(j\Omega)$ are $2\pi x_2(-j\Omega)$ are similar functions

Alternatively duality property is expressed as shown below.

If $\quad x_2(t) \Leftrightarrow X_1(j\Omega)$

then $X_2(j\Omega) \Leftrightarrow 2\pi x_1(-j\Omega)$

Proof :

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Let, $x_2(t)$ and $X_1(j\Omega)$ are similar in form.

$$\therefore\ x_2(t) = X_1(j\Omega)\big|_{j\Omega=t} \qquad .....(4.55)$$

By definition of inverse Fourier transform,

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\therefore\ x_1(-t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega)\, e^{-j\Omega t}\, d\Omega$$

Replacing t by –t

$$\therefore\ x_1(-t)\big|_{t=j\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(X_1(j\Omega)\big|_{j\Omega=t}\right) e^{-j\Omega t}\, d\Omega$$

interchanging jΩ and t

$$\therefore\ x_1(-j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, d\Omega$$

Using equation (4.55)

$$\therefore\ \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, d\Omega = 2\pi\, x_1(-j\Omega)$$

$$\therefore\ X_2(j\Omega) = 2\pi\, x_1(-j\Omega)$$

Using definition of Fourier transform

Note : For even function $x_1(-j\Omega) = x_1(j\Omega)$.

$$\therefore\ X_2(j\Omega) = 2\pi x_1(j\Omega)$$

14. Area under a time domain signal

$$\text{Area under } x(t) = \int_{-\infty}^{+\infty} x(t)\, dt$$

If $x(t)$ and $X(j\Omega)$ are Fourier transform pair,

$$\text{then, } \int_{-\infty}^{+\infty} x(t)\, dt = X(0)$$

$$\text{where, } X(0) = \underset{j\Omega \to 0}{\text{Lt}} X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore X(0) = \underset{j\Omega\to 0}{\text{Lt}} X(j\Omega) = \underset{j\Omega\to 0}{\text{Lt}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{0}\, dt = \int_{-\infty}^{+\infty} x(t)\, dt$$

$$\therefore \int_{-\infty}^{+\infty} x(t)\, dt = X(0)$$

15. Area under a frequency domain signal

$$\text{Area under } X(j\Omega) = \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$$

If $x(t)$ and $X(j\Omega)$ are Fourier transform pair,

$$\text{then, } \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$$

$$\text{where, } x(0) = \underset{t \to 0}{\text{Lt}}\, x(t)$$

Proof :

By definition of inverse Fourier transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\therefore x(0) = \underset{t\to 0}{\text{Lt}}\, x(t) = \underset{t\to 0}{\text{Lt}}\, \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{0}\, d\Omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$$

$$\therefore \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$$

Table 4.3 : Summary of Properties of Fourier Transform

Let, $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Property	Time domain signal	Frequency domain signal
Linearity	$a_1\,x_1(t) + a_2\,x_2(t)$	$a_1\,X_1(j\Omega) + a_2\,X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0}\,X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}\,X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t}\,x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt}\,x(t)$	$j\Omega\;X(j\Omega)$
Time integration	$\int\limits_{-\infty}^{t} x(\tau)\;d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi\,X(0)\;\delta(\Omega)$
Frequency differentiation	$t\;x(t)$	$j\frac{d}{d\Omega}X(j\Omega)$
Time convolution	$x_1(t) * x_2(t) = \int\limits_{-\infty}^{+\infty} x_1(\tau)\;x_2(t-\tau)d\tau$	$X_1(j\Omega)\;X_2(j\Omega)$
Frequency convolution (or Multiplication)	$x_1(t)\;x_2(t)$	$\frac{1}{2\pi}\int\limits_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\;X_2(j(\Omega-\lambda))\;d\lambda$
Symmetry of real signals	$x(t)$ is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega)\rvert = \lvert X(-j\Omega)\rvert$; $\angle X(j\Omega) = -\angle X(-j\Omega)$ $Re\{X(j\Omega)\} = Re\{X(-j\Omega)\}$ $Im\{X(j\Omega)\} = -Im\{X(-j\Omega)\}$
Real and even	$x(t)$ is real and even	$X(j\Omega)$ are real and even
Real and odd	$x(t)$ is real and odd	$X(j\Omega)$ are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e.,$X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under a frequency domain signal	$\int\limits_{-\infty}^{+\infty} X(j\Omega)\;d\Omega = 2\pi\;x(0)$	
Area under a time domain signal	$\int\limits_{-\infty}^{+\infty} x(t)\;dt = X(0)$	
Parseval's relation	Energy in time domain is, $E = \int\limits_{-\infty}^{+\infty} \lvert x(t)\rvert^2\,dt$	Energy in frequency domain is, $E = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\,d\Omega$
	$\int\limits_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\;d\Omega$	

4.11 Fourier Transform of Some Important Signals

Fourier Transform of Unit Impulse Signal

The impulse signal is defined as,

$$x(t) = \delta(t) = \infty \quad ; \quad t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0 \quad ; \quad t \neq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\Omega t}\, dt$$

$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1 \qquad \boxed{\delta(t) \text{ exists only for } t = 0}$$

$$\therefore \boxed{\mathcal{F}\{x(t)\} = 1}$$

The plot of impulse signal and its magnitude spectrum are shown in fig 4.18 and fig 4.19 respectively.

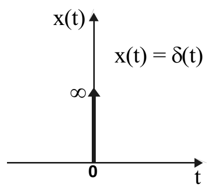


Fig 4.18 : Impulse signal.

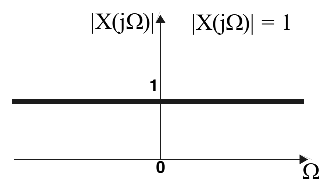


Fig 4.19 : Magnitude spectrum of impulse signal.

Fourier Transform of Single Sided Exponential Signal

The single sided exponential signal is defined as,

$$x(t) = A\, e^{-at} \quad ; \quad \text{for } t \geq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{0}^{+\infty} A\, e^{-at}\, e^{-j\Omega t}\, dt$$

$$= \int_{0}^{+\infty} A\, e^{-(a+j\Omega)t}\, dt = \left[\frac{A\, e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{+\infty}$$

$$= \left[\frac{A\, e^{-\infty}}{-(a+j\Omega)} - \frac{A\, e^{0}}{-(a+j\Omega)}\right] = \frac{A}{a+j\Omega} \qquad \boxed{e^{-\infty} = 0}$$

$$\therefore \boxed{\mathcal{F}\left\{A\, e^{-at}\, u(t)\right\} = \frac{A}{a+j\Omega}} \qquad .....(4.56)$$

The plot of exponential signal and its magnitude spectrum are shown in fig 4.20 and fig 4.21 respectively.

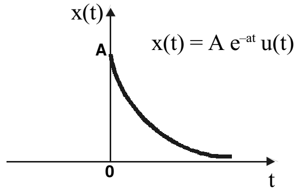


Fig 4.20: *Single sided exponential signal.*

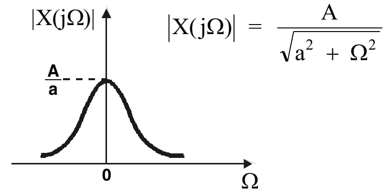


Fig 4.21 : *Magnitude spectrum of single sided exponential signal.*

Fourier Transform of Double Sided Exponential Signal

The double sided exponential signal is defined as,

$$x(t) = A\,e^{-a|t|} \quad ; \text{ for all } t$$

$$\therefore \quad x(t) = A\,e^{+at} \quad ; \text{ for } t = -\infty \text{ to } 0$$

$$= A\,e^{-at} \quad ; \text{ for } t = 0 \text{ to } +\infty$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{0} A\,e^{at}\,e^{-j\Omega t}\,dt + \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{0} A\,e^{(a-j\Omega)t}\,dt + \int_{0}^{\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{(a-j\Omega)t}}{a-j\Omega}\right]_{-\infty}^{0} + \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

$$= \frac{A\,e^{0}}{a-j\Omega} - \frac{A\,e^{-\infty}}{a-j\Omega} + \frac{A\,e^{-\infty}}{-(a+j\Omega)} - \frac{A\,e^{0}}{-(a+j\Omega)} = \frac{A}{a-j\Omega} + \frac{A}{a+j\Omega}$$

$$= \frac{A(a+j\Omega) + A(a-j\Omega)}{(a-j\Omega)\,(a+j\Omega)} = \frac{2aA}{a^2+\Omega^2}$$

$\boxed{e^{-\infty} = 0}$

$\boxed{(a+b)(a-b) = a^2-b^2 \quad | \quad j^2 = -1}$

$$\therefore \quad \boxed{\mathcal{F}\left\{A\,e^{-a|t|}\right\} = \frac{2aA}{a^2+\Omega^2}} \qquad .....(4.57)$$

The plot of double sided exponential signal and its magnitude spectrum are shown in fig 4.22 and fig 4.23 respectively.

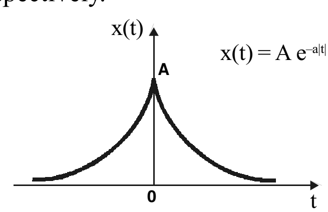


Fig 4.22 : *Double sided exponential signal.*

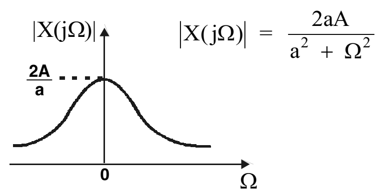


Fig 4.23 : *Magnitude spectrum of double sided exponential signal.*

Fourier Transform of a Constant

Let, x(t) = A, where A is a constant.

If definition of Fourier transform is directly applied, the constant will not satisfy the condition,

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

Hence the constant can be viewed as a double sided exponential with limit "a" tends to 0 as shown below.

Let $x_1(t)$ = Double sided exponential signal.

The double sided exponential signal is defined as,

$$x_1(t) = A\, e^{-a|t|}$$

$$\text{i.e,} \quad x_1(t) = A\, e^{at} \quad ; \quad \text{for } t = -\infty \text{ to } 0$$
$$= A\, e^{-at} \quad ; \quad \text{for } t = 0 \text{ to } +\infty$$

$$\therefore\ x(t) = A \underset{a \to 0}{\text{Lt}}\ x_1(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\underset{a \to 0}{\text{Lt}}\ x_1(t)\right\}$$

$$\mathcal{F}\{x(t)\} = \underset{a \to 0}{\text{Lt}}\ \mathcal{F}\{x_1(t)\}$$

$$X(j\Omega) = \underset{a \to 0}{\text{Lt}}\ \left[X_1(j\Omega)\right] \qquad \boxed{\mathcal{F}\{x(t)\} = X(j\Omega) \quad \mathcal{F}\{x_1(t)\} = X_1(j\Omega)}$$

$$= \underset{a \to 0}{\text{Lt}}\ \frac{2aA}{\Omega^2 + a^2} \qquad \boxed{\text{Using equation (4.57)}}$$

The above equation is 0 for all values of Ω except at Ω = 0.

At Ω = 0, the above equation represents an impulse of magnitude "k".

$$\therefore\ X(j\Omega) = k\,\delta(\Omega) \quad ; \quad \Omega = 0$$
$$= 0 \quad ; \quad \Omega \neq 0$$

The magnitude "k" can be evaluated as shown below.

$$k = \int_{-\infty}^{+\infty} \frac{2aA}{\Omega^2 + a^2}\, d\Omega = 2aA \int_{-\infty}^{+\infty} \frac{1}{\Omega^2 + a^2}\, d\Omega \qquad \boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a}}$$

$$= 2aA\left[\frac{1}{a}\tan^{-1}\left(\frac{\Omega}{a}\right)\right]_{-\infty}^{+\infty} = 2aA\left[\frac{1}{a}\tan^{-1}(+\infty) - \frac{1}{a}\tan^{-1}(-\infty)\right]$$

$$= 2aA\left[\frac{1}{a}\,\frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\therefore\ \boxed{\mathcal{F}\{A\} = 2\pi A\,\delta(\Omega)} \qquad .....(4.58)$$

The plot of constant and its magnitude spectrum are shown in fig 4.24 and fig 4.25 respectively.

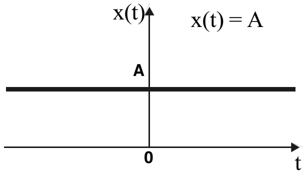


Fig 4.24 : *Constant.*

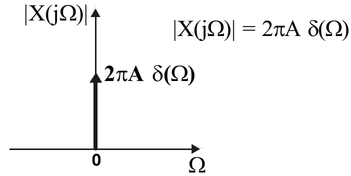


Fig 4.25 : *Magnitude spectrum of constant.*

Fourier Transform of Signum Function

The signum function is defined as,

$$x(t) = \mathrm{sgn}(t) = \begin{cases} 1 & ; \quad t > 0 \\ -1 & ; \quad t < 0 \end{cases}$$

The signum function can be expressed as a sum of two one sided exponential signal and taking limit "a" tends to 0 as shown below.

$$\therefore \ \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\, u(t) - e^{at}\, u(-t)\right]$$

$$\therefore \quad x(t) = \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\, u(t) - e^{at}\, u(-t)\right]$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at}\, u(t) - e^{at}\, u(-t)\right] e^{-j\Omega t}\, dt$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-at}\, e^{-j\Omega t}\, dt - \int_{-\infty}^{0} e^{at}\, e^{-j\Omega t}\, dt\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-(a + j\Omega)t}\, dt - \int_{-\infty}^{0} e^{+(a - j\Omega)t}\, dt\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\left[\frac{e^{-(a + j\Omega)t}}{-(a + j\Omega)}\right]_{0}^{\infty} - \left[\frac{e^{(a - j\Omega)t}}{(a - j\Omega)}\right]_{-\infty}^{0}\right] \qquad \boxed{e^0 = 1 \ ; \ e^{-\infty} = 0}$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{e^{-\infty}}{-(a + j\Omega)} - \frac{e^{0}}{-(a + j\Omega)} - \frac{e^{0}}{a - j\Omega} + \frac{e^{-\infty}}{a - j\Omega}\right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{1}{a + j\Omega} - \frac{1}{a - j\Omega}\right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

$$\therefore \boxed{\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\Omega}} \qquad .....(4.59)$$

The plot of signum function and its magnitude spectrum are shown in fig 4.26 and fig 4.27 respectively.

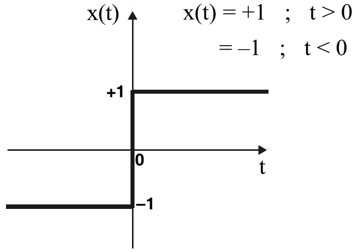


Fig 4.26 : *Signum function.*

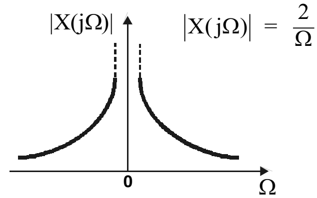


Fig 4.27 : *Magnitude spectrum of signum function.*

Fourier Transform of Unit Step Signal

The unit step signal is defined as,

$$u(t) = 1 \quad ; \quad t \geq 0$$
$$\quad = 0 \quad ; \quad t < 0$$

If can be proved that, $\text{sgn}(t) = 2u(t) - 1 \quad \Rightarrow \quad u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$

$$\therefore \; x(t) = u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{1}{2}\left[1 + \text{sgn}(t)\right]\right\}$$

$$\therefore \; X(j\Omega) = \mathcal{F}\left\{\frac{1}{2}\right\} + \mathcal{F}\left\{\frac{1}{2}\,\text{sgn}(t)\right\} = \frac{1}{2}\,\mathcal{F}\{1\} + \frac{1}{2}\,\mathcal{F}\{\text{sgn}(t)\}$$

$$= \frac{1}{2}\left[2\pi\,\delta(\Omega)\right] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] = \pi\,\delta(\Omega) + \frac{1}{j\Omega}$$

Using equations (4.58) and (4.59)

$$\therefore \; \boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(\Omega) + \frac{1}{j\Omega}} \qquad .....(4.60)$$

The plot of unit step signal and its magnitude spectrum are shown in fig 4.28 and fig 4.29 respectively.

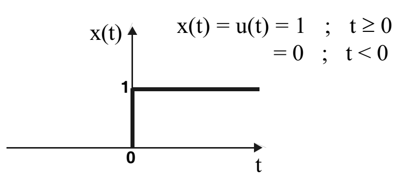


Fig 4.28 : *Unit step signal.*

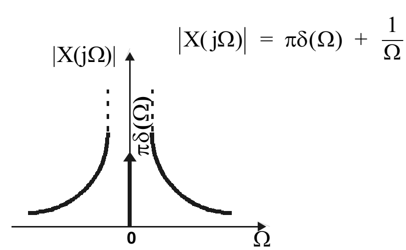


Fig 4.29 : *Magnitude spectrum of unit step signal.*

Fourier Transform of Complex Exponential Signal

The complex exponential signal is defined as,

$$x(t) = A\,e^{j\Omega_0 t}$$
$$= e^{j\Omega_0 t}\,A$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{e^{j\Omega_0 t}\,A\right\}$$

$$= \mathcal{F}\{A\}\big|_{\Omega = \Omega - \Omega_0}$$ Using frequency shifting property

$$= 2\pi\,\delta(\Omega)\big|_{\Omega = \Omega - \Omega_0}$$ Using equation (4.46)

$$= 2\pi\,\delta(\Omega - \Omega_0)$$

Frequency shifting property.
If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then,
$\mathcal{F}\left\{e^{j\Omega_0 t}\,x(t)\right\} = X(j(\Omega - \Omega_0))$

$$\therefore \boxed{\mathcal{F}\left\{A\,e^{j\Omega_0 t}\right\} = 2\pi A\,\delta(\Omega - \Omega_0)} \qquad .....(4.61)$$

$$\text{Similarly,} \quad \boxed{\mathcal{F}\left\{e^{-j\Omega_0 t}\,A\right\} = 2\pi A\,\delta(\Omega + \Omega_0)} \qquad .....(4.62)$$

The signal $A\,e^{-j\Omega_0 t}$ can be represented by a rotating vector of magnitude, "A", in clockwise direction in a complex plane with an angular speed of $\Omega_0 t$ as shown in fig.4.30 . The magnitude spectrum of $Ae^{-j\Omega_0 t}$ is shown in fig. 4.31.

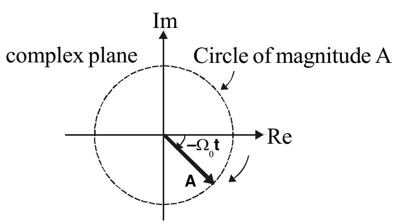


Fig 4.30 : *Complex exponential signal.*

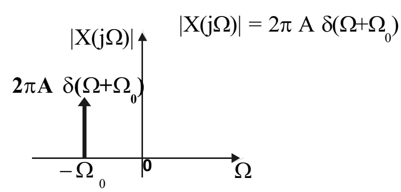


Fig 4.31 : *Magnitude spectrum of* $Ae^{-j\Omega_0 t}$.

The signal $Ae^{j\Omega_0 t}$ can be represented by a rotating vector of magnitude "A", in anticlockwise direction in a complex plane with an angular speed of $\Omega_0 t$ as shown in fig 4.32. The magnitude spectrum of $Ae^{+j\Omega_0 t}$ is shown in fig 4.33.

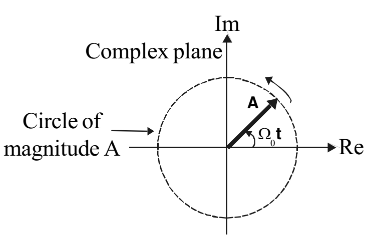


Fig 4.32 : *Complex exponential signal.*

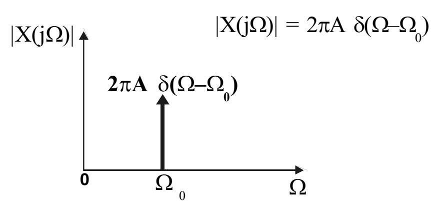


Fig 4.33: *Magnitude spectrum of* $Ae^{+j\Omega_0 t}$.

Fourier Transform of Sinusoidal Signal

The sinusoidal signal is defined as,

$$x(t) = A\sin\Omega_0 t = \frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2j}\left[\mathcal{F}\left\{e^{j\Omega_0 t}\right\} - \mathcal{F}\left\{e^{-j\Omega_0 t}\right\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2j}\left[2\pi\,\delta(\Omega - \Omega_0) - 2\pi\,\delta(\Omega + \Omega_0)\right] = \frac{A\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A\sin\Omega_0 t\} = \frac{A\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]} \quad .....(4.63)$$

The plot of sinusoidal signal and its spectrum are shown in fig 4.34 and fig 4.35.

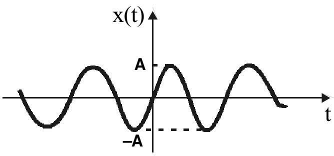


Fig 4.34 : Sinusoidal signal.

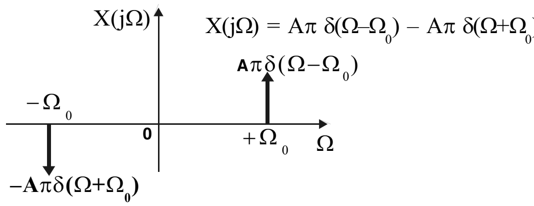


Fig 4.35 : Spectrum of sinusoidal signal.

Fourier Transform of Cosinusoidal Signal

The cosinusoidal signal is defined as,

$$x(t) = A\cos\Omega_0 t = \frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2}\left[\mathcal{F}\left\{e^{j\Omega_0 t}\right\} + \mathcal{F}\left\{e^{-j\Omega_0 t}\right\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2}\left[2\pi\,\delta(\Omega - \Omega_0) + 2\pi\,\delta(\Omega + \Omega_0)\right] = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A\cos\Omega_0 t\} = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]} \quad .....(4.64)$$

The plot of cosinusoidal signal and its magnitude spectrum are shown in fig 4.36 and fig 4.37.

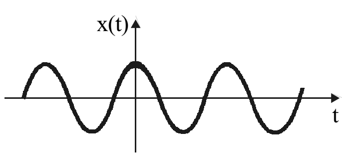


Fig 4.36 : Cosinusoidal signal.

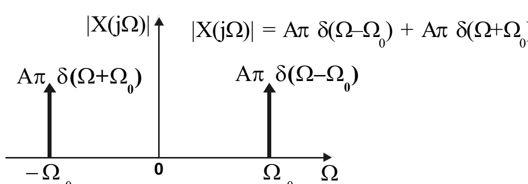


Fig 4.37 : Magnitude spectrum of cosinusoidal signal.

Table 4.4 : Fourier Transform of Standard Signals and their Magnitude Spectrum

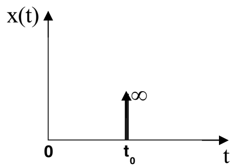
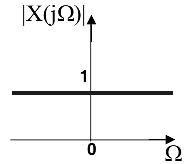
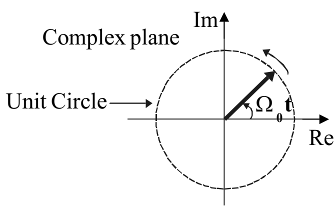
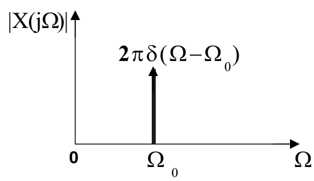
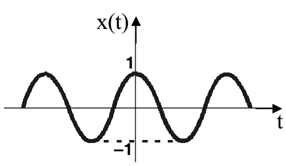
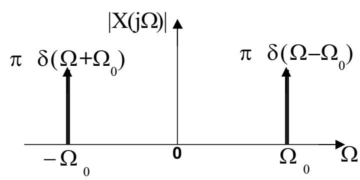
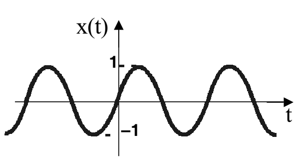
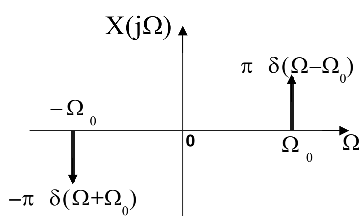
x(t)	X(jΩ) and Magnitude Spectrum
$x(t) = \delta(t-t_0)$  Shifted impulse signal	$X(j\Omega) = e^{-j\Omega t_0}$ 
$x(t) = e^{j\Omega_0 t}$  Complex exponential signal	$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$ 
$x(t) = \cos\Omega_0 t$  Cosinusoidal signal	$X(j\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$ 
$x(t) = \sin\Omega_0 t$  Sinusoidal signal	$X(j\Omega) = \frac{\pi}{j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$ 

Table 4.4 : Continued.....

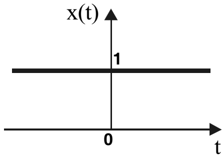
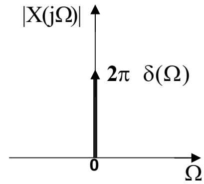
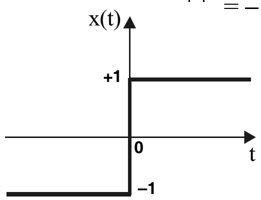
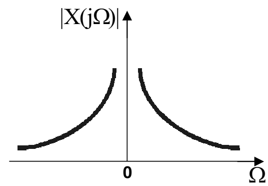
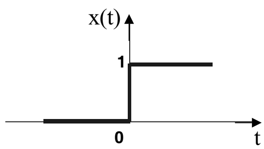
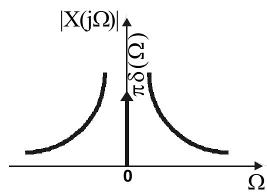
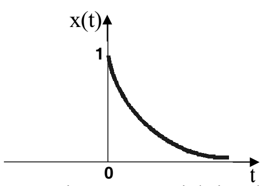
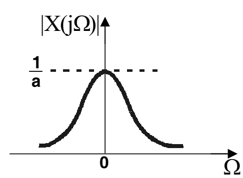
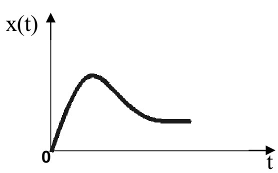
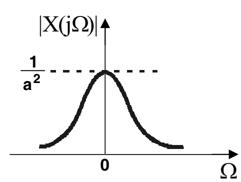
x(t)	**X(jΩ) and Magnitude Spectrum**
$x(t) = 1$  Constant	$X(j\Omega) = 2\pi\delta(\Omega)$ 
$x(t) = \mathrm{sgn}(t) = \frac{t}{\|t\|} = 1 \;;t > 0$ $= -1 \;;t < 0$  Signum signal	$X(j\Omega) = \frac{2}{j\Omega}$ 
$x(t) = u(t) = 1 \;;\; t \geq 0$ $= 0 \;;\; t < 0$  Unit step signal	$X(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega}$ 
$x(t) = e^{-at}\, u(t)$  Decaying exponential signal	$X(j\Omega) = \frac{1}{a + j\Omega}$ 
$x(t) = t\, e^{-at}\, u(t)$  Product of ramp and decaying exponential signal	$X(j\Omega) = \frac{1}{(a + j\Omega)^2}$ 

Table 4.4 : Continued.....

<table>
<tr><th>x(t)</th><th>X(jΩ) and Magnitude Spectrum</th></tr>
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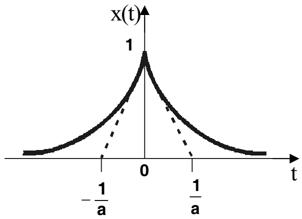
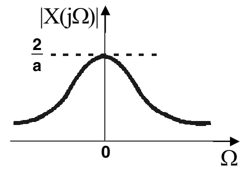
$x(t) = e^{-a|t|}$



Double exponential signal

</td><td>

$$X(j\Omega) = \frac{2a}{a^2 + \Omega^2}$$



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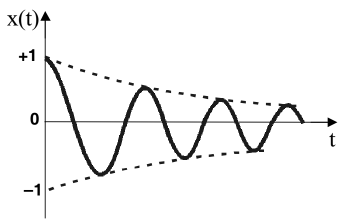
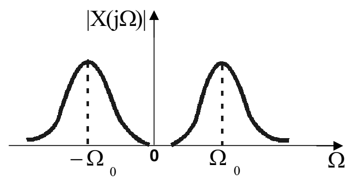


Exponentially decaying cosinusoidal signal

</td><td>

$$X(j\Omega) = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$$



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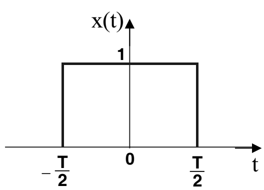
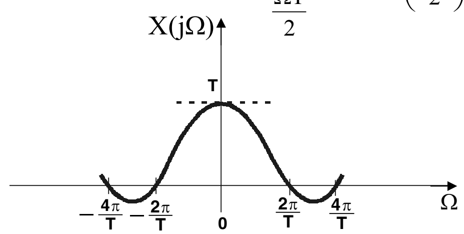


Rectangular pulse

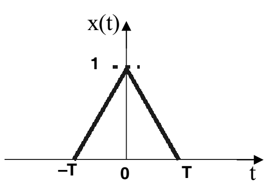
</td><td>

$$X(j\Omega) = T\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}} = T\,\mathrm{sinc}\left(\frac{\Omega T}{2}\right)$$



</td></tr>
<tr><td>

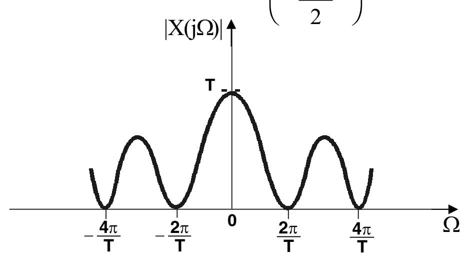
$$x(t) = 1 + \tfrac{t}{T}\ ;\ t = -T \text{ to } 0$$
$$= 1 - \tfrac{t}{T}\ ;\ t = 0 \text{ to } 0$$



Triangular pulse

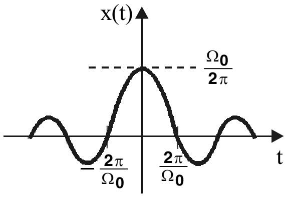
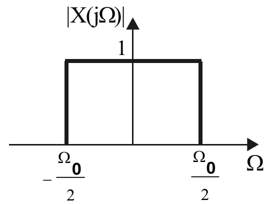
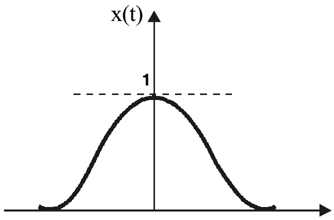
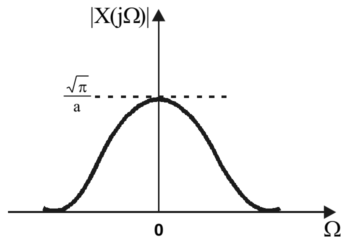
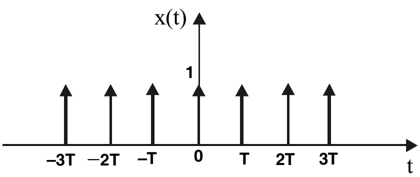
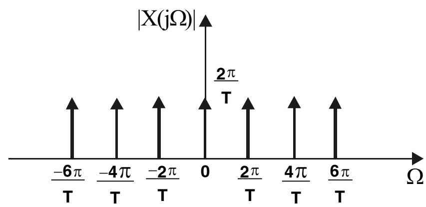
</td><td>

$$X(j\Omega) = T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2$$



</td></tr>
</table>

Table 4.4 : Continued.....

x(t)	**X(jΩ) and Magnitude Spectrum**
$x(t)=\frac{\Omega_0}{2\pi}\mathrm{sinc}\left(\frac{\Omega_0}{2}t\right)=\frac{1}{\pi}\frac{\sin\left(\frac{\Omega_0}{2}t\right)}{t}$  Sinc pulse	$X(j\Omega)=\left[u\left(\Omega+\frac{\Omega_0}{2}\right)-u\left(\Omega-\frac{\Omega_0}{2}\right)\right]$  $\boxed{\Omega_0=\frac{2\pi}{T}}$
$x(t)=e^{-a^2t^2}$  Gaussian pulse	$X(j\Omega)=\frac{\sqrt{\pi}}{a}e^{-\left(\frac{\Omega}{2a}\right)^2}$ 
$x(t)=\sum_{n=-\infty}^{+\infty}\delta(t-nT)$  Impulse train	$X(j\Omega)=\frac{2\pi}{T}\sum_{n=-\infty}^{+\infty}\delta\left(\Omega-\frac{2\pi n}{T}\right)$ 

From table 4.4 the following observations are made.

1. The Fourier transform of a Gaussian pulse will be another Gaussian pulse.
2. The Fourier transform of an impulse train will be another impulse train.
3. The Fourier transform of a rectangular pulse will be a sinc pulse and viceversa.
4. The Fourier transform of a triangular pulse will be a squared sinc pulse.
5. The Fourier transform of a constant will be an impulse and vice-versa.

Table 4.5 : Standard Fourier Tranform Pairs

x(t)	**X(jΩ)**
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\Omega t_0}$
A where, A is constant	$2\pi A\delta(\Omega)$
$u(t)$	$\pi\delta(\Omega) + \frac{1}{j\Omega}$
$\text{sgn}(t)$	$\frac{2}{j\Omega}$
$t\,u(t)$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\frac{1}{(j\Omega + a)^2}$
$Ae^{-a\|t\|}$	$\frac{2Aa}{a^2 + \Omega^2}$
$Ae^{j\Omega_0 t}$	$2\pi A\,\delta(\Omega - \Omega_0)$
$\sin\Omega_0 t$	$\frac{\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$
$\cos\Omega_0 t$	$\pi[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)]$

4.12 Fourier Transform of a Periodic Signal

Let, x(t) = Continuous time periodic signal

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \text{Fourier transform of } x(t)$$

The exponential form of Fourier series representation of x(t) is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \; e^{jn\Omega_0 t} \qquad \boxed{\text{From equation (4.9)}}$$

On taking Fourier transform of the above equation we get,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n \; e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n \; \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n \; 2\pi\,\delta(\Omega - n\Omega_0) = 2\pi \sum_{n=-\infty}^{+\infty} c_n \; 2\pi\,\delta(\Omega - n\Omega_0) \qquad \boxed{\text{Using equation (4.61)}}$$

$$= \ldots\ldots + 2\pi c_{-3}\,\delta(\Omega + 3\Omega_0) + 2\pi c_{-2}\,\delta(\Omega + 2\Omega_0) + 2\pi c_{-1}\,\delta(\Omega + \Omega_0)$$
$$+ 2\pi c_0\,\delta(\Omega) + 2\pi c_1\,\delta(\Omega - \Omega_0) + 2\pi c_2\,\delta(\Omega - 2\Omega_0)$$
$$+ 2\pi\, c_3\,\delta(\Omega - 3\Omega_0) + \ldots\ldots \qquad \ldots..(4.65)$$

The magnitude of each term of equation (4.65) represents an impulse, located at its harmonic frequency in the magnitude spectrum. Hence we can say that the Fourier transform of a periodic continuous time signal consists of impulses located at the harmonic frequencies of the signal. The magnitude of each impulse is 2π times the magnitude of Fourier coefficient, i.e., the magnitude of n^{th} impulse is $2\pi\ |c_n|$.

4.13. Analysis of LTI Continuous Time System Using Fourier Transform

4.13.1 Transfer Function of LTI Continuous Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI continuous time system in frequency domain.

Let, x(t) = Input to the continuous time system

y(t) = Output of the continuous time system

$\therefore$ $X(j\Omega)$ = Fourier transform of x(t)

$Y(j\Omega)$ = Fourier transform of y(t)

$$\text{Now, Transfer function} = \frac{Y(j\Omega)}{X(j\Omega)} \qquad \ldots..(4.66)$$

The transfer function of LTI continuous time system in frequency domain can be obtained from the differential equation governing the input-output relation of an LTI continuous time system, (refer chapter-2, equation (2.13)),

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + \ldots\ldots + a_{N-1}\frac{d}{dt}y(t) + a_N\,y(t) = b_0\frac{d^M}{dt^M}x(t)$$
$$+ b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \ldots\ldots + b_{M-1}\frac{d}{dt}x(t) + b_M x(t)$$

On taking Fourier transform of the above equation and rearranging the resultant equation as a ratio of $Y(j\Omega)$ and $X(j\Omega)$, the transfer function of LTI continuous time system in frequency domain is obtained.

Impulse Response and Transfer Function

Consider an LTI continuous time system, $\mathcal{H}$ shown in fig 4.38. Let x(t) and y(t) be the input and output of the system respectively.

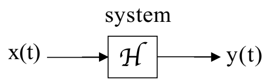


Fig 4.38.

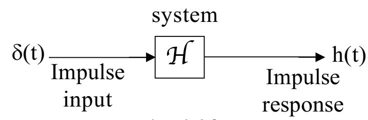


Fig 4.39.

For a continuous time system $\mathcal{H}$, if the input is impulse signal $\delta(t)$ as shown in fig 4.39, then the output is called ***impulse response***, which is denoted by h(t).

The importance of impulse response is that the response for any input to LTI system is given by convolution of input and impulse response.

Symbolically, the convolution operation is denoted as,

$$y(t) = x(t) * h(t) \qquad .....(4.67)$$

where, " $*$ " is the symbol for convolution.

Mathematically, the convolution operation is defined as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau$$

where, τ is the dummy variable for integration.

Let, $H(j\Omega)$ = Fourier transform of h(t)

$X(j\Omega)$ = Fourier transform of x(t)

$Y(j\Omega)$ = Fourier transform of y(t)

Now, by convolution property of Fourier transform we get,

$$\mathcal{F}\{x(t) * h(t)\} = X(j\Omega)\, H(j\Omega) \qquad .....(4.68)$$

Using equation (4.67), the equation (4.68) can be written as,

$$\mathcal{F}\{y(t)\} = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ Y(j\Omega) = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} \qquad .....(4.69)$$

From equations (4.66) and (4.69) we can say that the ***transfer function in frequency domain*** is given by Fourier transform of impulse response.

$$\boxed{\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}} \qquad .....(4.70)$$

4.13.2 Response of LTI Continuous Time System Using Fourier Transform

Consider the transfer function of LTI continuous time system, $H(j\Omega)$.

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

Now, response in frequency domain, $Y(j\Omega) = H(j\Omega)\, X(j\Omega)$

The response function $Y(j\Omega)$ will be a rational function of $j\Omega$, and so $Y(j\Omega)$ can be expressed as a ratio of two factorized polynomial in $j\Omega$ as shown below.

$$Y(j\Omega) = \frac{(j\Omega + z_1)\,(j\Omega + z_2)\,(j\Omega + z_3)\,.....}{(j\Omega + p_1)\,(j\Omega + p_2)\,(j\Omega + p_3)\,.....} \qquad .....(4.71)$$

By partial fraction expansion technique the equation (4.71) can be expressed as shown below.

$$Y(j\Omega) = \frac{k_1}{j\Omega + p_1} + \frac{k_2}{j\Omega + p_2} + \frac{k_3}{j\Omega + p_3} + \,..... \qquad .....(4.72)$$

where, $k_1, k_2, k_3, ...$ are residues.

Now the time domain response y(t) can be obtained by taking inverse Fourier transform of equation (4.72). The inverse Fourier transform of each term in equation (4.72) can be obtained by comparing the terms with standard Fourier transform pair listed in table 4.5.

$$\text{From table - 4.5, we get, } \mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{a + j\Omega} \qquad .....(4.73)$$

Using equation (4.73), the inverse Fourier transform of equation (4.72) can be obtained as shown below.

$$y(t) = k_1\, e^{-p_1 t}\, u(t) + k_2\, e^{-p_2 t}\, u(t) + k_3\, e^{-p_3 t}\, u(t) + \,..... \qquad .....(4.74)$$

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (4.74) is the time domain steady state (or forced) response of the LTI continuous time system.

Note: *Only steady state or forced response alone can be computed via frequency domain*

4.13.3 Frequency Response of LTI Continuous Time System

The output y(t) of an LTI continuous time system is given by convolution of h(t) and x(t).

$$\therefore\; y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\; x(t - \tau)\, d\tau \qquad .....(4.75)$$

Consider a special class of input (sinusoidal input), $\boxed{Ae^{j\Omega t} = A(\cos \Omega t + j \sin \Omega t)}$

$$x(t) = A\,e^{j\Omega t} \qquad .....(4.76)$$

where, A = Amplitude ; Ω = Angular frequency in rad/sec

$$\therefore\; x(t - \tau) = Ae^{j\Omega(t - \tau)} \qquad .....(4.77)$$

On substituting for $x(t - \tau)$ from equation (4.77) in equation (4.75) we get,

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\; Ae^{j\Omega(t - \tau)}\, d\tau$$

$$\therefore y(t) = \int_{-\infty}^{+\infty} h(\tau)\ Ae^{j\Omega t}\ e^{-j\Omega\tau} d\tau$$

$$= Ae^{j\Omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.78)$$

By the definition of Fourier transform,

Replace t by τ.

$$H(j\Omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.79)$$

Using equations (4.76) and (4.79), the equation (4.78) can be written as,

$$y(t) = x(t)\ H(j\Omega) \qquad .....(4.80)$$

From equation (4.80) we can say that if a complex sinusoidal signal is given as input signal to an LTI continuous time system, then the output is also a sinusoidal signal of the same frequency modified by $H(j\Omega)$. Hence $H(j\Omega)$ is called the ***frequency response*** of the continuous time system.

Since the $H(j\Omega)$ is a complex function of Ω, the multiplication of $H(j\Omega)$ with input produces a change in the amplitude and phase of the input signal, and the modified input signal is the output signal. Therefore, an LTI system is characterized in the frequency domain by its frequency response.

The function $H(j\Omega)$ is a complex quantity and so it can be expressed as magnitude function and phase function.

$$\therefore H(j\Omega) = |H(j\Omega)|\ \angle H(j\Omega)$$

where, $|H(j\Omega)|$ = Magnitude function

$\angle H(j\Omega)$ = Phase function

The sketch of magnitude function and phase function with respect to Ω will give the frequency response graphically.

Let, $H(j\Omega) = H_r(j\Omega) + jH_i(j\Omega)$

where, $H_r(j\Omega)$ = Real part of $H(j\Omega)$

$H_i(j\Omega)$ = Imaginary part of $H(j\Omega)$

The ***magnitude function*** is defined as,

$$|H(j\Omega)|^2 = H(j\Omega)\ H^*(j\Omega) = [H_r(j\Omega) + jH_i(j\Omega)]\ [H_r(j\Omega) - jH_i(j\Omega)]$$

where, $H^*(j\Omega)$ is complex conjugate of $H(j\Omega)$

$$\therefore |H(j\Omega)|^2 = H_r^2(j\Omega) + H_i^2(j\Omega) \quad \Rightarrow \quad |H(j\Omega)| = \sqrt{H_r^2(j\Omega) + H_i^2(j\Omega)}$$

The ***phase function*** is defined as,

$$\angle H(j\Omega) = \mathrm{Arg}[H(j\Omega)] = \tan^{-1}\left[\frac{H_i(j\Omega)}{H_r(j\Omega)}\right]$$

From equation (4.70) we can say that the frequency response $H(j\Omega)$ of an LTI continuous time system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\boxed{\therefore \text{ Frequency response, } H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}} \quad .....(4.81)$$

Advantages of frequency response analysis

1. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments .
2. The transfer function of complicated systems can be determined experimentally by frequency response tests.
3. The design and parameter adjustment is carried out more easily in frequency domain.
4. In frequency domain, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
5. The frequency response analysis and designs can be extended to certain nonlinear systems.

4.14 Relation Between Fourier and Laplace Transform

Let x(t) be a continuous time signal, defined for all t.

The definition of Laplace tranform of x(t) is,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

On substituting $s = \sigma + j\Omega$ in the above definition of Laplace tranform we get,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-(\sigma + j\Omega)t}\, dt \quad .....(4.82)$$

The definition of Fourier transform of x(t) is,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \quad .....(4.83)$$

On comparing equations (4.82) and (4.83) we can say that, the Fourier transform of a continuous time signal, is obtained by letting $\sigma = 0$ (i.e., $s = j\Omega$) in the Laplace transform. Summary of this relation is presented in table 4.6, for causal signals.

$$\boxed{\therefore X(j\Omega) = X(s)\big|_{s = j\Omega}}$$

Since $s = \sigma + j\Omega$, we can say that, the Laplace transform is a generalized transform and Fourier transform is a particular transform when $s = j\Omega$. Since $s = j\Omega$ represents the points on an imaginary axis in the s-plane, we can say that, the Fourier transform is an evaluation of the Laplace transform along the imaginary axis in the s-plane.

Since Fourier transform is evaluation of Laplace transform along imaginary axis, the ROC of X(s) should include the imaginary axis. For all causal signals, the imaginary axis is included in ROC. Therefore for all causal signals the Fourier transform exist.

Table 4.6 : Summary of Laplace and Fourier Tranform for Causal Signals

x(t) for **t = 0 to ∞**	**X(s)**	**X(jΩ)** $\left[X(j\Omega) = X(s)\big\|_{s=j\Omega}\right]$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{1}{j\Omega}$
$t\,u(t)$	$\frac{1}{s^2}$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{s^n}$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{s^{n+1}}$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\frac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\frac{1}{(j\Omega+a)^2}$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2+\Omega_0^2} = \frac{\Omega_0}{\Omega_0^2-\Omega^2}$
$\cos\Omega_0 t\;u(t)$	$\frac{s}{s^2+\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2+\Omega_0^2} = \frac{j\Omega}{\Omega_0^2-\Omega^2}$
$\sinh\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2-\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2-\Omega_0^2} = \frac{-\Omega_0}{\Omega^2+\Omega_0^2}$
$\cosh\Omega_0 t\,u(t))$	$\frac{s}{s^2-\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2-\Omega_0^2} = \frac{-j\Omega}{\Omega^2+\Omega_0^2}$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega+a)^2+\Omega_0^2}$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$\frac{j\Omega+a}{(j\Omega+a)^2+\Omega_0^2}$

4.15 Solved Problems in Fourier Transform

Example 4.13

Determine the Fourier transform of following continuous time domain signals.

a) $x(t) = 1 - t^2$; for $|t| < 1$
$= 0$; for $|t| > 1$

b) $x(t) = e^{-at}\cos\Omega_0 t\, u(t)$

Solution

a) Given that, $x(t) = 1 - t^2$; for $|t| < 1$

$$\therefore x(t) = 1 - t^2; \text{ for } t = -1 \text{ to } +1$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} (1 - t^2)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} e^{-j\Omega t}\, dt - \int_{-1}^{+1} t^2\, e^{-j\Omega t}\, dt$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[t^2\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 2t\,\frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-1}^{1} \qquad \boxed{\int uv = u\int v - \int\left[du\int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\int t\, e^{-j\Omega t}\, dt\right]_{-1}^{1}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\left(t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right)\right]_{-1}^{1} \qquad \boxed{\int uv = u\int v - \int\left[du\int v\right]}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{(j\Omega)^2}\left(-\,t\, e^{-j\Omega t} + \int e^{-j\Omega t}\, dt\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} - \frac{2}{\Omega^2}\left(-\,t\, e^{-j\Omega t} + \frac{e^{-j\Omega t}}{-j\Omega}\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2t\, e^{-j\Omega t}}{\Omega^2} + \frac{2e^{-j\Omega t}}{j\Omega^3}\right]_{-1}^{1}$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} - \left[-\frac{e^{-j\Omega}}{j\Omega} + \frac{2\,e^{-j\Omega}}{\Omega^2} + \frac{2\,e^{-j\Omega}}{j\Omega^3} + \frac{e^{j\Omega}}{j\Omega} + \frac{2\,e^{j\Omega}}{\Omega^2} - \frac{2e^{j\Omega}}{j\Omega^3}\right]$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} + \frac{e^{-j\Omega}}{j\Omega} - \frac{2\,e^{-j\Omega}}{\Omega^2} - \frac{2\,e^{-j\Omega}}{j\Omega^3} - \frac{e^{j\Omega}}{j\Omega} - \frac{2\,e^{j\Omega}}{\Omega^2} + \frac{2\,e^{j\Omega}}{j\Omega^3}$$

$$= -\frac{2}{\Omega^2}\left(e^{j\Omega} + e^{-j\Omega}\right) + \frac{2}{j\Omega^3}\left(e^{j\Omega} - e^{-j\Omega}\right) \qquad \boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}} \quad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= -\frac{2}{\Omega^2}\, 2\cos\Omega + \frac{2}{j\Omega^3}\, 2j\sin\Omega$$

$$= -\frac{4\cos\Omega}{\Omega^2} + \frac{4\sin\Omega}{\Omega^3}$$

$$= \frac{4}{\Omega^2}\left(\frac{\sin\Omega}{\Omega} - \cos\Omega\right)$$

(b) Given that, $\mathbf{x(t) = e^{-at} \cos\Omega_0 t\, u(t)}$

Since $u(t) = 1$, for $t \geq 0$, we can write,

$$x(t) = e^{-at} \cos\Omega_0 t \quad ; \quad \text{for} \quad t \geq 0$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at} \cos\Omega_0 t\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at}\left(\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\right) e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-at}\, e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-at}\, e^{-j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-(a - j\Omega_0 + j\Omega)t}\, dt + \frac{1}{2}\int_0^{\infty} e^{-(a + j\Omega_0 + j\Omega)t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-(a - j\Omega_0 + j\Omega)t}}{-(a - j\Omega_0 + j\Omega)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-(a + j\Omega_0 + j\Omega)t}}{-(a + j\Omega_0 + j\Omega)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(a - j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a - j\Omega_0 + j\Omega)}\right] + \frac{1}{2}\left[\frac{e^{-\infty}}{-(a + j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a + j\Omega_0 + j\Omega)}\right]$$

$$= \frac{1}{2}\left[0 + \frac{1}{a - j\Omega_0 + j\Omega}\right] + \frac{1}{2}\left[0 + \frac{1}{a + j\Omega_0 + j\Omega}\right]$$

$$\boxed{e^{-\infty} = 0\ ;\ e^{0} = 1}$$

$$= \frac{1}{2}\left[\frac{1}{(a + j\Omega) - j\Omega_0} + \frac{1}{(a + j\Omega) + j\Omega_0}\right]$$

$$\boxed{(a+b)(a-b) = a^2 - b^2 \;\big|\; j^2 = -1}$$

$$= \frac{1}{2}\left[\frac{(a + j\Omega) + j\Omega_0 + (a + j\Omega) - j\Omega_0}{(a + j\Omega)^2 + \Omega_0^2}\right]$$

$$= \frac{1}{2}\,\frac{2\,(a + j\Omega)}{(a + j\Omega)^2 + \Omega_0^2} = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$$

Example 4.14

Determine the Fourier transform of the rectangular pulse shown in fig 4.14.1.

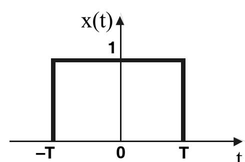


Fig 4.14.1.

Solution

The mathematical equation of the rectangular pulse is,

$$x(t) = 1 \quad ; \quad \text{for } t = -T \text{ to } +T$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{+T} 1 \times e^{-j\Omega t}\, dt = \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{+T}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{e^{-j\Omega T}}{-j\Omega} - \frac{e^{j\Omega T}}{-j\Omega} = \frac{1}{j\Omega}\left(e^{j\Omega T} - e^{-j\Omega T}\right) = \frac{1}{j\Omega}\, 2j\, \sin\Omega T$$

$$= 2\,\frac{\sin\Omega T}{\Omega} = 2T\,\frac{\sin\Omega T}{\Omega T}$$

$$\boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

$$= 2T\, \operatorname{sinc}\Omega T$$

Example 4.15

Determine the Fourier transform of the triangular pulse shown in fig 4.15.1.

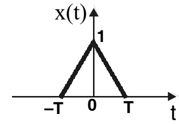


Fig 4.15.1.

Solution

The mathematical equation of triangular pulse is,

$$x(t) = 1 + \frac{t}{T} \quad ; \quad \text{for } t = -T \text{ to } 0$$
$$= 1 - \frac{t}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

(Please refer example 4.11 for the mathematical equation of triangular pulse).

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-j\Omega t}\, dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-j\Omega t}\, dt$$

$$= \int_{-T}^{0} e^{-j\Omega t}\, dt + \frac{1}{T}\int_{-T}^{0} t\, e^{-j\Omega t}\, dt + \int_{0}^{T} e^{-j\Omega t}\, dt - \frac{1}{T}\int_{0}^{T} t\, e^{-j\Omega t}\, dt \qquad \boxed{\int uv = u\int v - \int\left[du\int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} + \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-T}^{0} + \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T} - \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \int e^{-j\Omega t}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\, e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{0} - e^{j\Omega T}\right] - \frac{1}{j\Omega T}\left[0 - \frac{e^{0}}{-j\Omega} + T\, e^{j\Omega T} + \frac{e^{j\Omega T}}{-j\Omega}\right] - \frac{1}{j\Omega}\left[e^{-j\Omega T} - e^{0}\right]$$
$$+ \frac{1}{j\Omega T}\left[T\, e^{-j\Omega T} - \frac{e^{-j\Omega T}}{-j\Omega} - 0 + \frac{e^{0}}{-j\Omega}\right]$$

$$= -\frac{1}{j\Omega} + \frac{e^{j\Omega T}}{j\Omega} - 0 + \frac{1}{T\Omega^2} - \frac{e^{j\Omega T}}{j\Omega} - \frac{e^{j\Omega T}}{T\Omega^2} - \frac{e^{-j\Omega T}}{j\Omega} + \frac{1}{j\Omega} + \frac{e^{-j\Omega T}}{j\Omega} - \frac{e^{-j\Omega T}}{T\Omega^2} - 0 + \frac{1}{T\Omega^2}$$

$$= \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\left(e^{j\Omega T} + e^{-j\Omega T}\right) = \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\, 2\cos\Omega T \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right)$$

Alternatively the above result can be expressed as shown below.

$$\mathcal{F}\{x(t)\} = \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right) = \frac{2}{T\Omega^2}\left(1 - \cos 2\left(\frac{\Omega T}{2}\right)\right)$$

$$= \frac{2}{T\Omega^2}\left(2\sin^2\frac{\Omega T}{2}\right) = T\,\frac{4}{T^2\Omega^2}\sin^2\frac{\Omega T}{2} = T\,\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)^2} \qquad \boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$= T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2 = T\left(\operatorname{sinc}\frac{\Omega T}{2}\right)^2 \qquad \boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

Example 4.16

Determine the inverse Fourier transform of the following functions, using partial fraction expansion technique.

a) $X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}$ b) $X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}$

Solution

a) Given that, $\mathbf{X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}} = \dfrac{3(j\Omega) + 14}{(j\Omega + 3)\,(j\Omega + 4)}$

By partial fraction expansion technique we can write,

$$X(j\Omega) = \frac{3(j\Omega) + 14}{(j\Omega + 3)\,(j\Omega + 4)} = \frac{k_1}{j\Omega + 3} + \frac{k_2}{j\Omega + 4}$$

$$k_1 = \frac{3(j\Omega) + 14}{(j\Omega + 3)\,(j\Omega + 4)} \times (j\Omega + 3)\Bigg|_{j\Omega = -3} = \frac{3(-3) + 14}{-3 + 4} = 5$$

$$k_2 = \frac{3(j\Omega) + 14}{(j\Omega + 3)\,(j\Omega + 4)} \times (j\Omega + 4)\Bigg|_{j\Omega = -4} = \frac{3(-4) + 14}{-4 + 3} = -2$$

$$\therefore X(j\Omega) = \frac{5}{j\Omega + 3} - \frac{2}{j\Omega + 4} \qquad .....(1)$$

We know that, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{j\Omega + a}$ .....(2)

Using equation (2), the inverse Fourier transform of equation (1) is,

$$x(t) = 5\,e^{-3t}\,u(t) - 2\,e^{-4t}\,u(t)$$

---

b) Given that, $\mathbf{X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}}$

By partial fraction expansion technique $X(j\Omega)$ can be written as,

$$\therefore X(j\Omega) = \frac{k_1}{(j\Omega + 3)^2} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2\Bigg|_{j\Omega = -3} = -3 + 7 = 4$$

$$k_2 = \frac{d}{d(j\Omega)}\left[\frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2\right]\Bigg|_{j\Omega = -3} = \frac{d}{d(j\Omega)}\left[j\Omega + 7\right]\Bigg|_{j\Omega = -3} = 1$$

$$\therefore X(j\Omega) = \frac{4}{(j\Omega + 3)^2} + \frac{1}{j\Omega + 3} \qquad .....(3)$$

We know that, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{j\Omega + a}$ .....(4)

$$\mathcal{F}\left\{t\,e^{-at}\,u(t)\right\} = \frac{1}{(j\Omega + a)^2} \qquad .....(5)$$

Using equations (4) and (5), the inverse Fourier transform of equation (3) is,

$$x(t) = 4t\,e^{-3t}\,u(t) + e^{-3t}\,u(t) = (4t + 1)\,e^{-3t}\,u(t)$$

Example 4.17

Determine the convolution of $x_1(t) = e^{-2t}\,u(t)$ and $x_2(t) = e^{-6t}\,u(t)$, using Fourier transform.

Solution

Let, $X_1(j\Omega)$ = Fourier transform of $x_1(t)$

$X_2(j\Omega)$ = Fourier transform of $x_2(t)$

By convolution property of Fourier transform,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\,X_2(j\Omega)$$

Let,
$$X(j\Omega) = X_1(j\Omega)\,X_2(j\Omega)$$
$$= \mathcal{F}\left\{e^{-2t}\,u(t)\right\} \times \mathcal{F}\left\{e^{-6t}\,u(t)\right\}$$
$$= \frac{1}{j\Omega + 2} \times \frac{1}{j\Omega + 6}$$

By partial fraction expansion technique $X(j\Omega)$ can be expressed as,

$$X(j\Omega) = \frac{1}{(j\Omega + 2)\,(j\Omega + 6)} = \frac{k_1}{j\Omega + 2} + \frac{k_2}{j\Omega + 6}$$

$$k_1 = \frac{1}{(j\Omega + 2)\,(j\Omega + 6)} \times (j\Omega + 2)\Bigg|_{j\Omega = -2} = \frac{1}{-2 + 6} = \frac{1}{4} = 0.25$$

$$k_2 = \frac{1}{(j\Omega + 2)\,(j\Omega + 6)} \times (j\Omega + 6)\Bigg|_{j\Omega = -6} = \frac{1}{-6 + 2} = -\frac{1}{4} = -0.25$$

$$\therefore\ X(j\Omega) = \frac{0.25}{j\Omega + 2} - \frac{0.25}{j\Omega + 6}$$

On taking inverse Fourier transform of the above equation we get,

$$x(t) = 0.25\,e^{-2t}\,u(t) - 0.25\,e^{-6t}\,u(t)$$
$$= 0.25(e^{-2t} - e^{-6t})\,u(t)$$

$$\boxed{\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{j\Omega + a}}$$

Example 4.18

The impulse response of an LTI system is $h(t) = 2\,e^{-3t}\,u(t)$.

Find the response of the system for the input $x(t) = 2e^{-5t}\,u(t)$, using Fourier transform.

Solution

Given that, $x(t) = 2\,e^{-5t}\,u(t)$.

$$\therefore\ X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{2\,e^{-5t}\,u(t)\right\} = \frac{2}{j\Omega + 5} \quad .....(1)$$

Given that, $h(t) = 2\,e^{-3t}\,u(t)$.

$$\boxed{\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{j\Omega + a}}$$

$$\therefore\ H(j\Omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{2\,e^{-3t}\,u(t)\right\} = \frac{2}{j\Omega + 3} \quad .....(2)$$

For LTI system, the response, $y(t) = = x(t) * h(t)$ .....(3)

On taking Fourier transform of equation (3) we get,

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\}$$

Let, $\mathcal{F}\{y(t)\} = Y(j\Omega)$.

$$\therefore Y(j\Omega) = \mathcal{F}\{x(t) * h(t)\}$$

Using convolution property of Fourier transform.

$$= X(j\Omega)\, H(j\Omega)$$

$$= \frac{2}{j\Omega + 5} \times \frac{2}{j\Omega + 3} = \frac{4}{(j\Omega + 5)\,(j\Omega + 3)}$$

Using equations (1) and (2)

By partial fraction expansion technique, the above equation can be written as,

$$Y(j\Omega) = \frac{4}{(j\Omega + 5)\,(j\Omega + 3)} = \frac{k_1}{j\Omega + 5} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{4}{(j\Omega + 5)\,(j\Omega + 3)} \times (j\Omega + 5)\Bigg|_{j\Omega = -5} = \frac{4}{-5 + 3} = -2$$

$$k_2 = \frac{4}{(j\Omega + 5)\,(j\Omega + 3)} \times (j\Omega + 3)\Bigg|_{j\Omega = -3} = \frac{4}{-3 + 5} = 2$$

$$\therefore Y(j\Omega) = -\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}$$

On taking inverse Fourier transform of $Y(j\Omega)$ we get $y(t)$.

$$y(t) = \mathcal{F}^{-1}\{Y(j\Omega)\} = \mathcal{F}^{-1}\left\{-\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}\right\}$$

$$= -2\,e^{-5t}\,u(t) + 2\,e^{-3t}\,u(t) = 2\left(e^{-3t} - e^{-5t}\right)u(t)$$

Example 4.19

Determine the Fourier transform of the periodic pulse function shown in fig 4.19.1.

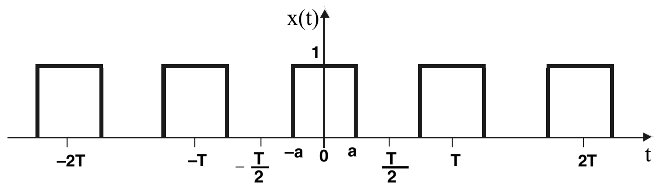


Fig 4.19.1.

Solution

The mathematical equation for one period of the periodic pulse function is,

$$x(t) = 1 \;;\quad t = -a \text{ to } +a$$

$$= 0 \;;\quad t = -\frac{T}{2} \text{ to } -a \quad \text{and} \quad t = a \text{ to } \frac{T}{2}$$

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_{-a}^{+a} e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-a}^{+a} = \frac{1}{T}\left[\frac{e^{-jn\Omega_0 t}}{-jn\Omega_0}\right]_{-a}^{+a}$$

$$\boxed{\Omega_0 = \frac{2\pi}{T}}$$

$$= \frac{1}{T}\left[\frac{e^{-jn\Omega_0 a}}{-jn\Omega_0} - \frac{e^{jn\Omega_0 a}}{-jn\Omega_0}\right] = \frac{1}{T}\,\frac{2}{n\Omega_0}\left[\frac{e^{jn\Omega_0 a} - e^{-jn\Omega_0 a}}{2j}\right]$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{T}\,\frac{2}{n}\,\frac{T}{2\pi}\,\sin(n\Omega_0 a) = \frac{1}{n\pi}\sin(an\Omega_0) \qquad \text{....(1)}$$

The exponential Fourier series representation of the periodic pulse function is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}\right\}$$

$$\therefore X(j\Omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$ $\boxed{\mathcal{F}\{x(t)\} = X(j\Omega)}$

$$= \sum_{n=-\infty}^{+\infty} c_n 2\pi\,\delta(\Omega - n\Omega_0)$$ $\boxed{\mathcal{F}\left\{e^{jn\Omega_0 t}\right\} = 2\pi\,\delta(\Omega - n\Omega_0)}$

$$= \sum_{n=-\infty}^{+\infty} \frac{1}{n\pi}\sin(an\Omega_0)\ 2\pi\,\delta(\Omega - n\Omega_0)$$ Substituting for c_n from equation (1).

$$= \sum_{n=-\infty}^{+\infty} \frac{2\sin(an\Omega_0)}{n}\,\delta(\Omega - n\Omega_0)$$ $\boxed{\frac{\sin\theta}{\theta} = \text{sinc}\,\theta}$

$$= \sum_{n=-\infty}^{+\infty} 2a\Omega_0\left(\frac{\sin an\Omega_0}{an\Omega_0}\right)\delta(\Omega - n\Omega_0) = \sum_{n=-\infty}^{+\infty} 2a\Omega_0\,\text{sinc}(an\Omega_0)\ \delta(\Omega - n\Omega_0)$$

Example 4.20

Determine the Fourier transform of the periodic impulse function shown in fig 4.20.1.

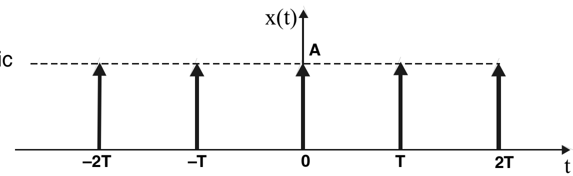


Fig 4.20.1.

Solution

The mathematical equation for one period of the periodic impulse function is,

$$x(t) = A\,\delta(t) \quad ; \quad \text{for } t = -\frac{T}{2} \text{ to } +\frac{T}{2}$$

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\,e^{-jn\Omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{+T/2} A\,\delta(t)\,e^{-jn\Omega_0 t}\,dt = \frac{A}{T}\,e^{-jn\Omega_0 t}\Big|_{t=0} = \frac{A}{T}$$ $\boxed{\Omega_0 = \frac{2\pi}{T}}$ ....(1)

The Exponential Fourier series representation of the periodic impulse train is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t}\right\}$$

$$\therefore X(j\Omega) = \sum_{n=-\infty}^{+\infty} c_n \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$ $\boxed{\mathcal{F}\{x(t)\} = X(j\Omega)}$

$$= \sum_{n=-\infty}^{+\infty} c_n\ 2\pi\,\delta(\Omega - n\Omega_0)$$ $\boxed{\mathcal{F}\left\{e^{jn\Omega_0 t}\right\} = 2\pi\,\delta(\Omega - n\Omega_0)}$

$$= \sum_{n=-\infty}^{+\infty} \frac{A}{T}\ 2\pi\,\delta(\Omega - n\Omega_0) = \sum_{n=-\infty}^{+\infty} A\Omega_0\ \delta(\Omega - n\Omega_0)$$ On substituting for c_n from equation (1)

The magnitude spectrum of $X(j\Omega)$ is shown in fig 1, which is also a periodic impulse function of Ω.

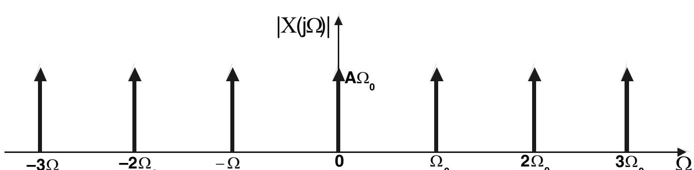


Fig 1.

Example 4.21

Find the Fourier transform and sketch the magnitude and phase spectrum for the signal, $x(t) = e^{-at}\,u(t)$.

Solution

Given that, $x(t) = e^{-at}\,u(t)$.

The Fourier transform of x(t) is, (refer section 4.11 for Fourier transform of $e^{-at}\,u(t)$).

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{a+j\Omega}$$

$$= \frac{1}{a+j\Omega} \times \frac{a-j\Omega}{a-j\Omega} = \frac{a-j\Omega}{(a+j\Omega)\,(a-j\Omega)}$$

$$= \frac{a-j\Omega}{a^2+\Omega^2} = \frac{a}{a^2+\Omega^2} - j\frac{\Omega}{a^2+\Omega^2}$$

The $X(j\Omega)$ is calculated for a = 0.5 and a = 1.0 and tabluted in table 1and table 2 respectively. Using the values listed in table 1and table 2, the magnitude and phase spectrum are sketched as shown in fig 1 and fig 2 respectively.

> ***Note*** *: The function $X(j\Omega)$ is calculated using complex mode of calculator, the magnitude and phase are calculated using rectangular to polar conversion technique.*

Table 1 : Frequency Spectrum for a = 0.5

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
– 8	$0.0080 + j\,0.12 = 0.12\ \angle 1.506 = 0.12\ \angle 0.48\pi$	0.12	0.48π
– 6	$0.014 + j\,0.167 = 0.166\ \angle 1.4866 = 0.166\ \angle 0.473\pi$	0.166	0.473π
– 4	$0.03 + j\,0.246 = 0.248\ \angle 1.45 = 0.248\ \angle 0.46\pi$	0.248	0.46π
– 3	$0.054 + j\,0.324 = 0.328\ \angle 1.40 = 0.328\ \angle 0.45\pi$	0.328	0.45π
– 2	$0.118 + j\,0.47 = 0.485\ \angle 1.325 = 0.485\ \angle 0.422\pi$	0.485	$0.422\,\pi$
– 1	$0.4 + j\,0.8 = 0.89\ \angle 1.11 = 0.89\ \angle 0.353\pi$	0.89	0.353π
0	$2 + j\,0 = 2\ \angle 0 = 2\ \angle 0$	2.0	0
1	$0.4 - j\,0.8 = 0.89\ \angle -1.11 = 0.89\ \angle -0.353\pi$	0.89	-0.353π
2	$0.118 - j\,0.47 = 0.485\ \angle -1.325 = 0.485\ \angle -0.422\pi$	0.485	-0.422π
3	$0.054 - j\,0.324 = 0.328\ \angle -1.40 = 0.282\ \angle -0.45\,\pi$	0.282	-0.45π
4	$0.03 - j\,0.246 = 0.248\ \angle -1.45 = 0.248\ \angle -0.46\,\pi$	0.248	-0.46π
6	$0.014 - j\,0.167 = 0.166\ \angle -1.4866 = 0.078\ \angle -0.473\pi$	0.166	-0.473π
8	$0.0080 - j\,0.12 = 0.12\ \angle -1.506 = 0.12\ \angle -0.48\,\pi$	0.12	$-0.48\,\pi$

Table 2 : Frequency Spectrum for a = 1

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
−8	$0.015 + j\,0.123 = 0.124\angle 1.45 = 0.124\angle 0.461\pi$	0.124	0.461π
−6	$0.03 + j\,0.162 = 0.165\angle 1.39 = 0.165\angle 0.442\pi$	0.165	0.442π
−4	$0.059 + j\,0.24 = 0.25\angle 1.33 = 0.25\angle 0.423\pi$	0.25	0.423π
−3	$0.1 + j\,0.3 = 0.316\angle 1.25 = 0.316\angle 0.398\pi$	0.316	0.398π
−2	$0.2 + j\,0.4 = 0.45\angle 1.11 = 0.45\angle 0.353\pi$	0.45	0.353π
−1	$0.5 + j\,0.5 = 0.707\angle 0.785 = 0.707\angle 0.25\pi$	0.707	0.25π
0	$1 + j\,0 = 1\angle 0 = 1\angle 0$	1.0	0
1	$0.5 - j\,0.5 = 0.707\angle -0.785 = 0.707\angle -0.25\pi$	0.707	-0.25π
2	$0.2 - j\,0.4 = 0.45\angle -1.11 = 0.45\angle -0.353\pi$	0.45	-0.353π
3	$0.1 - j\,0.3 = 0.316\angle -1.25 = 0.316\angle -0.398\pi$	0.316	-0.398π
4	$0.059 - j\,0.24 = 0.25\angle -1.33 = 0.25\angle -0.423\pi$	0.25	-0.423π
6	$0.03 - j\,0.162 = 0.165\angle -1.39 = 0.165\angle -0.442\pi$	0.165	-0.442π
8	$0.015 - j\,0.123 = 0.124\angle -1.45 = 0.124\angle -0.461\pi$	0.124	-0.461π

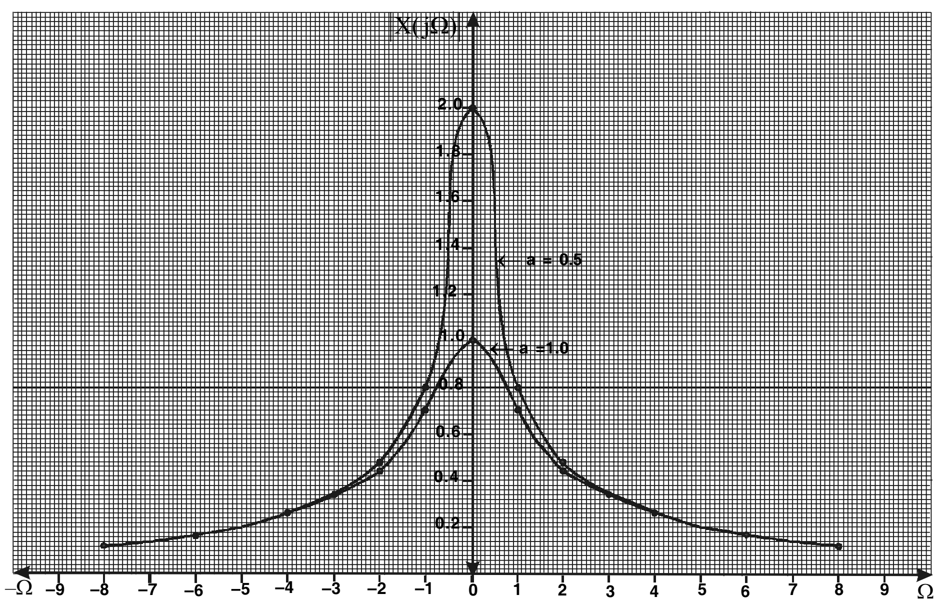


Fig 1 : Magnitude spectrum of $X(j\Omega)$.

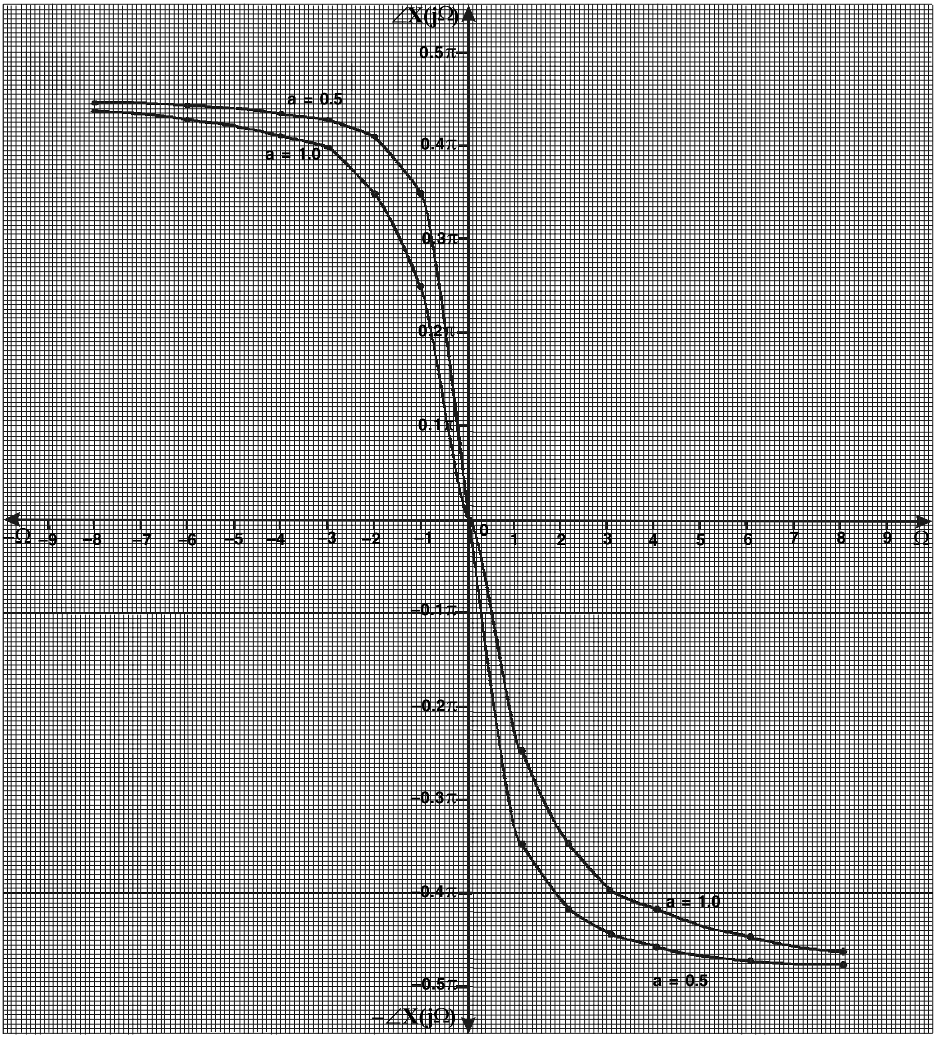


Fig 2: *Phase spectrum of X(jΩ).*

4.16 Summary of Important Concepts

1. The Fourier series is frequency domain representation of periodic signals.
2. The Fourier series exists only if Dirichlet's conditions are satisfied.
3. The signals with negative frequency are required for mathematical representation of real signals in terms of complex exponential signals.
4. In exponential form of Fourier series, $|c_n|$ represents the magnitude of n^{th} harmonic component.
5. In exponential form of Fourier series, $\angle c_n$ represents the phase of the n^{th} harmonic component.
6. The plot of harmonic magnitude/phase versus harmonic number "n" (or harmonic frequency) is called frequency spectrum.
7. The frequency spectrum obtained from Fourier series is also called line spectrum.
8. The plot of magnitude versus n (or $n\Omega_0$) is called magnitude (line) spectrum.

9. The plot of phase versus n (or $n\Omega_0$) is called phase (line) spectrum.
10. For signals with even symmetry, the Fourier coefficients b_n are zero.
11. For signals with odd symmetry, the Fourier coefficients a_0 and a_n are zero.
12. For signals with half wave symmetry, the Fourier series will consists of odd harmonic terms alone.
13. A signal with half wave symmetry, if in addition has even/odd symmetry then it is said to have quarter wave symmetry.
14. For signals with quarter wave symmetry, the Fourier series will consists of either odd harmonics of sine terms or odd harmonics of cosine terms.
15. The Fourier transform has been developed from Fourier series by considering the fundamental period T as infinity.
16. The Fourier transform is used to obtain the frequency domain representation of non-periodic as well as periodic signals.
17. The Fourier transform of a signal exists only if the signal is absolutely integral.
18. The Fourier transform of a signal is also called analysis of the signal.
19. The inverse Fourier transform of a signal is also called synthesis of the signal.
20. The frequency spectrum of non-periodic signals will be continuous, whereas frequency spectrum of periodic signals will be discrete.
21. The magnitude spectrum will have even symmetry and phase spectrum will have odd symmetry.
22. The Fourier transform of a periodic continuous time signal will have impulses located at the harmonic frequencies of the signal.
23. The ratio of Fourier transform of output and input signal of a system is called transfer function in frequency domain.
24. The Fourier transform of impulse response gives the frequency domain transfer function.
25. The Fourier transform is evaluation of Laplace transform along imaginary axis in s-plane.

4.17 Short Questions and Answers

Q4.1 ***What is the value of x(t) at $t = t_0$ in the waveform shown in fig Q4.1.***

Solution:

In the waveform shown in fig Q4.1, $t = t_0$ is a point of discontinuity. If $t = t_0$ is a point of discontinuity, then the value of x(t) at $t = t_0$ is given by,

Fig Q4.1.

$$x(t_0) = \frac{x(t_0^+) + x(t_0^-)}{2}$$

Here, $x(t_0^+) = 4$, $\quad x(t_0^-) = 0 \qquad \therefore x(t_0) = \frac{4+0}{2} = 2$

Q4.2 ***Find the constant component of the periodic pulse signal shown in fig Q4.2.***

Solution:

The constant component of any periodic time domain signal is $a_0/2$, where,

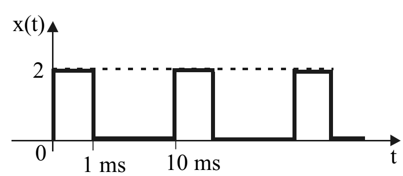


Fig Q4.2.

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt$$

Here, $x(t) = 2 \quad ; \quad$ for t = 0 to 1 ms, and T = 10 ms

$$\therefore a_0 = \frac{2}{10}\int_0^1 2\,dt = \frac{2}{10}\,[2t]_0^1 = \frac{2}{10}\,[2 - 0] = \frac{4}{10} = 0.4$$

$$\therefore \text{Constant component} = \frac{a_0}{2} = \frac{0.4}{2} = 0.2$$

Q4.3 ***Determine the magnitude of the fundamental frequency component of the periodic pulse signal shown in fig Q4.3.***

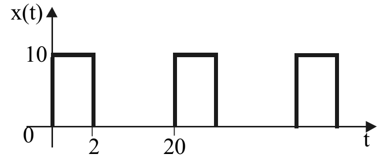


Fig Q4.3.

Solution:

The Fourier coefficient of n^{th} harmonic component is given by,

$$c_n = \frac{1}{T}\int_0^T x(t)\, e^{-jn\Omega_0 t}dt \quad ; \quad \text{where } \Omega_0 = \frac{2\pi}{T}$$

The Fourier coefficient of fundamental component is obtained when n = 1.

$$\therefore\ c_1 = \frac{1}{T}\int_0^T x(t)\, e^{\frac{-j2\pi t}{T}}dt$$

Here, x(t) = 10 ; for t = 0 to 2, and T = 20

$$\therefore c_1 = \frac{1}{20}\int_0^2 10\, e^{\frac{-j2\pi}{20}t}dt = \frac{1}{2}\int_0^2 e^{\frac{-j\pi t}{10}}dt = \frac{1}{2}\left[\frac{e^{\frac{-j\pi t}{10}}}{\frac{-j\pi}{10}}\right]_0^2 = \frac{1}{2}\left[\frac{e^{\frac{-j\pi 2}{10}}}{\frac{-j\pi}{10}} - \frac{e^0}{\frac{-j\pi}{10}}\right]$$

$$= \frac{5}{-j\pi}\left(e^{-j\frac{\pi}{5}} - 1\right) = \frac{5}{-j\pi}\left(\cos\frac{\pi}{5} - j\sin\frac{\pi}{5} - 1\right)$$

$$= j\,1.5915\,(0.8090 - j\,0.5878 - 1) = 0.9355 - j\,0.304 = 0.9836\angle -0.314 \text{ rad}$$

Magnitude of fundamental component, $|c_1| = 0.9836$

Q 4.4 ***What is Magnitude and Phase spectrum?***

Let x(t) be a time domain signal and X(jΩ) be the Fourier transform of x(t).

The X(jΩ) is a complex function of Ω and so it can be expressed as,

$$X(j\Omega) = |X(j\Omega)|\ \angle X(j\Omega)$$

where, $|X(j\Omega)|$ = Magnitude function or Magnitude spectrum.

$\angle X(j\Omega)$ = Phase function or phase spectrum.

Q 4.5 ***The Fourier transform of the signal shown in fig Q4.5.1 is,***

$$X(j\Omega) = \frac{1}{\Omega^2}\left(e^{j\Omega} - j\Omega e^{j\Omega} - 1\right).$$

Using the properties of Fourier transform find the Fourier transform of the signal shown in fig Q4.5.2 and Q4.5.3.

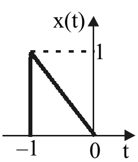


Fig Q4.5.1.

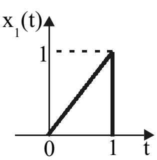


Fig Q4.5.2.

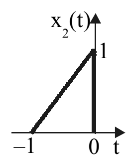


Fig Q4.5.3.

Solution:

The signal shown in fig Q4.5.2 is the folded version of the signal shown in fig Q4.5.1.

i.e., $x_1(t) = x(-t)$

Given that , $\mathcal{F}\{x(t)\} = X(j\Omega) = \frac{1}{\Omega^2}\left(e^{j\Omega} - j\Omega e^{j\Omega} - 1\right)$

Using time reversal property of Fourier transform we can write,

$$\mathcal{F}\{x_1(t)\} = X_1(j\Omega) = X(-j\Omega) = \frac{1}{\Omega^2}\left(e^{-j\Omega} + je^{-j\Omega} - 1\right)$$

The signal shown in fig Q4.5.3 is the shifted version of the signal shown in fig Q4.5.2.

i.e., $x_2(t) = x_1(t - t_0)$; and $t_0 = -1$

Using time shifting property of Fourier transform we can write,

$$\mathcal{F}\{x_2(t)\} = X_2(j\Omega) = e^{j\Omega t_0}X_1(j\Omega) = e^{-j\Omega}\,\frac{1}{\Omega^2}\left(e^{-j\Omega} + je^{-j\Omega} - 1\right) = \frac{e^{-j\Omega}}{\Omega^2}\left(e^{-j\Omega} + je^{-j\Omega} - 1\right)$$

Q 4.6 ***If Fourier transform of $e^{-t}\,u(t)$ is $\dfrac{1}{1+j\Omega}$ then find the Fourier transform of $\dfrac{1}{1+t}$ using duality property.***

Solution:

Given, $x_1(t) = e^{-t}\,u(t)$ and $X_1(j\Omega) = \dfrac{1}{1+j\Omega}$

$$x_2(t) = \frac{1}{1+t}$$

Here, $x_2(t)$ and $X_1(j\Omega)$ are similar functions.

$\therefore$ By duality property,

$$X_2(j\Omega) = 2\pi\left(x_1(t)\big|_{t=-j\Omega}\right) = 2\pi\left(e^{-t}\,u(t)\big|_{t=-j\Omega}\right) = 2\pi e^{j\Omega}\,u(-j\Omega)$$

Q 4.7 ***Find the Fourier constant a_0 for the continuous time signal defined as,***

$$x(t) = Kt \quad ; \quad 0 \le t \le \frac{T}{2}$$
$$= K(T-t)\,; \quad \frac{T}{2} \le t \le T$$

Solution:

Given that, $x(t) = Kt \quad ; \quad 0 \le t \le \dfrac{T}{2}$

$$= K(T-t)\,; \quad \frac{T}{2} \le t \le T$$

$$a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^{\frac{T}{2}} Kt\,dt + \frac{2}{T}\int_{\frac{T}{2}}^{T} K(T-t)\,dt$$

$$= \frac{2}{T}\left[\frac{Kt^2}{2}\right]_0^{\frac{T}{2}} + \frac{2}{T}\left[KTt - \frac{Kt^2}{2}\right]_{\frac{T}{2}}^{T} = \frac{2}{T}\left[\frac{KT^2}{8} - 0\right] + \frac{2}{T}\left[KT^2 - \frac{KT^2}{2} - \frac{KT^2}{2} + \frac{KT^2}{8}\right]$$

$$= \frac{2}{T}\left[\frac{KT^2}{8}\right] + \frac{2}{T}\left[\frac{KT^2}{8}\right] = \frac{4}{T}\,\frac{KT^2}{8} = \frac{KT}{2}$$

Q 4.8 ***A periodic signal x(t) is defined as $x(t) = (1-t)^2\,;\ 0 \le t \le T$. Find the Fourier coefficient b_n.***

Solution

Given that, $x(t) = (1-t)^2\,;\ 0 \le t \le T$.

Now, $x(-t) = (1-(-t)^2) = 1 - t^2$.

Here $x(t) = x(-t)$ and so the given signal is even signal.

For even signals, $b_n = 0$.

Q 4.9 ***A continuous time signal varies exponentially in the interval 0 to T. Find the Fourier constant $\dfrac{a_0}{2}$ of the signal.***

Solution

Given that, $x(t) = e^t\,;\ 0 \le t \le T$.

$$\text{Now, } a_0 = \frac{2}{T}\int_0^T x(t)\,dt = \frac{2}{T}\int_0^T e^t\,dt = \frac{2}{T}\left[e^t\right]_0^T = \frac{2}{T}\left[e^T - e^0\right] = \frac{2}{T}\left(e^T - 1\right)$$

$$\therefore \frac{a_0}{2} = \frac{e^T - 1}{T}$$

Q 4.10 ***Find the Fourier transform of the signal $e^{-3|t|}\,u(t)$.***

Solution

$$x(t) = e^{-3|t|} = e^{-3t} \text{ for } t > 0$$
$$= e^{3t} \text{ for } t < 0$$

The Fourier transform of x(t) is,

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{0} e^{3t}\, e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-3t}\, e^{-j\Omega t}\, dt = \int_{-\infty}^{0} e^{(3-j\Omega)t}\, dt + \int_{0}^{\infty} e^{-(3+j\Omega)t}\, dt$$

$$= \left[\frac{e^{(3-j\Omega)t}}{3-j\Omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(3+j\Omega)t}}{-(3+j\Omega)}\right]_{0}^{\infty} = \left[\frac{e^{0}}{3-j\Omega} - \frac{e^{-\infty}}{3-j\Omega}\right] + \left[-\frac{e^{-\infty}}{3+j\Omega} + \frac{e^{0}}{3+j\Omega}\right]$$

$\boxed{e^{-\infty} = 0\ ;\ e^{0} = 1}$

$$= \frac{1}{3-j\Omega} + \frac{1}{3+j\Omega} = \frac{3+j\Omega+3-j\Omega}{3^2+\Omega^2} = \frac{6}{3^2+\Omega^2}$$

$\boxed{(a+b)(a-b) = a^2 - b^2}$ $\boxed{j^2 = -1}$

Q 4.11 ***Determine the Fourier transform of x(t) using time shifting property, $x(t) = e^{-3|t-t_0|} + e^{3|t+t_0|}$.***

Solution

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{e^{-3|t-t_0|} + e^{3|t+t_0|}\right\}$$
$$= \mathcal{F}\left\{e^{-3|t-t_0|}\right\} + \mathcal{F}\left\{e^{3|t+t_0|}\right\}$$
$$= \frac{3\times 2}{3^2+\Omega^2} \times e^{-j\Omega t_0} + \frac{3\times 2}{3^2+\Omega^2}\times e^{j\Omega t_0} = \frac{6}{3^2+\Omega^2}\left(e^{-j\Omega t_0} + e^{j\Omega t_0}\right)$$
$$= \frac{12\cos\Omega t_0}{3^2+\Omega^2}$$

$\boxed{\mathcal{F}\left\{e^{-a|t|}\right\} = \frac{2a}{a^2+\Omega^2}}$

By time shifting property,
$$F\left\{e^{-a|t-t_0|}\right\} = \frac{2}{a^2+\Omega^2}\, e^{-j\Omega t_0}$$

$\boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$

Q 4.12 ***For the signal shown in fig Q4.12. Find a) X(j0) b) $\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$.***

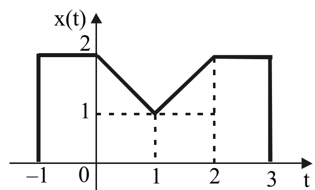


Fig Q4.12.

Solution

a) $X(j0) = \int_{-\infty}^{+\infty} x(t)\, dt$ = Area of the signal

= Area of rectangle – Area of triangle $= 4\times 2 - \frac{1}{2}\times 2\times 1 = 8 - 1 = 7$

b) By definition of inverse Fourier transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

On letting $t = 0$ in the above equation we get,

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{0}\, d\Omega$$

$\boxed{e^{0} = 1}$

$\boxed{\text{from fig Q4.12 , } x(0) = 2}$

$$\therefore \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi \times x(0) = 2\pi\times 2 = 4\pi$$

Q 4.13 ***Find energy in frequency domain for the signal shown in fig Q4.13.1.***

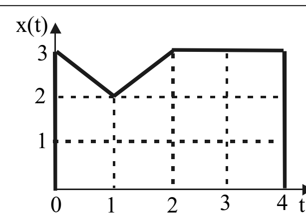


Fig Q4.13.1.

Solution

The square of the given signal is shown in fig Q4.13.2.

The energy E in frequency domain is given by,

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\Omega)|^2\,d\Omega$$

$$= \int_{-\infty}^{+\infty}|x(t)|^2\,dt \qquad \boxed{\text{Using Parseval's relation}}$$

$$= \text{Area of } x^2(t) = \text{Area of rectangle } - \text{Area of triangle}$$

$$= 9\times 4 - \frac{1}{2}\times 2\times 5 = 36 - 5 = 31\,\text{joules}$$

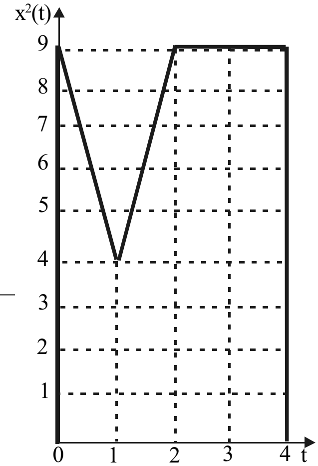


Fig Q4.13.2.

Q4.14 Determine which of the following real signals shown in fig Q4.14 have Fourier transforms that satisfy the following conditions.

a) $Re\{X(j\Omega)\} = 0$

b) $Im\{X(j\Omega)\} = 0$

c) $\int_{-\infty}^{+\infty} X(j\Omega)\,d\Omega = 0$

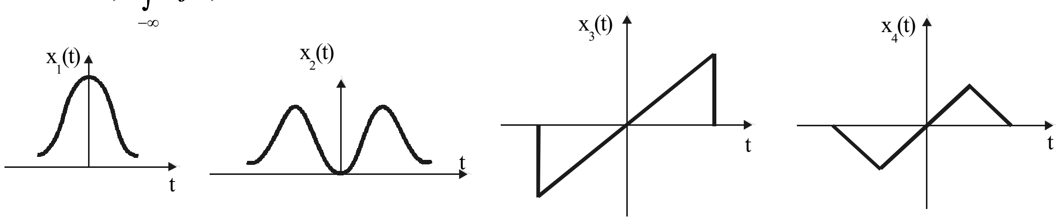


Fig a. *Fig b.* *Fig c.* *Fig d.*

Fig Q4.14.

Solution

a) Given that $Re\{X(j\Omega)\} = 0$

In order to satisfy the given condition, the time domain signal should be real and odd.

$\therefore$ The signals shown in fig c and d will satisfy the condition, $Re\{X(j\Omega)\} = 0$.

b) Given that $Im\{j\Omega\} = 0$

In order to satisfy the given condition, the time domain signal should be real and even.

$\therefore$ The signals shown in fig a and b will satisfy the condition, $Im\{X(j\Omega\} = 0$

c) Given that $\int_{-\infty}^{+\infty} X(j\Omega)\,d\Omega = 0$

We know that, $\int_{-\infty}^{+\infty} X(j\Omega)\,d\Omega = 2\pi\, x(0)$

$\therefore$ If $x(0) = 0$, then $\int_{-\infty}^{+\infty} X(j\Omega)\,d\Omega = 0$

Here the signals shown in fig b, c and d has $x(0) = 0$, and so they will satisfy the condition $\int_{-\infty}^{+\infty} X(j\Omega)\,d\Omega = 0$

Q 4.15 ***For the signal shown in fig Q4.15.1., find energy in frequency domain.***

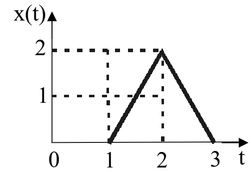


Fig Q4.15.1.

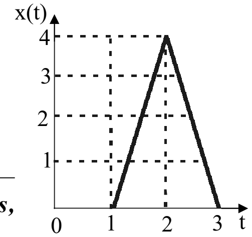


Fig Q4.15.2.

Solution

The square of the given signal is shown in fig Q4.15.2.

The energy E in frequency domain is given by,

$$E = \frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\Omega)|^2\, d\Omega$$

$$= \int_{-\infty}^{+\infty}|x(t)|^2\, dt$$

Using Parseval's relation

= Area of $x^2(t)$ = Area of triangle in fig Q4.15.2.

$$= \frac{1}{2}\times 2\times 4 = 4 \text{ joules}$$

Q 4.16 ***The impulse response of a system in frequency domain is,***

$$H(j\Omega) = +j \quad ; \quad \Omega < 0$$
$$= -j \quad ; \quad \Omega > 0$$

Find the response for the input, x(t) = cos t.

Solution

Given that, x(t) = cos t

$$\therefore X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{\cos t\} = \pi[\delta(\Omega+1)+\delta(\Omega-1)]$$

$$\therefore X(j\Omega) = \pi\,\delta(\Omega+1) \ ; \ \Omega < 0$$
$$= \pi\,\delta(\Omega-1) \ ; \ \Omega > 0$$

Let y(t) be response and h(t) be impulse response in time domain.

We know that, $y(t) = x(t) * h(t)$

On taking Fourier transform of above equation we get,

$$Y(j\Omega) = \mathcal{F}\{x(t) * h(t)\}$$
$$= X(j\Omega)\ H(j\Omega)$$

Using convolution property of Fourier transform

$$\therefore Y(j\Omega) = j\times\pi\,\delta(\Omega+1) \ ; \ \Omega < 0$$
$$= -j\times\pi\,\delta(\Omega-1) \ ; \ \Omega > 0$$

$$\therefore Y(j\Omega) = j\pi\,\delta(\Omega+1) - j\pi\,\delta(\Omega-1) = \frac{\pi}{j}[\delta(\Omega-1)-\delta(\Omega+1)]$$

Response in time domain, $y(t) = \mathcal{F}^{-1}\{Y(j\Omega)\} = \mathcal{F}^{-1}\left\{\frac{\pi}{j}[\delta(\Omega-1)-\delta(\Omega+1)]\right\} = \sin t$

Q 4.17 ***Find the output of LTI system with impulse response*** $h(t) = \dfrac{\sin 5(t-1)}{\pi(t-1)}$ ***and input*** $x(t) = \dfrac{\sin 5(t+1)}{\pi(t+1)}$

Solution

$$H(j\Omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{\frac{\sin 5(t-1)}{\pi(t-1)}\right\} = \mathcal{F}\left\{\frac{\sin 5t}{\pi t}\right\}e^{-j\Omega}$$

Using time shifting property

$$= 1\times e^{-j\Omega} \ ; \ |\Omega| < 5 \quad = e^{-j\Omega} \ ; \ |\Omega| < 5$$

$$\mathcal{F}\left\{\frac{\sin\Omega_0 t}{\pi t}\right\} = \mathcal{F}\left\{\frac{\Omega_0}{\pi}\frac{\sin\Omega_0 t}{\Omega_0 t}\right\}$$
$$= \mathcal{F}\left\{\frac{\Omega_0}{\pi}\ \text{sinc}\ \Omega_0 t\right\}$$
$$= 1 \ ; \ |\Omega| < \Omega_0$$

Similarly, $X(j\Omega) = e^{+j\Omega} \ ; \ |\Omega| < 5$

We know that,

Response, $y(t) = x(t) * h(t)$ .....(1)

On taking Fourier transform of equation (1) we get,

$$Y(j\Omega) = \mathcal{F}\{x(t) * h(t)\} = X(j\Omega)H(j\Omega)$$

Using convolution property

$$= e^{j\Omega}\, e^{-j\Omega} \;;\; |\Omega| < 5 \quad = 1 \;\;;\; |\Omega| < 5$$

$$\therefore y(t) = \mathcal{F}^{-1}\{Y(j\Omega)\} = \frac{5}{\pi}\ \mathrm{sinc}\,5t = \frac{5}{\pi}\frac{\sin 5t}{5t} = \frac{\sin 5t}{\pi t}$$

Q 4.18 ***Find the inverse Fourier transform of,*** $X(j\Omega) = \dfrac{1}{(4+j\Omega)^2}$***. Using convolution property.***

Solution

Let, $\quad X(j\Omega) = \dfrac{1}{(4+j\Omega)^2} = \dfrac{1}{4+j\Omega} \times \dfrac{1}{4+j\Omega} = X_1(j\Omega) \times X_2(j\Omega)$

where, $X_1(j\Omega) = \dfrac{1}{4+j\Omega}$ and $X_2(j\Omega) = \dfrac{1}{4+j\Omega}$

$$\therefore\; x_1(t) = \mathcal{F}^{-1}\{X_1(j\Omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{4+j\Omega}\right\} = e^{-4t}\, u(t)$$

$$x_2(t) = e^{-4t}\, u(t)$$

By time convolution property,

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_1(t-\tau)\, d\tau$$

$$= \int_0^t e^{-4\tau}\, e^{-4(t-\tau)}\, d\tau = \int_0^t e^{-4t}\, e^{-4\tau} e^{4\tau}\, d\tau \quad = e^{-4t}\int_0^t e^{-4\tau+4\tau}\, d\tau$$

$$= e^{-4t}\int_0^t d\tau \;= e^{-4t}[T]_0^t = e^{-4t}[t-0] \;= te^{-4t} \;\;;\;\; t \ge 0 \quad = te^{-4t}\, u(t)$$

Using frequency shifting property

Q 4.19 ***If x(t) and X(jΩ) are Fourier transform pair. Determine the Fourier transform of,*** $x(t)\sin\Omega_0 t$.

Solution

$$\mathcal{F}\{x(t)\,\sin\Omega_0 t\} = \mathcal{F}\left\{x(t)\ \frac{e^{j\Omega_0 t} - e^{-j\Omega_0 t}}{2j}\right\} = \frac{1}{2j}\mathcal{F}\{x(t)\, e^{j\Omega_0 t}\} - \frac{1}{2j}\mathcal{F}\{x(t)\, e^{-j\Omega_0 t}\}$$

$$= \frac{1}{2j}X(\Omega-\Omega_0) - \frac{1}{2j}X(\Omega+\Omega_0) = \frac{1}{2j}\left[X(\Omega-\Omega_0) - X(\Omega+\Omega_0)\right]$$

Q 4.20 ***Determine the exponential Fourier series representation of the following signals.***

a) $x(t) = \cos 4t + \sin 6t$

b) $x(t) = \sin^2 t$

Solution

a) $\mathbf{x(t) = \cos 4t + \sin 6t}$

$$= \frac{e^{j4t} + e^{-j4t}}{2} + \frac{e^{j6t} - e^{-j6t}}{2j} = -\frac{1}{2j}e^{-j6t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j4t} + \frac{1}{2j}e^{j6t}$$

b) $\mathbf{x(t) = \sin^2 t}$

$$= \frac{1-\cos 2t}{2} = \frac{1}{2} - \frac{1}{2}\ \cos 2t = \frac{1}{2} - \frac{1}{2}\left(\frac{e^{j2t} + e^{-j2t}}{2}\right)$$

$$= \frac{1}{2} - \frac{1}{4}e^{j2t} - \frac{1}{4}e^{-j2t} = -\frac{1}{4}e^{-j2t} + \frac{1}{2} - \frac{1}{4}e^{j2t}$$

4.18 MATLAB Programs

Program 4.1

Write a MATLAB program to find Fourier transform of the following signals.

a) A b) u(t) c) $Ae^{-t}u(t)$ d) At $e^{-bt}u(t)$ e) A $\cos\Omega_0 t$

```
% Program to find fourier transform of given time domain signals

%Let t, A, b, o be any real variables
syms t real; syms A real;  syms b real;  syms o real;

%(a)
x = A;
disp('(a) Fourier transform of "A" is');
X=fourier(x)

%(b)
x = heaviside(t);      %heaviside(t) is unit step signal
disp('(b) Fourier transform of  "u(t)" is');
X=fourier(x)

%(c)
x = A*exp(-t)*heaviside(t);
disp('(c) Fourier transform of  "A exp(-t) u(t)" is');
X=fourier(x)

%(d)
x=A*t*exp(-b*t)*heaviside(t);
disp('(d) Fourier transform of  "At exp(-b*t) u(t)" is');
X=fourier(x)

%(e)
x=A*cos(o*t);
disp('(e) Fourier transform of "A cos(o*t)" is');
X=fourier(x)
```

OUTPUT

```
(a) Fourier transform of "A" is
      X =
          2*i*pi*dirac(1,w)
(b) Fourier transform of  "u(t)" is
      X =
          pi*dirac(w)-i/w
(c) Fourier transform of  "A exp(-t) u(t)" is
      X =
         A/(1+i*w)
(d) Fourier transform of  "At exp(-b*t) u(t)" is
      X =
         A/(b+i*w)^2
(e) Fourier transform of "A cos(o*t)" is
      X =
          A*pi*(dirac(w-o)+dirac(w+o))
```

Program 4.2

Write a MATLAB program to find inverse Fourier transform of the following frequency domain signals.

a) $X_1(j\Omega) = A\pi[\delta(\Omega-\Omega_0)+\delta(\Omega+\Omega_0)]$ b) $X_2(j\Omega) = \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0)-\delta(\Omega+\Omega_0)\right]$

c) $X_3(j\Omega) = \frac{A}{1+j\Omega}$ d) $X_4(j\Omega) = \frac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}$

```
% Program to find inverse Fourier transform of frequency domain signals

%Let t, A, b, o, w be any real variables
syms t real; syms A real; syms o real; syms w real;

%(a)
X1=A*pi*(dirac(w-o)+dirac(w+o));
disp('(a) Inverse Fourier transform of X1 is');
x1=ifourier(X1,t)

%(b)
X2=A*pi*(dirac(w-o)-dirac(w+o))/i;
disp('(b) Inverse Fourier transform of X2 is');
x2=ifourier(X2,t)

%(c)
X3=A/(1+i*w);
disp('(c) Inverse Fourier transform of X3 is');
x3=ifourier(X3,t)

%(d)
X4=(3*w+14)/(w^2+7*w+12);
disp('(d) Inverse Fourier transform of X4 is');
x4=ifourier(X4,t)
```

OUTPUT

```
(a) Inverse Fourier transform of X1 is
      x1 =
             A*cos(o*t)
(b) Inverse Fourier transform of X2 is
      x2 =
             A*sin(o*t)
(c) Inverse Fourier transform of X3 is
      x3 =
              A*exp(-t)*heaviside(t)
(d) Inverse Fourier transform of X4 is
      x4 =
              1/2*i*(2*heaviside(t)-1)*(-2*exp(-4*i*t)+5*exp(-3*i*t))
```

Program 4.3

Write a MATLAB program to determine response of LTI system whose impulse response is $h(t)= 2e^{-3t}u(t)$ for the input is $x(t)=2e^{-5t}u(t)$,using Fourier transform.

```
% Program to find the  response of LTI system using Fourier transform

syms t real;
h=2*exp(-3*t).*heaviside(t);
x=2*exp(-5*t).*heaviside(t);
```

```
disp('Fourier transform of the impulse response is');
H=fourier(h)

disp('Fourier transform of the input is');
X=fourier(x)

Y=H*X;
disp('The response of the given system is');
y_res=ifourier(Y,t)
```

OUTPUT

```
Fourier transform of the impulse response is
      H =
         2/(3+i*w)
Fourier transform of the input is
      X =
         2/(5+i*w)
The response of the given system is
      y_res =
         2*heaviside(t)*(exp(-3*t)-exp(-5*t))
```

Program 4.4

Write a MATLAB program to perform convolution of signals, $x_1(t) = e^{-2t}u(t)$ and $x_2(t)=e^{-6t}u(t)$, using Fourier transform.

```
% Program to perform convolution using Fourier transform

syms t real;
x1=exp(-2*t).*heaviside(t); %heaviside(t) is unit step signal
x2=exp(-6*t).*heaviside(t);

disp('Fourier transform of x1(t) is');
X1=fourier(x1)
disp('Fourier transform of x2(t) is');
X2=fourier(x2)
Y=X1*X2;

disp('Let x3 be convolution of x1(t) and x2(t).');
x3=ifourier(Y,t)
```

OUTPUT

```
Fourier transform of x1(t) is
   X1 =
      1/(2+i*w)
Fourier transform of x2(t) is
   X2 =
      1/(6+i*w)
Let x3 be convolution of x1(t) and x2(t).
   x3 =
      1/4*heaviside(t)*(exp(-2*t)-exp(-6*t))
```

Program 4.5

Write a MATLAB program to reconstruct the following periodic signal represented by its Fourier series, by considering only 3,5 and 59 terms.

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin n\Omega_0 t; \;\; b_n = \frac{2}{\pi n}; \;\; \Omega_0 = 2\pi F; \;\; F = 1.$$

```
% Program to reconstruct periodic square pulse signal using
% partial sum of fourier series

syms t real;

N=input('Enter number of signals to reconstruct');
n_har=input('Enter no. of harmonics in each signal as array');

t=-1:.002:1;
omega_o=2*pi;
for k=1:N
    n=[];
    n=[1:2:n_har(k)];
    b_n=2./(pi*n);
    L_n=length(n);
    x=0.5+b_n*sin(omega_o*n'*t);
    subplot(N,1,k),plot(t,x),xlabel('t'),ylabel('recons signal');
    axis( [-1 1 -0.5 1.5]);
    text(.55, 1.0,['no. of har. =',num2str(n_har(k))]);
end
```

OUTPUT

```
Enter number of signals to reconstruct 3
Enter no. of harmonics in each signal as array [3 5 59]
```

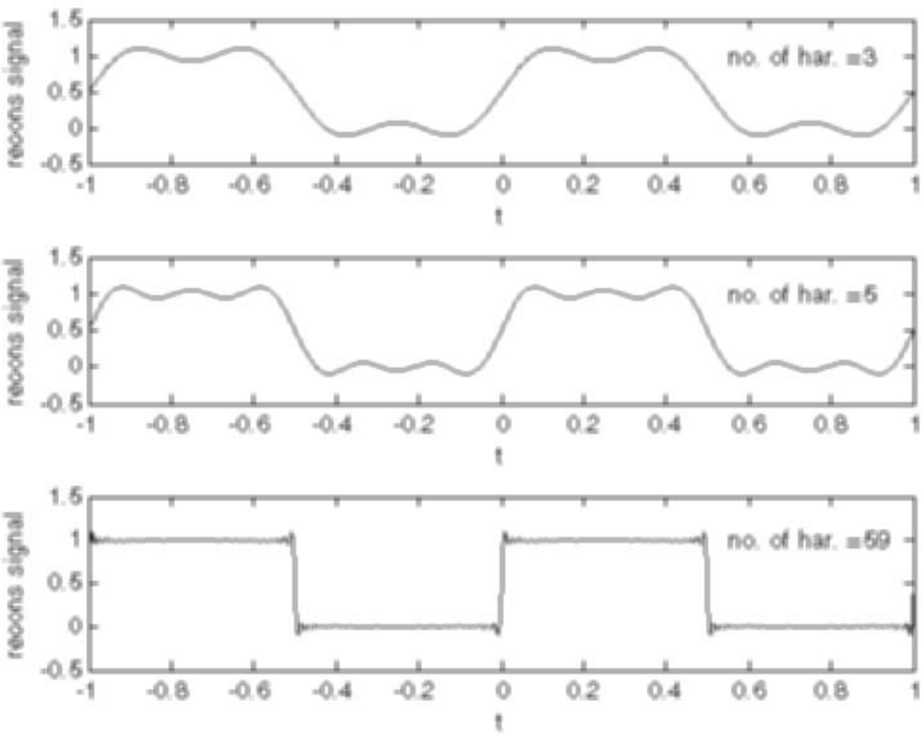


Fig P4.5 : *Reconstructed signals using only 3, 5 and 59 terms of Fourier series.*

4.19 Exercises

I. Fill in the blanks with appropriate words

1. In Fourier series representation of a signal, if Ω_0 is the fundamental frequency then $n\Omega_0$ are called _________ frequencies.
2. Fourier series is useful for frequency domain analysis of __________ signals.
3. The plot of magnitude of Fourier coefficient c_n with respect to "n" is called__________.
4. For a signal x(t), the condition to be satisfied for half wave symmetry is__________.
5. For a signal x(t), the condition to be satisfied for quarter wave symmetry is__________ and __________.
6. If x(t) and $X(j\Omega)$ are Fourier transform pair then magnitude of $X(j\Omega)$ is called__________.
7. If x(t) and $X(j\Omega)$ are Fourier transform pair then phase of $X(j\Omega)$ is called__________.
8. The ratio of Fourier transform of output and input of a system is called__________.
9. The Fourier transform of impulse response will be equal to__________.
10. If x(t) is real and odd, then $X(j\Omega)$ will be__________.
11. Fourier transform of periodic continuous time signal consists of __________ located at __________ frequencies of the signal.
12. Fourier transform is evalution of Laplace transform along the __________ axis in s-plane.

Answers

1. harmonic	2. periodic	3. magnitude line spectrum
4. $x(t \pm T/2) = -x(t)$,	5. $x(t \pm T/2) = -x(t)$; $x(-t) = \pm x(t)$	6. magnitude spectrum
7. phase spectrum	8. transfer function	9. transfer function
10. imaginary and odd	11. impulses, harmonic	12. imaginary

II. State whether the following statements are True/False

1. The magnitude line spectrum is symmetric and phase line spectrum is anti-symmetric.
2. For waveforms with even symmetry, the Fourier coefficient a_0 is always zero.
3. For waveforms with odd symmetry, the Fourier coefficients a_0 and a_n are always zero.
4. An alternating waveform will always have even harmonics only.
5. An even signal with half wave symmetry will always have even harmonics of cosine terms.
6. An odd signal with half wave symmetry will always have odd harmonics of sine terms.
7. The shifting of vertical axis of a waveform, will not affect the even/odd symmetry.
8. In frequency spectrum using Fourier transform, the magnitude spectrum will have even symmetry and phase spectrum will have odd symmetry.
9. The Fourier transform of a periodic signal will be summation of impulses.
10. The total average power in a periodic signal is equal to the sum of power in all of its harmonics.
11. Fourier transform is useful for frequency domain analysis of both periodic and non-periodic signals.
12. Laplace transform is a generalized transform and Fourier transform is a particular transform.

Answers

1. True	2. False	3. True	4. False	5. False	6. True
7. False	8. True	9. True	10. True	11. True	12. True

III. Choose the right answer for the following questions

1. ***The Fourier coefficient 'a_n' can be evaluated as,***

   a) $a_n = \frac{2}{T}\int_{-\infty}^{+\infty} x(t)\cos n\Omega_0 t\, dt$

   b) $a_n = \frac{2}{T}\int_{0}^{T} x(t)\cos n\Omega_0 t\, dt$

   c) $a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin n\Omega_0 t\, dt$

   d) $a_n = \frac{2}{T}\int_{0}^{\infty} x(t)\cos n\Omega_0 t\, dt$

2. ***The exponential Fourier coefficient of x(t) = $\sin^2 t$ is,***

   a) $C_1 = \frac{1}{\pi},\quad C_0 = 0,\quad C_{-1} = -\frac{1}{\pi}$

   b) $C_{-1} = -\frac{1}{4},\quad C_0 = \frac{1}{2},\quad C_1 = -\frac{1}{4}$

   c) $C_{-1} = -\frac{1}{2},\quad C_0 = 1,\quad C_1 = \frac{1}{2}$

   d) $C_{-1} = -2,\quad C_0 = 1,\quad C_1 = -2$

3. ***Fourier transform of the signum function, x(t)=sgn(t) is,***

   a) $2/\Omega$ **b)** $2/j\Omega$ **c)** $2/(j\Omega)^2$ **d)** $-2/j\Omega$

4. ***The Unit step signal u(t) can also be expressed as,***

   a) $1+\mathrm{sgn}(t)$ **b)** $1+\delta(t)$ **c)** $\frac{1}{2}+\frac{1}{2}\,\mathrm{sgn}(t)$ **d)** $2\mathrm{sgn}(t)$

5. ***The necessary and sufficient condition for x(t) to be real is,***

   a) $X^*(j\Omega) = X(j\Omega)$ **b)** $X^*(j\Omega) = X(-j\Omega)$ **c)** $X(j\Omega) = X^*(-j\Omega)$ **d)** none of the above

6. ***Fourier series of any periodic signal x(t) can only be obtained if,***

   a) finite number of discontinuities within finite time interval, T.

   b) finite number of positive and negative maxima in the period, T.

   c) well defined at infinite number of points.

   d) both (a) and (b).

7. ***The inverse Fourier Transform of X(jΩ) is defined as,***

   a) $\frac{1}{2\pi}\int_{0}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$ **b)** $\frac{1}{2\pi}\int_{+\infty}^{-\infty} X(j\Omega)\, e^{-j\Omega t}\, dt$ **c)** $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$ **d)** $\frac{1}{2\pi}\int_{0}^{\infty} X(j\Omega)\, e^{-j\Omega t}\, d\Omega$

8. ***The frequency convolution property of Fourier transform says that, $\mathcal{F}\{x_1(t)\, x_2(t)\}$ is given by,***

   a) $X_1(j\Omega)\, X_2(j\Omega)$

   b) $\frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\tau$

   c) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\Omega)\, X_2(j\Omega)\, d\Omega$

   d) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |X_1(j\Omega)|^2\, |X_2(j\Omega)|^2\, d\Omega$

9. ***Fourier transform of a sinusoidal signal $2\sin\Omega_0 t$ is,***

   a) $\frac{2\pi}{j}[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)]$

   b) $\frac{\pi}{2j}[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)]$

   c) $\frac{2\pi}{j}[\delta(\Omega+\Omega_0) - \delta(\Omega-\Omega_0)]$

   d) $\frac{\pi}{j}[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)]$

10. In the Fourier transform of a periodic signal, the magnitude of n^{th} impulse is,

a) $\frac{|C_n|}{2\pi}$ **b)** $2\pi|C_n|$ **c)** $|C_n|$ **d)** $\frac{|C_n|}{2}$

11. Fourier transform of the causal signal, $x(t) = \cos\Omega_0 t\ u(t)$ is,

a) $\frac{\Omega_0}{\Omega_0^2 - \Omega^2}$ **b)** $\frac{j\Omega}{\Omega_0^2 - \Omega^2}$ **c)** $\frac{-\Omega_0}{\Omega^2 + \Omega_0^2}$ **d)** $\frac{j\Omega}{\Omega_0^2 + \Omega^2}$

12. If the signal x(t) has odd and half wave symmetry, then, the Fourier series will have only,

a) odd harmonics of sine terms. **b)** constant term and odd harmonics of cosine terms.
c) even harmonics of sine terms **d)** odd harmonics of cosine terms.

13. Differentiation of signum function is, $\left(i.e., \frac{d}{dt}\ sgn(t)\right)$

a) $2\delta(t)$ **b)** $\delta(t)$ **c)** $\frac{1}{2}\delta(t)$ **d)** $2u(t)$

14. Fourier transform of $\cos\Omega_0 t$ is,

a) $X(\Omega - \Omega_0) + X(\Omega + \Omega_0)$ **b)** $\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
c) $\frac{1}{2}X(\Omega - \Omega_0) + \frac{1}{2}X(\Omega + \Omega_0)$ **d)** $\frac{\pi}{2}\left[X(\Omega - \Omega_0) + X(\Omega + \Omega_0)\right]$

15. The Fourier transform of x(t) exists only if,

a) $\int_0^{\infty} x(t)\ dt < \infty$ **b)** $\int_{-\infty}^{+\infty} x(t)\ dt > 0$ **c)** $\int_{-\infty}^{+\infty} x(t)\ e^{j\Omega t}\ dt < \infty$ **d)** $\int_{-\infty}^{+\infty} x(t)\ dt < \infty$

16. By convolution property of Fourier transform, $\mathcal{F}\{x(t) * h(t)\}$ is,

a) $\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, H(j\Omega)\ d\Omega$ **b)** $X(j\Omega)\, H(j\Omega)$
c) $X(j\Omega) * H(j\Omega)$ **d)** None of the above

17. A periodic signal x(t) of period T_0 is given by, $x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & T_1 < |t| < \frac{T_0}{2} \end{cases}$. The constant component of x(t) is

a) $\frac{T_0}{T_1}$ **b)** $\frac{T_1}{T_0}$ **c)** $\frac{2T_1}{T_0}$ **d)** $\frac{T_1}{2T_0}$

18. For two periodic waveforms, square and triangular, the magnitude of the n^{th} Fourier series coefficients, for $n > 0$, are respectively proportional to,

a) $|n^{-3}|$ and $|n^{-2}|$ **b)** $|n^{-2}|$ and $|n^{-3}|$ **c)** $|n^{-1}|$ and $|n^{-2}|$ **d)** $|n^{-4}|$ and $|n^{-2}|$

19. When the waveform has parabolic structure/wiggles, the magnitude of higher harmonics,

a) increases rapidly **b)** decreases more rapidly
c) remain same **d)** become zero

20. The initial value of a continuous time signal in frequency domain is,

a) $X(0) = \int_0^{\infty} x(t)\,dt$ **b)** $X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(j\Omega)\,dt$ **c)** $X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x(t)\,d\Omega$ **d)** $X(0) = \int_{-\infty}^{\infty} x(t)\,dt$

21. *Which of the following cannot be the Fourier series expansion of a periodic signal?*

a) $x(t) = 2\cos t + 3\cos 3t$ **b)** $x(t) = 2\cos \pi t + 7\cos t$

c) $x(t) = \cos t + 0.5$ **d)** $x(t) = 2\cos 1.5\pi t + \sin 3.5\pi t$

22. *Let C_n be a Fourier coefficient in exponential form. It is given that, $C_n = 2 + j7$ then C_{-n} is,*

a) $-2 - j7$ **b)** $2 - j7$ **c)** $-2 + j7$ **d)** $2 + j7$

23. *The Fourier transform of a function x(t) is $X(j\Omega)$. Then, the Fourier transform of $\frac{d}{dx}x(t)$ will be,*

a) $\frac{dX(\Omega)}{dF}$ **b)** $j2\pi\Omega\, X(\Omega)$ **c)** $j\Omega\, X(j\Omega)$ **d)** $\frac{X(\Omega)}{j\Omega}$

24. *If Fourier transform of the signal $e^{-|t|}$ is $\frac{2}{1+\Omega^2}$ then, the Fourier transform of the signal $\frac{2}{1+t^2}$, using duality property is,*

a) $2\pi\, e^{|j\Omega|}$ **b)** $2\pi\, e^{-|j\Omega|}$ **c)** $2\pi\,(-j\Omega)$ **d)** $2\pi\,(j\Omega)$

25. *Fourier transform of gaussian pulse will be,*

a) another gaussian pulse **b)** squared sinc pulse

c) sinc pulse **d)** impulse train

26. *The signal x(t) = u(6t) has a Fourier transform of,*

a) $\frac{1}{j\Omega} + 6\pi\,\delta(\Omega)$ **b)** $\frac{6}{j\Omega} + 6\pi\,\delta(\Omega)$ **c)** $\frac{1}{j\Omega} + \pi\,\delta(\Omega)$ **d)** $\frac{1}{j6\Omega} + \pi\,\delta(6\Omega)$

27. *Consider the signal, x(t) = t(T-t); $0 \le t \le T$. The Fourier coefficient a_0 for this signal is,*

a) $\frac{T^3}{4}$ **b)** $\frac{T^2}{3}$ **c)** $\frac{3}{T^2}$ **d)** $\frac{T^3}{12}$

28. *The Fourier transform of the signal $x(t) = e^{7t}\,u(-t)$ is,*

a) $\frac{1}{7 + j\Omega}$ **b)** $\frac{7}{1 + j\Omega}$ **c)** $\frac{7}{1 - j\Omega}$ **d)** $\frac{1}{7 - j\Omega}$

29. *For a signal $x(t) = e^{-|t|}$, the Fourier transform is,*

a) $\frac{2}{1+\Omega^2}$ **b)** $\frac{2}{1-\Omega^2}$ **c)** $\frac{1}{2-\Omega^2}$ **d)** $\frac{1}{2+\Omega^2}$

30. *If Fourier transform of $x_1(t)$ is $\frac{a}{\Omega - a}$ and Fourier transform of $x_2(t)$ is $\frac{a}{\Omega + a}$, then $\mathcal{F}\{x_1(t) * x_2(t)\}$ is,*

a) $\frac{\Omega - a}{\Omega + a}$ **b)** $\frac{\Omega + a}{\Omega - a}$ **c)** $\frac{a^2}{\Omega^2 - a^2}$ **d)** $\frac{a^2}{\Omega^2 + a^2}$

31. *If Fourier transform of x(t) is $X(\Omega)$ then Fourier transform of $e^{-j\Omega_0 t}\,x(t)$ is,*

a) $\frac{X(\Omega)}{\Omega_0}$ **b)** $X(\Omega - \Omega_0)$ **c)** $X(\Omega + \Omega_0)$ **d)** $\Omega_0\, X(\Omega)$

32. *If a signal x(t) is differentiated 'm' times to produce an impulse then its Fourier coefficients will be proportional to,*

a) n^m **b)** $\frac{1}{n^{m-1}}$ **c)** n^{m-1} **d)** $\frac{1}{n^m}$

33. If Fourier transform of x(t) is $\dfrac{2a}{a^2+\Omega^2}$ ***, then Fourier transform of*** $\dfrac{d^2}{dt^2}x(t)$ ***is,***

a) $\dfrac{-2a\Omega^2}{a^2+\Omega^2}$ **b)** $\dfrac{4a^2}{(a^2+\Omega^2)^2}$ **c)** $\dfrac{2a\Omega}{(a^2+\Omega^2)^2}$ **d)** $\dfrac{a^2+\Omega^2}{2a}$

34. The Fourier transform of the signal $x(t)=e^{-4|t-t_0|}$ ***is,***

a) $\left(\dfrac{4}{4^2+\Omega^2}\right)e^{-j\Omega t_0}$ **b)** $\dfrac{-4}{4^2+\Omega^2}e^{-j\Omega t_0}$ **c)** $\dfrac{8}{4^2+\Omega^2}e^{-j\Omega t_0}$ **d)** $\dfrac{-8}{4^2+\Omega^2}e^{-j\Omega t_0}$

35. A signal x(t) has a magnitude of 5 at t =0.Then value of $\int_{-\infty}^{+\infty} X(j\Omega)d\Omega$ ***will be,***

a) 10 **b)** 10π **c)** 5π **d)** $\dfrac{10}{\pi}$

Answers

1. b	6. d	11. b	16. b	21. b	26. c	31. b
2. b	7. c	12. a	17. c	22. b	27. b	32. d
3. b	8. b	13. a	18. c	23. c	28. d	33. a
4. c	9. a	14. b	19. b	24. b	29. a	34. c
5. a	10. b	15. d	20. d	25. a	30. c	35. b

IV. Answer the following questions

1. Write the conditions for existence of Fourier series.
2. Write the trigonometric form of Fourier series representation of a periodic signal and explain.
3. Write the exponential form of Fourier series representation of a periodic signal and explain.
4. What is the relation between Fourier coefficients of trigonometric and exponential form?
5. Write a short note on negative frequency.
6. Define frequency spectrum or line spectrum.
7. Write short note on Fourier coefficients of signals with even symmetry and odd symmetry.
8. What is the effect of half wave symmetry on Fourier coefficients of a signal?
9. What is the effect of quarter wave symmetry on Fourier coefficients of a signal?
10. Write any two properties of Fourier series.
11. Write the Parseval's relation for continuous time periodic signal.
12. Explain how Fourier transform is obtained from Fourier series.
13. Define Fourier transform and inverse Fourier transform of a signal.
14. Write any two properties of Fourier transform.
15. State and prove time differentiation property of Fourier transform.
16. State and prove frequency differentiation property of Fourier transform.
17. Write the convolution theorem of Fourier transform.
18. What is the relation between Fourier transform and Laplace transform?
19. Show that Fourier transform of a periodic signal will be impulses.
20. Define the frequency domain transfer function.

V. Solve the following problems

E 4.1 Determine the trigonometric Fourier series representation of the signal shown in fig E4.1.

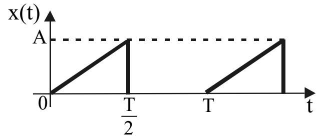


Fig E4.1.

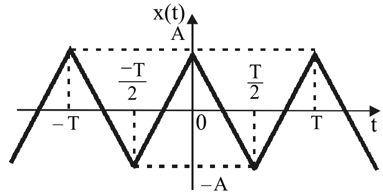


Fig E4.2.

E 4.2 Determine the trigonometric Fourier series representation of the signal shown in fig E4.2.

E 4.3 Determine the exponential Fourier series representation of the signal shown in fig E4.3 and hence obtain the trigonometric Fourier series.

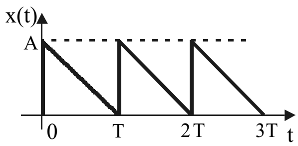


Fig E4.3.

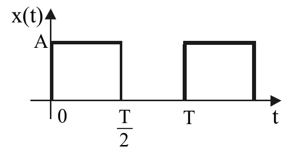


Fig E4.4.

E 4.4 Determine the exponential Fourier series representation of the signal shown in fig E4.4 and hence obtain the trigonometric Fourier series.

E 4.5 Determine the exponential Fourier series representation of the signal shown in fig E4.5.

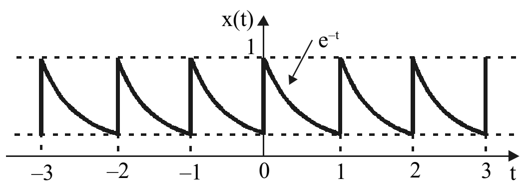


Fig E4.5.

E 4.6 Find the Fourier transform of the time domain signals given below.

i) $x(t) = t \sin\Omega_0 t$; $t = 0$ to $+\infty$

ii) $x(t) = t \cos \Omega_0 t$; $t = 0$ to $+\infty$

E 4.7 Determine the Fourier transform of the rectangular pulse shown in fig E4.7.

E 4.8 Determine the Fourier transform of the triangular pulse shown in fig E4.8.

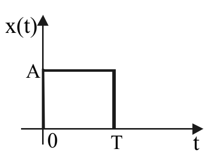


Fig E4.7.

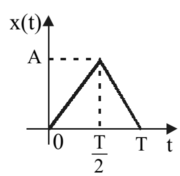


Fig E4.8.

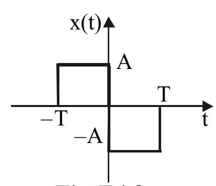


Fig E4.9.

E 4.9 Determine the Fourier transform of the signal shown in fig E4.9.

E 4.10 Determine Fourier transform of the periodic signal shown in fig E4.10.

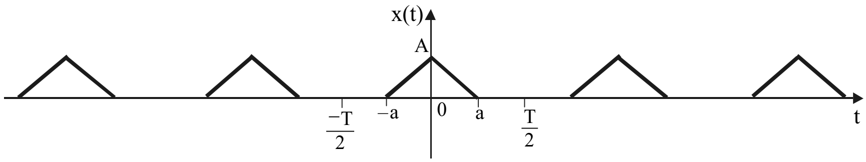


Fig E4.10.

Answers

E4.1 $x(t) = \frac{A}{4} - \frac{2A}{\pi^2}\left(\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right) + \frac{A}{\pi}\left(\frac{\sin\Omega_0 t}{1} - \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3}.....\right)$

E4.2 $x(t) = \frac{8A}{\pi^2}\left(\frac{\cos\Omega_0 t}{1^2} + \frac{\cos 3\Omega_0 t}{3^2} + \frac{\cos 5\Omega_0 t}{5^2} + ...\right)$

E4.3 $x(t) = \frac{A}{j2\pi}\left(.......-\frac{e^{-j3\Omega_0 t}}{3} - \frac{e^{-j2\Omega_0 t}}{2} - \frac{e^{-j\Omega_0 t}}{1} + j\pi + \frac{e^{j\Omega_0 t}}{1} + \frac{e^{j2\Omega_0 t}}{2} + \frac{e^{j3\Omega_0 t}}{3} + ...\right)$

$x(t) = \frac{A}{2} + \frac{A}{\pi}\left(\frac{\sin\Omega_0 t}{1} + \frac{\sin 2\Omega_0 t}{2} + \frac{\sin 3\Omega_0 t}{3} + ...\right)$

E4.4 $x(t) = \frac{A}{j\pi}\left(.....-\frac{e^{-j5\Omega_0 t}}{5} - \frac{e^{-j3\Omega_0 t}}{3} - \frac{e^{-j\Omega_0 t}}{1} + \frac{j\pi}{2} + \frac{e^{j\Omega_0 t}}{1} + \frac{e^{j3\Omega_0 t}}{3} + \frac{e^{j5\Omega_0 t}}{5} + ...\right)$

$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\frac{\sin\Omega_0 t}{1} + \frac{\sin 3\Omega_0 t}{3} + \frac{\sin 5\Omega_0 t}{5} + ...\right)$

E4.5 $x(t) = \left(1 - e^{-1}\right)\left[......+\frac{e^{-j3\Omega_0 t}}{1 - j6\pi} + \frac{e^{-j2\Omega_0 t}}{1 - j4\pi} + \frac{e^{-j\Omega_0 t}}{1 - j2\pi} + 1 + \frac{e^{j\Omega_0 t}}{1 + j2\pi} + \frac{e^{j2\Omega_0 t}}{1 + j4\pi} + \frac{e^{j3\Omega_0 t}}{1 + j6\pi} + ...\right]$

E4.6 i) $X(j\Omega) = \frac{-2\Omega\Omega_0}{(\Omega^2 + \Omega_0^2)^2}$ ii) $X(j\Omega) = \frac{\Omega_0^{\ 2} - \Omega^2}{(\Omega_0^{\ 2} + \Omega^2)^2}$

E4.7 $X(j\Omega) = \frac{A}{j\Omega}\,(1 - e^{-j\Omega T})$

E4.8 $X(j\Omega) = \frac{2A}{T\Omega^2}\,(\,2\,e^{-j\Omega T/2} - e^{-j\Omega T} - 1)$

E4.9 $X(j\Omega) = j\Omega A T^2 \operatorname{sinc}\left(\frac{\Omega T}{2\pi}\right)$

E4.10 $X(j\Omega) = \sum_{n=-\infty}^{+\infty} \frac{A}{2\pi^2 n^2}\left[1 - \cos 2n\pi\right]$

CHAPTER 5

State Space Analysis of Continuous Time Systems

5.1 Introduction

The state variable approach is a powerful tool / technique for the analysis and design of systems. The analysis and design of any system such as linear system, non-linear system, time invariant system, time varying system, multiple input and multiple output system can be carried out using state space method. The state space analysis can be carried with or without initial conditions.

The state space analysis is a modern approach and is also easy to analyse using digital computers. The conventional (or old) methods of analysis employ the transfer function of the system. The drawbacks in the transfer function model and analysis are,

1. Transfer function is defined under zero initial conditions.
2. Transfer function is applicable to linear time invariant systems.
3. Transfer function analysis is restricted to single input and single output systems.
4. Transfer function does not provide information regarding the internal state of the system.

5.2 State Model of a Continuous Time System

State and State Variables

The ***state*** of a continuous time system refers to the condition of the system at any time instant, which is described using a minimum set of variables called state variables. The knowledge of these variables at t = 0 together with knowledge of inputs for t > 0, completely describes the behaviour of the system for $t \geq 0$.

The ***state variables*** are a set of variables that completely describe the state or condition of a system at any time instant.

State Equations

Consider a continuous time system with M-inputs, N-state variables, and P-outputs.

Let, $q_1(t), q_2(t), q_3(t), \ldots\ldots, q_N(t)$ be N-state variables of the continuous time system,

$x_1(t), x_2(t), x_3(t), \ldots\ldots, x_M(t)$ be M-inputs of the continuous time system,

$y_1(t), y_2(t), y_3(t), \ldots\ldots, y_P(t)$ be P-outputs of the continuous time system.

Now, the continuous time system can be represented by the block diagram shown in fig 5.1.

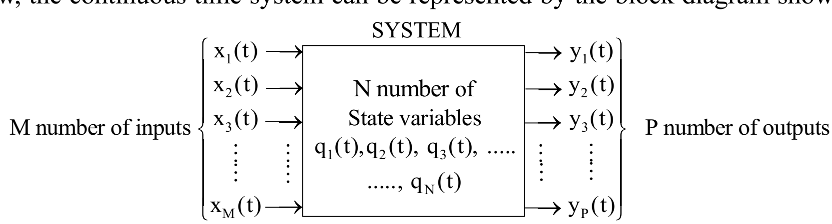


Fig 5.1 : *State space representation of a continuous time system.*

The state equations can be formed by taking first derivative of state variables as function of state variables and inputs. Therefore, the N-state variables can be expressed as N-number of first order differential equations as shown below.

$$
\begin{aligned}
\dot{q}_1(t) &= F\left[q_1(t),\, q_2(t),\, q_3(t),\, ...,\, q_N(t),\;\; x_1(t),\, x_2(t),\, x_3(t),\, ...,\, x_M(t)\right] \\
\dot{q}_2(t) &= F\left[q_1(t),\, q_2(t),\, q_3(t),\, ...,\, q_N(t),\;\; x_1(t),\, x_2(t),\, x_3(t),\, ...,\, x_M(t)\right] \\
\dot{q}_3(t) &= F\left[q_1(t),\, q_2(t),\, q_3(t),\, ...,\, q_N(t),\;\; x_1(t),\, x_2(t),\, x_3(t),\, ...,\, x_M(t)\right] \\
&\;\vdots \\
\dot{q}_N(t) &= F\left[q_1(t),\, q_2(t),\, q_3(t),\, ...,\, q_N(t),\;\; x_1(t),\, x_2(t),\, x_3(t),\, ...,\, x_M(t)\right]
\end{aligned}
$$

Mathematically, the above functional form of first derivative of the N-state variables can be expressed as shown below.

$$
\begin{aligned}
\dot{q}_1(t) &= a_{11}\, q_1(t) + a_{12}\, q_2(t) + a_{13}\, q_3(t) + ... + a_{1N}\, q_N(t) \\
&\qquad + b_{11}\, x_1(t) + b_{12}\, x_2(t) + b_{13}\, x_3(t) + ... + b_{1M}\, x_M(t) \\
\dot{q}_2(t) &= a_{21}\, q_1(t) + a_{22}\, q_2(t) + a_{23}\, q_3(t) + ... + a_{2N}\, q_N(t) \\
&\qquad + b_{21}\, x_1(t) + b_{22}\, x_2(t) + b_{23}\, x_3(t) + ... + b_{2M}\, x_M(t) \\
\dot{q}_3(t) &= a_{31}\, q_1(t) + a_{32}\, q_2(t) + a_{33}\, q_3(t) + ... + a_{3N}\, q_N(t) \\
&\qquad + b_{31}\, x_1(t) + b_{32}\, x_2(t) + b_{33}\, x_3(t) + ... + b_{3M}\, x_M(t) \\
&\;\vdots \\
\dot{q}_N(t) &= a_{N1}\, q_1(t) + a_{N2}\, q_2(t) + a_{N3}\, q_3(t) + ... + a_{NN}\, q_N(t) \\
&\qquad + b_{N1}\, x_1(t) + b_{N2}\, x_2(t) + b_{N3}\, x_3(t) + ... + b_{NM}\, x_M(t)
\end{aligned}
$$

The above equations of first derivative of the N-state variables are called ***state equations*** and can be expressed in the matrix form as shown in the matrix equation (5.1).

$$
\underbrace{\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \vdots \end{bmatrix}}_{\dot{\mathbf{Q}}(t)}
=
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13}\cdots \\ a_{21} & a_{22} & a_{23}\cdots \\ a_{31} & a_{32} & a_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \end{bmatrix}}_{\mathbf{Q}(t)}
+
\underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13}\cdots \\ b_{21} & b_{22} & b_{23}\cdots \\ b_{31} & b_{32} & b_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{B}}
\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{bmatrix}}_{\mathbf{X}(t)}
\qquad ....(5.1)
$$

where, **A** = System Matrix

B = Input Matrix

Q(t) = State Vector

X(t) = Input Vector

Therefore, the Matrix form of the state equation can be written as shown in equation (5.2).

$$
\boxed{\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)} \qquad ....(5.2)
$$

Output Equations

The output equations can be formed by taking the outputs as function of state variables and inputs. Therefore, the P-outputs can be expressed as shown below.

$$\begin{aligned}
y_1(t) &= F\big[q_1(t),\ q_2(t),\ q_3(t),\ .....,\ q_N(t),\ \ x_1(t),\ x_2(t),\ x_3(t),\ .....,\ x_M(t)\big] \\
y_2(t) &= F\big[q_1(t),\ q_2(t),\ q_3(t),\ .....,\ q_N(t),\ \ x_1(t),\ x_2(t),\ x_3(t),\ .....,\ x_M(t)\big] \\
y_3(t) &= F\big[q_1(t),\ q_2(t),\ q_3(t),\ .....,\ q_N(t),\ \ x_1(t),\ x_2(t),\ x_3(t),\ .....,\ x_M(t)\big] \\
&\ \vdots \\
y_P(t) &= F\big[q_1(t),\ q_2(t),\ q_3(t),\ .....,\ q_N(t),\ \ x_1(t),\ x_2(t),\ x_3(t),\ .....,\ x_M(t)\big]
\end{aligned}$$

Mathematically, the above functional form of the outputs can be expressed as shown below.

$$\begin{aligned}
y_1(t) &= c_{11}\,q_1(t) + c_{12}\,q_2(t) + c_{13}\,q_3(t) + ... + c_{1N}\,q_N(t) \\
&\qquad + d_{11}\,x_1(t) + d_{12}\,x_2(t) + d_{13}\,x_3(t) + ... + d_{1M}\,x_M(t) \\
y_2(t) &= c_{21}\,q_1(t) + c_{22}\,q_2(t) + c_{23}\,q_3(t) + ... + c_{2N}\,q_N(t) \\
&\qquad + d_{21}\,x_1(t) + d_{22}\,x_2(t) + d_{23}\,x_3(t) + ... + d_{2M}\,x_M(t) \\
y_3(t) &= c_{31}\,q_1(t) + c_{32}\,q_2(t) + c_{33}\,q_3(t) + ... + c_{3N}\,q_N(t) \\
&\qquad + d_{31}\,x_1(t) + d_{32}\,x_2(t) + d_{33}\,x_3(t) + ... + d_{3M}\,x_M(t) \\
&\ \vdots \\
y_P(t) &= c_{P1}\,q_1(t) + c_{P2}\,q_2(t) + c_{P3}\,q_3(t) + ... + c_{PN}\,q_N(t) \\
&\qquad + d_{P1}\,x_1(t) + d_{P2}\,x_2(t) + d_{P3}\,x_3(t) + ... + d_{PM}\,x_M(t)
\end{aligned}$$

The above equations are called ***output equations*** and can be expressed in the matrix form as shown in equation (5.3).

$$\underbrace{\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \end{bmatrix}}_{\mathbf{Y}(t)} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} & ...... \\ c_{21} & c_{22} & c_{23} & ...... \\ c_{31} & c_{32} & c_{33} & ...... \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ \vdots \end{bmatrix}}_{\mathbf{Q}(t)} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13} & ...... \\ d_{21} & d_{22} & d_{23} & ...... \\ d_{31} & d_{32} & d_{33} & ...... \\ \vdots & \vdots & \vdots & \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \end{bmatrix}}_{\mathbf{X}(t)} \qquad ....(5.3)$$

where, $\mathbf{C}$ = Output Matrix ; $\mathbf{D}$ = Transmission Matrix
$\mathbf{Q}(t)$ = State Vector ; $\mathbf{X}(t)$ = Input Vector

Therefore the Matrix form of output equation can be written as shown in equation (5.4).

$$\boxed{\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)} \qquad ....(5.4)$$

State Model

The ***state model*** of a continuous time system is given by state equations and output equations.

$$\boxed{\begin{array}{ll} \text{State equations} \quad : & \dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t) \\ \text{Output equations :} & \mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t) \end{array}}$$

5.3 State Model of a Continuous Time System from Direct Form-II Structure

Consider the equation of a 3rd order continuous time LTI system,

$$\frac{d^3}{dt^3}y(t) + a_1 \frac{d^2}{dt^2}y(t) + a_2 \frac{d}{dt}y(t) + a_3 y(t) = b_0 \frac{d^3}{dt^3}x(t) + b_1 \frac{d^2}{dt^2}x(t) + b_2 \frac{d}{dt}x(t) + b_3\, x(t)$$

The direct form-II structure of the system described by the above equation is shown in fig 5.2.

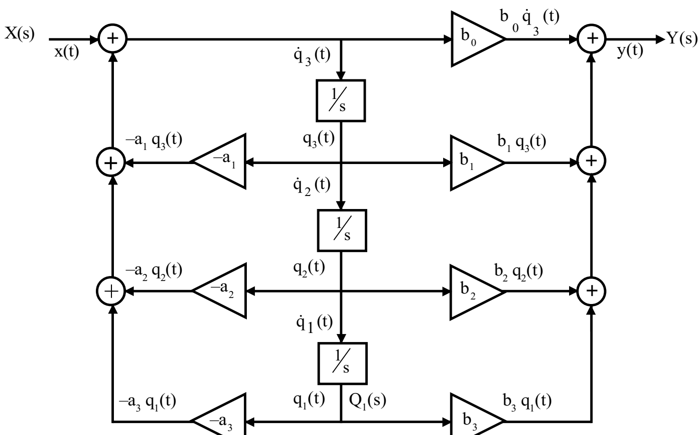


***Fig 5.2 :** Direct form - II structure of a continuous time system.*

In a state model of a system, *the choice of the number of state variables is equal to the number of integrators*. The direct form-II structure shown in fig 5.2 has one input x(t), one output y(t) and three integrators, and so let us choose three state variables. Let, $q_1(t)$, $q_2(t)$ and $q_3(t)$ be the three state variables. Assign state variables at the output of every integrator. Hence the first derivative of a state variable will be available at the input of the integrator.

State Equations

The state equations are formed by equating the sum of incoming signals of the integrator to first derivative of the state variable, as shown below.

$$\left.\begin{aligned}\dot{q}_1(t) &= q_2(t)\\ \dot{q}_2(t) &= q_3(t)\\ \dot{q}_3(t) &= -a_3\, q_1(t) - a_2\, q_2(t) - a_1\, q_3(t) + x(t)\end{aligned}\right\}\text{State equations}$$

$$\Downarrow$$

$$\left.\begin{aligned}\dot{q}_1(t) &= 0\times q_1(t) + 1\times q_2(t) + 0\times q_3(t) + 0\times x(t)\\ \dot{q}_2(t) &= 0\times q_1(t) + 0\times q_2(t) + 1\times q_3(t) + 0\times x(t)\\ \dot{q}_3(t) &= -a_3\times q_1(t) - a_2\times q_2(t) - a_1\times q_3(t) + 1\times x(t)\end{aligned}\right\}\text{State equations}$$

On arranging the state equations in the matrix form we get,

$$\underbrace{\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} \big[x(t)\big]$$

Output Equations

The output equation y(t) is formed by equating the incoming signals of output node / summing point to y(t) as shown below.

$$y(t) = b_0\, \dot{q}_3(t) + b_3\, q_1(t) + b_2\, q_2(t) + b_1\, q_3(t)$$

On substituting for $\dot{q}_3(t)$ from state equation we get,

$$y(t) = b_0\big[-a_3\, q_1(t) - a_2\, q_2(t) - a_1\, q_3(t) + x(t)\big] + b_3\, q_1(t) + b_2\, q_2(t) + b_1\, q_3(t)$$

$$= (b_3 - b_0\, a_3)\, q_1(t) + (b_2 - b_0\, a_2)\, q_2(t) + (b_1 - b_0 a_1)\, q_3(t) + b_0\, x(t)$$

On arranging the output equation in the matrix form we get,

$$y(t) = \underbrace{\big[b_3 - b_0\, a_3 \quad b_2 - b_0\, a_2 \quad b_1 - b_0\, a_1\big]}_{\mathbf{C}} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \underbrace{\big[b_0\big]}_{\mathbf{D}}\big[x(t)\big]$$

State Model

State model of a continuous time system is given by state equations and output equations.

State equations : $\dot{\mathbf{Q}}(t) = \mathbf{A}\, \mathbf{Q}(t) + \mathbf{B}\, \mathbf{X}(t)$

$$\Downarrow$$

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big[x(t)\big]$$

Output equations : $\mathbf{Y}(t) = \mathbf{C}\, \mathbf{Q}(t) + \mathbf{D}\, \mathbf{X}(t)$

$$\Downarrow$$

$$y(t) = \big[b_3 - b_0\, a_3 \quad b_2 - b_0\, a_2 \quad b_1 - b_0\, a_1\big] \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \big[b_0\big]\big[x(t)\big]$$

5.4 Transfer Function of a Continuous Time System from State Model

Consider the state equation,

$$\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)$$

On taking Laplace transform of the above equation with zero initial condition we get,

$$s\mathbf{Q}(s) = \mathbf{A}\,\mathbf{Q}(s) + \mathbf{B}\,\mathbf{X}(s)$$

$$s\mathbf{Q}(s) - \mathbf{A}\,\mathbf{Q}(s) = \mathbf{B}\,\mathbf{X}(s)$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{Q}(s) = \mathbf{B}\,\mathbf{X}(s), \text{ where } \mathbf{I} \text{ is the unit matrix.}$$

On premultiplying the above equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we get,

$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s) \qquad .....(5.5)$$

Consider the output equation,

$$\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)$$

On taking Laplace transform of the above equation we get,

$$\mathbf{Y}(s) = \mathbf{C}\,\mathbf{Q}(s) + \mathbf{D}\,\mathbf{X}(s) \qquad .....(5.6)$$

On substituting for $\mathbf{Q}(s)$ from equation (5.5) in (5.6) we get,

$$\mathbf{Y}(s) = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s) + \mathbf{D}\,\mathbf{X}(s)$$

$$\therefore\ \mathbf{Y}(s) = [\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}]\,\mathbf{X}(s)$$

For single input and single output system,

$$\boxed{\frac{Y(s)}{X(s)} = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}} \qquad .....(5.7)$$

The equation (5.7) is the ***transfer function*** of the continuous time system.

5.5 Solution of State Equations and Response of Continuous Time System

Solution of State Equations in Time Domain

The state equation of N^{th} order continuous time system is given by,

$$\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)$$

$$\therefore\ \dot{\mathbf{Q}}(t) - \mathbf{A}\,\mathbf{Q}(t) = \mathbf{B}\,\mathbf{X}(t)$$

On premultiplying the above equation by $e^{-\mathbf{A}t}$ we get,

$$e^{-\mathbf{A}t}\left[\dot{\mathbf{Q}}(t) - \mathbf{A}\,\mathbf{Q}(t)\right] = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{X}(t)$$

$$e^{-\mathbf{A}t}\,\dot{\mathbf{Q}}(t) + e^{-\mathbf{A}t}(-\mathbf{A})\,\mathbf{Q}(t) = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{X}(t) \qquad .....(5.8)$$

Consider the differential of $e^{-\mathbf{A}t}\mathbf{Q}(t)$. $\boxed{d(uv) = u\,dv + du\,v}$

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{Q}(t)\right) = e^{-\mathbf{A}t}\,\dot{\mathbf{Q}}(t) + e^{-\mathbf{A}t}(-\mathbf{A})\,\mathbf{Q}(t) \qquad .....(5.9)$$

Using equation (5.9), the equation (5.8) can be written as shown below.

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{Q}(t)\right) = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{X}(t)$$

$$\therefore\; d\left(e^{-\mathbf{A}t}\mathbf{Q}(t)\right) = e^{-\mathbf{A}t}\,\mathbf{B}\,\mathbf{X}(t)\,dt$$

On integrating the above equation between limits 0 to t we get,

$$e^{-\mathbf{A}t}\,\mathbf{Q}(t) = \int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau + \mathbf{Q}(0)$$

Q(0) is initial condition vector

τ = Dummy variable substituted for t

On premultiplying the above equation by $e^{\mathbf{A}t}$ we get,

$$e^{\mathbf{A}t}\,e^{-\mathbf{A}t}\,\mathbf{Q}(t) = e^{\mathbf{A}t}\,\mathbf{Q}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau$$

$$\therefore\; \mathbf{Q}(t) = e^{\mathbf{A}t}\,\mathbf{Q}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau$$

$e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{\mathbf{A}t-\mathbf{A}t} = e^0 = \mathbf{I}$ = Unit matrix

$$= e^{\mathbf{A}t}\,\mathbf{Q}(0) + \int_0^t e^{\mathbf{A}t}\,e^{-\mathbf{A}\tau}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau$$

$e^{\mathbf{A}t}$ is independent of integral variable τ.

$$= e^{\mathbf{A}t}\,\mathbf{Q}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau$$

$$\boxed{\mathbf{Q}(t) = \underbrace{e^{\mathbf{A}t}\,\mathbf{Q}(0)}_{\text{Solution due to initial condition}} + \underbrace{\int_0^t e^{\mathbf{A}(t-\tau)}\,\mathbf{B}\,\mathbf{X}(\tau)\,d\tau}_{\text{Solution due to input}}} \qquad .....(5.10)$$

The equation (5.10) is the time domain solution of state equations of the continuous time system. Here, the matrix $e^{\mathbf{A}t}$ is called ***state transition matrix of*** continuous time system.

Solution of State Equations using Laplace Transform

Consider the state equation of the continuous time system.

$$\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)$$

On taking Laplace transform of the above equation we get,

$$s\mathbf{Q}(s) - \mathbf{Q}(0) = \mathbf{A}\,\mathbf{Q}(s) + \mathbf{B}\,\mathbf{X}(s)$$

Q(0) is initial condition vector

$$s\mathbf{Q}(s) - \mathbf{A}\,\mathbf{Q}(s) = \mathbf{Q}(0) + \mathbf{B}\,\mathbf{X}(s)$$

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{Q}(s) = \mathbf{Q}(0) + \mathbf{B}\,\mathbf{X}(s)$$

I is the unit matrix

On premultiplying the above equation by $(s\mathbf{I} - \mathbf{A})^{-1}$, we get,

$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Q}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s) \qquad ......(5.11)$$

The equation (5.11) is the solution of state equations of the continuous time system in s-domain. The solution of state equations in time domain is given by the inverse Laplace transform of **Q**(s).

On taking inverse Laplace transform of equation (5.11) we get,

$$\mathbf{Q}(t) = \mathcal{L}^{-1}\{\mathbf{Q}(s)\}$$

$$\therefore\ \mathbf{Q}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{Q}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B}\,\mathbf{X}(s)\right\}$$

$$\boxed{\mathbf{Q}(t) = \underbrace{\mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{Q}(0)}_{\text{Solution due to initial condition}} + \underbrace{\mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B}\,\mathbf{X}(s)\right\}}_{\text{Solution due to input}}} \qquad ......(5.12)$$

The equation (5.12) is the time domain solution of state equations of the continuous time system.

On comparing equations (5.10) and (5.12) we get,

$$\text{State Transition Matrix,}\quad e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\} \qquad ......(5.13)$$

The equation (5.13) is used to compute state transition matrix via Laplace transform.

Response of Continuous Time System

The response, $\mathbf{Y}(t)$ of continuous time system can be computed by substituting the solution of state equations $\mathbf{Q}(t)$, from equation (5.10) or (5.12), in the output equation shown below.

$$\boxed{\text{Response of continuous time system, } \mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)} \qquad ......(5.14)$$

The response, when there is no input is called ***zero-input response***, $\mathbf{Y}_{zi}(t)$ and this can be obtained from equations (5.12) and (5.14) by taking $\mathbf{X}(t)$ and $\mathbf{X}(s)$ as zero. The zero-input response is due to initial conditions.

$$\boxed{\begin{aligned}&\text{Zero-input response, } \mathbf{Y}_{zi}(t) = \mathbf{C}\,\mathbf{Q}(t)\\ &\qquad\text{where, } \mathbf{Q}(t) = \mathcal{L}^{-1}\{\mathbf{Q}(s)\} = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}\mathbf{Q}(0)\end{aligned}} \qquad ......(5.15)$$

The response, when there is no initial conditions is called ***zero-state response***, $\mathbf{Y}_{zs}(t)$ and this can be obtained from equations (5.12) and (5.14) by taking $\mathbf{Q}(0)$ as zero. The zero-state response is due to input.

$$\boxed{\begin{aligned}&\text{Zero-state response, } \mathbf{Y}_{zs}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)\\ &\qquad\text{where, } \mathbf{Q}(t) = \mathcal{L}^{-1}\{\mathbf{Q}(s)\} = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{B}\,\mathbf{X}(s)\right\}\end{aligned}} \qquad ......(5.16)$$

5.6 Solved Problems in State Space Analysis

Example 5.1

Determine the state model of the continuous time system governed by the equation,

$$\frac{d^3}{dt^3}y(t) + 3\frac{d^2}{dt^2}y(t) - 2\frac{d}{dt}y(t) + 0.5\,y(t) = 4\frac{d^3}{dt^3}x(t) + 2.5\frac{d^2}{dt^2}x(t) + 1.5\frac{d}{dt}x(t) + 2x(t)$$

Solution

The direct form-II structure of the continuous time system described by the given equation is shown in fig 1.

The choice of number of state variables is equal to number of integrators. The direct form-II structure has one input x(t), one output y(t) and three integrators and so let us choose three state variables.

Let, $q_1(t)$, $q_2(t)$ and $q_3(t)$ be the three state variables. Assign state variables at the output of every integrator. Hence the first derivative of state variable will be available at the input of integrator.

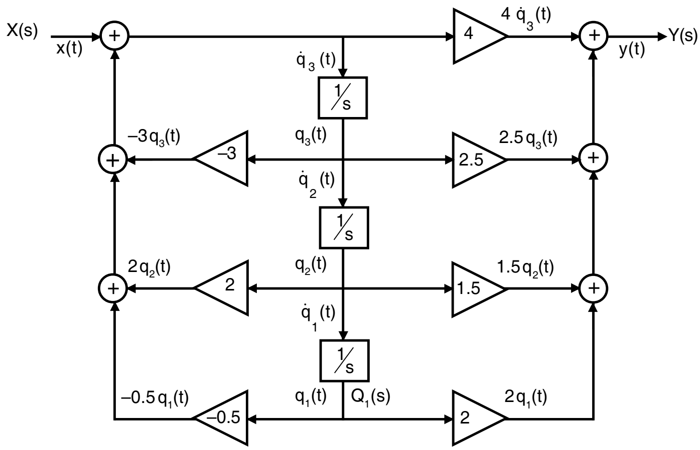


***Fig 1 :** Direct form - II structure.*

State Equations

The state equations are formed by equating the sum of incoming signals of integrator to the first derivative of state variable, as shown below.

$$\dot{q}_1(t) = q_2(t)$$

$$\dot{q}_2(t) = q_3(t)$$

$$\dot{q}_3(t) = -0.5\, q_1(t) + 2\, q_2(t) - 3\, q_3(t) + x(t)$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & 2 & -3 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big[x(t)\big]$$

Output Equations

Output equation y(t) is formed by equating incoming signals of output node / summing point to y(t) as shown below.

$$y(t) = 4\, \dot{q}_3(t) + 2\, q_1(t) + 1.5\, q_2(t) + 2.5\, q_3(t)$$

On substituting for $\dot{q}_3(t)$ from state equation we get,

$$y(t) = 4\big[-0.5\, q_1(t) + 2\, q_2(t) - 3\, q_3(t) + x(t)\big] + 2\, q_1(t) + 1.5\, q_2(t) + 2.5\, q_3(t)$$

$$= -2\, q_1(t) + 8\, q_2(t) - 12\, q_3(t) + 4\, x(t) + 2\, q_1(t) + 1.5\, q_2(t) + 2.5\, q_3(t)$$

$$\therefore\; y(t) = 9.5\, q_2(t) - 9.5\, q_3(t) + 4\, x(t)$$

On arranging the output equation in the matrix form we get,

$$y(t) = \begin{bmatrix} 0 & 9.5 & -9.5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \big[4\big]\big[x(t)\big]$$

State Model

State model of a continuous time system is given by state equations and output equations.

State equations : $\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)$

$$\Downarrow$$

$$\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & 2 & -3 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big[x(t)\big]$$

Output equations : $\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)$

$$\Downarrow$$

$$y(t) = \begin{bmatrix} 0 & 9.5 & -9.5 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{bmatrix} + [4]\,[x(t)]$$

Example 5.2

The transfer function of a continuous time system is given by,

$$H(s) = \frac{1.5\,s^3 + 2\,s^2 + 3\,s + 2}{s^3 + 3\,s^2 + 2\,s + 4}$$

Determine the state model of the continuous time system.

Solution

Given that, $H(s) = \dfrac{1.5\,s^3 + 2\,s^2 + 3\,s + 2}{s^3 + 3\,s^2 + 2\,s + 4}$

On dividing the numerator and denominator by s^3 we get,

$$H(s) = \frac{1.5 + 2\,\frac{1}{s} + 3\,\frac{1}{s^2} + 2\,\frac{1}{s^3}}{1 + 3\,\frac{1}{s} + 2\,\frac{1}{s^2} + 4\,\frac{1}{s^3}}$$

Let, $H(s) = \dfrac{Y(s)}{X(s)} = \dfrac{1.5 + 2\,\frac{1}{s} + 3\,\frac{1}{s^2} + 2\,\frac{1}{s^3}}{1 + 3\,\frac{1}{s} + 2\,\frac{1}{s^2} + 4\,\frac{1}{s^3}}$

On cross multiplication we get,

$$Y(s) + 3\,\frac{Y(s)}{s} + 2\,\frac{Y(s)}{s^2} + 4\,\frac{Y(s)}{s^3} = 1.5\,X(s) + 2\,\frac{X(s)}{s} + 3\,\frac{X(s)}{s^2} + 2\,\frac{X(s)}{s^3}$$

$$\therefore\ Y(s) = -3\,\frac{Y(s)}{s} - 2\,\frac{Y(s)}{s^2} - 4\,\frac{Y(s)}{s^3} + 1.5\,X(s) + 2\,\frac{X(s)}{s} + 3\,\frac{X(s)}{s^2} + 2\,\frac{X(s)}{s^3}$$

The direct form-II structure of the continuous time system described by the above equation is shown in fig 1.

The choice of number of state variables is equal to the number of integrators. The direct form-II structure has one input x(t), one output y(t) and three integrators, and so let us choose three state variables. Let, $q_1(t)$, $q_2(t)$ and $q_3(t)$ be the three state variables. Assign state variables at the output of every integrator. Hence the first derivative of state variable will be available at the input of integrator.

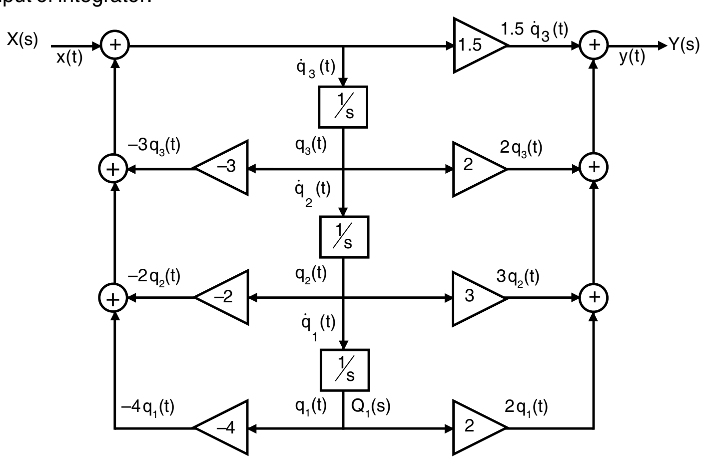


Fig 1 : Direct form - II structure.

State Equations

The state equations are formed by equating the sum of incoming signals of integrator to first derivative of state variable, as shown below.

$$\dot{q}_1(t) = q_2(t)$$
$$\dot{q}_2(t) = q_3(t)$$
$$\dot{q}_3(t) = -4\, q_1(t) - 2\, q_2(t) - 3\, q_3(t) + x(t)$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -2 & -3\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[x(t)]$$

Output Equations

Output equation y(t) is formed by equating incoming signals of output node / summing point to y(t) as shown below.

$$y(t) = 1.5\, \dot{q}_3(t) + 2\, q_1(t) + 3\, q_2(t) + 2\, q_3(t)$$

On substituting for $\dot{q}_3(t)$ from state equation we get,

$$\begin{aligned} y(t) &= 1.5\left[-4\, q_1(t) - 2\, q_2(t) - 3\, q_3(t) + x(t)\right] + 2\, q_1(t) + 3\, q_2(t) + 2\, q_3(t)\\ &= -6\, q_1(t) - 3\, q_2(t) - 4.5\, q_3(t) + 1.5\, x(t) + 2\, q_1(t) + 3\, q_2(t) + 2\, q_3(t)\\ &= -4\, q_1(t) - 2.5\, q_3(t) + 1.5\, x(t)\end{aligned}$$

On arranging the output equation in the matrix form we get,

$$y(t) = \begin{bmatrix}-4 & 0 & -2.5\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} + [1.5]\,[x(t)]$$

State Model

State model of a continuous time system is given by state equations and output equations.

State equations : $\dot{\mathbf{Q}}(t) = \mathbf{A}\,\mathbf{Q}(t) + \mathbf{B}\,\mathbf{X}(t)$

$$\Downarrow$$

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -4 & -2 & -3\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}[x(t)]$$

Output equations : $\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)$

$$\Downarrow$$

$$y(t) = \begin{bmatrix}-4 & 0 & -2.5\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix} + [1.5]\,[x(t)]$$

Example 5.3

The state space representation of a continuous time system is given by,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \quad ; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ; \quad \mathbf{C} = \begin{bmatrix} 1 & 3 \end{bmatrix} \quad ; \quad \mathbf{D} = [3]$$

Derive the transfer function of the continuous time system.

> Let, **P** be a square matrix.
>
> Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$
>
> If, **P** is a square matrix of size 2×2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.
>
> $$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$
>
> $$\therefore \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

Solution

Transfer function of a continuous time system is given by,

$$\frac{Y(s)}{X(s)} = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 1 \\ -3 & s-1 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix} s-2 & 1 \\ -3 & s-1 \end{vmatrix}} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix} = \frac{1}{(s-2)\,(s-1) - (-3) \times 1} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix}$$

$$= \frac{1}{s^2 - 3s + 5} \begin{bmatrix} s-1 & -1 \\ 3 & s-2 \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2-3s+5} & \dfrac{-1}{s^2-3s+5} \\ \dfrac{3}{s^2-3s+5} & \dfrac{s-2}{s^2-3s+5} \end{bmatrix}$$

$$\therefore \frac{Y(s)}{X(s)} = \mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} \dfrac{s-1}{s^2-3s+5} & \dfrac{-1}{s^2-3s+5} \\ \dfrac{3}{s^2-3s+5} & \dfrac{s-2}{s^2-3s+5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3]$$

$$= \begin{bmatrix} \dfrac{s-1+9}{s^2-3s+5} & \dfrac{-1+3(s-2)}{s^2-3s+5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3] = \left[\frac{s-1+9}{s^2-3s+5} + \frac{2(-1+3(s-2))}{s^2-3s+5} \right] + [3]$$

$$= \frac{s-1+9 + 2(-1+3(s-2)) + 3(s^2-3s+5)}{s^2 - 3s + 5}$$

$$= \frac{s-1+9-2+6s-12+3s^2-9s+15}{s^2-3s+5} = \frac{3s^2-2s+9}{s^2-3s+5}$$

Example 5.4

Find the state transition matrix for the continuous time system parameter matrix, $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$

Solution

The state transition matrix, $e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\,\mathbf{I} - \mathbf{A})^{-1}\}$

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix} s+3 & 0 \\ 0 & s+2 \end{vmatrix}} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}$$

$$= \frac{1}{(s+3)\,(s+2)} \begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left\{(s\,\mathbf{I} - \mathbf{A})^{-1}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} & 0 \\ 0 & \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \end{bmatrix} = \begin{bmatrix} e^{-3t}\,u(t) & 0 \\ 0 & e^{-2t}\,u(t) \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\therefore\ \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

Example 5.5

The state equation of an LTI continuous time system is given by, $\begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$

Compute the solution of state equation by assuming the initial state vector as, $\begin{bmatrix} q_1(0) \\ q_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

The solution of state equations are given by,

$$\mathbf{Q}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Q}(0)\right\} + \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)\right\}$$

Here $\mathbf{X}(s) = 0$, (because there is no input).

$$\therefore\ \mathbf{Q}(t) = \mathcal{L}^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Q}(0)\right\}$$

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix}} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\therefore\ \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

Now, $\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Q}(0)$

$$= \begin{bmatrix} \dfrac{1}{s-1} & 0 \\ \dfrac{1}{(s-1)^2} & \dfrac{1}{s-1} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s-1} \\ \dfrac{1}{(s-1)^2} \end{bmatrix}$$

$$\mathbf{Q}(t) = \mathcal{L}^{-1}\{\mathbf{Q}(s)\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2}\right\} \end{bmatrix} = \begin{bmatrix} e^t\,u(t) \\ t\,e^t\,u(t) \end{bmatrix}$$

Example 5.6

The state model of a continuous time system is given by,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} \;;\; \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \;;\; C = \begin{bmatrix} 1 & 3 \end{bmatrix} \;;\; \mathbf{D} = 3$$

Find the response of the system for unit step input. Assume zero initial condition.

Solution

The response, $\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)$

$$\text{where, } \mathbf{Q}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{Q}(0)\} + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)\}$$

Here $\mathbf{Q}(0) = 0$, (because initial conditions are zero).

$$\therefore\; \mathbf{Q}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)\}$$

$$s\mathbf{I} - \mathbf{A} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} s-2 & 0 \\ -3 & s-1 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix} s-2 & 0 \\ -3 & s-1 \end{vmatrix}}\begin{bmatrix} s-1 & 0 \\ 3 & s-2 \end{bmatrix}$$

$$= \frac{1}{(s-2)\,(s-1)}\begin{bmatrix} s-1 & 0 \\ 3 & s-2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s-2} & 0 \\ \dfrac{3}{(s-1)\,(s-2)} & \dfrac{1}{s-1} \end{bmatrix}$$

> Let, **P** be a square matrix.
>
> Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$
>
> If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.
>
> $$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$
>
> $$\therefore\; \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

Given that, $x(t) = u(t)$.

$$\therefore\; X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{u(t)\} = \frac{1}{s}$$

Now, $\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)$

$$= \begin{bmatrix} \dfrac{1}{s-2} & 0 \\ \dfrac{3}{(s-1)\,(s-2)} & \dfrac{1}{s-1} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} \dfrac{1}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s-2} + 0 \\ \dfrac{3}{(s-1)\,(s-2)} + \dfrac{2}{s-1} \end{bmatrix}\begin{bmatrix} \dfrac{1}{s} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{s-2} \\ \dfrac{3+2\,(s-2)}{(s-1)\,(s-2)} \end{bmatrix}\begin{bmatrix} \dfrac{1}{s} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s(s-2)} \\ \dfrac{2s-1}{s(s-1)\,(s-2)} \end{bmatrix}$$

$$\mathbf{Q}(t) = \mathcal{L}^{-1}\{\mathbf{Q}(s)\} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{1}{s\,(s-2)} \\ \dfrac{2s-1}{s(s-1)\,(s-2)} \end{bmatrix} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{1}{s\,(s-2)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{2s-1}{s(s-1)\,(s-2)}\right\} \end{bmatrix}$$

Let, $\dfrac{1}{s(s-2)} = \dfrac{k_1}{s} + \dfrac{k_2}{s-2}$

$$\text{where, } k_1 = \frac{1}{s(s-2)} \times s\Big|_{s=0} = \frac{1}{0-2} = -\frac{1}{2}$$

$$k_2 = \frac{1}{s(s-2)} \times (s-2)\Big|_{s=2} = \frac{1}{2}$$

$$\therefore \frac{1}{s(s-2)} = -\frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{s-2}$$

Let, $\dfrac{2s-1}{s(s-1)\,(s-2)} = \dfrac{k_1}{s} + \dfrac{k_2}{s-1} + \dfrac{k_3}{s-2}$

$$\text{where, } k_1 = \frac{2s-1}{s(s-1)\,(s-2)} \times s\Big|_{s=0} = \frac{0-1}{(0-1)\,(0-2)} = -\frac{1}{2}$$

$$k_2 = \frac{2s-1}{s(s-1)\,(s-2)} \times (s-1)\Big|_{s=1} = \frac{2\times 1-1}{1\times(1-2)} = -1$$

$$k_3 = \frac{2s-1}{s(s-1)\,(s-2)} \times (s-2)\Big|_{s=2} = \frac{2\times 2-1}{2\times(2-1)} = \frac{3}{2}$$

$$\therefore \frac{2s-1}{s(s-1)\,(s-2)} = -\frac{1}{2}\frac{1}{s} - \frac{1}{s-1} + \frac{3}{2}\frac{1}{s-2}$$

$$\mathbf{Q}(t) = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{1}{s(s-2)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{2s-1}{s(s-1)\,(s-2)}\right\} \end{bmatrix} = \begin{bmatrix} \mathcal{L}^{-1}\left\{-\dfrac{1}{2}\dfrac{1}{s} + \dfrac{1}{2}\dfrac{1}{s-2}\right\} \\ \mathcal{L}^{-1}\left\{-\dfrac{1}{2}\dfrac{1}{s} - \dfrac{1}{s-1} + \dfrac{3}{2}\dfrac{1}{s-2}\right\} \end{bmatrix}$$

$$= \begin{bmatrix} -\dfrac{1}{2}u(t) + \dfrac{1}{2}e^{2t}u(t) \\ -\dfrac{1}{2}u(t) - e^{t}u(t) + \dfrac{3}{2}e^{2t}u(t) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2}\left(e^{2t}-1\right)u(t) \\ \dfrac{1}{2}\left(3e^{2t}-e^{t}-1\right)u(t) \end{bmatrix}$$

Now, the response of the continuous time system is given by,

Response, $\mathbf{Y}(t) = \mathbf{C\,Q}(t) + \mathbf{D\,X}(t)$

$$= \begin{bmatrix}1 & 3\end{bmatrix}\begin{bmatrix} \dfrac{1}{2}\left(e^{2t}-1\right)u(t) \\ \dfrac{1}{2}\left(3e^{2t}-e^{t}-1\right)u(t) \end{bmatrix} + [3]\,[u(t)]$$

$$= 0.5\left(e^{2t}-1\right)u(t) + 1.5\left(3e^{2t}-e^{t}-1\right)u(t) + 3u(t)$$

$$= \left(0.5e^{2t} - 0.5 + 4.5e^{2t} - 1.5e^{t} - 1.5 + 3\right)u(t) = \left(5\,e^{2t} - 1.5e^{t} + 1\right)u(t)$$

$$\therefore \text{ Response, } y(t) = \left(5e^{2t} - 1.5e^{t} + 1\right)u(t)$$

5.7 Summary of Important Concepts

1. The state of a continuous time system refers to the condition of continuous time system at any time instant.
2. The state variables of a continuous time system are a set of variables that completely describe the state of continuous time system at any time instant.
3. The state equations of a continuous time system are a set of N-number of first order differential equations.
4. The state equations of a continuous time system are formed by taking first derivative of state variables as function of state variables and inputs.
5. The output equations of a continuous time system are a set of P-number of algebraic equations formed by taking outputs as function of state variables and inputs.
6. The state model of a continuous time system is given by state equations and output equations.
7. The transfer function of a continuous time system can be obtained from its state model using the equation $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$.
8. The state transition matrix of a continuous time system is, $e^{\mathbf{A}t} = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\}$.
9. The solution of state equations of a continuous time system due to initial condition (and with no input) is, $\mathbf{Q}(t) = e^{\mathbf{A}t}\,\mathbf{Q}(0)$.
10. The solution of state equations of a continuous time system due to input (and with zero initial conditions) is, $\mathbf{Q}(t) = \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)\}$.
11. The solution of state equations of a continuous time system due to input and initial conditions is,
    $\mathbf{Q}(t) = e^{\mathbf{A}t}\,\mathbf{Q}(0) + \mathcal{L}^{-1}\{(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(s)\}$.
12. The response of a continuous time system using its state model can be obtained from the equation,
    $\mathbf{Y}(t) = \mathbf{C}\,\mathbf{Q}(t) + \mathbf{D}\,\mathbf{X}(t)$

5.8 Short Questions and Answers

Q5.1 What are the advantages of state space analysis?

1. The state space analysis is applicable to any type of systems. They can be used for modelling and analysis of linear and non-linear systems, time invariant and time variant systems and multiple input and multiple output systems.
2. The state space analysis can be performed with initial conditions.
3. The variables used to represent the system can be any variables in the system.
4. Using this analysis the internal states of the system at any time instant can be predicted.

Q5.2 What is a state vector?

The state vector is an (N × 1) column matrix (or vector) whose elements are state variables of the system, (where N is the order of the system). It is denoted by $\mathbf{Q}(t)$.

Q5.3 What is state space?

The set of all possible values which the state vector $\mathbf{Q}(t)$ can have (or assume) at time **t** forms the state space of the system.

Q5.4 What is input and output space?

The set of all possible values which the input vector $\mathbf{X}(t)$ can have (or assume) at time **t** forms the input space of the system.

The set of all possible values which the output vector $\mathbf{Y}(t)$ can have (or assume) at time **t** forms the output space.

Q5.5 ***Determine system matrix of the system governed by the differential equation,*** $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+4y=0.$

Solution

Given that, $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}+4y=0$. The given system is a second order system and so let us choose two state variables q_1 and q_2. Let us equate the state variables to the system variable y and its derivative dy/dt.

$$\therefore\ q_1 = y \quad \Rightarrow \quad \dot{q}_1 = \dot{y} = q_2$$

$$q_2 = \dot{y} \quad \Rightarrow \quad \dot{q}_2 = \ddot{y}$$

On substituting the state variables and their derivatives in the given system equation we get,

$$\dot{q}_2 + 2q_2 + 4q_1 = 0 \qquad \Rightarrow \qquad \dot{q}_2 = -4q_1 - 2q_2$$

Now the state equations are,

$$\dot{q}_1 = q_2$$

$$\dot{q}_2 = -4q_1 - 2q_2$$

Now, the state equations in the matrix form is,

$$\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -4 & -2\end{bmatrix}\begin{bmatrix}q_1\\ q_2\end{bmatrix} \qquad \Rightarrow \qquad \dot{\mathbf{Q}} = \mathbf{A\,Q}$$

Therefore the system matrix is, $A = \begin{bmatrix}0 & 1\\ -4 & -2\end{bmatrix}$

Q5.6 ***Determine state model of the system shown in fiq Q5.6.1.***

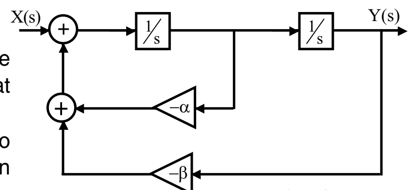


Fig Q5.6.1.

Solution

Let us assign first derivative of state variables at the input of the integrator as shown in fig Q5.6.2, so that the state variables exists at the output of integrators.

The state equations are formed by equating the incoming signals to input of the integrator to the first derivative of state variable as shown below.

$$\dot{q}_1(t) = q_2(t)$$

$$\dot{q}_2(t) = -b\,q_1(t) - a\,q_2(t) + x(t)$$

Here the output equation is, $y(t) = q_1(t)$.

Now, the state model in matrix form is,

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}x(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

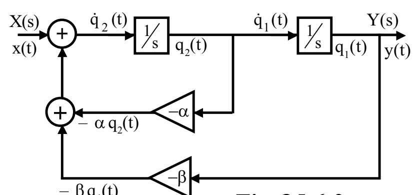


Fig Q5.6.2.

Q5.7 ***Given that,*** $\phi(s) = \mathcal{L}\left\{e^{At}\right\} = \begin{bmatrix}\dfrac{s}{s^2+5} & \dfrac{1}{s^2+5}\\ \dfrac{-5}{s^2+5} & \dfrac{s}{s^2+5}\end{bmatrix}$. ***Determine the system matrix.***

Solution

We know that , $\phi(s) = [\,s\mathbf{I} - \mathbf{A}\,]^{-1}$

On taking inverse on both sides of above equation we get,

$$[\phi(s)]^{-1} = s\mathbf{I} - \mathbf{A} \qquad \Rightarrow \qquad \mathbf{A} = s\mathbf{I} - [\phi(s)]^{-1}$$

$$[\phi(s)]^{-1} = \frac{1}{|\phi(s)|}\begin{bmatrix}\dfrac{s}{s^2+5} & \dfrac{-1}{s^2+5}\\ \dfrac{5}{s^2+5} & \dfrac{s}{s^2+5}\end{bmatrix}$$

Here, $|\phi(s)| = \frac{s}{s^2+5} \times \frac{s}{s^2+5} - \frac{-5}{s^2+5} \times \frac{1}{s^2+5} = \frac{s^2+5}{\left(s^2+5\right)^2} = \frac{1}{s^2+5}$

$$\therefore\ [\phi(s)]^{-1} = \left(s^2+5\right)\begin{bmatrix} \frac{s}{s^2+5} & \frac{-1}{s^2+5} \\ \frac{5}{s^2+5} & \frac{s}{s^2+5} \end{bmatrix} = \begin{bmatrix} s & -1 \\ 5 & s \end{bmatrix}$$

$$\therefore \text{ System matrix, } A = sI - [\phi(s)]^{-1} = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} s & -1 \\ 5 & s \end{bmatrix} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} s & -1 \\ 5 & s \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$$

Q5.8 ***Determine the transfer function of the system described by the following state model.***

$$\dot{q}(t) = -2q(t) + 0.5\,x(t)$$
$$y(t) = 0.7\,q(t)$$

Solution

Given that, $\dot{q}(t) = -2q(t) + 0.5x(t)$

On taking Laplace transform of above equation we get, $\boxed{Q(0) = 0}$

$$sQ(s) = -2Q(s) + 0.5X(s) \quad \Rightarrow \quad sQ(s) + 2Q(s) = 0.5X(s) \quad \Rightarrow \quad (s+2)\,Q(s) = 0.5X(s)$$

$$\therefore\ Q(s) = \frac{0.5}{s+2}\,X(s) \qquad .....(1)$$

Given that, $y(t) = 0.7q(t)$

On taking Laplace transform of above equation we get,

$$Y(s) = 0.7\,Q(s)$$
$$= 0.7 \times \frac{0.5}{s+2}\,X(s) \qquad \boxed{\text{Using equation (1)}}$$

From equations (1) and (2) we can write,

$$\therefore \text{ Transfer function, } \frac{Y(s)}{X(s)} = 0.7 \times \frac{0.5}{s+2} = \frac{0.35}{s+2}$$

Q5.9 ***The output equation of a system is, y(t) = [1 2] Q(t).***

If the state transition matrix, $e^{At} = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-2t} & 0 \end{bmatrix}$ ***and the initial condition vector,*** $Q(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

Find the zero-input response of the system.

Solution

When the input is zero, the state equations, $\mathbf{Q}(t) = e^{At}\,\mathbf{Q}(0)$.

$$\therefore \text{Zero-input response, } y_{zi}(t) = [1 \quad 2]\ Q(t) = [1 \quad 2]\ e^{At}\ Q(0)$$
$$= [1 \quad 2]\begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-2t} & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \left[e^{-t} - 2e^{-2t} \quad e^{-2t}\right]\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$= 2\left(e^{-t} - 2e^{-2t}\right) + e^{-2t} = 2e^{-t} - 3e^{-2t};\ \text{for } t > 0$$
$$= \left(2e^{-t} - 3e^{-2t}\right) u(t)$$

Q5.10 ***The output equation of a system is, y(t) = [1 2] Q(t) + [2] X(t).***

If the state vector, $Q(t) = \begin{bmatrix} e^{-2t} \\ e^{-t} \end{bmatrix}$ ***.Find the zero-state response of the system for unit step input.***

Solution

$$\text{Zero-state response, } y_{zs}(t) = \begin{bmatrix}1 & 2\end{bmatrix} \mathbf{Q}(t) + [2]\,\mathbf{X}(t) = \begin{bmatrix}1 & 2\end{bmatrix} \begin{bmatrix}e^{-2t} \\ e^{-t}\end{bmatrix} + [2]\ u(t)$$

$$= \left[e^{-2t} + 2e^{-t}\right] + 2 \times 1 = e^{-2t} + 2e^{-t} + 2\text{; for } t > 0$$

$$= \left(e^{-2t} + 2e^{-t} + 2\right) u(t)$$

5.9 MATLAB Programs

Program 5.1

Write a MATLAB program to find the state model of the system governed by the transfer function, $H(s)= (1.5s^3+2s^2+3s+2)/(s^3+3s^2+2s+4)$.

```
%Program to determine the state model from the transfer function

clear all
s=tf('s');
H=(1.5*s^3+2*s^2+3*s+2)/(s^3+3*s^2+2*s+4);
disp('The given transfer function is');
tf(H)                          %display the given transfer function

disp('Enter numerator coefficients of given transfer function');
b=input('');
disp('Enter denominator coefficients of given transfer function');
a=input('');

[A,B,C,D]=tf2ss(b,a)           %compute matrices A,B,C,D of state model
```

OUTPUT

```
The given transfer function is
     Transfer function:
                         1.5 s^3 + 2 s^2 + 3 s + 2
                          -------------------------
                             s^3 + 3 s^2 + 2 s + 4

Enter numerator coefficients of given transfer function
     [1.5   2   3   2]

Enter denominator coefficients of given transfer function
     [1   3   2   4]

   A =
     -3     -2     -4
      1      0      0
      0      1      0

   B =
     1
     0
     0

  C =
     -2.5000          0    -4.0000

  D =
     1.5000
```

Program 5.2

Write a MATLAB program to determine the transfer function of a system by getting the state model from the user through keyboard.

```
%Program to determine the transfer function of Nth order system.

disp('Enter elements of the system matrix A of size NxN');
disp('A=');
A=input('');
disp('Enter elements of the input matrix B of size Nx1');
disp('B=');
B=input('');
disp('Enter elements of the output matrix C of size 1xN');
disp('C=');
C=input('');
disp('Enter elements of the transmission matrix D of size 1x1');
disp('D=');
D=input('');
[b,a]=ss2tf(A,B,C,D)
disp('The transfer function of the state model is');
tf(b,a)
```

OUTPUT

```
Enter elements of the system matrix A of size NxN
         A =
            [2 -1; 3 1]
Enter elements of the input matrix B of size Nx1
         B =
            [1; 2]
Enter elements of the output matrix C of size 1xN
         C=
            [1 3]
Enter elements of the transmission matrix D of size 1x1
         D =
            [3]

B =
    3.0000    -2.0000     9.0000

A =
   1.0000    -3.0000     5.0000

The transfer function of the state model is

       Transfer Function:
                      3 S^2 - 2 S + 9
                      ---------------
                       S^2 - 3 S + 5
```

5.10 Exercises

I. Fill in the blanks with appropriate words

1. A set of variables that completely describe a system at any time instant are called ————.
2. The ———— of a continuous time system consists of the state equations and output equations.
3. The pictorial representation of the state model of the system is called ———— .
4. The number of state variables in the state diagram of a continuous time system is equal to the number of ————.
5. In state space modelling the number of state variables will decide ———— of the system.

6. In the matrix form of state equation, **A** represents ————.
7. In the matrix form of state equation, **B** represents ————.
8. In the matrix form of output equation, **C** represents ————.
9. In the matrix form of output equation, **D** represents ————.
10. The transfer function model does not provide information regarding ———— of the system.

Answers

1. state variables	2. state model	3. state diagram	4. integrators	5. order
6. system matrix	7. input matrix	8.output matrix	9. transmission matrix	10. internal state

II. State whether the following statements are True/False

1. The state variable analysis can be applied for any type of system (linear/nonlinear and time varying/invarying system).
2. The transfer function analysis is applicable to nonlinear systems.
3. The transfer function analysis can be carried with initial conditions.
4. The state space analysis can be carried with initial conditions and on multiple input and output systems.
5. In transfer function analysis only one input and output is considered at any one time.
6. In state space analysis multiple inputs and outputs can be allowed at any one time.
7. The state model of a system is nonunique but the transfer function is unique.
8. The state and output equations of a system are functions of state variables and inputs.
9. The state diagram of a continuous time system cannot be used for simulation of the system in analog computers.
10. The state diagram of a continuous time system provides a direct relation between time domain and s-domain.

Answers

1. True	2. False	3. False	4. True	5. True
6. True	7. True	8. True	9. False	10. True

III. Choose the right answer for the following questions

1. In the state equation of a continuous time system, $\dot{Q}(t) = A\,Q(t) + B\,X(t)$ the input matrix is,

a) **B** **b)** **X**(t) **c)** **A** **d)** **Q**(t)

2. A continuous time system is described by four numbers of first order differential equations. The number of integrators present in the direct form-II structure is,

a) 2 **b)** 4 **c)** 1 **d)** 8

3. If there is no input, the time domain solution of state equation of a continuous time system is equal to,

a) $\mathbf{Q}(t) = e^{At}$ **b)** $\mathbf{Q}(t) = e^{At}\,\mathbf{X}(0)$ **c)** $\mathbf{Q}(t) = e^{At}\,\mathbf{Q}(0)$ **d)** $\mathbf{Q}(t) = e^{-At}\,\mathbf{Q}(0)$

4. Inverse Laplace transform of $(sI - A)^{-1}$ is called

a) state equation in matrix form **b)** state transition matrix
c) transfer function **d)** response of continuous time system

5. The transfer functions analysis has a drawback that it cannot be applied to

a) single input and single output system **b)** linear time varying system
c) linear time invariant system **d)** none of the above

6. ***The transfer function of a continuous time system is described by,*** $\dfrac{Y(s)}{X(s)} = \dfrac{s^4 + 4s + 3}{s^5 + 4s^3 + 1}$. ***The number of state variables in the state model of the system is,***

   a) 1 **b)** 5 **c)** 4 **d)** 2

7. ***The transfer function of the continuous time system having the following state variable description is,***

   $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$

   a) $\dfrac{2}{(s-1)^2}$ **b)** $\dfrac{1}{s+1}$ **c)** $\dfrac{1}{s-1}$ **d)** $\dfrac{1}{(s+1)^2}$

8. ***The state transition matrix of a system is given by,*** $e^{At} = \begin{bmatrix} t\,e^{-4t}u(t) & 0 \\ e^{t}\,u(t) & 2t\,e^{-4t}u(t) \end{bmatrix}$. ***The value of*** $(sI - A)^{-1}$ ***is,***

   a) $\begin{bmatrix} \dfrac{1}{(s-4)^2} & 0 \\ \dfrac{1}{(s+1)} & \dfrac{2}{(s-1)^2} \end{bmatrix}$ **b)** $\begin{bmatrix} \dfrac{2}{(s-4)^2} & 0 \\ \dfrac{1}{(s+1)} & \dfrac{2}{(s-1)^2} \end{bmatrix}$ **c)** $\begin{bmatrix} \dfrac{2}{(s+4)} & 0 \\ \dfrac{1}{(s-1)} & \dfrac{2}{(s+1)^2} \end{bmatrix}$ **d)** $\begin{bmatrix} \dfrac{1}{(s+4)^2} & 0 \\ \dfrac{1}{(s-1)} & \dfrac{2}{(s+1)^2} \end{bmatrix}$

9. ***The system matrix of a continuous time system is given by,*** $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. ***The state transition matrix is,***

   a) $\begin{bmatrix} \dfrac{1}{s-1} & 0 \\ -1 & \dfrac{1}{s-1} \end{bmatrix}$ **b)** $\begin{bmatrix} \dfrac{1}{s-1} & \dfrac{1}{s-1} \\ (s+1)^2 & \dfrac{1}{s^2} \end{bmatrix}$ **c)** $\begin{bmatrix} \dfrac{1}{s-1} & 0 \\ \dfrac{-1}{(s-1)^2} & \dfrac{1}{s-1} \end{bmatrix}$ **d)** $\begin{bmatrix} \dfrac{1}{(s+1)^2} & 0 \\ \dfrac{1}{(s-1)} & \dfrac{1}{(s+1)^2} \end{bmatrix}$

10. ***The state transition matrix of a system is*** $\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}$, ***and initial condition vector is*** $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

    The state of the system at the end of 1-second without any external input is,

    a) $\begin{bmatrix} 0.271 \\ 1.100 \end{bmatrix}$ **b)** $\begin{bmatrix} 0.135 \\ 0.368 \end{bmatrix}$ **c)** $\begin{bmatrix} 0.271 \\ 0.736 \end{bmatrix}$ **d)** $\begin{bmatrix} 0.135 \\ 1.100 \end{bmatrix}$

Answers

1. a	2. b	3. c	4. b	5. b
6. b	7. c	8.d	9. c	10. a

IV. Answer the following questions

1. Compare state space analysis and transfer function model analysis of continuous time system.
2. What are the drawbacks in transfer function model analysis of continuous time system?
3. Define state and state variables of continuous time system.
4. What is state model of a continuous time system?
5. Write the state model of N^{th} order continuous time system.
6. Write the state equations of N^{th} order continuous time system.
7. Write the output equations of N^{th} order continuous time system.

8. How will you determine the transfer function of a continuous time system from state model ?
9. Write the expression to compute state transition matrix of a continuous time system via Laplace transform.
10. Write the expression for time domain solution of state equations of a continuous time system in terms of state transition matrix and using Laplace transform.

V. Solve the following problems

E 5.1. Determine the state model of the continuous time systems governed by the following differential equations.

a) $$\frac{d^3y(t)}{dt^3}+3.5\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}-4\,y(t)=2.5\frac{d^3x(t)}{dt^3}+1.2\frac{d^2x(t)}{dt^2}-5\frac{dx(t)}{dt}+x(t)$$

b) $$\frac{d^2y(t)}{dt^2}+6\frac{dy(t)}{dt}+4\,y(t)=4\frac{dx(t)}{dt}+x(t)$$

E 5.2. Compute the state transition matrix e^{At} for the following system matrices.

a) $\mathbf{A}=\begin{bmatrix}0 & 1\\ -5 & -6\end{bmatrix}$ **b)** $\mathbf{A}=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}$ **c)** $\mathbf{A}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ **d)** $\mathbf{A}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}$

E 5.3. Obtain the state model of the continuous time systems whose transfer functions are given below.

a) $H(s)=\dfrac{1}{s^2+3s+2}$ **b)** $H(s)=\dfrac{s+3}{s^2+3s+2}$ **c)** $H(s)=\dfrac{s^2+4s+3}{s^2+9s+20}$ **d)** $H(s)=\dfrac{s^2+6s+8}{(s+3)\,(s^2+2s+2)}$

E 5.4. Determine the transfer function of the continuous time systems having the following state variable description.

a) $\mathbf{A}=\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}1\\ 0\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}3 & 1\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}2\end{bmatrix}$

b) $\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}0\\ 2\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}3 & 0\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}0\end{bmatrix}$

E 5.5. Compute the solution of the following state equations representing LTI continuous time systems.

a) $\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}3 & 1\\ 0 & -4\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$; $\begin{bmatrix}q_1(0)\\ q_2(0)\end{bmatrix}=\begin{bmatrix}0\\ -2\end{bmatrix}$

b) $\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 2 & -5\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$; $\begin{bmatrix}q_1(0)\\ q_2(0)\end{bmatrix}=\begin{bmatrix}-0.5\\ 1\end{bmatrix}$

E 5.6. Find the response of LTI continuous time systems having the state model and inputs as given below. Assume zero initial conditions.

a) $\mathbf{A}=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}3\\ 5\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}2 & -1\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}1\end{bmatrix}$; Input, $x(t)=e^{-2t}\,u(t)$

b) $\mathbf{A}=\begin{bmatrix}1 & 0\\ 4 & 3\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}-1\\ -2\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}4 & 3\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}2\end{bmatrix}$; Input, $x_1(t)=2\,u(t)$ and $x_2(t)=e^{t}\,u(t)$

Answers

E5.1 a) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 4 & -2 & -3.5\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[11 \quad -10 \quad -7.55\big]\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}+\big[2.5\big]\big[x(t)\big]$$

b) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -4 & -6\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[1 \quad 4\big]\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

E5.2 a) $$e^{At}=\begin{bmatrix}\frac{1}{4}\left(5e^{-t}-e^{-5t}\right) & \frac{1}{4}\left(e^{-t}-e^{-5t}\right)\\ \frac{5}{4}\left(-e^{-t}+e^{-5t}\right) & \frac{1}{4}\left(5e^{-5t}-e^{-t}\right)\end{bmatrix}u(t)$$

b) $$e^{At}=\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-3t}\end{bmatrix}u(t)$$

c) $$e^{At}=\begin{bmatrix}e^{-2t}\cosh t & e^{-2t}\sinh t\\ e^{-2t}\sinh t & e^{-2t}\cosh t\end{bmatrix}u(t)$$

d) $$e^{At}=\begin{bmatrix}e^{-2t} & 0\\ 0 & e^{-t}\end{bmatrix}u(t)$$

E5.3 a) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[1 \quad 0\big]\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

b) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[3 \quad 1\big]\begin{bmatrix}q_1(t)\\ q_2(t)\end{bmatrix}$$

c) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -20 & -9 & 0\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[3 \quad 4 \quad 1\big]\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}$$

d) State equation

$$\begin{bmatrix}\dot{q}_1(t)\\ \dot{q}_2(t)\\ \dot{q}_3(t)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -8 & -5\end{bmatrix}\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\big[x(t)\big]$$

Output equation

$$y(t)=\big[8 \quad 6 \quad 1\big]\begin{bmatrix}q_1(t)\\ q_2(t)\\ q_3(t)\end{bmatrix}$$

E5.4 a) $$\frac{Y(s)}{X(s)}=\frac{2s^2-s+3}{(s-1)^2}$$

b) $$\frac{Y(s)}{X(s)}=\frac{6}{s^2-s+1}$$

E5.5 a) $$\mathbf{Q}(t)=\begin{bmatrix}\frac{2}{7}\left(e^{-4t}-e^{3t}\right)\\ -2e^{-4t}\end{bmatrix}u(t)$$

b) $$\mathbf{Q}(t)=\frac{1}{4}\begin{bmatrix}-2e^{-t}\\ -e^{-t}+5e^{-5t}\end{bmatrix}u(t)$$

E5.6 a) $$\mathbf{Q}(t)=\begin{bmatrix}\frac{1}{3}\left(-5te^{t}+\frac{14}{3}e^{t}-\frac{14}{3}e^{-2t}\right)\\ \frac{5}{3}\left(e^{t}-e^{-2t}\right)\end{bmatrix}u(t)$$

$$y(t)=\left[-\frac{10}{3}te^{t}+\frac{13}{9}e^{t}-\frac{34}{9}e^{-2t}\right]u(t)$$

b) $$\mathbf{Q}(t)=\begin{bmatrix}2\left(1-e^{3t}\right)\\ 4\left(-\frac{1}{3}-\frac{2}{3}e^{3t}+e^{t}\right)\end{bmatrix}u(t)$$

$$y(t)=\big[8+12e^{t}-16e^{3t}\big]u(t)$$

CHAPTER 6

Discrete Time Signals and Systems

6.1 Discrete and Digital Signals

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n). The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range. Here both the value of the signal and the independent variable are discrete.

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is quantized uniformly using the relation $t = nT$, where T is the sampling time period. (The sampling time period is inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

The digital signal is same as discrete signal except that the magnitude of the signal is quantized. The magnitude of the signal can take one of the values in a set of quantized values. Here quantization is necessary to represent the signal in binary codes.

The discrete or digital signals have a sequence of numbers (or values) defined for integer values of the independent variable. Hence the discrete or digital signals are also known as ***discrete sequence***. In this book the term sequence and signal are used synonymously. Also in this book the discrete signal is referred as discrete time signal.

6.1.1 Generation of Discrete Signals

A discrete signal can be generated in the following three methods.

The methods 1 and 2 are independent of any time frame but method 3 depends critically on time.

1. Generate a set of numbers and arrange them as a sequence.

Example :

The numbers 0, 1, 2, ..., (N – 1) form the ramp like sequence and can be expressed as,

$$x(n) = n \quad ; \quad 0 \le n \le (N-1)$$

2. Evaluation of a numerical recursion relation will generate a discrete signal.

Example :

$x(n) = \dfrac{x(n-1)}{3}$ with initial condition $x(0) = 1$, gives the sequence, $x(n) = \left(\dfrac{1}{3}\right)^n \; ; \; 0 \le n < \infty$

When $n = 0$; $x(0) = 1$ (initial condition) $= \left(\dfrac{1}{3}\right)^0$

When $n = 1$; $x(1) = \dfrac{x(1-1)}{3} = \dfrac{x(0)}{3} = \dfrac{1}{3} = \left(\dfrac{1}{3}\right)^1$

$$\text{When } n = 2\ ;\ x(2) = \frac{x(2-1)}{3} = \frac{x(1)}{3} = \frac{1}{9} = \left(\frac{1}{3}\right)^2$$

$$\text{When } n = 3\ ;\ x(3) = \frac{x(3-1)}{3} = \frac{x(2)}{3} = \frac{1}{27} = \left(\frac{1}{3}\right)^3 \text{ and so on}$$

$$\therefore x(n) = \left(\frac{1}{3}\right)^n \quad ; \quad 0 \le n < \infty$$

3. A third method is by uniformly sampling a continuous time signal and using the amplitudes of the samples to form a sequence.

   Let, x(t) = Continuous time signal

   $$\text{Now, Discrete signal, } x(nT) = x(t)\big|_{t=nT} \quad ; \quad -\infty < n < \infty$$

   where, T is the sampling interval

The generation of discrete signal by sampling an analog ramp signal is shown in fig 6.1.

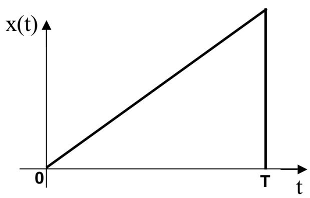


***Fig 6.1a.** Analog signal.*

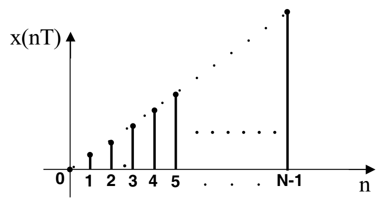


***Fig 6.1b.** Discrete time signal.*

***Fig 6.1 :** Generation of a discrete time signal.*

6.1.2 Representation of Discrete Time Signals

The discrete time signal can be represented by the following methods.

1. Functional representation

In functional representation the signal is represented as mathematical equation, as shown in the following example.

$$
\begin{aligned}
x(n) &= -1 &&; \quad n = -2\\
&= 2 &&; \quad n = -1\\
&= 1.5 &&; \quad n = 0\\
&= -0.9 &&; \quad n = 1\\
&= 1.4 &&; \quad n = 2\\
&= 1.6 &&; \quad n = 3\\
&= 0 &&; \quad \text{other } n
\end{aligned}
$$

2. Graphical representation

In graphical representation the signal is represented in a two dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 6.2.

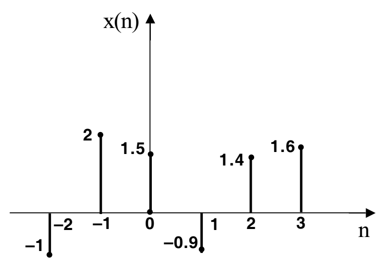


***Fig 6.2 :** Graphical representation of a discrete time signal.*

3. Tabular representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following example.

n	....	–2	–1	0	1	2	3	....
x(n)	....	–1	2	1.5	–0.9	1.4	1.6	....

4. Sequence representation

In sequence representation, the discrete time signal is represented as one dimensional array as shown in the following examples.

An infinite duration discrete time signal with the time origin, n = 0, indicated by the symbol ↑ is represented as,

x(n) = { ..... – 1, 2, 1.5, – 0.9, 1.4, 1.6, ... }
↑

An infinite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

x(n) = { 1.5, – 0.9, 1.4, 1.6, ... } or x(n) = {1.5, – 0.9, 1.4, 1.6, ... }
↑

A finite duration discrete time signal with the time origin, n = 0, indicated by the symbol ↑ is represented as,

x(n) = { – 1, 2, 1.5, – 0.9, 1.4, 1.6 }
↑

A finite duration discrete time signal that satisfies the condition x(n) = 0 for n < 0 is represented as,

x(n) = { 1.5, – 0.9, 1.4, 1.6 } or x(n) = { 1.5, – 0.9, 1.4, 1.6}
↑

6.2 Standard Discrete Time Signals

1. Digital impulse signal or Unit sample sequence

$$\text{Impulse signal, } \delta(n) = 1 \;;\; n = 0$$
$$= 0 \;;\; n \neq 0$$

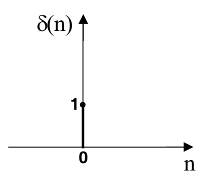


Fig 6.3 : Digital impulse signal.

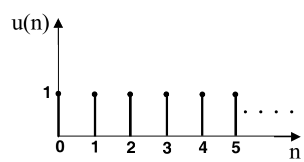


Fig 6.4 : Unit step signal.

2. Unit step signal

$$\text{Unit step signal, } u(n) = 1 \;;\; n \geq 0$$
$$= 0 \;;\; n < 0$$

3. Ramp signal

$$\text{Ramp signal, } u_r(n) = n \;;\; n \geq 0$$
$$= 0 \;;\; n < 0$$

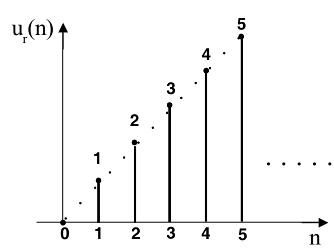


Fig 6.5 : Ramp signal.

4. Exponential signal

Exponential signal, $g(n) = a^n \;;\; n \geq 0$

$= 0 \;;\; n < 0$

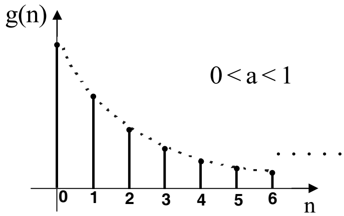


Fig 6.6a : *Decreasing exponential signal.*

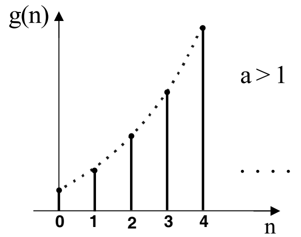


Fig 6.6b : *Increasing exponential signal.*

Fig 6.6 : *Exponential signal.*

5. Discrete time sinusoidal signal

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta) \;;\; \text{for n in the range } -\infty < n < +\infty$$

$$x(n) = A\sin(\omega_0 n + \theta) \;;\; \text{for n in the range } -\infty < n < +\infty$$

where, ω_0 = Frequency in radians/sample ; θ = Phase in radians

$$f_0 = \frac{\omega_0}{2\pi} = \text{Frequency in cycles/sample}$$

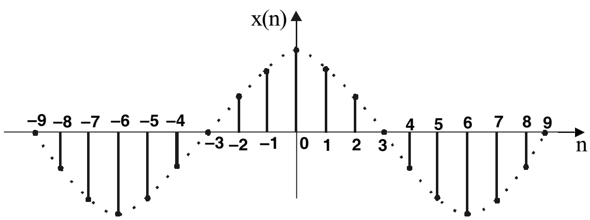


Fig 6.7a : *Discrete time sinusoidal signal represented by equation $x(n) = A\cos(\omega_0 n)$.*

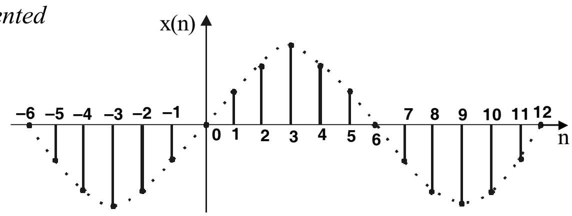


Fig 6.7b : *Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.*

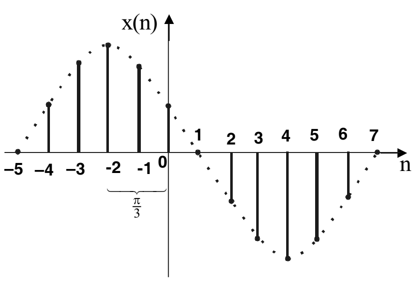


Fig 6.7c : *Discrete time sinusoidal signal represented by equation $x(n) = A\cos\left(\frac{\pi}{6}n + \frac{\pi}{3}\right)$; $\omega_0 = \frac{\pi}{6}$; $\theta = \frac{\pi}{3}$.*

Fig 6.7 : *Discrete time sinusoidal signals.*

Properties of Discrete Time Sinusoid

1. A discrete time sinusoid is periodic only if its frequency f is a rational number, (i.e., ratio of two integers).
2. Discrete time sinusoids whose frequencies are separated by integer multiples of 2π are identical.

$\therefore x(n) = A\cos[(\omega_0 + 2\pi k)\,n + \theta\,]$, for $k = 0,1,2$..........are identical in the interval $-\pi \le \omega_0 \le \pi$ and so they are indistinguishable.

Proof :

$$\cos[(\omega_0 + 2\pi k)\,n + \theta] = \cos(\omega_0 n + 2\pi nk + \theta) = \cos[(\omega_0 n + \theta) + 2\pi nk]$$

$$= \cos(\omega_0 n + \theta)\cos 2\pi nk - \sin(\omega_0 n + \theta)\sin 2\pi nk$$

Since n and k are integers, $\cos 2\pi nk = 1$ & $\sin 2\pi nk = 0$

$\therefore \cos[(\omega_0 + 2\pi k)\,n + \theta] = \cos(\omega_0 n + \theta)$, for $k = 1, 2, 3, \ldots\ldots$

Conclusion

1. The sequences of any two sinusoids with frequencies in the range, $-\pi \le \omega_o \le \pi$, or, $-1/2 \le f_0 \le 1/2$, are distinct.

$$[-\pi \le \omega \le \pi \xrightarrow{\text{divide by } 2\pi} -1/2 \le f \le 1/2]$$

2. Any discrete time sinusoid with frequency $\omega_0 > |\pi|$ (or $f_0 > |1/2|$) will be identical to another discrete time sinusoid with frequency $\omega_0 < |\pi|$ (or $f_0 < |1/2|$).

6. Discrete time complex exponential signal

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n + \theta)} = a^n[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)]$$

$$= a^n\cos(\omega_0 n + \theta) + j\,a^n\sin(\omega_0 n + \theta) = x_r(n) + j\,x_i(n)$$

where, $x_r(n)$ = Real part of $x(n) = a^n\cos(\omega_0 n + \theta)$

$x_i(n)$ = Imaginary part of $x(n) = a^n\sin(\omega_0 n + \theta)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for $a > 1$ and exponentially decreasing cosinusoid sequence for $0 < a < 1$.

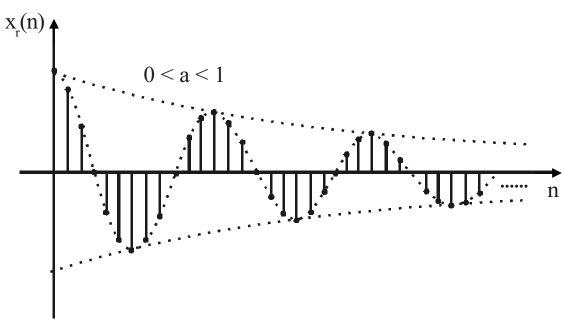


Fig 6.8a : *The discrete time sequence represented by the equation, $x_r(n) = a_n \cos\omega_0 n$ for $0 < a < 1$.*

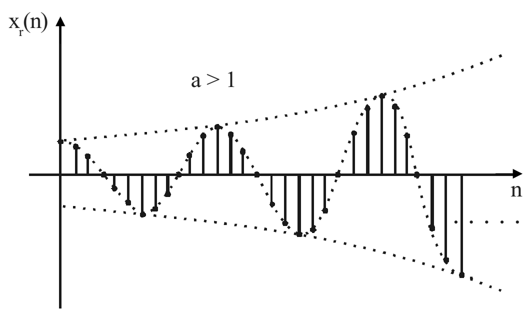


Fig 6.8b : *The discrete time sequence represented by the equation $x_r(n) = a_n \cos\omega_0 n$ for $a > 1$.*

Fig 6.8 : *Real part of complex exponential signal.*

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

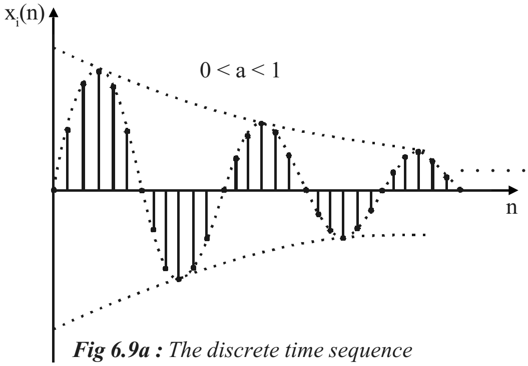


Fig 6.9a : *The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $0 < a < 1$.*

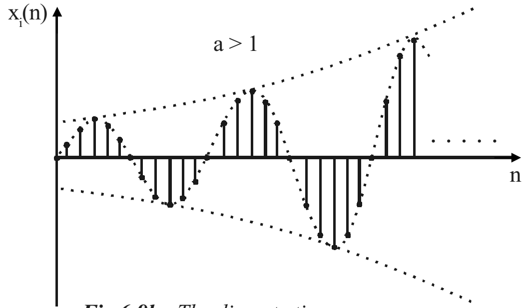


Fig 6.9b : *The discrete time sequence represented by the equation $x_i(n) = a^n \sin \omega_0 n$ for $a > 1$.*

Fig 6.9 : *Imaginary part of complex exponential signal.*

6.3 Sampling of Continuous Time (Analog) Signals

The ***sampling*** is the process of conversion of a continuous time signal into a discrete time signal. The sampling is performed by taking samples of continuous time signal at definite intervals of time. Usually, the time interval between two successive samples will be same and such type of sampling is called ***periodic or uniform sampling***.

The time interval between successive samples is called ***sampling time*** (or sampling period or sampling interval), and it is denoted by "T". The unit of sampling period is second (s). (The lower units are milli-second (ms) and micro-second (μs)).

The inverse of sampling period is called ***sampling frequency*** (or sampling rate), and it is denoted by F_s. The unit of sampling frequency is Hertz (Hz). (The higher units are kHz and MHz).

Let, $x_a(t)$ = Analog / Continuous time signal.

x(n) = Discrete time signal obtained by sampling $x_a(t)$.

Mathematically the relation between x(n) and $x_a(t)$ can be expressed as,

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right); \qquad \text{for n in the range } -\infty < n < \infty$$

where, T = Sampling period or interval in seconds

$$F_s = \frac{1}{T} = \text{Sampling rate or Sampling frequency in Hertz}$$

Example : Let, $x_a(t) = A\cos(\Omega_0 t + \theta) = A\cos(2\pi F_0 t + \theta)$

where, Ω_0 = Frequency of analog signal in rad/s

$$F_0 = \frac{\Omega_0}{2\pi} = \text{Frequency of analog signal in Hz}$$

Let $x_a(t)$ be sampled at intervals of T seconds to get x(n), where $T = \frac{1}{F_s}$

$$\therefore\ x(n) = x_a(t)\big|_{t=nT} = A\cos(\Omega_0 t + \theta)\big|_{t=nT}$$

$$= A\cos(\Omega_0 nT + \theta) = A\cos\left(\frac{2\pi F_0}{F_s}n + \theta\right) = A\cos\left(2\pi f_o n + \theta\right) = A\cos\left(\omega_o n + \theta\right)$$

where, $f_0 = \frac{F_0}{F_s}$ = Frequency of discrete sinusoid in cycles/sample

$\omega_0 = 2\pi f_0$ = Frequency of discrete sinusoid in radians/sample

6.3.1 Sampling and Aliasing

In section 6.2 it is observed that any two sinusoid signals with frequencies in the range $-1/2 \leq f \leq +1/2$ are distinct and a discrete sinusoid with frequency, $f > 1/2$ will be identical to another discrete sinusoid with frequency, $f < 1/2$. Therefore we can conclude that range of discrete frequency is $-1/2$ to $+1/2$. But the range of analog frequency is $-\infty$ to $+\infty$. While sampling analog signals, the infinite frequency range continuous time signals are mapped or converted to finite range discrete time signals.

The relation between analog and digital frequency is,

$$f = \frac{F}{F_s} \qquad .....(6.1)$$

The range of discrete frequency is,

$$-\frac{1}{2} \leq f \leq \frac{1}{2} \qquad .....(6.2)$$

On substituting for f from equation (6.1) in equation (6.2) we get,

$$-\frac{1}{2} \leq \frac{F}{F_s} \leq \frac{1}{2} \qquad .....(6.3)$$

On multiplying equation (6.3) by F_s we get,

$$-\frac{F_s}{2} \leq F \leq \frac{F_s}{2} \qquad .....(6.4)$$

From equation (6.4) we can say that when an analog signal is sampled at a frequency F_s, the highest analog frequency that can be uniquely represented by a discrete time signal will be $F_s/2$. The continuous time signal with frequency above $F_s/2$ will be represented as a signal within the range $+F_s/2$ to $-F_s/2$. Hence the signal with frequency above $F_s/2$ will have an identical signal with frequency below $F_s/2$ in the discrete form.

Hence infinite number of high frequency continuous time signals will be represented by a single discrete time signal. Such signals are called ***alias.*** While sampling at F_s, the frequency above $F_s/2$ will have alias with frequency below $F_s/2$. Hence the point of reflection is $F_s/2$, and the frequency $F_s/2$ is called ***folding frequency***.

The discrete time sinusoids, A sin((2πf + 2πk)n), will be alias for integer values of k. It is also observed that, a sinusoidal signal with frequency F_1 will be an alias of sinusoidal signal with frequency F_2 if it is sampled at a frequency $F_s = F_1 - F_2$. In general if the sampling frequency is any multiple of $F_1 - F_2$, (i.e., $F_s = k(F_1 - F_2)$ where k = 1, 2, 3, ...) the signal with frequency F_2 will be an alias of the signal with frequency F_1. The phenomenon of high frequency component getting the identity of low frequency component during sampling is called ***aliasing.***

Let, F_{max} be maximum analog frequency that can be uniquely represented as discrete time signal when sampled at a frequency F_s.

$$\text{Now, } F_{max} = \frac{F_s}{2} \qquad .....(6.5)$$

$$\therefore F_s = 2F_{max} \qquad .....(6.6)$$

The equation (6.6) gives a choice for selecting sampling frequency. From equation (6.6) we can say that for unique representation of analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.

$$\text{i.e., to avoid aliasing } F_s \geq 2F_{max} \qquad ..... (6.7)$$

When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called ***Nyquist rate***.

It is observed that a nonshifted sinusoidal signal when sampled at Nyquist rate, will produce zero sample sequence (i.e., discrete sequence with all zeros), (because the sinusoidal signal is sampled at its zero crossings). (Refer example 6.3). Hence to avoid zero sampling of sinewave, the sampling frequency F_s should be greater than $2F_{max}$, where F_{max} is the maximum frequency in the analog signal.

A discrete signal obtained by sampling can be reconstructed to analog signal, only when it is sampled without aliasing. The above concepts of sampling analog signals are summarized as the sampling theorem, given below.

Sampling Theorem : *A band limited continuous time signal with highest frequency (bandwidth) F_m hertz can be uniquely recovered from its samples provided that the sampling rate F_s is greater than or equal to $2F_m$ samples per second.*

> ***Note :*** *The effects of aliasing in frequency spectrum are discussed in Chapter-8.*

Example 6.1

Consider the analog signals, $x_1(t) = 2\cos 2\pi(10t)$ and $x_2(t) = 2\cos 2\pi(50t)$.
Find a sampling frequency so that 50Hz signal is an alias of the 10Hz signal?

Solution

Let, the sampling frequency, $F_s = 50 - 10 = 40$Hz.

$$\therefore x_1(n) = x_1(t)\Big|_{t = nT = \frac{n}{F_s}} = 2\cos 2\pi(10t)\Big|_{t = \frac{n}{F_s}} = 2\cos 2\pi\left(\frac{10 \times n}{40}\right)$$

$$= 2\cos\frac{\pi}{2}n$$

$$x_2(n) = x_2(t)\Big|_{t = nT = \frac{n}{F_s}} = 2\cos 2\pi(50t)\Big|_{t = \frac{n}{F_s}} = 2\cos 2\pi\left(\frac{50 \times n}{40}\right)$$

$$= 2\cos\frac{5\pi}{2}n = 2\cos\left(2\pi n + \frac{\pi}{2}n\right) = 2\cos\frac{\pi}{2}n$$

For integer values of n
$\cos(2\pi n + \theta) = \cos\theta$

From the above analysis we observe that $x_1(n)$ and $x_2(n)$ are identical, and so $x_2(t)$ is an alias of $x_1(t)$ when sampled at a frequency of 40Hz.

Example 6.2

Let an analog signal, $x_a(t) = 10\cos 100\pi t$. If the sampling frequency is 75Hz, find the discrete time signal x(n). Also find an alias frequency corresponding to $F_s = 75$Hz.

Solution

$$x(n) = x_a(t)\Big|_{t = nT = \frac{n}{F_s}} = 10\cos 100\pi t\Big|_{t = \frac{n}{F_s}} = 10\cos 100\pi \times \frac{n}{F_s}$$

$$= 10\cos\frac{100\pi \times n}{75} = 10\cos\frac{4\pi}{3}n = 10\cos\left(2\pi - \frac{2\pi}{3}\right)n = 10\cos\frac{2\pi}{3}n = 10\cos 2\pi\frac{1}{3}n$$

We know that the discrete time sinusoids whose frequencies are separated by integer multiple of 2π are identical.

$$\therefore 10\cos\frac{2\pi}{3}n = 10\cos\left(\frac{2\pi}{3} + 2\pi\right)n = 10\cos\frac{8\pi}{3}n = 10\cos 2\pi\frac{4}{3}n$$

Now, $10\cos 2\pi\frac{4}{3}n$ is an alias of $10\cos\frac{2\pi}{3}n$. Here the frequency of the signal, $10\cos 2\pi\frac{4}{3}n$ is,

$$f = \frac{4}{3} \text{ cycles / sample}$$

We know that, $f = \frac{F}{F_s} \Rightarrow F = f F_s = \frac{4}{3} \times 75 = 100\text{Hz}$

$\therefore$ When, $F_s = 75$Hz, $F = 100$ Hz is an alias frequency.

Example 6.3

Consider the analog signal, $x_a(t) = 5\cos 50\pi t + 2\sin 200\pi t - 2\cos 100\pi t$.

Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points. Comment on the result.

Solution

The given analog signal can be written as shown below.

$$x_a(t) = 5\cos 50\pi t + 2\sin 200\pi t - 2\cos 100\pi t = 5\cos 2\pi F_1 t + 2\sin 2\pi F_2 t - 2\cos 2\pi F_3 t$$

$$\text{Where, } 2\pi F_1 = 50\pi \Rightarrow F_1 = 25\text{Hz}$$

$$2\pi F_2 = 200\pi \Rightarrow F_2 = 100\text{Hz}$$

$$2\pi F_3 = 100\pi \Rightarrow F_3 = 50\text{Hz}$$

The maximum analog frequency in the signal is 100Hz. The sampling frequency should be twice that of this maximum analog frequency.

i.e., $F_s \geq 2F_{max} \Rightarrow F_s \geq 2 \times 100$

Let, sampling frequency, $F_s = 200$ Hz

$$\therefore x_a(nT) = x_a(t)\big|_{t = nT} = x_a(t)\Big|_{t = \frac{n}{F_s}}$$

$$= 5\cos\frac{50\pi n}{200} + 2\sin\frac{200\pi n}{200} - 2\cos\frac{100\pi n}{200} = 5\cos\frac{\pi n}{4} + 2\sin\pi n - 2\cos\frac{\pi n}{2}$$

For integer values of n, $\sin\pi n = 0$.

$$\therefore x_a(nT) = 5\cos\frac{\pi n}{4} - 2\cos\frac{\pi n}{2}$$

The components of analog waveform and the sampling points are shown in fig1.

Comment : *In the sampled version of analog signal $x_a(nT)$, the component $2\sin 200\pi t$ will give always zero samples when sampled at 200Hz for any value of n. This is the drawback in sampling at Nyquist rate (i.e., sampling at $F_s = 2F_{max}$).*

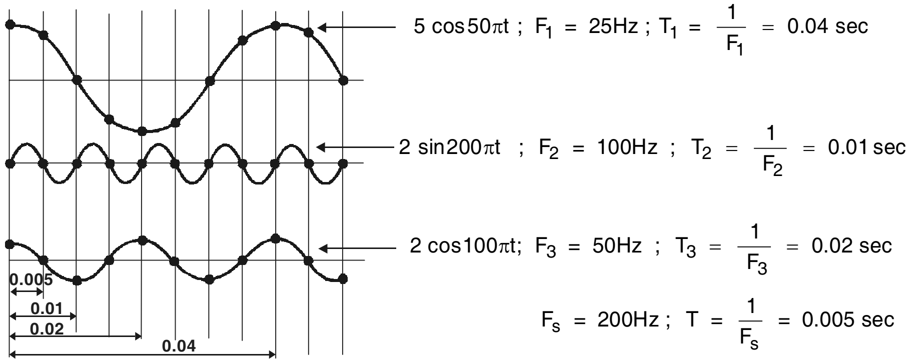


Fig 1. *Sampling points of the components of the signal $x_a(t)$.*

6.4 Classification of Discrete Time Signals

The discrete time signals are classified depending on their characteristics. Some ways of classifying discrete time signals are,

1. Deterministic and nondeterministic signals
2. Periodic and aperiodic signals
3. Symmetric and antisymmetric signals.
4. Energy signals and power signals
5. Causal and noncausal signals

6.4.1 Deterministic and Nondeterministic Signals

The signals that can be completely specified by mathematical equations are called ***deterministic signals***. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

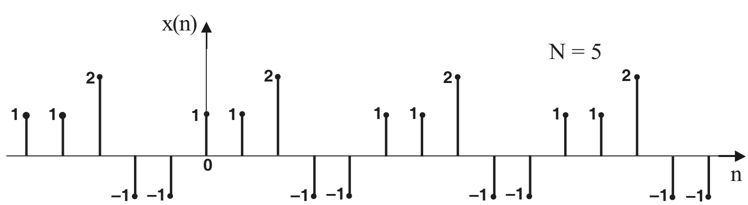
The signals whose characteristics are random in nature are called ***nondeterministic signals.*** The noise signals from various sources are best examples of nondeterministic signals.

6.4.2 Periodic and Aperiodic Signals

A signal x(n) is ***periodic*** with periodicity of N samples (where N is an integer) if and only if

$$\boxed{x(n + N) = x(n) \quad ; \ \text{for all n}}$$

The smallest value of N for which the above equation is true is called ***fundamental period***. If there is no value of N that satisfies the above equation, then it is called ***aperiodic*** or ***nonperiodic*** signal. When N is the fundamental period, the periodic signals will also satisfy the condition x(n + kN) = x(n), where k is an integer. Periodic signals are power signals. The sinusoidal and complex exponential signals are periodic signals when their fundamental frequency, f_0 is a rational number.



$$x(n) = \{\ldots\ldots 1, 1, 2, -1, -1, \underset{\uparrow}{1}, 1, 2, -1, -1, 1, 1, 2, -1, -1, \ldots\ldots\}$$

Fig 6.10 : *Periodic discrete time signal.*

Example 6.4

Determine whether following signals are periodic or not. If periodic find the fundamental period.

a) $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$ b) $x(n) = \cos\left(\frac{n}{8} - \pi\right)$ c) $x(n) = \cos\frac{\pi}{8}n^2$ d) $x(n) = e^{j7\pi n}$

Solution

a) Given that, $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$

Let N and M be two integers.

Now, $x(n + N) = \sin\left(\frac{6\pi}{7}(n + N) + 1\right) = \sin\left(\frac{6\pi n}{7} + 1 + \frac{6\pi}{7}N\right)$

Since $\sin(\theta + 2\pi M) = \sin\theta$, for periodicity $\frac{6\pi}{7}N$ should be integral multiple of 2π.

Let, $\frac{6\pi}{7}N = M \times 2\pi$, where M and N are integers.

$$\therefore N = M \times 2\pi \times \frac{7}{6\pi} = \frac{7M}{3}$$

Here N is an integer if, $M = 3, 6, 9, 12, ...$

Let, $M = 3$; $\therefore N = 7$

When $N = 7$; $x(n + N) = \sin\left(\frac{6\pi n}{7} + 1 + \frac{6\pi}{7} \times 7\right) = \sin\left(\frac{6\pi n}{7} + 1 + 6\pi\right) = \sin\left(\frac{6\pi n}{7} + 1\right) = x(n)$

Hence x(n) is periodic with fundamental period of 7 samples.

---

b) Given that, $x(n) = \cos\left(\frac{n}{8} - \pi\right)$

Let N and M be two integers.

Now, $x(n + N) = \cos\left(\frac{n + N}{8} - \pi\right) = \cos\left(\frac{n}{8} + \frac{N}{8} - \pi\right) = \cos\left(\frac{n}{8} - \pi + \frac{N}{8}\right)$

Since $\cos(\theta + 2\pi M) = \cos\theta$, for periodicity $\frac{N}{8}$ should be equal to integral multiple of 2π.

Let, $\frac{N}{8} = M \times 2\pi$; where M and N are integers.

$$\therefore N = 16\pi M$$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

---

c) Given that, $x(n) = \cos\left(\frac{\pi}{8}n^2\right)$

The general form of discrete time cosinusoid is $x(n) = \cos(2\pi f_o n)$.

Let, $2\pi f_o n = \frac{\pi}{8}n^2$

$$\therefore f_o = \frac{\pi}{8}n^2 \times \frac{1}{2\pi n} = \frac{n}{16}$$

Since n is an integer, f_0 is a rational number and so $\cos\left(\frac{\pi}{8}n^2\right)$ is periodic.

Let N be the fundamental period and M be an integer.

Now for periodicity, $\frac{\pi}{8}N^2 = M \times 2\pi \quad \Rightarrow \quad N^2 = M \times 2\pi \times \frac{8}{\pi} = 16M \quad \Rightarrow \quad N = 4\sqrt{M}$

Here N is integer, when $M = 1^2, 2^2, 3^2, ...$

Let, $M = 1 \; ; \; \therefore \; N = 4$

Hence x(n) is periodic with fundamental period of 4 samples.

d) Given that, $x(n) = e^{j7\pi n}$

Let N and M be two integers.

Now, $x(n + N) = e^{j7\pi(n+N)} = e^{j7\pi n} \, e^{j7\pi N}$

Since $e^{j2\pi M} = 1$, for periodicity 7π N should be integral multiple of 2π.

Let, $7\pi N = M \times 2\pi$,

$$\therefore \; N = M \times 2\pi \times \frac{1}{7\pi} = \frac{2M}{7}$$

Here, N is integer, when $M = 7, 14, 21, ...$

When $M = 7 \; ; \; N = 2$

$\therefore$ x(n) is periodic with fundamental period of 2 samples.

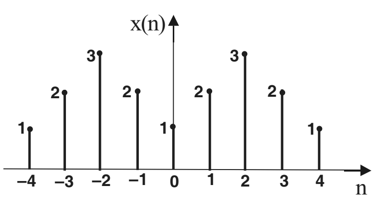
6.4.3 Symmetric (Even) and Antisymmetric (Odd) Signals

The signals may exhibit symmetry or antisymmetry with respect to n = 0. When a signal exhibits symmetry with respect to n = 0 then it is called an ***even signal.*** Therefore the even signal satisfies the condition,

$$x(-n) = x(n)$$

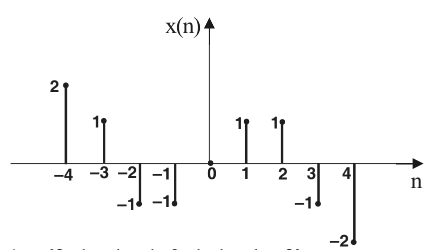
When a signal exhibits antisymmetry with respect to n = 0, then it is called an ***odd signal***. Therefore the odd signal satisfies the condition,

$$x(-n) = -x(n)$$



$$\mathbf{x(n) = \{1, 2, 3, 2, \underset{\uparrow}{1}, 2, 3, 2, 1\}}$$

Fig 6.11a : *Symmetric (or even) signal.*



$$\mathbf{x(n) = \{2, 1, -1, -1, \underset{\uparrow}{0}, 1, 1, -1, -2\}}$$

Fig 6.11b : *Antisymmetric (or odd) signal.*

Fig 6.11 : *Symmetric and antisymmetric discrete time signal.*

A discrete signal x(n) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $\quad x(n) = x_e(n) + x_o(n)$

where, $x_e(n)$ = Even part of x(n) ; $\quad x_o(n)$ = Odd part of x(n)

Note : *If x(n) is even then its odd part will be zero. If x(n) is odd then its even part will be zero.*

Now, it can be proved that,

$$\boxed{\begin{aligned} x_e(n) &= \frac{1}{2}\left[x(n) + x(-n)\right] \\ x_o(n) &= \frac{1}{2}\left[x(n) - x(-n)\right] \end{aligned}}$$

Proof :

Let, $x(n) = x_e(n) + x_o(n)$ ......(6.8)

On replacing n by –n in equation (6.8) we get,

$x(-n) = x_e(-n) + x_o(-n)$ ......(6.9)

Since x_e (n) is even, $x_e(-n) = x_e(n)$

Since x_o (n) is odd, $x_o(-n) = -x_o(n)$

Hence the equation (6.9) can be written as,

$x(-n) = x_e(n) - x_o(n)$ ......(6.10)

On adding equation (6.8) and (6.10) we get,

$$x(n) + x(-n) = 2\,x_e(n)$$

$$\therefore\ x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right]$$

On subtracting equation (6.10) from equation (6.8) we get,

$$x(n) - x(-n) = 2\,x_o(n)$$

$$\therefore\ x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right]$$

Example 6.5

Determine the even and odd parts of the signals.

a) $x(n) = a^n$ b) $x(n) = 2\,e^{j\frac{\pi}{3}n}$ c) $x(n) = \{4,\ -4,\ 2,\ -2\}$

Solution

a) Given that, $x(n) = a^n$

$$\therefore\ x(-n) = a^{-n}$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[a^n + a^{-n}]$

Odd part, $x_o(-n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[a^n - a^{-n}]$

b) Given that, $x(n) = 2\,e^{j\frac{\pi}{3}n}$

$$x(n) = 2\,e^{j\frac{\pi}{3}n} = 2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n$$

$$\therefore x(-n) = 2\,e^{-j\frac{\pi}{3}n} = 2\cos\frac{\pi}{3}n - j2\sin\frac{\pi}{3}n$$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$$= \frac{1}{2}\left[2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n + 2\cos\frac{\pi}{3}n - j2\sin\frac{\pi}{3}n\right] = \frac{1}{2}\left[4\cos\frac{\pi}{3}n\right] = 2\cos\frac{\pi}{3}n$$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$$= \frac{1}{2}\left[2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n - 2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n\right] = \frac{1}{2}\left[j4\sin\frac{\pi}{3}n\right] = j2\sin\frac{\pi}{3}n$$

c) Given that, $x(n) = \{\underset{\uparrow}{4}, -4, 2, -2\}$

Given that, $x(n) = \{\underset{\uparrow}{4}, -4, 2, -2\}$, $\quad \therefore\ x(0) = 4 \ ;\ x(1) = -4 \ ;\ x(2) = 2 \ ;\ x(3) = -2$

$x(-n) = \{-2, 2, -4, \underset{\uparrow}{4}\}$, $\quad \therefore\ x(-3) = -2 \ ;\ x(-2) = 2 \ ;\ x(-1) = -4 \ ;\ x(0) = 4$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

At $n = -3$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = -2$; $x(n) + x(-n) = 0 + 2 = 2$

At $n = -1$; $x(n) + x(-n) = 0 + (-4) = -4$

At $n = 0$; $x(n) + x(-n) = 4 + 4 = 8$

At $n = 1$; $x(n) + x(-n) = -4 + 0 = -4$

At $n = 2$; $x(n) + x(-n) = 2 + 0 = 2$

At $n = 3$; $x(n) + x(-n) = -2 + 0 = -2$

$\therefore\ x(n) + x(-n) = \{-2, 2, -4, \underset{\uparrow}{8}, -4, 2, -2\}$

$\therefore\ x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$= \{-1, 1, -2, \underset{\uparrow}{4}, -2, 1, -1\}$

Odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

At $n = -3$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = -2$; $x(n) - x(-n) = 0 - 2 = -2$

At $n = -1$; $x(n) - x(-n) = 0 - (-4) = 4$

At $n = 0$; $x(n) - x(-n) = 4 - 4 = 0$

At $n = 1$; $x(n) - x(-n) = -4 - 0 = -4$

At $n = 2$; $x(n) - x(-n) = 2 - 0 = 2$

At $n = 3$; $x(n) - x(-n) = -2 - 0 = -2$

$\therefore\ x(n) - x(-n) = \{2, -2, 4, \underset{\uparrow}{0}, -4, 2, -2\}$

$\therefore\ x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$= \{1, -1, 2, \underset{\uparrow}{0}, -2, 1, -1\}$

6.4.4 Energy and Power Signals

The ***energy*** E of a discrete time signal x(n) is defined as,

$$\boxed{\text{Energy, } E = \sum_{n=-\infty}^{\infty} |x(n)|^2} \quad .....(6.11)$$

The energy of a signal may be finite or infinite, and can be applied to complex valued and real valued signals.

If energy E of a signal is finite and non-zero, then the signal is called an ***energy signal.*** The exponential signals are examples of energy signals.

The average ***power*** of a discrete time signal x(n) is defined as,

$$\boxed{\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2} \quad .....(6.12)$$

If power P of a signal is finite and non-zero, then the signal is called a ***power signal***. The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero. For power signals the average power is finite and energy will be infinite.

i.e., For energy signal, $0 < E < \infty$ and $P = 0$

For power signal, $0 < P < \infty$ and $E = \infty$

Example 6.6

Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ b) $x(n) = \cos\left(\frac{\pi}{3}n\right)$ c) $x(n) = u(n)$

Solution

a) Given that, $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Here, $x(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$$\therefore\ x(n) = \left(\frac{1}{4}\right)^n = 0.25^n\ ;\quad n \geq 0$$

Infinite geometric series sum formula.
$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left|(0.25)^n\right|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1-0.0625} = 1.067 \text{ joules}$$

Using infinite geometric series sum formula

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left|(0.25)^n\right|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{(0.0625)^{N+1}-1}{0.0625-1}$$

Using finite geometric series sum formula

$$= \frac{1}{\infty} \times \frac{0.0625-1}{0.0625-1} = 0$$

Finite geometric series sum formula.
$$\sum_{n=0}^{\infty} C^n = \frac{C^{N+1}-1}{C-1}$$

Here E is finite and P is zero and so x(n) is an energy signal.

---

b) Given that, $x(n) = \cos\left(\frac{\pi}{3}n\right)$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} \cos^2\left(\frac{\pi}{3}n\right) = \sum_{n=-\infty}^{+\infty} \frac{1+\cos\frac{2\pi}{3}n}{2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\left(1+\cos\frac{2\pi}{3}n\right)\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty} 1^n + \sum_{n=-\infty}^{+\infty} \cos\frac{2\pi}{3}n\right) = \frac{1}{2}(\infty + 0) = \infty$$

Note *: Sum of infinite 1's is infinity. Sum of samples of one period of sinusoidal signal is zero.*

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2\frac{\pi n}{3}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{\left(1+\cos\frac{2\pi}{3}n\right)}{2}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}\left[\sum_{n=-N}^{N} 1^n + \sum_{n=-N}^{N} \cos\frac{2\pi}{3}n\right]$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \frac{1}{2} \left[\underbrace{1 + 1 + \dots\dots + 1}_{\text{N terms}} + 1 + \underbrace{1 + \dots\dots + 1 + 1}_{\text{N terms}} + 0 \right]$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \frac{1}{2} [2N+1] = \underset{N\to\infty}{\text{Lt}} \frac{1}{2} = \frac{1}{2} \text{ watts}$$

Since P is finite and E is inifinite, x(n) is a power signal.

> ***Note:*** *The term* $\cos \frac{2\pi}{3}n$ *is periodic with periodicity of 3 samples. Samples of* $\cos \frac{2\pi}{3}n$ *for two periods are given below. It can be observed that sum of samples of a period is zero.*
>
> When $n = 0$; $\cos \frac{2\pi}{3}n = 1$, When $n = 1$; $\cos \frac{2\pi}{3}n = -0.5$, When $n = 2$; $\cos \frac{2\pi}{3}n = -0.5$
>
> When $n = 3$; $\cos \frac{2\pi}{3}n = 1$, When $n = 4$; $\cos \frac{2\pi}{3}n = -0.5$, When $n = 5$; $\cos \frac{2\pi}{3}n = -0.5$

c) Given that, x(n) = u(n)

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{+\infty} (u(n))^2 = \sum_{n=0}^{+\infty} u(n) = 1 + 1 + 1 \dots\dots\dots\infty = \infty$$

$$P = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \sum_{n=0}^{N} u(n) = \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} \left(\underbrace{1 + 1 + 1 + \dots\dots\dots + 1}_{\text{N +1 terms}} \right)$$

$$= \underset{N\to\infty}{\text{Lt}} \frac{1}{2N+1} (N+1) = \underset{N\to\infty}{\text{Lt}} \frac{N\left(1 + \frac{1}{N}\right)}{N\left(2 + \frac{1}{N}\right)} = \frac{1 + \frac{1}{\infty}}{2 + \frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2} \text{ watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

6.4.5 Causal, Noncausal and Anticausal signals

A signal is said to be ***causal***, if it is defined for $n \geq 0$. Therefore if x(n) is causal, then x(n) = 0 for $n < 0$.

A signal is said to be ***noncausal***, if it is defined for either $n \leq 0$, or for both $n \leq 0$ and $n > 0$. Therefore if x(n) is noncausal, then $x(n) \neq 0$ for $n < 0$. A noncausal signal can be converted to causal signal by multiplying the noncausal signal by a unit step signal, u(n).

When a noncausal signal is defined only for $n \leq 0$, it is called an ***anticausal signal***.

Examples of Causal and Noncausal Signals

$$\left.\begin{array}{l} x(n) = \{\underset{\uparrow}{1}, -1, 2, -2, 3, -3\} \\ x(n) = \{\underset{\uparrow}{2}, 2, 3, 3, \dots\dots\dots\dots\} \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{l} x(n) = \{1, -1, 2, -2, 3, \underset{\uparrow}{-3}\} \\ x(n) = \{\dots\dots\dots\dots, 2, 2, 3, \underset{\uparrow}{3}\} \\ x(n) = \{2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2\} \\ x(n) = \{\dots\dots, 2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2, \dots\dots\} \end{array}\right\} \text{Noncausal signals}$$

6.5 Mathematical Operations on Discrete Time Signals

6.5.1 Scaling of Discrete Time Signals

Amplitude Scaling or Scalar Multiplication

Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by the constant A.

Example :

Let y(n) be amplitude scaled signal of x(n), then $y(n) = A\,x(n)$

Let, $x(n) = 20$; $n = 0$ and $A = 0.1$,
$= 36$; $n = 1$
$= 40$; $n = 2$
$= -15$; $n = 3$

When $n = 0$; $y(0) = A\,x(0) = 0.1 \times 20 = 2.0$
When $n = 1$; $y(1) = A\,x(1) = 0.1 \times 36 = 6.6$
When $n = 2$; $y(2) = A\,x(2) = 0.1 \times 40 = 4.0$
When $n = 3$; $y(3) = A\,x(3) = 0.1 \times (-15) = -1.5$

Time Scaling (or Down Sampling and Up Sampling)

There are two ways of time scaling a discrete time signal. They are down sampling and up sampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called ***down sampling***.

In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***up sampling***.

Example :

If $x(n) = a^n$; $n \geq 0$; $0 < a < 1$

then $x_1(n) = x(2n)$ will be a down sampled version of x(n) and $x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on x(n).

When $n = 0$; $x_1(0) = x(0) = a^0$
When $n = 1$; $x_1(1) = x(2) = a^2$
When $n = 2$; $x_1(2) = x(4) = a^4$
When $n = 3$; $x_1(3) = x(6) = a^6$ and so on.

When $n = 0$; $x_2(0) = x\left(\frac{0}{2}\right) = x(0) = a^0$
When $n = 1$; $x_2(1) = x\left(\frac{1}{2}\right) = 0$
When $n = 2$; $x_2(2) = x\left(\frac{2}{2}\right) = x(1) = a^1$
When $n = 3$; $x_2(3) = x\left(\frac{3}{2}\right) = 0$
When $n = 4$; $x_2(4) = x\left(\frac{4}{2}\right) = x(2) = a^2$ and so on.

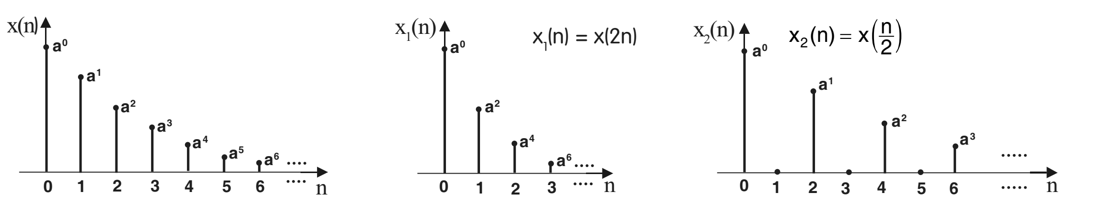


Fig 6.12a : A discrete time signal x(n). *Fig 6.12b : Down sampled signal of x(n).* *Fig 6.12c : Up sampled signal of x(n).*

Fig 6.12 : A discrete time signal and its time scaled version.

6.5.2 Folding (Reflection or transpose) of Discrete Time Signals

The ***folding*** of a signal x(n) is performed by changing the sign of the time base n in x(n). The folding operation produces a signal x(–n) which is a mirror image of x(n) with respect to time origin n = 0.

Example :

Let x(n) = n ; $-3 \le n \le 3$. Now the folded signal, $x_1(n) = x(-n) = -n$; $-3 \le n \le 3$

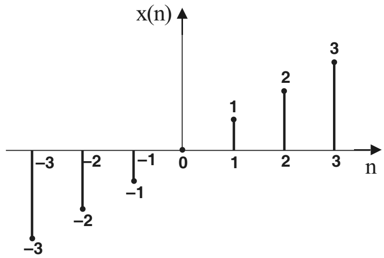


Fig 6.13a : *A discrete time signal x(n).*

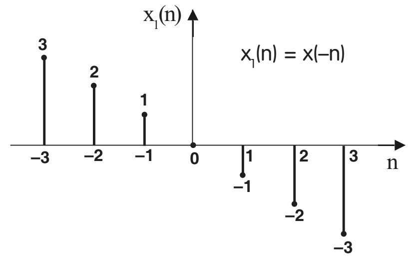


Fig 6.13b : *Folded signal of x(n).*

Fig 6.13 : *A discrete time signal and its folded version.*

6.5.3 Time shifting of Discrete Time Signals

A signal x(n) may be shifted in time by replacing the independent variable n by n – m, where m is an integer. If m is a positive integer, the time shift results in a delay by m units of time. If m is a negative integer, the time shift results in an advance of the signal by |m| units in time. The ***delay*** results in shifting each sample of x(n) to the right. The ***advance*** results in shifting each sample of x(n) to the left.

Example :

$$x(n) = \begin{cases} 1 & ; n = 2 \\ 2 & ; n = 3 \\ 3 & ; n = 4 \\ 0 & ; \text{for other } n \end{cases}$$

Let, $x_1(n) = x(n-2)$, where $x_1(n)$ is delayed signal of x(n)

When n = 4 ; $x_1(4) = x(4-2) = x(2) = 1$

When n = 5 ; $x_1(5) = x(5-2) = x(3) = 2$

When n = 6 ; $x_1(6) = x(6-2) = x(4) = 3$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 4 in $x_1(n)$. Similarly every sample of x(n) is delayed by two sampling times.

Let, $x_2(n) = x(n+2)$, where $x_2(n)$ is an advanced signal of x(n)

When n = 0 ; $x_2(n) = x(0+2) = x(2) = 1$

When n = 1 ; $x_2(n) = x(1+2) = x(3) = 2$

When n = 2 ; $x_2(n) = x(2+2) = x(4) = 3$

The sample x(2) is available at n = 2 in the original sequence x(n), but the same sample is available at n = 0 in $x_2(n)$. Similarly every sample of x(n) is advanced by two sampling times. Hence the signal $x_2(n)$ is an advanced version of x(n).

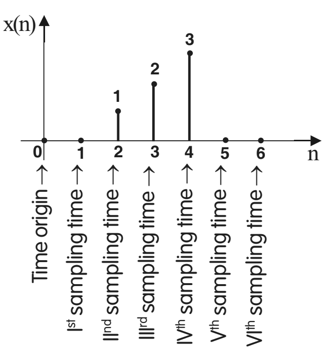


Fig 6.14a : *A discrete time signal x(n).*

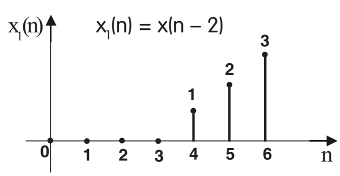


Fig 6.14b : *Delayed signal of x(n).*

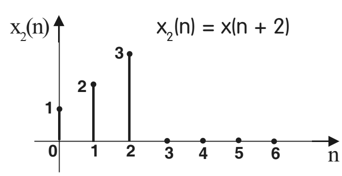


Fig 6.14c : *Advanced signal of x(n).*

Fig 6.14 : *A discrete time signal and its shifted version.*

Delayed Unit Impulse Signal

The unit impulse signal is defined as,

$$\delta(n) = 1 \;;\; \text{for } n = 0$$
$$= 0 \;;\; \text{for } n \neq 0$$

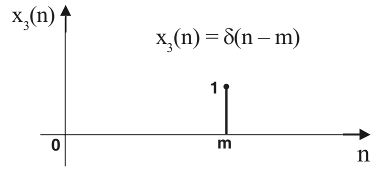


Fig 6.15 : *Delayed unit impulse.*

The unit impulse signal delayed by m units of time is denoted as $\delta(n - m)$.

Now, $\delta(n - m) = 1 \;;\; n = m$

$$= 0 \;;\; n \neq m$$

Delayed Unit Step Signal

The unit step signal is defined as,

$$u(n) = 1 \;;\; \text{for } n \geq 0$$
$$= 0 \;;\; \text{for } n < 0$$

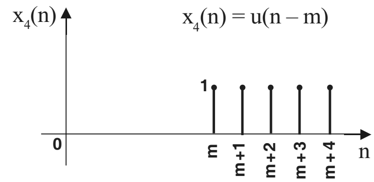


Fig 6.16 : *Delayed unit step signal.*

The unit step signal delayed by m units of time is denoted as $u(n - m)$.

Now, $u(n - m) = 1 \;;\; n \geq m$

$$= 0 \;;\; n < m$$

6.5.4 Addition of Discrete Time Signals

The ***addition*** of two signals is performed on a sample-by-sample basis.

The sum of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the samples of these two signals at that instant.

i.e., $y(n) = x_1(n) + x_2(n) \;;\; -\infty < n < \infty$.

Example :

Let, $x_1(n) = \{1, 2, -1\}$ and $x_2(n) = \{-2, 1, 3\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 1 + (-2) = -1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 3 = 2$

$\therefore\; y(n) = x_1(n) + x_2(n) = \{-1, 3, 2\}$

6.5.5 Multiplication of Discrete Time Signals

The ***multiplication*** of two signals is performed on a sample-by-sample basis.The product of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the product of the samples of these two signals at that instant. The product is also called ***modulation***.

Example :

Let, $x_1(n) = \{ 1, 2, -1 \}$ and $x_2(n) = \{ -2, 1, 3 \}$

When $n = 0$; $y(0) = x_1(0) \times x_2(0) = 1 \times (-2) = -2$

When $n = 1$; $y(1) = x_1(1) \times x_2(1) = 2 \times 1 = 2$

When $n = 2$; $y(2) = x_1(2) \times x_2(2) = -1 \times 3 = -3$

$\therefore\; y(n) = x_1(n) \times x_2(n) = \{-2, 2, -3\}$

6.6 Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal x(n) is transformed by the system into a signal y(n), and the transformation can be expressed mathematically as shown in equation (6.13).The diagrammatic representation of discrete time system is shown in fig 6.17.

$$\text{Response,} \quad y(n) = \mathcal{H}\{x(n)\} \qquad .....(6.13)$$

where, $\mathcal{H}$ denotes the transformation (also called an operator).

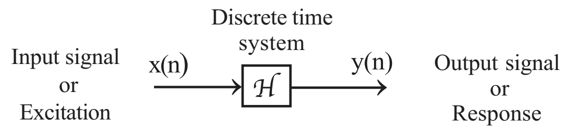


Fig 6.17 : *Representation of discrete time system.*

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship do not change with time. When a discrete time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a discrete time system is a unit impulse δ(n) then the output is called an ***impulse response*** of the system and is denoted by h(n).

$$\therefore \text{ Impulse Response,} \quad h(n) = \mathcal{H}\{\delta(n)\} \qquad .....(6.14)$$

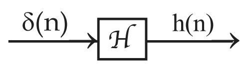


Fig 6.18 : *Discrete time system with impulse input.*

6.6.1 Mathematical Equation Governing Discrete Time System

The mathematical equation governing the discrete time system can be developed as shown below.

The response of a discrete time system at any time instant depends on the present input, past inputs and past outputs.

Let us consider the response at n = 0. Let us assume a relaxed system and so at n = 0, there is no past input or output. Therefore the response at n = 0, is a function of present input alone.

i.e., $y(0) = F[x(0)]$

Let us consider the response at n =1. Now the present input is x(1), the past input is x(0) and past output is y(0). Therefore the response at n = 1, is a function of x(1), x(0), y(0).

i.e., $y(1) = F[y(0), x(1), x(0)]$

Let us consider the response at n = 2. Now the present input is x(2), the past inputs are x(1) and x(0), and past outputs are y(1) and y(0). Therefore the response at n = 2, is a function of x(2), x(1), x(0), y(1), y(0).

i.e., $y(2) = F[y(1), y(0), x(2), x(1), x(0)]$

Similarly, at n = 3, $y(3) = F[y(2), y(1), y(0), x(3), x(2), x(1), x(0)]$

at n = 4, $y(4) = F[y(3), y(2), y(1), y(0), x(4), x(3), x(2), x(1),$ and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), ...y(1), y(0), x(n), x(n-1), x(n-2), x(n-3) ..... x(1), x(0)] \qquad .....(6.15)$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore the equation (6.15) can be written as,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - .... + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + ... \qquad ..... (6.16)$$

where, $a_1, a_2, a_3, ...$ and $b_0, b_1, b_2, b_3, ...$ are constants.

> ***Note :*** *Negative constants are inserted for output signals, because output signals are feedback from output to input. Positive constants are inserted for input signals, because input signals are feed forward from input to output.*

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where $M \le N$. Hence the equation (6.16) can be written as,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) - .... - a_N y(n-N)$$

$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) + ... + b_M x(n-M)$$

$$\therefore \quad y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m) \qquad .....(6.17)$$

The equation (6.17) is a constant coefficient ***difference equation***, governing the input-output relation of an LTI discrete time system.

In equation (6.17) the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1^{st} order system

If N = 2, the discrete time system is called 2^{nd} order system

If N = 3, the discrete time system is called 3^{rd} order system , and so on.

The general difference equation governing 1^{st} order discrete time LTI system is,

$$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$$

The general difference equation governing 2^{nd} order discrete time LTI system is,

$$y(n) = -a_2 y(n-2) - a_1 y(n-1) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

6.6.2 Block Diagram and Signal Flow Graph Representation of Discrete Time System

The discrete time system can be represented diagrammatically by ***block diagram*** or ***signal flow graph***. These diagrammatic representations are useful for physical implementation of discrete time system in hardware or software.

The basic elements employed in block diagram or signal flow graph are Adder, Constant multiplier, Unit delay element and Unit advance element.

Adder : An adder is used to represent addition of two discrete time sequences.

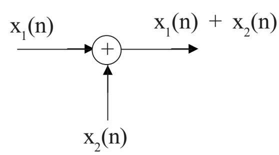
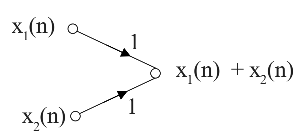
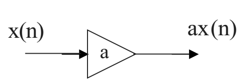
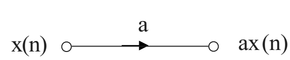
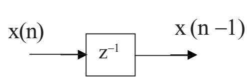
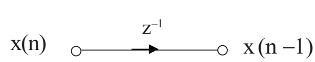
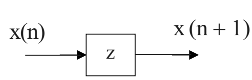
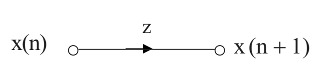
Constant Multiplier : A constant multiplier is used to represent multiplication of a scaling factor (constant) to a discrete time sequence.

Unit Delay Element : A unit delay element is used to represent the delay of samples of a discrete time sequence by one sampling time.

Unit Advance Element : A unit advance element is used to represent the advance of samples of a discrete time sequence by one sampling time.

The symbolic representation of the basic elements of block diagram and signal flow graph are listed in table 6.1.

Table 6.1 : Basic Elements of Block Diagram and Signal Flow Graph

Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 6.7

Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 0.5\,x(n) + 0.5\,x(n-1)$

b) $y(n) = 0.5\,y(n-1) + x(n) - 2\,x(n-2)$

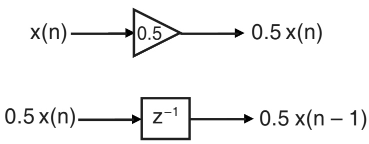
c) $y(n) = 0.25\,y(n-1) + 0.5\,x(n) + 0.75\,x(n-1)$

Solution

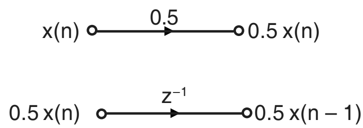
a) Given that, $y(n) = 0.5\,x(n) + 0.5\,x(n-1)$

The individual terms of the given equation are $0.5\,x(n)$ and $0.5\,x(n-1)$. They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is $x(n)$ and the output of the system is $y(n)$. The above elements are connected as shown below to get the output $y(n)$.

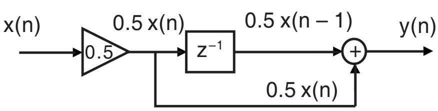


***Fig 1 :** Block diagram of the system* $y(n) = 0.5\,x(n) + 0.5\,x(n-1)$.

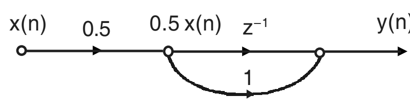


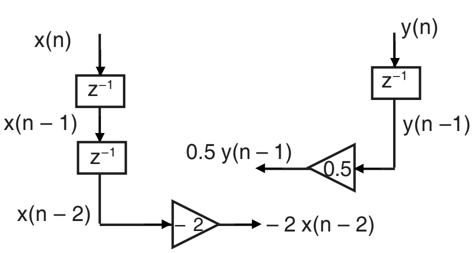
***Fig 2 :** Signal flow graph of the system* $y(n) = 0.5\,x(n) + 0.5\,x(n-1)$.

---

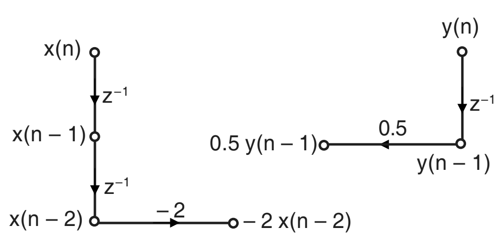
b) Given that, $y(n) = 0.5\,y(n-1) + x(n) - 2\,x(n-2)$

The individual terms of the given equation are $0.5\,y(n-1)$ and $-2\,x(n-2)$. They are represented by basic elements as shown below.

Block diagram representation



Signal flow graph representation



The input to the system is $x(n)$ and the output of the system is $y(n)$. The above elements are connected as shown below to get the output $y(n)$.

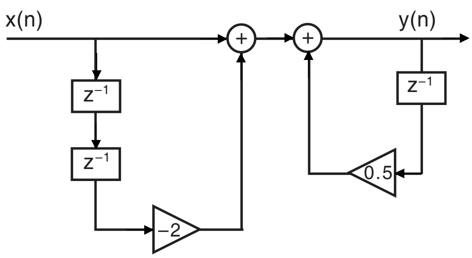


***Fig 3 :** Block diagram of the system described by the equation $y(n) = 0.5\ y(n-1) + x(n) - 2\ x(n-2)$.*

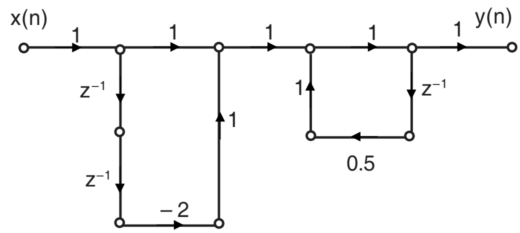
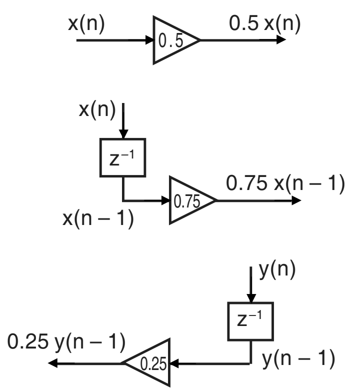


***Fig 4 :** Signal flow graph of the system described by the equation $y(n) = 0.5\ y(n-1) + x(n) - 2\ x(n-2)$.*

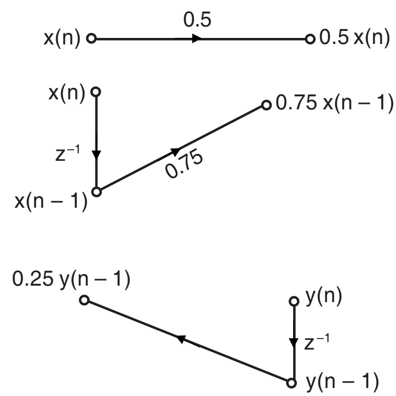
c) Given that, $y(n) = 0.25\ y(n-1) + 0.5\ x(n) + 0.75\ x(n-1)$

The individual terms of the given equation are $0.25\ y(n-1)$, $0.5\ x(n)$ and $0.75\ x(n-1)$. They are represented by basic elements as shown below.

<u>Block diagram representation</u>



<u>Signal flow graph representation</u>



The input to the system is x(n) and the output of the system is y(n). The above elements are connected as shown below to get the output y(n).

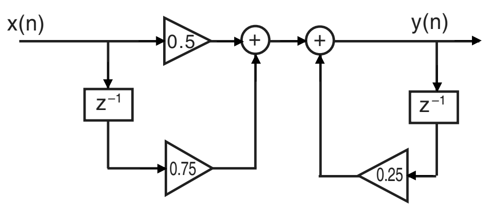


***Fig 5 :** Block diagram of the system described by the equation $y(n) = 0.25\ y(n-1) + 0.5\ x(n) + 0.75\ x(n-1)$.*

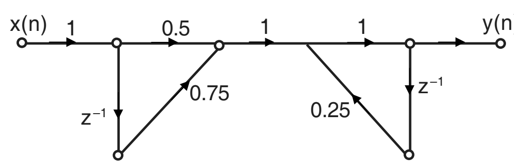


***Fig 6 :** Signal flow graph of the system described by the equation $y(n) = 0.25\ y(n-1) + 0.5\ x(n) + 0.75\ x(n-1)$.*

6.7 Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\therefore \quad y(n) + \sum_{m=1}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \ \text{ with } a_o = 1 \qquad .....(6.18)$$

The solution of the difference equation (6.18) is the ***response*** y(n) of LTI system, which consists of two parts. In mathematics, the two parts of the solution y(n) are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$.

$$\boxed{\therefore \text{ Response, } \ y(n) = y_h(n) + y_p(n)} \qquad .....(6.19)$$

The homogeneous solution is the response of the system when there is no input.The particular solution $y_p(n)$ is the solution of difference equation for specific input signal x(n) for $n \geq 0$.

In signals and systems, the two parts of the solution y(n) are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore \text{ Response, } \ y(n) = y_{zi}(n) + y_{zs}(n)} \qquad .....(6.20)$$

The ***zero-input response*** is mainly due to initial conditions (or initial stored energy) in the system. Hence zero-input response is also called ***free response*** or ***natural response***. The ***zero-input response*** is given by homogeneous solution with constants evaluated using initial conditions.

The ***zero-state response*** is the response of the system due to input signal and with zero initial condition. Hence the zero-state response is called forced response.The ***zero-state response*** or ***forced response*** is given by the sum of homogeneous solution and particular solution with zero initial conditions.

6.7.1 Zero-Input Response or Homogeneous Solution

The ***zero-input response*** is obtained from homogeneous solution $y_h(n)$ with constants evaluated using initial condition.

$$\therefore \quad \text{Zero - input response, } y_{zi}(n) = y_h(n)\Big|_{\text{with constants evaluated using initial conditions}}$$

The ***homogeneous solution*** is obtained when x(n) = 0. Therefore the homogeneous solution is the solution of the equation,

$$\sum_{m=0}^{N} a_m\, y(n-m) = 0 \qquad .....(6.21)$$

Let us assume that the solution of equation (6.21) is in the form of an exponential.

i.e., $y(n) = \lambda^n$

On substituting $y(n) = \lambda^n$ in equation (6.21) we get,

$$\sum_{m=0}^{N} a_m \lambda^{n-m} = 0$$

On expanding the above equation (by taking $a_0 = 1$), we get,

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_{N-1} \lambda^{n-(N-1)} + a_N \lambda^{n-N} = 0$$
$$\lambda^{n-N} (\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_{N-1} \lambda + a_N) = 0$$

Now, the characteristic polynomial of the system is given by,

$$\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_{N-1} \lambda + a_N = 0$$

The characteristic polynomial has N roots, which are denoted as $\lambda_1, \lambda_2, ... \lambda_N$.

The roots of the characteristic polynomial may be distinct real roots, repeated real roots or complex. The assumed solutions for various types of roots are given below.

Distinct Real Roots

Let the roots $\lambda_1, \lambda_2, \lambda_3, ... \lambda_N$ be distinct real roots. Now the homogeneous solution will be in the form,

$$y_h(n) = C_1 \lambda_1^n + C_2 \lambda_2^n + C_3 \lambda_3^n + ... + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, ... C_N$ are constants that can be evaluated using initial conditions.

Repeated Real Roots

Let one of the real roots λ_1 repeats p times and the remaining (N – p) roots are distinct real roots. Now, the homogeneous solution is in the form,

$$y_h(n) = (C_1 + C_2 n + C_3 n^2 + ... + C_p n^{p-1}) \lambda_1^n + C_{p+1} \lambda_{p+1}^n + ... + C_N \lambda_N^n$$

where, $C_1, C_2, C_3, ... C_N$ are constants that can be evaluated using initial conditions.

Complex Roots

Let the characteristic polynomial has a pair of complex roots λ and λ^* and the remaining (N – 2) roots be distinct real roots. Now, the homogeneous solution will be in the form,

$$y_h(n) = r^n [C_1 \cos n\theta + C_2 \sin n\theta] + C_3 \lambda_3^n + C_4 \lambda_4^n + ... + C_N \lambda_N^n$$

where, $\lambda = a + jb, \quad \lambda^* = a - jb, \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1} \dfrac{b}{a}$

$C_1, C_2, C_3 ... C_N$ are constants that can be evaluated using initial conditions.

6.7.2 Particular Solution

The ***particular solution***, $y_p(n)$ is the solution of the difference equation for specific input signal $x(n)$ for $n \geq 0$. Since the input signal may have different form, the particular solution depends on the form or type of the input signal $x(n)$.

If $x(n)$ is constant, then $y_p(n)$ is also a constant.

Example :

Let, $x(n) = u(n)$; now, $y_p(n) = K\, u(n)$

If x(n) is exponential, then $y_p(n)$ is also an exponential.

> **Example :**
> Let, $x(n) = a^n u(n)$; now, $y_p(n) = K a^n u(n)$

If x(n) is sinusoid, then $y_p(n)$ is also a sinusoid.

> **Example :**
> Let, $x(n) = A \cos \omega_0 n$; now, $y_p(n) = K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

The general form of particular solution for various types of inputs are listed in table 6.2.

Table 6.2 : Particular Solution

Input signal, x(n)	Particular solution, $y_p(n)$
A	K
AB^n	KB^n
An^B	$K_0 n^B + K_1 n^{(B-1)} + ... + K_B$
$A^n n^B$	$A^n (K_0 n^B + K_1 n^{(B-1)} + ...+ K_B)$
$\left.\begin{matrix} A \cos \omega_0 n \\ A \sin \omega_0 n \end{matrix}\right\}$	$K_1 \cos \omega_0 n + K_2 \sin \omega_0 n$

6.7.3 Zero-State Response

The ***zero-state response*** or ***forced response*** is obtained from the sum of homogeneous solution and particular solution and evaluating the constants with zero initial conditions.

$$\therefore \text{ Zero - state response, } y_{zs}(n) = y_h(n) + y_p(n)\Big|_{\text{with constants } C_1,\, C_2,\ldots\, C_N \text{ evaluated with zero initial conditions}}$$

6.7.4 Total Response

The total response of discrete time system can be obtained by the following two methods.

Method-1

The ***total response*** is given by sum of homogeneous solution and particular solution.

$$\therefore \text{ Total response, } y(n) = y_h(n) + y_p(n)$$

Procedure to Determine Total Response by Method-1

1. Determine the homogeneous solution $y_h(n)$ with constants C_1, C_2, ...C_N.
2. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of y(n) vanishes.
3. Now the total response is given by the sum of $y_h(n)$ and $y_p(n)$.
   $$\therefore \text{ Total response, } y(n) = y_h(n) + y_p(n)$$
4. The total response will have N number of constants C_1, C_2, ...C_N. Evaluate the given equation and the total response for n = 0, 1, 2, ...N – 1 and form two sets of N number of equations and solve the constants C_1, C_2, ...C_N.

Method-2

The ***total response*** is given by sum of zero-input response and zero-state response.

$\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Procedure to Determine Total Response by Method-2

1. Determine the homogeneous solution $y_h(n)$ with constants $C_1, C_2, ...C_N$.
2. Determine the zero-input response, which is obtained from the homogeneous solution $y_h(n)$ and evaluating the constants $C_1, C_2, ...C_N$ using the initial conditions.
3. Determine the particular solution $y_p(n)$ and evaluate the constants K for any value of $n \geq 1$ so that no term of y(n) vanishes.
4. Determine the zero-state response, $y_{zs}(n)$ which is given by sum of homogeneous solution and particular soulution and evaluating the constants $C_1, C_2, ...C_N$ with zero initial conditions.
5. Now, the total response is given by sum of zero input response and zero state response.

   $\therefore$ Total response, $y(n) = y_{zi}(n) + y_{zs}(n)$

Example 6.8

Determine the response of first order discrete time system governed by the difference equation,

$$y(n) = -\,0.5\, y(n-1) + x(n)$$

When the input is unit step, and with initial condition a) $y(-1) = 0$ b) $y(-1) = 1/3$.

Solution

Given that, $y(n) = -\,0.5\, y(n-1) + x(n)$

$$\therefore\quad y(n) + 0.5\, y(n-1) = x(n) \qquad .....(1)$$

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when $x(n) = 0$.

$$\therefore\quad y(n) + 0.5\, y(n-1) = 0 \qquad .....(2)$$

Put, $y(n) = \lambda^n$ in equation (2).

$$\therefore\quad \lambda^n + 0.5\,\lambda^{(n-1)} = 0$$

$$\lambda^{(n-1)}\,(\lambda + 0.5) = 0 \qquad \Rightarrow \qquad \lambda = -\,0.5$$

The homogeneous solution $y_h(n)$ is given by,

$$y_h(n) = C\,\lambda^n = C\,(-\,0.5)^n \;;\quad \text{for } n \geq 0 \qquad .....(3)$$

Particular Solution

Given that the input is unit step and so the particular solution will be in the form,

$$y(n) = K\,u(n) \qquad .....(4)$$

On substituting for y(n) from equation (4) in equation (1) we get,

$$K\,u(n) + 0.5\,K\,u(n-1) = u(n) \qquad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for $n = 1$, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when n = 1, we get,

$$K + 0.5\,K = 1$$

$$1.5\,K = 1$$

$$\therefore \quad K = \frac{1}{1.5} = \frac{10}{15} = \frac{2}{3}$$

The particular solution $y_p(n)$ is given by,

$$y_p(n) = K\,u(n) = \frac{2}{3}\,u(n) \;; \quad \text{for all n}$$

$$= \frac{2}{3} \quad ; \text{ for } \quad n \ge 0$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$\therefore$ Response, $y(n) = y_h(n) + y_p(n)$

$$= C(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for } n \ge 0 \qquad .....(6)$$

At n = 0, from equation (1), we get, $y(0) + 0.5\,y(-1) = 1$

$$\therefore \quad y(0) = 1 - 0.5\,y(-1) \qquad ....(7)$$

At n = 0, from equation (6), we get, $\quad y(0) = C + \frac{2}{3}$ .....(8)

On equating (7) and (8) we get, $\quad C + \frac{2}{3} = 1 - 0.5\,y(-1)$

$$\therefore\; C = 1 - 0.5\,y(-1) - \frac{2}{3}$$

$$= \frac{1}{3} - 0.5\,y(-1) \qquad .....(9)$$

On substituting for C from equation (9) in equation (6) we get,

$$y(n) = \left(\frac{1}{3} - 0.5\,y(-1)\right)(-0.5)^n + \frac{2}{3}$$

a) When y(–1) = 0

$$y(-1) = 0$$

$$\therefore\; y(n) = \frac{1}{3}(-0.5)^n + \frac{2}{3} \quad ; \quad \text{for} \quad n \ge 0$$

b) When y(–1) = 1/3

$$y(-1) = \frac{1}{3}$$

$$\therefore\; y(n) = \left(\frac{1}{3} - 0.5 \times \frac{1}{3}\right)(-0.5)^n + \frac{2}{3}$$

$$= \frac{0.5}{3}(-0.5)^n + \frac{2}{3}$$

$$= \frac{1}{6}(-0.5)^n + \frac{2}{3} \;; \text{ for } n \ge 0$$

Example 6.9

Determine the response y(n), $n \geq 0$ of the system described by the second order difference equation,

$$y(n) - 2\,y(n-1) - 3\,y(n-2) = x(n) + 4\,x(n-1),$$

when the input signal is, $x(n) = 2^n\,u(n)$ and with initial conditions $y(-2) = 0$, $y(-1) = 5$.

Solution

Given that, $y(n) - 2\,y(n-1) - 3\,y(n-2) = x(n) + 4\,x(n-1)$ .....(1)

Homogeneous Solution

The homogeneous equation is the solution of equation (1) when x(n) = 0.

$$\therefore\ y(n) - 2\,y(n-1) - 3\,y(n-2) = 0 \quad .....(2)$$

Put $y(n) = \lambda^n$ in equation (2).

$$\therefore\ \lambda^n - 2\,\lambda^{n-1} - 3\,\lambda^{n-2} = 0$$

$$\lambda^{n-2}\,(\lambda^2 - 2\lambda - 3) = 0$$

The characteristic equation is,

$$\lambda^2 - 2\,\lambda - 3 = 0 \qquad \Rightarrow \qquad (\lambda - 3)\,(\lambda + 1) = 0$$

$\therefore$ The roots are, $\lambda = 3, -1$

The homogeneous solution, $y_h(n)$ is given by,

$$\begin{aligned} y_h(n) &= C_1\,\lambda_1^n + C_2\,\lambda_2^n \\ &= C_1(3)^n + C_2(-1)^n\,; \quad \text{for } n \geq 0 \end{aligned} \quad .....(3)$$

Particular Solution

Given that the input is an exponential signal, $2^n\,u(n)$ and so the particular solution will be in the form,

$$y(n) = K\,2^n\,u(n) \quad .....(4)$$

On substituting for y(n) from equation (4) in equation (1) we get,

$$K\,2^n\,u(n) - 2\,K\,2^{(n-1)}\,u(n-1) - 3\,K\,2^{(n-2)}\,u(n-2) = 2^n\,u(n) + 4 \times 2^{(n-1)}\,u(n-1) \quad .....(5)$$

In order to determine the value of K, let us evaluate equation (5) for n = 2, ($\because$ we have to evaluate equation (5) for any $n \geq 1$, such that none of the term vanishes).

From equation (5) when n = 2, we get,

$$K\,2^2 - 2K \times 2^1 - 3K \times 2^0 = 2^2 + 4 \times 2^1$$

$$4K - 4K - 3K = 12$$

$$-3K = 12$$

$$\therefore\ K = -\frac{12}{3} = -4$$

The particular solution $y_p(n)$ is given by,

$$y_p\,(n) = K\,2^n\,u(n) = (-4)\,2^n\,u(n)$$

Total Response

The total response y(n) of the system is given by sum of homogeneous and particular solution.

$$\therefore\ \text{Response, } y(n) = y_h\,(n) + y_p\,(n)$$

$$= C_1\,3^n + C_2\,(-1)^n + (-4)\,2^n\,; \quad \text{for } n \geq 0 \quad .....(6)$$

When $n = 0$,

From equation (1) we get,

$$y(0) - 2\,y(-1) - 3\,y(-2) = x(0) + 4\,x(-1) \qquad .....(7)$$

Given that, $y(-1) = 5,\ y(-2) = 0$

$$x(n) = 2^n\,u(n), \quad \therefore \quad x(0) = 2^0 = 1$$

$$x(-1) = 0$$

On substituting the above conditions in equation (7) we get,

$$y(0) - 2 \times 5 - 3 \times 0 = 1 + 0$$

$$\therefore \quad y(0) = 11 \qquad .....(8)$$

When $n = 1$,

From equation (1) we get,

$$y(1) - 2\,y(0) - 3\,y(-1) = x(1) + 4\,x(0) \qquad .....(9)$$

We know that, $y(0) = 11, \quad y(-1) = 5, \quad y(-2) = 0$

Given that, $\quad x(n) = 2^n\,u(n), \quad \therefore \ x(0) = 2^0 = 1$

$$x(1) = 2^1 = 2$$

On substituting the above conditions in equation (9) we get,

$$y(1) - 2 \times 11 - 3 \times 5 = 2 + 4 \times 1$$

$$\therefore \quad y(1) = 6 + 37 = 43 \qquad .....(10)$$

When $n = 0$,

From equation (6) we get,

$$y(0) = C_1\,3^0 + C_2\,(-1)^0 + (-4)\,2^0 = C_1 + C_2 - 4 \qquad .....(11)$$

From equations (8) and (11) we can write,

$$C_1 + C_2 - 4 = 11$$

$$\therefore \quad C_1 + C_2 = 15 \qquad .....(12)$$

When $n = 1$,

From equation (6) we get,

$$y(1) = C_1 \times 3 + C_2\,(-1) + (-4)\,2 = 3\,C_1 - C_2 - 8 \qquad .....(13)$$

From equations (10) and (13) we can write,

$$3\,C_1 - C_2 - 8 = 43$$

$$\therefore \quad 3\,C_1 - C_2 = 51 \qquad .....(14)$$

On adding equations (12) and (14) we get,

$$4C_1 = 66$$

$$\therefore \ C_1 = \frac{66}{4} = \frac{33}{2}$$

From equation (12), $C_2 = 15 - C_1 = 15 - \dfrac{33}{2} = \dfrac{30-33}{2} = -\dfrac{3}{2}$

$$\therefore \quad y(n) = \frac{33}{2}\,(3)^n - \frac{3}{2}\,(-1)^n + (-4)\,2^n \quad ; \ \text{for } n \geq 0$$

$$= \left[\frac{33}{2}\,3^n - \frac{3}{2}\,(-1)^n - 4\,(2)^n\right] u(n)\ ; \quad \text{for all n.}$$

6.8 Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

6.8.1 Static and Dynamic Systems

A discrete time system is called ***static*** or ***memoryless*** if its output at any instant n depends at most on the input sample at the same time but not on the past or future samples of the input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{aligned} y(n) &= a\,x(n) \\ y(n) &= n\,x(n) + 6\,x^3(n) \end{aligned}\right\} \text{Static systems}$$

$$\left.\begin{aligned} &\left.\begin{aligned} y(n) &= x(n) + 3\,x(n-1) \\ y(n) &= \sum_{m=0}^{N} x(n-m) \end{aligned}\right\} \text{Finite memory is required} \\ &\left.\begin{aligned} y(n) &= \sum_{m=0}^{\infty} x(n-m) \end{aligned}\right\} \text{Infinite memory is required} \end{aligned}\right\} \text{Dynamic systems}$$

6.8.2 Time Invariant and Time Variant Systems

A system is said to be ***time invariant*** if its input-output characteristics do not change with time.

Definition : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$x(n) \xrightarrow{\mathcal{H}} y(n) \text{ implies that, } x(n-m) \xrightarrow{\mathcal{H}} y(n-m)$$

for every input signal x(n) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(n) = \mathcal{H}\{x(n)\}}$ then $\mathbf{y(n-m) = \mathcal{H}\{x(n-m)\}}$.

Alternative Definition for Time Invariance

A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

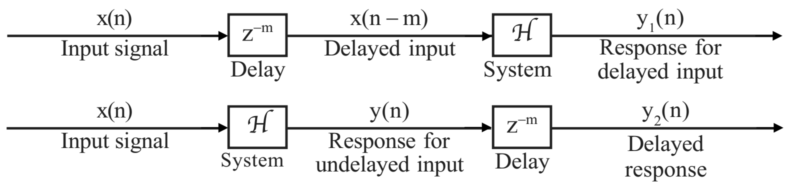
i.e., In a time invariant system, $\mathcal{H}\{\mathbf{x(n-m)}\} = \mathbf{z^{-m}}\ \mathcal{H}\{\mathbf{x(n)}\}$**;** for all values of m .....(6.22)

The operator z^{-m} represents a signal delay of m samples.

The diagrammatic explanation of the above definition of time invariance is shown in fig 6.19.

Procedure to test for time invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be $y_1(n)$.
2. Delay the response of the system for undelayed input by m units of time. Let this delayed response be $y_2(n)$.
3. Check whether $y_1(n) = y_2(n)$. If they are equal then the system is time invariant. Otherwise the system is time variant.



If, $y_1(n) = y_2(n)$, then the system is time invariant

***Fig 6.19 :** Diagrammatic explanation of time invariance.*

Example 6.10

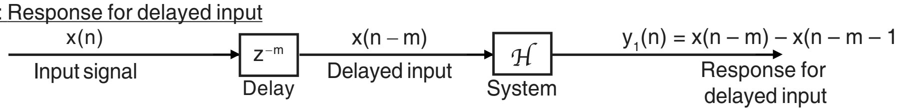
Test the following systems for time invariance.

a) $y(n) = x(n) - x(n-1)$ b) $y(n) = n\,x(n)$ c) $y(n) = x(-n)$ d) $y(n) = x(n) - b\,x(n-1)$

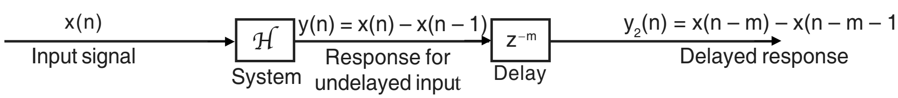
Solution

a) Given that, $y(n) = x(n) - x(n-1)$

Test 1 : Response for delayed input



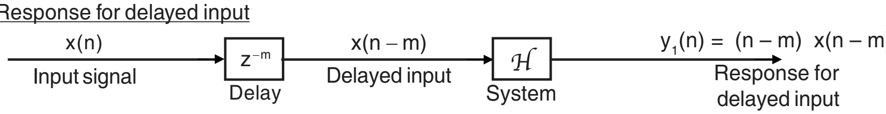
Test 2 : Delayed response



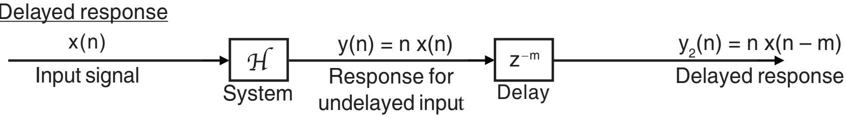
Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

b) Given that, $y(n) = n\,x(n)$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

c) Given that, $y(n) = x(-n)$

Test 1 : Response for delayed input

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed input}}{\xrightarrow{x(n-m)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for delayed input}}{\xrightarrow{y_1(n) = x(-(n-m)) = x(-n+m)}}$$

Test 2 : Delayed response

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for undelayed input}}{\xrightarrow{y(n) = x(-n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed response}}{\xrightarrow{y_2(n) = x(-n-m)}}$$

Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

---

d) Given that, $y(n) = x(n) - b\,x(n-1)$

Test 1 : Response for delayed input

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed input}}{\xrightarrow{x(n-m)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for delayed input}}{\xrightarrow{y_1(n) = x(n-m) - b\,x(n-m-1)}}$$

Test 2 : Delayed response

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for undelayed input}}{\xrightarrow{y(n) = x(n) - b\,x(n-1)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed response}}{\xrightarrow{y_2(n) = x(n-m) - b\,x(n-m-1)}}$$

Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

Example 6.11

Test the following systems for time invariance.

a) $y(n) = x(n) + C$ b) $y(n) = n\,x^2(n)$ c) $y(n) = a^{x(n)}$ d) $y(n) = \sum_{k=0}^{M} b_k\,x(n-k) - \sum_{k=1}^{N} a_k\,y(n-k)$

Solution

a) Given that, $y(n) = x(n) + C$

Test 1 : Response for delayed input

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed input}}{\xrightarrow{x(n-m)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for delayed input}}{\xrightarrow{y_1(n) = x(n-m) + C}}$$

Test 2 : Delayed response

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for undelayed input}}{\xrightarrow{y(n) = x(n) + C}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed response}}{\xrightarrow{y_2(n) = x(n-m) + C}}$$

Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

---

b) Given that, $y(n) = n\,x^2(n)$

Test 1 : Response for delayed input

$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed input}}{\xrightarrow{x(n-m)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for delayed input}}{\xrightarrow{y_1(n) = (n-m)\,x^2(n-m)}}$$

Test 2 : Delayed response

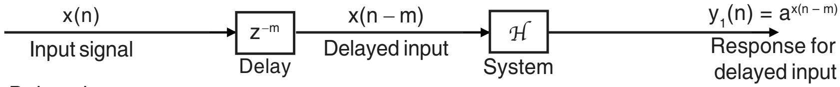
$$\underset{\text{Input signal}}{\xrightarrow{x(n)}} \underset{\text{System}}{\boxed{\mathcal{H}}} \underset{\text{Response for undelayed input}}{\xrightarrow{y(n) = n\,x^2(n)}} \underset{\text{Delay}}{\boxed{z^{-m}}} \underset{\text{Delayed response}}{\xrightarrow{y_2(n) = n\,x^2(n-m)}}$$

Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

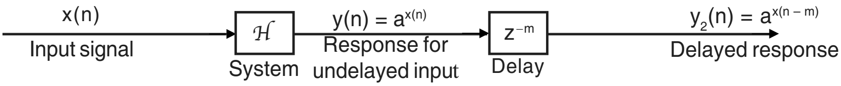
---

c) Given that, $y(n) = a^{x(n)}$

Test 1 : Response for delayed input



Test 2 : Delayed response

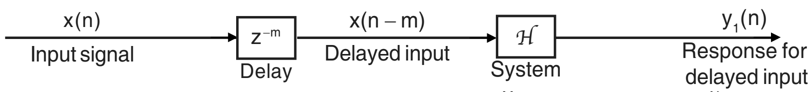


Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

---

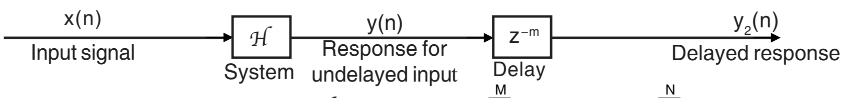
d) Given that, $y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)$

Test 1 : Response for delayed input



Response for delayed input, $y_1(n) = \mathcal{H}\{x(n-m)\} = \sum_{k=0}^{M} b_k\, x(n-m-k) - \sum_{k=1}^{N} a_k\, y(n-m-k)$

Test 2 : Delayed response



Response for undelayed input $= \mathcal{H}\{x(n)\} = y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)$

Delayed response, $y_2(n) = z^{-m}\, \mathcal{H}\{x(n)\}$

$$= z^{-m}\left[\sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k)\right]$$

$$= \sum_{k=0}^{M} b_k\, x(n-m-k) - \sum_{k=1}^{N} a_k\, y(n-m-k)$$

Conclusion : Here, $y_1(n) = y_2(n)$, therefore the system is time invariant.

6.8.3 Linear and Nonlinear Systems

A ***linear system*** is one that satisfies the superposition principle. The ***principle of superposition*** requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

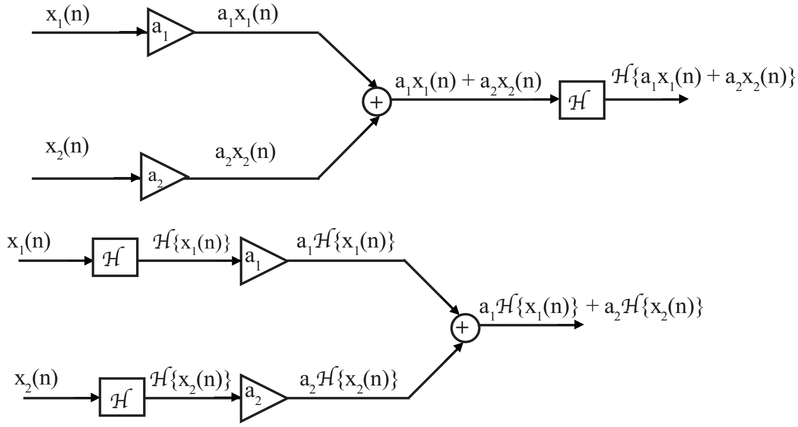
$$\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\} \qquad .....(6.23)$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, the system is ***nonlinear***.The diagrammatic explanation of linearity is shown in fig 6.20.

Procedure to test for linearity

1. Let $x_1(n)$ and $x_2(n)$ be two inputs to system $\mathcal{H}$, and $y_1(n)$ and $y_2(n)$ be corresponding responses.
2. Consider a signal, $x_3(n) = a_1\, x_1(n) + a_2\, x_2(n)$ which is a weighed sum of $x_1(n)$ and $x_2(n)$.
3. Let $y_3(n)$ be the response for $x_3(n)$.
4. Check whether $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. If they are equal then the system is linear, otherwise it is nonlinear.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = a_1\,\mathcal{H}\{x_1(n)\} + a_2\,\mathcal{H}\{x_2(n)\}$

***Fig 6.20 :** Diagrammatic explanation of linearity.*

Example 6.12

Test the following systems for linearity.

a) $y(n) = n\,x(n)$, b) $y(n) = x(n^2)$, c) $y(n) = x^2(n)$, d) $y(n) = A\,x(n) + B$, e) $y(n) = e^{x(n)}$.

Solution

a) Given that, y(n) = n x(n)

Let H be the system represented by the equation, $y(n) = n\,x(n)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = n\,x(n)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1(n)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2(n)$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1(n) + a_2\,n\,x_2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)] = a_1\,n\,x_1(n) + a_2\,n\,x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is linear.

b) Given that, y(n) = x(n²)

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n^2)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = x(n^2)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n^2)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n^2)$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1\,x_1(n^2) + a_2\,x_2(n^2) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = a_1\,x_1(n^2) + a_2\,x_2(n^2) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is linear.

---

c) Given that, $y(n) = x^2(n)$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x^2(n)$ and the system $\mathcal{H}$ operates on x(n) to produce, $y(n) = \mathcal{H}\{x(n)\} = x^2(n)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = x_1^{\,2}(n)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2^{\,2}(n)$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1\,x_1^{\,2}(n) + a_2\,x_2^{\,2}(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = [a_1\,x_1(n) + a_2\,x_2(n)]^2$$

$$= a_1^{\,2}\,x_1^{\,2}(n) + a_2^{\,2}\,x_2^{\,2}(n) + 2\,a_1\,a_2\,x_1(n)x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

---

d) Given that, $y(n) = A\,x(n) + B$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = A\,x(n) + B$ and the system $\mathcal{H}$ operates on x(n) to produce, $y(n) = \mathcal{H}\{x(n)\} = A\,x(n) + B$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = A\,x_1(n) + B$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = A\,x_2(n) + B$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1[A\,x_1(n) + B] + a_2[A\,x_2(n) + B]$$

$$= A\,a_1\,x_1(n) + B\,a_1 + A\,a_2\,x_2(n) + B\,a_2 \qquad .....(1)$$

Consider a linear combination of inputs, $a_1\,x_1(n) + a_2\,x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = A\,[a_1\,x_1(n) + a_2\,x_2(n)] + B$$

$$= A\,a_1\,x_1(n) + A\,a_2\,x_2(n) + B \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

---

e) Given that, $y(n) = e^{x(n)}$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = e^{x(n)}$ and the system $\mathcal{H}$ operates on x(n) to produce, $y(n) = \mathcal{H}\{x(n)\} = e^{x(n)}$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = e^{x_1(n)}$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = e^{x_2(n)}$$

$$\therefore\ a_1 y_1(n) + a_2 y_2(n) = a_1 e^{x_1(n)} + a_2 e^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\}$$

$$= e^{[a_1 x_1(n) + a_2 x_2(n)]} = e^{a_1 x_1(n)}\ e^{a_2 x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

Example 6.13

Test the following systems for linearity.

a) $y(n) = x(n) + C$, b) $y(n) = a^{x(n)}$, c) $y(n) = n\, x^2(n)$.

Solution

a) Given that, $y(n) = x(n) + C$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n) + C$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = x(n) + C$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n) + C$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n) + C$$

$$\therefore\ a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) + a_1 C + a_2 x_2(n) + a_2 C \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 x_1(n) + a_2 x_2(n) + C \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

b) Given that, $y(n) = a^{x(n)}$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = a^{x(n)}$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = a^{x(n)}$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = a^{x_1(n)}$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = a^{x_2(n)}$$

$$\therefore\ a_1 y_1(n) + a_2 y_2(n) = a_1 a^{x_1(n)} + a_2 a^{x_2(n)} \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = a^{[a_1 x_1(n) + a_2 x_2(n)]}$$

$$= a^{a_1 x_1(n)}\ a^{a_2 x_2(n)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

c) Given that, $y(n) = n\,x^2(n)$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = n\,x^2(n)$ and the system $\mathcal{H}$ operates on x(n) to produce, $y(n) = \mathcal{H}\{x(n)\} = n\,x^2(n)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system H for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1^2(n)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2^2(n)$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1\,n\,x_1^2(n) + a_2\,n\,x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\} = n[a_1\,x_1(n) + a_2\,x_2(n)]^2$$

$$= n\,a_1^2\,x_1^2(n) + n\,a_2^2\,x_2^2(n) + 2\,n\,a_1\,a_2\,x_1(n)\,x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

Example 6.14

Test the following systems for linearity.

a) $y(n) = 2\,x(n) + \dfrac{1}{x(n-1)}$ b) $y(n) = x(n) - b\,x(n-1)$ c) $y(n) = \sum\limits_{m=0}^{M} b_m\,x(n-m) - \sum\limits_{m=1}^{N} c_m\,y(n-m)$

Solution

a) Given that, $y(n) = 2x(n) + \dfrac{1}{x(n-1)}$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = 2x(n) + \dfrac{1}{x(n-1)}$ and the system $\mathcal{H}$ operates on x(n) to produce, $y(n) = \mathcal{H}\{x(n)\} = 2x(n) + \dfrac{1}{x(n-1)}$

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\ y_1(n) = \mathcal{H}\{x_1(n)\} = 2x_1(n) + \frac{1}{x_1(n-1)}$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = 2x_2(n) + \frac{1}{x_2(n-1)}$$

$$\therefore\ a_1\,y_1(n) + a_2\,y_2(n) = a_1\left(2x_1(n) + \frac{1}{x_1(n-1)}\right) + a_2\left(2x_2(n) + \frac{1}{x_2(n-1)}\right) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore\ y_3(n) = \mathcal{H}\{a_1\,x_1(n) + a_2\,x_2(n)\}$$

$$= 2[a_1 x_1(n) + a_2 x_2(n)] + \frac{1}{a_1 x_1(n-1) + a_2 x_2(n-1)} \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\,y_1(n) + a_2\,y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1\,y_1(n) + a_2\,y_2(n)$. Hence the system is nonlinear.

b) Given that, $y(n) = x(n) - b\,x(n-1)$

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n) - b\,x(n-1)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = x(n) - b\,x(n-1)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n) - b\,x_1(n-1)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n) - b\,x_2(n-1)$$

$$\therefore a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) - a_1 b\,x_1(n-1) + a_2 x_2(n) - a_2 b\,x_2(n-1) \quad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 x_1(n) + a_2 x_2(n) - b[a_1 x_1(n-1) + a_2 x_2(n-1)]$$

$$= a_1 x_1(n) - a_1 b\,x_1(n-1) + a_2 x_2(n) - a_2 b\,x_2(n-1) \quad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$. Hence the system is linear.

---

c) Given that, $y(n) = \sum\limits_{m=0}^{M} b_m\,x(n-m) - \sum\limits_{m=1}^{N} c_m\,y(n-m)$

Let $\mathcal{H}$ be the system represented by the equation,

$$y(n) = \sum_{m=0}^{M} b_m\,x(n-m) - \sum_{m=1}^{N} c_m\,y(n-m)$$

$$\left.\begin{array}{l}\text{The response of the system}\\ \mathcal{H}\text{ for the input } x(n)\end{array}\right\} = \mathcal{H}\{x(n)\} = y(n) = \sum_{m=0}^{M} b_m\,x(n-m) - \sum_{m=1}^{N} c_m\,y(n-m)$$

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore y_1(n) = \mathcal{H}\{x_1(n)\} = \sum_{m=0}^{M} b_m\,x_1(n-m) - \sum_{m=1}^{N} c_m\,y_1(n-m)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = \sum_{m=0}^{M} b_m\,x_2(n-m) - \sum_{m=1}^{N} c_m\,y_2(n-m)$$

$$\therefore a_1 y_1(n) + a_2 y_2(n) = a_1\left(\sum_{m=0}^{M} b_m\,x_1(n-m) - \sum_{m=1}^{N} c_m\,y_1(n-m)\right)$$

$$+ a_2\left(\sum_{m=0}^{M} b_m\,x_2(n-m) - \sum_{m=1}^{N} c_m\,y_2(n-m)\right) \quad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\}$$

$$= \sum_{m=0}^{M} b_m\,(a_1 x_1(n-m) + a_2 x_2(n-m)) - \sum_{m=1}^{N} c_m\,y_3(n-m)$$

$$= a_1 \sum_{m=0}^{M} b_m\,x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\,x_2(n-m) - \sum_{m=1}^{N} c_m\,y_3(n-m) \quad .....(2)$$

By time invariant property,

If $y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\}$ then $y_3(n-m) = \mathcal{H}\{a_1 x_1(n-m) + a_2 x_2(n-m)\}$

If $y_2(n) = \mathcal{H}\{x_2(n)\}$ then $y_2(n-m) = \mathcal{H}\{x_2(n-m)\}$

If $y_1(n) = \mathcal{H}\{x_1(n)\}$ then $y_1(n-m) = \mathcal{H}\{x_1(n-m)\}$

$$\therefore y_3(n-m) = \mathcal{H}\{a_1 x_1(n-m) + a_2 x_2(n-m)\} = a_1\, \mathcal{H}\{x_1(n-m)\} + a_2\, \mathcal{H}\{x_2(n-m)\}$$

$$= a_1 y_1(n-m) + a_2 y_2(n-m)] \qquad .....(3)$$

Using equation (3), the equation (2) can be written as,

$$y_3(n) = a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, [a_1\, y_1(n-m) + a_2\, y_2(n-m)]$$

$$= a_1 \sum_{m=0}^{M} b_m\, x_1(n-m) + a_2 \sum_{m=0}^{M} b_m\, x_2(n-m) - a_1 \sum_{m=1}^{N} c_m\, y_1(n-m) - a_2 \sum_{m=1}^{N} c_m\, y_2(n-m)$$

$$= a_1 \left(\sum_{m=0}^{M} b_m\, x_1(n-m) - \sum_{m=1}^{N} c_m\, y_1(n-m) \right) + a_2 \left(\sum_{m=0}^{M} b_m\, x_2(n-m) - \sum_{m=1}^{N} c_m\, y_2(n-m) \right) \qquad .....(4)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$.

From equations (1) and (4) we can say that the condition for linearity is satisfied. Therefore the system is linear.

6.8.4 Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time n depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time n depends on future inputs or outputs then the system is called ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $n < 0$, (i.e., $h(n) = 0$ for $n < 0$).

Let, $x(n)$ = Present input and $y(n)$ = Present output

$\therefore x(n-1), x(n-2), ...,$ are past inputs

$y(n-1), y(n-2), ...,$ are past outputs

In mathematical terms the output of a causal system satisfies the equation of the form

$$y(n) = F\,[x(n),\, x(n-1),\, x(n-2),\, ...\,,\, y(-1),\, y(n-2)\, ...]$$

where, $F[\cdot]$ is some arbitrary function.

Example 6.15

Test the casuality of the following systems.

a) $y(n) = x(n) - x(n-1)$ b) $y(n) = \sum\limits_{m=-\infty}^{n} x(m)$ c) $y(n) = a\, x(n)$ d) $y(n) = n\, x(n)$

Solution

a) Given that, $y(n) = x(n) - x(n-1)$

When $n = 0$, $y(0) = x(0) - x(-1)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-1)$

When $n = 1$, $y(1) = x(1) - x(0)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(0)$.

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

b) Given that, $y(n) = \sum_{m=-\infty}^{n} x(m)$

When n = 0, $y(0) = \sum_{m=-\infty}^{0} x(m)$

$= \ldots x(-2) + x(-1) + x(0)$ $\Rightarrow$ The response at n = 0, i.e., y(0) depends on the present input x(0) and past inputs x(–1), x(–2),.....

When n = 1, $y(1) = \sum_{m=-\infty}^{1} x(m)$

$= \ldots x(-2) + x(-1) + x(0) + x(1)$ $\Rightarrow$ The response at n = 1, i.e., y(1) depends on the present input x(1) and past inputs x(0), x(–1), x(–2),..

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

c) Given that, y(n) = a x(n)

When n = 0, y(0) = a x(0) $\Rightarrow$ The response at n = 0, i.e., y(0) depends on the present input x(0).

When n = 1, y(1) = a x(1) $\Rightarrow$ The response at n = 1, i.e., y(1) depends on the present input x(1).

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

d) Given that, y(n) = n x(n)

When n = 0, $y(0) = 0 \times x(0)$ $\Rightarrow$ The response at n = 0, i.e., y(0) depends on the present input x(0).

When n = 1, $y(1) = 1 \times x(1)$ $\Rightarrow$ The response at n = 1, i.e., y(1) depends on the present input x(1).

When n = 2, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at n = 2, i.e., y(2) depends on the present input x(2).

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

Example 6.16

Test the causality of the following systems.

a) y(n) = x(n) + 3 x(n + 4) b) $y(n) = x(n^2)$ c) y(n) = x(2n) d) y(n) = x(–n)

Solution

a) Given that, y(n) = x(n) + 3 x(n + 4)

When n = 0, y(0) = x(0) + 3 x(4) $\Rightarrow$ The response at n = 0, i.e., y(0) depends on the present input x(0) and future input x(4).

When n = 1, y(1) = x(1) + 3 x(5) $\Rightarrow$ The response at n = 1, i.e., y(1) depends on the present input x(1) and future input x(5).

From the above analysis we can say that the response for any value of n depends on present and future inputs. Hence the system is noncausal.

b) Givent that, $y(n) = x(n^2)$

When n = –1 ; y(–1) = x(1) $\Rightarrow$ The response at n = –1, depends on the future input x(1).

When n = 0 ; y(0) = x(0) $\Rightarrow$ The response at n = 0, depends on the present input x(0).

When n = 1 ; y(1) = x(1) $\Rightarrow$ The response at n = 1, depends on the present input x(1).

When n = 2 ; y(2) = x(4) $\Rightarrow$ The response at n = 2, depends on the future input x(4).

From the above analysis we can say that the response for any value of n (except n = 0 and n = 1) depends on future inputs. Hence the system is noncausal.

c) Given that, y(n) = x(2n)

When n = –1 ; y(–1) = x(–2) $\Rightarrow$ The response at n = –1, depends on the past input x(–2).

When n = 0 ; y(0) = x(0) $\Rightarrow$ The response at n = 0, depends on the present input x(0).

When n = 1 ; y(1) = x(2) $\Rightarrow$ The response at n = 1, depends on the future input x(2).

From the above analysis we can say that the response of the system for n > 0, depends on future inputs. Hence the system is noncausal.

d) Given that, y(n) = x(–n)

When n = –2 ; y(–2) = x(2) ⇒ The response at n = –2, depends on the future input x(2).
When n = –1 ; y(–1) = x(1) ⇒ The response at n = –1, depends on the future input x(1).
When n = 0 ; y(0) = x(0) ⇒ The response at n = 0, depends on the present input x(0).
When n = 1 ; y(1) = x(–1) ⇒ The response at n = 1, depends on the past input x(–1).

From the above analysis we can say that the response of the system for n < 0 depends on future inputs. Hence the system is noncausal.

6.8.5 Stable and Unstable Systems

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(n) be the input of discrete time system and y(n) be the response or output for x(n).The term ***bounded input*** refers to finite value of the input signal x(n) for any value of n. Hence if input x(n) is bounded then there exits a constant M_x such that $|x(n)| \le M_x$ and $M_x < \infty$, for all n.

Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of n. Hence if output y(n) is bounded then there exists a constant M_y such that $|y(n)| \le M_y$ and $M_y < \infty$, for all n.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable.For LTI (Linear Time Invariant) systems the condition for BIBO stability can be transformed to a condition on impulse response as shown below.

Condition for Stability of LTI System

By convolution sum formula, the response of LTI system is given by,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) \qquad .....(6.24)$$

Taking absolute values on both sides of equation (6.24) we get,

$$|y(n)| = \left| \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) \right| = \sum_{m=-\infty}^{\infty} |h(m)|\, |x(n-m)| \qquad .....(6.25)$$

If the input is bounded then $|x(n-m)| = M_x$. Hence equation (6.25) can be written as,

$$|y(n)| = \sum_{m=-\infty}^{\infty} |h(m)|\, M_x = M_x \sum_{m=-\infty}^{\infty} |h(m)| \qquad .....(6.26)$$

From equation (6.26) we can say that, the output is bounded if the impulse response of the system satisfies the condition,

$$\sum_{m=-\infty}^{\infty} |h(m)| < \infty$$

Since m is a dummy variable used in convolution operation we can change m by n in the above equation.

$$\boxed{\therefore \sum_{n=-\infty}^{+\infty} |h(n)| < \infty} \qquad .....(6.27)$$

Hence from equation (6.27) we can say that, an LTI system is ***stable*** if the impulse response is absolutely summable.

Example 6.17

Test the stability of the following systems.

a) $y(n) = \cos[x(n)]$ b) $y(n) = x(-n-2)$ c) $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = \cos[x(n)]$

The given system is nonlinear system, and so the test for stability should be performed for specific inputs.

The value of $\cos\theta$ lies between –1 to +1 for any value of θ. Therefore the output y(n) is bounded for any value of input x(n). Hence the given system is stable.

b) Given that, $y(n) = x(-n-2)$

The given system is time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, $y(n) = n\,x(n)$

The given system is time variant system, and so the test for stability should be performed for specific inputs.

Case i : If x(n) tends to infinity or constant, as "n" tends to infinity, then $y(n) = n\,x(n)$ will be infinite as "n" tends to infinity. So the ststem is unstable.

Case ii : If x(n) tends to zero as "n" tends to infinity, then $y(n) = n\,x(n)$ will be zero as "n" tends to infinity. So the system is stable.

Example 6.18

Determine the range of values of "a" and "b" for the stability of LTI system with impulse response,

$$h(n) = \begin{cases} b^n & ; \quad n < 0 \\ a^n & ; \quad n \ge 0 \end{cases}$$

Solution

The condition to be satisfied for the stability of the system is, $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.

Given that, $h(n) = a^n \;;\; n \ge 0$
$\qquad\qquad = b^n \;;\; n < 0$

$$\therefore \sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{-1} |b|^n + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=1}^{\infty} |b|^{-n} + \sum_{n=0}^{\infty} |a|^n$$

$$= \sum_{n=0}^{\infty} |b|^{-n} - 1 + \sum_{n=0}^{\infty} |a|^n \qquad \boxed{|b|^0 = 1}$$

$$= \sum_{n=0}^{\infty} \left(|b|^{-1}\right)^n - 1 + \sum_{n=0}^{\infty} |a|^n$$

The summation of infinite terms in the above equation converges if, $0 < |a| < 1$ and $0 < |b|^{-1} < 1$. Hence by using infinite geometric series formula,

$$\sum_{n=-\infty}^{+\infty} |h(n)| = \frac{1}{1-|b|^{-1}} - 1 + \frac{1}{1-|a|}$$

$$= \text{constant}$$

Therefore, the system is stable if $|a| < 1$ and $|b| > 1$.

Infinite qeometric series sum formula

$$\sum_{n=0}^{\infty} c^n = \frac{1}{1-c}$$

if $0 < |c| < 1$

6.8.6 FIR and IIR Systems

In ***FIR system*** (**F**inite duration **I**mpulse **R**esponse system), the impulse response consists of finite number of samples. The convolution formula for FIR system is given by,

$$y(n) = \sum_{m=0}^{N-1} h(m)\, x(n-m) \qquad .....(6.28)$$

where, $h(n) = 0$; for $n < 0$ and $n \geq N$

From equation (6.28) it can be concluded that the impulse response selects only N samples of the input signal.In effect, the system acts as a window that views only the most recent N input signal samples in forming the ouput. It neglects or simply forgets all prior input samples. Thus a FIR system requires memory of length N. In general, a FIR system is described by the difference equation,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m)$$

where, $b_m = h(m)$; for $m = 0$ to $N-1$

In ***IIR system*** (**I**nfinite duration **I**mpulse **R**esponse system), the impulse response has infinite number of samples. The convolution formula for IIR systems is given by,

$$y(n) = \sum_{m=0}^{\infty} h(m)\, x(n-m)$$

Since this weighted sum involves the present and all the past input sample, we can say that the IIR system requires infinite memory. In general, an IIR system is described by the difference equation,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

6.8.7 Recursive and Nonrecursive Systems

A system whose output y(n) at time n depends on any number of past output values as well as present and past inputs is called a ***recursive system.*** The past outputs are y(n–1), y(n–2), y(n–3), etc.,.

Hence for recursive system, the output y(n) is given by,

$$y(n) = F\,[y(n-1),\ y(n-2),...y(n-N),\ x(n),\ x(n-1),...x(n-M)]$$

A system whose output does not depend on past output but depends only on the present and past input is called a ***nonrecursive system.***

Hence for nonrecursive system, the output y(n) is given by,

$$y(n) = F\,[x(n),\ x(n-1)\ ,.....,\ x(n-M)]$$

In a recursive system, in order to compute $y(n_0)$, we need to compute all the previous values y(0), y(1) ,......., $y(n_0-1)$ before calculating $y(n_0)$. Hence the output samples of a recursive system has to be computed in order [i.e., y(0), y(1), y(2), ...]. The IIR systems are recursive systems.

In nonrecursive system, $y(n_0)$ can be computed immediately without having $y(n_0-1)$, $y(n_0-2)$..... Hence the output samples of nonrecursive system can be computed in any order [i.e. y(50), y(5), y(2), y(100),....]. The FIR systems are nonrecursive systems.

6.9 Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad .....(6.29)$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$
m is a dummy variable

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples. The convolution results in a nonperiodic sequence. Hence this convolution is also called ***aperiodic convolution.***

The convolution relation of equation (6.29) can be symbolically expressed as

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad ..... (6.30)$$

where, the symbol $*$ indicates convolution operation.

Procedure For Evaluating Linear Convolution

Let, $x_1(n)$ = Discrete time sequence with N_1 samples
$x_2(n)$ = Discrete time sequence with N_2 samples

Now, the convolution of $x_1(n)$ and $x_2(n)$ will produce a sequence $x_3(n)$ consisting of N_1+N_2-1 samples. Each sample of $x_3(n)$ can be computed using the equation (6.29). The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in equation (6.29).

$$\therefore\ x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(6.31)$$

The evaluation of equation (6.31) to determine the value of $x_3(n)$ at $n = q$, involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Shifting : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q-m)$.

4. Multiplication : Multiply $x_1(m)$ by $x_2(q-m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q-m)$.

5. Summation : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value $x_3(n)$ at a single time instant say $n = q$. In general, we are interested in evaluating the values of the seqence $x_3(n)$ over all the time instants in the range $-\infty < n < \infty$. Hence the steps 3, 4 and 5 given above must be repeated, for all possible time shifts in the range $-\infty < n < \infty$.

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero.The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

6.9.1 Representation of Discrete Time Signal as Summation of Impulses

A discrete time signal can be expressed as summation of impulses and this concept will be useful to prove that the response of discrete time LTI system can be determined using discrete convolution.

Let, $x(n)$ = Discrete time signal

$\delta(n)$ = Unit impulse signal

$\delta(n - m)$ = Delayed impulse signal

We know that,

$$\delta(n) = 1 \; ; \text{ at } n = 0$$
$$= 0 \; ; \text{ when } n \neq 0$$

and,

$$\delta(n - m) = 1 \; ; \text{ at } n = m$$
$$= 0 \; ; \text{ when } n \neq m$$

If we multiply the signal $x(n)$ with the delayed impulse $\delta(n - m)$ then the product is non-zero only at $n = m$ and zero for all other values of n. Also at $n = m$, the value of product signal is m^{th} sample $x(m)$ of the signal $x(n)$.

$$\therefore \; x(n)\; \delta(n - m) = x(m)$$

Each multiplication of the signal $x(n)$ by an unit impulse at some delay m, in essence picks out the single value $x(m)$ of the signal $x(n)$ at $n = m$, where the unit impulse is non-zero. Consequently if we repeat this multiplication for all possible delays in the range $-\infty < m < \infty$ and add all the product sequences, the result will be a sequence that is equal to the sequence $x(n)$.

For example,

$$x(n)\; \delta(n - (-2)) = x(-2)$$
$$x(n)\; \delta(n - (-1)) = x(-1)$$
$$x(n)\; \delta(n) = x(0)$$
$$x(n)\; \delta(n - 1) = x(1)$$
$$x(n)\; \delta(n - 2) = x(2)$$

From the above products we can say that each sample of $x(n)$ can be expressed as a product of the sample and delayed impulse, as shown below.

$$\therefore \; x(-2) = x(-2)\; \delta(n-(-2))$$
$$x(-1) = x(-1)\; \delta(-(-1))$$
$$x(0) = x(0)\; \delta(n)$$
$$x(1) = x(1)\; \delta(n - 1)$$
$$x(2) = x(2)\; \delta(n - 2)$$

$$\therefore\ x(n) = \cdots\cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots\cdots\cdots\cdots$$

$$= \cdots\cdots + x(-2)\ \delta(n - (-2)) + x(-1)\ \delta(n - (-1)) + x(0)\ \delta(n) + x\ (1)\ \delta(n-1) + x(2)\ \delta(n-2) + \cdots\cdots\cdots\cdots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m) \qquad .....(6.32)$$

In equation (6.32) each product x(m) δ(n – m) is an impulse and the summation of impulses gives the sequence x(n).

6.9.2 Response of LTI Discrete Time System Using Discrete Convolution

In an LTI system, the response y(n) of the system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system. It is expressed as,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m) \qquad .....(6.33)$$

where, the symbol $*$ represents convolution operation.

Proof :

Let y(n) be the response of system $\mathcal{H}$ for an input x(n)

$$\therefore\ y(n) = \mathcal{H}\{x(n)\} \qquad .....(6.34)$$

From equation (6.32) we know that the signal x(n) can be expressed as a summation of impulses,

$$\text{i.e., } x(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m) \qquad .....(6.35)$$

where, δ(n – m) is the delayed unit impulse signal.

From equation (6.34) and (6.35) we get,

$$y(n) = \mathcal{H}\left\{\sum_{m=-\infty}^{+\infty} x(m)\ \delta(n-m)\right\} \qquad .....(6.36)$$

The system $\mathcal{H}$ is a function of n and not a function of m. Hence by linearity property the equation (6.36) can be written as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ \mathcal{H}\{\delta(n-m)\} \qquad .....(6.37)$$

Let the response of the LTI system to the unit impulse input δ(n) be denoted by h(n),

$$\therefore\ h(n) = \mathcal{H}\{\delta(n)\}$$

Then by time invariance property the response of the system to the delayed unit impulse input δ(n – m) is given by,

$$h(n-m) = \mathcal{H}\{\delta(n-m)\} \qquad .....(6.38)$$

Using equation (6.38), the equation (6.37) can be expressed as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\ h(n-m)$$

The above equation represents the convolution of input x(n) with the impulse response h(n) to yield the output y(n). Hence it is proved that the response y(n) of LTI discrete time system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system.

6.9.3 Properties of Linear Convolution

The Discrete convolution will satisfy the following properties.

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

Proof of Commutative Property :

Consider convolution of $x_1(n)$ and $x_2(n)$.

By commutative property we can write,

$$\underset{\text{(LHS)}}{x_1(n) * x_2(n)} = \underset{\text{(RHS)}}{x_2(n) * x_1(n)}$$

LHS = $x_1(n) * x_2(n)$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad .....(6.39)$$

where, m is a dummy variable used for convolution operation.

Let, $n - m = p$ | when $m = -\infty$, $p = n - m = n + \infty = +\infty$

$\therefore m = n - p$ | when $m = +\infty$, $p = n - m = n - \infty = -\infty$

On replacing m by (n – p) and (n – m) by p in equation (6.39) we get,

$$\text{LHS} = \sum_{p=-\infty}^{+\infty} x_1(n-p)\, x_2(p)$$

$$= \sum_{p=-\infty}^{+\infty} x_2(p)\, x_1(n-p)$$

$$= x_2(n) * x_1(n)$$

p is a dummy variable used for convolution operation

= RHS

Proof of Associative Property :

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$. By associative property we can write,

$$\underset{\text{LHS}}{[x_1(n) * x_2(n)] * x_3(n)} = \underset{\text{RHS}}{x_1(n) * [x_2(n) * x_3(n)]}$$

Let, $y_1(n) = x_1(n) * x_2(n)$ .....(6.40)

Let us replace n by p

$\therefore y_1(p) = x_1(p) * x_2(p)$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m) \qquad .....(6.41)$$

Let, $y_2(n) = x_2(n) * x_3(n)$ .....(6.42)

$$\therefore y_2(n) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q)$$

$$\therefore \; y_2(n-m) = \sum_{q=-\infty}^{+\infty} x_1(q)\, x_2(n-q-m) \quad .....(6.43)$$

where p, m and q are dummy variables used for convolution operation.

$$\text{LHS} = [x_1(n) * x_2(n)] * x_3(n)$$

$$= y_1(n) * x_3(n) \qquad \boxed{\text{Using equation (6.40)}}$$

$$= \sum_{p=-\infty}^{+\infty} y_1(p)\, x_3(n-p)$$

$$= \sum_{p=-\infty}^{+\infty} \; \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3\,(n-p) \qquad \boxed{\text{Using equation (6.41)}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{p=-\infty}^{+\infty} x_2\,(p-m)\, x_3(n-p) \quad .....(6.44)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore \; p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

On replacing (p – m) by q, and p by (q + m) in the equation (6.44) we get,

$$\text{LHS} = \sum_{m=-\infty}^{+\infty} x_1(m) \sum_{q=-\infty}^{+\infty} x_2(q)\; x_3\,(n-q-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, y_2\,(n-m) \qquad \boxed{\text{Using equation (6.43)}}$$

$$= x_1\,(n) * y_2(n)$$

$$= x_1(n) * [x_2(n) * x_3(n)] \qquad \boxed{\text{Using equation (6.42)}}$$

$$= \text{RHS}$$

<u>Proof of Distributive Property :</u>

Consider the discrete time signals $x_1(n)$, $x_2(n)$ and $x_3(n)$. By distributive property we can write,

$$\underbrace{x_1\,(n) * [x_2(n) + x_3(n)]}_{\text{LHS}} = \underbrace{[x_1(n) * x_2\,(n)] + [x_1(n) * x_3\,(n)]}_{\text{RHS}}$$

$$\text{LHS} = x_1(n) * [x_2(n) + x_3(n)]$$

$$= x_1(n) * x_4(n) \qquad \boxed{x_4(n) = x_2(n) + x_3(n)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_4(n-m) \qquad \boxed{\text{m is a dummy variable used for convolution operation}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, [x_2(n-m) + x_3(n-m)]$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) + \sum_{m=-\infty}^{+\infty} x_1(m)\, x_3(n-m)$$

$$= [x_1\,(n) * x_2(n)] + [x_1(n) * x_3\,(n)]$$

$$= \text{RHS}$$

6.9.4 Interconnections of Discrete Time Systems

Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are shown in fig 6.21.

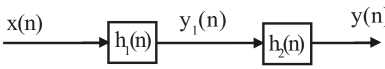


Fig 6.21a : *Cascade connection.*

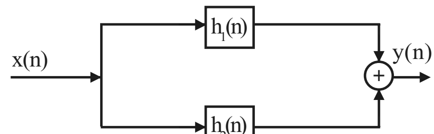


Fig 6.21b : *Parallel connection.*

Fig 6.21 : *Interconnection of discrete time systems.*

Cascade Connected Discrete Time System

Two cascade connected discrete time systems with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by convolution of individual impulse responses.

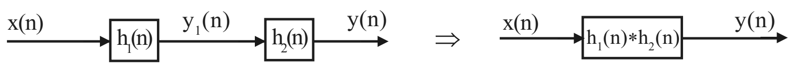


Fig 6.22 : *Cascade connected discrete time systems and their equivalent.*

Proof:

With reference to fig 6.22 we can write,

$$y_1(n) = x(n) * h_1(n) \quad .....(6.45)$$

$$y(n) = y_1(n) * h_2(n) \quad .....(6.46)$$

Using equation (6.45), the equation (6.46) can be written as,

$$y(n) = x(n) * h_1(n) * h_2(n)$$

$$= x(n) * [h_1(n) * h_2(n)]$$

$$= x(n) * h(n) \quad .....(6.47)$$

where, $h(n) = h_1(n) * h_2(n)$

From equation (6.47) we can say that the overall impulse response of two cascaded discrete time systems is given by convolution of individual impulse responses.

Parallel Connected Discrete Time Systems

Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by sum of individual impulse responses.

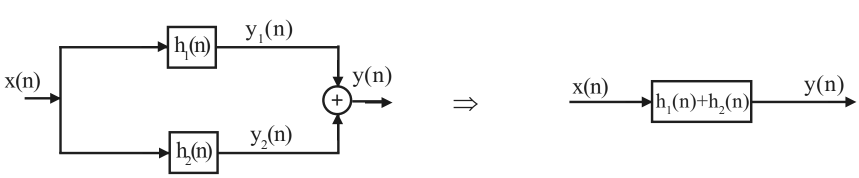


Fig 6.23 : *Parallel connected discrete time systems and their equivalent.*

Proof:

With reference to fig 6.23 we can write,

$$y_1(n) = x(n) * h_1(n) \qquad .....(6.48)$$

$$y_2(n) = x(n) * h_2(n) \qquad .....(6.49)$$

$$y(n) = y_1(n) + y_2(n) \qquad .....(6.50)$$

On substituting for $y_1(n)$ and $y_2(n)$ from equations (6.48) and (6.49) in equation (6.50) we get,

$$y(n) = [\, x(n) * h_1(n)] + [\, x(n) * h_2(n)] \qquad .....(6.51)$$

By using distributive property of convolution the equation (6.51) can be written as shown below,

$$y(n) = x(n) * [h_1(n) + h_2(n)]$$

$$= x(n) * h(n) \qquad .....(6.52)$$

where, $h(n) = h_1(n) + h_2(n)$

From equation (6.52) we can say that the overall impulse response of two parallel connected discrete time system is given by sum of individual impulse responses.

Example 6.19

Determine the impulse reponse for the cascade of two LTI systems having impulse responses,

$$h_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } h_2(n) = \left(\frac{1}{4}\right)^n u(n).$$

Solution

Let h(n) be the impulse response of the cascade system. Now h(n) is given by convolution of $h_1(n)$ and $h_2(n)$.

$$\therefore\ h(n) = h_1(n) * h_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} h_1(m)\, h_2(n-m) \qquad \text{where, m is a dummy variable used for convolution operation}$$

The product $h_1(m)\, h_2(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation is changed to m = 0 to n.

$$\therefore\ h(n) = \sum_{m=0}^{n} h_1(m)\, h_2(n-m) = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-m} = \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 4^m$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{4}{2}\right)^m$$

Finite geometric series sum formula

$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} 2^m$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right)$$

Using finite geometric series sum formula

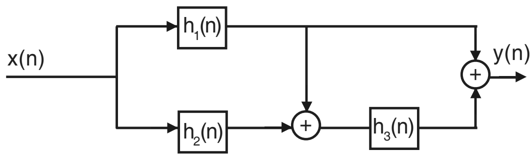
$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1) \quad ; \text{ for } n \ge 0$$

$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1)\, u(n) \quad ; \text{ for all n}$$

Example 6.20

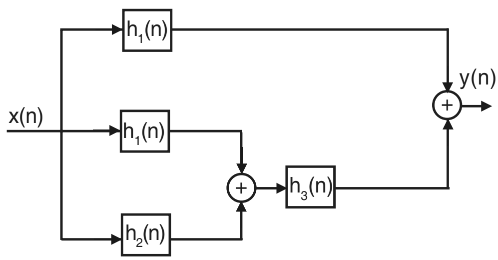
Determine the overall impulse response of the interconnected discrete time system shown below,

where, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)$, $h_2(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h_3(n) = \left(\frac{1}{5}\right)^n u(n)$.

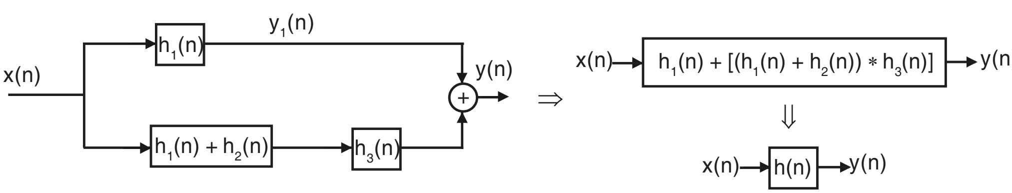


Solution

The given system can be redrawn as shown below.



The above system can be reduced to single equivalent system as shown below.



Here, $h(n) = h_1(n) + [(h_1(n) + h_2(n)) * h_3(n)]$

$= h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$ $\boxed{\text{Using distributive property}}$

Let us evaluate the convolution of $h_1(n)$ and $h_3(n)$.

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

The product of $h_1(m)\, h_3(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be changed to $m = 0$ to n.

$$\therefore\ h_1(n) * h_3(n) = \sum_{m=0}^{n} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{5}{3} - 1}$$ Using finite geometric series sum formula

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^n \frac{5}{3} - 1}{\frac{5-3}{3}} = \left(\frac{1}{5}\right)^n \left[\frac{3}{2}\left(\frac{5}{3}\right)^n \frac{5}{3} - \frac{3}{2}\right]$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \frac{5}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n = \frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n \ ; \text{for } n \geq 0$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \ ; \text{ for all } n$$

Let us evaluate the convolution of $h_2(n)$ and $h_3(n)$.

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n - m)$$

The product of $h_2(m)$ and $h_3(n–m)$ will be non-zero in the range $0 \leq m \leq n$. Therefore the summation index in the above equation can be change to m = 0 to n.

$$\therefore\ h_2(n) * h_3(n) = \sum_{m=0}^{n} h_2(m)\, h_3(n - m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{2}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^{n+1} - 1}{\frac{5}{2} - 1}$$ Using finite geometric series sum formula

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^n \frac{5}{2} - 1}{\frac{5-2}{2}} = \left(\frac{1}{5}\right)^n \left[\frac{2}{3}\left(\frac{5}{2}\right)^n \frac{5}{2} - \frac{2}{3}\right]$$

$$= \frac{5}{3}\left(\frac{1}{5}\right)^n \left(\frac{5}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n = \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n \text{ for } n \geq 0$$

$$= \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n) \text{ for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = h_1(n) + \left[h_1(n) * h_3(n)\right] + \left[h_2(n) * h_3(n)\right]$$

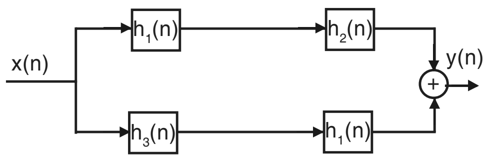
$$= \left(\frac{1}{3}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n)$$

$$= \left(1 + \frac{5}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{3}{2} + \frac{2}{3}\right)\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$= \left[\frac{7}{2}\left(\frac{1}{3}\right)^n - \frac{13}{6}\left(\frac{1}{5}\right)^n + \frac{5}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

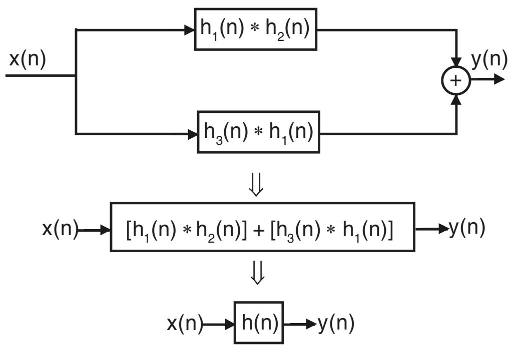
Example 6.21

Find the overall impulse response of the interconnected system shown below. Given that $h_1(n) = a^n\, u(n)$, $h_2(n) = \delta(n-1)$, $h_3(n) = \delta(n-2)$.



Solution

The given system can be reduced to single equivalent system as shown below.



Here, $h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

Let us evaluate the convolution of $h_1(n)$ and $h_2(n)$.

$$h_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m) \qquad \boxed{\text{Using commutative property}}$$

$$= \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{-m}$$

The product of $\delta(m-1)$ and a^{-m} in the above equation will be non-zero only when $m = 1$.

$$\therefore\ h_1(n) * h_2(n) = a^n\, a^{-1} = a^{n-1} \quad ; \text{ for } n \geq 1$$

$$= a^{n-1}\, u(n-1) \quad ; \text{ for all } n.$$

Let us evaluate the convolution of $h_3(n)$ and $h_1(n)$.

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} h_3(m)\, h_1(n-m)$$

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{-m}$$

The product of $\delta(m-2)$ and a^{-m} in the above equation will be non-zero only when $m = 2$.

$$\therefore h_1(n) * h_2(n) = a^n\, a^{-2} = a^{n-2} \quad ; \quad \text{for } n \geq 2$$

$$= a^{n-2} u(n-2) \quad ; \quad \text{for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$$

$$= a^{(n-1)}\, u(n-1) + a^{(n-2)}\, u(n-2)$$

6.9.5 Methods of Performing Linear Convolution

Method -1: Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one non-zero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
6. To get subsequent samples of output sequence, the step-5 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method -3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $x_3(n)$. (The sum of the diagonal elements are shown below for reference).

	$x_2(0)$	$x_2(1)$	$x_2(2)$	$x_2(3)$
$x_1(0)$	$x_1(0)x_2(0)$	$x_1(0)x_2(1)$	$x_1(0)x_2(2)$	$x_1(0)x_2(3)$
$x_1(1)$	$x_1(1)x_2(0)$	$x_1(1)x_2(1)$	$x_1(1)x_2(2)$	$x_1(1)x_2(3)$
$x_1(2)$	$x_1(2)x_2(0)$	$x_1(2)x_2(1)$	$x_1(2)x_2(2)$	$x_1(2)x_2(3)$
$x_1(3)$	$x_1(3)x_2(0)$	$x_1(3)x_2(1)$	$x_1(3)x_2(2)$	$x_1(3)x_2(3)$

$$x_3(0) = \ldots\ldots + x_1(0)\,x_2(0) + \ldots\ldots$$

$$x_3(1) = \ldots\ldots + x_1(1)\,x_2(0) + x_1(0)\,x_2(1) + \ldots\ldots$$

$$x_3(2) = \ldots\ldots + x_1(2)\,x_2(0) + x_1(1)\,x_2(1) + x_1(0)\,x_2(2) + \ldots\ldots$$

$$x_3(3) = \ldots\ldots + x_1(3)\,x_2(0) + x_1(2)\,x_2(1) + x_1(1)\,x_2(2) + x_1(0)\,x_2(3) + \ldots\ldots$$

$$\vdots$$

$$\therefore\ x_3(n) = \{\ldots\ldots\ x_3(0),\ x_3(1),\ x_3(2),\ x_3(3),\ \ldots\ldots\}$$

Example 6.22

Determine the response of the LTI system whose input x(n) and impulse response h(n) are given by, $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m)$$

In this example the convolution operation is performed by three methods.

The Input sequence starts at n = 0 and the impulse response sequence starts at n = –1. Therefore the output sequence starts at n = 0 + (–1) = –1.

The input and impulse response consists of 4 samples, so the output consists of 4 + 4 – 1 = 7 samples.

Method 1 : Graphical Method

The graphical representation of x(n) and h(n) after replacing n by m are shown below. The sequence h(m) is folded with respect to m = 0 to obtain h(–m).

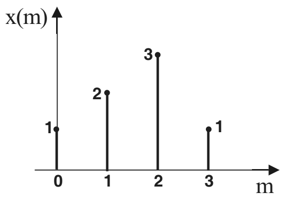


***Fig 1** : Input sequence.*

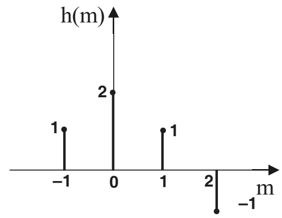


***Fig 2** : Impulse response.*

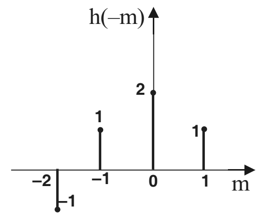


***Fig 3** : Folded impulse response.*

The samples of y(n) are computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m)\ ;\ \text{where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

When n = –1 ; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

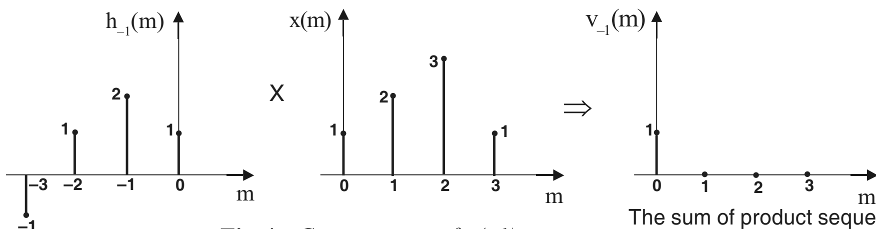


***Fig 4 : ** Computation of y(–1).*

The sum of product sequence $v_{-1}(m)$ gives y(–1). ∴ y(–1) = 1

When n = 0 ; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

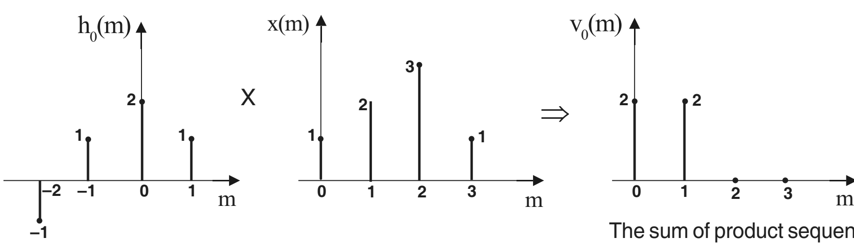
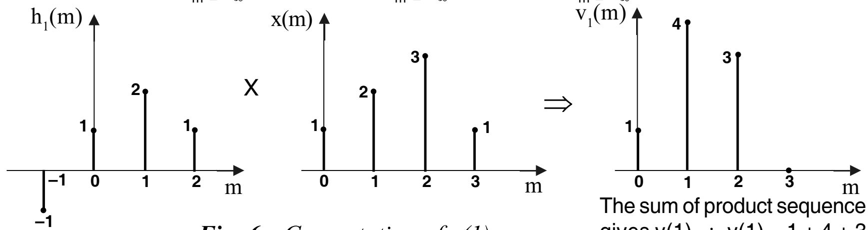


***Fig 5 : ** Computation of y(0).*

The sum of product sequence $v_0(m)$ gives y(0). ∴ y(0) = 2 + 2 = 4

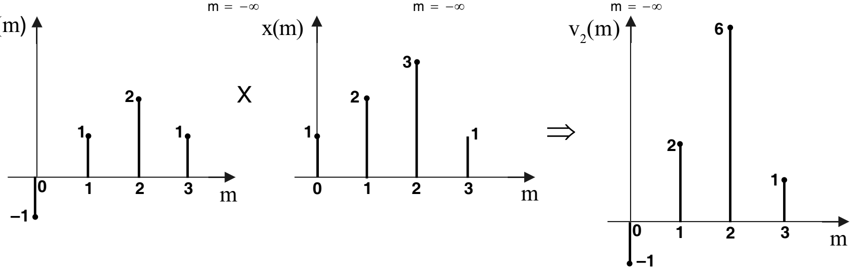
When n = 1 ; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$



The sum of product sequence $v_1(m)$ gives y(1). $\therefore$ $y(1) = 1 + 4 + 3 = 8$

Fig 6 : Computation of y(1).

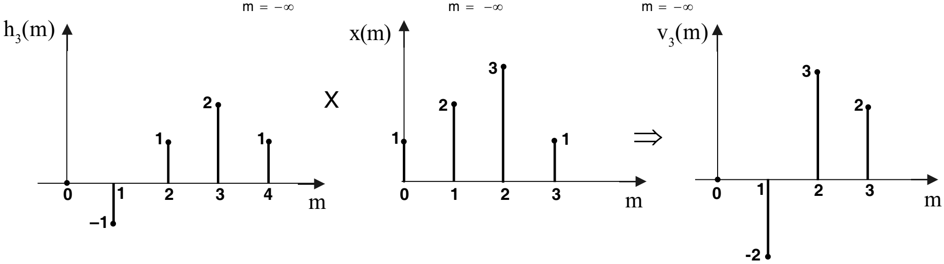
When n = 2 ; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$



The sum of product sequence $v_2(m)$ gives y(2). $\therefore$ $y(2) = -1 + 2 + 6 + 1 = 8$

Fig 7 : Computation of y(2).

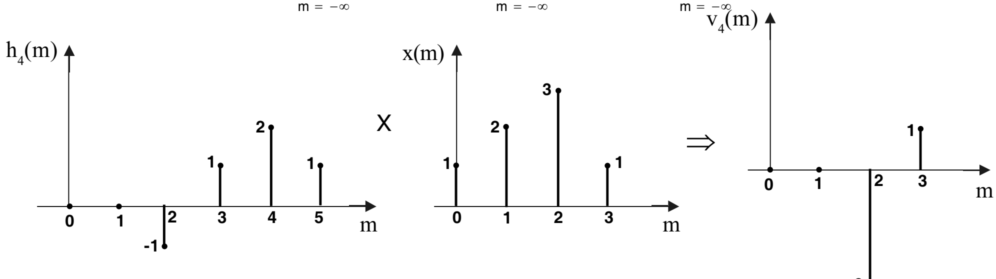
When n = 3 ; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$



The sum of product sequence $v_3(m)$ gives y(3). $\therefore$ $y(3) = -2 + 3 + 2 = 3$

Fig 8 : Computation of y(3).

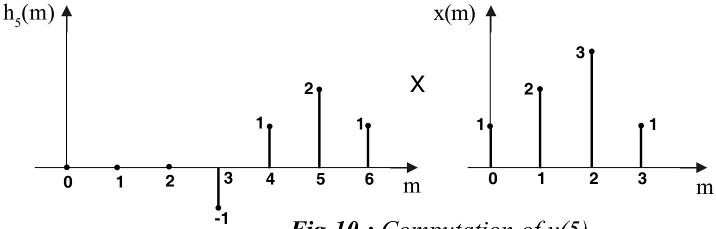
When n = 4 ; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

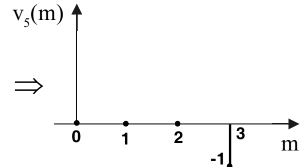


The sum of product sequence $v_4(m)$ gives y(4). $\therefore$ $y(4) = -3 + 1 = -2$

Fig 9 : Computation of y(4).

When $n = 5$; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$





The sum of product sequence $v_5(m)$ gives y(5). $\therefore$ $y(5) = -1$

***Fig 10 :** Computation of y(5).*

The output sequence, $y(n) = \left\{1,\ \underset{\uparrow}{4},\ 8,\ 8,\ 3,\ -2,\ -1\right\}$

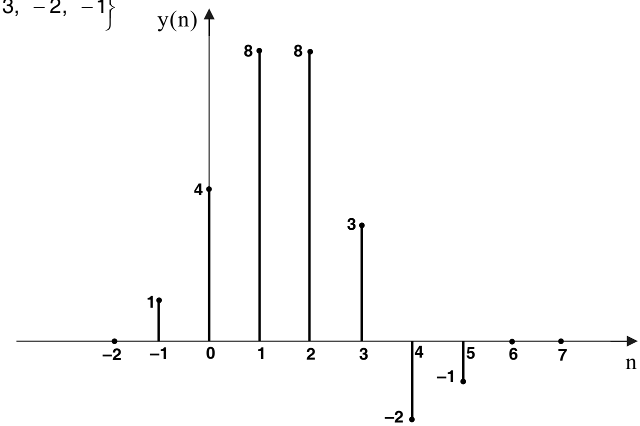


***Fig 11 :** Graphical representation of y(n).*

Method - 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

***Note :** The unfilled boxes in the table are considered as zeros.*

m	–3	–2	–1	0	1	2	3	4	5	6
x(m)				1	2	3	1			
h(m)			1	2	1	–1				
h(–m)		–1	1	2	1					
$h(-1-m) = h_{-1}(m)$	–1	1	2	1						
$h(0-m) = h_0(m)$		–1	1	2	1					
$h(1-m) = h_1(m)$			–1	1	2	1				
$h(2-m) = h_2(m)$				–1	1	2	1			
$h(3-m) = h_3(m)$					–1	1	2	1		
$h(4-m) = h_4(m)$						–1	1	2	1	
$h(5-m) = h_5(m)$							–1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

$$\text{When } n=-1\ ;\ y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m) \qquad \boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3}$$

$$= x(-3)\, h_{-1}(-3) + x(-2) h_{-1}(-2) + x(-1) h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1)$$
$$+ x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$
$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-2}^{3} x(m)\, h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-1}^{3} x(m)\, h_1(m) = 0 + 1 + 4 + 3 + 0 = 8$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=0}^{3} x(m)\, h_2(m) = -1 + 2 + 6 + 1 = 8$$

$$\text{When } n = 3\ ;\ y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = 0 - 2 + 3 + 2 + 0 = 3$$

$$\text{When } n = 4\ ;\ y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0 - 3 + 1 + 0 + 0 = -2$$

$$\text{When } n = 5\ ;\ y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$$

$$\text{The output sequence, } y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$$

Method - 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

$$\begin{array}{c|cccc} {}_{x(n)\downarrow}\diagdown^{h(n)\rightarrow} & 1 & 2 & 1 & -1 \\ \hline 1 & 1\times 1 & 1\times 2 & 1\times 1 & 1\times(-1) \\ 2 & 2\times 1 & 2\times 2 & 2\times 1 & 2\times(-1) \\ 3 & 3\times 1 & 3\times 2 & 3\times 1 & 3\times(-1) \\ 1 & 1\times 1 & 1\times 2 & 1\times 1 & 1\times(-1) \end{array} \Rightarrow \begin{array}{c|cccc} {}_{x(n)\downarrow}\diagdown^{h(n)\rightarrow} & 1 & 2 & 1 & 1 \\ \hline 1 & 1 & 2 & 1 & -1 \\ 2 & 2 & 4 & 2 & -2 \\ 3 & 3 & 6 & 3 & -3 \\ 1 & 1 & 2 & 1 & -1 \end{array}$$

y(–1) = 1

y(0) = 2 + 2 = 4

y(1) = 3 + 4 + 1 = 8

y(2) = 1 + 6 + 2 + (–1) = 8

y(3) = 2 + 3 + (–2) = 3

y(4) = 1 + (–3) = –2

y(5) = –1

$$\therefore\ y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$$

Example 6.23

Determine the output y(n) of a relaxed LTI system with impulse response,

$h(n) = a^n\, u(n)$; where $|a| < 1$ and

When input is a unit step sequence, i.e., $x(n) = u(n)$.

Solution

The graphical representation of x(n) and h(n) after replacing n by m are shown below. Also the sequence x(m) is folded to get x(–m)

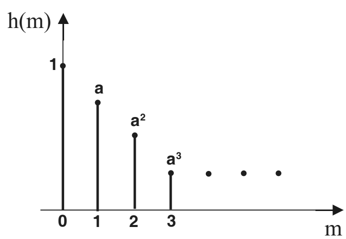


Fig 1 : *Impulse response.*

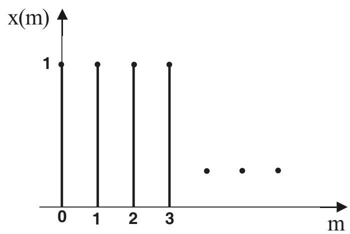


Fig 2 : *Input sequence.*

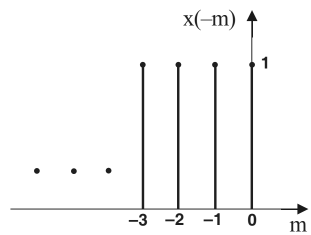


Fig 3 : *Folded input sequence.*

Here both h(m) and x(m) are infinite duration sequences starting at n = 0. Hence the output sequence y(n) will also be an infinite duration sequence starting at n = 0

By convolution formula,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=0}^{\infty} h(m)\, x_n(m)\ ; \quad \text{where } x_n(m) = x(n-m)$$

The computation of some samples of y(n) using the above equation are graphically shown below.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=0}^{\infty} h(m)\, x(m) = \sum_{m=0}^{\infty} h(m)\, x_0(m) = \sum_{m=0}^{\infty} v_0(m)$$

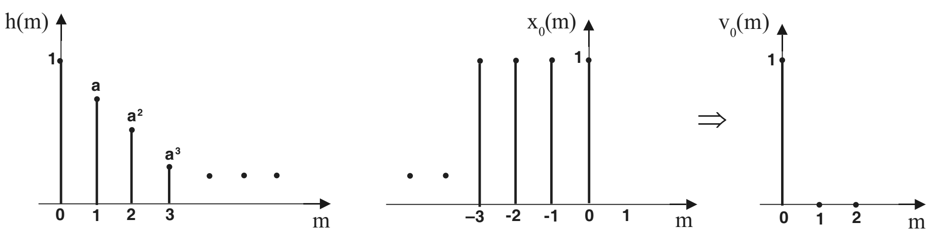


Fig 4 : *Computation of y(0).*

y(0) = 1

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=0}^{\infty} h(m)\, x(1-m) = \sum_{m=0}^{\infty} h(m)\, x_1(m) = \sum_{m=0}^{\infty} v_1(m)$$

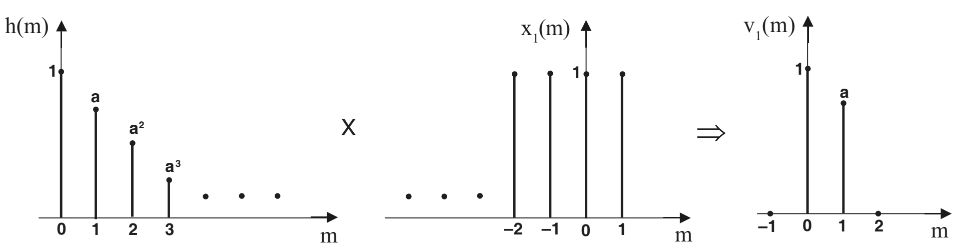


Fig 5 : *Computation of y(1).*

y(1) = 1 + a

$$\text{When } n = 2 \;;\; y(2) = \sum_{m=0}^{\infty} h(m)\, x(2-m) = \sum_{m=0}^{\infty} h(m)\, x_2(m) = \sum_{m=0}^{\infty} v_2(m)$$

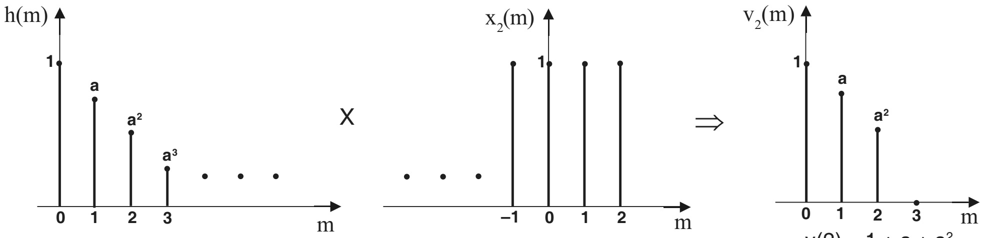


Fig 6 : Computation of y(2).

Solving similarly for other values of n, we can write y(n) for any value of n as shown below.

$$y(n) = 1 + a + a^2 + \ldots\ldots + a^n = \sum_{p=0}^{n} a^p \quad ; \text{ for } n \geq 0$$

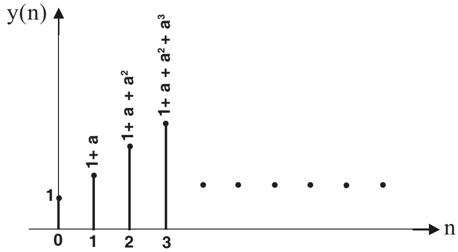


Fig 7 : Graphical representation of y(n).

6.10 Circular Convolution

6.10.1 Circular Representation and Circular Shift of Discrete Time Signal

Consider a finite duration sequence x(n) and its periodic extension $x_p(n)$. The periodic extension of x(n) can be expressed as $x_p(n) = x(n + N)$, where N is the periodicity. Let N = 4. The sequence x(n) and its periodic extension are shown in fig 6.24.

$$\begin{aligned} \text{Let, } x(n) &= 1 \;;\; n = 0 \\ &= 2 \;;\; n = 1 \\ &= 3 \;;\; n = 2 \\ &= 4 \;;\; n = 3 \end{aligned}$$

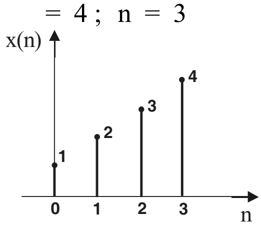


Fig 6.24a : Finite duration sequence x(n).

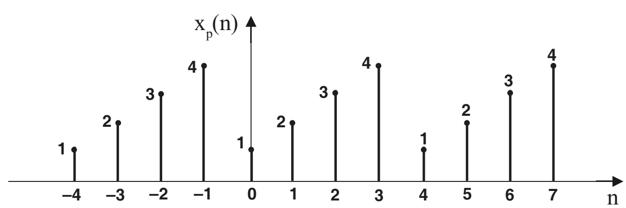


Fig 6.24b : Periodic extension of x(n).

Fig 6.24 : A finite duration sequence and its periodic extension.

Let us delay the periodic sequence $x_p(n)$ by two units of time as shown in fig 6.25(a). (For delay the sequence is shifted right). Let us denote one period of this delayed sequence by $x_1(n)$. One period of the delayed sequence is shown in figure 6.25(b).

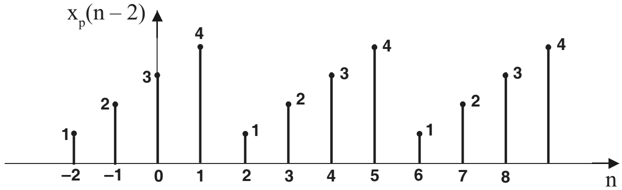


Fig 6.25a : $x_p(n)$ delayed by two units of time.

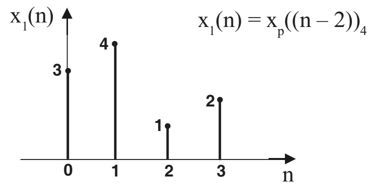


Fig 6.25b : One period of $x_p(n-2)$.

Fig 6.25 : Delayed version of $x_p(n)$.

The sequence $x_1(n)$ can be represented by $x_p(n-2, (\text{mod } 4))$, or $x_p((n-2))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the delayed sequence $x_1(n)$ are shown below.

$$x_1(n) = x_p(n-2, (\text{mod } 4)) = x_p((n-2))_4$$

$$\therefore \text{ When } n = 0;\ x_1(0) = x_p((0-2))_4 = x_p((-2))_4 = x(2) = 3$$

$$\text{When } n = 1;\ x_1(1) = x_p((1-2))_4 = x_p((-1))_4 = x(3) = 4$$

$$\text{When } n = 2;\ x_1(2) = x_p((2-2))_4 = x_p((0))_4 = x(0) = 1$$

$$\text{When } n = 3;\ x_1(3) = x_p((3-2))_4 = x_p((1))_4 = x(1) = 2$$

The sequences x(n) and $x_1(n)$ can be represented as points on a circle as shown in fig 6.26. From fig 6.26 we can say that, $x_1(n)$ is simply x(n) shifted circularly by two units in time, where the counter clockwise (anticlockwise) direction has been arbitrarily selected for right shift or delay.

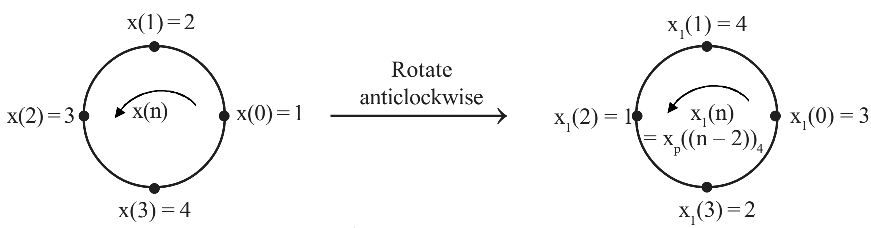


Fig 6.26a : Circular representation of x(n). *Fig 6.26b : Circular representation of $x_1(n)$.*

Fig 6.26 : Circular representation of a signal and its delayed version.

Let us advance the periodic sequence $x_p(n)$ by three units of time as shown in fig 6.27(a). Let us denote one period of this advanced sequence by $x_2(n)$. One period of the advanced sequence is shown in fig 6.27(b).

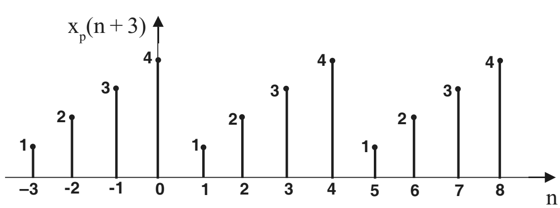


Fig 6.27a : $x_p(n)$ advanced by three units of time.

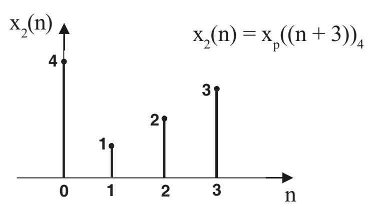


Fig 6.27b : One period of $x_p(n+3)$.

Fig 6.27 : Advanced version of $x_p(n)$.

The sequence $x_2(n)$ can be represented by $x_p(n+3, (\text{mod } 4))$ or $x_p((n+3))_4$, where mod 4 indicates that the sequence repeats after 4 samples. The relation between the original sequence x(n) and one period of the advanced sequence $x_2(n)$ are shown below.

$$x_2(n) = x_p(n+3, (\text{mod } 4)) = x_p((n+3))_4$$

$$\therefore \text{ When } n = 0;\ x_2(0) = x_p((0+3))_4 = x_p((3))_4 = x(3) = 4$$

$$\text{When } n = 1;\ x_2(1) = x_p((1+3))_4 = x_p((4))_4 = x(0) = 1$$

$$\text{When } n = 2;\ x_2(2) = x_p((2+3))_4 = x_p((5))_4 = x(1) = 2$$

$$\text{When } n = 3;\ x_2(3) = x_p((3+3))_4 = x_p((6))_4 = x(2) = 3$$

The sequences x(n) and $x_2(n)$ can be represented as points on a circle as shown in fig 6.28. From fig 6.28 we can say that $x_2(n)$ is simply x(n) shifted circularly by three units in time where clockwise direction has been selected for left shift or advance.

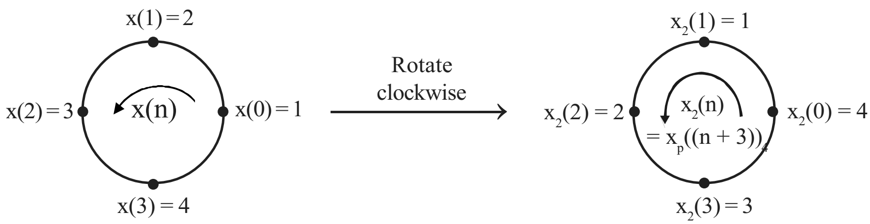


Fig 6.28a : Circular representation of x(n). *Fig 6.28b : Circular representationof $x_2(n)$.*

Fig 6.28 : Circular representationof a signal and its advanced version.

Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and viceversa. If a nonperiodic N-point sequence is represented on the circumference of a circle then it becomes a periodic sequence of periodicity N. When the sequence is shifted circularly, the samples repeat after N shifts. This is similar to modulo-N operation. Hence, in general, the circular shift may be represented by the index mod-N. Let x(n) be an N-point sequence represented on a circle and $x'(n)$ be its shifted sequence by m units of time.

Now, $x'(n) = x(n-m, \text{mod } N) \equiv x((n-m))_N$ ..... (6.53)

When m is positive, the equation (6.53) represents delayed sequence and when m is negative, the equation (6.53) represents advanced sequence.

6.10.2 Circular Symmetrics of Discrete Time Signal

The circular representation of a sequence and the resulting periodicity gives rise to new definitions for even symmetry, odd symmetry and the time reversal of the sequence.

An N-point sequence is called even if it is symmetric about the point zero on the circle. This implies that,

$$x(N-n) = x(n) \quad ; \quad \text{for} \quad 0 \le n \le N-1 \qquad \text{...... (6.54)}$$

An N-point sequence is called odd if it is antisymmetric about the point zero on the circle. This implies that,

$$x(N-n) = -\,x(n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad \text{...... (6.55)}$$

The time reversal of a N-point sequence is obtained by reversing its sample about the point zero on the circle. Thus the sequence x(–n, (mod N)) is simply written as,

$$x\,(-n, (\text{mod } N)) = x(N-n) \quad ; \text{ for } \quad 0 \le n \le N-1 \qquad \text{...... (6.56)}$$

This time reversal is equivalent to plotting x(n) in a clockwise direction on a circle, as shown in fig 6.29.

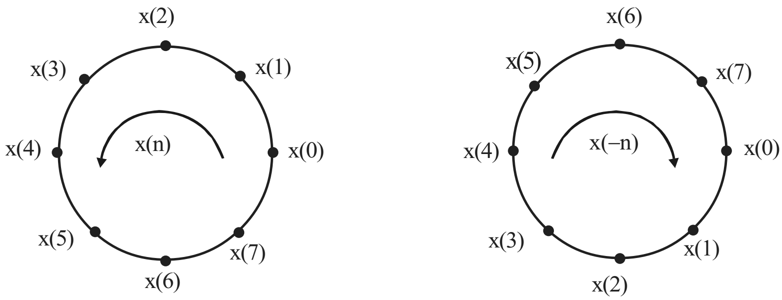


***Fig 6.29 :** Circular representation of an 8 point sequence and its folded sequence.*

6.10.3 Definition of Circular Convolution

The ***Circular convolution*** of two periodic discrete time sequences $x_1(n)$ and $x_2(n)$ with periodicity of N samples is defined as,

$$\boxed{x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=0}^{N-1} x_2(m)\, x_1((n-m))_N} \qquad \text{.....(6.57)}$$

where, $x_3(n)$ is the sequence obtained by circular convolution,

$x_1((n-m))_N$ represents circular shift of $x_1(n)$

$x_2((n-m))_N$ represents circular shift of $x_2(n)$

m is a dummy variable.

The output sequence $x_3(n)$ obtained by circular convolution is also a periodic sequence with periodicity of N samples. Hence this convolution is also called periodic convolution.

The convolution relation of equation (6.57) can be symbolically expressed as

$$x_3(n) = x_1(n) \circledast x_2(n) = x_2(n) \circledast x_1(n) \qquad \text{..... (6.58)}$$

where, the symbol $\circledast$ indicates circular convolution operation.

The circular convolution is defined for periodic sequences. But circular convolution can be performed with non-periodic sequences by periodically extending them.The circular convolution of two sequences requires that, at least one of the sequences should be periodic. Hence it is sufficient if one of the sequences is periodically extended in order to perform circular convolution.

The circular convolution of finite duration sequences can be performed only if both the sequences consist of the same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the length of larger size sequence by appending zeros.

Circular convolution basically involves the same four steps as that for linear convolution, namely, folding one sequence, shifting the folded sequence, multiplying the two sequences and finally summing the values of the product sequence. Like linear convolution, any one of the sequence is folded and rotated in circular convolution.

The difference between the two is that, in circular convolution the folding and shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In linear convolution there is no modulo-N operation.

6.10.4 Procedure for Evaluating Circular Convolution

Let, $x_1(n)$ and $x_2(n)$ be periodic discrete time sequences with periodicity of N-samples. If $x_1(n)$ and $x_2(n)$ are non-periodic then convert the sequences to N-sample sequences and periodically extend the sequence $x_2(n)$ with periodicity of N-samples.

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will produce a periodic sequence $x_3(n)$ with periodicity of N-samples. The samples of one period of $x_3(n)$ can be computed using the equation (6.57). The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in equation (6.57).

$$\therefore\ x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N \qquad .....(6.59)$$

The evaluation of equation (6.59) to determine the value of $x_3(n)$ at $n = q$ involves the following five steps.

1. Change of index : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, in order to get the sequences $x_1(m)$ and $x_2(m)$. Represent the samples of one period of the sequences on circles.

2. Folding : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.

3. Rotation : Rotate $x_2(-m)$ by q times in anti-clockwise if q is positive, rotate $x_2(-m)$ by q times in clockwise if q is negative to obtain $x_2((q-m))_N$.

4. Multiplication : Multiply $x_1(m)$ by $x_2((q-m))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2((q-m))_N$.

5. Summation : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value of $x_3(n)$ at a single time instant say $n = q$. In general we are interested in evaluating the values of the sequence $x_3(n)$ in the range $0 < n < N-1$. Hence the steps 3 , 4 and 5 given above must be repeated, for all possible time shifts in the range $0 < n < N-1$.

6.10.5 Linear Convolution via Circular Convolution

When two numbers of N-point sequences are circularly convolved, it produces another N-point sequence. For circular convolution, one of the sequence should be periodically extended. Also the resultant sequence is periodic with period N.

The linear convolution of two sequences of length N_1 and N_2 produces an output sequence of length $N_1 + N_2 - 1$. To perform linear convolution via circular convolution both the sequences should be converted to $N_1 + N_2 - 1$ point sequences by padding with zeros. Then perform circular convolution of $N_1 + N_2 - 1$ point sequences. The resultant sequence will be same as that of linear convolution of N_1 and N_2 point sequences.

6.10.6 Methods of Computing Circular Convolution

Method 1 : Graphical Method

In graphical method the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. One of the sequence is folded and shifted circularly. Let $x_1(n)$ and $x_2(n)$ be the given sequences. Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to get a sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences on circles.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Rotate (or shift) the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
4. The sample of $x_3(q)$ at $n = q$ is given by,

$$x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$$

$$\text{where, } x_{2,q}(m) = x_2((q-m))_N$$

Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at $n = q$].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 2 : Tabular Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. Let $x_3(n)$ be the N-point sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$. The following procedure can be used to obtain one sample of $x_3(n)$ at $n = q$.

1. Change the index n in the sequences $x_1(n)$ and $x_2(n)$ to get $x_1(m)$ and $x_2(m)$ and then represent the sequences as two rows of tabular array.
2. Fold one of the sequence. Let us fold $x_2(m)$ to get $x_2(-m)$.
3. Periodically extend $x_2(-m)$. Here the periodicity is N, where N is the length of the given sequences.
4. Shift the sequence $x_2(-m)$, q times to get the sequence $x_2((q-m))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.
5. The sample of $x_3(q)$ at n = q is given by

$$x_3(q) = \sum_{m=0}^{N-1} x_1(m)\, x_2((q-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,q}(m)$$

$$\text{where } x_{2,q}(m) = x_2((q-m))_N$$

   Determine the product sequence $x_1(m)\, x_{2,q}(m)$ for one period.

6. The sum of the samples of the product sequence gives the sample $x_3(q)$ [i.e., $x_3(n)$ at n = q].

The above procedure is repeated for all possible values of n to get the sequence $x_3(n)$.

Method 3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the given N-point sequences. The circular convolution of $x_1(n)$ and $x_2(n)$ yields another N-point sequence $x_3(n)$.

In this method an (N × N) matrix is formed using one of the sequence as shown below. Another sequence is arranged as a column vector (column matrix) of order (N × 1). The product of the two matrices gives the resultant sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(N-1) & x_2(N-2) & \cdots\cdots & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(N-1) & \cdots\cdots & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & \cdots\cdots & x_2(4) & x_2(3) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_2(N-2) & x_2(N-3) & x_2(N-4) & \cdots\cdots & x_2(0) & x_2(N-1) \\ x_2(N-1) & x_2(N-2) & x_2(N-3) & \cdots\cdots & x_2(1) & x_2(0) \end{bmatrix} \times \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ \vdots \\ \vdots \\ x_1(N-2) \\ x_1(N-1) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ \vdots \\ \vdots \\ x_3(N-2) \\ x_3(N-1) \end{bmatrix}$$

Example 6.24

Perform circular convolution of the two sequences,

$$x_1(n) = \{\underset{\uparrow}{2}, 1, 2, 1\} \text{ and } x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Solution

Method 1:Graphical Method of Computing Circular Convolution

Let $x_3(n)$ be the sequence obtained by circular convolution of $x_1(n)$ and $x_2(n)$.

The circular convolution of $x_1(n)$ and $x_2(n)$ is given by,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,n}(m)$$

where $x_{2,n}(m) = x_2((n-m))_N$ and m is the dummy variable used for convolution.

The index n in the given sequences are changed to m and each sequence is represented as points on a circle as shown below. The folded sequence $x_2(-m)$ is also represented on the circle.

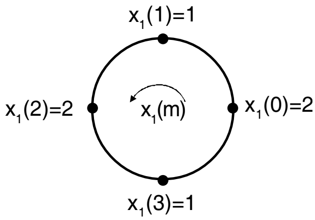


Fig 1.

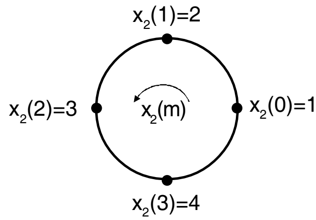


Fig 2.

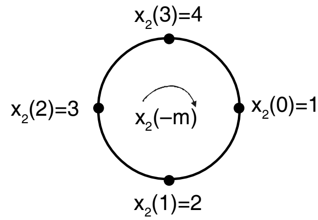


Fig 3.

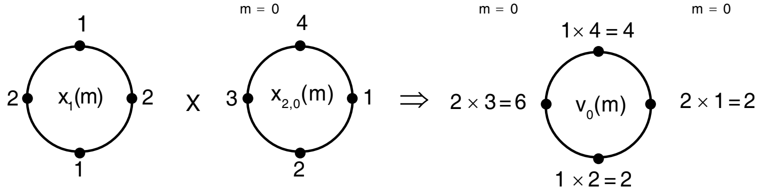
The given sequences are 4-point sequences . $\therefore N = 4$.

Each sample of $x_3(n)$ is given by sum of the samples of product sequence defined by the equation,

$$x_3(n) = \sum_{m=0}^{3} x_1(m)\; x_{2,n}(m) = \sum_{m=0}^{3} v_n(m) \quad ; \quad \text{where } v_n(m) = x_1(m)\, x_{2,n}(m)$$

Using the above equation, graphical method of computing each sample of $x_3(n)$ are shown in fig 4 to fig 7.

When $n = 0$; $x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_2(-m) = \sum_{m=0}^{3} x_1(m)\, x_{2,0}(m) = \sum_{m=0}^{3} v_0(m)$

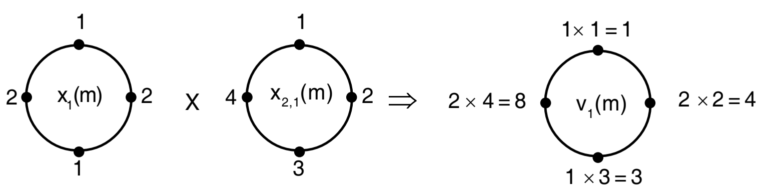


The sum of samples of $v_0(m)$ gives $x_3(0)$

$\therefore x_3(0) = 2 + 4 + 6 + 2 = 14$

Fig 4 : Computation of $x_3(0)$.

When $n = 1$; $x_3(1) = \sum_{m=0}^{3} x_1(m)\, x_2(1-m) = \sum_{m=0}^{3} x_1(m)\, x_{2,1}(m) = \sum_{m=0}^{3} v_1(m)$



The sum of samples of $v_1(m)$ gives $x_3(1)$

$\therefore x_3(1) = 4 + 1 + 8 + 3 = 16$

Fig 5 : Computation of $x_3(1)$.

When $n = 2$; $x_3(2) = \sum_{m=0}^{3} x_1(m)\, x_2(2-m) = \sum_{m=0}^{3} x_1(m)\, x_{2,2}(m) = \sum_{m=0}^{3} v_2(m)$

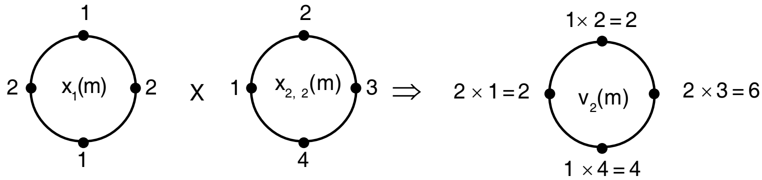


Fig 6 : Computation of $x_3(2)$.

The sum of samples of $v_2(m)$ gives $x_3(2)$

$\therefore\ x_3(2) = 6 + 2 + 2 + 4 = 14$

When $n = 3$; $x_3(3) = \sum_{m=0}^{3} x_1(m)\, x_2(3-m) = \sum_{m=0}^{3} x_1(m)\, x_{2,3}(m) = \sum_{m=0}^{3} v_3(m)$

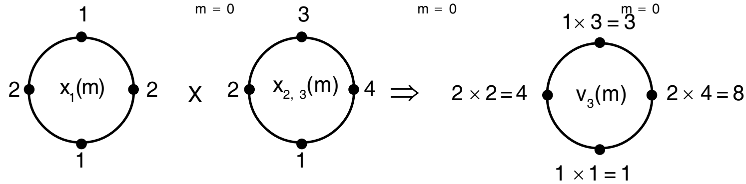


Fig 7 : Computation of $x_3(3)$.

The sum of samples of $v_3(m)$ gives $x_3(3)$

$\therefore\ x_3(3) = 8 + 3 + 4 + 1 = 16$

$$\therefore\ x_3(n) = \{\underset{\uparrow}{14}, 16, 14, 16\}$$

Method 2 : Circular Convolution Using Tabular Array

The index n in the given sequences are changed to m and then, the given sequences can be represented in the tabular array as shown below. Here the shifted sequences $x_{2,n}$ are periodically extended with a periodicity of $N = 4$. Let $x_3(n)$ be the sequence obtained by convolution of $x_1(n)$ and $x_2(n)$. Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N = \sum_{m=0}^{N-1} x_1(m)\, x_{2,m}(m), \text{ where } x_{2,m}(m) = x_2((n-m))_N$$

> ***Note :*** *The bold faced numbers are samples obtained by periodic extension.*

m	–3	–2	–1	0	1	2	3
$x_1(m)$				2	1	2	1
$x_2(m)$				1	2	3	4
$x_2(-m) = x_{2,0}(m)$	4	3	2	1	**4**	**3**	**2**
$x_2(1-m) = x_{2,1}(m)$		4	3	2	1	**4**	**3**
$x_2(2-m) = x_{2,2}(m)$			4	3	2	1	**4**
$x_3(3-m) = x_{2,3}(m)$				4	3	2	1

To determine a sample of $x_3(n)$ at $n = q$, multiply the sequence, $x_1(m)$ and $x_{2,q}(m)$, to get a product sequence $x_1(m)\, x_{2,q}(m)$. (i.e., multiply the corresponding elements of the row $x_1(m)$ and $x_{2,q}(m)$). The sum of all the samples of the product sequence gives $x_3(q)$.

$$\text{When } n = 0\ ;\ x_3(0) = \sum_{m=0}^{3} x_1(m)\, x_{2,0}(m)$$

$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$

$$= 2 + 4 + 6 + 2 = 14$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{3} x_1(m)\ x_{2,1}(m) = 4 + 1 + 8 + 3 = 16$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{3} x_1(m)\ x_{2,2}(m) = 6 + 2 + 2 + 4 = 14$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{3} x_1(m)\ x_{2,3}(m) = 8 + 3 + 4 + 1 = 16$$

$$\therefore\ x_3(n) = \{\underset{\uparrow}{14},\ 16,\ 14,\ 16\}$$

Method 3 : Circular Convolution Using Matrices

The sequence $x_1(n)$ can be arranged as a column vector of order $N \times 1$ and using the samples of $x_2(n)$ the $N \times N$ matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(3) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix}$$

$$\therefore\ x_3(n) = \{\underset{\uparrow}{14},\ 16,\ 14,\ 16\}$$

Example 6.25

Perform the circular convolution of the two sequences $x_1(n)$ and $x_2(n)$, where,

$$x_1(n) = \{\underset{\uparrow}{0.2},\ 0.4,\ 0.6,\ 0.8,\ 1.0,\ 1.2,\ 1.4,\ 1.6\}$$

$$x_2(n) = \{\underset{\uparrow}{0.1},\ 0.3,\ 0.5,\ 0.7,\ 0.9,\ 1.1,\ 1.3,\ 1.5\}$$

Solution

Let $x_3(n)$ be the result of the circular convolution of $x_1(n)$ and $x_2(n)$. The given sequences consists of eight samples. The $x_3(n)$ will also have 8 samples.

The sequences are represented in the tabular array as shown below after replacing n by m. The sequence $x_2(m)$ is folded and shifted.

The shifted sequences $x_{2,m}(m)$ are periodically extended with a periodicity of N = 8.

Note : *The bold faced numbers are samples obtained by periodic extension.*

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_1(m)$								0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$x_2(m)$								0.1	0.3	0.5	0.7	0.9	1.1	1.3	1.5
$x_2(-m) = x_{2,0}(m)$	1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**	**0.3**
$x_2(1-m) = x_{2,1}(m)$		1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**	**1.1**	**0.9**	**0.7**	**0.5**
$x_2(2-m) = x_{2,2}(m)$			1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**	**1.1**	**0.9**	**0.7**
$x_2(3-m) = x_{2,3}(m)$				1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**	**1.1**	**0.9**
$x_2(4-m) = x_{2,4}(m)$					1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**	**1.1**
$x_2(5-m) = x_{2,5}(m)$						1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**	**1.3**
$x_2(6-m) = x_{2,6}(m)$							1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1	**1.5**
$x_2(7-m) = x_{2,7}(m)$								1.5	1.3	1.1	0.9	0.7	0.5	0.3	0.1

Each sample of $x_3(n)$ is given by the equation,

$$x_3(n) = \sum_{m=0}^{7} x_1(m)\, x_2((n-m))_8 = \sum_{m=0}^{7} x_1(m)\, x_{2,n}(m)\ ; \quad \text{where } x_{2,n}(m) = x_2((n-m))_8$$

The samples of $x_3(0)$ are calculated as shown below.

$$\text{When } n = 0\ ;\ x_3(0) = \sum_{m=0}^{7} x_1(m)\ x_2(-m) = \sum_{m=0}^{7} x_1(m)\ x_{2,0}(m)$$

$$= x_1(0)\, x_{2,0}(0) + x_1(1)\, x_{2,0}(1) + x_1(2)\, x_{2,0}(2) + x_1(3)\, x_{2,0}(3)$$

$$+ x_1(4)\, x_{2,0}(4) + x_1(5)\, x_{2,0}(5) + x_1(6)\, x_{2,0}(6) + x_1(7)\, x_{2,0}(7)$$

$$= 0.02 + 0.6 + 0.78 + 0.88 + 0.9 + 0.84 + 0.7 + 0.48 = 5.20$$

The samples of $x_3(n)$ for other values of n are calculated as shown for n = 0.

$$\text{When } n = 1;\ x_3(1) = \sum_{m=0}^{7} x_1(m)\ x_2(1-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,1}(m) = 6.00$$

$$\text{When } n = 2;\ x_3(2) = \sum_{m=0}^{7} x_1(m)\ x_2(2-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,2}(m) = 6.48$$

$$\text{When } n = 3;\ x_3(3) = \sum_{m=0}^{7} x_1(m)\ x_2(3-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,3}(m) = 6.64$$

$$\text{When } n = 4;\ x_3(4) = \sum_{m=0}^{7} x_1(m)\, x_2(4-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,4}(m) = 6.48$$

$$\text{When } n = 5;\ x_3(5) = \sum_{m=0}^{7} x_1(m)\, x_2(5-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,5}(m) = 6.00$$

$$\text{When } n = 6;\ x_3(6) = \sum_{m=0}^{7} x_1(m)\, x_2(6-m) = \sum_{m=0}^{7} x_1(m)\ x_{2,6}(m) = 5.20$$

$$\text{When } n = 7;\ x_3(7) = \sum_{m=0}^{7} x_1(m)\, x_2(7-m) = \sum_{m=0}^{7} x_1(m)\, x_{2,7}(m) = 4.08$$

$$\therefore x_3(n) = \{\underset{\uparrow}{5.20},\ 6.00,\ 6.48,\ 6.64,\ 6.48,\ 6.00,\ 5.20,\ 4.08\}$$

Example 6.26

Find the linear and circular convolution of the sequences, $x(n) = \{\underset{\uparrow}{1},\ 0.5\}$ and $h(n) = \{\underset{\uparrow}{0.5},\ 1\}$.

Solution

Linear Convolution by Tabular Array

Let, $y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m)$; where m is a dummy variable for convolution.

Since both x(n) and h(n) starts at n = 0, the output sequence y(n) will also start at n = 0.

The length of y(n) is 2 + 2 – 1 = 3.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented in the tabular array as shown below.

Note : *The unfilled boxes in the table are considered as zeros.*

m	–1	0	1	2
x(m)		1	0.5	
h(m)		0.5	1	
$h(-m) = h_0(m)$	1	0.5		
$h(1-m) = h_1(m)$		1	0.5	
$h(2-m) = h_2(m)$			1	0.5

Each sample of y(n) is given by the relation,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\,h(n-m) = \sum_{m=-\infty}^{\infty} x(m)\,h_n(m) \quad ; \text{ where } h_n(m) = h(n-m)$$

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-\infty}^{\infty} x(m)\,h(-m) = \sum_{m=-1}^{1} x(m)\,h_0(m)$$

$$= x(-1)\,h_0(-1) + x(0)\,h_0(0) + x(1)\,h_0(1) = 0 + 0.5 + 0 = 0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-\infty}^{\infty} x(m)\,h(1-m) = \sum_{m=0}^{1} x(m)\,h_1(m) = 1 + 0.25 = 1.25$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-\infty}^{\infty} x(m)\,h(2-m) = \sum_{m=0}^{2} x(m)\,h_2(m) = 0 + 0.5 + 0 = 0.5$$

$$\therefore y(n) = \{\underset{\uparrow}{0.5},\ 1.25,\ 0.5\}$$

Circular Convolution by Tabular Array

Let, $y(n) = x(n) \circledast h(n) = \sum_{m=0}^{N-1} x(m)\,h((n-m))_N$; where m is a dummy variable for convolution.

The index n in the sequences are changed to m and the sequences are represented in the tabular array as shown below. The shifted sequence $h_n(m)$ is periodically extended with periodicity N = 2.

Note : *The bold faced number is the sample obtained by periodic extension.*

m	–1	0	1
x(m)		1	0.5
h(m)		0.5	1
$h(-m) = h_0(m)$	1	0.5	**1**
$h(1-m) = h_1(m)$		1	0.5

Each sample of y(n) is given by the equation,

$$y(n) = \sum_{m=0}^{N-1} x(m)\, h((n-m))_N = \sum_{m=0}^{N-1} x(m)\, h_n(m);\ \text{where } h_n(m) = h((n-m))_N$$

$$\text{When } n = 0\,;\ y(0) = \sum_{m=0}^{N-1} x(m)\, h(-m) = \sum_{m=0}^{1} x(m)\, h_0(m)$$

$$= x(0)\, h_0(0) + x(1)\, h_0(1) = 0.5 + 0.5 = 1.0$$

$$\text{When } n = 1\,;\ y(1) = \sum_{m=0}^{N-1} x(m)\, h(1-m) = \sum_{m=0}^{1} x(m)\, h_1(m)$$

$$= x(0)\, h_1(0) + x(1)\, h_1(1) = 1 + 0.25 = 1.25$$

$$\therefore y(n) = \{\underset{\uparrow}{1.0},\ 1.25\}$$

Example 6.27

The input x(n) and impulse response h(n) of a LTI system are given by,

$$x(n) = \{\underset{\uparrow}{-1}, 1, 2, -2\}\ ;\ h(n) = \{0.5, \underset{\uparrow}{1}, -1, 2, 0.75\}$$

Determine the response of the system a) Using linear convolution and b) using circular convolution.

Solution

a) Response of LTI system using linear convolution

Let y(n) be the response of LTI system. By convolution sum formula,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \quad ; \quad \text{where m is a dummy variable used for convolution.}$$

The sequence x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Also y(n) ends at n = 0 + (–1) + (4 + 5 –2) = 6.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented on the tabular array as shown below. Let us fold h(m) to get h(–m) and shift h(–m) to perform convolution operation.

Note : *The unfilled boxes in the table are considered as zeros.*

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)					–1	1	2	–2				
h(m)				0.5	1	–1	2	0.75				
h(–m)		0.75	2	–1	1	0.5						
$h(-1-m) = h_{-1}(m)$	0.75	2	–1	1	0.5							
$h(-m) = h_0(m)$		0.75	2	–1	1	0.5						
$h(1-m) = h_1(m)$			0.75	2	–1	1	0.5					
$h(2-m) = h_2(m)$				0.75	2	–1	1	0.5				
$h(3-m) = h_3(m)$					0.75	2	–1	1	0.5			
$h(4-m) = h_4(m)$						0.75	2	–1	1	0.5		
$h(5-m) = h_5(m)$							0.75	2	–1	1	0.5	
$h(6-m) = h_6(m)$								0.75	2	–1	1	0.5

Each sample of y(n) is given by summation of the product sequence, x(m) h(n – m). To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

$$\text{i.e., } y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h(m)$$

$$\text{When } n = -1\ ;\ y(-1) = \sum_{m=-4}^{3} x(m)\, h_{-1}(m)$$

$$= x(-4)\, h_{-1}(-4) + x(-3)\, h_{-1}(-3) + x(-2)\, h_{-1}(-2) + x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 0 + (-0.5) + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\ ;\ y(0) = \sum_{m=-3}^{3} x(m)\, h_0(m) = 0 + 0 + 0 + (-1) + 0.5 + 0 + 0 = -0.5$$

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=-2}^{3} x(m)\, h_1(m) = 0 + 0 + 1 + 1 + 1 + 0 = 3$$

$$\text{When } n = 2\ ;\ y(2) = \sum_{m=-1}^{3} x(m)\, h_2(m) = 0 + (-2) + (-1) + 2 + (-1) = -2$$

$$\text{When } n = 3\ ;\ y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = -0.75 + 2 + (-2) + (-2) + 0 = -2.75$$

$$\text{When } n = 4\ ;\ y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0.75 + 4 + 2 + 0 + 0 = 6.75$$

$$\text{When } n = 5\ ;\ y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$$

$$\text{When } n = 6\ ;\ y(6) = \sum_{m=0}^{7} x(m)\, h_6(m) = 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 + 0 = -1.5$$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

---

b) Response of LTI System Using Circular Convolution

The response of LTI system is given by linear convolution of x(m) and h(m). Let y(n) be the response sequence of LTI system. To get the result of linear convolution from circular convolution, both the sequences should be converted to the size of y(n) and perform circular convolution of the converted sequences. Also the converted sequences should start and end at the same value of n as that of y(n).

The length of x(n) is 4 and the length of h(n) is 5. Hence the length of y(n) is (4 + 5 – 1) = 8. Therefore both the sequences should be converted to 8-point sequences.

The x(n) starts at n = 0 and h(n) starts at n = –1. Hence y(n) will start at n = 0 + (–1) = –1. The y(n) will end at n = [0 +(–1)] + (4 + 5 – 2) = 6. Therefore the converted sequences should start at n = –1 and end at n = 6.

$$\therefore\ x(n) = \{0, \underset{\uparrow}{-1}, 1, 2, -2, 0, 0, 0\} \text{ and } h(n) = \{0.5, \underset{\uparrow}{1}, -1, 2, 0.75, 0, 0, 0\}$$

The converted sequences x(n) and h(n) are represented on the tabular array after replacing the index n by m as shown below. The sequence h(m) is folded and shifted.

The shifted sequences $h_n(m)$ are periodically extended with a periodicity of N = 8.

***Note :** The bold faced numbers are samples obtained by periodic extension of the sequences.*

m	–7	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	7
x(m)							0	–1	1	2	–2	0	0	0	
h(m)							0.5	1	–1	2	0.75	0	0	0	
h(–m)		0	0	0	0.75	2	–1	1	0.5						
$h(-1-m) = h_{-1}(m)$	0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**	**1**
$h(-m) = h_0(m)$		0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**	**–1**
$h(1-m) = h_1(m)$			0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**	**2**
$h(2-m) = h_2(m)$				0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**	**0.75**
$h(3-m) = h_3(m)$					0	0	0	0.75	2	–1	1	0.5	**0**	**0**	**0**
$h(4-m) = h_4(m)$						0	0	0	0.75	2	–1	1	0.5	**0**	**0**
$h(5-m) = h_5(m)$							0	0	0	0.75	2	–1	1	0.5	**0**
$h(6-m) = h_6(m)$	**0**	**0**	**0.75**	**2**	**–1**	**1**	**0.5**	0	0	0	0.75	2	–1	1	0.5

Let y(n) be the sequence obtained by circular convolution of x(n) and h(n). Each sample of y(n) is given by,

$$y(n) = \sum_{m=-1}^{6} x(m)\, h((n-m))_8 = \sum_{m=-1}^{6} x(m)\, h_n(m) \quad ; \quad \text{where } h_n(m) = h((n-m))_8$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence x(m) $h_q(m)$, (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

$$\text{When } n = -1\,;\; y(-1) = \sum_{m=-1}^{6} x(m)\, h_{-1}(n) = x(-1)\, h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2)$$

$$+\, x(3)\, h_{-1}(3) + x(4)\, h_{-1}(4) + x(5)\, h_{-1}(5) + x(6)\, h_{-1}(6)$$

$$= 0 + (-0.5) + 0 + 0 + 0 + 0 + 0 + 0 = -0.5$$

The samples of y(n) for other values of n are calculated as shown for n = –1.

$$\text{When } n = 0\,;\; y(0) = \sum_{m=-1}^{6} x(m)\, h_0 m = 0 + (-1) + 0.5 + 0 + 0 + 0 + 0 + 0 = -0.5$$

$$\text{When } n = 1\,;\; y(1) = \sum_{m=-1}^{6} x(m)\, h_1 m = 0 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 3$$

$$\text{When } n = 2\,;\; y(2) = \sum_{m=-1}^{6} x(m)\, h_2 m = 0 + (-2) + (-1) + 2 + (-1) + 0 + 0 + 0 = -2$$

$$\text{When } n = 3\,;\; y(3) = \sum_{m=-1}^{6} x(m)\, h_3 m = 0 + (-0.75) + 2 + (-2) + (-2) + 0 + 0 + 0 = -2.75$$

$$\text{When } n = 4\,;\; y(4) = \sum_{m=-1}^{6} x(m)\, h_4 m = 0 + 0 + 0.75 + 4 + 2 + 0 + 0 + 0 = 6.75$$

$$\text{When } n = 5\,;\; y(5) = \sum_{m=-1}^{6} x(m)\, h_5 m = 0 + 0 + 0 + 1.5 + (-4) + 0 + 0 + 0 = -2.5$$

$$\text{When } n = 6\,;\; y(6) = \sum_{m=-1}^{6} x(m)\, h_6 m = 0 + 0 + 0 + 0 + (-1.5) + 0 + 0 + 0 = -1.5$$

The response of LTI system y(n) is,

$$y(n) = \{-0.5, \underset{\uparrow}{-0.5}, 3, -2, -2.75, 6.75, -2.5, -1.5\}$$

***Note :** 1. Since circular convolution is periodic, the convolution is performed for any one period.*

2. It can be observed that the results of both the methods are same.

6.11 Sectioned Convolution

The response of an LTI system for any arbitrary input is given by linear convolution of the input and the impulse response of the system. If one of the sequences (either the input sequence or impulse response sequence) is very much larger than the other, then it is very difficult to compute the linear convolution for the following reasons.

1. The entire sequence should be available before convolution can be carried out. This makes long delay in getting the output.
2. Large amounts of memory is required to store the sequences.

The above problems can be overcome in the sectioned convolutions. In this technique the larger sequence is sectioned (or splitted) into the size of smaller sequence. Then the linear convolution of each section of longer sequence and the smaller sequence is performed. The output sequences obtained from the convolutions of all the sections are combined to get the overall output sequence. There are two methods of sectioned convolutions. They are overlap add method and overlap save method.

6.11.1 Overlap Add Method

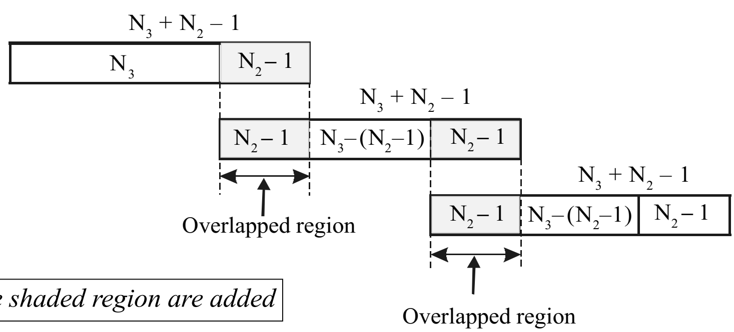
In ***overlap add method*** the longer sequence is divided into smaller sequences. Then linear convolution of each section of longer sequence and smaller sequence is performed. The overall output sequence is obtained by combining the output of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

Note : *Normally the longer sequence is divided into sections of size same as that of smaller sequence.*



Note : *Samples in the shaded region are added*

Fig 6.30 : *Overlapping of output sequence of sectioned convolution by overlap add method.*

The linear convolution of each section with smaller sequence will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this method the last $N_2 - 1$ samples of each output sequence overlaps with the first $N_2 - 1$ samples of next section. (i.e., there will be a region of $N_2 - 1$ samples over which the output sequence of q^{th} convolution overlaps the output sequence of $(q+1)^{th}$ convolution). While combining the output sequences of the various sectioned convolutions, the corresponding samples of overlapped regions are added and the samples of non-overlapped regions are retained as such.

6.11.2 Overlap Save Method

In overlap save method the results of linear convolution of the various sections are obtained using circular convolution. In this method the longer sequence is divided into smaller sequences. Each section of the longer sequence and the smaller sequence are converted to the size of the output sequence of sectioned convolution. The circular convolution of each section of the longer sequence and the smaller sequence is performed. The overall output sequence is obtained by combining the outputs of the sectioned convolution.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

Let the longer sequence be divided into sections of size N_3 samples.

***Note :** Normally the longer sequence is divided into sections of size same as that of smaller sequence.*

In ***overlap save method*** the results of linear convolution are obtained by circular convolution. Hence each section of longer sequence and the smaller sequence are converted to the size of output sequence of size $N_3 + N_2 - 1$ samples.The smaller sequence is converted to size of $N_3 + N_2 - 1$ samples, by appending with zeros.The convertion of each section of longer sequence to the size $N_3 + N_2 - 1$ samples can be performed in two different methods.

Method-1

In this method the first $N_2 - 1$ samples of a section is appended as last $N_2 - 1$ samples of the previous section (i.e., the overlapping samples are placed at the beginning of the section). The circular convolution of each section will produce an output sequence of size $N_3 + N_2 - 1$ samples. In this output the first $N_2 - 1$ samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

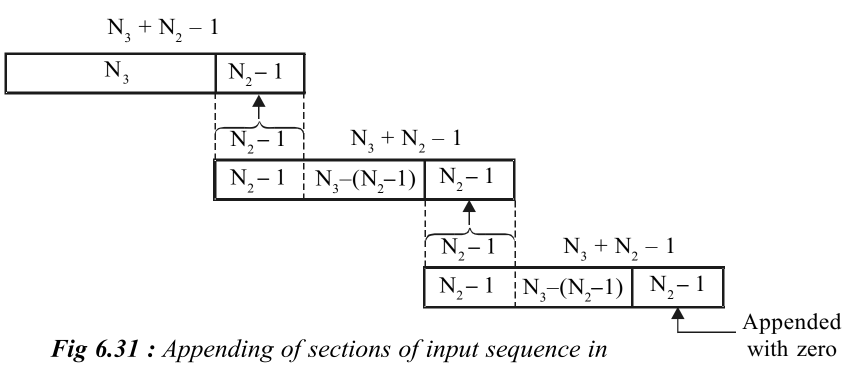
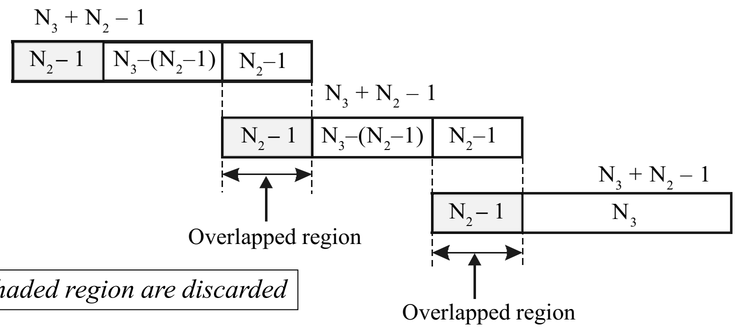


***Fig 6.31 :** Appending of sections of input sequence in method-1 of overlap save method.*



Note : Samples in the shaded region are discarded

Fig 6.32 : *Overlapping of output sequence of sectioned convolution by method-1 of overlap save method.*

Method-2

In this method the last N_2–1 samples of a section is appended as last N_2 –1 samples of the next section (i.e, the overlapping samples are placed at the end of the sections). The circular convolution of each section will produce an output sequence of size $N_3 + N_2$ –1 samples. In this output the last N_2 –1 samples are discarded and the remaining samples of the output of sectioned convolutions are saved as the overall output sequence.

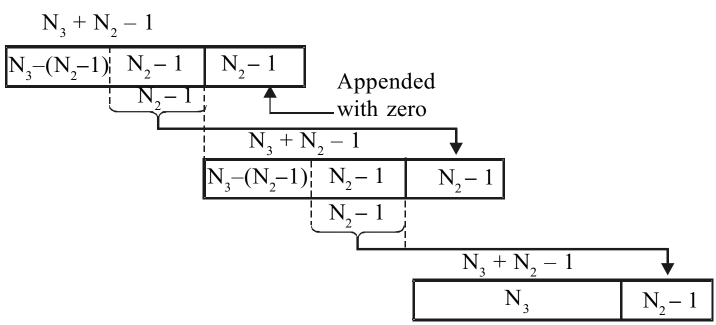
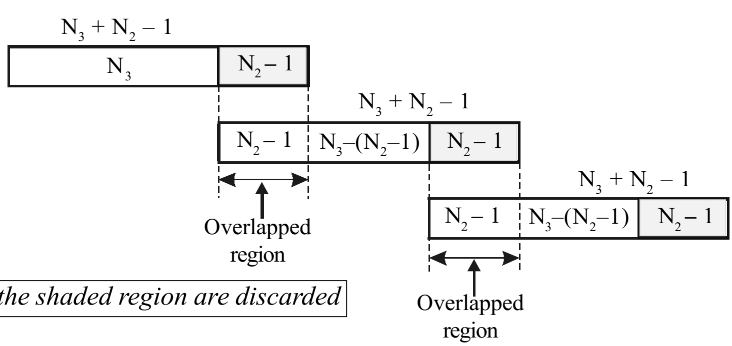


Fig 6.33 : *Appending of sections of input sequence in method-2 of overlap save method.*



Note : Samples in the shaded region are discarded

Fig 6.34 : *Overlapping of output sequence of sectioned convolution by method-2 of overlap save method.*

Example 6.28

Perform the linear convolution of the following sequences by a) Overlap add method and b) Overlap save method.

$x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$; $h(n) = \{-1, 1\}$

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n).

Given that, $x(n) = \{1, -1, 2, -2, 3, -3, 4, -4\}$

Let x(n) can be sectioned into four sequences, each consisting of two samples of x(n) as shown below.

$x_1(n) = 1$; $n = 0$	$x_2(n) = 2$; $n = 2$	$x_3(n) = 3$; $n = 4$	$x_4(n) = 4$; $n = 6$
$= -1$; $n = 1$	$= -2$; $n = 3$	$= -3$; $n = 5$	$= -4$; $n = 7$

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n) respectively.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\quad\therefore\ y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\quad\therefore\ y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\quad\therefore\ y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\quad\therefore\ y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences each consisting of 2 samples. Hence each convolution output will consists of $2 + 2 - 1 = 3$ samples. The convolution of each section is performed by tabular method as shown below.

> ***Note :***
>
> *1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\quad\therefore (N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$*
>
> *2. The unfilled boxes in the tables are considered as zero.*
>
> *3. For convenience of convolution operation the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and h(n).*

Convolution of Section - 1

m	–1	0	1	2
$x_1(m)$		1	–1	
h(m)		–1	1	
$h(-m) = h_o(m)$	1	–1		
$h(1-m) = h_1(m)$		1	–1	
$h(2-m) = h_2(m)$			1	–1

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, h_n(m)\ ;\ n = 0, 1, 2$$

where $h_n(m) = h(n-m)$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = 0 - 1 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section - 2

m	–1	0	1	2	3	4
$x_2(m)$				2	–2	
h(m)		–1	1			
h(–m)	1	–1				
$h(2-m) = h_2(m)$			1	–1		
$h(3-m) = h_3(m)$				1	–1	
$h(4-m) = h_4(m)$					1	–1

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_2(m)\, h_n(m) \quad ; n = 2, 3, 4$$

where $h_n(m) = h(n - m)$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = 0 - 2 + 0 = -2$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 0 = -2$

Convolution of Section - 3

m	–1	0	1	2	3	4	5	6
$x_3(m)$						3	–3	
h(m)		–1	1					
h(–m)	1	–1						
$h(4-m) = h_4(m)$					1	–1		
$h(5-m) = h_5(m)$						1	–1	
$h(6-m) = h_6(m)$							1	–1

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_3(m)\, h_n(m) \quad ; n = 4, 5, 6$$

where $h_n(m) = h(n-m)$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = 0 - 3 + 0 = -3$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 0 = -3$

Convolution of Section - 4

m	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$								4	–4	
h(m)		–1	1							
h(–m)	1	–1								
$h(6-m) = h_6(m)$							1	–1		
$h(7-m) = h_7(m)$								1	–1	
$h(8-m) = h_8(m)$									1	–1

$$y_4(n) = x_4(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_4(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x_4(m)\, h_n(m) \quad ; n = 6, 7, 8$$

where $h_n(m) = h(n-m)$

When $n = 6$; $y_4(6) = \sum x_4(m)\, h_6(m) = 0 - 4 + 0 = -4$

When $n = 7$; $y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 = 8$

When $n = 8$; $y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 0 = -4$

To Combine the Output of Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In this method the overall output is obtained by combining the outputs of the convolution of all sections. The overlapped portions (or samples) are added while combining the output .

The output of all sections can be represented in a table as shown below. Then the samples corresponding to same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	–1	2	–1						
$y_2(n)$			–2	4	–2				
$y_3(n)$					–3	6	–3		
$y_4(n)$							–4	8	–4
$y_5(n)$	–1	2	–3	4	–5	6	–7	8	–4

$\therefore$ y(n) = x(n) $*$ h(n) = {–1, 2, –3, 4, –5, 6, –7, 8, –4}

b) Overlap Save Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of longer sequence is converted to the size of output sequence using the samples of original longer sequence.

The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n). Given that, x(n) = {1, –1, 2, –2, 3, –3, 4, –4}

Let x(n) be sectioned into four sequences each consisting of two samples of x(n) as shown below.

$x_1(n)$ = 1 ; n = 0	$x_2(n)$ = 2 ; n = 2	$x_3(n)$ = 3 ; n = 4	$x_4(n)$ = 4 ; n = 6
= –1 ; n = 1	= –2 ; n = 3	= –3 ; n = 5	= –4 ; n = 7

Let $y_1(n)$, $y_2(n)$, $y_3(n)$ and $y_4(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$, $x_3(n)$ and $x_4(n)$ with h(n) respectively. Here linear convolution of each section will result in an output sequence consisting of 2 + 2 – 1 = 3 samples.

The sequence h(n) is converted to 3-sample sequence by appending with zero.

$\therefore$ h(n) = {–1, 1, 0}

Method - 1

In method - 1 the overlapping samples are placed at the beginning of the sections. Each section of longer sequence is converted to 3-sample sequence using the samples of original longer sequence as shown below. It can be observed that the first sample of $x_2(n)$ is placed as overlapping sample at the end of $x_1(n)$. The first sample of $x_3(n)$ is placed as overlapping sample at the end of $x_2(n)$. The first sample of $x_4(n)$ is placed as overlapping sample at the end of $x_3(n)$. Since there is no fifth section, the overlapping sample of $x_4(n)$ is taken as zero.

$x_1(n)$ = 1 ; n = 0	$x_2(n)$ = 2 ; n = 2	$x_3(n)$ = 3 ; n = 4	$x_4(n)$ = 4 ; n = 6
= –1 ; n = 1	= –2 ; n = 3	= –3 ; n = 5	= –4 ; n = 7
= 2 ; n = 2	= 3 ; n = 4	= 4 ; n = 6	= 0 ; n = 8

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at n = n_h = 0

$x_1(n)$ starts at n = n_1 = 0, $\therefore$ $y_1(n)$ will start at n = $n_1 + n_h$ = 0 + 0 = 0

$x_2(n)$ starts at n = n_2 = 2, $\therefore$ $y_2(n)$ will start at n = $n_2 + n_h$ = 2 + 0 = 2

$x_3(n)$ starts at n = n_3 = 4, $\therefore$ $y_3(n)$ will start at n = $n_3 + n_h$ = 4 + 0 = 4

$x_4(n)$ starts at n = n_4 = 6, $\therefore$ $y_4(n)$ will start at n = $n_4 + n_h$ = 6 + 0 = 6

Note : *1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$*

2. The bold faced numbers in the tables are obtained by periodic extension.

3. For convenience of convolution operation, the index n in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and h(n) are replaced by m.

Convolution of Section - 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	2
$h(m)$			–1	1	0
$h(-m) = h_0(m)$	0	1	–1	**0**	**1**
$h(1-m) = h_1(m)$		0	1	–1	**0**
$h(2-m) = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m) \; ; \quad n = 0, 1, 2,$$

$$\text{where } h_n(m) = h(n - m)$$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 2 = 1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 - 2 = -3$

Convolution of Section - 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	3
$h(m)$			–1	1	0		
$h(-m)$	0	1	–1				
$h(2-m) = h_2(m)$			0	1	–1	**0**	**1**
$h(3-m) = h_3(m)$				0	1	–1	**0**
$h(4-m) = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h((n-m))_N$$

$$= \sum_{m=2}^{4} x_2(m)\, h_n(m) \; ; \quad n = 2, 3, 4$$

$$\text{where } h_n(m) = h((n-m))_N$$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = -2 + 0 + 3 = 1$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 + 0 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 + -2 - 3 = -5$

Convolution of Section - 3

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	4
$h(m)$			–1	1	0				
$h(-m)$	0	1	–1						
$h(4-m) = h_4(m)$					0	1	–1	**0**	**1**
$h(5-m) = h_5(m)$						0	1	–1	**0**
$h(6-m) = h_6(m)$							0	1	–1

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=4}^{6} x_3(m)\, h_n(m) \; ; \quad n = 4, 5, 6$$

$$\text{where } h_n(m) = h((n-m))_N$$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = -3 + 0 + 4 = 1$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 + 0 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 - 4 = -7$

Convolution of section - 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	0
$h(m)$			–1	1	0						
$h(-m)$	0	1	–1								
$h(6-m) = h_6(m)$							0	1	–1	**0**	**1**
$h(7-m) = h_7(m)$								0	1	–1	**0**
$h(8-m) = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_4(m)\, h((n-m))_N = \sum_{m=6}^{8} x_4(m)\, h_n(m)\ ;\ n = 6,\ 7,\ 8$$

where $h_n(m) = h((n-m))_N$

When $n = 6\ ;\ y_4(6) = \sum x_4(m)\, h_6(m) = -4 + 0 + 0 = -4$

When $n = 7\ ;\ y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 + 0 = 8$

When $n = 8\ ;\ y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 0 = -4$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs, the overlapped first sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	~~1~~	2	–3						
$y_2(n)$			~~1~~	4	–5				
$y_3(n)$					~~1~~	6	–7		
$y_4(n)$							~~–4~~	8	–4
$y(n)$	∗	2	–3	4	–5	6	–7	8	–4

$y(n) = x(n) * h(n) = \{*, 2, -3, -4, -5, 6, -7, 8, -4\}$

> ***Note :*** *Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same, except the first $N_2 - 1$ samples.*

Method - 2

In method - 2 the overlapping samples are placed at the end of the section.Each section of longer sequence is converted to 3-sample sequence, using the samples of original longer sequence as shown below. It can be observed that the last sample of $x_1(n)$ is placed as overlapping sample at the end of $x_2(n)$. The last sample of $x_2(n)$ is placed as overlapping sample at the end of $x_3(n)$. The last sample of $x_3(n)$ is placed as overlapping sample at the end of $x_4(n)$. Since there is no previous section for $x_1(n)$, the overlapping sample of $x_1(n)$ is taken as zero.

$x_1(n) = 1\ ;\ n = 0$	$x_2(n) = 2\ ;\ n = 2$	$x_3(n) = 3\ ;\ n = 4$	$x_4(n) = 4\ ;\ n = 6$
$= -1\ ;\ n = 1$	$= -2\ ;\ n = 3$	$= -3\ ;\ n = 5$	$= -4\ ;\ n = 7$
$= 0\ ;\ n = 2$	$= -1\ ;\ n = 4$	$= -2\ ;\ n = 6$	$= -3\ ;\ n = 8$

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have three samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 2$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 2 + 0 = 2$

$x_3(n)$ starts at $n = n_3 = 4$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 4 + 0 = 4$

$x_4(n)$ starts at $n = n_4 = 6$, $\therefore$ $y_4(n)$ will start at $n = n_4 + n_h = 6 + 0 = 6$

> ***Note :*** *1. Here $N_1 = 8$, $N_2 = 2$, $N_3 = 2$. $\therefore$ $(N_2 - 1) = 2 - 1 = 1$ and $(N_2 + N_3 - 1) = 2 + 2 - 1 = 3$*
>
> *2. The bold faced numbers in the tables are obtained by periodic extension.*
>
> *3. For convenience of convolution the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$, $x_4(n)$ and $h(n)$.*

Convolution of Section - 1

m	–2	–1	0	1	2
$x_1(m)$			1	–1	0
h(m)			–1	1	0
$h(-m) = h_0(m)$	0	1	–1	**0**	**1**
$h(1-m) = h_1(m)$		0	1	–1	**0**
$h(2-m) = h_2(m)$			0	1	–1

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_1(m)\, h((n-m))_N$$

$$= \sum_{m=0}^{2} x_1(m)\, h_n(m) \; ; \quad n = 0, 1, 2$$

where $h_n(m) = h((n-m))_N$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_0(m) = -1 + 0 + 0 = -1$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 1 + 1 + 0 = 2$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = 0 - 1 + 0 = -1$

Convolution of Section - 2

m	–2	–1	0	1	2	3	4
$x_2(m)$					2	–2	–1
h(m)			–1	1	0		
h(–m)	0	1	–1				
$h(2-m) = h_2(m)$			0	1	–1	**0**	**1**
$h(3-m) = h_3(m)$				0	1	–1	**0**
$h(4-m) = h_4(m)$					0	1	–1

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_2(m)\, h((n-m))_N = \sum_{m=2}^{4} x_2(m)\, h_n(m); \; n = 2, 3, 4,$$

where $h_n(m) = h((n-m))_N$

When $n = 2$; $y_2(2) = \sum x_2(m)\, h_2(m) = -2 + 0 - 1 = -3$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 2 + 2 + 0 = 4$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 2 + 1 = -1$

Convolution of Section - 3

m	–2	–1	0	1	2	3	4	5	6
$x_3(m)$							3	–3	–2
h(m)			–1	1	0				
h(–m)	0	1	–1						
$h(4-m) = h_4(m)$					0	1	–1	**0**	**1**
$h(5-m) = h_5(m)$						0	1	–1	**0**
$h(6-m) = h_6(m)$							0	1	–1

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=4}^{6} x_3(m)\, h_n(m)\ ; \quad n = 4, 5, 6$$

where $h_n(m) = h((n-m))_N$

When $n = 4$; $y_3(4) = \sum x_3(m)\, h_4(m) = -3 + 0 - 2 = -5$

When $n = 5$; $y_3(5) = \sum x_3(m)\, h_5(m) = 3 + 3 + 0 = 6$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 - 3 + 2 = -1$

Convolution of Section - 4

m	–2	–1	0	1	2	3	4	5	6	7	8
$x_4(m)$									4	–4	–3
$h(m)$			–1	1	0						
$h(-m)$	0	1	–1								
$h(6-m) = h_6(m)$							0	1	–1	**0**	**1**
$h(7-m) = h_7(m)$								0	1	–1	**0**
$h(8-m) = h_8(m)$									0	1	–1

$$y_4(n) = x_4(n) \circledast h(n) = \sum_{m = m_i}^{m_f} x_4(m)\, h((n-m))_N = \sum_{m=6}^{8} x_4(m)\, h_n(m)\ ; \ n = 6, 7, 8$$

where $h_n(m) = h((n-m))_N$

When $n = 6$; $y_4(6) = \sum x_4(m)\, h_6(m) = -4 + 0 - 3 = -7$

When $n = 7$; $y_4(7) = \sum x_4(m)\, h_7(m) = 4 + 4 + 0 = 8$

When $n = 8$; $y_4(8) = \sum x_4(m)\, h_8(m) = 0 - 4 + 3 = -1$

To Combine the Output of the Convolution of Each Section

It can be observed that the last sample in an output sequence overlaps with the first sample of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs the overlapped last sample of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8
$y_1(n)$	–1	2	~~–1~~						
$y_2(n)$			–3	4	~~–1~~				
$y_3(n)$					–5	6	~~–1~~		
$y_4(n)$							–7	8	–1
$y(n)$	–1	2	–3	4	–5	6	–7	8	*

Note :

Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the last N_2–1 samples.

$\therefore y(n) = x(n) * h(n) = \{-1, 2, -3, 4, -5, 6, -7, 8, *\}$

Example 6.29

Perform the linear convolution of the following sequences by a) Overlap add method and b) Overlap save method.

$x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\}$ and $h(n) = \{2, 1, -1\}$

Solution

a) Overlap Add Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n).

Given that $x(n) = \{1, 2, 3, -1, -2, -3, 4, 5, 6\}$. Let x(n) can be sectioned into three sequences, each consisting of three samples of x(n) as shown below.

$x_1(n) = 1 \; ; n = 0$ | $x_2(n) = -1 \; ; n = 3$ | $x_3(n) = 4 \; ; n = 6$

$= 2 \; ; n = 1$ | $= -2 \; ; n = 4$ | $= 5 \; ; n = 7$

$= 3 \; ; n = 2$ | $= -3 \; ; n = 5$ | $= 6 \; ; n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with $h(n)$ respectively.

Here $h(n)$ starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, $\therefore$ $y_2(n)$ will start at $n = n_3 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, $\therefore$ $y_3(n)$ will start at $n = n_6 + n_h = 6 + 0 = 6$

Here linear convolution of each section is performed between two sequences each consisting of three samples. Hence each convolution output will consists of 3 + 3 – 1 = 5 samples. The convolution of each section is performed by tabular method as shown below.

> ***Note :*** *1. Here $N_1 = 9$, $N_2 = 3$, $N_3 = 3$, $\therefore$ $(N_2 - 1) = 3 - 1 = 2$ and $(N_2 + N_3 - 1) = 3 + 3 - 1 = 5$.*
>
> *2. The unfilled boxes in the table are considered as zero.*
>
> *3. For convenience of convolution operation, the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and $h(n)$.*

Convolution of Section -1

m	–2	–1	0	1	2	3	4
$x_1(m)$			1	2	3		
$h(m)$			2	1	–1		
$h(-m) = h_0(m)$	–1	1	2				
$h(1-m) = h_1(m)$		–1	1	2			
$h(2-m) = h_2(m)$			–1	1	2		
$h(3-m) = h_3(m)$				–1	1	2	
$h(4-m) = h_4(m)$					–1	1	2

$$y_1(n) = x_1(n) * h(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, h(n-m)$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, h_n(m)$$

for $n = 0, 1, 2, 3, 4$

where $h_n(m) = h(n-m)$

When $n = 0$; $y_1(0) = \sum x_1(m)\, h_o(m) = 0 + 0 + 2 + 0 + 0 = 2$

When $n = 1$; $y_1(1) = \sum x_1(m)\, h_1(m) = 0 + 1 + 4 + 0 = 5$

When $n = 2$; $y_1(2) = \sum x_1(m)\, h_2(m) = -1 + 2 + 6 = 7$

When $n = 3$; $y_1(3) = \sum x_1(m)\, h_3(m) = 0 - 2 + 3 + 0 = 1$

When $n = 4$; $y_1(4) = \sum x_1(m)\, h_4(m) = 0 + 0 - 3 + 0 + 0 = -3$

Convolution of Section - 2

m	–2	–1	0	1	2	3	4	5	6	7
$x_2(m)$						–1	–2	–3		
$h(m)$			2	1	–1					
$h(-m) = h_0(m)$	–1	1	2							
$h(3-m) = h_3(m)$				–1	1	2				
$h(4-m) = h_4(m)$					–1	1	2			
$h(5-m) = h_5(m)$						–1	1	2		
$h(6-m) = h_6(m)$							–1	1	2	
$h(7-m) = h_7(m)$								–1	1	2

$$y_2(n) = x_2(n) * h(n) = \sum_{m=-\infty}^{\infty} x_2(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x_2(m)\, h_n(m);\ n = 3, 4, 5, 6, 7$$

where $h_n(m) = h(n-m)$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = 0 + 0 - 2 + 0 + 0 = -2$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = 0 - 1 - 4 + 0 = -5$

When $n = 5$; $y_2(5) = \sum x_2(m)\, h_5(m) = 1 - 2 - 6 = -7$

When $n = 6$; $y_2(6) = \sum x_2(m)\, h_6(m) = 0 + 2 - 3 + 0 = -1$

When $n = 7$; $y_2(7) = \sum x_2(m)\, h_7(m) = 0 + 0 + 3 + 0 + 0 = 3$

Convolution of Section - 3

m	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$									4	5	6		
$h(m)$			2	1	–1								
$h(-m) = h_0(m)$	–1	1	2										
$h(6-m) = h_6(m)$							–1	1	2				
$h(7-m) = h_7(m)$								–1	1	2			
$h(8-m) = h_8(m)$									–1	1	2		
$h(9-m) = h_9(m)$										–1	1	2	
$h(10-m) = h_{10}(m)$											–1	1	2

$$y_3(n) = x_3(n) * h(n) = \sum_{m=-\infty}^{\infty} x_3(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x_3(m)\, h_n(m)\ ;\quad n = 6, 7, 8, 9, 10$$

where $h_n(m) = h(n-m)$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 0 + 0 + 8 + 0 + 0 = 8$

When $n = 7$; $y_3(7) = \sum x_3(m)\, h_7(m) = 0 + 4 + 10 + 0 = 14$

When $n = 8$; $y_3(8) = \sum x_3(m)\, h_8(m) = -4 + 5 + 12 = 13$

When $n = 9$; $y_3(9) = \sum x_3(m)\, h_9(m) = 0 - 5 + 6 + 0 = 1$

When $n = 10$; $y_3(10) = \sum x_3(m)\, h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

To Combine the Output of the Convolution of Each Section

It can be observed that the last $N_2 - 1$ sample in an output sequence overlaps with the first $N_2 - 1$ sample of next output sequence. In this method the overall output is obtained by combining the outputs of the convolution of all sections. The overlapped portions (or samples) are added while combining the output.

The output of all sections can be represented in a table as shown below. Then the samples corresponding to same value of n are added to get the overall output.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	2	5	7	1	–3						
$y_2(n)$				–2	–5	–7	–1	3			
$y_3(n)$							8	14	13	1	– 6
$y(n)$	2	5	7	–1	–8	–7	7	17	13	1	– 6

$\therefore y(n) = x(n) * h(n) = \{2, 5, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

b) Overlap Save Method

In this method the longer sequence is sectioned into sequences of size equal to smaller sequence. The number of samples that will be obtained in the output of linear convolution of each section is determined. Then each section of longer sequence is converted to the size of output sequence using the samples of original longer sequences. The smaller sequence is also converted to the size of output sequence by appending with zeros. Then the circular convolution of each section is performed.

Here x(n) is a longer sequence when compared to h(n). Hence x(n) is sectioned into sequences of size equal to h(n). Given that x(n) = {1, 2, 3, –1, –2, –3, 4, 5, 6}.

Let x(n) be sectioned into three sequences each consisting of three samples as shown below.

Let, N_1 = Length of longer sequence

N_2 = Length of smaller sequence

$N_3 = N_2$ = Length of each section of longer sequence.

$x_1(n) = 1 \; ; n = 0$	$x_2(n) = -1 \; ; n = 3$	$x_3(n) = 4 \; ; n = 6$
$= 2 \; ; n = 1$	$= -2 \; ; n = 4$	$= 5 \; ; n = 7$
$= 3 \; ; n = 2$	$= -3 \; ; n = 5$	$= 6 \; ; n = 8$

Let $y_1(n)$, $y_2(n)$ and $y_3(n)$ be the output of linear convolution of $x_1(n)$, $x_2(n)$ and $x_3(n)$ with h(n) respectively. Here linear convolution of each section will result in an output sequence consisting of 3 + 3 – 1 = 5 samples.

Hence each section of longer sequence is converted to five sample sequence, using the samples of original longer sequence as shown below. It can be observed that the first $N_2 - 1$ samples of $x_2(n)$ is placed as overlapping sample at the end of $x_1(n)$. The first $N_2 - 1$ samples of $x_3(n)$ is placed as overlapping sample at the end of $x_2(n)$. Since there is no fourth section, the overlapping samples of $x_3(n)$ are considered as zeros.

$x_1(n) = 1 \; ; n = 0$	$x_2(n) = -1 \; ; n = 3$	$x_3(n) = 4 \; ; n = 6$
$= 2 \; ; n = 1$	$= -2 \; ; n = 4$	$= 5 \; ; n = 7$
$= 3 \; ; n = 2$	$= -3 \; ; n = 5$	$= 6 \; ; n = 8$
$= -1 \; ; n = 3$	$= 4 \; ; n = 6$	$= 0 \; ; n = 9$
$= -2 \; ; n = 4$	$= 5 \; ; n = 7$	$= 0 \; ; n = 10$

The sequence h(n) is also converted to five sample sequence by appending with zeros.

$\therefore$ h(n) = {2, 1, –1, 0, 0}

Now perform circular convolution of each section with h(n). The output sequence obtained from circular convolution will have five samples. The circular convolution of each section is performed by tabular method as shown below.

Here h(n) starts at $n = n_h = 0$

$x_1(n)$ starts at $n = n_1 = 0$, $\therefore$ $y_1(n)$ will start at $n = n_1 + n_h = 0 + 0 = 0$

$x_2(n)$ starts at $n = n_2 = 3$, $\therefore$ $y_2(n)$ will start at $n = n_2 + n_h = 3 + 0 = 3$

$x_3(n)$ starts at $n = n_3 = 6$, $\therefore$ $y_3(n)$ will start at $n = n_3 + n_h = 6 + 0 = 6$

> ***Note :*** *1. Here $N_1 = 9$, $N_2 = 3$, $N_3 = 3$ $\quad \therefore (N_2 - 1) = 3 - 1 = 2$ and $[N_2 + N_3 - 1] = 3 + 3 - 1 = 5$ samples.*
>
> *2. The bold faced numbers in the table are obtained by periodic extension.*
>
> *3. For convenience of convolution operation the index n is replaced by m in $x_1(n)$, $x_2(n)$, $x_3(n)$ and h(n).*

Convolution of Section - 1

m	–4	–3	–2	–1	0	1	2	3	4
$x_1(m)$					1	2	3	–1	–2
h(m)					2	1	–1	0	0
$h(-m) = h_0(m)$	0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h(1-m) = h_1(m)$		0	0	–1	1	2	**0**	**0**	**–1**
$h(2-m) = h_2(m)$			0	0	–1	1	2	**0**	**0**
$h(3-m) = h_3(m)$				0	0	–1	1	2	**0**
$h(4-m) = h_4(m)$					0	0	–1	1	2

$$y_1(n) = x_1(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_1(m)\, h((n-m))_N = \sum_{m=0}^{4} x_1(m)\, h_n(m);\ n = 0, 1, 2, 3, 4$$

where $h_n(m) = h((n-m))_N$

When $n = 0$; $y_1(0) = \sum x_1(m)h_o(m) = 2 + 0 + 0 + 1 - 2 = 1$

When $n = 1$; $y_1(0) = \sum x_1(m)h_1(m) = 1 + 4 + 0 + 0 + 2 = 7$

When $n = 2$; $y_1(2) = \sum x_1(m)h_2(m) = -1 + 2 + 6 + 0 + 0 = 7$

When $n = 3$; $y_1(3) = \sum x_1(m)h_3(m) = 0 - 2 + 3 - 2 + 0 = -1$

When $n = 4$; $y_1(4) = \sum x_1(m)h_4(m) = 0 + 0 - 3 - 1 - 4 = -8$

Convolution of Section - 2

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7
$x_2(m)$								–1	–2	–3	4	5
$h(m)$					2	1	–1	0	0			
$h(-m) = h_0(m)$	0	0	–1	1	2							
$h(3-m) = h_3(m)$				0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h(4-m) = h_4(m)$					0	0	–1	1	2	**0**	**0**	**–1**
$h(5-m) = h_5(m)$						0	0	–1	1	2	**0**	**0**
$h(6-m) = h_6(m)$							0	0	–1	1	2	**0**
$h(7-m) = h_7(m)$								0	0	–1	1	2

$$y_2(n) = x_2(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_2(m)\, h((n-m))_N = \sum_{m=3}^{7} x_2(m)\, h_n(m);\ n = 3, 4, 5, 6, 7$$

where $h_n(m) = h((n-m))_N$

When $n = 3$; $y_2(3) = \sum x_2(m)\, h_3(m) = -2 + 0 + 0 - 4 + 5 = -1$

When $n = 4$; $y_2(4) = \sum x_2(m)\, h_4(m) = -1 - 4 + 0 + 0 - 5 = -10$

When $n = 5$; $y_2(5) = \sum x_2(m)\, h_5(m) = 1 - 2 - 6 + 0 + 0 = -7$

When $n = 6$; $y_2(6) = \sum x_2(m)\, h_6(m) = 0 + 2 - 3 + 8 + 0 = 7$

When $n = 7$; $y_2(7) = \sum x_2(m)\, h_7(m) = 0 + 0 + 3 + 4 + 10 = 17$

Convolution of Section - 3

m	–4	–3	–2	–1	0	1	2	3	4	5	6	7	8	9	10
$x_3(m)$											4	5	6	0	0
$h(m)$					2	1	–1	0	0						
$h(-m) = h_0(m)$	0	0	–1	1	2										
$h(6-m) = h_6(m)$							0	0	–1	1	2	**0**	**0**	**–1**	**1**
$h(7-m) = h_7(m)$								0	0	–1	1	2	**0**	**0**	**–1**
$h(8-m) = h_8(m)$									0	0	–1	1	2	**0**	**0**
$h(9-m) = h_9(m)$										0	0	–1	1	2	**0**
$h(10-m) = h_{10}(m)$											0	0	–1	1	2

$$y_3(n) = x_3(n) \circledast h(n) = \sum_{m=m_i}^{m_f} x_3(m)\, h((n-m))_N = \sum_{m=6}^{10} x_3(m)\, h_n(m)\ ; \quad n = 6, 7, 8, 9, 10$$

where $h_n(m) = h((n-m))_N$

When $n = 6$; $y_3(6) = \sum x_3(m)\, h_6(m) = 8 + 0 + 0 + 0 + 0 = 8$

When $n = 7$; $y_3(7) = \sum x_3(m)\, h_7(m) = 4 + 10 + 0 + 0 + 0 = 14$

When $n = 8$; $y_3(8) = \sum x_3(m)\, h_8(m) = -4 + 5 + 12 + 0 + 0 = 13$

When $n = 9$; $y_3(9) = \sum x_3(m)\, h_9(m) = 0 - 5 + 6 + 0 + 0 = 1$

When $n = 10$; $y_3(10) = \sum x_3(m)\, h_{10}(m) = 0 + 0 - 6 + 0 + 0 = -6$

To Combine the Output of Convolution of Each Section

It can be observed that the last N_2-1 samples in an output sequence overlaps with the first N_2-1 samples of next output sequence. In overlap save method the overall output is obtained by combining the outputs of the convolution of all sections. While combining the outputs, the overlapped first N_2-1 samples of every output sequence is discarded and the remaining samples are simply saved as samples of y(n) as shown in the following table.

n	0	1	2	3	4	5	6	7	8	9	10
$y_1(n)$	~~1~~	~~7~~	7	–1	–8						
$y_2(n)$				~~–1~~	~~10~~	–7	7	17			
$y_3(n)$							~~8~~	~~14~~	13	1	–6
y(n)	*	*	7	–1	–8	–7	7	17	13	1	–6

$\therefore y(n) = x(n) * h(n) = \{*, *, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$

> ***Note :*** *Here y(n) is linear convolution of x(n) and h(n). It can be observed that the results of both the methods are same except the first $N_2 - 1$ samples.*

6.12 Inverse System and Deconvolution

6.12.1 Inverse System

The ***inverse system*** is used to recover the input from the response of a system. For a given system, the inverse system exists, if distinct inputs to a system leads to distinct outputs. The inverse systems exists for all LTI systems.

The inverse system is denoted by $\mathcal{H}^{-1}$. If x(n) is input and y(n) is the output of a system, then y(n) is the input and x(n) is the output of its inverse system.

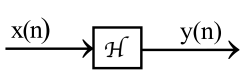


Fig 6.35a : *System.*

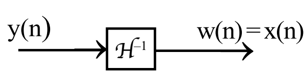


Fig 6.35b : *Inverse system.*

Fig 6.35 : *A system and its inverse system.*

Let h(n) be the impulse response of a system and h′(n) be the impulse response of inverse system. Let us connect the system and its inverse in cascade as shown in fig 6.36.

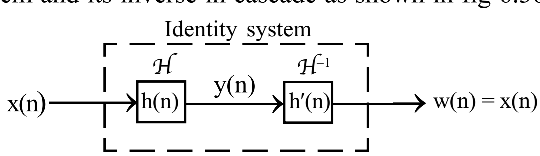


Fig 6.36 : *Cascade connection of a system and its inverse.*

Now it can be proved that,

$$h(n) * h'(n) = \delta(n) \qquad .....(6.60)$$

Therefore the cascade of a system and its inverse is identity system.

Proof :

With reference to fig 6.36 we can write,

$$y(n) = x(n) * h(n) \qquad .....(6.61)$$

$$w(n) = y(n) * h'(n) \qquad .....(6.62)$$

On substituting for y(n) from equation (6.61) in equation (6.62) we get,

$$w(n) = x(n) * h(n) * h'(n) \qquad .....(6.63)$$

In equation (6.63),

if, $h(n) * h'(n) = \delta(n)$, then, $x(n) * \delta(n) = x(n)$

In a inverse system, w(n) = x(n), and so,

$h(n) * h'(n) = \delta(n)$. Hence proved.

6.12.2 Deconvolution

In an LTI system the response y(n) is given by convolution of input x(n) and impulse response h(n).

i.e., $y(n) = x(n) * h(n)$

The process of recovering the input from the response of a system is called ***deconvolution***. (or the process of recovering x(n) from x(n) ∗ h(n) is called ***deconvolution***).

When the response y(n) and impulse response h(n) are available the input x(n) can be computed using the equation (6.64).

$$x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\,h(n-m)\right] \qquad .....(6.64)$$

Proof :

Let x(n) and h(n) be finite duration sequences starting from n = 0. Consider the matrix method of convolution of x(n) and h(n) shown below.

	h(0)	h(1)	h(2)	h(3)	......
x(0)	x(0)h(0)	x(0)h(1)	x(0)h(2)	x(0)h(3)	......
x(1)	x(1)h(0)	x(1)h(1)	x(1)h(2)	x(1)h(3)	......
x(2)	x(2)h(0)	x(2)h(1)	x(2)h(2)	x(2)h(3)	......
x(3)	x(3)h(0)	x(3)h(1)	x(3)h(2)	x(3)h(3)	......
⋮	⋮	⋮	⋮	⋮	⋮

From the above two dimensional array we can write,

$$y(n) = x(0)\,h(0) \quad \Rightarrow \quad x(0) = \frac{y(0)}{h(0)}$$

$$y(1) = x(1)\,h(0) + x(0)\,h(1) \quad \Rightarrow \quad x(1) = \frac{y(1) - x(0)\,h(1)}{h(0)}$$

$$y(2) = x(2)\,h(0) + x(1)\,h(1) + x(0)\,h(2) \quad \Rightarrow \quad x(2) = \frac{y(2) - x(0)\,h(2) - x(1)\,h(1)}{h(0)}$$

$$y(3) = x(3)\,h(0) + x(2)\,h(1) + x(1)\,h(2) + x(0)\,h(3) \quad \Rightarrow \quad x(3) = \frac{y(3) - x(0)\,h(3) - x(1)\,h(2) - x(2)\,h(1)}{h(0)}$$

and so on.

From the above analysis, in general for any value of n, the x(n) is given by,

$$x(n) = \frac{y(n) - x(0)\,h(n) - x(1)\,h(n-1) - \ldots\ldots - x(n-1)\,h(1)}{h(0)}$$

$$\therefore \; x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\,h(n-m)\right]$$

Example 6.30

A discrete time system is defined by the equation, $y(n) = \sum\limits_{m=0}^{n} x(m)$; for $n \geq 0$. Find the inverse system.

Solution

Given that, $y(n) = \sum\limits_{m=0}^{n} x(m)$

$$\text{When } n = 0;\; y(0) = \sum_{m=0}^{0} x(m) = x(0)$$

$$\text{When } n = 1;\; y(1) = \sum_{m=0}^{1} x(m) = x(0) + x(1) = y(0) + x(1)$$

$$\text{When } n = 2;\; y(2) = \sum_{m=0}^{2} x(m) = x(0) + x(1) + x(2) = y(1) + x(2)$$

$$\text{When } n = 3;\; y(3) = \sum_{m=0}^{3} x(m) = x(0) + x(1) + x(2) + x(3) = y(2) + x(3)$$

and so on,

From the above analysis we can write,

$x(0) = y(0)$; $x(1) = y(1) - y(0)$; $x(2) = y(2) - y(1)$; $x(3) = y(3) - y(2)$ and so on,

In general for any value of n, the signal x(n) can be written as,

$$x(n) = y(n) - y(n-1)$$

Therefore the inverse system is defined by the equation,

$$x(n) = y(n) - y(n-1)$$

Example 6.31

When a discrete time system is excited by an input x(n) the response is, $y(n) = \{\underset{\uparrow}{2}, 5, 11, 17, 13, 12\}$

If the impulse response of the system is, $h(n) = \{\underset{\uparrow}{2}, 1, 3\}$, then what will be the input to the system?

Solution

Let N_1 be number of samples in x(n) and N_2 be number of samples in h(n), then the number of samples N_3 in y(n) is given by,

$$N_3 = N_1 + N_2 - 1$$

$$\therefore N_1 = N_3 - N_2 + 1 = 6 - 3 + 1 = 4 \text{ samples}$$

Therefore x(n) is 4 sample sequence.

Each sample of x(n) is given by,

$$x(n) = \frac{1}{h(0)}\left[y(n) - \sum_{m=0}^{n-1} x(m)\, h(n-m)\right]$$

$$\text{When } n = 0;\ x(0) = \frac{y(0)}{h(0)} = \frac{2}{2} = 1$$

$$\text{When } n = 1;\quad x(1) = \frac{1}{h(0)}\left[y(1) - \sum_{m=0} x(m)\ h(1-m)\right]$$

$$= \frac{1}{h(0)}\left[y(1) - x(0)\, h(1)\right] = \frac{1}{2}\left[5 - 1 \times 1\right] = 2$$

$$\text{When } n = 2;\ x(2) = \frac{1}{h(0)}\left[y(2) - \sum_{m=0}^{1} x(m)\, h(2-m)\right]$$

$$= \frac{1}{h(0)}\left[y(2) - x(0)\, h(2) - x(1)\, h(1)\right]$$

$$= \frac{1}{2}\left[11 - 1 \times 3 - 2 \times 1\right] = 3$$

$$\text{When } n = 3;\ x(3) = \frac{1}{h(0)}\left[y(3) - \sum_{m=0}^{2} x(m)\, h(3-m)\right]$$

$$= \frac{1}{h(0)}\left[y(3) - x(0)\, h(3) - x(1)\, h(2) - x(2)\, h(1)\right]$$

$$= \frac{1}{2}\left[17 - 1 \times 0 - 2 \times 3 - 3 \times 1\right] = 4$$

$$\therefore \quad x(n) = \{x(0),\ x(1),\ x(2),\ x(3)\} = \{\underset{\uparrow}{1},\ 2,\ 3,\ 4\}$$

6.13 Correlation, Crosscorrelation and Autocorrelation

The ***correlation*** of two discrete time sequences x(n) and y(n) is defined as,

$$\boxed{r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)} \qquad .....(6.65)$$

where $r_{xy}(m)$ is the correlation sequence obtained by correlation of x(n) and y(n) and m is the variable used for time shift. The correlation of two different sequences is called ***crosscorrelation*** and the correlation of a sequence with itself is called ***autocorrelation***. Hence autocorrelation of a discrete time sequence is defined as,

$$\boxed{r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)} \qquad .....(6.66)$$

If the sequence x(n) has N_1 samples and sequence y(n) has N_2 samples then the crosscorrelation sequence $r_{xy}(m)$ will be a finite duration sequence consisting of $N_1 + N_2 - 1$ samples. If the sequence x(n) has N samples, then the autocorrelation sequence $r_{xx}(m)$ will be a finite duration sequence consisting of $2N - 1$ samples.

In the equation (6.65) the sequence x(n) is unshifted and the sequence y(n) is shifted by m units of time for correlation operation. The same results can be obtained if the sequence y(n) is unshifted and the sequence x(n) is shifted opposite to that of earlier case by m units of time, hence the crosscorrelation operation can also be expressed as,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n+m)\, y(n) \qquad ....(6.67)$$

6.13.1 Procedure for Evaluating Correlation

Let, $x(n)$ = Discrete time sequence with N_1 samples

$y(n)$ = Discrete time sequence with N_2 samples

Now the correlation of x(n) and y(n) will produce a sequence $r_{xy}(m)$ consisting of N_1+N_2-1 samples. Each sample of $r_{xy}(m)$ can be computed using the equation (6.65). The value of $r_{xy}(m)$ at $m = q$ is obtained by replacing m by q, in equation (6.65).

$$\therefore\ r_{xy}(q) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-q) \qquad .....(6.68)$$

The evaluation of equation (6.68) to determine the value of $r_{xy}(m)$ at $m = q$ involves the following three steps.

1. **Shifting** : Shift y(n) by q times to the right if q is positive, shift y(n) by q times to the left if q is negative to obtain $y(n-q)$.
2. **Multiplication** : Multiply x(n) by $y(n-q)$ to get a product sequence. Let the product sequence be $v_q(n)$. Now, $v_q(n) = x(n) \times y(n-q)$.
3. **Summation** : Sum all the values of the product sequence $v_q(n)$ to obtain the value of $r_{xy}(m)$ at $m = q$. [i.e., $r_{xy}(q)$].

The above procedure will give the value $r_{xy}(m)$ at a single time instant say $m = q$. In general we are interested in evaluating the values of the sequence $r_{xy}(m)$ over all the time instants in the range $-\infty < m < \infty$. Hence the steps 1, 2 and 3 given above must be repeated, for all possible time shifts in the range $-\infty < m < \infty$.

In the correlation of finite duration sequences it is possible to predict the start and end of the resultant sequence. If x(n) is N-point sequence and starts at $n = n_1$ and if y(n) is N_2-point sequence and starts at $n = n_2$ then, the initial value of $m = m_i$ for $r_{xy}(m)$ is $\mathbf{m_i = n_1 - (n_2 + N_2 - 1)}$. The value of x(n) for $n < n_1$ and the value of y(n) for $n < n_2$ are then assumed to be zero.The final value of $m = m_f$ for $r_{xy}(m)$ is $\mathbf{m_f = m_i + (N_1+N_2-2)}$.

The correlation operation involves all the steps in convolution operation except the folding. Hence it can be proved that the convolution of x(n) and folded sequence y(–n) will generate the crosscorrelation sequence $r_{xy}(m)$.

i.e., $r_{xy}(m) = x(n) * y(-n)$ .....(6.69)

The procedure given above can be used for computing autocorrelation of x(n). For computing autocorrelation using equation (6.68) replace y(n – q) by x(n – q). Similarly when equation (6.69) is used, replace y(–n) by x(–n).

The autocorrelation of N-point sequence x(n) will give 2N–1 point autocorrelation sequence. If x(n) starts at $n = n_x$ then initial value of $m = m_i$ for $r_{xx}(m)$ is $\mathbf{m_i = -(N-1)}$. The final value of $m = m_f$ for $r_{xx}(m)$ is $\mathbf{m_f = m_i + (2N-2)}$.

Properties of Correlation

1. The crosscorrelation sequence $r_{xy}(m)$ is simply folded version of $r_{yx}(m)$,

   i.e., $r_{xy}(m) = r_{yx}(-m)$

   Similarly for autocorrelation sequence,

   $r_{xx}(m) = r_{xx}(-m)$

   Hence autocorrelation is an even function.

2. The crosscorrelation sequence satisfies the condition,

   $$\left|r_{xy}(m)\right| \le \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x\, E_y}$$

   where, E_x and E_y are energy of x(n) and y(n) respectively.

   On applying the above condition to autocorrelation sequence we get,

   $$\left|r_{xx}(m)\right| \le r_{xx}(0) = E_x$$

   From the above equations we infer that the crosscorrelation sequence and autocorrelation sequences attain their respective maximum values at zero shift/lag.

3. Using the maximum value of crosscorrelation sequence, the normalized crosscorrelation sequence is defined as,

   $$\rho_{xy}(m) \le \frac{r_{xy}(m)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

   Using the maximum value of autocorrelation sequence, the normalized autocorrelation sequence is defined as,

   $$\rho_{xx}(m) \le \frac{r_{xx}(m)}{r_{xx}(0)}$$

Methods of Computing Correlation

Method -1: Graphical Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence.

1. Sketch the graphical representation of the input sequences x(n) and y(n).
2. Shift the sequence y(n) to the left graphically so that the product of x(n) and shifted y(n) gives only one non-zero sample. Now multiply x(n) and shifted y(n) to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
3. To get the next sample of output sequence, shift y(n) of previous step to one position right and multiply the shifted sequence with x(n) to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
4. To get subsequent samples of output sequence, the step-3 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence and shifted sequence is represented on a row in a table.

Method -3: Matrix Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence. We know that the convolution of x(n) and folded sequence y(–n) will generate the crosscorrelation sequence $r_{xy}(m)$. Hence fold y(n) to get y(–n), and compute convolution of x(n) and y(–n) by matrix method.

In matrix method one of the sequence is represented as a row and the other as a column as shown below.

	....	y(0)	y(−1)	y(−2)	y(−3)	....
⋮		⋮	⋮	⋮	⋮	
x(0)	....	x(0)y(0)	x(0)y(−1)	x(0)y(−2)	x(0)y(−3)	....
x(1)	....	x(1)y(0)	x(1)y(−1)	x(1)y(−2)	x(1)y(−3)	....
x(2)	....	x(2)y(0)	x(2)y(−1)	x(2)y(−2)	x(2)y(−3)	....
x(3)	....	x(3)y(0)	x(3)y(−1)	x(3)y(−2)	x(3)y(−3)	....
⋮		⋮	⋮	⋮	⋮	

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $r_{xy}(m)$. (The sum of the diagonal elements are shown below for reference).

$$\vdots$$

$$r_{xy}(0) = \ldots.. + x(0)\ y(0) + \ldots..$$

$$r_{xy}(1) = \ldots.. + x(1)\ y(0) + x(0\ )\ y(-1) + \ldots..$$

$$r_{xy}(2) = \ldots.. + x(2)\ y(0) + x(1)\ y(-1) + x(0)\ y(-2) + \ldots..$$

$$r_{xy}(3) = \ldots.. + x(3)\ y(0) + x(2)\ y(-1) + x(1)\ y(-2) + x(0)\ y(-3) + \ldots..$$

$$\vdots$$

Example 6.32

Perform crosscorrelation of the sequences, $x(n) = \{1, 1, 2, 2\}$ and $y(n) = \{1, 3, 1\}$.

Solution

Let $r_{xy}(m)$ be the crosscorrelation sequence obtained by crosscorrelation of x(n) and y(n).

The crosscorrelation sequence $r_{xy}(m)$ is given by,

$$r_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\ y(n-m)$$

The x(n) starts at n = 0 and has 4 samples.

$$\therefore\ n_1 = 0,\ N_1 = 4$$

The y(n) starts at n = 0 and has 3 samples.

$$\therefore\ n_2 = 0,\ N_2 = 3$$

Now, $r_{xy}(m)$ will have $N_1 + N_2 - 1 = 4 + 3 - 1 = 6$ samples.

The initial value of $m = m_i = n_1 - (n_2 + N_2 - 1)$

$$= 0 - (0 + 3 - 1) = -2$$

The final value of $m = m_f = m_i + (N_1 + N_2 - 2)$

$$= -2 + (4 + 3 - 2) = 3$$

In this example the correlation operation is performed by three methods.

Method-1 : Graphical Method

The graphical representation of x(n) and y(n) are shown below.

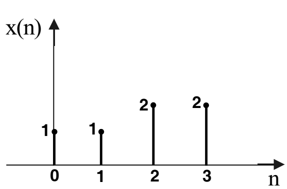


Fig 1.

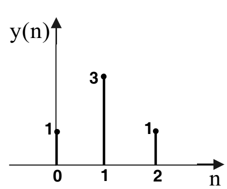


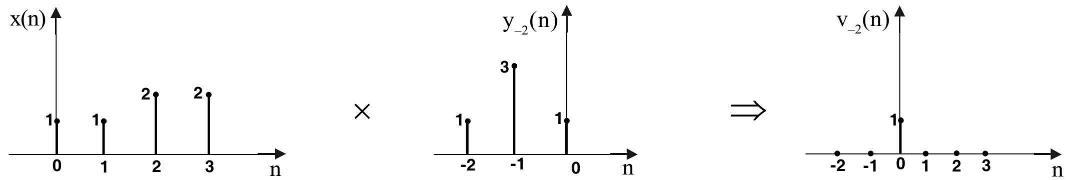
Fig 2.

The 6 samples of $r_{xy}(m)$ are computed using the equation,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\ y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\ y_m(n)\ ;\ \text{where}\ y_m(n) = y(n-m)$$

The computation of each sample of $r_{xy}(n)$ using the above equation are graphically shown in fig 3 to fig 8. The graphical representation of output sequence is shown in fig 9.

When $m = -2$; $r_{xy}(-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-2)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-2}(n) = \sum_{n=-\infty}^{+\infty} v_{-2}(n)$

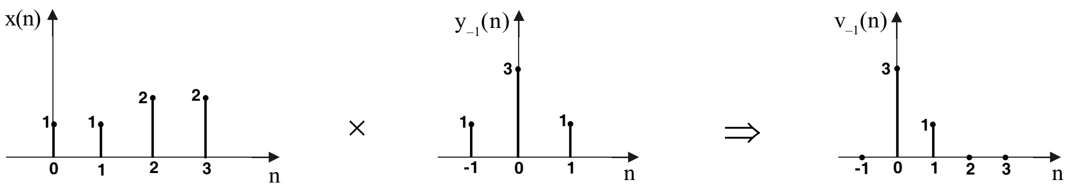


The sum of product sequence $v_{-2}(n)$ gives $r_{xy}(-2)$

Fig 3 : *Computation of $r_{xy}(-2)$.*

$\therefore r_{xy}(-2) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

When $m = -1$; $r_{xy}(-1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-1)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-1}(n) = \sum_{n=-\infty}^{+\infty} v_{-1}(n)$

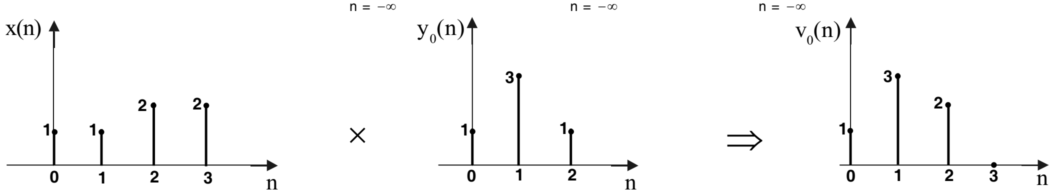


The sum of product sequence $v_{-1}(n)$ gives $r_{xy}(-1)$

Fig 4 : *Computation of $r_{xy}(-1)$.*

$\therefore r_{xy}(-1) = 0 + 3 + 1 + 0 + 0 = 4$

When $m = 0$; $r_{xy}(0) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n) = \sum_{n=-\infty}^{+\infty} x(n)\, y_0(n) = \sum_{n=-\infty}^{+\infty} v_0(n)$

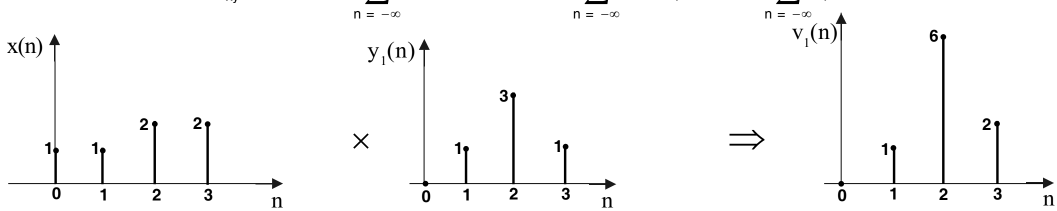


The sum of product sequence $v_0(n)$ gives $r_{xy}(0)$

Fig 5 : *Computation of $r_{xy}(0)$.*

$\therefore r_{xy}(0) = 1 + 3 + 2 + 0 = 6$

When $m = 1$; $r_{xy}(1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-1) = \sum_{n=-\infty}^{+\infty} x(n)\, y_1(n) = \sum_{n=-\infty}^{+\infty} v_1(n)$

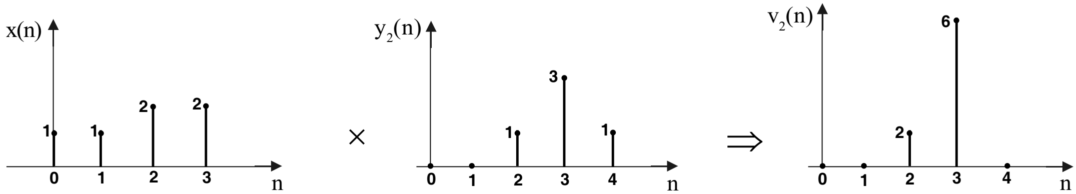


The sum of product sequence $v_1(n)$ gives $r_{xy}(1)$

Fig 6 : *Computation of $r_{xy}(1)$.*

$\therefore r_{xy}(1) = 0 + 1 + 6 + 2 = 9$

When m = 2 ; $r_{xy}(2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y_2(n) = \sum_{n=-\infty}^{+\infty} v_2(n)$

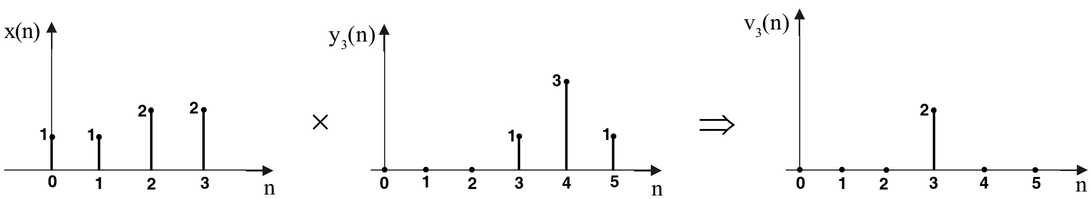


The sum of product sequence $v_2(n)$ gives $r_{xy}(2)$

$\therefore\ r_{xy}(2) = 0 + 0 + 2 + 6 + 0 = 8$

***Fig 7 :** Computation of $r_{xy}(2)$.*

When m = 3 ; $r_{xy}(3) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-3) = \sum_{n=-\infty}^{+\infty} x(n)\, y_3(n) = \sum_{n=-\infty}^{+\infty} v_3(n)$



The sum of product sequence $v_3(n)$ gives $r_{xy}(3)$

$\therefore\ r_{xy}(3) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

***Fig 8 :** Computation of $r_{xy}(3)$.*

The crosscorrelation sequence, $r_{xy}(m) = \{1, 4, \underset{\uparrow}{6}, 9, 8, 2\}$

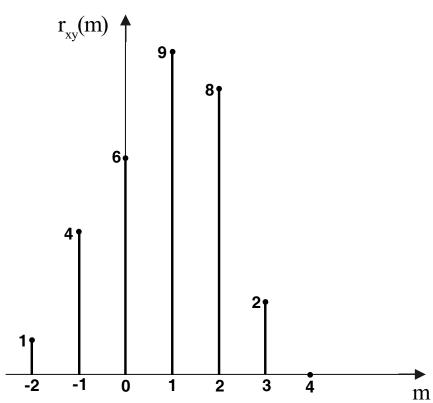


***Fig 9 :** Graphical representation of $r_{xy}(m)$.*

Method - 2: Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

n	–2	–1	0	1	2	3	4	5
$x(n)$			1	1	2	2		
$y(n)$			1	3	1			
$y(n-(-2)) = y_{-2}(n)$	1	3	1					
$y(n-(-1)) = y_{-1}(n)$		1	3	1				
$y(n) = y_0(n)$			1	3	1			
$y(n-1) = y_1(n)$				1	3	1		
$y(n-2) = y_2(n)$					1	3	1	
$y(n-3) = y_3(n)$						1	3	1

Note: *The unfilled boxes in the table are considered as zeros.*

Each sample of $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\;;\text{ where } y_m(n) = y(n-m)$$

To determine a sample of $r_{xy}(m)$ at m = q, multiply the sequence x(n) and $y_q(n)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(n) and $y_q(n)$). The sum of all the samples of the product sequence gives $r_{xy}(q)$.

$$\text{When } m = -2 \;;\; r_{xy}(-2) = \sum_{n=-2}^{3} x(n)\, y_{-2}(n) = 0 + 0 + 1 + 0 + 0 + 0 = 1$$

$$\text{When } m = -1 \;;\; r_{xy}(-1) = \sum_{n=-1}^{3} x(n)\, y_{-1}(n) = 0 + 3 + 1 + 0 + 0 = 4$$

$$\text{When } m = 0 \;;\; r_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = 1 + 3 + 2 + 0 = 6$$

$$\text{When } m = 1 \;;\; r_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 1 + 6 + 2 + 0 = 9$$

$$\text{When } m = 2 \;;\; r_{xy}(2) = \sum_{n=0}^{4} x(n)\, y_2(n) = 0 + 0 + 2 + 6 + 0 = 8$$

$$\text{When } m = 3 \;;\; r_{xy}(3) = \sum_{n=0}^{5} x(n)\, y_3(n) = 0 + 0 + 0 + 2 + 0 + 0 = 2$$

$$\therefore \text{Crosscorrelation sequence, } r_{xy}(m) = \{1, 4, \underset{\uparrow}{6}, 9, 8, 2\}$$

Method - 3: Matrix Method

Given that, $x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$

$y(n) = \{\underset{\uparrow}{1}, 3, 1\}$

$\therefore y(-n) = \{1, 3, \underset{\uparrow}{1}\}$

The sequence x(n) is arranged as a column and the folded sequence y(–n) is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

$x(n) \downarrow$ \ $y(-n) \rightarrow$	1	3	1
1	1×1	1×3	1×1
1	1×1	1×3	1×1
2	2×1	2×3	2×1
2	2×1	2×3	2×1

$\Rightarrow$

$x(n) \downarrow$ \ $y(-n) \rightarrow$	1	3	1
1	1	3	1
1	1	3	1
2	2	6	2
2	2	6	2

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 1 + 3 = 4$; $r_{xy}(0) = 2 + 3 + 1 = 6$

$r_{xy}(1) = 2 + 6 + 1 = 9$; $r_{xy}(2) = 6 + 2 = 8$; $r_{xy}(3) = 2$

$\therefore r_{xy}(m) = \{1, 4, \underset{\uparrow}{6}, 9, 8, 2\}$

Example 6.33

Determine the autocorrelation sequence for $x(n) = \{1, 2, 3, 4\}$.

Solution

Let, $r_{xx}(m)$ be the autocorrelation sequence.

The autocorrelation sequence $r_{xx}(m)$ is given by,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n - m)$$

The x(n) starts at n = 0 and has 4 samples.

$$\therefore n_x = 0 \quad \text{and} \quad N = 4$$

Now, $r_{xx}(m)$ will have, $2N - 1 = 2 \times 4 - 1 = 7$ samples.

The initial value of $m = m_i = -(N - 1) = -(4 - 1) = -3$

The final value of $m = m_f = m_i + (2N - 2) = -3 + (2 \times 4 - 2) = 3$

The autocorrelation is computed by tabular method. Hence the sequence x(n) and the shifted sequences of x(n) are tabulated in the following table.

n	–3	–2	–1	0	1	2	3	4	5	6
$x(n)$				1	2	3	4			
$x(n-(-3)) = x_{-3}(n)$	1	2	3	4						
$x(n-(-2)) = x_{-2}(n)$		1	2	3	4					
$x(n-(-1)) = x_{-1}(n)$			1	2	3	4				
$x(n) = x_0(n)$				1	2	3	4			
$x(n-1) = x_1(n)$					1	2	3	4		
$x(n-2) = x_2(n)$						1	2	3	4	
$x(n-3) = x_3(n)$							1	2	3	4

Each sample of $r_{xx}(m)$ is given by,

$$r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, x_m(n) \quad ; \quad \text{where } x_m(n) = x(n-m)$$

To determine a sample of $r_{xx}(m)$ at m = q, multiply the sequence x(n) and $x_q(n)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(n) and $x_q(n)$). The sum of all the samples of the product sequence gives $r_{xx}(q)$.

When m = –3 ; $r_{xx}(-3) = \sum_{n=-3}^{3} x(n)\, x_{-3}(n) = 0 + 0 + 0 + 4 + 0 + 0 + 0 = 4$

When m = –2 ; $r_{xx}(-2) = \sum_{n=-2}^{3} x(n)\, x_{-2}(n) = 0 + 0 + 3 + 8 + 0 + 0 = 11$

When m = –1 ; $r_{xx}(-1) = \sum_{n=-1}^{3} x(n)\, x_{-1}(n) = 0 + 2 + 6 + 12 + 0 = 20$

When m = 0 ; $r_{xx}(0) = \sum_{n=0}^{3} x(n)\, x_{0}(n) = 1 + 4 + 9 + 16 = 30$

When m = 1 ; $r_{xx}(1) = \sum_{n=0}^{4} x(n)\, x_{1}(n) = 0 + 2 + 6 + 12 + 0 = 20$

When m = 2 ; $r_{xx}(2) = \sum_{n=0}^{5} x(n)\, x_{2}(n) = 0 + 0 + 3 + 8 + 0 + 0 = 11$

When m = 3 ; $r_{xx}(3) = \sum_{n=0}^{6} x(n)\, x_{3}(n) = 0 + 0 + 0 + 4 + 0 + 0 + 0 = 4$

$\therefore$ Autocorrelation sequence, $r_{xx}(m) = \{4,\ 11,\ 20,\ \underset{\uparrow}{30},\ 20,\ 11,\ 4\}$

6.14 Circular Correlation

The ***circular correlation*** of two periodic discrete time sequences x(n) and y(n) with periodicity of N samples is defined as,

$$\boxed{\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\; y^*((n-m))_N} \qquad .....(6.70)$$

where, $\bar{r}_{xy}(m)$ is the sequence obtained by circular correlation

$y^*((n-m))_N$ represents circular shift of y*(n)

m is a variable used for time shift

The circular correlation of two different sequences is called ***circular crosscorrelation*** and the circular correlation of a sequence with itself is called ***circular autocorrelation***. Hence circular autocorrelation of a discrete time sequence is defined as,

$$\boxed{\bar{r}_{xx}(m) = \sum_{n=0}^{N-1} x(n)\; x^*((n-m))_N} \qquad .....(6.71)$$

The output sequence obtained by circular correlation is also periodic sequence with periodicity of N samples. Hence this correlation is also called periodic correlation.The circular correlation is defined for periodic sequences. But circular correlation can be performed with non-periodic sequences by periodically extending them.The circular correlation of two sequences requires that, at least one of the sequences should be periodic. Hence it is sufficient if one of the sequences is periodically extended in order to perform circular correlation.

The circular correlation of finite duration sequences can be performed only if both the sequences consists of same number of samples. If the sequences have different number of samples, then convert the smaller size sequence to the size of larger size sequence by appending zeros.

In the equation (6.70), the sequence x(n) is unshifted and the sequence y*(n) is circularly shifted by m units of time for correlation operation. The same results can be obtained if the sequence y*(n) is unshifted and the sequence x(n) is circularly shifted opposite to that of earlier case by m units of time, hence the circular correlation operation can also be expressed as,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x((n+m))_N \; y^*(n) \qquad .....(6.72)$$

Circular correlation basically involves the same three steps as that for correlation namely shifting one of the sequence, multiplying the two sequences and finally summing the values of product sequence. The difference between the two is that in circular correlation the shifting (rotating) operations are performed in a circular fashion by computing the index of one of the sequences by modulo-N operation. In correlation there is no modulo-N operation.

6.14.1 Procedure for Evaluating Circular Correlation

Let, x(n) and y(n) be periodic discrete time sequences with periodicity of N-samples. If x(n) and y(n) are non-periodic then convert the sequences to N-sample sequence and periodically extend the sequence y(n) with periodicity of N-samples.

Now the circular correlation of x(n) and y(n) will produce a periodic sequence $\bar{r}_{xy}(m)$ with periodicity of N-samples. The samples of one period of $\bar{r}_{xy}(m)$ can be computed using the equation (6.70).

The value of $\bar{r}_{xy}(m)$ at $m = q$ is obtained by replacing m by q, in equation (6.70), as shown in equation (6.73).

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n) \; y^*((n-q))_N \qquad .....(6.73)$$

The evaluation of equation (6.73) to determine the value of $\bar{r}_{xy}(m)$ at $m = q$ involves the following four steps.

1. **Conjugation** : Take conjugate of y(n) to get y*(n). If y(n) is a real sequence then y*(n) will be same as y(n). Represent the samples of one period of the sequences on circles.
2. **Rotation** : Rotate y*(n) by q times in anticlockwise if q is positive, rotate y*(n) by q times in clockwise if q is negative to obtain $y^*((n-q))_N$.
3. **Multiplication** : Multiply x(n) by $y^*((n-q))_N$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x(n) \times y^*((n-q))_N$.
4. **Summation** : Sum up the samples of one period of the product sequence $v_q(m)$ to obtain the value of $\bar{r}_{xy}(m)$ at $m = q$. [i.e., $\bar{r}_{xy}(q)$].

The above procedure will give the value of $\bar{r}_{xy}(m)$ at a single time instant say $m = q$. In general we are interested in evaluating the values of the sequence $\bar{r}_{xy}(m)$ in the range $0 < m < N - 1$. Hence the steps 2 , 3 and 4 given above must be repeated, for all possible time shifts in the range $0 < m < N - 1$.

6.14.2 Methods of Computing Circular Correlation

Method 1 : Graphical Method

In graphical method the given sequences are converted to same size and represented on circles. In case of periodic sequences, the samples of one period are represented on circles. Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at $m = q$.

1. Represent the sequences x(n) and y(n) on circles.
2. Rotate (or shift) the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then rotate (or shift) the sequence in anticlockwise direction and if q is negative then rotate (or shift) the sequence in clockwise direction.
3. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

$$\text{where, } y_q(n) = y((n-q))_N$$

Determine the product sequence $x(n)y_q(n)$ for one period.

4. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 2 : Using Tabular Array

Let x(n) and y(n) be the given real sequences. Let $\bar{r}_{xy}(m)$ be the sequence obtained by circular correlation of x(n) and y(n). The following procedure can be used to get a sample of $\bar{r}_{xy}(m)$ at m = q.

1. Represent the sequences x(n) and y(n) as two rows of tabular array.
2. Periodically extend y(n). Here the periodicity is N, where N is the length of the given sequences.
3. Shift the sequence y(n), q times to get the sequence $y((n-q))_N$. If q is positive then shift the sequence to the right and if q is negative then shift the sequence to the left.
4. The sample of $\bar{r}_{xy}(q)$ at m = q is given by,

$$\bar{r}_{xy}(q) = \sum_{n=0}^{N-1} x(n)\, y((n-q))_N = \sum_{n=0}^{N-1} x(n)\, y_q(n)$$

$$\text{where, } y_q(n) = y((n-q))_N$$

Determine the product sequence $x(n)y_q(n)$ for one period.

5. The sum of all the samples of the product sequence gives the sample $\bar{r}_{xy}(q)$ [i.e., $\bar{r}_{xy}(m)$ at m = q].

The above procedure is repeated for all possible values of m to get the sequence $\bar{r}_{xy}(m)$.

Method 3: Using Matrices

Let x(n) and y(n) be the given N-point sequences. The circular correlation of x(n) and y(n) yields another N-point sequence $\bar{r}_{xy}(m)$.

In this method an N×N matrix is formed using the sequence y(n) as shown below. The sequence x(n) is arranged as a column vector (column matrix) of order N×1. The product of the two matrices gives the resultant sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) & ..... & y(N-1) & y(N) \\ y(N) & y(0) & y(1) & ..... & y(N-2) & y(N-1) \\ y(N-1) & y(N) & y(0) & ..... & y(N-3) & y(N-2) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y(2) & y(3) & y(4) & ..... & y(0) & y(1) \\ y(1) & y(2) & y(3) & ..... & y(N) & y(0) \end{bmatrix} \times \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ \vdots \\ x(N-2) \\ x(N-1) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \\ \vdots \\ \vdots \\ \bar{r}_{xy}(N-2) \\ \bar{r}_{xy}(N-1) \end{bmatrix}$$

Example 6.34

Perform circular correlation of the two sequences,

$$x(n) = \{\underset{\uparrow}{1}, 1, 2, 1\} \quad \text{and} \quad y(n) = \{\underset{\uparrow}{2}, 3, 1, 1\}$$

Solution

Method 1:Graphical Method of Computing Circular Correlation

The given sequences are represented as points on circles as shown in fig 1 and 2.

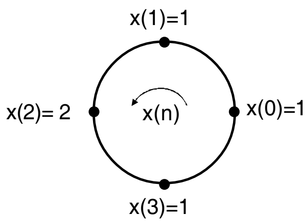


Fig 1.

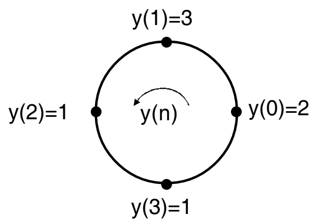


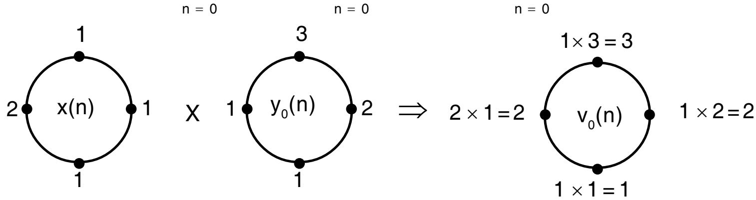
Fig 2.

Let $\bar{r}_{xy}(m)$ be the sequence obtained by correlation of x(n) and y(n). The given sequences are 4 sample sequences and so N = 4. Each sample of $\bar{r}_{xy}(m)$ is given by the equation,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y((n-m))_N = \sum_{n=0}^{N-1} x(n)\, y_m(n), \quad \text{where } y_m(n) = y((n-m))_N$$

Using the above equation, graphical method of computing each sample of $\bar{r}_{xy}(m)$ are shown in fig 3 to fig 6.

When $m = 0$; $\bar{r}_{xy}(0) = \sum_{n=0}^{3} x(n)\, y(n) = \sum_{n=0}^{3} x(n)\, y_0(n) = \sum_{n=0}^{3} v_0(n)$

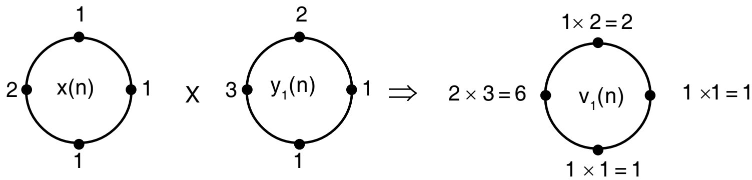


The sum of samples of $v_0(n)$ gives $\bar{r}_{xy}(0)$

$\therefore\ \bar{r}_{xy}(0) = 2+3+2+1 = 8$

Fig 3 : *Computation of* $\bar{r}_{xy}(0)$.

When $m = 1$; $\bar{r}_{xy}(1) = \sum_{n=0}^{3} x(n)\, y(n-1) = \sum_{n=0}^{3} x(n)\, y_1(n) = \sum_{n=0}^{3} v_1(n)$

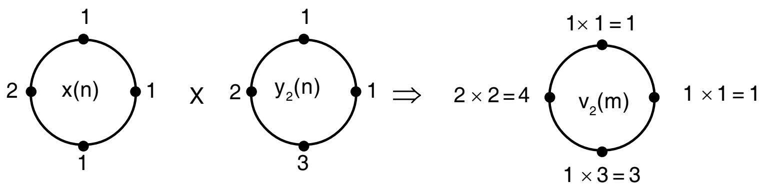


The sum of samples of $v_1(n)$ gives $\bar{r}_{xy}(1)$

$\therefore\ \bar{r}_{xy}(1) = 1 + 2 + 6 + 1 = 10$

***Fig 4 :** Computation of* $\bar{r}_{xy}(1)$.

When $m = 2$; $\bar{r}_{xy}(2) = \sum_{n=0}^{3} x(n)\, y(n-2) = \sum_{n=0}^{3} x(n)\, y_2(n) = \sum_{n=0}^{3} v_2(n)$



The sum of samples of $v_2(m)$ gives $\bar{r}_{xy}(2)$

$\therefore\ \bar{r}_{xy}(2) = 1 + 1 + 4 + 3 = 9$

***Fig 5 :** Computation of* $\bar{r}_{xy}(2)$.

When $m = 3$; $\bar{r}_{xy}(3) = \sum_{n=0}^{3} x(n)\, y(n-3) = \sum_{n=0}^{3} x(n)\, y_3(n) = \sum_{n=0}^{3} v_3(n)$

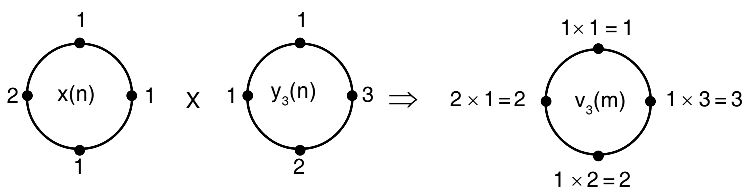


***Fig 6 :** Computation of* $\bar{r}_{xy}(3)$.

The sum of samples of $v_3(m)$ gives $\bar{r}_{xy}(3)$

$\therefore\ \bar{r}_{xy}(3) = 3 + 1 + 2 + 2 = 8$

$$\therefore\ \bar{r}_{xy}(m) = \{\underset{\uparrow}{8}, 10, 9, 8\}$$

Method 2 : Circular Correlation Using Tabular Array

The given sequences are represented in the tabular array as shown below. Here the shifted sequences $y_m(n)$ are periodically extended with a periodicity of $N = 4$. Let $\bar{r}_{xy}(m)$ be the sequence obtained by correlation of x(n) and y(n). Each sample of $\bar{r}_{xy}(m)$ is given by the equation,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y((n-m))_N = \sum_{n=0}^{N-1} x(n)\, y_m(n), \quad \text{where } y_m(n) = y((n-m))_N$$

***Note :** The bold faced numbers are samples obtained by periodic extension.*

n	0	1	2	3	4	5	6
x(n)	1	1	2	1			
y(n)	2	3	1	1			
$y_0(n-0) = y_0(n)$	2	3	1	1	**2**	**3**	**1**
$y_1(n-1) = y_1(n)$	**1**	2	3	1	1	**2**	**3**
$y_2(n-2) = y_2(n)$	**1**	**1**	2	3	1	1	**2**
$y_3(n-3) = y_3(n)$	**3**	**1**	**1**	2	3	1	1

To determine a sample of $\bar{r}_{xy}(m)$ at m = q, multiply the sequence, x(n) and $y_q(n)$, to get a product sequence $x(n)\, x_q(n)$. (i.e., multiply the corresponding elements of the row x(n) and $y_q(n)$). The sum of all the samples of the product sequence gives $\bar{r}_{xy}(m)$.

$$\text{When } m = 0\,;\ \bar{r}_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n)$$
$$= x(0)\, y_0(0) + x(1)\, y_0(1) + x(2)\, y_0(2) + x(3)\, y_0(3)$$
$$= 2 + 3 + 2 + 1 = 8$$

The samples of $\bar{r}_{xy}(m)$ for other values of m are calculated as shown for m = 0.

$$\text{When } m = 1;\ \bar{r}_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 1 + 2 + 6 + 1 = 10$$

$$\text{When } m = 2;\ \bar{r}_{xy}(2) = \sum_{n=0}^{3} x(n)\, y_2(n) = 1 + 1 + 4 + 3 = 9$$

$$\text{When } m = 3;\ \bar{r}_{xy}(3) = \sum_{n=0}^{3} x(n)\, y_3(n) = 3 + 1 + 2 + 2 = 8$$

$$\therefore\ \bar{r}_{xy}(m) = \{\underset{\uparrow}{8}, 10, 9, 8\}$$

Method 3 : Circular Correlation Using Matrices

The sequence $\bar{r}_{xy}(m)$ can be arranged as a column vector of order N × 1 and using the samples of y(n) the N × N matrix is formed as shown below. The product of the two matrices gives the sequence $\bar{r}_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) & y(3) \\ y(3) & y(0) & y(1) & y(2) \\ y(2) & y(3) & y(0) & y(1) \\ y(1) & y(2) & y(3) & y(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} \bar{r}_{xy}(0) \\ \bar{r}_{xy}(1) \\ \bar{r}_{xy}(2) \\ \bar{r}_{xy}(3) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \\ 3 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 10 \\ 9 \\ 8 \end{bmatrix}$$

$$\therefore\ \bar{r}_{xy}(m) = \{\underset{\uparrow}{8}, 10, 9, 8\}$$

6.15. Summary of Important Concepts

1. The discrete signal is a function of a discrete independent variable.
2. In a discrete time signal, the value of discrete time signal and the independent variable time are discrete.
3. The digital signal is same as discrete signal except that the magnitude of the signal is quantized.
4. A discrete time sinusoid is periodic only if its frequency is a rational number.
5. Discrete time sinusoids whose frequencies are separated by an integer multiple of 2π are identical.
6. The sampling is the process of conversion of continuous time signal into discrete time signal.
7. The time interval between successive samples is called sampling time or sampling period.
8. The inverse of sampling period is called sampling frequency.
9. The phenomenon of high frequency component getting the identity of low frequency component during sampling is called aliasing.
10. For analog signal with maximum frequency F_{max}, the sampling frequency should be greater than $2F_{max}$.
11. When sampling frequency F_s is equal to $2F_{max}$, the sampling rate is called Nyquist rate.
12. The signals that can be completely specified by mathematical equations are called deterministic signals.
13. The signals whose characteristics are random in nature are called nondeterministic signals.
14. A signal x(n) is periodic with periodicity of N samples if $x(n + N) = x(n)$.
15. When a signal exhibits symmetry with respect to $n = 0$ then it is called an even signal.
16. When a signal exhibits antisymmetry with respect to $n = 0$, then it is called an odd signal.
17. When the energy E of a signal is finite and non-zero, the signal is called energy signal.
18. When the power P of a signal is finite and non-zero, the signal is called power signal.
19. For energy signals, the energy will be finite and average power will be zero.
20. For power signals the average power is finite and energy will be infinite.
21. A signal is said to be causal, if it is defined for $n \geq 0$.
22. A signal is said to be noncausal, if it is defined for both $n \leq 0$ and $n > 0$.
23. A discrete time system is a device or algorithm that operates on a discrete time signal.
24. When a discrete time system satisfies the properties of linearity and time invariance, it is called an LTI system.
25. When the input to a discrete time system is unit impulse $\delta(n)$, the output is called impulse response, h(n).
26. In a static or memoryless system, the output at any instant n depends on input at the same time.
27. A system is said to be time invariant if its input-output characteristics do not change with time.
28. A linear system is one that satisfies the superposition principle.
29. A system is said to be causal if the output does not depends on future inputs/outputs.
30. When a system output at any time n depends on future inputs/outputs, it is called noncausal system.
31. System is said to be BIBO stable if and only if every bounded input produces a bounded output.
32. When a system output at any time n depends on past outputs, it is called a recursive system.
33. A system whose output does not depends on past outputs is called a nonrecursive system.
34. The convolution of N_1 sample and N_2 sample sequence produce a sequence consisting of N_1+N_2-1 samples.
35. In an LTI system, response for an arbitrary input is given by convolution of input with impulse response h(n).
36. The output sequence of circular convolution is also periodic sequence with periodicity of N samples.
37. The inverse system is used to recover the input from the response of a system.
38. The process of recovering the input from the response of a system is called deconvolution.
39. The correlation of two different sequences is called crosscorrelation.
40. The correlation of a sequence with itself is called autocorrelation.

6.16. Short Questions and Answers

Q6.1 ***Perform addition of the discrete time signals, $x_1(n) = \{2, 2, 1, 2\}$ and $x_2(n) = \{-2, -1, 3, 2\}$.***

Solution

In addition operation the samples corresponding to same value of n are added.

When n = 0, $x_1(0) + x_2(0) = 2 + (-2) = 0$

When n = 1, $x_1(1) + x_2(1) = 2 + (-1) = 1$

When n = 2, $x_1(2) + x_2(2) = 1 + 3 = 4$

When n = 3, $x_1(3) + x_2(3) = 2 + 2 = 4$

$\therefore x_1(n) + x_2(n) = \{0, 1, 4, 4\}$

Q6.2 ***Perform multiplication of discrete time signals, $x_1(n) = \{2, 2, 1, 2\}$ and $x_2(n) = \{-2, -1, 3, 2\}$.***

Solution

In multiplication operation the samples corresponding to same value of n are multiplied.

When n = 0, $x_1(0) \times x_2(0) = 2\times(-2) = -4$

When n = 1, $x_1(1) \times x_2(1) = 2\times(-1) = -2$

When n = 2, $x_1(2) \times x_2(2) = 1 \times 3 = 3$

When n = 3, $x_1(3) \times x_2(3) = 2 \times 2 = 4$

$\therefore x_1(n) \times x_2(n) = \{-4, -2, 3, 4\}$

Q6.3 ***Express the discrete time signal x(n) as a summation of impulses.***

If we multiply a signal x(n) by a delayed unit impulse $\delta(n - m)$, then the product is x(m), where x(m) is the signal sample at n = m (because $\delta(n - m)$ is 1 only at n = m and zero for other values of n). Therefore, if we repeat this multiplication over all possible delays in the range $-\infty < m < \infty$ and sum all the product sequences, then the result will be a sequence that is equal to the sequence x(n).

$$\therefore x(n) = \dots x(-2)\,\delta(n + 2) + x(-1)\,\delta(n + 1) + x(0)\,\delta(n) + x(1)\,\delta(n - 1) + x(2)\,\delta(n - 2) + \dots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n - m)$$

Q6.4 ***What are the basic elements used to construct the block diagram of discrete time system?***

The basic elements used to construct the block diagram of discrete time system are Adder, Constant multiplier and Unit delay element.

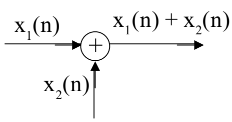


***Fig a:** Adder.*

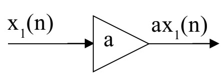


***Fig b:** Constant multiplier.*

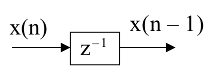


***Fig c:** Unit delay element.*

Q6.5 ***Let, x(n) = {1, 2, 3, 4}, be one period of a periodic sequence. What is x(n – 2, mod4)?***

The x(n) can be represented on the circle as shown in fig Q6.5a. The x(n – 2, mod4) is circularly shifted sequence of x(n) by two units of time as shown in fig Q6.5b.(Here, mod 4 stands for periodicity of 4).

$\therefore$ x(n – 2, mod4) = {3, 4, 1, 2}

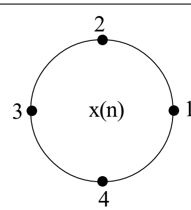


Fig Q6.5a.

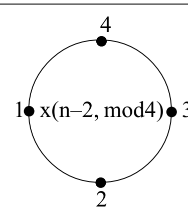


Fig Q6.5b.

Q6.6 ***Why linear convolution is important in signals and systems?***

The response or output of an LTI discrete time system for any input x(n) is given by linear convolution of the input x(n) and the impulse response h(n) of the system. (This means that if the impulse response of a system is known, then the response of the system for any input can be determined by convolution operation).

Q6.7 ***In y(n) = x(n)∗h(n), how will you determine the start and end point of y(n)? What will be the length of y(n)?***

Let, length of x(n) be N_1 and starts at $n = n_x$. Let, length of h(n) be N_2 and starts at $n = n_h$.

Now, y(n) will start at $n = n_x + n_h$

y(n) will end at $n = (n_x + n_h) + (N_1 + N_2 - 2)$

The length of y(n) is $N_1 + N_2 - 1$.

Q6.8 ***What is zero padding? Why it is needed?***

Appending zeros to a sequence in order to increase the size or length of the sequence is called zero padding.

In circular convolution, when the two input sequences are of different size, then they are converted to equal size by zero padding.

Q6.9 ***List the differences between linear convolution and circular convolution.***

Linear convolution	*Circular convolution*
1. The length of the input sequence can be different	1. The length of the input sequences should be same.
2. Zero padding is not required.	2. If the length of the input sequences are different, then zero padding is required.
3. The input sequences need not be periodic.	3. Atleast one of the input sequence should be periodic or should be periodically extended.
4. The output sequence is non-periodic.	4. The output sequence is periodic. The periodicity is same as that of input sequence.
5. The length of output sequence will be greater than the length of input sequences.	5. The length of the input and output sequences are same.

Q6.10 ***Perform the circular convolution of the two sequences $x_1(n)$ = {1, 2, 3} and $x_2(n)$ = {4, 5, 6}.***

Solution

Let $x_3(n)$ be the sequence obtained from circular convolution of $x_1(n)$ and $x_2(n)$. The sequence $x_1(n)$ can be arranged as a column vector of order 3×1 and using the samples of $x_2(n)$ a 3×3 matrix is formed as shown below. The product of two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 4 & 6 & 5 \\ 5 & 4 & 6 \\ 6 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 31 \\ 31 \\ 28 \end{bmatrix}$$

$\therefore x_3(n) = x_1(n) \circledast x_2(n) = \{31, 31, 28\}$.

Q6.11 ***Perform the linear convolution of the two sequences $x_1(n)$ = {1, 2} and $x_2(n)$ = {3, 4} via circular convolution.***

Solution

Let $x_3(n)$ be the sequence obtained from linear convolution of $x_1(n)$ and $x_2(n)$. The length of $x_3(n)$ will be $2 + 2 - 1 = 3$. Let us convert $x_1(n)$ and $x_2(n)$ into three sample sequences by padding with zeros as shown below.

$x_1(n) = \{1, 2, 0\}$ and $x_2(n) = \{3, 4, 0\}$

Now the circular convolution of $x_1(n)$ and $x_2(n)$ will give $x_3(n)$. The sequence $x_1(n)$ is arranged as a column vector and using the sequence $x_2(n)$, a 3×3 matrix is formed as shown below. The product of the two matrices gives the sequence $x_3(n)$.

$$\begin{bmatrix} x_2(0) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 0 & 4 \\ 4 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 8 \end{bmatrix}$$

$$\therefore x_3(n) = x_1(n) * x_2(n) = \{3, 10, 8\}$$

Q6.12 ***Compare the overlap add and overlap save method of sectioned convolutions.***

Overlap add method	Overlap save method
1. Linear convolution of each section of longer sequence with smaller sequence is performed.	1. Circular convolution of each section of longer sequence with smaller sequence is performed. (after converting them to the size of output sequence).
2. Zero padding is not required.	2. Zero padding is required to convert the input sequences to the size of output sequence.
3. Overlapping of samples of input sections are not required.	3. The N_2–1 samples of an input section of longer sequence is overlapped with next input section.
4. The overlapped samples in the output of sectioned convolutions are added to get the overall output.	4. Depending on method of overlapping input samples, either last N_2–1 samples or first N_2–1 samples of output sequence of each sectioned convolution are discarded.

Q6.13 ***In what way zero padding is implemented in overlap save method?***

In overlap save method, the zero padding is employed to convert the smaller input sequence to the size of the output sequence of each sectioned convolution. The zero padding is also employed to convert either the last section or the first section of the longer input sequence to the size of the output sequence of each sectioned convolution. (This depends on the method of overlapping input samples).

Q6.14 ***List the similarities and differences in convolution and correlation of two sequences.***

Similarities

1. Both convolution and correlation operation involves shifting, multiplication and summation of product sequence.
2. Both convolution and correlation operation produce same size of output sequence.

Differences

1. Correlation operation does not involve change of index and folding of one of the input sequence.
2. The convolution operation is commutative, [i.e., x(n)∗y(n)=y(n)∗x(n)], whereas in correlation operation in order to satisfy commutative property, while performing correlation of y(n) and x(n), the shifting has to performed in opposite direction to that of performing correlation of x(n) and y(n).

Q6.15 ***Let $r_{xy}(m)$ be the correlation sequence obtained by correlation of x(n) and y(n), how will you determine the start and end point of $r_{xy}(m)$? What will be the length of $r_{xy}(m)$?***

Let, length of x(n) be N_1 and starts at $n = n_1$. Let length of y(n) be N_2 and starts at $n = n_2$.

Now, $r_{xy}(m)$ will start at $m_i = n_1 - (n_2 + N_2 - 1)$

$r_{xy}(m)$ will end at $m_f = m_i + (N_1 + N_2 - 2)$

The length of $r_{xy}(m)$ is $N_1 + N_2 - 1$.

Q6.16 ***What are the differences between crosscorrelation and autocorrelation.***

1. Crosscorrelation operation is correlation of two different sequences, whereas autocorrelation is correlation of a sequence with itself.
2. Autocorrelation operation is an even function, whereas crosscorrelation is not an even function.

Q6.17 ***Perform the correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Given that, $x(n) = \{1, \underset{\uparrow}{2}, 3\}$ and $y(n) = \{\underset{\uparrow}{2}, 4, 1\}$. $\therefore\ y(-n) = \{1, 4, \underset{\uparrow}{2}\}$

The sequence x(n) is arranged as a column and the folded sequence y(–n) is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

x(n) ↓ \ y(–n) →	1	4	2
1	1×1	1×4	1×2
2	2×1	2×4	2×2
3	3×1	3×4	3×2

⇒

x(n) ↓ \ y(–n) →	1	4	2
1	1	4	2
2	2	8	4
3	3	12	6

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 2 + 4 = 6$; $r_{xy}(0) = 3 + 8 + 2 = 13$; $r_{xy}(1) = 12 + 4 = 16$; $r_{xy}(2) = 6$;

$\therefore\ r_{xy}(m) = \{1, 6, \underset{\uparrow}{13}, 16, 6\}$

Q6.18 ***Perform the circular correlation of the two sequences, x(n) = {1, 2, 3} and y(n) = {2, 4, 1}.***

Solution

Let $r_{xy}(m)$ be the sequence obtained from circular correlation of x(n) and y(n). The sequence x(n) can be arranged as a column vector of order 3×1 and using the samples of y(n) a 3×3 matrix is formed as shown below. The product of two matrices gives the sequence $r_{xy}(m)$.

$$\begin{bmatrix} y(0) & y(1) & y(2) \\ y(2) & y(0) & y(1) \\ y(1) & y(2) & y(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} r_{xy}(0) \\ r_{xy}(1) \\ r_{xy}(2) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 4 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 17 \\ 12 \end{bmatrix}$$

$\therefore\ r_{xy}(m) = \{13, 17, 12\}$

Q6.19 ***Perform circular autocorrelation of the sequence, x(n) = {1, 2, 3, 4}.***

Solution

Let $r_{xx}(m)$ be the sequence obtained from circular autocorrelation of x(n). The sequence x(n) can be arranged as a column vector of order 4×1 and again by using the samples of x(n) a 4×4 matrix is formed as shown below. The product of two matrices gives the sequence $r_{xx}(m)$.

$$\begin{bmatrix} x(0) & x(1) & x(2) & x(3) \\ x(3) & x(0) & x(1) & x(2) \\ x(2) & x(3) & x(0) & x(1) \\ x(1) & x(2) & x(3) & x(0) \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \begin{bmatrix} r_{xx}(0) \\ r_{xx}(1) \\ r_{xx}(2) \\ r_{xx}(3) \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 30 \\ 24 \\ 22 \\ 24 \end{bmatrix}$$

$\therefore\ r_{xx}(m) = \{30, 24, 22, 24\}$

Q6.20 ***What is the difference between circular crosscorrelation and circular autocorrelation.***

Circular crosscorrelation operation is circular correlation of two different sequences, whereas circular autocorrelation is circular correlation of a sequence with itself.

6.17 MATLAB Programs

Program 6.1

Write a MATLAB program to generate the standard discrete time signals unit impulse, unit step and unit ramp signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                 %specify the range of n

%****************** unit impulse signal
x1=1;
x2=0;
x=x1.*(n==0)+x2.*(n~=0);        %generate unit impulse signal
subplot(3,1,1);stem(n,x);       %plot the generated unit impulse signal
xlabel('n');ylabel('x(n)');title('unit impulse signal');

%****************** unit step signal
x1=1;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);         %generate unit step signal
subplot(3,1,2);stem(n,x);       %plot the generated unit step signal
xlabel('n');ylabel('x(n)');title('unit step signal');

%****************** unit ramp signal
x1=n;
x2=0;
x=x1.*(n>=0)+x2.*(n<0);         %generate unit ramp signal
subplot(3,1,3);stem(n,x);       %plot the generated unit ramp signal
xlabel('n');ylabel('x(n)');title('unit ramp signal');
```

OUTPUT

The output waveforms of program 6.1 are shown in fig P6.1.

Program 6.2

Write a MATLAB program to generate the standard discrete time signals exponential and sinusoidal signals.

```
%****************** program to plot some standard signals

n=-20 : 1 : 20;                 %specify the range of n

%****************** exponential signal
A=0.95;
x=A.^n;                         %generate exponential signal
subplot(2,1,1);stem(n,x);       %plot the generated exponential signal
xlabel('n');ylabel('x(n)');title('exponential signal');

%****************** sinusoidal signal
N=20;                           %declare periodicity
f=1/20;                         %compute frequency
x=sin(2*pi*f*n);                %generate sinusoidal signal
subplot(2,1,2);stem(n,x);       %plot the generated sinusoidal signal
xlabel('n');ylabel('x(n)');title('sinusoidal signal');
```

OUTPUT

The output waveforms of program 6.2 are shown in fig P6.2.

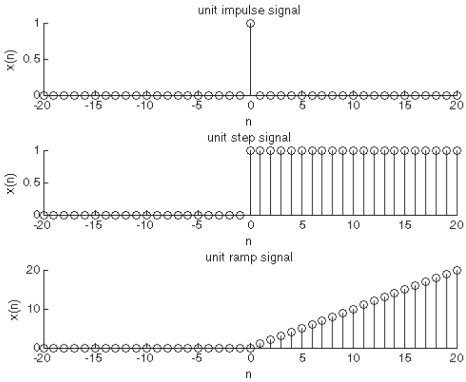


***Fig P6.1** : Output waveforms of program 6.1.*

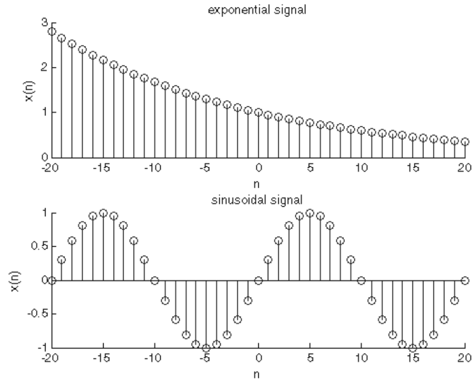


***Fig P6.2** : Output waveforms of program 6.2.*

Program 6.3

Write a MATLAB program to find the even and odd part of the signal $x(n)=0.8^n$.

```
%To find the even and odd parts of the signal, x(n)= 0.8^n

n= -5 :1 :5;        %specify the range of n
A=0.8;
x1=A.^n;            %generate the given signal
x2=A.^(-n);         %generate the folded signal

if(x2==x1)
      disp(‘“The given signal is even signal”’);
else if (x2==(-x1))
      disp(‘“The given signal is odd signal”’);
else
      disp(‘“The given signal is neither even nor odd signal”’);
end
end

xe=(x1+x2)/2;       %compute even part
xo=(x1-x2)/2;       %compute odd part

subplot(2,2,1);stem(n,x1);
xlabel(‘n’);ylabel(‘x1(n)’);title(‘signal x(n)’);

subplot(2,2,2);stem(n,x2);
xlabel(‘n’);ylabel(‘x2(n)’);title(‘signal x(-n)’);

subplot(2,2,3);stem(n,xe);
xlabel(‘n’);ylabel(‘xe(n)’);title(‘even part of x(n)’);

subplot(2,2,4);stem(n,xo);
xlabel(‘n’);ylabel(‘xo(n)’);title(‘odd part of x(n)’);
```

OUTPUT

“The given signal is neither even nor odd signal”

The output waveforms of program 6.3 are shown in fig P6.3.

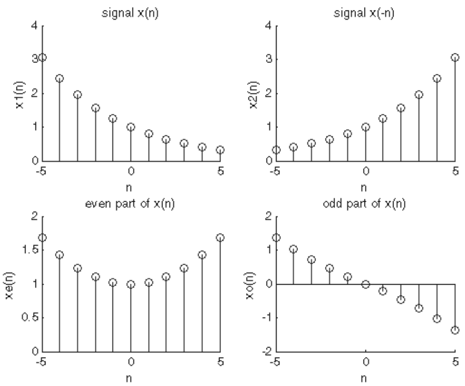


Fig P6.3 : *Output waveforms of program 6.3.*

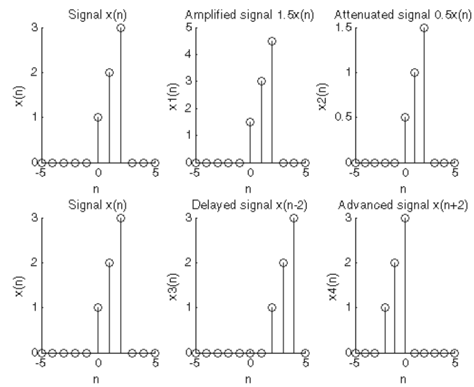


Fig P6.4 : *Output waveforms of program 6.4.*

Program 6.4

Write a MATLAB program to perform Amplitude scaling and Time shift on the signal x(n) = 1+n; for n = 0 to 2.

Program to declare the given signal as function y(n)

```
% declare the given signal as function y(n)

function x = y(n)
x=(1.0 + n).*(n>=0 & n<=2);
```

Note: The above program should be stored as a separate file in the current working directory

Program to perform amplitude scaling and time shift on y(n)

```
%To Perform Amplitude scaling and Time shift on signal x(n)=1+n; for
n= 0 to 2
%include y.m file in current work directory which declare given
signal as function y(n)

n=-5:1:5;          %specify range of n

y0 =y(n);          %assign the given signal as y0
y1 =1.5*y(n);      %compute the amplified version of x(n)
y2 =0.5*y(n);      %compute the attenuated version of x(n)
y3 =y(n-2);        %compute the delayed version of x(n)
y4 =y(n+2);        %compute the advanced version of x(n)

%plot the given signal and amplitude scaled signal
subplot(2,3,1);stem(n,y0);
xlabel('n');ylabel('x(n)');title('Signal x(n)');
subplot(2,3,2);stem(n,y1);
xlabel('n');ylabel('x1(n)');title('Amplified signal 1.5x(n)');
subplot(2,3,3);stem(n,y2);
xlabel('n');ylabel('x2(n)');title('Attenuated signal 0.5x(n)');

%plot the given signal and time shifted signal
subplot(2,3,4);stem(n,y0);
```

```
xlabel('n');ylabel('x(n)');title('Signal  x(n)');
subplot(2,3,5);stem(n,y3);
xlabel('n');ylabel('x3(n)');title('Delayed  signal  x(n-2)');
subplot(2,3,6);stem(n,y4);
xlabel('n');ylabel('x4(n)');title('Advanced  signal  x(n+2)');
```

OUTPUT

The input and output waveforms of program 6.4 are shown in fig P6.4.

Program 6.5

Write a MATLAB program to perform convolution of the following two discrete time signals.

x1(n)=1; 1<n<10 x2(n)=1; 2<n<10

```
%******************Program  to  perform  convolution  of  two  signals
%******************x1(N)=1;  n=  1  to  10  and  x2(n)=1;  n=  2  to  10

n  =  0  :  1  :  15;          %specify  range  of  n

x1=1.*(n>=1  &  n<=10);    %generate  signal  x1(n)
x2=1.*(n>=2  &  n<=10);    %generate  signal  x2(n)
N1=length(x1);
N2=length(x2);
x3=conv(x1,x2);            %perform  convolution  of  signals  x1(n)  and  x2(n)
n1=0  :  1  :  N1+N2-2;      %specify  range  of  n  for  x3(n)

subplot(3,1,1);stem(n,x1);
xlabel('n');ylabel('x1(n)');
title('signal  x1(n)');

subplot(3,1,2);stem(n,x2);
xlabel('n');ylabel('x2(n)');
title('signal  x2(n)');

subplot(3,1,3);stem(n1,x3);
xlabel('n');ylabel('x3(n)');
title('signal,  x3(n)  =  x1(n)*x2(n)');
```

OUTPUT

The input and output waveforms of program 2.5 are shown in fig P2.5.

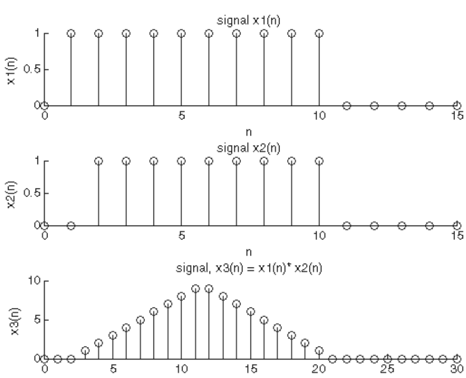


***Fig P6.5** : Output waveforms of program 6.5.*

6.18 Exercises

I Fill in the blanks with appropriate words

1. A signal x(n) may be shifted in time by m units by replacing the independent variable n by _______.
2. The _______ of a signal x(n) is performed by changing the sign of the time base n.
3. If the average power of a signal is finite then it is called _______.
4. The smallest value of N for which x(n + N) = x(n) is true is called _______.
5. In a discrete time signal x(n), if x(n) = x(–n) then it is called _______ signal.
6. In a discrete time signal x(n), if x(–n) = –x(n) then it is called _______ signal.
7. The output of the system with zero input is called _______.
8. A discrete time system is _______ if it obeys the principle of superposition.
9. A discrete time system is _______ if its input-output relationship do not change with time.
10. The response of an LTI system is given by _______ of input and impulse response.
11. If the output of a system depends only on present input then it is called _______.
12. A system is said to be _______ if the output does not depends on future inputs and outputs.
13. An LTI system is causal if and only if its impulse response is _______ for negative values of n.
14. When a system output at any time n depends on past output values, it is called _______ system.
15. An N-point sequence is called _______ if it is symmetric about point zero on the circle.
16. An N-point sequence is called _______ if it is antisymmetric about point zero on the circle.
17. The _______ is called aperiodic convolution.
18. The _______ is called periodic convolution.
19. Appending zeros to a sequence in order to increase its length is called _______.
20. The two methods of sectioned convolutions are _______ and _______method.
21. In _______ method of sectioned convolution, overlapped samples of output sequences are _______.
22. In _______ method, the overlapped samples in one of the output sequences are discarded.
23. The correlation of two different discrete time sequences is called _______ .
24. The cascade of a system and its inverse is _______.
25. The process of recovering the input from the response of a system is called ______ .

Answers

1. n – m	8. linear	15. even	22. overlap save
2. folding	9. time invariant	16. odd	23. cross correlation
3. power signal	10. convolution	17. linear convolution	24. identity system
4. fundamental period	11. memoryless or static	18. circular convolution	25. deconvolution
5. symmetric	12. causal	19. zero padding	
6. antisymmetric	13. zero	20. overlap add, overlap save	
7. natural response	14. recursive	21. overlap add, added	

II. State whether the following statements are True/False

1. The discrete signals are continuous function of an independent variable.
2. In digital signal the magnitudes of the signal are unquantized.
3. A discrete time signal x(n) is defined for noninteger values of n.
4. An impulse signal has a nonzero sample only for one value of n.
5. When we multiply a discrete time signal by unit step signal, the signal is converted to one sided signal.

6. Shifting a signal to left is called delay and shifting to right is called advance.
7. Any discrete time signal can be expressed as a summation of impulses.
8. Periodic signals are power signals.
9. When the energy of a signal is infinite, it is called energy signal.
10. The output of a system for impulse input is called impulse response.
11. A system can be realized in real time only if it is noncausal and stable.
12. Dynamic systems does not require memory but static systems require memory.
13. A system is time invariant if the response to a shifted version of the input is identical to a shifted version of the response based on the unshifted input.
14. An LTI system is unstable if the impulse response is absolutely summable.
15. A system whose output depends only on the present and past input is called a recursive system.
16. The circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension.
17. For an N-point sequence represented on a circle, the time reversal is obtained by reversing its sample about the point zero on the circle.
18. When a nonperiodic N-point sequence is represented on a circle then it becomes periodic with periodicity N.
19. In linear convolution the length of the input sequences should be same.
20. In circular convolution the length of the input sequences need not be same.
21. In circular correlation the length of the input and output sequences are same.
22. The correlation operation, $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n)\,y(n-m)$ is not same as $r_{xy}(m) = \sum_{n=-\infty}^{\infty} x(n+m)\,y(n)$.
23. The cross correlation sequence $r_{xy}(m)$ is folded version of $r_{yx}(m)$.
24. The inverse systems exist for all LTI systems.
25. The final value of m in autocorrelation sequence of N-point sequence is, $m_f = m_i + (2N - 1)$.

Answers

1.	False	6.	False	11.	False	16.	True	21.	True
2.	False	7.	True	12.	False	17.	True	22.	False
3.	False	8.	True	13.	True	18.	True	23.	True
4.	True	9.	False	14.	False	19.	False	24.	True
5.	True	10.	True	15.	False	20.	False	25.	False

III. Choose the right answer for the following questions

1. $x(n) = \dfrac{x(n-1)}{4}$ ***with initial condition x(0) = –1, gives the sequence,***

a) $x(n) = \left(\dfrac{1}{4}\right)^n$ **b)** $x(n) = -\left(\dfrac{1}{4}\right)^n$ **c)** $x(n) = \left(\dfrac{1}{4}\right)^{-n}$ **d)** $x(n) = \left(\dfrac{-1}{4}\right)^{-n}$

2. The process of conversion of continuous time signal into discrete time signal is known as,

a) aliasing **b)** sampling **c)** convolution **d)** none of the above

3. If F_s is sampling frequency then the relation between analog frequency F and digital frequency f is,

a) $f = \dfrac{F}{2F_s}$ **b)** $f = \dfrac{F_s}{F}$ **c)** $f = \dfrac{F}{F_s}$ **d)** $f = \dfrac{2F}{F_s}$

4. ***If F_s is sampling frequency then the highest analog frequency that can be uniquely represented in its sampled version of discrete time signal is,***

a) $\frac{F_s}{2}$ **b)** $2F_s$ **c)** F_s **d)** $\frac{1}{F_s}$

5. ***The sampling frequency of the following analog signal, $x(t) = 4 \sin 150\pi t + 2 \cos 50\pi t$ should be,***

a) greater than 75 Hz **b)** greater than 150 Hz **c)** less than 150 Hz **d)** greater than 50 Hz

6. ***Which of the following signal is the example for deterministic signal?***

a) step **b)** ramp **c)** exponential **d)** all of the above

7. ***For Energy signals, the energy will be finite and the average power will be,***

a) infinite **b)** finite **c)** zero **d)** cannot be defined

8. ***In a signal x(n), if 'n' is replaced by $\frac{n}{3}$, then it is called,***

a) upsampling **b)** folded version **c)** downsampling **d)** shifted version

9. ***The unit step signal u(n) delayed by 3 units of time is denoted as,***

a) $u(n+3) = 1;\ n \geq 3$, $= 0;\ n < 3$ **b)** $u(3-n) = 1;\ n \geq 3$, $= 0;\ n < 3$ **c)** $u(n-3) = 1;\ n \geq 3$, $= 0;\ n < 3$ **d)** $u(3n) = 1;\ n > 3$, $= 0;\ n < 3$

10. ***The zero input response (or) natural response is mainly due to,***

a) Initial stored energy in the system **b)** Initial conditions in the system
c) Specific input signal **d)** both a and b

11. ***If $x(n) = a^n u(n)$ is the input signal, then the particular solution $y_p(n)$ will be,***

a) $K^n a^n u(n)$ **b)** $K a^n u(n)$
c) $K_1 a^n u(n) + K_2 a^n u(n)$ **d)** $K a^{-n} u(n)$

12. ***The discrete time system, $y(n) = x(n-3) - 4x(n-10)$ is a,***

a) dynamic system **b)** memoryless system **c)** time varying system **d)** none of the above

13. ***An LTI discrete time system is causal if and only if,***

a) $h(n) \neq 0$ for $n < 0$ **b)** $h(n) = 0$ for $n < 0$ **c)** $h(n) \neq \infty$ for $n < 0$ **d)** $h(n) \neq 0$ for $n > 0$

14. ***Which of the following system is causal?***

a) $h(n) = n\left(\frac{1}{2}\right)^n u(n+1)$ **b)** $y(n) = x^2(n) - x(n+1)$ **c)** $y(n) = x(-n) + x(2n-1)$ **d)** $h(n) = n\left(\frac{1}{2}\right)^n u(n)$

15. ***An LTI system is stable, if the impulse response is,***

a) $\sum_{n=-\infty}^{\infty} |h(n)| = 0$ **b)** $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$ **c)** $\sum_{n=-\infty}^{\infty} |h(n)| \neq 0$ **d)** either a or b

16. ***The system $y(n) = \sin[x(n)]$ is,***

a) stable **b)** BIBO stable **c)** unstable **d)** none of the above

17. ***Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system with impulse response,***

a) $h_1(n) * h_2(n)$ **b)** $h_1(n) + h_2(n)$ **c)** $h_1(n) - h_2(n)$ **d)** $h_1(n) * [h_1(n) + h_2(n)]$

18. ***Sectioned convolution is performed if one of the sequence is very much larger than the other in order to overcome,***

a) long delay in getting output **b)** larger memory space requirment
c) both a and b **d)** none of the above

19. In overlap save method, the convolution of various sections are performed by,

a) zero padding **b)** linear convolution **c)** circular convolution **d)** both b and c

20. If x(n) is N_1-point sequence starts at n = n_1, if y(n) is N_2-point sequence starts at n = n_2, if $r_{xy}(m)$ is the correlation sequence then the value of m corresponding, to last sample of $r_{xy}(m)$ is,

a) $m_f = m_i + (N_1 + N_2 - 2)$ **b)** $m_f = m_i + (2N - 2)$ **c)** $m_f = m_i + (N_1 + N_2 - 1)$ **d)** $m_f = m_i + (2N + 1)$

21. For a system, y(n) = nx(n), the inverse system will be,

a) $y\left(\frac{1}{n}\right)$ **b)** $\frac{1}{n}\,y(n)$ **c)** $ny(n)$ **d)** $n^{-1}y(n)$

22. For a system y(n) = x(n–3) the impulse response of the system and the inverse system will be ——— and ——— respectively.

a) $h(n) = \delta(n + 3),\ h'(n) = \delta(n - 3)$

b) $h(n) = \delta(3n),\ h'(n) = \delta\left(\frac{n}{3}\right)$

c) $h(n) = \delta(n - 3),\ h'(n) = \delta(n + 3)$

d) $h(n) = \delta(n + 3),\ h'(n) = \delta(n - 6)$

23. The circular correlation $\bar{r}_{xy}(q)$ of the sequence $x_1(n)$ and $x_2(n)$ of length 'N' can be defined by the equation,

a) $\sum_{n=-\infty}^{\infty} x_1(n)\, x_2(n-q)$

b) $\sum_{n=0}^{N-1} x_1(n)\, x_2^*(n-q)$

c) $\sum_{n=0}^{N-1} x_1(n)\, x_2^*((n-q))_N$

d) $\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*((n-q))_N$

24. The evaluation of correlation involves,

a) shifting, rotating and summation

b) shifting, multiplication and summation

c) change of index, folding and summation

d) change of index, folding, shifting & multiplication

25. The circular correlation of N-point sequences is evaluated in the range,

a) $-N < m < N$ **b)** $-N < m < 0$ **c)** $0 < m < N$ **d)** $0 < m < N-1$

Answers

1. b	6. d	11. b	16. a	21. b
2. b	7. c	12. a	17. b	22. c
3. c	8. a	13. b	18. c	23. c
4. a	9. c	14. d	19. c	24. b
5. b	10. a	15. d	20. a	25. d

IV. Answer the following questions

1. Define discrete and digital signal.
2. Explain briefly, the various methods of representing discrete time signal with examples.
3. Define sampling and aliasing.
4. What is Nyquist rate?
5. State sampling theorem.
6. Define the impulse and unit step signal.
7. Express the discrete time signal x(n) as a summation of impulses.
8. How will you classify the discrete time signals?
9. What are energy and power signals?

10. When a discrete time signal is called periodic?
11. What is discrete time system?
12. What is impulse response? Explain its significance.
13. Write the difference equation governing the N^{th} order LTI system.
14. Write the expression for discrete convolution.
15. List the various methods of classifying discrete time systems.
16. Define time invariant system.
17. What is linear and nonlinear systems?
18. What is the importance of causality?
19. What is BIBO stability? What is the condition to be satisfied for stability?
20. What are FIR and IIR systems?
21. Write the convolution sum formula for FIR and IIR systems.
22. What is recursive and non recursive system? Give example.
23. Write the properties of linear convolution.
24. Prove the distributive property of linear convolution.
25. What are the two ways of interconnecting LTI systems ?
26. Define circular convolution.
27. What is the importance of linear and circular convolution in signals & systems ?
28. How will you perform linear convolution via circular convolution?
29. What is sectioned convolution? Why it is performed?
30. What are the two methods of sectioned convolution?
31. What is inverse system? What is its importance?
32. Define deconvolution.
33. Define cross correlation and autocorrelation?
34. What are the properties of correlation?
35. What is circular correlation?

V Solve the following Questions

E6.1 Determine whether the following signals are periodic or not. If periodic find the fundamental period.

a) $x(n) = \sin\left(\frac{3\pi}{8}n + 4\right)$ **b)** $x(n) = \cos\left(\frac{5n}{3} + \pi\right)$ **c)** $x(n) = \cos\left(\frac{4\pi n}{12}\right)$

d) $x(n) = \sin\left(\frac{\pi}{18}n^2\right)$ **e)** $x(n) = e^{j5n}$

E6.2 Determine the even and odd part of the signals.

a) $x(n) = \frac{2}{a^n}$ **b)** $x(n) = 7e^{-j\frac{\pi}{5}n}$ **c)** $x(n) = \{\underset{\uparrow}{3}, \ -6, \ 2, \ -4\}$

E6.3 **a)** Consider the analog signal x(t) = 5 sin50πt. If the sampling frequency is 60Hz, find the sampled version of discrete time signal x(n). Also find an alias frequency corresponding to F_s = 60Hz.

b) Consider the analog signals, $x_1(t) = 3\cos 2\pi(40t)$ and $x_2(t) = 3\cos 2\pi(5t)$. Find a sampling frequency so that 40Hz signal is an alias of 5Hz signal.

c) Consider the analog signal, $x(t) = 2\sin 40\pi t - 3\sin 100\pi t + \cos 50\pi t$. Determine the minimum sampling frequency and the sampled version of analog signal at this frequency. Sketch the waveform and show the sampling points. Comment on the result.

E6.4 Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{3}{8}\right)^n u(n)$ **b)** $x(n) = \left(\sin\frac{3\pi}{4}n\right)$ **c)** $x(n) = u(2n)$ **d)** $x(n) = 2\,u(3-n)$

E6.5 Construct the block diagram and signal flow graph of the discrete time systems whose input-output relations are described by the following difference equations.

a) $y(n) = 2y(n-1) + 2.5\,x(n-2) + 0.5\,x(n-3)$

b) $y(n) = 3.2\,x(n-2) + 0.7\,x(n) + 5y(n-1) + 0.35y(n-2)$

E6.6 Determine the response of the discrete time systems governed by the following difference equations.

a) $y(n) = 0.2y(n-1) + x(n-1) + 0.5x(n)$; $x(n) = 2^{-n}\,u(n)$; $y(-1) = -1$

b) $y(n) + 2.3y(n-1) + 0.6y(n-2) = x(n) + 0.56x(n-1)$; $x(n) = u(n)$; $y(-2) = 1$; $y(-1) = -3$

E.6.7 Test the following systems for time invariance.

a) $y(n) = x(n) + x(n+2)$ **b)** $y(n) = na^{x(n)}$ **c)** $y(n) = x^2(n-1) + C$ **d)** $y(n) = (n-1)\,x^2(n) + C$

E6.8 Test the following systems for linearity.

a) $y(n) = x(n) + x^2(n-1)$ **b)** $y(n) = bx(n) + ne^{x(n)}$ **c)** $y(n) = a\sqrt{x(n)} + b\,x(n)$

d) $y(n) = \sqrt{x(n)} + \frac{1}{\sqrt{x(n)}}$ **e)** $y(n) = \sum_{m=-1}^{N} b_m\,x(n+m) + \sum_{m=0}^{M} c_m\,y(n+m)$

E6.9 Test the causality of the following systems.

a) $y(n) = x(n) - x(-n-1) + x(n-1)$ **b)** $y(n) = a\,x(2n) + x(n^2)$

c) $y(n) = \sum_{m=-1}^{n} x(m) + \sum_{m=-\infty}^{n} x(2m)$ **d)** $h(n) = (0.1)^n\,u(n+2)$ **e)** $y(n) = \sum_{k=-4}^{2} x(n-k)$

E6.10 Test the stability of the following discrete time systems.

a) $y(n) = x^2(n) + x(n+1)$ **b)** $y(n) = nx(n-1)$ **c)** $h(n) = (0.2)^n\,u(n+3)$

d) $h(n) = (4)^n\,u(4-n)$ **e)** $y(n) = x(n-5)$

E6.11 Determine the range of values of 'a' and 'b' for the stability of an LTI system with impulse response,

$$h(n) = \begin{cases} (-2a)^n & ; \quad n \geq 0 \\ b^{-n} & ; \quad n < 0 \end{cases}$$

E6.12 a) Determine the impulse response for the cascade of two LTI systems having impulse responses,

$$h_1(n) = \left(\frac{1}{5}\right)^n u(n) \quad \text{and} \quad h_2(n) = \delta(n-2)$$

b) Determine the overall impulse response of the interconnected discrete time system shown in fig E6.12.

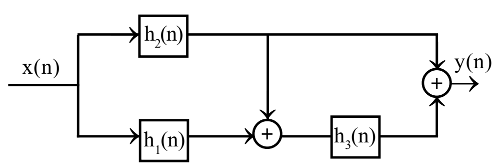


Fig E6.12.

Take, $h_1(n) = \left(\frac{1}{7}\right)^n u(n)$; $h_2(n) = \left(\frac{1}{3}\right)^n u(n)$; $h_3(n) = \left(\frac{1}{9}\right)^n u(n)$

E6.13 Determine the response of an LTI system whose impulse response h(n) and input x(n) are given by,

a) $h(n) = \{1,\ 2,\ \underset{\uparrow}{1},\ -2,\ -1\}$, $x(n) = \{1,\ \underset{\uparrow}{2},\ 3,\ -1,\ -3\}$

b) $h(n) = \begin{cases} 1 & ; \quad 0 \leq n \leq 3 \\ 0 & ; \quad n \geq 4 \end{cases}$, $x(n) = a^n u(n)$; $|a| < 1$

E6.14 Perform circular convolution.

a) $x_1(n) = \{1,\ \ 2,\ \ -1,\ \ 1\}$; $\qquad x_2(n) = \{2,\ \ 4,\ \ 2,\ \ 1\}$

b) $x_1(n) = \{0,\ \ 0.5,\ \ 1,\ \ 1.5,\ \ 2\}$; $\qquad x_2(n) = \{-2,\ \ 2,\ \ 0.4,\ \ 0.6,\ \ 0.8\}$

E6.15 The input x(n) and impulse response h(n) of an LTI system are given by,

$x(n) = \{-1,\ \ \underset{\uparrow}{1},\ \ -2,\ \ -1,\ \ 1,\ \ 2\}$; $\qquad h(n) = \{-0.5,\ \ \underset{\uparrow}{0.5},\ \ -1,\ \ 0.25,\ \ -1,\ \ -2\}$

Find the response of the system using **a)** Linear convolution, **b)** Circular convolution.

E6.16 Perform linear convolution of the following sequences by,

a) Overlap add method **b)** Overlap save method

$x(n) = \{-1,\ \ 1,\ \ 2,\ \ -1,\ \ 1,\ \ 2,\ \ -1,\ \ 1,\ \ -1\}$; $\qquad h(n) = \{2,\ \ 3,\ \ -2\}$

E6.17 Perform crosscorrelation.

$x(n) = \{\underset{\uparrow}{-1},\ \ 1\ \ 3,\ \ -4,\}$; $\qquad h(n) = \{2,\ \ \underset{\uparrow}{-1},\ \ -2,\}$

E6.18 Determine autocorrelation sequence for $x(n) = \{1,\ \ \underset{\uparrow}{4},\ \ -3,\ \ 2,\ \ 1\}$.

E6.19 Find the inverse system for the discrete time system,

$$y(n) = \sum_{m=0}^{n} a^m\, x(m-1) \ ; \quad \text{for} \quad n \ge 0$$

E6.20 The response of a discrete time system is, $y(n) = \{\underset{\uparrow}{4},\ \ 9,\ \ 6,\ \ 7.5,\ \ 3,\ \ 30,\ \ -7\}$ when excited by an input x(n). If the impulse response of the system is $h(n) = \{2,\ \ \underset{\uparrow}{4},\ \ -1\}$ then what will be the input to the system?

E6.21 Preform circular correlation of the sequences, $x(n) = \{-1,\ \ 1,\ \ 2,\ \ -5\}$ and $y(n) = \{4,\ \ 3,\ \ -1,\ \ 1\}$.

Answers

E6.1 **a)** periodic; N = 16 **b)** nonperiodic **c)** periodic; N = 6 **d)** periodic; N = 6 **e)** nonperiodic.

E6.2 **a)** $x_e(n) = a^n + a^{-n}$

$x_0(n) = a^{-n} - a^n$

b) $x_e(n) = 7\cos\frac{\pi}{5}n$

$x_0(n) = \frac{7}{j}\sin\frac{\pi}{5}n$

c) $x_e(n) = \{-2,\ 1,\ -3,\ \underset{\uparrow}{3},\ -3,\ 1,\ -2\}$

$x_0(n) = \{2,\ -1,\ 3,\ \underset{\uparrow}{0},\ -3,\ 1,\ -2\}$

E6.3 **a)** $x(n) = 5\sin\frac{5\pi n}{6}$; Alias frequency = 85Hz **b)** $F_s = 35\text{Hz}$

c) $F_{s,min} = 100\text{Hz}$; $x(nT) = 2\sin\frac{2\pi n}{5} + \cos\frac{\pi}{2}n$ $\quad (\sin\pi n = 0,\ \text{for integer } n)$

The component $3\sin 100\pi t$ will give always zero samples when sampled at 100Hz for any value of n (Refer fig E6.3c).

E6.4 **a)** E = 1.6 ; P = 0 ; Energy signal.

b) $E = \infty$; P = 0.5 ; Power signal.

c) $E = \infty$; P = 0.25 ; Power signal.

d) $E = \infty$; P = 2 ; Power signal.

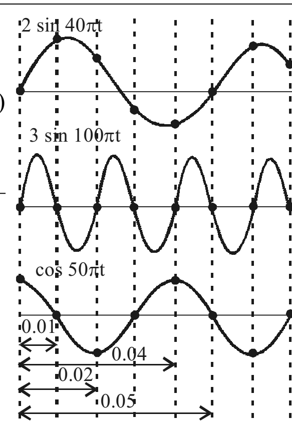


Fig E6.3c : Sampling points.

E6.5 a)

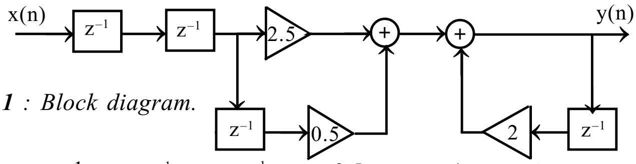


Fig E6.5a.1 : Block diagram.

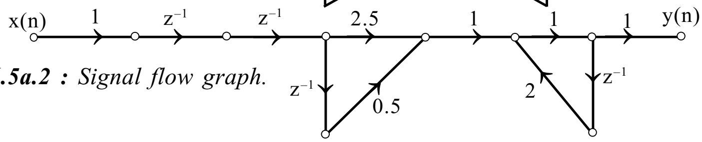


Fig E6.5a.2 : Signal flow graph.

E6.5 b)

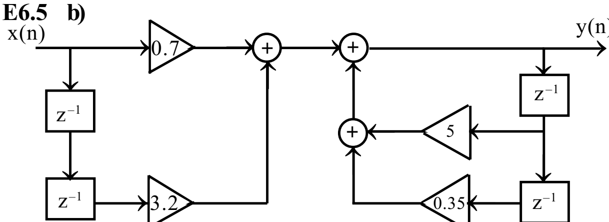


Fig E6.5b.1 : Block diagram.

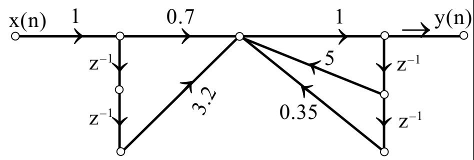


Fig E6.5b.2 : Signal flow graph.

E6.6 **a)** $y(n) = \left[\frac{1}{2}\left(\frac{1}{2}\right)^n - (0.2)^n\right] u(n)$ **b)** $y(n) = \left[0.4 - 0.0176\ (-0.3)^n + 6.9176\ (-2)^n\right] u(n)$

E6.7 **a) c)** Time invariant. **b) d)** Time variant

E6.8 **a) e)** Linear **b) c) d)** Nonlinear

E6.9 **a)** Causal **b) c) d) e)** Noncausal

E6.10 **a) c) d) e)** Stable system **b)** Unstable system

E6.11 For stability, $0 < |a| < \frac{1}{2}$ and $0 < |b| < 1$.

E6.12 **a)** $h(n) = \left(\frac{1}{5}\right)^{(n-2)} u(n-2)$ **b)** $h(n) = \left[\frac{5}{2}\left(\frac{1}{3}\right)^n - 4\left(\frac{1}{9}\right)^n + \left(\frac{9}{2}\right)\left(\frac{1}{7}\right)^n\right] u(n)$

E6.13 **a)** $y(n) = \{1,\ 4,\ 8,\ \underset{\uparrow}{5},\ -7,\ -15,\ -4,\ 7,\ 3\}$ **b)** $y(n) = \sum_{k=n-3}^{n} a^k$; for $n \ge 0$

E6.14 **a)** $x_3(n) = \{\underset{\uparrow}{6},\ 9,\ 9,\ 3\}$ b) $x_3(n) = \{\underset{\uparrow}{5.6},\ 1.5,\ 1.4,\ 0.8,\ -0.3\}$

E6.15 $y(n) = \{0.5\quad \underset{\uparrow}{-1}\quad 2.5\quad -1.75\quad 2.25\quad 1\quad -0.25\quad 3.25\quad 1.5\quad -4\quad -4\}$

E6.16 **a)** Overlap add method : $y(n) = \{-2,\ -1,\ 9,\ 2,\ -5,\ 9,\ 2,\ -5,\ 3,\ -5,\ 2\}$
b) Overlap save method : $y(n) = \{\ *,\ \ *,\ 9,\ 2,\ -5,\ 9,\ 2,\ -5,\ 3,\ -5,\ 2\}$

E6.17 $r_{xy}(m) = \{2,\ \underset{\uparrow}{-1},\ -9,\ 7,\ 10,\ -8,\}$

E6.18 $r_{xx}(m) = \{1,\ 6,\ 2,\ -12,\ \underset{\uparrow}{31},\ -12,\ 2,\ 6,\ 1\}$

E6.19 $x(n) = \frac{1}{a^{n+1}}\ [y(n+1) - y(n)]$

E6.20 $x(n) = \{\underset{\uparrow}{2},\ 0.5,\ 3,\ -2,\ 7\}$

E6.21 $r_{xy}(m) = \{-8,\ 14,\ -5,\ -22\}$

CHAPTER 7

Ƶ-Transform

7.1 Introduction

Transform techniques are an important tool in the analysis of signals and systems. The Laplace transforms are popularly used for analysis of continuous time signals and systems. Similarly Ƶ-transform plays an important role in analysis and representation of discrete time signals and systems. The Ƶ-transform provides a method for the analysis of discrete time signals and systems in the frequency domain which is generally more efficient than its time domain analysis.

The Ƶ-transform of x(n) will convert the time domain signal x(n) to z-domain signal X(z), where the signal becomes a function of complex variable z.

The complex variable z is defined as,

$$z = u + jv = r\,e^{j\omega}$$

where, u = Real part of z ; v = Imaginary part of z

$$r = \sqrt{u^2 + v^2} = \text{Magnitude of } z$$

$$\omega = \tan^{-1}\frac{v}{u} = \text{Phase or Argument of } z$$

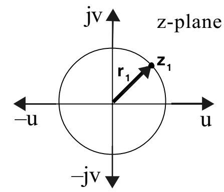


Fig 7.1 : *z-plane.*

The u and v take values from $-\infty$ to $+\infty$. A two dimensional complex plane with values of u on horizontal axis and values of v on vertical axis as shown in fig 7.1 is called z-plane. A circle with radius r_1 in z-plane represents all values of z_1 having same magnitude r_1 with variable phase (i.e., $\omega = 0$ to 2π).

History of Ƶ-Transform

A transform of a sampled signal or sequence was defined in 1947 by W. Hurewicz as,

$$z[f(kT)] = \sum_{k=0}^{\infty} f(kT)\,z^{-k}$$

which was later denoted in 1952 as Ƶ-transform by a sampled-data control group at Columbia University led by professor John R. Raggazini and including L.A. Zadeh, E.I. Jury, R.E. Kalman, J.E. Bertram, B. Friedland and G.F. Franklin, (Source : www.ling.upenn.edu).

Definition of Ƶ-Transform

Let, x(n) = Discrete time signal

X(z) = Ƶ-transform of x(n)

The ***Ƶ-transform*** of a discrete time signal, x(n) is defined as,

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}} \quad ; \quad \text{where, z is a complex variable.} \qquad .....(7.1)$$

The Ƶ-transform of x(n) is symbolically denoted as,

Ƶ{x(n)} ; where, Ƶ is the operator that represents Ƶ-transform.

$$\boxed{\therefore\ X(z) = Ƶ\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,z^{-n}}$$

Since the time index n is defined for both positive and negative values, the discrete time signal x(n) in equation (7.1) is considered to be two sided and the transform is called ***two sided Z-transform***. If the signal x(n) is one sided signal, (i.e., x(n) is defined only for positive value of n) then the Z-transform is called ***one sided Z-transform.***

The one sided Z-transform of x(n) is defined as,

$$\boxed{X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n}} \quad .....(7.2)$$

The computation of X(z) involves summation of infinite terms which are functions of z. Hence it is possible that the infinite series may not converge to finite value for certain values of z. Therefore for every X(z) there will be a set of values of z for which X(z) can be computed. Such a set of values will lie in a particular region of z-plane and this region is called Region Of Convergence (ROC) of X(z).

Inverse Z-Transform

Let, X(z) be Z-transform of x(n). Now the signal x(n) can be uniquely determined from X(z) and its region of convergence (ROC).

The ***inverse Z-transform*** of X(z) is defined as,

$$\boxed{x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz} \quad .....(7.3)$$

The inverse Z-transform of X(z) is symbolically denoted as,

$\mathcal{Z}^{-1}\{X(z)\}$; where, $\mathcal{Z}^{-1}$ is the operator that represents the inverse Z-transform

$$\boxed{\therefore\ x(n) = \mathcal{Z}^{-1}\{X(z)\} = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz}$$

We also refer x(n) and X(z) as a Z-transform pair and this relation is expressed as,

$$\boxed{x(n) \underset{\mathcal{Z}^{-1}}{\overset{\mathcal{Z}}{\longleftrightarrow}} X(z)}$$

Proof :

Consider the definition of Z-transform of x(n),

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k} \qquad \boxed{\text{Let } n \to k}$$

$$X(z)\, z^{n-1} = \sum_{k=-\infty}^{+\infty} x(k)\, z^{-k}\, z^{n-1} \qquad \boxed{\text{Multiply both sides by } z^{n-1}}$$

Let us integrate the above equation on both sides over a closed contour "C" within the ROC of X(z) which encloses the origin.

$$\therefore \oint_c X(z)\, z^{n-1}\, dz = \oint_c \sum_{k=-\infty}^{+\infty} x(k)\, z^{n-1-k} dz$$

$$= \sum_{k=-\infty}^{+\infty} x(k) \oint_c z^{n-1-k} dz \qquad \boxed{\text{Interchanging the order of summation and integration}}$$

$$= 2\pi j \sum_{k=-\infty}^{+\infty} x(k)\, \frac{1}{2\pi j} \oint_c z^{n-1-k} dz \qquad \boxed{\text{Multiply and divide by } 2\pi j} \quad .....(7.4)$$

By Cauchy integral theorem,

$$\frac{1}{2\pi j}\oint_c z^{n-1-k}\, dz = 1 \quad ; \quad k = n$$

$$= 0 \quad ; \quad k \neq n$$

On applying Cauchy integral theorem the equation (7.4) reduces to,

$$\oint_c X(z)\, z^{n-1}\, dz = 2\pi j\, x(n) \qquad \boxed{\sum_{k=-\infty}^{+\infty} x(k)\Big|_{n=k} = x(n)}$$

$$\therefore\ x(n) = \frac{1}{2\pi j}\oint_c X(z)\, z^{n-1}\, dz$$

Geometric Series

The $\mathcal{Z}$-transform of a discrete time signal involves convergence of geometric series. Hence the following two geometric series sum formula will be useful in evaluating $\mathcal{Z}$-transform.

1. Infinite geometric series sum formula.

   If C is a complex constant and $0 < |C| < 1$, then,

$$\boxed{\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}} \qquad .....(7.5)$$

2. Finite geometric series sum formula.

   If C is a complex constant and,

$$\boxed{\text{When } C \neq 1, \quad \sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C} = \frac{C^N-1}{C-1}} \qquad .....(7.6)$$

$$\boxed{\text{When } C = 1, \quad \sum_{n=0}^{N-1} C^n = N} \qquad .....(7.7)$$

> ***Note :*** *The infinite geometric series sum formula requires that the magnitude of C be strictly less than unity, but the finite geometric series sum formula is valid for any value of C.*

7.2 Region of Convergence

Since the $\mathcal{Z}$-transform is an infinite power series, it exists only for those values of z for which the series converges. The ***region of convergence, (ROC)*** of X(z) is the set of all values of z, for which X(z) attains a finite value. The ROC for the following six types of signals are discussed here.

Case i : Finite duration, right sided (causal) signal

Case ii : Finite duration, left sided (anticausal) signal

Case iii : Finite duration, two sided (noncausal) signal

Case iv : Infinite duration, right sided (causal) signal

Case v : Infinite duration, left sided (anticausal) signal

Case vi : Inifinite duration, two sided (noncausal) signal

Case i : Finite duration, right sided (causal) signal

Let, x(n) be a finite duration signal with N-samples, defined in the range $0 \le n \le (N-1)$.

$$\therefore\ x(n) = \{x(0),\ x(1),\ x(2), ...x(N-1)\}$$

Now, the $\mathcal{Z}$ - transform of x(n) is,

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + ...x(N-1)\, z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + ...+ \frac{x(N-1)}{z^{N-1}}$$

In the above summation, when z = 0, all the terms except the first term become infinite. Hence the X(z) exists for all values of z, except z = 0. *Therefore, the ROC for finite duration right sided (or causal signal) is entire z-plane except z = 0.*

Case ii : Finite duration, left sided (anticausal) signal

Let, x(n) be a finite duration signal with N-samples, defined in the range $-(N-1) \le n \le 0$.

$$\therefore\; x(n) = \{x(-(N-1)),.....,x(-2),\ x(-1),\ x(0)\}$$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-(N-1)}^{0} x(n)\, z^{-n}$$
$$= x(-(N-1))\, z^{(N-1)} + ... + x(-2)z^2 + x(-1)z + x(0)$$

In the above summation, when $z = \infty$, all the terms except the last term become infinite. Hence the X(z) exists for all values of z, except, $z = \infty$. *Therefore, the ROC of X(z) is entire z-plane, except $z = \infty$.*

Case iii : Finite duration, two sided (noncausal) signal

Let, x(n) be a finite duration signal with N-samples, defined in the range $-M \le n \le +M$, where, $M = \dfrac{N-1}{2}$

$$\therefore\; x(n) = \left\{x(-M),......., x(-2),\ x(-1),\ x(0),\ x(1),\ x(2),\ ....x(M)\right\}$$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-M}^{+M} x(n)\, z^{-n}$$
$$= x(-M)\, z^{M} + ... + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + ... + x(M)\, z^{-M}$$
$$= x(-M)\, z^{M} + ... + x(-2)z^2 + x(-1)z + x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + ... + \frac{x(M)}{z^M}$$

In the above summation, when z = 0, the terms with negative power of z attain infinity and when $z = \infty$, the terms with positive power of z attain infinity. Hence X(z) converges for all values of z, except z = 0 and $z = \infty$. *Therefore, the ROC is entire z-plane, except z = 0 and $z = \infty$.*

Case - iv : Infinite duration, right sided (causal) signal

Let, $x(n) = r_1^{\,n}$; $n \ge 0$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} r_1^n\, z^{-n} = \sum_{n=0}^{\infty} \left(r_1\, z^{-1}\right)^n$$

If, $0 < |r_1 z^{-1}| < 1$, then $\sum\limits_{n=0}^{\infty} (r_1 z^{-1})^n = \dfrac{1}{1-r_1 z^{-1}}$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

if, $0 < |C| < 1$

$$\therefore\ X(z) = \frac{1}{1-r_1 z^{-1}} = \frac{1}{1-\dfrac{r_1}{z}} = \frac{1}{\dfrac{z-r_1}{z}} = \frac{z}{z-r_1}$$

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < |r_1 z^{-1}| < 1$$

$$\therefore\ |r_1 z^{-1}| < 1$$

$$\frac{|r_1|}{|z|} < 1 \quad \Rightarrow \quad |z| > |r_1|$$

The term $|r_1|$ represent a circle of radius r_1 in z-plane as shown in fig 7.2. From the above analysis we can say that, X(z) converges for all points external to the circle of radius r_1 in z-plane. *Therefore, the ROC of X(z) is exterior of the circle of radius r_1 in z-plane as shown in fig 7.2.*

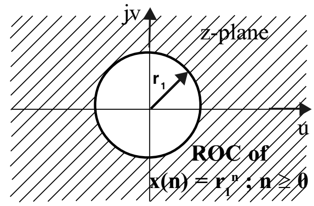


Fig 7.2 : *ROC of infinite duration right sided signal.*

Case v : Infinite duration, left sided (anticausal) signal

Let, $x(n) = r_2^n \ ; n \le 0$

Now, the Ƶ-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} r_2^n\, z^{-n} = \sum_{n=0}^{+\infty} r_2^{-n}\, z^n = \sum_{n=0}^{+\infty} (r_2^{-1} z)^n$$

If, $0 < |r_2^{-1} z| < 1$, then $\sum\limits_{n=0}^{\infty} (r_2^{-1} z)^n = \dfrac{1}{1-r_2^{-1} z}$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

if, $0 < |C| < 1$

$$\therefore\ X(z) = \frac{1}{1-r_2^{-1} z} = \frac{1}{1-\dfrac{z}{r_2}} = \frac{1}{\dfrac{r_2-z}{r_2}} = \frac{r_2}{r_2-z}$$

$$= -\frac{r_2}{z-r_2}$$

Here the condition to be satisfied for the convergence of X(z) is,

$$0 < |r_2^{-1} z^{-1}| < 1$$

$$\therefore\ |r_2^{-1} z| < 1$$

$$\frac{|z|}{|r_2|} < 1 \quad \Rightarrow \quad |z| < |r_2|$$

The term $|r_2|$ represent a circle of radius r_2 in z-plane as shown in fig 7.3. From the above analysis we can say that X(z) converges for all points internal to the circle of radius r_2 in z-plane. *Therefore, the ROC of X(z) is interior of the circle of radius r_2 as shown in fig 7.3.*

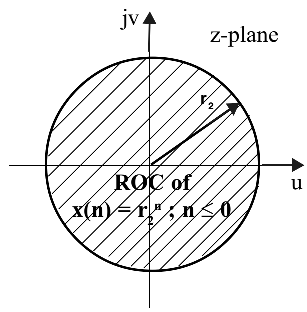


Fig 7.3 : *ROC of infinite duration left sided signal.*

***Case vi:* Infinite duration, two sided (anticausal) signal**

Let, $x(n) = r_1^n\, u(n) + r_2^n\, u(-n)$

Now, the Z-transform of x(n) is,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{0} r_2^n\, z^{-n} + \sum_{n=0}^{+\infty} r_1^n\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} r_2^{-n}\, z^{n} + \sum_{n=0}^{+\infty} r_1^n\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} (r_2^{-1} z)^n + \sum_{n=0}^{+\infty} (r_1 z^{-1})^n$$

$$= \frac{1}{1 - r_2^{-1} z} + \frac{1}{1 - r_1 z^{-1}}$$

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$$
if, $0 < |C| < 1$

Using infinite geometric series sum formula
if, $0 < |r_2^{-1} z| < 1$, and, $0 < |r_1 z^{-1}| < 1$

The term $\sum_{n=0}^{\infty} (r_2^{-1} z)^n$ converges if,

$$0 < \left|r_2^{-1} z\right| < 1$$

$$\therefore \quad \left|r_2^{-1} z\right| < 1$$

$$\frac{|z|}{|r_2|} < 1 \quad \Rightarrow \quad |z| < |r_2|$$

The term $\sum_{n=0}^{\infty} (r_1 z^{-1})$ converges if,

$$0 < \left|r_1^{-1} z\right| < 1$$

$$\therefore \quad \left|r_1 z^{-1}\right| < 1$$

$$\frac{|r_1|}{|z|} < 1 \quad \Rightarrow \quad |z| > |r_1|$$

The term $|r_2|$ represents a circle of radius r_2 and $|r_1|$ represents a circle of radius r_1 in z-plane. If $|r_2| > |r_1|$ then there will be a region between two circles as shown in fig 7.4. Now the X(z) will converge for all points in the region between two circles (because the first term of X(z) converges for $|z| < |r_2|$ and the second term of X(z) converges for $|z| > |r_1|$). Hence the *ROC is the region between two circles of radius r_1 and r_2* as shown in fig 7.4.

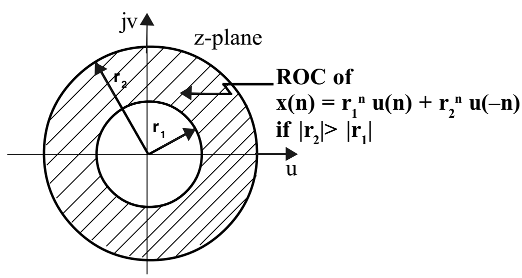
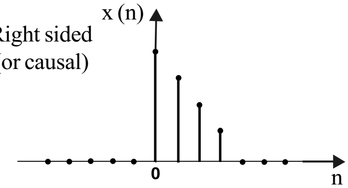
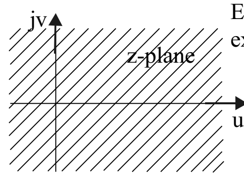


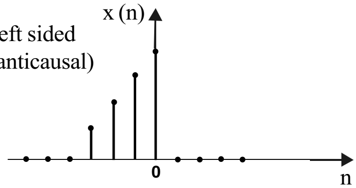
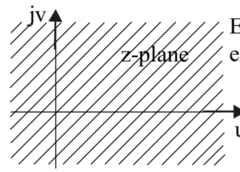
Fig 7.4 : *ROC of infinite duration two sided signal..*

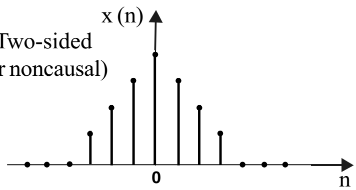
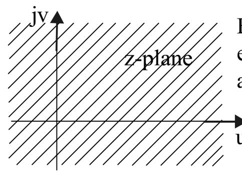
Table 7.1 Summary of ROC of Discrete Time Signals

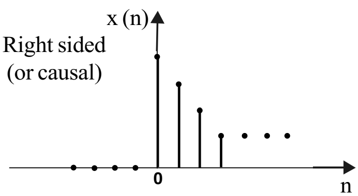
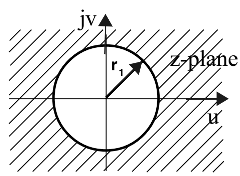
Sequence	ROC
Finite, right sided (causal)	Entire z-plane except z = 0
Finite, left sided (anticausal)	Entire z-plane except z = ∞
Finite, two sided (noncausal)	Entire z-plane except z = 0 and z = ∞
Infinite, right sided (causal)	Exterior of circle of radius r_1, where $\|z\| > r_1$
Infinite, left sided (anticausal)	Interior of circle of radius r_2, where $\|z\| < r_2$
Infinite, two sided (noncausal)	The area between two circles of radius r_2 and r_1 where, $r_2 > r_1$, and $r_1 < \|z\| < r_2$, (i.e., $\|z\| > r_1$, and, $\|z\| < r_2$)

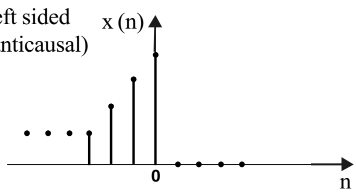
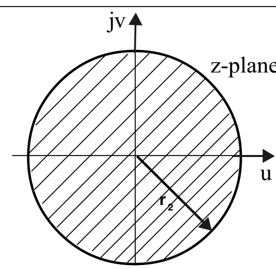
Table-7.2 : Characteristic Families of Signals and Corresponding ROC

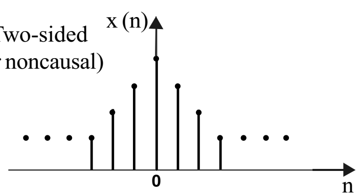
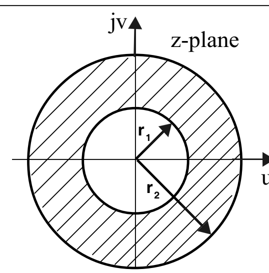
<table>
<tr><th>Signal</th><th>ROC in z-plane</th></tr>
<tr><td colspan="2">Finite Duration Signals</td></tr>
<tr><td>Right sided
(or causal)

</td><td>

Entire z-plane except z = 0</td></tr>
<tr><td>Left sided
(or anticausal)

</td><td>

Entire z-plane except z = ∞</td></tr>
<tr><td>Two-sided
(or noncausal)

</td><td>

Entire z-plane except z = 0 and z = ∞</td></tr>
<tr><td colspan="2">Infinite Duration Signals</td></tr>
<tr><td>Right sided
(or causal)

</td><td>

$|z| > r_1$</td></tr>
<tr><td>Left sided
(or anticausal)

</td><td>

$|z| < r_2$</td></tr>
<tr><td>Two-sided
(or noncausal)

</td><td>

$r_1 < |z| < r_2$
$[|z| > r_1$
and $|z| < r_2]$</td></tr>
</table>

Example 7.1

Determine the $\mathcal{Z}$-transform and their ROC of the following discrete time signals.

a) $x(n) = \{\underset{\uparrow}{3}, 2, 5, 7\}$ b) $x(n) = \{6, 4, 5, \underset{\uparrow}{3}\}$ c) $x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 3\}$

Solution

a) Given that, $x(n) = \{\underset{\uparrow}{3}, 2, 5, 7\}$

i.e., $x(0) = 3$; $x(1) = 2$; $x(2) = 5$; $x(3) = 7$; and $x(n) = 0$ for $n < 0$ and for $n > 3$.

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = 0 to 3, hence the limits of summation is changed to n = 0 to n = 3.

$$\therefore\ X(z) = \sum_{n=0}^{3} x(n)\, z^{-n}$$

$$= x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3}$$

$$= 3 + 2z^{-1} + 5z^{-2} + 7z^{-3}$$

$$= 3 + \frac{2}{z} + \frac{5}{z^2} + \frac{7}{z^3}$$

In X(z), when z = 0, except the first terms all other terms will become infinite. Hence X(z) will be finite for all values of z, except z = 0. Therefore, the ROC is entire z-plane except z = 0.

---

b) Given that, $x(n) = \{6, 4, 5, \underset{\uparrow}{3}\}$

i.e, $x(-3) = 6$; $x(-2) = 4$; $x(-1) = 5$; $x(0) = 3$; and $x(n) = 0$ for $n < -3$ and for $n > 0$.

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = –3 to 0, hence the limits of summation is changed to n = –3 to 0.

$$\therefore\ X(z) = \sum_{n=-3}^{0} x(n)\, z^{-n}$$

$$= x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z + x(0)$$

$$= 6z^{3} + 4z^{2} + 5z + 3$$

In X(z), when $z = \infty$, except the last term all other terms become infinite. Hence X(z) will be finite for all values of z, except $z = \infty$. Therefore, the ROC is entire z-plane except $z = \infty$.

---

c) Given that, $x(n) = \{2, 4, \underset{\uparrow}{5}, 7, 3\}$

i.e, $x(-2) = 2$; $x(-1) = 4$; $x(0) = 5$; $x(1) = 7$; $x(2) = 3$ and $x(n) = 0$ for $n < -2$ and for $n > 2$.

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

The given sequence is a finite duration sequence defined in the range n = –2 to +2, hence the limits of summation is changed to n = –2 to n = 2.

$$\therefore\ X(z) = \sum_{n=-2}^{2} x(n)\, z^{-n}$$

$$= x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2}$$

$$= 2z^2 + 4z + 5 + 7z^{-1} + 3z^{-2}$$

$$= 2z^2 + 4z + 5 + \frac{7}{z} + \frac{3}{z^2}$$

In X(z), when z = 0, the terms with negative power of z will become infinite and when z = ∞, the terms with positive power of z will become infinite. Hence X(z) will be finite for all values of z except when z = 0 and z = ∞. Therefore, the ROC is entire z-plane except z = 0 and z = ∞.

Example 7.2

Determine the $\mathcal{Z}$-transform and their ROC of the following discrete time signals.

a) $x(n) = u(n)$ b) $x(n) = 0.5^n\, u(n)$ c) $x(n) = 0.2^n\, u(-n-1)$ d) $x(n) = 0.5^n\, u(n) + 0.8^n\, u(-n-1)$

Solution

a) Given that, x(n) = u(n)

The u(n) is a discrete unit step signal, which is defined as,

$$u(n) = 1\ ;\ \text{for } n \ge 0$$
$$= 0\ ;\ \text{for } n < 0$$

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = \frac{1}{1-z^{-1}}$$

$$= \frac{1}{1-1/z} = \frac{z}{z-1}$$

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \quad \text{if, } 0 < |C| < 1$$

Using infinite geometric series sum formula

Here the condition for convergence is, $0 < |z^{-1}| < 1$.

$$\therefore\ |z^{-1}| < 1 \quad \Rightarrow \quad \frac{1}{|z|} < 1 \quad \Rightarrow \quad |z| > 1$$

The term |z| = 1 represents a circle of unit radius in z-plane. Therefore, the ROC is exterior of unit circle in z-plane.

b) Given that, $x(n) = 0.5^n\, u(n)$

The u(n) is a discrete unit step signal, which is defined as,

$$u(n) = 1\ ;\ \text{for } n \ge 0$$
$$= 0\ ;\ \text{for } n < 0$$
$$\therefore\ x(n) = 0.5^n\ ;\ \text{for } n \ge 0$$
$$= 0\ ;\ \text{for } n < 0$$

By the definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} 0.5^n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(0.5z^{-1}\right)^n = \frac{1}{1 - 0.5z^{-1}}$$

Using infinite geometric series sum formula

$$\therefore\ X(z) = \frac{1}{1 - 0.5\,\frac{1}{z}} = \frac{z}{z - 0.5}$$

Here the condition for convergence is, $0 < |0.5\, z^{-1}| < 1$.

$$\therefore\ |0.5\, z^{-1}| < 1 \quad \Rightarrow \quad \frac{0.5}{|z|} < 1 \quad \Rightarrow \quad |z| > 0.5$$

The term $|z| = 0.5$ represents a circle of radius 0.5 in z-plane. Therefore, the ROC is exterior of circle with radius 0.5 in z-plane.

---

c) Given that, $x(n) = 0.8^n\, u(-n-1)$

The $u(-n-1)$ is a discrete unit step signal, which is defined as,

$$\begin{aligned} u(-n-1) &= 0 \quad ; \text{ for } n \ge 0 \\ &= 1 \quad ; \text{ for } n \le -1 \\ \therefore\ x(n) &= 0 \quad ; \text{ for } n \ge 0 \\ &= 0.8^n \ ; \text{ for } n \le -1 \end{aligned}$$

By the definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=-\infty}^{-1} 0.8^n\, z^{-n}$$

$$= \sum_{n=1}^{\infty} 0.8^{-n} z^n = \sum_{n=1}^{\infty} (0.8^{-1} z)^n = \sum_{n=0}^{\infty} (0.8^{-1} z)^n - 1 \qquad \boxed{(0.8^{-1} z)^0 = 1}$$

$$= \frac{1}{1 - (0.8^{-1} z)} - 1 \qquad \boxed{\text{Using infinite geometric series sum formula}}$$

$$= \frac{1}{1 - \frac{z}{0.8}} - 1 = \frac{0.8}{0.8 - z} - 1 = \frac{0.8 - 0.8 + z}{0.8 - z} = \frac{z}{0.8 - z} = -\frac{z}{z - 0.8}$$

Here the condition for convergence is, $0 < |0.8^{-1} z| < 1$.

$$\therefore\ |0.8^{-1} z| < 1 \quad \Rightarrow \quad \frac{|z|}{0.8} < 1 \quad \Rightarrow \quad |z| < 0.8$$

The term $|z| = 0.8$, represents a circle of radius 0.8 in z-plane. Therefore, the ROC is interior of the circle of radius 0.8 in z-plane.

---

d) Given that, $x(n) = 0.5^n\, u(n) + 0.8^n\, u(-n-1)$

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.5^n\, u(n) + 0.8^n\, u(-n-1)\}$$

$$= \mathcal{Z}\{0.5^n\, u(n)\} + \mathcal{Z}\{0.8^n\, u(-n-1)\} \qquad \boxed{\text{Using linearity property}}$$

$$= \frac{z}{z - 0.5} - \frac{z}{z - 0.8} \qquad \boxed{\text{Using the results of (b) and (c)}}$$

$$= \frac{z(z - 0.8) - z(z - 0.5)}{(z - 0.5)\,(z - 0.8)} = \frac{z^2 - 0.8z - z^2 + 0.5z}{z^2 - 0.8z - 0.5z + 0.4} = \frac{-0.3z}{z^2 - 1.3z + 0.4}$$

Here the condition for convergence of $0.5^n\, u(n)$ is,

$$0 < |0.5\, z^{-1}| < 1 \quad \Rightarrow \quad |z| > 0.5$$

and the condition for convergence of $0.8^n\, u(-n-1)$ is,

$$0 < |0.8^{-1} z| < 1 \quad \Rightarrow \quad |z| < 0.8$$

The term $|z| = 0.8$, represents a circle of radius 0.8 in z-plane and the term $|z| = 0.5$ represents a circle of radius 0.5 in z-plane. Hence the common region of convergence for both the terms of x(n) is the region in between the circles of radius $|z| = 0.8$ and $|z| = 0.5$ in the z-plane.

7.3 Properties of $\mathcal{Z}$-Transform

1. Linearity property

The linearity property of $\mathcal{Z}$-transform states that the $\mathcal{Z}$-transform of linear weighted combination of discrete time signals is equal to similar linear weighted combination of $\mathcal{Z}$-transform of individual discrete time signals.

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$ then by linearity property,

$$\mathcal{Z}\{a_1x_1(n) + a_2x_2(n)\} = a_1X_1(z) + a_2X_2(z) \qquad ; \quad \text{where, } a_1 \text{ and } a_2 \text{ are constants.}$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X_1(z) = \mathcal{Z}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad .....(7.8)$$

$$X_2(z) = \mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(7.9)$$

$$\mathcal{Z}\{a_1x_1(n) + a_2x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n) + a_2x_2(n)\right] z^{-n} = \sum_{n=-\infty}^{+\infty} \left[a_1x_1(n)\, z^{-n} + a_2x_2(n)\, z^{-n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1x_1(n)\, z^{-n} + \sum_{n=-\infty}^{+\infty} a_2x_2(n) z^{-n} = a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n) z^{-n}$$

$$= a_1\, X_1(z) + a_2X_2(z) \qquad \boxed{\text{Using equations (7.8) and (7.9)}}$$

2. Shifting property

Case i: *Two sided $\mathcal{Z}$-transform*

The shifting property of $\mathcal{Z}$-transform states that, $\mathcal{Z}$-transform of a shifted signal shifted by m-units of time is obtained by multiplying z^m to $\mathcal{Z}$-transform of unshifted signal.

Let, $\mathcal{Z}\{x(n)\} = X(z)$

Now, by shifting property,

$$\mathcal{Z}\{x(n-m)\} = z^{-m}\, X(z)$$

$$\mathcal{Z}\{x(n+m)\} = z^{m}\, X(z)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(7.10)$$

$$\mathcal{Z}\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, z^{-n}$$

$$\boxed{\begin{array}{ll} \text{Let, } n-m=p, & \therefore n=p+m \\ \text{when } n\to-\infty, & p\to-\infty \\ \text{when } n\to+\infty, & p\to+\infty \end{array}}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-(m+p)}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-m}\, z^{-p}$$

$$= z^{-m} \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} = z^{-m} \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad \boxed{\text{Let, } p \to n}$$

$$= z^{-m}\, X(z) \qquad \boxed{\text{Using equation (7.10)}}$$

$$\mathcal{Z}\{x(n+m)\} = \sum_{n=-\infty}^{+\infty} x(n+m)\, z^{-m}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-(p-m)}$$

Let, $n+m=p, \quad \therefore n=p-m$
when $n \to -\infty, \quad p \to -\infty$
when $n \to +\infty, \quad p \to +\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} z^{m}$$

$$= z^{m} \sum_{p=-\infty}^{+\infty} x(p)\, z^{-p} = z^{m} \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

Let, $p \to n$

$$= z^{m} X(z)$$

Using equation (7.10)

Case ii: ***One sided Z-transform***

Let x(n) be a discrete time signal defined in the range $0 < n < \infty$.

Let, $\mathcal{Z}\{x(n)\} = X(z)$

Now by shifting property,

$$\mathcal{Z}\{x(n-m)\} = z^{-m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$$

$$\mathcal{Z}\{x(n+m)\} = z^{m} X(z) - \sum_{i=0}^{m-1} x(i)\, z^{(m-i)}$$

Proof :

By definition of one sided Z-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n} \quad .....(7.11)$$

$$\mathcal{Z}\{x(n-m)\} = \sum_{n=0}^{+\infty} x(n-m)\, z^{-n}$$

$$= \sum_{n=0}^{+\infty} x(n-m)\, z^{-n} z^{m} z^{-m}$$

Multiply by z^{m} and z^{-m}

$$= z^{-m} \sum_{n=0}^{+\infty} x(n-m)\, z^{-(n-m)}$$

$$= z^{-m} \sum_{p=-m}^{+\infty} x(p)\, z^{-p}$$

Let, $n-m=p$,
when $n \to 0, \quad p \to -m$
when $n \to +\infty, \quad p \to +\infty$

$$= z^{-m} \sum_{p=0}^{+\infty} x(p)\, z^{-p} + z^{-m} \sum_{p=-m}^{-1} x(p)\, z^{-p}$$

$$= z^{-m} \sum_{p=0}^{+\infty} x(p)\, z^{-p} + z^{-m} \sum_{p=1}^{m} x(-p)\, z^{p}$$

Let $p = n$, in first summation
Let $p = i$, in second summation

$$= z^{-m} \sum_{n=0}^{+\infty} x(n)\, z^{-n} + z^{-m} \sum_{i=1}^{m} x(-i)\, z^{i}$$

Using equation (7.11)

$$= z^{m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)} \quad .....(7.12)$$

Note : *In equation (7.12) if x(–i) for i = 1 to m are zero then the shifting property for delayed signal will be same as that for two sided Z-transform.*

$$\mathcal{Z}\{x(n+m)\} = \sum_{n=0}^{+\infty} x(n+m)\,z^{-n}$$

$$= \sum_{n=0}^{+\infty} x(n+m)\,z^{-n}\,z^{m}\,z^{-m}$$ Multiply by z^m and z^{-m}

$$= z^m \sum_{n=0}^{+\infty} x(n+m)\,z^{-(n+m)}$$

Let, $n + m = p$,
when $n \to 0, \quad p \to m$
when $n \to +\infty, \quad p \to +\infty$

$$= z^m \sum_{p=m}^{+\infty} x(p)\,z^{-p}$$

$$= z^m \sum_{p=0}^{+\infty} x(p)\,z^{-p} - z^m \sum_{p=0}^{m-1} x(p)\,z^{-p}$$

$$= z^m \sum_{n=0}^{+\infty} x(n)\,z^{-n} - z^m \sum_{i=0}^{m-1} x(i)\,z^{-i}$$ Let $p = n$, in first summation. Let $p = i$, in second summation

$$= z^m X(z) - \sum_{i=0}^{m-1} x(i)\,z^{m-i} \quad .....(7.13)$$ Using equation (7.11)

Note : *In equation (7.13) if x(i) for i = 0 to m−1 are zero then the shifting property for advanced signal will be same as that for two sided $\mathcal{Z}$-transform.*

3. Multiplication by n (or Differentiation in z-domain)

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\{nx(n)\} = -z\dfrac{d}{dz}X(z)$

In general,

$$\mathcal{Z}\{n^m\,x(n)\} = \left(-z\frac{d}{dz}\right)^m X(z)$$

$$= \underbrace{-z\frac{d}{dz}\left(-z\frac{d}{dz}\left(.....\left(-z\frac{d}{dz}\left(-z\frac{d}{dz}\right.\right.\right.\right.}_{m\,-\,\text{times}}\left.\left.X(z)\right)\right)\,.....\,\Big)\Big)$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,z^{-n} \quad .....(7.14)$$

$$\mathcal{Z}\{n\,x(n)\} = \sum_{n=-\infty}^{+\infty} n\,x(n)\,z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} n\,x(n)\,z^{-n}\,z\,z^{-1}$$ Multiply by z and z^{-1}

$$= -z \sum_{n=-\infty}^{+\infty} x(n)\left[-n\,z^{-n-1}\right]$$

$$= -z \sum_{n=-\infty}^{+\infty} x(n)\left[\frac{d}{dz}z^{-n}\right]$$ $\dfrac{d}{dz}z^{-n} = -n\,z^{-n-1}$

$$= -z\frac{d}{dz}\sum_{n=-\infty}^{+\infty} x(n)\,z^{-n}$$ Interchanging summation and differentiation

$$= -z\frac{d}{dz}X(z)$$ Using equation (7.14)

4. Multiplication by an exponential sequence, a^n (or Scaling in z-domain)

If $\quad Z\{x(n)\} = X(z)$

then $Z\left\{a^n\, x(n)\right\} = X(a^{-1}z)$

Proof :

By definition of Z-transform,

$$Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(7.15)$$

$$Z\{a^n\, x(n)\} = \sum_{n=-\infty}^{+\infty} a^n\, x(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, (a^{-1}z)^{-n} \qquad .....(7.16)$$

$$= X(a^{-1}z)$$

The equation (7.16) is similar to the form of equation (7.15)

5. Time reversal

If $\quad Z\{x(n)\} = X(z)$

then $Z\{x(-n)\} = X(z^{-1})$

Proof :

By definition of Z-transform,

$$Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(7.17)$$

$$Z\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n)\, z^{-n}$$

Let, $p = -n$
when $n \to -\infty, \quad p \to +\infty$
when $n \to +\infty, \quad p \to -\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, z^{p}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, (z^{-1})^{-p} \qquad .....(7.18)$$

$$= X(z^{-1})$$

The equation (7.18) is similar to the form of equation (7.17)

6. Conjugation

If $\quad Z\{x(n)\} = X(z)$

then $Z\{x^*(n)\} = X^*(z^*)$

Proof :

By definition of Z-transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad .....(7.19)$$

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{+\infty} x^*(n)\, z^{-n}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, (z^*)^{-n}\right]^* \qquad .....(7.20)$$

$$= \left[X(z^*)\right]^*$$

$$= X^*(z^*)$$

The equation (7.20) is similar to the form of equation (7.19)

7. Convolution theorem

If $\mathcal{Z}\{x_1(n)\} = X_1(z)$

and $\mathcal{Z}\{x_2(n)\} = X_2(z)$

then $\mathcal{Z}\{x_1(n) * x_2(n)\} = X_1(z)\, X_2(z)$

$$\text{where, } x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad \text{.....(7.21)}$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X_1(z) = \mathcal{Z}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n} \qquad \text{.....(7.22)}$$

$$X_2(z) = \mathcal{Z}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad \text{.....(7.23)}$$

$$\mathcal{Z}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[x_1(n) * x_2(n)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] z^{-n} \qquad \boxed{\text{Using equation (7.21)}}$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, z^{-n}\, z^{-m}\, z^{m} \qquad \boxed{\text{Multiply by } z^{m} \text{ and } z^{-m}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, z^{-(n-m)}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} \sum_{p=-\infty}^{+\infty} x_2(p)\, z^{-p} \qquad \boxed{\begin{array}{l}\text{Let, } n-m=p \\ \text{when } n \to -\infty,\ p \to -\infty \\ \text{when } n \to +\infty,\ p \to +\infty\end{array}}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, z^{-n}\right]\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n}\right] \qquad \boxed{\begin{array}{l}\text{Let } m = n, \text{ in first summation} \\ \text{Let } p = n, \text{ in second summation}\end{array}}$$

$$= X_1(z)\, X_2(z) \qquad \boxed{\text{Using equations (7.22) and (7.23)}}$$

8. Correlation property

If $\mathcal{Z}\{x(n)\} = X(z)$ and $\mathcal{Z}\{y(n)\} = Y(z)$

then $\mathcal{Z}\{r_{xy}(m)\} = X(z)\, Y(z^{-1})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad \text{.....(7.24)}$$

Proof :

By definition of $\mathcal{Z}$-transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \qquad \text{.....(7.25)}$$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} \qquad \text{.....(7.26)}$$

$$\mathcal{Z}\{r_{xy}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xy}(m)\, z^{-m}$$

$$= \sum_{m=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \right] z^{-m}$$ Using equation (7.24)

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\, z^{-m}\, z^{-n}\, z^{n}$$ Multiply by z^n and z^{-n}

$$= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \sum_{m=-\infty}^{+\infty} y(n-m)\, z^{(n-m)}$$

Let, $n-m=p$ $\quad \therefore m=n-p$
when $m \to -\infty$, $p \to +\infty$,
when $m \to +\infty$, $p \to -\infty$.

$$= \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \sum_{p=-\infty}^{+\infty} y(p)\, z^{p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} \right] \left[\sum_{p=-\infty}^{+\infty} y(p)\, (z^{-1})^{-p} \right]$$

$$= X(z)\; Y(z^{-1})$$ Using equations (7.25) and (7.26)

9. Initial value theorem

Let x(n) be an one sided signal defined in the range $0 \le n \le \infty$.

Now, if $\mathcal{Z}\{x(n)\} = X(z)$,

then the initial value of x(n) (i.e., x(0)) is given by,

$$x(0) = \underset{z \to \infty}{\text{Lt}}\; X(z)$$

Proof :

By definition of one sided Ƶ - transfrom,

$$X(z) = \sum_{n=0}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + \ldots\ldots$$

$$\therefore\; X(z) = x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \ldots\ldots$$

On taking limit $z \to \infty$ in the above equation we get,

$$\underset{z \to \infty}{\text{Lt}}\; X(z) = \underset{z \to \infty}{\text{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \ldots\ldots \right]$$

$$= x(0) + 0 + 0 + 0 + \ldots\ldots$$

$$\therefore\; x(0) = \underset{z \to \infty}{\text{Lt}}\; X(z)$$

10. Final value theorem

Let x(n) be an one sided signal defined in the range $0 \le n \le \infty$.

Now, if $\mathcal{Z}\{x(n)\} = X(z)$,

then the final value of x(n) (i.e., $x(\infty)$) is given by,

$$x(\infty) = \underset{z \to 1}{\text{Lt}} (1 - z^{-1})\, X(z) \quad \text{or} \quad x(\infty) = \underset{z \to 1}{\text{Lt}} \left(\frac{z-1}{z}\right) X(z)$$

Proof :

By definition of one sided $\mathcal{Z}$ - transfrom,

$$\mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n)\, z^{-n} \qquad .....(7.27)$$

$$\therefore \underbrace{\mathcal{Z}\{x(n-1) - x(n)\}}_{\text{(RHS)}} = \sum_{n=0}^{+\infty} \underbrace{[x(n-1) - x(n)]\, z^{-n}}_{\text{(LHS)}}$$

$$\begin{aligned}
\text{RHS} &= \mathcal{Z}\{x(n-1) - x(n)\} \\
&= \mathcal{Z}\{x(n-1)\} - \mathcal{Z}\{x(n)\} && \boxed{\text{Using linearity property}} \\
&= z^{-1}\, X(z) + x(-1) - X(z) && \boxed{\text{Using shifting property and equation 7.27}} \\
&= x(-1) - (1 - z^{-1})\, X(z) \\
&= \underset{z \to 1}{\text{Lt}} \left[x(-1) - (1 - z^{-1})\, X(z)\right] && \boxed{\text{Taking limit } z \to 1} \\
&= x(-1) - \underset{z \to 1}{\text{Lt}} (1 - z^{-1})\, X(z) && .....(7.28)
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \sum_{n=0}^{+\infty} [x(n-1) - x(n)]\, z^{-n} \\
&= \underset{z \to 1}{\text{Lt}} \sum_{n=0}^{+\infty} [x(n-1) - x(n)]\, z^{-n} && \boxed{\text{Taking limit } z \to 1} \\
&= \sum_{n=0}^{+\infty} [x(n-1) - x(n)] && \boxed{\text{On applying limit } z \to 1 \text{, the term } z^{-n} \text{ becomes unity}} \\
&= \underset{p \to \infty}{\text{Lt}} \sum_{n=0}^{p} [x(n-1) - x(n)] && \boxed{\text{Changing the summation index from 0 to p and then taking limit } p \to \infty} \\
&= \underset{p \to \infty}{\text{Lt}} \begin{bmatrix} [x(-1) - x(0)] + [x(0) - x(1)] + [x(1) - x(2)] + \;..... \\ .....\; + [x(p-2) - x(p-1)] + [x(p-1) - x(p)] \end{bmatrix} \\
&= \underset{p \to \infty}{\text{Lt}} [x(-1) - x(p)] \\
&= x(-1) - x(\infty) && .....(7.29)
\end{aligned}$$

On equating equation (7.29) with (7.28) we get,

$$x(-1) - x(\infty) = x(-1) - \underset{z \to 1}{\text{Lt}} (1 - z^{-1})\, X(z)$$

$$\therefore\; x(\infty) = \underset{z \to 1}{\text{Lt}} (1 - z^{-1})\, X(z)$$

11. Complex convolution theorem (or Multiplication in time domain)

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, the complex convolution theorem states that,

$$\mathcal{Z}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi j}\oint_C X_1(v)\ X_2\left(\frac{z}{v}\right) v^{-1}\, dv$$

where, v is a dummy variable used for contour integration

Proof :

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, by definition of inverse $\mathcal{Z}$-transform,

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv \qquad \boxed{\text{let, } z = v} \qquad .....(7.30)$$

Now, by definition of $\mathcal{Z}$-transform,

$$X_2(z) = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(7.31)$$

Using the definition of $\mathcal{Z}$-transform, the $\mathcal{Z}\{x_1(n)\, x_2(n)\}$ can be written as,

$$\mathcal{Z}\{x_1(n)\, x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv\right] x_2(n)\, z^{-n} \qquad \boxed{\text{Using equation (7.30)}}$$

$$= \frac{1}{2\pi j}\oint_C X_1(v) \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n}\, v^{n}\, v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2(n)\left(\frac{z}{v}\right)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\, X_2\left(\frac{z}{v}\right) v^{-1}\, dv \qquad \boxed{\text{Using equation (7.31)}}$$

12. Parseval's relation

If $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Then the Parseval's relation states that,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\ x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, X_2^*\left(\frac{1}{z^*}\right) z^{-1}\, dz$$

Proof :

Let, $\mathcal{Z}\{x_1(n)\} = X_1(z)$ and $\mathcal{Z}\{x_2(n)\} = X_2(z)$.

Now, by definition of inverse $\mathcal{Z}$-transform,

$$x_1(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, z^{n-1}\, dz = \frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv \qquad \boxed{\text{let, } z = v} \qquad .....(7.32)$$

Now, by definition of $\mathcal{Z}$-transform,

$$\mathcal{Z}\left\{x_2(n)\right\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, z^{-n} \qquad .....(7.33)$$

Using the definition of $\mathcal{Z}$-transform, the $\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\}$ can be written as,

$$\mathcal{Z}\left\{x_1(n)\, x_2^*(n)\right\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} \qquad .....(7.34)$$

On substituting for $x_1(n)$ from equation (7.32) in equation (7.34) we can write,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi j}\oint_C X_1(v)\, v^{n-1}\, dv\right] x_2^*(n)\, z^{-n}$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2^*(n)\, z^{-n}\, v^{n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2^*(n)\left(\frac{z}{v}\right)^{-n}\right] v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2(n)\left(\frac{z^*}{v^*}\right)^{-n}\right]^* v^{-1}\, dv$$

$$= \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv \qquad \boxed{\text{using equation (7.33)}}$$

Let us take limit $z \to 1$ in the above equation,

$$\therefore \lim_{z\to 1}\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)\, z^{-n} = \lim_{z\to 1}\frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{z^*}{v^*}\right) v^{-1}\, dv$$

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)\, X_2^*\left(\frac{1}{v^*}\right) v^{-1}\, dv$$

$$\therefore \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, X_2^*\left(\frac{1}{v^*}\right) z^{-1}\, dz \qquad \boxed{\text{let } v = z}$$

Table-7.3 : Summary of Properties of Z-Transform

Note : $X(z) = \mathcal{Z}\{x(n)\}$; $X_1(z) = \mathcal{Z}\{x_1(n)\}$; $X_2(z) = \mathcal{Z}\{x_2(n)\}$; $Y(z) = \mathcal{Z}\{y(n)\}$

Property		Discrete time signal	Z-transform
Linearity		$a_1\, x_1(n) + a_2\, x_2(n)$	$a_1\, X_1(z) + a_2\, X_2(z)$
Shifting ($m \ge 0$)	$x(n)$ for $n \ge 0$	$x(n-m)$	$z^{-m}\, X(z) + \sum\limits_{i=1}^{m} x(-i)\, z^{-(m-i)}$
		$x(n+m)$	$z^{m}\, X(z) - \sum\limits_{i=0}^{m-1} x(i)\, z^{m-i}$
	$x(n)$ for all n	$x(n-m)$	$z^{-m}\, X(z)$
		$x(n+m)$	$z^{m}\, X(z)$
Multiplication by n^m (or differentiation in z-domain)		$n^m\, x(n)$	$\left(-z\,\frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)		$a^n\, x(n)$	$X(a^{-1} z)$
Time reversal		$x(-n)$	$X(z^{-1})$
Conjugation		$x^*(n)$	$X^*(z^*)$
Convolution		$x_1(n) * x_2(n) = \sum\limits_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(z)\, X_2(z)$
Corrrelation		$r_{xy}(m) = \sum\limits_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(z)\, Y(z^{-1})$
Initial value		$x(0) = \underset{z\to\infty}{\text{Lt}}\; X(z)$	
Final value		$x(\infty) = \underset{z\to 1}{\text{Lt}}\,(1 - z^{-1})\, X(z)$ $= \underset{z\to 1}{\text{Lt}}\,\frac{(z-1)}{z}\, X(z)$ if X(z) is analytic for $\lvert z\rvert > 1$	
Complex convolution theorem		$x_1(n)\, x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)\, X_2\!\left(\frac{z}{v}\right) v^{-1}\, dv$
Parseval's relation		$\sum\limits_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\, X_2^*\!\left(\frac{1}{z^*}\right) z^{-1}\, dz$	

Example 7.3

Find the one sided $\mathcal{Z}$-transform of the following discrete time signals.

a) $x(n) = n\, a^{(n-1)}$ b) $x(n) = n^2$

Solution

a) Given that, $x(n) = n\, a^{(n-1)}$

Let, $x_1(n) = a^n$

By definition of one sided $\mathcal{Z}$-transform,

$$X_1(z) = \sum_{n=0}^{\infty} x_1(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(a\, z^{-1}\right)^n$$

$$= \frac{1}{1 - a z^{-1}} = \frac{z}{z - a}$$

Using infinite geometric series sum formula

Let, $x_1(n-1) = a^{n-1}$

By shifting property,

$$\mathcal{Z}\{x_1(n-1)\} = z^{-1} X_1(z) = z^{-1}\frac{z}{z-a} = \frac{1}{z-a}$$

Given that, $x(n) = n\, a^{n-1}$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{n\, a^{n-1}\} = \mathcal{Z}\{n\, x_1(n-1)\} = -z\frac{d}{dz}X_1(z)$$

$$= -z\frac{d}{dz}\frac{1}{z-a} = -z \times \frac{-1}{(z-a)^2} = \frac{z}{(z-a)^2}$$

If $\mathcal{Z}\{x(n)\} = X(z)$

then $\mathcal{Z}\{n\, x(n)\} = -z\dfrac{d}{dz}X(z)$

---

b) Given that, $x(n) = n^2$

Let us multiply the given discrete time signal by a discrete unit step signal,

$\therefore\ x(n) = n^2\, u(n)$

Note : *Multiplying a one sided sequence by u(n) will not alter its value.*

By the property of $\mathcal{Z}$-transform, we get,

$$\mathcal{Z}\{n^m u(n)\} = \left(-z\frac{d}{dz}\right)^m U(z)$$

$$\text{where, } U(z) = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

$$\therefore\ -z\frac{d}{dz}U(z) = -z\left[\frac{d}{dz}\left(\frac{z}{z-1}\right)\right] = -z\left[\frac{z-1-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}$$

$$d\frac{u}{v} = \frac{v\,du - u\,dv}{v^2}$$

$$\left(-z\frac{d}{dz}\right)^2 U(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}U(z)\right]$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) = -z\left(\frac{(z-1)^2 - z \times 2(z-1)}{(z-1)^4}\right)$$

$$= -z\left(\frac{(z-1)(z-1-2z)}{(z-1)^4}\right) = -z\left(\frac{-(z+1)}{(z-1)^3}\right) = \frac{z(z+1)}{(z-1)^3}$$

$$\therefore\quad \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2 u(n)\} = \left(-z\frac{d}{dz}\right)^2 U(z) = \frac{z(z+1)}{(z-1)^3}$$

Example 7.4

Find the one sided Z-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) t^2 b) $\sin\Omega_o t$ c) $\cos\Omega_o t$

Solution

a) Given that, $x(t) = t^2$

The discrete time signals is generated by replacing t by nT, where T is the sampling time period.

$\therefore$ $x(n) = (nT)^2 = n^2 T^2 = n^2 g(n)$

where, $g(n) = T^2$

By the definition of one sided Z-transform we get,

$$G(z) = \mathcal{Z}\{g(n)\} = \mathcal{Z}\{T^2\} = \sum_{n=0}^{\infty} T^2 z^{-n} = T^2 \sum_{n=0}^{n} (z^{-1})^n = T^2\left(\frac{1}{1-z^{-1}}\right) = \frac{T^2 z}{z-1}$$

By the property of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{n^2 g(n)\} = \left(-z\frac{d}{dz}\right)^2 G(z) = -z\frac{d}{dz}\left(-z\frac{d}{dz}G(z)\right)$$

$$= -z\frac{d}{dz}\left(-z\frac{d}{dz}\frac{T^2 z}{z-1}\right) = -z\frac{d}{dz}\left(-z \times \frac{(z-1)T^2 - T^2 z}{(z-1)^2}\right)$$

$$= -z\frac{d}{dz}\left(\frac{zT^2}{(z-1)^2}\right) = -z \times \frac{(z-1)^2 T^2 - zT^2 \times 2(z-1)}{(z-1)^4}$$

$$= -z \times \frac{(z-1)(zT^2 - T^2 - 2zT^2)}{(z-1)^4} = -z \times \frac{-zT^2 - T^2}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$$

b) Given that, $x(t) = \sin\Omega_o t$

The discrete time signals is generated by replacing t by nT, where T is the sampling time period.

$\therefore$ $x(n) = \sin(\Omega_0 nT) = \sin\omega n$; where $\omega = \Omega_0 T$

By the definition of one sided Z-transform,

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \sin\omega n \times z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} - e^{-j\omega n}}{2j} z^{-n} = \frac{1}{2j}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \frac{1}{2j}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{j\omega} z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-j\omega} z^{-1}\right)^n$$

$$= \frac{1}{2j}\frac{1}{1-e^{j\omega}z^{-1}} - \frac{1}{2j}\frac{1}{1-e^{-j\omega}z^{-1}}$$

Using infinite geometric series sum formula

$$= \frac{1}{2j}\frac{z}{z-e^{j\omega}} - \frac{1}{2j}\frac{z}{z-e^{-j\omega}}$$

$$= \frac{z(z-e^{-j\omega}) - z(z-e^{j\omega})}{2j(z-e^{j\omega})(z-e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} - z^2 + z e^{j\omega}}{2j(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{z(e^{j\omega} - e^{-j\omega})/2j}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z\sin\omega T}{z^2 - 2z\cos\omega T + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$
$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

c) Given that, $x(t) = \cos\Omega_o t$

The discrete time signal is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = \cos(\Omega_0 nT) = \cos\omega n$; where $\omega = \Omega_0 T$

By the definition of one sided $\mathcal{Z}$-transform,

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} \cos\omega n \times z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} + e^{-j\omega n}}{2} z^{-n} = \frac{1}{2}\sum_{n=0}^{\infty} e^{j\omega n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n + \frac{1}{2}\sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{j\omega} z^{-1}} + \frac{1}{2}\,\frac{1}{1 - e^{-j\omega} z^{-1}}$$

Using infinite geometric series sum formula

$$= \frac{1}{2}\,\frac{z}{z - e^{j\omega}} + \frac{1}{2}\,\frac{z}{z - e^{-j\omega}}$$

$$= \frac{z(z - e^{-j\omega}) + z(z - e^{j\omega})}{2(z - e^{j\omega})(z - e^{-j\omega})} = \frac{z^2 - z e^{-j\omega} + z^2 - z e^{j\omega}}{2(z^2 - z e^{-j\omega} - z e^{j\omega} + e^{j\omega} e^{-j\omega})}$$

$$= \frac{2z^2 - z(e^{j\omega} + e^{-j\omega})}{2[z^2 - z(e^{j\omega} + e^{-j\omega}) + 1]} = \frac{z^2 - z(e^{j\omega} + e^{-j\omega})/2}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1}$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Example 7.5

Find the one sided $\mathcal{Z}$-transform of the discrete time signals generated by mathematically sampling the following continuous time signals.

a) $e^{-at}\cos\Omega_o t$ b) $e^{-at}\sin\Omega_o t$

Solution

a) Given that, $x(t) = e^{-at}\cos\Omega_o t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = e^{-anT}\cos\Omega_0 nT = e^{-anT}\cos\omega n$; where $\omega = \Omega_0 T$

By the definition of one sided $\mathcal{Z}$-transform we get,

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT}\cos\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} + e^{-j\omega n}}{2}\right) z^{-n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty} \left(e^{-aT} e^{j\omega} z^{-1}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty} \left(e^{-aT} e^{-j\omega} z^{-1}\right)^n$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{-aT} e^{j\omega} z^{-1}} + \frac{1}{2}\,\frac{1}{1 - e^{-aT} e^{-j\omega} z^{-1}}$$

$$= \frac{1}{2}\,\frac{1}{1 - e^{j\omega}/z e^{aT}} + \frac{1}{2}\,\frac{1}{1 - e^{-j\omega}/z e^{aT}}$$

$$= \frac{1}{2}\left[\frac{z e^{aT}}{z e^{aT} - e^{j\omega}} + \frac{z e^{aT}}{z e^{aT} - e^{-j\omega}}\right]$$

$$= \frac{1}{2}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) + z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{z\,e^{aT} - e^{-j\omega} + z\,e^{aT} - e^{j\omega}}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}\,e^{-j\omega}}\right]$$

$$= \frac{z\,e^{aT}}{2}\left[\frac{2z\,e^{aT} - \left(e^{j\omega} + e^{-j\omega}\right)}{z^2\,e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \left[\frac{z\,e^{aT}\left(z\,e^{aT} - \cos\omega\right)}{z^2\,e^{2aT} - 2z\,e^{aT}\cos\omega + 1}\right] \quad ; \quad \text{where } \omega = \Omega_0 T$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

b) Given that, $x(t) = e^{-at}\sin\Omega_o t$

The discrete time signal x(n) is generated by replacing t by nT, where T is the sampling time period.

$\therefore\ x(n) = e^{-anT}\sin\Omega_0\, n\,T = e^{-anT}\sin\omega n \ ;\ \text{where } \omega = \Omega_0 T$

By the definition of one sided Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT}\sin\omega n\, z^{-n} = \sum_{n=0}^{\infty} e^{-anT}\left(\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right) z^{-n}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{j\omega}z^{-1}\right)^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left(e^{-aT}e^{-j\omega}z^{-1}\right)^n$$

$$\boxed{\text{Infinite geometric series sum formula } \sum_{n=0}^{\infty} C^n = \frac{1}{1-C}}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{-aT}\,e^{j\omega}z^{-1}} - \frac{1}{2j}\,\frac{1}{1 - e^{-aT}e^{-j\omega}z^{-1}}$$

$$= \frac{1}{2j}\,\frac{1}{1 - e^{j\omega}/z\,e^{aT}} - \frac{1}{2j}\,\frac{1}{1 - e^{-j\omega}/z\,e^{aT}}$$

$$= \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{j\omega}} - \frac{1}{2j}\,\frac{z\,e^{aT}}{z\,e^{aT} - e^{-j\omega}}$$

$$= \frac{1}{2j}\left[\frac{z\,e^{aT}\left(z\,e^{aT} - e^{-j\omega}\right) - z\,e^{aT}\left(z\,e^{aT} - e^{j\omega}\right)}{\left(z\,e^{aT} - e^{j\omega}\right)\left(z\,e^{aT} - e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2j}\left[\frac{\left(z\,e^{aT}\right)[z\,e^{aT} - e^{-j\omega} - z\,e^{aT} + e^{j\omega}]}{\left(z\,e^{aT}\right)^2 - z\,e^{aT}e^{-j\omega} - z\,e^{aT}e^{j\omega} + e^{j\omega}\,e^{-j\omega}}\right]$$

$$= \left[\frac{z\,e^{aT}\left[e^{j\omega} - e^{-j\omega}\right]/2j}{z^2\,e^{2aT} - z\,e^{aT}\left(e^{j\omega} + e^{-j\omega}\right) + 1}\right]$$

$$= \frac{z\,e^{aT}\sin\omega}{z^2\,e^{2aT} - 2ze^{aT}\cos\omega + 1} \quad ; \quad \text{where } \omega = \Omega_0 T$$

Example 7.6

Find the initial value, x(0) and final value, x(∞) of the following z-domain signals.

a) $X(z) = \dfrac{1}{1 - z^{-2}}$ b) $\dfrac{1}{1 + 2z^{-1} - 3z^{-2}}$ c) $X(z) = \dfrac{2z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}}$

Solution

a) Given that, $X(z) = \dfrac{1}{1 - z^{-2}}$

By initial value theorem of $\mathcal{Z}$-transform we get,

$$x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z) = \underset{z \to \infty}{\text{Lt}}\ \frac{1}{1 - z^{-2}} = \underset{z \to \infty}{\text{Lt}}\ \frac{1}{1 - \dfrac{1}{z^2}} = \frac{1}{1 - \dfrac{1}{\infty}} = \frac{1}{1 - 0} = 1$$

By final value theorem of $\mathcal{Z}$-transform we get,

$$x(\infty) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1})\, X(z) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1}) \frac{1}{1 - z^{-2}}$$

$$= \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1}) \frac{1}{(1 - z^{-1})\,(1 + z^{-1})} = \underset{z \to 1}{\text{Lt}}\ \frac{1}{(1 + z^{-1})} = \frac{1}{1 + 1^{-1}} = \frac{1}{2}$$

---

b) Given that, $X(z) = \dfrac{1}{1 + 2z^{-1} - 3z^{-2}}$

By initial value theorem of $\mathcal{Z}$-transform we get,

$$x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z) = \underset{z \to \infty}{\text{Lt}}\ \frac{1}{1 + 2z^{-1} - 3z^{-2}} = \underset{z \to \infty}{\text{Lt}}\ \frac{1}{1 + \dfrac{2}{z} - \dfrac{3}{z^2}}$$

$$= \frac{1}{1 + \dfrac{2}{\infty} - \dfrac{3}{\infty}} = \frac{1}{1 + 0 + 0} = 1$$

By final value theorem of $\mathcal{Z}$-transform we get,

$$x(\infty) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1})\, X(z) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1}) \frac{1}{1 + 2z^{-1} - 3z^{-2}}$$

$$= \underset{z \to 1}{\text{Lt}}\ \frac{z^{-1}(z - 1)}{z^{-2}(z^2 + 2z - 3)} = \underset{z \to 1}{\text{Lt}}\ \frac{z(z - 1)}{(z - 1)\,(z + 3)} = \underset{z \to 1}{\text{Lt}}\ \frac{z}{z + 3} = \frac{1}{1 + 3} = \frac{1}{4}$$

---

c) Given that, $X(z) = \dfrac{2\,z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}}$

By initial value theorem of $\mathcal{Z}$-transform we get,

$$x(0) = \underset{z \to \infty}{\text{Lt}}\ X(z) = \underset{z \to \infty}{\text{Lt}}\ \frac{2z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}} = \underset{z \to \infty}{\text{Lt}}\ \frac{\dfrac{2}{z}}{1 - \dfrac{1.8}{z} + \dfrac{0.8}{z^2}}$$

$$= \frac{\dfrac{2}{\infty}}{1 - \dfrac{1.8}{\infty} + \dfrac{0.8}{\infty}} = \frac{0}{1 - 0 + 0} = 0$$

By final value theorem of $\mathcal{Z}$-transform we get,

$$x(\infty) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1})\, X(z) = \underset{z \to 1}{\text{Lt}}\ (1 - z^{-1}) \frac{2z^{-1}}{1 - 1.8z^{-1} + 0.8z^{-2}}$$

$$= \underset{z \to 1}{\text{Lt}}\ \frac{z^{-1}(z - 1)\, 2z^{-1}}{z^{-2}(z^2 - 1.8z + 0.8)} = \underset{z \to 1}{\text{Lt}}\ \frac{2(z - 1)}{(z - 1)\,(z - 0.8)}$$

$$= \underset{z \to 1}{\text{Lt}}\ \frac{2}{z - 0.8} = \frac{2}{1 - 0.8} = 10$$

Table - 7.4 : Some Common Z-transform Pairs

x(t)	x(n)	X(z) With positive power of z	X(z) With negative power of z	ROC
	$\delta(n)$	1	1	Entire z-plane
	u(n) or 1	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
	$a^n\, u(n)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
	$n\, a^n\, u(n)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
	$n^2\, a^n\, u(n)$	$\frac{az\,(z+a)}{(z-a)^3}$	$\frac{az^{-1}\,(1 + az^{-1})}{(1 - az^{-1})^3}$	$\|z\| > \|a\|$
	$-\, a^n\, u(-n-1)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
	$-na^n\, u(-n-1)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$t\, u(t)$	$nT\, u(nT)$	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
$t^2\, u(t)$	$(nT)^2\, u(nT)$	$\frac{T^2 z\,(z+1)}{(z-1)^3}$	$\frac{T^2 z^{-1}\,(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
$e^{-at}\, u(t)$	$e^{-anT}\, u(nT)$	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}\, z^{-1}}$	$\|z\| > \|e^{-aT}\|$
$te^{-at}\, u(t)$	$nTe^{-anT}\, u(nT)$	$\frac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\frac{z^{-1}\,T\,e^{-aT}}{(1-e^{-aT}\, z^{-1})^2}$	$\|z\| > \|e^{-aT}\|$
$\sin\Omega_0 t\, u(t)$	$\sin\Omega_0 nT\;\; u(nT)$ $= \sin\,\omega n\; u(nT)$ where, $\omega = \Omega_0 T$	$\frac{z\,\sin\omega}{z^2 - 2z\cos\omega + 1}$	$\frac{z^{-1}\,\sin\omega}{1-2z^{-1}\,\cos\omega + z^{-2}}$	$\|z\| > 1$
$\cos\Omega_0 t\, u(t)$	$\cos\Omega_0 nT\, u(nT)$ $= \cos\,\omega n\; u(nT)$ where, $\omega = \Omega_0 T$	$\frac{z\,(z-\cos\omega)}{z^2 - 2z\cos\omega + 1}$	$\frac{1-z^{-1}\,\cos\omega}{1-2z^{-1}\,\cos\omega + z^{-2}}$	$\|z\| > 1$

Note : *1. The signals multiplied by u(n) are causal signals (defined for n ≥ 0).*
2. The signals multiplied by u(–n – 1) are anticausal signals (defined for n ≤ 0).

7.4 Poles and Zeros of Rational Function of z

Let, X(z) be Ƶ-transform of x(n). When X(z) is expressed as a ratio of two polynomials in z or z^{-1}, then X(z) is called a ***rational function*** of z.

Let X(z) be expressed as a ratio of two polynomials in z, as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}} \quad .....(7.35)$$

where, N(z) = Numerator polynomial of X(z)

D(z) = Denominator polynomial of X(z)

In equation (7.35) let us scale the coefficients of numerator polynomial by b_0 and that of denominator polynomial by a_0, and then convert the polynomials to positive power of z as shown below.

$$X(z) = \frac{b_0\left(1 + \frac{b_1}{b_0} z^{-1} + \frac{b_2}{b_0} z^{-2} + \frac{b_3}{b_0} z^{-3} + ... + \frac{b_M}{b_0} z^{-M}\right)}{a_0\left(1 + \frac{a_1}{a_0} z^{-1} + \frac{a_2}{a_0} z^{-2} + \frac{a_3}{a_0} z^{-3} + ... + \frac{a_N}{a_0} z^{-N}\right)}$$

$$= G\,\frac{z^{-M}\left(z^M + \frac{b_1}{b_0} z^{M-1} + \frac{b_2}{b_0} z^{M-2} + \frac{b_3}{b_0} z^{M-3} + ... \frac{b_M}{b_0}\right)}{z^{-N}\left(z^N + \frac{a_1}{a_0} z^{N-1} + \frac{a_2}{a_0} z^{N-2} + \frac{a_3}{a_0} z^{N-3} + ... \frac{a_N}{a_0}\right)} \qquad \boxed{\text{Let, } M = N}$$

$$= G\,\frac{(z - z_1)(z - z_2)(z - z_3)\,.....\,(z - z_N)}{(z - p_1)(z - p_2)(z - p_3)\,.....\,(z - p_N)} \quad .....(7.36)$$

where, $z_1, z_2, z_3, ...z_N$ are roots of numerator polynomial

$p_1, p_2, p_3, ...p_N$ are roots of denominator polynomial

G is a scaling factor.

In equation (7.36) if the value of z is equal to one of the roots of the numerator polynomial, then the function X(z) will become zero.

Therefore the roots of numerator polynomial $z_1, z_2, z_3, ...z_N$ are called zeros of X(z). Hence the ***zeros*** are defined as values z at which the function X(z) become zero.

In equation (7.36) if the value of z is equal to one of the roots of the denominator polynomial then the funcion X(z) will become infinite. Therefore the roots of denominator polynomial $p_1, p_2, p_3, ...p_N$ are called poles of X(z). Hence the ***poles*** are defined as values of z at which the function X(z) become infinite.

Since the function X(z) attains infinte values at poles, the ROC of X(z) does not include poles.

In a realizable system, the number of zeros will be less than or equal to number of poles. Also for every zero, we can associate one pole (the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate |X(z)| for various values of z, then |X(z)| will be zero for $z = z_i$ and infnite for $z = p_i$. Hence the plot of |X(z)| in a three dimensional plane will look like a pole (or pillar like structure) and so the point $z = p_i$ is called a pole.

7.4.1 Representation of Poles and Zeros in z-Plane

The complex variable, z is defined as,

$$z = u + jv$$

where, u = Real part of z

v = Imaginary part of z

Hence the z-plane is a complex plane, with u on real axis and v on imaginary axis (Refer fig 7.1 in section 7.1). In the z-plane, the zeros are marked by small circle "o" and the poles are marked by letter "x".

For example consider a rational function of z shown below.

$$X(z) = \frac{1.25 - 1.25\,z^{-1} + 0.2\,z^{-2}}{2 + 2\,z^{-1} + z^{-2}}$$

$$= \frac{1.25\left(1 - z^{-1} + \dfrac{0.2}{1.25}\,z^{-2}\right)}{2\left(1 + z^{-1} + \dfrac{1}{2}\,z^{-2}\right)} = \frac{0.625\,(1 - z^{-1} + 0.16\,z^{-2})}{(1 + z^{-1} + 0.5\,z^{-2})}$$

$$= \frac{0.625\,z^{-2}(z^2 - z + 0.16)}{z^{-2}(z^2 + z + 0.5)} = \frac{0.625\,(z - 0.8)\,(z - 0.2)}{(z + 0.5 + j0.5)\,(z + 0.5 - j0.5)} \qquad .....(7.37)$$

The roots of quadratic, $z^2 - z + 0.16 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.16}}{2} = \frac{1 \pm 0.6}{2} = 0.8,\ 0.2$$

$$\therefore \quad z^2 - z + 0.16 = (z - 0.8)\,(z - 0.2)$$

The roots of quadratic, $z^2 + z + 0.5 = 0$ are,

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 0.5}}{2} = \frac{-1 \pm j}{2} = -0.5 \pm j\,0.5$$

$$\therefore \quad z^2 + z + 0.5 = (z + 0.5 + j0.5)\,(z + 0.5 - j0.5)$$

The zeros of X(z) are roots of numerator polynomial, which has two roots.

Therefore, the zeros of X(z) are,

$$z_1 = 0.8,\quad z_2 = 0.2$$

The poles of X(z) are roots of denominator polynomial, which has two roots.

Therefore, the poles of X(z) are,

$$p_1 = -0.5 - j0.5, \quad p_2 = -0.5 + j0.5$$

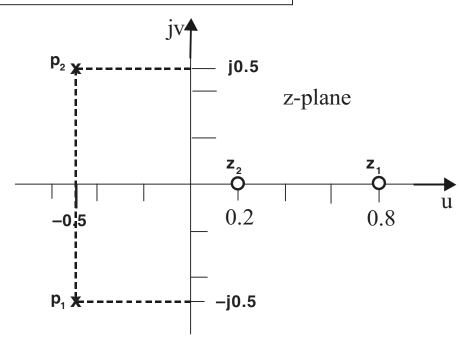


Fig 7.5 : *Pole-zero plot of X(z) of equation (7.37).*

The pole-zero plot of X(z) is shown in fig 7.5.

7.4.2 ROC of Rational Function of z

***Case i:* Right sided (causal) signal**

Let x(n) be a right sided (causal) signal defined as,

$$x(n) = r_1^n\, u(n) + r_2^n\, u(n) + r_3^n\, u(n) \quad ; \quad \text{where } r_1 < r_2 < r_3$$

Now, the Ƶ-transform of x(n) is,

$$\boxed{Ƶ\{a^n\, u(n)\} = \frac{z}{z-a} \text{ with ROC } |z| > |a|}$$

$$X(z) = \frac{z}{z-r_1} + \frac{z}{z-r_2} + \frac{z}{z-r_3}$$

$$= \frac{N(z)}{(z-r_1)\,(z-r_2)\,(z-r_3)}$$

where, $N(z) = z(z - r_2)\,(z - r_3) + z(z - r_1)\,(z - r_3) + z(z - r_1)\,(z - r_2)$

The poles of X(z) are,

$$p_1 = r_1,\ \ p_2 = r_2,\ p_3 = r_3$$

The convergence criteria for X(z) are,

$$|z| > |r_1| \quad ; \quad |z| > |r_2| \quad ; \quad |z| > |r_3|$$

Since $r_1 < r_2 < r_3$, the ROC is exterior of the circle of radius r_3 in z-plane as shown in fig.7.6. In terms of poles of X(z) we can say that the *ROC is exterior of a circle, whose radius is equal to the magnitude of outer most pole (i.e., pole with largest magnitude) of X(z).*

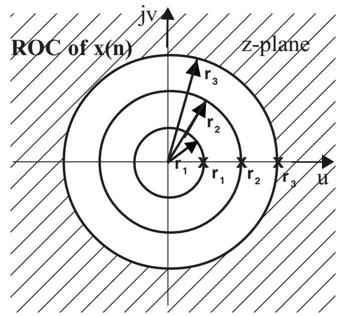


Fig 7.6 : *ROC of $x(n) = r_1\, u(n) + r_2\, u(n) + r_3\, u(n)$ where $r_1 < r_2 < r_3$.*

***Case ii:* Left sided (anticausal) signal**

Let x(n) be a left sided (anticausal) signal defined as,

$$x(n) = -r_1^n\, u(-n-1) - r_2^n\, u(-n-1) - r_3^n\, u(-n-1) \quad ; \quad \text{where } r_1 < r_2 < r_3$$

Now, the Ƶ-transform of x(n) is,

$$\boxed{Ƶ\{-a^n\, u(-n-1)\} = \frac{z}{z-a} \text{ with ROC } |z| < |a|}$$

$$X(z) = \frac{z}{z-r_1} + \frac{z}{z-r_2} + \frac{z}{z-r_3}$$

$$= \frac{N(z)}{(z-r_1)\,(z-r_2)\,(z-r_3)}$$

where, $N(z) = z(z - r_2)\,(z - r_3) + z(z - r_1)\,(z - r_3) + z(z - r_1)\,(z - r_2)$

The poles of X(z) are,

$$p_1 = r_1,\ \ p_2 = r_2,\ \ p_3 = r_3$$

The convergence criteria for X(z) are,

$$|z| < |r_1| \quad ; \quad |z| < |r_2| \quad ; \quad |z| < |r_3|$$

Since $r_1 < r_2 < r_3$, the ROC is interior of the circle of radius r_1 in z-plane as shown in fig.7.7. In terms of poles of X(z) we can say that the *ROC is interior of a circle, whose radius is equal to the magnitude of inner most pole (i.e., pole with smallest magnitude) of X(z).*

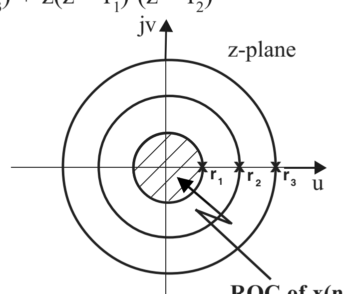


Fig 7.7 : *ROC of $x(n) = -r_1\, u(-n-1) - r_2\, u(-n-1) - r_3\, u(-n-1)$, where $r_1 < r_2 < r_3$.*

***Case iii:* Two sided (noncausal) signal**

Let x(n) be two sided signal defined as,

$$x(n) = r_1^n\, u(n) + r_2^n\, u(n) - r_3^n\, u(-n-1) - r_4^n\, u(-n-1) \quad ; \quad \text{where } r_1 < r_2 < r_3 < r_4$$

Now, the Z-transform of x(n) is,

$$X(z) = \frac{z}{z - r_1} + \frac{z}{z - r_2} + \frac{z}{z - r_3} + \frac{z}{z - r_4}$$

$$= \frac{N(z)}{(z - r_1)\,(z - r_2)\,(z - r_3)\,(z - r_4)}$$

$$\text{where, } N(z) = z(z - r_2)\,(z - r_3)\,(z - r_4) + z(z - r_1)\,(z - r_3)\,(z - r_4) + z(z - r_1)\,(z - r_2)\,(z - r_4) + z(z - r_1)\,(z - r_2)\,(z - r_3)$$

The poles of X(z) are,

$$p_1 = r_1\ ;\ p_2 = r_2\ ;\ p_3 = r_3\ ;\ p_4 = r_4$$

The convergence criteria for X(z) are,

$$|z| > |r_1|\ ;\ |z| > |r_2|\ ;\ |z| < |r_3|\ ;\ |z| < |r_4|$$

Since $r_1 < r_2 < r_3 < r_4$, the ROC is the region inbetween the circles of radius r_2 and r_3 as shown in fig 7.8. Let r_x be the magnitude of largest pole of causal signal and let r_y be the magnitude of smallest pole of anticausal signal and let $r_x < r_y$. Now in term of poles of X(z) we can say that the *ROC is the region in between two circles of radius r_x and r_y, where $r_x < r_y$.*

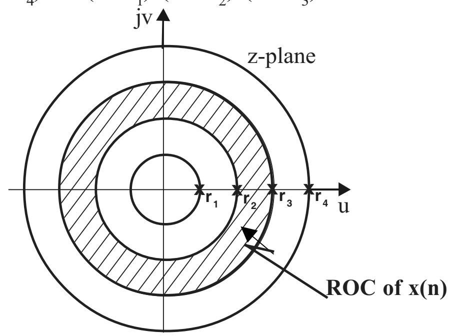


***Fig 7.8** : ROC of $x(n) = r_1\, u(n) + r_2\, u(n) - r_3\, u(-n-1), -r_4\, u(-n-1)$.*

7.4.3 Properties of ROC

The various concepts of ROC that has been discussed in sections 7.2 and 7.4.2 are summarized as properties of ROC and given below.

Property- 1: The ROC of X(z) is a ring or disk in z-plane, with centre at origin.

Property- 2: If x(n) is finite duration right sided (causal) signal, then the ROC is entire z- plane except z = 0.

Property- 3: If x(n) is finite duration left sided (anticausal) signal, then the ROC is entire z-plane except z = ∞.

Property-4: If x(n) is finite duration two sided (noncausal) signal, then the ROC is entire z-plane except z = 0 and z = ∞.

Property- 5 : If x(n) is infinite duration right sided (causal) signal, then the ROC is exterior of a circle of radius r_1.

Property- 6: If x(n) is infinite duration left sided (anticausal) signal, then the ROC is interior of a circle of radius r_2.

Property- 7: If x(n) is infinite duration two sided (noncausal) signal, then the ROC is the region in between two circles of radius r_1 and r_2.

Property- 8: If X(z) is rational, (where X(z) is Z-transform of x(n)), then the ROC does not include any poles of X(z).

Property- 9: If X(z) is rational, (where X(z) is Z-transform of x(n)), and if x(n) is right sided, then the ROC is exterior of a circle whose radius corresponds to the pole with largest magnitude.

Property-10: If X(z) is rational, (where X(z) is Z-transform of x(n)), and if x(n) is left sided, then the ROC is interior of a circle whose radius corresponds to the pole with smallest magnitude.

Property-11: If X(z) is rational, (where X(z) is Z-transform of x(n)), and if x(n) is two sided, then the ROC is region in between two circles whose radius corresponds to the pole of causal part with largest magnitude and the pole of anticausal part with smallest magnitude.

7.5 Inverse $\mathcal{Z}$-Transform

Let X(z) be $\mathcal{Z}$-transform of the discrete time signal x(n).The inverse $\mathcal{Z}$-transform is the process of recovering the discrete time signal x(n) from its $\mathcal{Z}$-transform X(z). The signal x(n) can be uniquely determined from X(z) and its ROC.

The inverse $\mathcal{Z}$-transform can be determined by the following three methods.

1. Direct evaluation by contour integration (or residue method)
2. Partial fraction expansion method.
3. Power series expansion method.

7.5.1 Inverse $\mathcal{Z}$-Transform by Contour Integration or Residue Method

Let, X(z) be $\mathcal{Z}$-transform of x(n).

Now by definition of inverse $\mathcal{Z}$-transform,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)\, z^{n-1} dz \qquad .....(7.38)$$

Using partial fraction expansion technique the function $X(z)\, z^{n-1}$ can be expressed as shown below.

$$X(z)\, z^{n-1} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + ... + \frac{A_N}{z-p_N} \qquad .....(7.39)$$

where, $p_1, p_2, p_3, ... p_N$ are poles of $X(z)\, z^{n-1}$ and $A_1, A_2, A_3, ... A_N$ are residues

The residue A_1 is obtained by multiplying the equation (7.39) by $(z-p_1)$ and letting $z = p_1$. Similarly other residues are evaluated.

$$\therefore\ A_1 = (z-p_1)\, X(z)\, z^{n-1}\Big|_{z=p_1} \qquad .....(7.40.1)$$

$$A_2 = (z-p_2)\, X(z)\, z^{n-1}\Big|_{z=p_2} \qquad .....(7.40.2)$$

$$A_3 = (z-p_3)\, X(z)\, z^{n-1}\Big|_{z=p_3} \qquad .....(7.40.3)$$

$$\vdots$$

$$A_N = (z-p_N)\, X(z)\, z^{n-1}\Big|_{z=p_N} \qquad .....(7.40.N)$$

Using equation (7.39) the equation (7.38) can be written as,

$$x(n) = \frac{1}{2\pi j}\oint_C \left[\frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + ... + \frac{A_N}{z-p_N}\right] dz$$

$$= \frac{1}{2\pi j}\left[A_1\oint_C \frac{dz}{z-p_1} + A_2\oint_C \frac{dz}{z-p_2} + A_3\oint_C \frac{dz}{z-p_3} + ... + A_N\oint_C \frac{dz}{z-p_N}\right] \qquad .....(7.41)$$

If, $G(z) = \dfrac{1}{z-p_0}$, then by ***Cauchy's integral theorem***,

$$\oint_C G(z)\,dz = \oint_C \frac{1}{z-p_0}\,dz = \begin{cases} 2\pi j \;;\; \text{if } p_0 \text{ is a point inside the contour C in z-plane} \\ 0 \quad ;\; \text{if } p_0 \text{ is a point outside the contour C in z-plane} \end{cases}$$

Using Cauchy's integral theorem, the equation (7.41) can be written as shown below.

$$x(n) = \frac{1}{2\pi j}\left[A_1\, 2\pi j + A_2\, 2\pi j + A_3\, 2\pi j + \ldots\ldots + A_N\, 2\pi j\right]$$

$$= A_1 + A_2 + A_3 + \ldots\ldots + A_N \qquad \ldots..(7.42)$$

$$= \text{Sum of residues of } X(z)\, z^{n-1}$$

On substituting for residues from equation (7.40.1) to (7.40.N) in equation(7.42) we get,

$$x(n) = (z-p_1)\,X(z)\,z^{n-1}\Big|_{z=p_1} + (z-p_2)\,X(z)\,z^{n-1}\Big|_{z=p_2}$$

$$+ (z-p_3)\,X(z)\,z^{n-1}\Big|_{z=p_3} + \ldots\ldots + (z-p_N)\,X(z)\,z^{n-1}\Big|_{z=p_N}$$

$$\boxed{\therefore\; x(n) = \sum_{i=1}^{N}\left[(z-p_i)\,X(z)\,z^{n-1}\Big|_{z=p_i}\right]} \qquad \ldots..(7.43)$$

where, N = Number or poles of $X(z)\,z^{n-1}$ lying inside the contour C.

Using equation (7.43), by considering only the poles lying inside the contour C, the inverse Ƶ-transform can be evaluated. For a stable system the contour C is the unit circle in z-plane.

7.5.2 Inverse Ƶ-Transform by Partial Fraction Expansion Method

Let X(z) be Ƶ-transform of x(n), and X(z) be a rational function of z. Now the function X(z) can be expressed as a ratio of two polynomials in z as shown below. (Refer equation 7.35).

$$X(z) = \frac{N(z)}{D(z)} \qquad \ldots..(7.44)$$

where, N(z) = Numerator polynomial of X(z)

D(z) = Denominator polynomial of X(z)

Let us divide both sides of equation (7.44) by z and express equation (7.44) as shown below.

$$\frac{X(z)}{z} = \frac{N(z)}{z\,D(z)}$$

$$\therefore\; \frac{X(z)}{z} = \frac{Q(z)}{D(z)} \qquad \ldots..(7.45)$$

where, $Q(z) = \dfrac{N(z)}{z}$

Note : *It is convenient, if we consider* $\dfrac{X(z)}{z}$ *rather than X(z) for inverse Ƶ-transform by partial fraction expansion method.*

On factorizing the denominator polynomial of equation (7.45) we get,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\,(z-p_3)\;.....\;(z-p_N)} \qquad .....(7.46)$$

where, $p_1, p_2, p_3, ...p_N$ are roots of denominator polynomial (as well as poles of X(z)).

The equation (7.46) can be expressed as a series of sum terms by partial fraction expansion technique as shown below.

$$\frac{X(z)}{z} = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + ... + \frac{A_N}{z-p_N}$$

where, $A_1, A_2, A_3, ... A_N$ are residues.

$$\therefore\; X(z) = A_1\frac{z}{z-p_1} + A_2\frac{z}{z-p_2} + A_3\frac{z}{z-p_3} + ...$$
$$+ A_N\frac{z}{z-p_N} \qquad .....(7.47)$$

Now, the inverse Ƶ-transform of equation (7.47) is obtained by comparing each term with standard Ƶ-transform pair. The two popular Ƶ-transform pairs useful for inverse Ƶ-transform of equation (7.47) are given below.

If a^n is a causal (or right sided) signal then,

$$\mathcal{Z}\left\{a^n\, u(n)\right\} = \frac{z}{z-a} \quad ; \text{ with ROC } |z| > |a|$$

If a^n is an anticausal (or left sided) signal then,

$$\mathcal{Z}\left\{-a^n\, u(-n-1)\right\} = \frac{z}{z-a} \quad ; \text{ with ROC } |z| < |a|$$

Let r_1 be the magnitude of the largest pole and let the ROC be $|z| > r_1$ (where r_1 is radius of a circle in z-plane), then each term of equation (7.47) gives rise to a causal sequence, and so the inverse Ƶ-transform of equation (7.47) will be as shown in equation (7.48).

$$x(n) = A_1\, p_1^n\, u(n) + A_2\, p_2^n\, u(n) + A_3\, p_3^n\, u(n) + ... + A_N\, p_N^n\, u(n) \qquad .....(7.48)$$

Let r_2 be the magnitude of the smallest pole and let ROC be $|z| < r_2$ (where r_2 is radius of a circle in z-plane), then each term of equation (7.47) give rise to an anticausal sequence, and so the inverse Ƶ-transform of equation (7.47) will be as shown in equation (7.49).

$$x(n) = -A_1\, p_1^n\, u(-n-1) - A_2\, p_2^n\, u(-n-1) - A_3\, p_3^n\, u(-n-1) - .....$$
$$- A_N\, p_N^n\, u(-n-1) \qquad .....(7.49)$$

Sometimes the specified ROC will be in between two circles of radius r_x and r_y, where $r_x < r_y$. (i.e., ROC is $r_x < |z| < r_y$). Now in this case, the terms with magnitude of pole less than r_x will give rise to causal signal and the terms with magnitude of pole greater than r_y will give rise to anticausal signal so that the inverse Ƶ-transform of X(z) will give a two sided signal. [Refer section 7.4.2, case iii]

Evaluation of Residues

The coefficients of the denominator polynomial D(z) are assumed real and so the roots of the denominator polynomial are real and/or complex conjugate pairs (i.e., complex roots will occur only in conjugate pairs). Hence on factorizing the denominator polynomial we get the following cases. (The roots of the denominator polynomial are poles of X(z)).

Case i *: When roots (or poles) are real and distinct*

Case ii *: When roots (or poles) have multiplicity*

Case iii *: When roots (or poles) are complex conjugate*

Case i : ***When roots (or poles) are real and distinct***

In this case $\frac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\;......\;(z-p_N)}$$

$$= \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \;.....\; + \frac{A_N}{(z-p_N)}$$

where, A_1, A_2 ...... A_N are residues and p_1, p_2, ... p_N are poles

The residue A_1 is evaluated by multiplying both sides of $\frac{X(z)}{z}$ by $(z–p_1)$ and letting $z = p_1$. Similarly other residues are evaluated.

$$\therefore\; A_1 = (z-p_1)\left.\frac{X(z)}{z}\right|_{z\,=\,p_1}$$

$$A_2 = (z-p_2)\left.\frac{X(z)}{z}\right|_{z\,=\,p_2}$$

$$\vdots$$

$$A_N = (z-p_N)\left.\frac{X(z)}{z}\right|_{z\,=\,p_N}$$

Case ii : ***When roots (or poles) have multiplicity***

Let one of pole has a multiplicity of q. (i.e., repeats q times). In this case $\frac{X(z)}{z}$ is expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)......(z-p_x)^q......(z-p_N)}$$

$$= \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \;.....\; + \frac{A_{x0}}{(z-p_x)^q}$$

$$+ \frac{A_{x1}}{(z-p_x)^{q-1}} + \;......\; + \frac{A_{x(q-1)}}{(z-p_x)} + \;.....\; + \frac{A_N}{(z-p_N)}$$

where, A_{x0}, A_{x1}, ... $A_{x(q-1)}$ are residues of repeated root (or pole), $z = p_x$

The residues of distinct real roots are evaluated as explained in case i.

The residue A_{xr} of repeated root is obtained as shown below.

$$A_{xr} = \frac{1}{r!}\frac{d^r}{dz^r}\left[(z-p_x)^q\,\frac{X(z)}{z}\right]\Bigg|_{z=p_x} \quad ; \text{ where } r = 0, 1, 2, \ldots.(q-1)$$

Case iii : When roots (or poles) are complex conjugate

Let $\dfrac{X(z)}{z}$ has one pair of complex conjugate pole. In this case $\dfrac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)\,(z-p_2)\ldots\ldots(z^2+az+b)\ldots\ldots(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \ldots.. + \frac{A_x}{z-(x+jy)} + \frac{A_x^*}{z-(x-jy)} + \ldots.. + \frac{A_N}{z-p_N}$$

The residues of real and non-repeated roots are evaluated as explained in case i.

The residue A_x is evaluated as that of case i and the residue A_x^* is the conjugate of A_x.

7.5.3 Inverse Z-Transform by Power Series Expansion Method

Let X(z) be Z-transform of x(n), and X(z) be a rational function of z as shown below.

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots.. + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \ldots.. + a_N z^{-N}}$$

On dividing the numerator polynomial N(z) by denominator polynomial D(z) we can express X(z) as a power series of z. It is possible to express X(z) as positive power of z or as negative power of z or with both positive and negative power of z as shown below.

Case i :

$$X(z) = \frac{N(z)}{D(z)} = c_0 + c_1 z^{-1} + c_2 z^{-2} + c_3 z^{-3} + \ldots.. \qquad \ldots..(7.50.1)$$

Case ii :

$$X(z) = \frac{N(z)}{D(z)} = d_0 + d_1 z^1 + d_2 z^2 + d_3 z^3 + \ldots.. \qquad \ldots..(7.50.2)$$

Case iii :

$$X(z) = \frac{N(z)}{D(z)} = \ldots.. + e_{-3} z^3 + e_{-2} z^2 + e_{-1} z + e_0$$

$$+ e_1 z^{-1} + e_2 z^{-2} + e_3 z^{-3} + \ldots.. \qquad \ldots..(7.50.3)$$

The case i power series of z is obtained when the ROC is exterior of a circle of radius r in z-plane (i.e., ROC is $|z| > r$). The case ii power series of z is obtained when the ROC is interior of a circle of radius r in z-plane (i.e., ROC is $|z| < r$). The case iii power series of z is obtained when the ROC is in between two circles of radius r_1 and r_2 in z-plane (i.e., ROC is $r_1 < |z| < r_2$).

By the definition of Z-transform, we get,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

On expanding the summation we get,

$$X(z) = \ldots\ldots\ x(-3)\, z^{3} + x(-2)\, z^{2} + x(-1)\, z^{1} + x(0)\, z^{0} + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + \ldots\ldots \qquad \ldots..(7.51)$$

On comparing the coefficients of z of equations (7.50) and (7.51), the samples of x(n) are determined. [i.e., the coefficient of z^i is the i^{th} sample, x(i) of the signal x(n)].

> ***Note :*** *The different methods of evaluation of inverse Z-transform of a funcion X(z) will result in different type of mathematical expressions. But the inverse Z-transform is unique for a specified ROC and so on evaluating the expressions for each value of n, we may get a same signal.*

Example 7.7

Determine the inverse Z-transform of the function, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$ by the following three methods and prove that the inverse Z-transform is unique.

1. Residue Method
2. Partial Fraction Expansion Method
3. Power Series Expansion Method

Solution

Method-1 : Residue Method

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \dfrac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \dfrac{3z^2 + 2z + 1}{z^2 - 3z + 2}$.

Let us divide the numerator polynomial by denominator polynomial and express X(z) as shown below.

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 - 3z + 2} = 3 + \frac{11z - 5}{z^2 - 3z + 2}$$

$$= 3 + \frac{11z - 5}{(z - 1)(z - 2)}$$

$$\begin{array}{r|l} & \quad 3 \\ \hline z^2 - 3z + 2 & 3z^2 + 2z + 1 \\ & 3z^2 - 9z + 6 \\ & (-) \quad (+) \quad (-) \\ \hline & \qquad 11z - 5 \end{array}$$

Let, $X_1(z) = 3$ and $X_2(z) = \dfrac{11z - 5}{z^2 - 3z + 2}$; $\therefore X(z) = X_1(z) + X_2(z)$

$$x(n) = Z^{-1}\{X(z)\} = Z^{-1}\{X_1(z)\} + Z^{-1}\{X_2(z)\}$$

$$= Z^{-1}\{3\} + Z^{-1}\{X_2(z)\}$$

Using residue theorem

$$= 3\,\delta(n) + \sum_{i=1}^{N} \left[(z - p_i)\, X_2(z)\, z^{n-1} \Big|_{z = p_i} \right]$$

$$= 3\,\delta(n) + (z - 1)\frac{11z - 5}{(z - 1)(z - 2)}\, z^{n-1}\Bigg|_{z=1} + (z - 2)\frac{11z - 5}{(z - 1)(z - 2)}\, z^{n-1}\Bigg|_{z=2}$$

$$= 3\,\delta(n) + \frac{11 - 5}{1 - 2}(1)^{n-1} + \frac{11 \times 2 - 5}{2 - 1}\, 2^{n-1}$$

$$\therefore x(n) = 3\,\delta(n) - 6\,u(n - 1) + 17(2)^{n-1}\, u(n - 1) = 3\,\delta(n) + \left[-6 + 17(2)^{n-1}\right] u(n - 1)$$

When $n = 0$, $x(0) = 3 - 0 + 0 = 3$

When $n = 1$, $x(1) = 0 - 6 + 17 \times 2^0 = 11$

When $n = 2$, $x(2) = 0 - 6 + 17 \times 2^1 = 28$

When $n = 3$, $x(3) = 0 - 6 + 17 \times 2^2 = 62$

When $n = 4$, $x(4) = 0 - 6 + 17 \times 2^3 = 130$

$\therefore x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, ...\}$

Method-2 : Partial Fraction Expansion Method

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \dfrac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \dfrac{3z^2 + 2z + 1}{(z - 1)(z - 2)}$.

$$\therefore \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}$$

Let, $\dfrac{X(z)}{z} = \dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)} = \dfrac{A_1}{z} + \dfrac{A_2}{z - 1} + \dfrac{A_3}{z - 2}$

Now, $A_1 = z\dfrac{X(z)}{z}\Big|_{z=0} = z\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=0} = \dfrac{1}{(-1)\times(-2)} = 0.5$

$A_2 = (z - 1)\dfrac{X(z)}{z}\Big|_{z=1} = (z - 1)\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=1} = \dfrac{3 + 2 + 1}{1 \times (1 - 2)} = -6$

$A_3 = (z - 2)\dfrac{X(z)}{z}\Big|_{z=2} = (z - 2)\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=2} = \dfrac{3 \times 2^2 + 2 \times 2 + 1}{2 \times (2 - 1)} = 8.5$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z - 1} + \frac{8.5}{z - 2}$$

$$\therefore X(z) = 0.5 - 6\,\frac{z}{z - 1} + 8.5\,\frac{z}{z - 2}$$

$\mathcal{Z}\{\delta(n)\} = 1$

$\mathcal{Z}\{u(n)\} = \dfrac{z}{z - 1}$

$\mathcal{Z}\{a^n n(n)\} = \dfrac{z}{z - a}$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5\,(2)^n\,u(n) = 0.5\,\delta(n) + [-6 + 8.5(2)^n]\,u(n)$$

When $n = 0$, $x(0) = 0.5 - 6 + 8.5 \times 2^0 = 3$

When $n = 1$, $x(1) = 0 - 6 + 8.5 \times 2^1 = 11$

When $n = 2$, $x(2) = 0 - 6 + 8.5 \times 2^2 = 28$

When $n = 3$, $x(3) = 0 - 6 + 8.5 \times 2^3 = 62$

When $n = 4$, $x(4) = 0 - 6 + 8.5 \times 2^4 = 130$

$\therefore x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, ...\}$

Method-3 : Power Series Expansion Method

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}}$

Let us divide the numerator polynomial by denominator polynomial as shown below.

$$
\begin{array}{r|l}
 & 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots\ldots \\
\hline
1 - 3z^{-1} + 2z^{-2} & 3 + 2z^{-1} + z^{-2} \\
 & 3 - 9z^{-1} + 6z^{-2} \\
 & \underline{(-)\ \ (+)\ \ \ (-)} \\
 & 11z^{-1} - 5z^{-2} \\
 & 11z^{-1} - 33z^{-2} + 22z^{-3} \\
 & \underline{(-)\ \ \ \ (+)\ \ \ \ (-)} \\
 & 28z^{-2} - 22z^{-3} \\
 & 28z^{-2} - 84z^{-3} + 56z^{-4} \\
 & \underline{(-)\ \ \ \ (+)\ \ \ \ (-)} \\
 & 62z^{-3} - 56z^{-4} \\
 & 62z^{-3} - 186z^{-4} + 124z^{-5} \\
 & \underline{(-)\ \ \ \ (+)\ \ \ \ (-)} \\
 & 130z^{-4} - 124z^{-5} \\
 & \vdots \\
 & \overline{\phantom{130z^{-4} - 124z^{-5}}}
\end{array}
$$

$$\therefore\ X(z) = \frac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = 3 + 11z^{-1} + 28z^{-2} + 62z^{-3} + 130z^{-4} + \ldots.. \qquad \ldots..(1)$$

Let x(n) be inverse Z-transform of X(z).

Now, by definition of Z-transform,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

$$= \ldots..+ x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + x(4)\, z^{-4} + \ldots\ldots \qquad \ldots..(2)$$

On comparing equations (1) and (2) we get,

$x(0) = 3$

$x(1) = 11$

$x(2) = 28$

$x(3) = 62$

$x(4) = 130$ and so on.

$$\therefore\ x(n) = \{\underset{\uparrow}{3}, 11, 28, 62, 130, \ldots..\}$$

Conclusion : It is observed that the results of all the three methods are same.

Example 7.8

Determine the inverse Z-transform of the following Z-domain functions.

a) $X(z) = \dfrac{3z^2 + 2z + 1}{z^2 + 3z + 2}$ b) $X(z) = \dfrac{z - 0.4}{z^2 + z + 2}$ c) $X(z) = \dfrac{z - 4}{(z - 1)(z - 2)^2}$

Solution

a) Given that, $\mathbf{X(z) = \dfrac{3z^2 + 2z + 1}{z^2 + 3z + 2}}$

On dividing the numerator by denominator, the X(z) can be expressed as shown below.

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2} = 3 + \frac{-7z - 5}{z^2 + 3z + 2} = 3 + \frac{-7z - 5}{(z + 1)(z + 2)}$$

$$
\begin{array}{r|l}
 & 3 \\
\hline
z^2 + 3z + 2 & 3z^2 + 2z + 1 \\
 & 3z^2 + 9z + 6 \\
 & \underline{(-)\ \ \ (-)\ \ \ (-)} \\
 & -7z - 5
\end{array}
$$

By partial fraction expansion we get, $X(z) = 3 + \dfrac{A_1}{z + 1} + \dfrac{A_2}{z + 2}$

$$A_1 = (z+1)\left.\frac{-7z-5}{(z+1)(z+2)}\right|_{z=-1} = \left.\frac{-7z-5}{z+2}\right|_{z=-1} = \frac{-7\times(-1)-5}{-1+2} = 2$$

$$A_2 = (z+2)\left.\frac{-7z-5}{(z+1)(z+2)}\right|_{z=-2} = \left.\frac{-7z-5}{z+1}\right|_{z=-2} = \frac{-7\times(-2)-5}{-2+1} = -9$$

$$\therefore\ X(z) = 3 + \frac{2}{z+1} - \frac{9}{z+2} = 3 + 2\,\frac{1}{z}\,\frac{z}{z-(-1)} - 9\,\frac{1}{z}\,\frac{z}{z-(-2)}$$

Multiply and dvide by z

$$= 3 + 2z^{-1}\,\frac{z}{z-(-1)} - 9z^{-1}\,\frac{z}{z-(-2)}$$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$\mathcal{Z}\{\delta(n)\} = 1$

$\mathcal{Z}\{a^n\, u(n)\} = \dfrac{z}{z-a}$

If $\mathcal{Z}\{a^n\, u(n)\} = \dfrac{z}{z-a}$

then by time shifting property

$\mathcal{Z}\{a^{(n-1)}\, u(n-1)\} = z^{-1}\,\dfrac{z}{z-a}$

$$x(n) = 3\,\delta(n) + 2(-1)^{n-1}\,u(n-1) - 9(-2)^{n-1}\,u(n-1)$$

$$= 3\,\delta(n) + \left[2(-1)^{n-1} - 9(-2)^{n-1}\right] u(n-1)$$

When $n = 0$, $x(0) = 3 + 0 + 0 = 3$

When $n = 1$, $x(1) = 0 + 2 - 9 = -7$

When $n = 2$, $x(2) = 0 - 2 - 9 \times (-2) = 16$

When $n = 3$, $x(3) = 0 + 2 - 9 \times (-2)^2 = -34$

When $n = 4$, $x(4) = 0 - 2 - 9 \times (-2)^3 = 70$

$$\therefore\ x(n) = \{\underset{\uparrow}{3}, -7, 16, -34, 70, \ldots..\}$$

Alternate Method

$$X(z) = \frac{3z^2 + 2z + 1}{z^2 + 3z + 2}$$

$$\therefore\ \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z^2 + 3z + 2)} = \frac{3z^2 + 2z + 1}{z(z+1)(z+2)}$$

By partial fraction expansion technique $\dfrac{X(z)}{z}$ can be expressed as,

$$\frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z+1)(z+2)} = \frac{A_1}{z} + \frac{A_2}{z+1} + \frac{A_3}{z+3}$$

$$A_1 = z\left.\frac{X(z)}{z}\right|_{z=0} = z\left.\frac{3z^2 + 2z + 1}{z(z+1)(z+2)}\right|_{z=0} = \frac{1}{1\times 2} = 0.5$$

$$A_2 = (z+1)\left.\frac{X(z)}{z}\right|_{z=-1} = (z+1)\left.\frac{3z^2 + 2z + 1}{z(z+1)(z+2)}\right|_{z=-1} = \frac{3(-1)^2 + 2(-1) + 1}{-1\times(-1+2)} = -2$$

$$A_3 = (z+2)\left.\frac{X(z)}{z}\right|_{z=-2} = (z+2)\left.\frac{3z^2 + 2z + 1}{z(z+1)(z+2)}\right|_{z=-2} = \frac{3(-2)^2 + 2(-2) + 1}{-2\times(-2+1)} = 4.5$$

$$\therefore\ \frac{X(z)}{z} = \frac{0.5}{z} - \frac{2}{z+1} + \frac{4.5}{z+2}$$

$$\therefore\ X(z) = 0.5 - \frac{2z}{z+1} + \frac{4.5\,z}{z+2}$$

$$= 0.5 - 2\,\frac{z}{z-(-1)} + 4.5\,\frac{z}{z-(-2)}$$

$\mathcal{Z}\{\delta(n)\} = 1$

$\mathcal{Z}\{a^n\, u(n)\} = \dfrac{z}{z-a}$

On taking inverse Z-transform of X(z), we get,

$$x(n) = 0.5\,\delta(n) - 2\,(-1)^n\,u(n) + 4.5\,(-2)^n\,u(n) = 0.5\,\delta(n) + [-2\,(-1)^n + 4.5\,(-2)^n]\,u(n)$$

When $n = 0$, $x(0) = 0.5 - 2 + 4.5 = 3$

When $n = 1$, $x(1) = 0 + 2 + 4.5 \times (-2) = -7$

When $n = 2$, $x(2) = 0 - 2 + 4.5 \times (-2)^2 = 16$

When $n = 3$, $x(3) = 0 + 2 + 4.5 \times (-2)^3 = -34$

When $n = 4$, $x(4) = 0 - 2 + 4.5 \times (-2)^4 = 70$

$$\therefore x(n) = \{\underset{\uparrow}{3}, -7, 16, -34, 70, ...\}$$

> ***Note:*** *The closed form expression of x(n) in the two methods look different, but on evaluating x(n) for various values of n we get same signal x(n).*

b) Given that, $\mathbf{X(z) = \dfrac{z - 0.4}{z^2 + z + 2}}$

> The roots of the quadratic $z^2 + z + 2 = 0$ are,
> $$z = \frac{-1 \pm \sqrt{1 - 4 \times 2}}{2} = -0.5 \pm j\sqrt{7}/2$$

$$X(z) = \frac{z - 0.4}{z^2 + z + 2} = \frac{z - 0.4}{(z + 0.5 - j\sqrt{7}/2)\,(z + 0.5 + j\sqrt{7}/2)}$$

By partial fraction expansion we get,

$$X(z) = \frac{A}{z + 0.5 - j\sqrt{7}/2} + \frac{A^*}{z + 0.5 + j\sqrt{7}/2}$$

$$A = (z + 0.5 - j\sqrt{7}/2)\left.\frac{z - 0.4}{(z + 0.5 - j\sqrt{7}/2)\,(z + 0.5 + j\sqrt{7}/2)}\right|_{z = -0.5 + j\sqrt{7}/2}$$

$$= \left.\frac{z - 0.4}{(z + 0.5 + j\sqrt{7}/2)}\right|_{z = -0.5 + j\sqrt{7}/2} = \frac{-0.5 + j\sqrt{7}/2 - 0.4}{-0.5 + j\sqrt{7}/2 + 0.5 + j\sqrt{7}/2}$$

$$= \frac{-0.9 + j\sqrt{7}/2}{j\sqrt{7}} = \frac{-0.9}{j\sqrt{7}} + \frac{j\sqrt{7}/2}{j\sqrt{7}} = 0.5 + \frac{j0.9}{\sqrt{7}}$$

$$\therefore \quad A^* = \left(0.5 + \frac{j0.9}{\sqrt{7}}\right)^* = \left(0.5 - \frac{j0.9}{\sqrt{7}}\right)$$

$$\therefore X(z) = \frac{0.5 + j0.9/\sqrt{7}}{z + 0.5 - j\sqrt{7}/2} + \frac{0.5 - j0.9/\sqrt{7}}{z + 0.5 + j\sqrt{7}/2}$$

> Multiply and dvide by z

$$= (0.5 + j0.9/\sqrt{7})\,\frac{1}{z}\,\frac{z}{z + 0.5 - j\sqrt{7}/2} + (0.5 - j0.9/\sqrt{7})\,\frac{1}{z}\,\frac{z}{z + 0.5 + j\sqrt{7}/2}$$

$$= (0.5 + j0.9/\sqrt{7})z^{-1}\,\frac{z}{z - (-0.5 + j\sqrt{7}/2)} + (0.5 - j0.9/\sqrt{7})\,z^{-1}\,\frac{z}{z - (-0.5 - j\sqrt{7}/2)}$$

On taking inverse Z-transform of X(z) we get,

> If $Z\{a^n\,u(n)\} = \dfrac{z}{z - a}$
> then by time shifting property,
> $Z\{a^{(n-1)}\,u(n - 1)\} = z^{-1}\,\dfrac{z}{z - a}$

$$x(n) = (0.5 + j0.9/\sqrt{7})\,(-0.5 + j\sqrt{7}/2)^{(n-1)}u(n - 1)$$
$$+ (0.5 - j0.9/\sqrt{7})\,(-0.5 - j\sqrt{7}/2)^{(n-1)}u(n - 1)$$

Alternatively the above result can be expressed as shown below.

Here, $0.5 + j0.9\sqrt{7} = 0.605 \angle 34.2^\circ = 0.605 \angle 0.19\pi$

$0.5 - j0.9\sqrt{7} = 0.605 \angle -34.2^\circ = 0.605 \angle -0.19\pi$

$-0.5 + j\sqrt{7}/2 = 1.414 \angle 110.7^\circ = 1.414 \angle 0.62\pi$

$-0.5 - j\sqrt{7}/2 = 1.414 \angle -110.7^\circ = 1.414 \angle -0.62\pi$

$\therefore x(n) = [0.605 \angle 0.19\pi]\ [1.414 \angle 0.62\pi]^{(n-1)}\ u(n-1) + [0.605 \angle -0.19\pi]\ [1.414 \angle -0.62\pi]^{(n-1)}\ u(n-1)$

$= [0.605 \angle 0.19\pi]\ [1.414^{(n-1)} \angle 0.62\pi\ (n-1)]\ u(n-1) + [0.605 \angle -0.19\pi]\ [1.414^{(n-1)} \angle -0.62\pi\ (n-1)]\ u(n-1)$

$= 0.605\ (1.414)^{(n-1)} \angle (0.19\pi + 0.62\pi n - 0.62\pi)\ u(n-1) + 0.605\ (1.414)^{(n-1)} \angle (-0.19\pi - 0.62\pi n + 0.62\pi)\ u(n-1)$

$= 0.605\ (1.414)^{(n-1)}\ [1 \angle (0.62n - 0.43)\pi + 1 \angle -(0.62n - 0.43)\pi]\ u(n-1)$

$= 0.605\ (1.414)^{(n-1)}\ [\cos((0.62n - 0.43)\pi) + j\sin((0.62n - 0.43)\pi) + \cos((0.62\,n - 0.43)\pi)$
$- j\sin((0.62n - 0.43)\pi)\ ]\ u(n-1)$

$= 0.605\ (1.414)^{(n-1)}\ 2\cos((0.62n - 0.43)\pi)\ u(n-1)$

$= 1.21\ (1.414)^{(n-1)} \cos((0.62n - 0.43)\pi)\ u(n-1)$

c) Given that, $\mathbf{X(z) = \dfrac{z - 4}{(z - 1)\,(z - 2)^2}}$

By partial fraction expansion we get,

$$X(z) = \frac{z - 4}{(z - 1)\,(z - 2)^2} = \frac{A_1}{z - 1} + \frac{A_2}{(z - 2)^2} + \frac{A_3}{(z - 2)}$$

$$A_1 = (z - 1)\left.\frac{z - 4}{(z - 1)\,(z - 2)^2}\right|_{z=1} = \left.\frac{z - 4}{(z - 2)^2}\right|_{z=1} = \frac{1 - 4}{(1 - 2)^2} = -3$$

$$A_2 = (z - 2)^2\left.\frac{z - 4}{(z - 1)\,(z - 2)^2}\right|_{z=2} = \left.\frac{z - 4}{z - 1}\right|_{z=2} = \frac{2 - 4}{2 - 1} = -2$$

$$A_3 = \left.\frac{d}{dz}\left[(z - 2)^2\,\frac{z - 4}{(z - 1)\,(z - 2)^2}\right]\right|_{z=2} = \left.\frac{d}{dz}\left[\frac{z - 4}{z - 1}\right]\right|_{z=2}$$

$$= \left.\frac{(z - 1) - (z - 4)}{(z - 1)^2}\right|_{z=2} = \left.\frac{3}{(z - 1)^2}\right|_{z=2} = \frac{3}{(2 - 1)^2} = 3$$

$$\therefore X(z) = \frac{-3}{z - 1} - \frac{2}{(z - 2)^2} + \frac{3}{z - 2} = -3\,\frac{1}{z}\,\frac{z}{z - 1} - \frac{1}{z}\,\frac{2z}{(z - 2)^2} + 3\,\frac{1}{z}\,\frac{z}{z - 2}$$

Multiply and dvide by z

$$= -3z^{-1}\,\frac{z}{z - 1} - z^{-1}\,\frac{2z}{(z - 2)^2} + 3z^{-1}\,\frac{z}{z - 2}$$

$\mathcal{Z}\{u(n)\} = \dfrac{z}{z - 1}$; $\mathcal{Z}\{a^n\,u(n)\} = \dfrac{z}{z - a}$; $\mathcal{Z}\{na^n\,u(n)\} = \dfrac{az}{(z - a)^2}$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by time shifting property $\mathcal{Z}\{x(n-1)\} = z^{-1}\,X(z)$

$\therefore\ \mathcal{Z}\{u(n-1)\} = z^{-1}\dfrac{z}{z - 1}$; $\mathcal{Z}\{a^{(n-1)}\,u(n - 1)\} = z^{-1}\dfrac{z}{z - a}$

and $\mathcal{Z}\{(n - 1)\,a^{(n-1)}\,u(n - 1)\} = z^{-1}\,\dfrac{az}{(z - a)^2}$

On taking inverse $\mathcal{Z}$-transform of X(z) using standard transform and shifting property we get,

$$x(n) = -3\,u(n-1) - (n-1)\,2^{n-1}\,u(n-1) + 3 \times 2^{n-1}\,u(n-1)$$

$$= \left[-3 - (n-1)\,2^{n-1} + 3(2)^{n-1}\right]u(n-1)$$

Example 7.9

Determine the inverse Z-transform of the following function.

a) $X(z) = \dfrac{1}{1 - 1.5\, z^{-1} + 0.5z^{-2}}$ b) $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

c) $X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$ d) $X(z) = \dfrac{1}{(1 + z^{-1})\,(1 - z^{-1})^2}$

Solution

a) Given that, $X(z) = \dfrac{1}{1 - 1.5\, z^{-1} + 0.5z^{-2}}$

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{1 - \dfrac{1.5}{z} + \dfrac{0.5}{z^2}} = \frac{z^2}{z^2 - 1.5z + 0.5} = \frac{z^2}{(z - 1)\,(z - 0.5)}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z}{(z - 1)\,(z - 0.5)}$$

By partial fraction expansion, X(z)/z can be expressed as,

$$\frac{X(z)}{z} = \frac{A_1}{z - 1} + \frac{A_2}{z - 0.5}$$

$$A_1 = (z - 1)\left.\frac{X(z)}{z}\right|_{z\,=\,1} = (z - 1)\left.\frac{z}{(z - 1)\,(z - 0.5)}\right|_{z\,=\,1} = \frac{1}{1 - 0.5} = 2$$

$$A_2 = (z - 0.5)\left.\frac{X(z)}{z}\right|_{z\,=\,0.5} = (z - 0.5)\left.\frac{z}{(z - 1)\,(z - 0.5)}\right|_{z\,=\,0.5} = \frac{0.5}{0.5 - 1} = -1$$

$$\therefore \quad \frac{X(z)}{z} = \frac{2}{z - 1} - \frac{1}{z - 0.5}$$

$$\therefore \quad X(z) = \frac{2z}{z - 1} - \frac{z}{z - 0.5}$$

$$\mathcal{Z}\{a^n\, u(n)\} = \frac{z}{z - a}\ ; \ \text{ROC } |z| > |a|$$

$$\mathcal{Z}\{u(n)\} = \frac{z}{z - 1}\ ; \ \text{ROC } |z| > 1$$

On taking inverse Z-transform of X(z), we get,

$$x(n) = 2\, u(n) - 0.5^n\, u(n) = [2 - 0.5^n]\, u(n)$$

---

b) Given that, $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

$$X(z) = \frac{z^2}{z^2 - z + 0.5} = \frac{z^2}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{z}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

The roots of quadratic $z^2 - z + 0.5 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2}$$
$$= 0.5 \pm j\,0.5$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$= (z - 0.5 - j0.5)\left.\frac{z}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$\therefore\ A = \frac{0.5 + j0.5}{0.5 + j0.5 - 0.5 + j0.5} = \frac{0.5 + j0.5}{j1.0} = 0.5 - j0.5$$

$$\therefore\ A^* = (0.5 - j0.5)^* = 0.5 + j0.5$$

$$\therefore\ \frac{X(z)}{z} = \frac{0.5 - j0.5}{z - 0.5 - j0.5} + \frac{0.5 + j0.5}{z - 0.5 + j0.5}$$

$$X(z) = \frac{(0.5 - j0.5)\, z}{z - (0.5 + j0.5)} + \frac{(0.5 + j0.5)\, z}{z - (0.5 - j0.5)}$$

$z\{a^n u(n)\} = \dfrac{z}{z - a}$; ROC $\lvert z\rvert > \lvert a\rvert$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$$x(n) = (0.5 - j0.5)\,(0.5 + j0.5)^n\, u(n) + (0.5 + j0.5)\,(0.5 - j0.5)^n\, u(n)$$

Alternatively the above result can be expressed as shown below.

Here, $0.5 + j0.5 = 0.707\ \angle 45^\circ = 0.707\ \angle 0.25\pi$

$0.5 - j0.5 = 0.707\ \angle{-45^\circ} = 0.707\ \angle{-0.25\pi}$

$$\therefore\ x(n) = [0.707\ \angle{-0.25\pi}]\,[0.707\ \angle 0.25\pi]^n\ u(n) + [0.707\ \angle 0.25\pi]\,[0.707\ \angle{-0.25\pi}]^n\ u(n)$$

$$= [0.707\ \angle{-0.25\pi}]\,[0.707^n\ \angle 0.25\pi n]\, u(n) + [0.707\ \angle 0.25\pi]\,[0.707^n\ \angle{-0.25\pi n}]\, u(n)$$

$$= 0.707^{(n+1)}\ \angle(0.25\pi\ (n-1))\, u(n) + 0.707^{(n+1)}\ \angle(-0.25\pi(n-1))\, u(n)$$

$$= 0.707^{(n+1)}\ [1\ \angle 0.25\pi\ (n-1) + 1\angle{-0.25\pi(n-1)}]u(n)$$

$$= 0.707^{(n+1)}\ [\cos(0.25\pi\ (n-1)) + j\sin(0.25\pi(n-1)) + \cos(0.25\pi(n-1)) - j\sin(0.25\pi(n-1))]\, u(n)$$

$$= 0.707^{(n+1)}\ 2\cos(0.25\pi\ (n-1))\, u(n)$$

---

c) Given that, $\mathbf{X(z)} = \dfrac{\mathbf{1 + z^{-1}}}{\mathbf{1 - z^{-1} + 0.5z^{-2}}}$

$$X(z) = \frac{1 + z^{-1}}{1 - z^{-1} + 0.5\,z^{-2}} = \frac{z^{-1}(z + 1)}{z^{-2}(z^2 - z + 0.5)}$$

$$= \frac{z\,(z + 1)}{(z^2 - z + 0.5)} = \frac{z\,(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

The roots of the quadratic $z^2 - z + 0.5 = 0$ are, $z = \dfrac{1 \pm \sqrt{1 - 4 \times 0.5}}{2} = 0.5 \pm j0.5$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$= (z - 0.5 - j0.5)\left.\frac{(z + 1)}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}\right|_{z\,=\,0.5\,+\,j0.5}$$

$$= \frac{0.5 + j0.5 + 1}{0.5 + j0.5 - 0.5 + j0.5} = \frac{1.5 + j0.5}{j1} = -j1.5 + 0.5 = 0.5 - j1.5$$

$$A^* = (0.5 - j1.5)^* = 0.5 + j1.5$$

$$\therefore\ \frac{X(z)}{z} = \frac{0.5 - j1.5}{z - 0.5 - j0.5} + \frac{0.5 + j1.5}{z - 0.5 + j0.5}$$

$$X(z) = (0.5 - j1.5)\,\frac{z}{z - (0.5 + j0.5)} + (0.5 + j1.5)\,\frac{z}{z - (0.5 - j0.5)}$$

$\mathcal{Z}\{a^n u(n)\} = \dfrac{z}{z - a}$

On taking inverse Z-transform of X(z) we get,

$$x(n) = (0.5 - j1.5)\,(0.5 + j0.5)^n\,u(n) + (0.5 + j1.5)\,(0.5 - j0.5)^n\,u(n)$$

Alternatively the above result can be expressed as shown below.

Here, $0.5 - j1.5 = 1.581\ \angle{-71.6^\circ} = 1.581\ \angle{-0.4\pi}$

$$0.5 + j1.5 = 1.581\ \angle 71.6^\circ = 1.581\ \angle 0.4\pi$$

$$0.5 + j0.5 = 0.707\ \angle 45^\circ = 0.707\ \angle 0.25\pi$$

$$0.5 - j0.5 = 0.707\ \angle{-45^\circ} = 0.707\ \angle{-0.25\pi}$$

$$\therefore\ x(n) = [1.581\ \angle{-0.4\pi}]\,[0.707\ \angle 0.25\pi]^n\,u(n) + [1.581\ \angle 0.4\pi]\,[0.707\ \angle{-0.25\pi}]^n\,u(n)$$

$$= [1.581\ \angle{-0.4\pi}]\,[0.707^n\ \angle 0.25\pi n]\,u(n) + [1.581\ \angle 0.4\pi]\,[0.707^n\ \angle{-0.25\pi n}]\,u(n)$$

$$= 1.581\ (0.707)^n\,[1\ \angle\pi(0.25n - 0.4) + 1\ \angle{-\pi(0.25n - 0.4)}]\,u(n)$$

$$= 1.581\,(0.707)^n\,[\cos(\pi(0.25n - 0.4)) + j\sin(\pi(0.25n - 0.4)) + \cos(\pi(0.25n - 0.4)) - j\sin(\pi(0.25n - 0.4))]\,u(n)$$

$$= 1.581\,(0.707)^n\,2\cos(\pi(0.25\,n - 0.4))\,u(n)$$

$$= 3.162\,(0.707)^n\cos(\pi(0.25n - 0.4))\,u(n)$$

d) Given that, $X(z) = \dfrac{1}{(1 + z^{-1})\,(1 - z^{-1})^2}$

$$X(z) = \frac{1}{(1 + z^{-1})\,(1 - z^{-1})^2} = \frac{1}{z^{-1}(z + 1)\,z^{-2}(z - 1)^2} = \frac{z^3}{(z + 1)\,(z - 1)^2}$$

$$\therefore\ \frac{X(z)}{z} = \frac{z^2}{(z + 1)\,(z - 1)^2}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z + 1} + \frac{A_2}{(z - 1)^2} + \frac{A_3}{z - 1}$$

$$A_1 = (z + 1)\left.\frac{X(z)}{z}\right|_{z = -1} = (z + 1)\left.\frac{z^2}{(z + 1)\,(z - 1)^2}\right|_{z = -1} = \left.\frac{z^2}{(z - 1)^2}\right|_{z = -1} = \frac{(-1)^2}{(-1 - 1)^2} = 0.25$$

$$A_2 = (z - 1)^2\left.\frac{X(z)}{z}\right|_{z = 1} = (z - 1)^2\left.\frac{z^2}{(z + 1)\,(z - 1)^2}\right|_{z = 1} = \left.\frac{z^2}{z + 1}\right|_{z = 1} = \frac{1}{1 + 1} = 0.5$$

$$A_3 = \frac{d}{dz}\left[(z - 1)^2\,\frac{X(z)}{z}\right]\Bigg|_{z = 1} = \frac{d}{dz}\left[(z - 1)^2\,\frac{z^2}{(z + 1)\,(z - 1)^2}\right]\Bigg|_{z = 1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{z + 1}\right]\Bigg|_{z = 1} = \left.\frac{(z + 1)\,2z - z^2}{(z + 1)^2}\right|_{z = 1} = \frac{(1 + 1) \times 2 - 1}{(1 + 1)^2} = \frac{3}{4} = 0.75$$

$$\therefore\ \frac{X(z)}{z} = \frac{0.25}{z + 1} + \frac{0.5}{(z - 1)^2} + \frac{0.75}{z - 1}$$

$$X(z) = 0.25\,\frac{z}{z + 1} + 0.5\,\frac{z}{(z - 1)^2} + 0.75\,\frac{z}{z - 1}$$

$$= 0.25\,\frac{z}{z - (-1)} + 0.5\,\frac{z}{(z - 1)^2} + 0.75\,\frac{z}{z - 1}$$

$$\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z - a}$$

$$\mathcal{Z}\{n\,u(n)\} = \frac{z}{(z - 1)^2}$$

$$\mathcal{Z}\{u(n)\} = \frac{z}{z - 1}$$

On taking inverse Z-transform of X(z) we get,

$$x(n) = 0.25(-1)^n + 0.5n\,u(n) + 0.75\,u(n)$$

$$= [0.25(-1)^n + 0.5n + 0.75]\,u(n)$$

Example 7.10

Determine the inverse $\mathcal{Z}$-transform of $X(z) = \dfrac{1}{1 - 1.5\,z^{-1} + 0.5\,z^{-2}}$

When (a) ROC : $|z| > 1.0$ and (b) ROC : $|z| < 0.5$.

Solution

a) Since the ROC is the exterior of a circle, we expect x(n) to be causal signal. Hence we can express X(z) as a power series expansion in negative powers of z. On dividing the numerator of X(z) by its denominator we get,

$$X(z) = \frac{1}{1 - 1.5\,z^{-1} + 0.5\,z^{-2}}$$

$$= 1 + 1.5\,z^{-1} + 1.75\,z^{-2} + 1.875\,z^{-3} + 1.9375\,z^{-4} + \,..... \qquad .....(1)$$

$$
\begin{array}{r|l}
 & 1 + 1.5\,z^{-1} + 1.75\,z^{-2} + 1.875\,z^{-3} + 1.9375\,z^{-4} + \,......... \\
\hline
1 - 1.5\,z^{-1} + 0.5\,z^{-2} & 1 \\
 & 1 - 1.5\,z^{-1} + 0.5\,z^{-2} \\
 & \underline{(-)\quad(+)\qquad\quad(-)} \\
 & \quad 1.5\,z^{-1} - 0.5\,z^{-2} \\
 & \quad 1.5\,z^{-1} - 2.25\,z^{-2} + 0.75\,z^{-3} \\
 & \quad \underline{(-)\qquad\quad(+)\qquad\quad(-)} \\
 & \qquad\qquad 1.75\,z^{-2} - 0.75\,z^{-3} \\
 & \qquad\qquad 1.75\,z^{-2} - 2.625\,z^{-3} + 0.875\,z^{-4} \\
 & \qquad\qquad \underline{(-)\qquad\quad(+)\qquad\quad(-)} \\
 & \qquad\qquad\qquad 1.875\,z^{-3} - 0.875\,z^{-4} \\
 & \qquad\qquad\qquad 1.875\,z^{-3} - 2.8125\,z^{-4} + 0.9375\,z^{-5} \\
 & \qquad\qquad\qquad \underline{(-)\qquad\quad(+)\qquad\quad(-)} \\
 & \qquad\qquad\qquad\qquad 1.9375\,z^{-4} - 0.9375\,z^{-5} \\
 & \qquad\qquad\qquad\qquad \vdots
\end{array}
$$

If X(z) is $\mathcal{Z}$-transform of x(n) then, by the definition of $\mathcal{Z}$-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\,z^{-n}$$

For a causal signal,

$$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n}$$

On expanding the summation we get,

$$X(z) = x(0)\,z^{0} + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3} + x(4)\,z^{-4} + \,........ \qquad .....(2)$$

On comparing the two power series of X(z) (equations (1) and (2)), we get,

$$x(0) = 1\,;\ x(1) = 1.5\,;\quad x(2) = 1.75\,;\ x(3) = 1.875\,;\ x(4) = 1.9375\,;\ ....$$

$$x(n) = \{\underset{\uparrow}{1},\ 1.5,\ 1.75,\ 1.875,\ 1.9375,....\}$$

---

b) Since the ROC is the interior of a circle, we expect x(n) to be anticausal signal. Hence we can express X(z) as a power series expansion in positive powers of z. Therefore, rewrite the denominator polynomial of X(z) in the reverse order and then the numerator, is divided by the denominator as shown below.

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots.. \\
\hline
0.5\,z^{-2} - 1.5\,z^{-1} + 1 & 1 \\
 & \underset{(-)\;(+)\;\;(-)}{1 - 3z + 2z^2} \\
\hline
 & \quad 3z - 2z^2 \\
 & \quad \underset{(-)\;\;(+)\;\;\;(-)}{3z - 9z^2 + 6z^3} \\
\hline
 & \qquad 7z^2 - 6z^3 \\
 & \qquad \underset{(-)\;\;(+)\;\;\;(-)}{7z^2 - 21z^3 + 14z^4} \\
\hline
 & \qquad\quad 15z^3 - 14z^4 \\
 & \qquad\quad \underset{(-)\;\;(+)\;\;\;(-)}{15z^3 - 45z^4 + 30z^5} \\
\hline
 & \qquad\qquad 31z^4 - 30z^5 \\
 & \qquad\qquad \vdots \\
\hline
\end{array}
$$

$$\therefore X(z) = \frac{1}{1 - 1.5\,z^{-1} + 0.5\,z^{-2}} = \frac{1}{0.5\,z^{-2} - 1.5\,z^{-1} + 1}$$

$$= 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \ldots\ldots \qquad \ldots..(3)$$

If X(z) is Z-transform of x(n) then, by the definition of Z-transform we get,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

For an anticausal signal, $X(z) = \sum_{n=-\infty}^{0} x(n)\, z^{-n}$

On expanding the summation we get,

$$X(z) = \ldots..\ x(-6)\, z^6 + x(-5)\, z^5 + x(-4)\, z^4 + x(-3)\, z^3 + x(-2)\, z^2 + x(-1)\, z + x(0) \qquad \ldots.\ (4)$$

On comparing the two power series of X(z) (equations (3) and (4)), we get,

$$x(0) = 0\ ;\ x(-1) = 0\ ;\ x(-2) = 2\ ;\ x(-3) = 6\ ;\ x(-4) = 14\ ;\ x(-5) = 30\ ;\ x(-6) = 62\ ;\ \ldots..$$

$$\therefore\ x(n) = \{\ldots\ldots.., 62, 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

Example 7.11

Determine the inverse Z-transform of $X(z) = \dfrac{1}{1 - 0.8\,z^{-1} + 0.12\,z^{-2}}$

a) if ROC is, $|z| > 0.6$ b) if ROC is, $|z| < 0.2$ c) if ROC is, $0.2 < |z| < 0.6$

Solution

Given that, $X(z) = \dfrac{1}{1 - 0.8\,z^{-1} + 0.12\,z^{-2}} = \dfrac{1}{z^{-2}(z^2 - 0.8z + 0.12)} = \dfrac{z^2}{(z - 0.6)\,(z - 0.2)}$

$$\therefore\ \frac{X(z)}{z} = \frac{z}{(z - 0.6)\,(z - 0.2)}$$

By partial fraction expansion technique we get,

$$\frac{X(z)}{z} = \frac{z}{(z - 0.6)\,(z - 0.2)} = \frac{A_1}{z - 0.6} + \frac{A_2}{z - 0.2}$$

The roots of quadratic $z^2 - 0.8z + 0.12 = 0$ are,

$$z = \frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.12}}{2} = \frac{0.8 \pm 0.4}{2} = 0.6,\ 0.2$$

$$A_1 = (z-0.6)\left.\frac{X(z)}{z}\right|_{z=0.6} = (z-0.6)\left.\frac{z}{(z-0.6)(z-0.2)}\right|_{z=0.6} = \frac{0.6}{0.6-0.2} = 1.5$$

$$A_2 = (z-0.2)\left.\frac{X(z)}{z}\right|_{z=0.2} = (z-0.2)\left.\frac{z}{(z-0.6)(z-0.2)}\right|_{z=0.2} = \frac{0.2}{0.2-0.6} = -0.5$$

$$\therefore \frac{X(z)}{z} = \frac{1.5}{z-0.6} - \frac{0.5}{z-0.2}$$

$$\therefore X(z) = 1.5\frac{z}{z-0.6} - 0.5\frac{z}{z-0.2}$$

Now, the poles of X(z) are $p_1 = 0.6$, $p_2 = 0.2$

a) ROC is |z| > 0.6

The specified ROC is exterior of the circle whose radius corresponds to the largest pole, hence x(n) will be a causal (or right sided) signal.

$$\therefore x(n) = 1.5(0.6)^n u(n) - 0.5(0.2)^n u(n)$$

$$\boxed{\mathcal{Z}\{a^n u(n)\} = \frac{z}{z-a}\ ;\ \text{ROC } |z| > |a|}$$

b) ROC is |z| < 0.2

The specified ROC is interior of the circle whose radius corresponds to the smallest pole, hence x(n) will be an anticausal (or left sided) signal.

$$\therefore x(n) = 1.5(-(0.6)^n u(-n-1)) - 0.5(-(0.2)^n u(-n-1))$$
$$= -1.5(0.6)^n u(-n-1) + 0.5(0.2)^n u(-n-1)$$

$$\boxed{\mathcal{Z}\{-a^n u(-n-1)\} = \frac{z}{z-a}\ ;\ \text{ROC } |z| < |a|}$$

c) ROC is 0.2 < |z| < 0.6

The specified ROC is the region in between two circles of radius 0.2 and 0.7. Hence the term corresponds to the pole, $p_1 = 0.6$ will be anticausal signal (because $|z| < 0.6$) and the term corresponds to the pole, $p_2 = 0.2$, will be a causal signal (because $|z| > 0.2$).

$$\therefore x(n) = 1.5(-(0.6)^n u(-n-1)) - 0.5(0.2)^n u(n)$$
$$= -1.5(0.6)^n u(-n-1)) - 0.5(0.2)^n u(n)$$

7.6 Analysis of LTI Discrete Time System Using $\mathcal{Z}$-Transform

7.6.1 Transfer Function of LTI Discrete Time System

Let x(n) be the input and y(n) be the output of an LTI discrete time system. The mathematical equation governing the input-output relation of an LTI discrete time system is given by, (refer chapter-6, equation (6.17)),

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad .....(7.52)$$

The equation (7.52) is a constant coefficient difference equation and N is the order of the system.

Let us take $\mathcal{Z}$-transform of equation (7.52) with zero initial conditions (i.e., $y(n) = 0$ for $n < 0$ and $x(n) = 0$ for $n < 0$).

$$\therefore \mathcal{Z}\{y(n)\} = \mathcal{Z}\left\{-\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)\right\}$$
$$= \mathcal{Z}\left\{-\sum_{m=1}^{N} a_m\, y(n-m)\right\} + \mathcal{Z}\left\{\sum_{m=0}^{M} b_m\, x(n-m)\right\}$$
$$= -\sum_{m=1}^{N} a_m\, \mathcal{Z}\{y(n-m)\} + \sum_{m=0}^{M} b_m\, \mathcal{Z}\{x(n-m)\} \qquad .....(7.53)$$

Let y(n) = 0 for n < 0, now if $\mathcal{Z}\{y((n)\} = Y(z)$, then $\mathcal{Z}\{y(n-m)\} = z^{-m}\,Y(z)$ (Using shifting property)
Let x(n) = 0 for n < 0, now if $\mathcal{Z}\{x((n)\} = X(z)$, then $\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$ (Using shifting property)

Using shifting property of $\mathcal{Z}$-transform, the equation (7.53) is written as shown below.

$$Y(z) = -\sum_{m=1}^{N} a_m\, z^{-m}\, Y(z) + \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$Y(z) + \sum_{m=1}^{N} a_m\, z^{-m}\, Y(z) = \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$Y(z)\left[1 + \sum_{m=1}^{N} a_m\, z^{-m}\right] = \sum_{m=0}^{M} b_m\, z^{-m}\, X(z)$$

$$\boxed{\therefore \quad \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m\, z^{-m}}{1 + \sum_{m=1}^{N} a_m\, z^{-m}}}$$

On expanding the summations in the above equation we get,

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1\, z^{-1} + b_2\, z^{-2} + b_3\, z^{-3} + \,.....\, + b_M\, z^{-M}}{1 + a_1\, z^{-1} + a_2\, z^{-2} + a_3\, z^{-3} + \,.....\, + a_N\, z^{-N}} \qquad .....(7.54)$$

The ***transfer function*** of a discrete time system is defined as the ratio of $\mathcal{Z}$-transform of output and $\mathcal{Z}$-transform of input. Hence the equation (7.54) is the transfer function of an LTI discrete time system.

The equation (7.54) is a rational function of z^{-1} (i.e., ratio of two polynomials in z^{-1}). The numerator and denominator polynomials of equation (7.54) are converted to positive power of z and then expressed in the factorized form as shown in equation (7.55). (Refer equation (7.36)).

Let M = N

$$\frac{Y(z)}{X(z)} = G\, \frac{(z - z_1)\,(z - z_2)\,(z - z_3)\, ...\, (z - z_N)}{(z - p_1)\,(z - p_2)\,(z - p_3)\, ...\, (z - p_N)} \qquad .....(7.55)$$

where, $z_1, z_2, z_3, \,.....\, z_N$ are roots of numerator polynomial (or zeros of discrete time system)

$p_1, p_2, p_3, \,.....\, p_N$ are roots of denominator polynomial (or poles of discrete time system)

7.6.2 Impulse Response and Transfer Function

Let, x(n) = Input of an LTI discrete time system

y(n) = Output / Response of the LTI discrete time system for the input x(n)

h(n) = Impulse response (i.e., response for impulse input)

Now, the response y(n) of the discrete time system is given by convolution of input and impulse response. (Refer chapter-6, equation (6.33)).

$$\therefore\ y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$$

Using convolution property of $\mathcal{Z}$-transform, the above equation can be written as,

If $\mathcal{Z}\{x(n)\} = X(z)$
and $\mathcal{Z}\{h(n)\} = H(z)$
then by convolution property,
$\mathcal{Z}\{x(n) * h(n)\} = X(z)\, H(z)$

$$Y(z) = X(z)\, H(z)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} \qquad .....(7.56)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = G\,\frac{(z - z_1)(z - z_2)(z - z_3)\,.....\,(z - z_N)}{(z - p_1)(z - p_2)(z - p_3)\,.....\,(z - p_N)}$$

Using equation (7.55)

From equation (7.56) we can conclude that the transfer function of an LTI discrete time system is also given by $\mathcal{Z}$-transform of the impulse response.

Alternatively, we can say that the inverse $\mathcal{Z}$-transform of transfer function is the impulse response of the system.

$$\therefore\ \text{Impulse reponse, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{Y(z)}{X(z)}\right\}$$

Using equation (7.56)

7.6.3 Response of LTI Discrete Time System Using $\mathcal{Z}$-Transform

In general, the input-output relation of an LTI (Linear Time Invariant) discrete time system is represented by the constant coefficient difference equation shown below, (equation (7.52)).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)} \qquad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \ \text{ with } a_o = 1 \qquad .....(7.57)$$

The solution of the above difference equation (equation (7.57)) is the (total) response y(n) of LTI discrete time system, which consists of two parts. In signals and systems the two parts of the solution y(n) are called zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore\ \text{Response, } y(n) = y_{zi}(n) + y_{zs}(n)} \qquad .....(7.58)$$

Zero-input Response (or Free Response or Natural Response) Using $\mathcal{Z}$-Transform

The ***zero-input response*** $y_{zi}(n)$ is mainly due to initial output (or initial stored energy) in the system. The zero-input response is obtained from system equation (equation (7.57)) when input x(n) = 0.

On substituting x(n) = 0 and $y(n) = y_{zi}(n)$ in equation (7.57) we get,

$$\sum_{m=0}^{N} a_m\, y_{zi}(n-m) = 0\,; \ \text{ with } a_o = 1$$

On taking $\mathcal{Z}$-transform of the above equation with non-zero initial conditions for output we can form an equation for $Y_{zi}(z)$. The zero-input response $y_{zi}(n)$ of a discrete time system is given by inverse $\mathcal{Z}$-transform of $Y_{zi}(z)$.

Zero-State Response (or Forced Response) Using $\mathcal{Z}$-Transform

The ***zero-state response*** $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output. The zero-state response is obtained from the difference equation governing the system (equation(7.57)) for specific input signal x(n) for n ≥ 0 and with zero initial output.

On substituting $y(n) = y_{zs}(n)$ in equation (7.57) we get,

$$\sum_{m=0}^{N} a_m \, y_{zs}(n-m) = \sum_{m=0}^{M} b_m \, x(n-m) \; ; \text{ with } a_o = 1$$

On taking Ƶ-transform of the above equation with zero initial conditions for output (i.e., $y_{zs}(n)$) and non-zero initial values for input (i.e., x(n)) we can form an equation for $Y_{zs}(z)$. The zero-state response $y_{zs}(n)$ of a discrete time system is given by inverse Ƶ-transform of $Y_{zs}(z)$.

Total Response

The ***total response*** y(n) is the response of the system due to input signal and initial output (or intial stored energy). The total response is obtained from the difference equation governing the system (equation(7.57)) for specific input signal x(n) for $n \geq 0$ and with non-zero initial conditions.

On taking Ƶ-transform of equation (7.57) with non-zero initial conditions for both input and output, and then substituting for X(z) we can form an equation for Y(z). The total response y(n) is given by inverse Ƶ-transform of Y(z). Alternatively, the total response y(n) is given by sum of zero-input response $y_{zi}(n)$ and zero-state response $y_{zs}(n)$.

$$\boxed{\therefore \text{ Total response, } \quad y(n) = y_{zi}(n) + y_{zs}(n)}$$

7.6.4 Convolution and Deconvolution Using Ƶ-Transform

Convolution

The ***convolution*** operation is performed to find the response y(n) of an LTI discrete time system from the input x(n) and impulse response h(n).

$\therefore$ Response, $y(n) = x(n) * h(n)$

On taking Ƶ-transform of the above equation we get,

$$Ƶ\{y(n)\} = Ƶ\{x(n) * h(n)\} \qquad \boxed{\text{Using convolution property}}$$

$$\therefore Y(z) = X(z)\,H(z) \qquad .....(7.59)$$

$\therefore$ Response, $y(n) = Ƶ^{-1}\{Y(z)\} = Ƶ^{-1}\{X(z)\,H(z)\}$

Procedure :
1. Take Ƶ-transform of x(n) to get X(z).
2. Take Ƶ-transform of h(n) to get H(z).
3. Get the product X(z) H(z).
4. Take inverse Ƶ-transform of the product X(z) H(z).

Deconvolution

The ***deconvolution*** operation is performed to extract the input x(n) of an LTI system from the response y(n) of the system.

From equation (7.59) get,

$$X(z) = \frac{Y(z)}{H(z)}$$

On taking inverse Ƶ-transform of the above equation we get,

$$\text{Input, } x(n) = Ƶ^{-1}\{X(z)\} = Ƶ^{-1}\left\{\frac{Y(z)}{H(z)}\right\}$$

Procedure :
1. Take Ƶ-transform of y(n) to get Y(z).
2. Take Ƶ-transform of h(n) to get H(z).
3. Divide Y(z) by H(z) to get X(z) (i.e., X(z) = Y(z) / H(z)).
4. Take inverse Ƶ-transform of X(z) to get x(n).

7.6.5 Stability in z-Domain

Location of Poles for Stability

Let, h(n) be the impulse response of an LTI discrete time system. Now, if h(n) satisfies the condition,

$$\sum_{n=-\infty}^{+\infty} |h(n)| < \infty \qquad .....(7.60)$$

then the LTI discrete time system is stable, (Refer equation 6.27 in chapter - 6).

The stability condition of equation (7.60) can be transformed as a condition on location of poles of transfer function of the LTI discrete time system in z-plane.

Let, $h(n) = a^n\, u(n)$

$$\text{Now,}\quad \sum_{n=-\infty}^{+\infty} |h(n)| = \sum_{n=-\infty}^{+\infty} \left|a^n\, u(n)\right| = \sum_{n=0}^{\infty} a^n$$

If |a| is such that, $0 < |a| < 1$, then $\sum_{n=0}^{\infty} a^n = \dfrac{1}{1-a}$ = constant, and so the system is stable.

If $|a| > 1$, then $\sum_{n=0}^{\infty} a^n = \infty$ and so the system is unstable.

$$\text{Now,}\quad H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\left\{a^n\, u(n)\right\} = \frac{z}{z-a}$$

Here H(z) has pole at $z = a$.

If $|a| < 1$, then the pole will lie inside the unit circle and if $|a| > 1$, then the pole will lie outside the unit circle. Therefore we can say that, *for a stable discrete time system the pole should lie inside the unit circle.* The various types of impulse response of LTI system and their transfer functions and the locations of poles are summarized in table 7.5.

ROC of a Stable System

Let, H(z) be $\mathcal{Z}$-transform of h(n). Now, by definition of $\mathcal{Z}$-transform we get,

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n}$$

Let us evaluate H(z) for $z = 1$.

$$\therefore\ H(z) = \sum_{n=-\infty}^{+\infty} h(n)$$

On taking absolute value on both sides we get,

$$|H(z)| = \left|\sum_{n=-\infty}^{+\infty} h(n)\right| \quad \Rightarrow \quad |H(z)| = \sum_{n=-\infty}^{+\infty} |h(n)|$$

For a stable LTI discrete time system,

$$\boxed{\sum_{n=-\infty}^{+\infty} |h(n)| < \infty} \quad \Rightarrow \quad \boxed{|H(z)| < \infty}$$

Therefore, we can conclude that $z = 1$ will be a point in the ROC of a stable system. Hence *for a stable discrete time system the ROC should include the unit circle.*

General Condition for Stability in z-plane

On combining the condition for location of poles and the ROC we can say that *for a stable LTI discrete time system the poles should lie inside the unit circle and the unit circle should be included in ROC.*

Table-7.5 : Impulse Response and Location of Poles

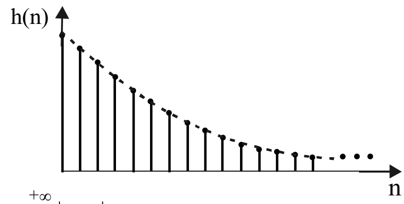
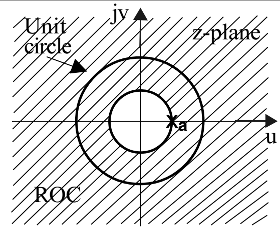
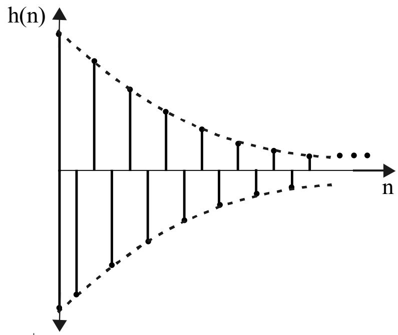
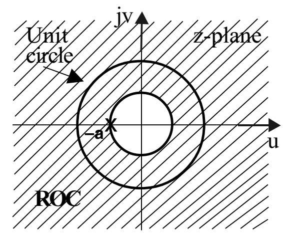
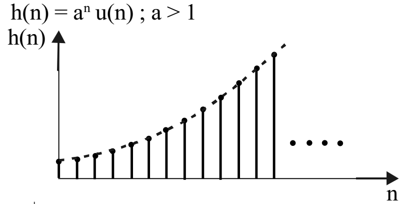
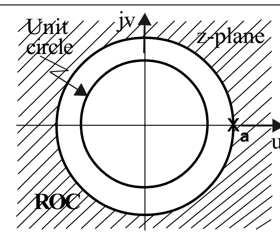
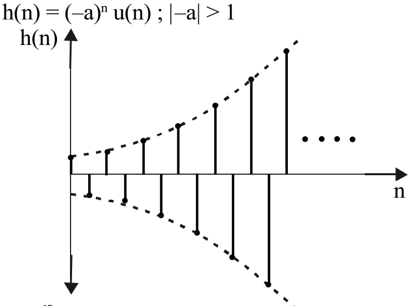
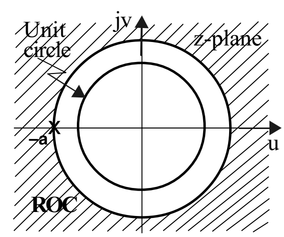
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = a^n\,u(n)\ ;\ 0 < a < 1$  $\sum\limits_{n=0}^{+\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{z}{z-a}$ ROC is $\lvert z\rvert > a$ pole at $z = a$	 Since $0 < a < 1$, the pole $z = a$, lies inside the unit circle. The ROC contains the unit circle.
$h(n) = (-a)^n\,u(n)\ ;\ 0 < \lvert -a\rvert < 1$  $\sum\limits_{n=0}^{+\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{z}{z+a}$ ROC is $\lvert z\rvert > \lvert -a\rvert$ pole at $z = -a$	 Since $0 < \lvert -a\rvert < 1$, the pole at $z = -a$, lies inside the unit circle. The ROC contains the unit circle.
$h(n) = a^n\,u(n)\ ;\ a > 1$  $\sum\limits_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{z-a}$ ROC is $\lvert z\rvert > a$ pole at $z = a$	 Since $a > 1$, the pole at $z = a$, lies outside the unit circle. The ROC does not contain the unit circle.
$h(n) = (-a)^n\,u(n)\ ;\ \lvert -a\rvert > 1$  $\sum\limits_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{z+a}$ ROC is $\lvert z\rvert > \lvert -a\rvert$ pole at $z = -a$	 Since $\lvert -a\rvert > 1$, the pole at $z = -a$, lies outside the unit circle. The ROC does not contain the unit circle.

Table-7.5 : Continued....

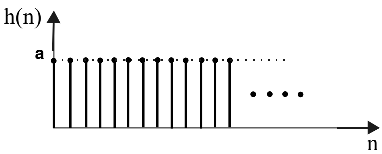
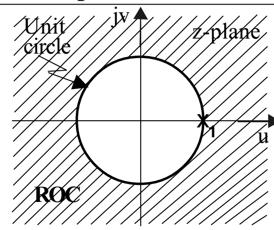
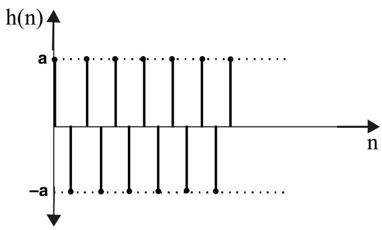
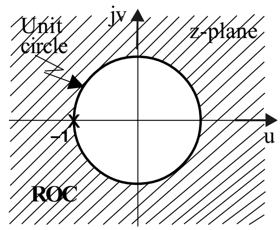
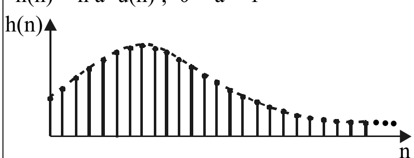
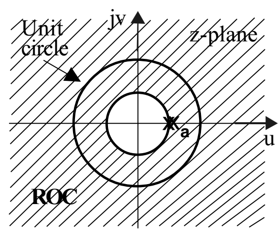
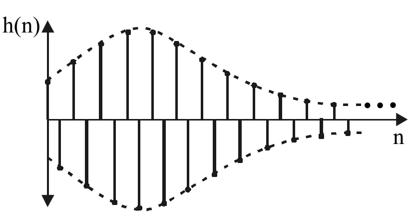
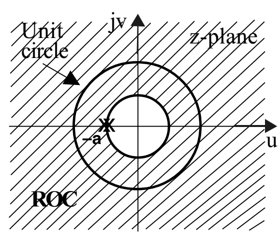
Impulse response $h(n)$	Transfer function $H(z)$	Location of poles in z-plane and ROC
$h(n) = a\,u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{z-1}$ ROC is $\lvert z\rvert > 1$ pole at $z = 1$	 The pole z = 1 lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = a(-1)^n u(n)$; $a > 0$ (i.e., a is positive)  $\sum_{n=0}^{+\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{z+1}$ ROC is $\lvert z\rvert > 1$ pole at $z = -1$	 The pole at z = –1 lies on the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,a^n u(n)$; $0 < a < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{az}{(z-a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = a$	 Since 0 < a < 1, the two poles at z = a, lie inside the unit circle. The ROC contains the unit circle.
$h(n) = n\,(-a)^n u(n)$; $0 < \lvert -a\rvert < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{az}{(z+a)^2}$ ROC is $\lvert z\rvert > a$ Two poles at $z = -a$	 Since 0 < |–a| < 1, the two poles at z = –a, lie inside the unit circle. The ROC contains the unit circle.

Table-7. 5 : Continued....

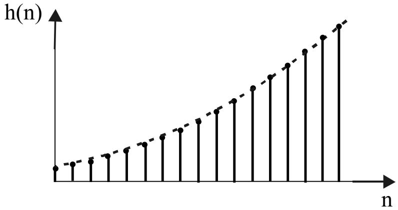
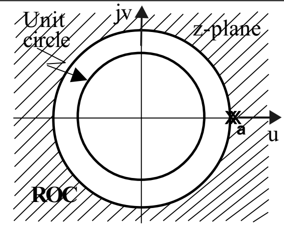
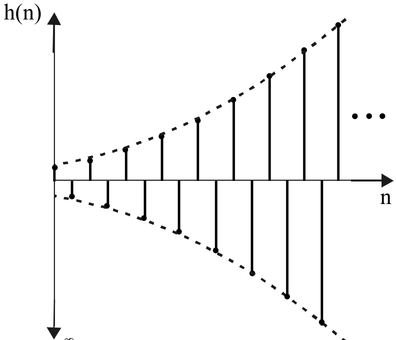
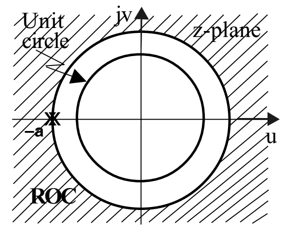
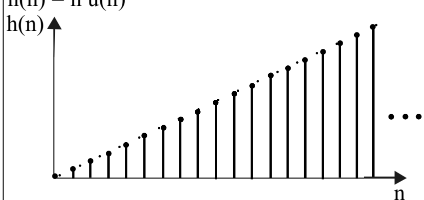
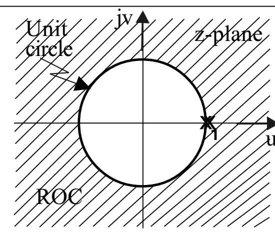
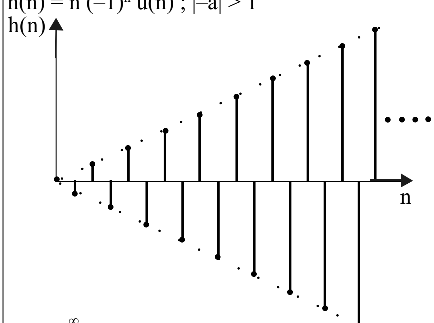
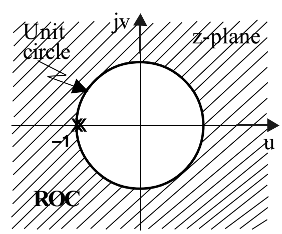
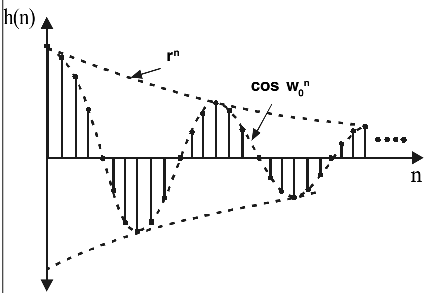
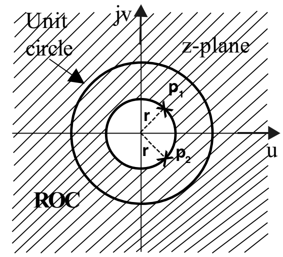
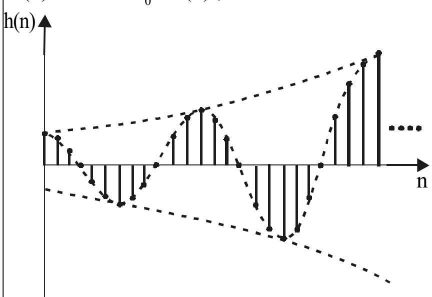
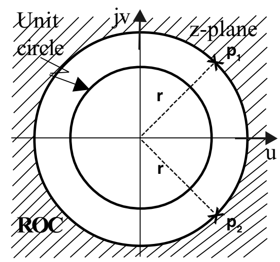
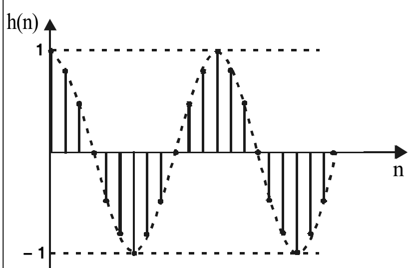
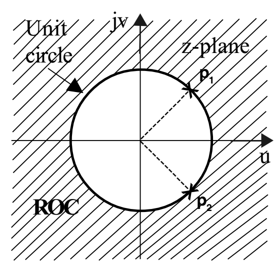
Impulse response h(n)	Transfer function H(z)	Location of poles in z-plane and ROC
$h(n) = n\,a^n\,u(n)\ ;\ a > 1$  $\sum\limits_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z-a)^2}$ ROC is $\lvert z \rvert > a$ Two poles at $z = a$	 Since $a > 1$, the two poles at $z = a$, lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,(-a)^n\,u(n)\ ;\ \lvert -a \rvert > 1$  $\sum\limits_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{az}{(z+a)^2}$ ROC is $\lvert z \rvert > \lvert -a \rvert$ Two poles at $z = -a$	 Since $\lvert -a \rvert > 1$, the two poles at $z = -a$, lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,u(n)$  $\sum\limits_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z-1)^2}$ ROC is $\lvert z \rvert > 1$ Two poles at $z = 1$	 The two poles at $z = 1$, lie on the unit circle. The ROC does not contain the unit circle.
$h(n) = n\,(-1)^n\,u(n)\ ;\ \lvert -a \rvert > 1$  $\sum\limits_{n=0}^{\infty} \lvert h(n) \rvert = \infty$; Unstable system	$H(z) = \dfrac{z}{(z+1)^2}$ ROC is $\lvert z \rvert > 1$ Two poles at $z = -1$	 The two poles at $z = -1$, lie on the unit circle. The ROC does not contain the unit circle.

Table-7.5 : Continued....

Impulse response $h(n)$	Transfer function $H(z)$	Location of poles in z-plane and ROC
$h(n) = r^n \cos \omega_0 n\, u(n)$; $0 < r < 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert < \infty$; Stable system	$H(z) = \dfrac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0 - jr\sin\omega_0)(z - r\cos\omega_0 + jr\sin\omega_0)}$ ROC is $\lvert z\rvert > r$. A pair of conjugate poles at $z = p_1 = r\cos\omega_0 + jr\sin\omega_0$ $z = p_2 = r\cos\omega_0 - jr\sin\omega_0$	 Since $0 < r < 1$, the conjugate pole pairs lie inside the unit circle. The ROC contains the unit circle
$h(n) = r^n \cos \omega_0 n\, u(n)$; $r > 1$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z(z - r\cos\omega_0)}{(z - r\cos\omega_0 - jr\sin\omega_0)(z - r\cos\omega_0 + jr\sin\omega_0)}$ ROC is $\lvert z\rvert > r$. A pair of conjugate poles at $z = p_1 = r\cos\omega_0 + jr\sin\omega_0$ $z = p_2 = r\cos\omega_0 - jr\sin\omega_0$	 Since $r > 1$, the conjugate pole pairs lie outside the unit circle. The ROC does not contain the unit circle.
$h(n) = \cos \omega_0 n\, u(n)$  $\sum_{n=0}^{\infty} \lvert h(n)\rvert = \infty$; Unstable system	$H(z) = \dfrac{z(z - \cos\omega_0)}{(z - \cos\omega_0 - j\sin\omega_0)(z - \cos\omega_0 + j\sin\omega_0)}$ ROC is $\lvert z\rvert > 1$. A pair of conjugate poles on unit circle at, $z = p_1 = \cos\omega_0 + j\sin\omega_0$ $z = p_2 = \cos\omega_0 - j\sin\omega_0$	 The conjugate pole pairs lie on the unit circle. The ROC does not contain the unit circle.

7.7 Relation Between Laplace Transform and Z-Transform

7.7.1 Impulse Train Sampling of Continuous Time Signal

Consider a periodic impulse train p(t) shown in fig 7.9a, with period T. The pulse train can be mathematically expressed as shown in equation (7.61).

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \qquad \text{..... (7.61)}$$

When a continuous time signal x(t) is multiplied by the impulse train p(t), the product signal will have impulses. A continuous time signal x(t) and the product of x(t) and p(t) are shown in fig 7.9b and fig 7.9c respectively. In fig 7.9c, the magnitudes of the impulses are equal to magnitude of x(t), and so the product signal is impulse sampled version of x(t), with sampling period T. Let us denote the product signal as $x_p(t)$ and it is mathematically expressed as shown in equation (7.62).

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT) \qquad \text{..... (7.62)}$$

where, x(nT) are samples of x(t) at t = nT

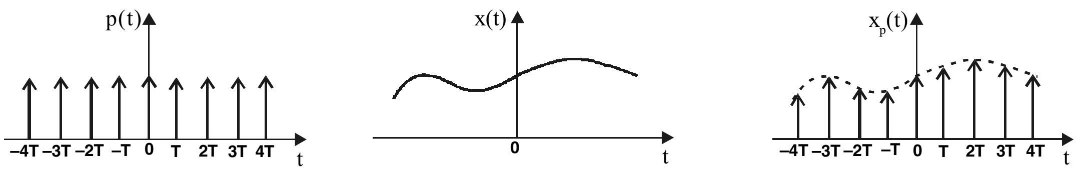


Fig 7.9a : *Impulse train.* ***Fig 7.9b :*** *Continuous time signal.* ***Fig 7.9c :*** *Samples of continuous time signal.*

Fig 7.9 : *Impulse sampling of continuous time signals.*

7.7.2 Transformation From Laplace Transform to Z-Transform

Let x(t) be a continuous time signal, and $x_p(t)$ be its impulse sampled version of discrete time signal. From equation (7.62) we get,

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

$\boxed{\mathcal{L}\{\delta(t)\} = 1}$

If $\mathcal{L}\{x(t)\} = X(s)$ then by time shifting property $\mathcal{L}\{x(t-a)\} = e^{-as}\, X(s)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{x_P(t)\} = X_P(s) = \mathcal{L}\left\{\sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)\right\} = \sum_{n=-\infty}^{\infty} x(nT)\, \mathcal{L}\{\delta(t - nT)\}$$

$$\therefore X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-nsT} = \sum_{n=-\infty}^{\infty} x(nT) \left(e^{sT}\right)^{-n} \qquad \text{..... (7.63)}$$

where $X_p(s)$ is Laplace transform of $x_p(t)$.

Let us take a transformation, $e^{sT} = z$.

On substituting, $e^{sT} = z$, in equation (7.63) we get,

$$X_p(s) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \qquad \text{.... (7.64)}$$

The $\mathcal{Z}$-transform of x(nT), using the definition of $\mathcal{Z}$-transform is given by,

$$X(z) = \sum_{n=-\infty}^{\infty} x(nT)\, z^{-n} \quad .... (7.65)$$

On comparing equations (7.64) and (7.65) we can say that, if a discrete time signal x(nT) is a sampled version of x(t), then *$\mathcal{Z}$-transform of the discrete time signal can be obtained from Laplace transform of sampled version of x(t), by choosing the transformation, $e^{sT} = z$.* This transformation is also called ***impulse invariant transformation***.

7.7.3 Relation Between s-Plane and z-Plane

Consider a point s_1 in s-plane as shown in fig 7.10. Now the transformation,

$$e^{s_1 T} = z_1 \quad ..... (7.66)$$

will transform the point s_1 to a corresponding point z_1 in z-plane.

Let the coordinates of s_1 be σ_1 and Ω_1 as shown in fig 7.10.

$$\therefore\; s_1 = \sigma_1 + j\Omega_1 \quad ..... (7.67)$$

Using equation (7.67) the equation (7.66) can be written as,

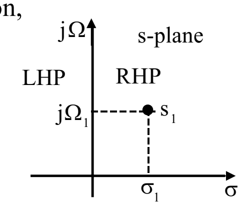


Fig 7.10 : s-plane.

$$z_1 = e^{(\sigma_1 + j\Omega_1)T} = e^{\sigma_1 T}\, e^{j\Omega_1 T} \quad ..... (7.68)$$

On separating the magnitude and phase of equation (7.68) we get,

$$z_1 = e^{(\sigma_1 + j\Omega_1)T} = e^{\sigma_1 T}\, e^{j\Omega_1 T} \quad ..... (7.69)$$

From equation (7.69) the following observations can be made.

1. If $\sigma_1 < 0$ (i.e., σ_1 is negative), then the point-s_1 lies on Left Half (LHP) of s-plane. In this case, $|z_1| < 1$, hence the corresponding point-z_1 will lie inside the unit circle in z-plane.
2. If $\sigma_1 = 0$ (i.e., real part is zero), then the point-s_1 lies on imaginary axis of s-pane. In this case, $|z_1| = 1$, hence the corresponding point-z_1 will lie on the unit circle in z-plane.
3. If $\sigma_1 > 0$ (i.e., σ_1 is positive), then the point-s_1 lies on the Right Half (RHP) of s-plane. In this case $|z_1| > 1$, hence the corresponding point-z_1 will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping of any point on s-plane to z-plane.

In general all points of s-plane, described by the equation,

$$s = \sigma_1 + j\Omega_1 + j\frac{2\pi k}{T}, \quad \text{for } k = 0, \pm 1, \pm 2 ..... \quad ..... (7.70)$$

map as a single point in the z-plane described by equation,

$$\boxed{e^{\pm j2\pi k} = 1 \;;\; \text{for integer k}}$$

$$z_1 = e^{\left(\sigma_1 + j\Omega_1 + \frac{j2\pi k}{T}\right)T} = e^{\sigma_1 T}\, e^{j\Omega_1 T}\, e^{j2\pi k} = e^{\sigma_1 T}\, e^{j\Omega_1 T} \quad ..... (7.71)$$

The equation (7.70) represents a strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $-\pi/T \le \Omega \le +\pi/T$ is mapped into the entire z-plane. Similarly the strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $\pi/T \le \Omega \le 3\pi/T$ is also mapped into the entire z-plane. Likewise the strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $-3\pi/T \le \Omega \le -\pi/T$ is also mapped into the entire z-plane.

In general any strip of width $2\pi/T$ in the s-plane for values of imaginary part of s in the range $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, is mapped into the entire z-plane. Therefore we can say that the transformation, $e^{sT} = z$, leads to many-to-one mapping, (and does not provide one-to-one mapping).

In this mapping, *the left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane* as shown in fig 7.11.

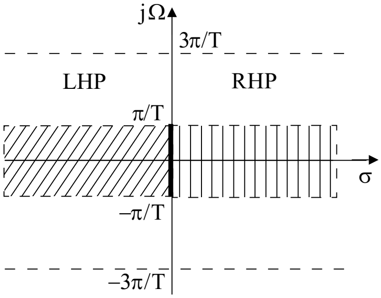


Fig 7.11a : *s-plane.*

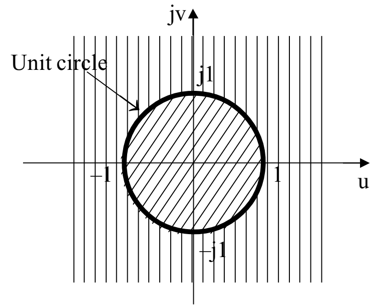


Fig 7.11b : *z-plane.*

Fig 7.11 : *Mapping of s-plane into z-plane.*

Relation Between Frequency of Continuous Time and Discrete Time Signal

Let, Ω = Frequency of continuous time signal in rad/sec.

ω = Frequency of discrete time signal in rad/sec.

Let, $z = re^{j\omega}$ be a point on z-plane, and $s = \sigma + j\Omega$, be a corresponding point in s-plane.

Consider the transformation,

$$z = e^{sT} \qquad \text{..... (7.72)}$$

Put, $z = r\,e^{j\omega}$ and $s = \sigma + j\Omega$ in equation (7.72)

$$\therefore\; r\,e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r\,e^{j\omega} = e^{\sigma T}\, e^{j\Omega T} \qquad \text{..... (7.73)}$$

On equating the imaginary part on either side of equation (7.73) we get,

$$\boxed{\omega = \Omega T \quad \text{or} \quad \Omega = \frac{\omega}{T}} \qquad \text{..... (7.74)}$$

When the transformation $e^{sT} = z$ is employed, the equation (7.74) can be used to compute the frequency of discrete time signal for a given frequency of continuous time signal and viceversa.The frequency of discrete time signal ω is unique over the range $(-\pi, +\pi)$, and so the mapping $\omega = \Omega T$ implies that the frequency of continuous time signal in the interval $-\pi/T \le \Omega \le +\pi/T$ maps into the corresponding values of frequency of discrete time signal in the interval $-\pi \le \omega \le +\pi$.

The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one. Therefore in general, the interval $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$, where k is an integer, maps into the corresponding values of $-\pi \le \omega \le +\pi$. Thus the mapping of the frequency of continuous time signal Ω to the frequency of discrete time signal ω is many-to-one. This reflects the effects of aliasing due to sampling.

7.8 Structures for Realization of LTI Discrete Time Systems in z-Domain

In time domain, the input-output relation of an LTI (Linear Time Invariant) discrete time system is represented by constant coefficient difference equation shown in equation (7.75).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad .....(7.75)$$

In z-domain, the input-output relation of an LTI (Linear Time Invariant) discrete time system is represented by the transfer function H(z), which is a rational function of z, as shown in equation (7.76).

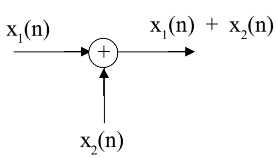
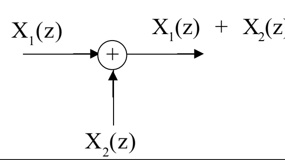
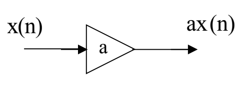
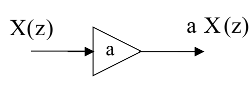
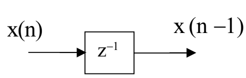
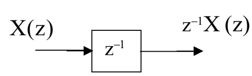
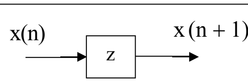
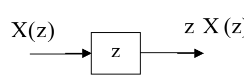
$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + ... + a_N z^{-N}} \qquad .....(7.76)$$

where, N = Order of the system, $M \le N$

The above two representations of discrete time system can be viewed as a computational procedure (or algorithm) to determine the output signal y(n) from the input signal x(n). The computations in the above equation can be arranged into various equivalent sets of difference equations.

For each set of equations, we can construct a block diagram consisting of Adder, Constant multiplier, Unit delay element and Unit advance element. Such block diagrams are referred to as realization of system or equivalently as structure for realizing system.The basic elements used to construct block diagrams are listed in table 7.6. The block diagram representation of discrete time system in time domain are discussed in chapter-6, section 6.6.2, and the block diagram representation of discrete time system in z-domain are discussed in chapter-10.

Table 7.6 : Basic elements of block diagram in time domain and z-domain

Elements of block diagram	**Time domain representation**	**z-domain representation**
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Example 7.12

Determine the impulse response h(n) for the system described by the second order difference equation,

$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$.

Solution

The difference equation governing the system is,

$$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$$

Let us take $\mathcal{Z}$-transform of the difference equation governing the system with zero initial conditions.

$$\therefore \quad \mathcal{Z}\{y(n) - 4y(n-1) + 4y(n-2)\} = \mathcal{Z}\{x(n-1)\}$$

$$\mathcal{Z}\{y(n)\} - 4\,\mathcal{Z}\{y(n-1)\} + 4\,\mathcal{Z}\{y(n-2)\} = \mathcal{Z}\{x(n-1)\}$$

$$Y(z) - 4z^{-1}\,Y(z) + 4z^{-2}\,Y(z) = z^{-1}\,X(z)$$

$$(1 - 4z^{-1} + 4z^{-2})\,Y(z) = z^{-1}\,X(z)$$

If $\mathcal{Z}\{x(n)\} = X(z)$
then by shifting property
$\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$

If $\mathcal{Z}\{y(n)\} = Y(z)$
then by shifting property
$\mathcal{Z}\{y(n-m)\} = z^{-m}\,Y(z)$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - 4z^{-1} + 4z^{-2}}$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore \quad H(z) = \frac{z^{-1}}{1 - 4z^{-1} + 4z^{-2}} = \frac{z^{-1}}{z^{-2}(z^2 - 4z + 4)} = \frac{z}{(z-2)^2} = \frac{1}{2}\,\frac{2z}{(z-2)^2}$$

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z).

$$\text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{2}\,\frac{2z}{(z-2)^2}\right\}$$

$$= (1/2)\,n2^n\,u(n) = n2^{(n-1)}\,u(n)$$

$$\mathcal{Z}\{na^n\,u(n)\} = \frac{az}{(z-a)^2}$$

Example 7.13

Find the transfer function and unit sample response of the second order difference equation with zero initial condition,

$y(n) = x(n) - 0.25y(n-2)$.

Solution

The difference equation governing the system is,

$$y(n) = x(n) - 0.25\,y(n-2)$$

Let us take $\mathcal{Z}$-transform of the difference equation governing the system with zero initial condition.

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n - 0.25\,y(n-2)\}$$

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n)\} - 0.25\,\mathcal{Z}\{y(n-2)\}$$

$$Y(z) = X(z) - 0.25\,z^{-2}\,Y(z)$$

$$Y(z) + 0.25z^{-2}\,Y(z) = X(z)$$

$$(1 + 0.25z^{-2})\,Y(z) = X(z)$$

$\mathcal{Z}\{x(n)\} = X(z)$

$\mathcal{Z}\{y(n)\} = Y(z)$
$\mathcal{Z}\{y(n-2)\} = z^{-2}\,Y(z)$
(Using shifting property)

$$\therefore \quad \text{Transfer function, } \frac{Y(z)}{X(z)} = \frac{1}{1 + 0.25z^{-2}}$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$(a+b)(a-b) = a^2 - b^2$	$j^2 = -1$

$$\therefore\ H(z) = \frac{1}{1 + 0.25\,z^{-2}} = \frac{1}{z^{-2}(z^2 + 0.25)} = \frac{z^2}{(z + j\,0.5)\,(z - j\,0.5)}$$

By partial fraction expansion we can write,

$$\frac{H(z)}{z} = \frac{z}{(z + j0.5)\,(z - j0.5)} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} \quad ; \quad \text{where } A^* \text{ is conjugate of A.}$$

$$A = (z + j0.5)\,\frac{H(z)}{z}\bigg|_{z\,=\,-j0.5} = (z + j0.5)\,\frac{z}{(z + j0.5)\,(z - j0.5)}\bigg|_{z\,=\,-j0.5}$$

$$= \frac{z}{z - j0.5}\bigg|_{z\,=\,-j0.5} = \frac{-j0.5}{-j0.5 - j0.5} = \frac{-j0.5}{2(-j0.5)} = \frac{1}{2} = 0.5$$

$$\therefore\ A^* = 0.5$$

$$\frac{H(z)}{z} = \frac{A}{z + j0.5} + \frac{A^*}{z - j0.5} = \frac{0.5}{z + j0.5} + \frac{0.5}{z - j0.5}$$

$$\therefore\ H(z) = \frac{0.5z}{z + j0.5} + \frac{0.5z}{z - j0.5} = \frac{0.5z}{z - (-j0.5)} + \frac{0.5z}{z - j0.5}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$$\therefore\ \text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{0.5z}{z - (-j0.5)} + \frac{0.5z}{z - j0.5}\right\}$$

$$= 0.5\left[\mathcal{Z}^{-1}\left\{\frac{z}{z - (-j0.5)}\right\} + \mathcal{Z}^{-1}\left\{\frac{z}{z - j0.5}\right\}\right]$$

$$\boxed{\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z - a}}$$

$$= 0.5\left[(-j0.5)^n\,u(n) + (j0.5)^n\,u(n)\right]$$

> Alternatively the impulse response can be expressed as shown below.
>
> $$\text{Here, } -j0.5 = 0.5\angle -90^\circ = 0.5\angle -\pi/2 = 0.5\,\angle -0.5\pi$$
> $$+j0.5 = 0.5\angle 90^\circ = 0.5\angle \pi/2 = 0.5\,\angle 0.5\pi$$
> $$\therefore\ h(n) = 0.5\left[(0.5\angle -0.5\pi)^n + (0.5\angle 0.5\pi)^n\right]u(n)$$
> $$= 0.5\left[0.5^n\angle -0.5n\pi + 0.5^n\angle 0.5n\pi\right]u(n)$$
> $$= 0.5\,(0.5)^n\left[\cos 0.5n\pi - j\sin 0.5n\pi + \cos 0.5n\pi + j\sin 0.5n\pi\right]u(n)$$
> $$= 0.5\,(0.5)^n\,2\cos 0.5n\pi\,u(n)$$
> $$= 0.5^n\,\cos(0.5n\pi)\,u(n)$$

Example 7.14

Determine the impulse response sequence of the discrete time LTI system defined by,

$y(n) - 2y(n-1) + y(n-2) = x(n) + 3x(n-3)$.

Solution

The difference equation governing the LTI system is,

$$y(n) - 2y(n-1) + y(n-2) = x(n) + 3x(n-3)$$

Let us assume that the initial conditions are zero.

Ƶ{x(n)} = X(z) ,	$\therefore$ Ƶ{ax(n – m} = $az^{-m}X(z)$
Ƶ{y(n)} = Y(z) ,	$\therefore$ Ƶ{ay(n – m} = $az^{-m}Y(z)$

On taking Ƶ-transform of the difference equation governing the system we get,

$$Ƶ\{y(n) - 2y(n-1) + y(n-2)\} = Ƶ\{x(n) + 3x(n-3)\}$$

$$Ƶ\{y(n)\} - 2\,Ƶ\{y(n-1)\} + Ƶ\{y(n-2)\} = Ƶ\{x(n)\} + 3\,Ƶ\{x(n-3)\}$$

$$Y(z) - 2z^{-1}\,Y(z) + z^{-2}\,Y(z) = X(z) + 3z^{-3}\,X(z)$$

$$(1 - 2z^{-1} + z^{-2})\,Y(z) = (1 + 3z^{-3})\,X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-3}}{1 - 2z^{-1} + z^{-2}}$$

$$Ƶ\{u(n)\} = \frac{z}{z-1}$$

$$Ƶ\{n\,u(n)\} = \frac{z}{(z-1)^2}$$

$$Ƶ\{(n+1)\,u(n+1)\} = z\,\frac{z}{(z-1)^2}$$

$$Ƶ\{(n-2)\,u(n-2)\} = z^{-2}\,\frac{z}{(z-1)^2}$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore H(z) = \frac{1 + 3z^{-3}}{1 - 2z^{-1} + z^{-2}} = \frac{1 + 3z^{-3}}{z^{-2}(z^2 - 2z + 1)} = \frac{z^2 + 3z^{-1}}{(z-1)^2}$$

$$= \frac{z^2}{(z-1)^2} + \frac{3z^{-1}}{(z-1)^2} = z\,\frac{z}{(z-1)^2} + 3z^{-2}\,\frac{z}{(z-1)^2}$$

The impulse response is obtained by taking inverse Ƶ-transform of H(z).

$$\therefore \text{Impulse response, } h(n) = Ƶ^{-1}\{H(z)\} = Ƶ^{-1}\left\{z\,\frac{z}{(z-1)^2} + 3z^{-2}\,\frac{z}{(z-1)^2}\right\}$$

$$= Ƶ^{-1}\left\{z\,\frac{z}{(z-1)^2}\right\} + 3Ƶ^{-1}\left\{z^{-2}\,\frac{z}{(z-1)^2}\right\}$$

$$= (n+1)\,u(n+1) + 3(n-2)\,u(n-2)$$

Example 7.15

Find the impulse response of the system described by the difference equation,
y(n) – 3y(n – 1) – 4y (n – 2) = x(n) + 2x(n – 1).

Solution

The difference equation governing the LTI system is,

Ƶ{y(n)} = Y(z) ;	$\therefore$ Ƶ{y(n – m)} = z^{-m} Y(z)
Ƶ{x(n)} = X(z) ;	$\therefore$ Ƶ{x(n – m)} = z^{-m} X(z)

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

On taking Ƶ-transform we get,

$$Y(z) - 3z^{-1}Y(z) - 4\,z^{-2}Y(z) = X(z) + 2z^{-1}\,X(z)$$

$$(1 - 3z^{-1} - 4z^{-2})\,Y(z) = (1 + 2z^{-1})\,X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}}$$

The roots of the quadratic,

$z^2 - 3z - 4 = 0$ are,

$$z = \frac{3 \pm \sqrt{3^2 + 4 \times 4}}{2} = 4 \text{ or } -1$$

We know that $\dfrac{Y(z)}{X(z)} = H(z)$

$$\therefore H(z) = \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z^{-2}(z^2 + 2z)}{z^{-2}(z^2 - 3z - 4)} = \frac{z^2 + 2z}{(z-4)\,(z+1)}$$

By partial fraction expansion technique,

$$\frac{H(z)}{z} = \frac{z+2}{(z-4)(z+1)} = \frac{A}{z-4} + \frac{B}{z+1}$$

$$A = (z-4)\left.\frac{H(z)}{z}\right|_{z=4} = (z-4)\left.\frac{z+2}{(z-4)(z+1)}\right|_{z=4} = \left.\frac{z+2}{z+1}\right|_{z=4} = \frac{4+2}{4+1} = \frac{6}{5} = 1.2$$

$$B = (z+1)\left.\frac{H(z)}{z}\right|_{z=-1} = (z+1)\left.\frac{z+2}{(z-4)(z+1)}\right|_{z=-1} = \left.\frac{z+2}{z-4}\right|_{z=-1} = \frac{-1+2}{-1-4} = \frac{1}{-5} = -0.2$$

$$\therefore \frac{H(z)}{z} = \frac{A}{z-4} + \frac{B}{z+1} = \frac{1.2}{z-4} - \frac{0.2}{z+1}$$

$$\therefore H(z) = 1.2\frac{z}{z-4} - 0.2\frac{z}{z+1} = 1.2\left(\frac{z}{z-4}\right) - 0.2\left(\frac{z}{z-(-1)}\right)$$

$$\boxed{\mathcal{Z}\left\{\frac{z}{z-a}\right\} = a^n}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$\therefore$ Impulse response, $h(n) = 1.2(4)^n u(n) - 0.2(-1)^n u(n)$

Example 7.16

Determine the steady state response for the system with impulse function $h(n) = (j0.5)^n\, u(n)$ for an input $x(n) = \cos(\pi n)\, u(n)$.

Solution

Let y(n) be the steady state response of the system, which is given by convolution of x(n) and h(n).

$\therefore$ Steady state response, $y(n) = x(n) * h(n)$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$$

$$\therefore Y(z) = X(z)\,H(z) \qquad \boxed{\text{Using convolution property}}$$

$$\therefore y(n) = \mathcal{Z}^{-1}\{X(z)\,H(z)\}$$

Given that, $h(n) = (j0.5)^n\, u(n)$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \frac{z}{z-j0.5}$$

$$\boxed{\begin{aligned}\mathcal{Z}\{a^n\, u(n)\} &= \frac{z}{z-a}\\ \mathcal{Z}\{\cos(\omega n)\, u(n)\} &= \frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}\end{aligned}}$$

Given that, $x(n) = \cos(\pi n)\, u(n)$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \frac{z(z-\cos\pi)}{z^2-2z\cos\pi+1} = \frac{z(z+1)}{z^2+2z+1} = \frac{z(z+1)}{(z+1)^2} = \frac{z}{z+1}$$

$$\therefore Y(z) = X(z)\,H(z) = \frac{z}{z+1} \times \frac{z}{z-j0.5} = \frac{z^2}{(z+1)(z-j0.5)}$$

By partial fraction expansion technique we can write,

$$\frac{Y(z)}{z} = \frac{z}{(z+1)(z-j0.5)} = \frac{A}{z+1} + \frac{B}{z-j0.5}$$

$$A = (z+1)\left.\frac{Y(z)}{z}\right|_{z=-1} = (z+1)\left.\frac{z}{(z+1)(z-j0.5)}\right|_{z=-1} = \left.\frac{z}{z-j0.5}\right|_{z=-1} = \frac{-1}{-1-j0.5}$$

$$= \frac{-1}{-1-j0.5} \times \frac{-1+j0.5}{-1+j0.5} = \frac{1-j0.5}{1^2+0.5^2} = \frac{1-j0.5}{1.25} = 0.8 - j0.4$$

$$B = (z - j0.5)\left.\frac{Y(z)}{z}\right|_{z = j0.5} = (z - j0.5)\left.\frac{z}{(z + 1)(z - j0.5)}\right|_{z = j0.5} = \left.\frac{z}{z + 1}\right|_{z = j0.5} = \frac{j0.5}{j0.5 + 1}$$

$$= \frac{j0.5}{1 + j0.5} \times \frac{1 - j0.5}{1 - j0.5} = \frac{j0.5 - (j0.5)^2}{1^2 + 0.5^2} = \frac{0.25 + j0.5}{1.25} = 0.2 + j0.4$$

$$\therefore \frac{Y(z)}{z} = \frac{A}{z + 1} + \frac{B}{z - j0.5} = \frac{0.8 - j0.4}{z + 1} + \frac{0.2 + j0.4}{z - j0.5}$$

$$\therefore Y(z) = (0.8 - j0.4)\frac{z}{z + 1} + (0.2 + j0.4)\frac{z}{z - j0.5}$$

$$= (0.8 - j0.4)\frac{z}{z - (-1)} + (0.2 + j0.4)\frac{z}{z - j0.5}$$

$$\boxed{\mathcal{Z}\{a^n\, u(n)\} = \frac{z}{z - a}}$$

The steady state response is obtained by taking inverse Z-transform of Y(z).

$\therefore$ Steady state response, $y(n) = (0.8 - j0.4)\,(-1)^n\, u(n) + (0.2 + j0.4)\,(j0.5)^n\, u(n)$

Alternatively the steady state response can be expressed as shown below.

$$\text{Here, } 0.8 - j0.4 = 0.894\,\angle -26.6^\circ = 0.894\,\angle -0.15\pi$$

$$0.2 + j0.4 = 0.447\,\angle 63.4^\circ = 0.447\,\angle 0.35\pi$$

$$-1 = 1\,\angle 180^\circ = 1\,\angle \pi$$

$$j0.5 = 0.5\,\angle 90^\circ = 0.5\,\angle 0.5\pi$$

$$\therefore y(n) = 0.894\,\angle -0.15\pi\; [1\,\angle \pi]^n\; u(n) + 0.447\angle 0.35\pi\; [0.5\angle 0.5\pi]^n\; u(n)$$

$$= 0.894\,\angle -0.15\pi\; 1^n\,\angle n\pi\; u(n) + 0.447\angle 0.35\pi\; 0.5^n\angle 0.5n\pi\; u(n)$$

$$= 0.894\,\angle (n - 0.15)\pi\; u(n) + 0.447\,(0.5)^n\angle (0.5n + 0.35)\pi\; u(n)$$

Example 7.17

Obtain and sketch the impulse response of shift invariant system described by,
$y(n) = 0.4\,x(n) + x(n-1) + 0.6\,x(n-2) + x(n-3) + 0.4\,x(n-4)$.

Solution

The difference equation governing the system is,

$y(n) = 0.4\,x(n) + x(n-1) + 0.6\,x(n-2) + x(n-3) + 0.4\,x(n-4)$

On taking Z-transform we get,

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property
$\mathcal{Z}\{x(n-k)\} = z^{-k}\, X(z)$

$$Y(z) = 0.4X(z) + z^{-1}X(z) + 0.6z^{-2}X(z) + z^{-3}X(z) + 0.4z^{-4}X(z)$$

$$Y(z) = [0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}]\, X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \left[0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4}\right]$$

We know that, $\dfrac{Y(z)}{X(z)} = H(z)$;

$$\therefore H(z) = 0.4 + z^{-1} + 0.6z^{-2} + z^{-3} + 0.4z^{-4} \qquad .....(1)$$

By the definition of one sided $\mathcal{Z}$-transform we get,

$$H(z) = \sum_{n=0}^{+\infty} h(n) z^{-n}$$

$$= h(0)\,z^0 + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3} + h(4)\,z^{-4} + ... \quad .....(2)$$

On comparing equations (1) & (2) we get,

$h(0) = 0.4$	$h(3) = 1$
$h(1) = 1$	$h(4) = 0.4$
$h(2) = 0.6$	$h(n) = 0 \quad ;$ for $n < 0$ and $n > 4$

$\therefore$ Impulse response, $h(n) = \{\underset{\uparrow}{0.4}, 1.0, 0.6, 1.0, 0.4\}$

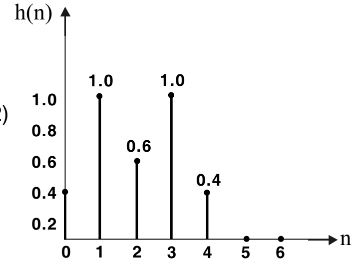


Fig 1 : Graphical representation of impulse response h(n).

Example 7.18

Determine the response of discrete time LTI system governed by the difference equation $y(n) = -0.5\,y(n-1) + x(n)$, when the input is unit step and initial condition, a) $y(-1) = 0$ and b) $y(-1) = 1/3$.

Solution

Given that, $x(n) = u(n)$; $\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{u(n)\} = \dfrac{z}{z-1}$ .....(1)

Given that, $y(n) = -0.5\,y(n-1) + x(n)$

$\therefore\ y(n) + 0.5\,y(n-1) = x(n)$

On taking $\mathcal{Z}$-transform of above equation we get,

If $\mathcal{Z}\{y(n)\} = Y(z)$
then $\mathcal{Z}\{y(n-1)\} = z^{-1}\,Y(z) - y(-1)$

$$Y(z) + 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

Using equation (1)

$$Y(z)\left[1 + 0.5z^{-1}\right] + 0.5\,y(-1) = \frac{z}{z-1}$$

$$Y(z)\left(1 + \frac{0.5}{z}\right) = \frac{z}{z-1} - 0.5\,y(-1)$$

$$Y(z)\left(\frac{z+0.5}{z}\right) = \frac{z}{z-1} - 0.5\,y(-1)$$

$$\therefore\ Y(z) = \frac{z^2}{(z-1)\,(z+0.5)} - \frac{1}{2}\,\frac{z\,y(-1)}{z+0.5}$$

Let, $P(z) = \dfrac{z^2}{(z-1)\,(z+0.5)} \quad \Rightarrow \quad \dfrac{P(z)}{z} = \dfrac{z}{(z-1)\,(z+0.5)}$

Let, $\dfrac{z}{(z-1)\,(z+0.5)} = \dfrac{A}{z-1} + \dfrac{B}{z+0.5}$

$$A = \frac{z}{(z-1)\,(z+0.5)} \times (z-1)\Big|_{z=1} = \frac{1}{1+0.5} = \frac{1}{1.5} = \frac{2}{3}$$

$$B = \frac{z}{(z-1)\,(z+0.5)} \times (z+0.5)\Big|_{z=-0.5} = \frac{-0.5}{-0.5-1} = \frac{-0.5}{-1.5} = \frac{5}{15} = \frac{1}{3}$$

$$\therefore\ \frac{P(z)}{z} = \frac{2}{3}\,\frac{1}{z-1} + \frac{1}{3}\,\frac{1}{z+0.5} \quad \Rightarrow \quad P(z) = \frac{2}{3}\,\frac{z}{z-1} + \frac{1}{3}\,\frac{z}{z+0.5}$$

$$\therefore\ Y(z) = \frac{2}{3}\,\frac{z}{z-1} + \frac{1}{3}\,\frac{z}{z+0.5} - \frac{1}{2}\,\frac{z\,y(-1)}{z+0.5} \quad .....(2)$$

a) When y(–1) = 0

From equation (2), when $y(-1) = 0$, we get,

$$Y(z) = \frac{2}{3}\,\frac{z}{z-1} + \frac{1}{3}\,\frac{z}{z+0.5}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{2}{3}\,\frac{z}{z-1} + \frac{1}{3}\,\frac{z}{z+0.5}\right\} = \frac{2}{3}\,u(n) + \frac{1}{3}\,(-0.5)^n\,u(n) = \frac{1}{3}\left[2 + (-0.5)^n\right] u(n)$$

b) When y(–1) = 1/3

From equation (2), when y(–1) = 1/3, we get,

$$Y(z) = \frac{2}{3}\;\frac{z}{z-1} + \frac{1}{3}\;\frac{z}{z+0.5} - \frac{1}{2}\times\frac{1}{3}\;\frac{z}{z+0.5}$$

$$= \frac{2}{3}\;\frac{z}{z-1} + \frac{1}{2}\times\frac{1}{3}\;\frac{z}{z+0.5} = \frac{2}{3}\;\frac{z}{z-1} + \frac{1}{6}\;\frac{z}{z+0.5}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\frac{2}{3}\;\frac{z}{z-1} + \frac{1}{6}\;\frac{z}{z+0.5}\right\}$$

$$= \frac{2}{3}\;u(n) + \frac{1}{6}\;(-0.5)^n\;u(n) = \frac{1}{6}\left[4 + (-0.5)^n\right]u(n)$$

***Note :** Compare the results of example 7.18 with example 6.8 of chapter-6.*

Example 7.19

Determine the response of LTI discrete time system governed by the difference equation, y(n) – 2 y(n – 1) – 3 y(n – 2) = x(n) + 4 x(n – 1) for the input $x(n) = 2^n\,u(n)$ and with initial condition y(–2)= 0, y(–1) = 5.

Solution

Given that, $x(n) = 2^n\,u(n)$; $\quad \therefore\; X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{2^n\,u(n)\} = \dfrac{z}{z-2}$ .....(1)

Given that, y(n) – 2 y(n – 1) – 3 y(n – 2) = x(n) + 4 x(n – 1)

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 2\left[z^{-1}Y(z) + y(-1)\right] - 3\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = X(z) + 4\left[z^{-1}X(z) + x(-1)\right] \quad .....(2)$$

If $\mathcal{Z}\{y(n)\} = Y(z)$, then $\mathcal{Z}\{y(n-1)\} = z^{-1}\,Y(z) + y(-1)$
and $\mathcal{Z}\{y(n-2\} = z^{-2}\,Y(z) + z^{-1}\,y(-1) + y(-2)$

Given that, y(–2)= 0, y(–1) = 5

$$\begin{aligned} x(n) = 2^n\;u(n) &= 2^n \quad \text{for} \quad n \ge 0 \\ &= 0 \quad \text{for} \quad n < 0 \end{aligned} \quad \Rightarrow \quad x(-1) = 0$$

On substituting the above initial conditions in equation (2) we get,

$$Y(z) - 2z^{-1}\,Y(z) - 2\times 5 - 3z^{-2}\,Y(z) - 3z^{-1}\times 5 + 0 = X(z) + 4z^{-1}\,X(z) + 0$$

$$Y(z) - \frac{2}{z}Y(z) - 10 - \frac{3}{z^2}Y(z) - \frac{15}{z} = X(z) + \frac{4}{z}X(z) \quad \Rightarrow \quad Y(z)\left(1 - \frac{2}{z} - \frac{3}{z^2}\right) - \left(\frac{15}{z} + 10\right) = X(z)\left(1 + \frac{4}{z}\right)$$

$$Y(z)\left(\frac{z^2 - 2z - 3}{z^2}\right) - \left(\frac{15 + 10z}{z}\right) = \left(\frac{z}{z-2}\right)\left(\frac{z+4}{z}\right) \qquad \boxed{\text{Using equation (1)}}$$

$$Y(z)\frac{(z-3)\,(z+1)}{z^2} = \frac{z+4}{z-2} + \frac{15+10z}{z} \quad \Rightarrow \quad Y(z)\frac{(z-3)\,(z+1)}{z^2} = \frac{z(z+4)\;+(15+10z)\,(z-2)}{z(z-2)}$$

$$Y(z)\frac{(z-3)\,(z+1)}{z^2} = \frac{z^2 + 4z + 15z - 30 + 10z^2 - 20z}{z(z-2)}$$

$$Y(z)\frac{(z-3)\,(z+1)}{z^2} = \frac{11z^2 - z - 30}{z(z-2)}$$

$$Y(z) = \frac{11z^2 - z - 30}{z(z-2)} \times \frac{z^2}{(z-3)\,(z+1)} = \frac{z(11z^2 - z - 30)}{(z-2)\,(z-3)\,(z+1)}$$

Let, $\frac{Y(z)}{z} = \frac{11z^2 - z - 30}{(z-2)\,(z-3)\,(z+1)} = \frac{A}{z-2} + \frac{B}{z-3} + \frac{C}{z+1}$

$$A = \frac{11z^2 - z - 30}{(z-2)\,(z-3)\,(z+1)} \times (z-2)\Big|_{z=2} = \left.\frac{11z^2 - z - 30}{(z-3)\,(z+1)}\right|_{z=2} = \frac{11 \times 2^2 - 2 - 30}{(2-3)\,(2+1)} = \frac{12}{-3} = -4$$

$$B = \frac{11z^2 - z - 30}{(z-2)\,(z-3)\,(z+1)} \times (z-3)\Big|_{z=3} = \left.\frac{11z^2 - z - 30}{(z-2)\,(z+1)}\right|_{z=3} = \frac{11 \times 3^2 - 3 - 30}{(3-2)\,(3+1)} = \frac{66}{4} = \frac{33}{2}$$

$$C = \frac{11z^2 - z - 30}{(z-2)\,(z-3)\,(z+1)} \times (z+1)\Big|_{z=-1} = \left.\frac{11z^2 - z - 30}{(z-2)\,(z-3)}\right|_{z=-1} = \frac{11 \times (-1)^2 + 1 - 30}{(-1-2)\,(-1-3)} = \frac{-18}{12} = \frac{-3}{2}$$

$$\therefore \frac{Y(z)}{z} = \frac{-4}{z-2} + \frac{33}{2}\,\frac{1}{z-3} - \frac{3}{2}\,\frac{1}{z+1} \quad \Rightarrow \quad Y(z) = -4\,\frac{z}{z-2} + \frac{33}{2}\,\frac{z}{z-3} - \frac{3}{2}\,\frac{z}{z+1}$$

$$\therefore \text{Response, } y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{-4\,\frac{z}{z-2} + \frac{33}{2}\,\frac{z}{z-3} - \frac{3}{2}\,\frac{z}{z+1}\right\}$$

$$= -4\,(2)^n\,u(n) + \frac{33}{2}(3)^n\,u(n) - \frac{3}{2}(-1)^n\,u(n) = \left[-4\,(2)^n + \frac{33}{2}(3)^n - \frac{3}{2}(-1)^n\right] u(n)$$

Note : *Compare the result of example 7.19 with example 6.9 of chapter-6.*

Example 7.20

Find the response of the time invariant system with impulse response h(n) = {1, 2, 1, –1} to an input signal x(n) = {1, 2, 3, 1}.

Solution

Let y(n) = Response or Output of an LTI system.

The response of an LTI system is given by the convolution of input signal and impulse response.

$$\therefore y(n) = x(n) * h(n)$$

On taking $\mathcal{Z}$-transform we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{x(n) * h(n)\}$$

$$\therefore Y(z) = X(z)\,H(z)$$

By convolution property
$\mathcal{Z}\{x(n) * h(n)\} = X(z)\,H(z)$

Given that, x(n) = {1, 2, 3, 1}

By definition of one sided $\mathcal{Z}$-transform,

$$X(z) = \sum_{n=0}^{\infty} x(n)\,z^{-n} = \sum_{n=0}^{3} x(n)\,z^{-n} = x(0)\,z^0 + x(1)\,z^{-1} + x(2)\,z^{-2} + x(3)\,z^{-3}$$

$$= 1 + 2z^{-1} + 3z^{-2} + z^{-3}$$

Given that, h(n) = { 1, 2, 1, –1}

By definition of one sided $\mathcal{Z}$-transform,

$$H(z) = \sum_{n=0}^{\infty} h(n)\,z^{-n} = \sum_{n=0}^{3} h(n)\,z^{-n} = h(0)\,z^0 + h(1)\,z^{-1} + h(2)\,z^{-2} + h(3)\,z^{-3}$$

$$= 1 + 2z^{-1} + z^{-2} - z^{-3}$$

$$\begin{aligned}
\therefore Y(z) &= X(z)\,H(z) \\
&= [1 + 2z^{-1} + 3z^{-2} + z^{-3}]\,[1 + 2z^{-1} + z^{-2} - z^{-3}] \\
&= 1 + 2z^{-1} + z^{-2} - z^{-3} \\
&\quad + 2z^{-1} + 4z^{-2} + 2z^{-3} - 2z^{-4} \\
&\quad + 3z^{-2} + 6z^{-3} + 3z^{-4} - 3z^{-5} \\
&\quad + z^{-3} + 2z^{-4} + z^{-5} - z^{-6} \\
&= 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6} \qquad .....(1)
\end{aligned}$$

By definition of one sided Z-transform we get,

$$Y(z) = \sum_{n=0}^{\infty} y(n)\, z^{-n}$$

$$= y(0)\, z^{0} + y(1)\, z^{-1} + y(2)\, z^{-2} + y(3)\, z^{-3} + y(4)\, z^{-4} + y(5)\, z^{-5} + \ldots \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$y(0) = 1$	$y(2) = 8$	$y(4) = 3$	$y(6) = -1$
$y(1) = 4$	$y(3) = 8$	$y(5) = -2$	

$\therefore$ The response of the system, $y(n) = \{\underset{\uparrow}{1}, 4, 8, 8, 3, -2, -1\}$

Example 7.21

Using Z-transform, perform deconvolution of the response $y(n) = \{1,\ 4,\ 8,\ 8,\ 3,\ -2,\ -1\}$ and impulse response $h(n) = \{1,\ 2,\ 1,\ -1\}$ to extract the input x(n).

Solution

Given that, $y(n) = \{1,\ 4,\ 8,\ 8,\ 3,\ -2,\ -1\}$

$$\therefore Y(z) = Z\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} = \sum_{n=0}^{6} y(n)\, z^{-n}$$

$$= y(0) + y(1)\, z^{-1} + y(2)\, z^{-2} + y(3)\, z^{-3} + y(4)\, z^{-4} + y(5)\, z^{-5} + y(6)\, z^{-6}$$

$$= 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6}$$

Given that, $h(n) = \{1,\ 2,\ 1,\ -1\}$

$$\therefore H(z) = Z\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} = \sum_{n=0}^{3} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} = 1 + 2z^{-1} + z^{-2} - z^{-3}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6}}{1 + 2z^{-1} + z^{-2} - z^{-3}}$$

$$= 1 + 2z^{-1} + 3z^{-2} + z^{-3} \qquad .....(1)$$

$$\begin{array}{r|l}
 & 1 + 2z^{-1} + 3z^{-2} + z^{-3} \\
\hline
1 + 2z^{-1} + z^{-2} - z^{-3} & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} + 3z^{-4} - 2z^{-5} - z^{-6} \\
 & \underset{(-)}{1} \underset{(-)}{+} 2z^{-1} \underset{(-)}{+} z^{-2} \underset{(+)}{-} z^{-3} \\
\cline{2-2}
 & 2z^{-1} + 7z^{-2} + 9z^{-3} + 3z^{-4} \\
 & \underset{(-)}{2z^{-1}} \underset{(-)}{+} 4z^{-2} \underset{(-)}{+} 2z^{-3} \underset{(+)}{-} 2z^{-4} \\
\cline{2-2}
 & 3z^{-2} + 7z^{-3} + 5z^{-4} - 2z^{-5} \\
 & \underset{(-)}{3z^{-2}} \underset{(-)}{+} 6z^{-3} \underset{(-)}{+} 3z^{-4} \underset{(+)}{-} 3z^{-5} \\
\cline{2-2}
 & z^{-3} + 2z^{-4} + z^{-5} - z^{-6} \\
 & \underset{(-)}{z^{-3}} \underset{(-)}{+} 2z^{-4} \underset{(-)}{+} z^{-5} \underset{(+)}{-} z^{-6} \\
\cline{2-2}
 & 0 \\
\cline{2-2}
\end{array}$$

By the definition of Z-transform,

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$X(z) = \ldots\ldots + x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + x(3)\, z^{-3} + \ldots\ldots \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$x(0) = 1$; $x(1) = 2$; $x(2) = 3$; $x(3) = 1$

$\therefore$ Input, $x(n) = \{1,\ 2,\ 3,\ 1\}$

Example 7.22

An LTI system is described by the equation y(n) = x(n) + 0.8 x(n – 1) + 0.8 x(n – 2) – 0.49 y(n – 2). Determine the transfer function of the system. Sketch the poles and zeros on the z-plane.

Solution

$\mathcal{Z}\{y(n)\} = Y(z)$; $\therefore \mathcal{Z}\{y(n-m)\} = z^{-m}Y(z)$
$\mathcal{Z}\{x(n)\} = X(z)$; $\therefore \mathcal{Z}\{x(n-m)\} = z^{-m}X(z)$

Given that, y(n) = x(n) + 0.8 x(n – 1) + 0.8 x(n – 2) – 0.49 y(n – 2)

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z) - 0.49z^{-2}Y(z)$$

$$Y(z) + 0.49z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z) + 0.8z^{-2}X(z)$$

$$(1 + 0.49z^{-2})\,Y(z) = (1 + 0.8z^{-1} + 0.8z^{-2})\,X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} \qquad .....(1)$$

The equation(1) is the transfer function of the LTI system.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1} + 0.8z^{-2}}{1 + 0.49z^{-2}} = \frac{z^{-2}(z^2 + 0.8z + 0.8)}{z^{-2}(z^2 + 0.49)}$$

$$= \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49}$$

The poles are the roots of the denominator polynomial,

$$z^2 + 0.49 = 0$$

$$\therefore z^2 = -0.49$$

$$z = \pm\sqrt{-0.49} = \pm j0.7$$

$\therefore$ The poles are $p_1 = j0.7$, $p_2 = -j0.7$

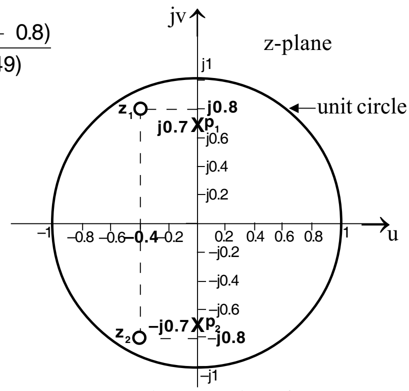


Fig 1 : *Pole-zero plot of LTI system.*

The zeros are the roots of the numerator polynomial,

$$z^2 + 0.8z + 0.8 = 0$$

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4 \times 0.8}}{2} = \frac{-0.8 \pm \sqrt{-2.56}}{2} = \frac{-0.8 \pm j0.16}{2} = -0.4 \pm j0.8$$

$\therefore$ The zeros are $z_1 = -0.4 + j0.8$ and $z_2 = -0.4 - j0.8$

$$\therefore H(z) = \frac{z^2 + 0.8z + 0.8}{z^2 + 0.49} = \frac{(z + 0.4 - j0.8)(z + 0.4 + j0.8)}{(z - j0.7)(z + j0.7)}$$

The fig1 Shows the location of poles and zeros on the z-plane. The poles are marked as "**X**" and Zeros as "**O**".

Example 7.23

Determine the step response of an LTI system whose impulse response h(n) is given by $h(n) = a^{-n}\,u(-n)$; $0 < a < 1$.

Solution

$\because u(-n) = 1 \; ; n \le 0$
$= 0 \; ; n > 0$

On taking $\mathcal{Z}$-transform of impulse response h(n) we get,

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)\,z^{-n} = \sum_{n=-\infty}^{0} a^{-n}\,z^{-n} = \sum_{n=0}^{\infty} a^n z^n = \sum_{n=0}^{\infty} (az)^n$$

$$= \frac{1}{1 - az} = -\frac{1/a}{z - 1/a} \quad ; \quad \text{ROC } |z| < |1/a| \qquad .....(1)$$

The step input, $u(n) = 1 \; ; \; n \ge 0$

$= 0 \; ; \; n < 0$

Note : *Since a < 1, 1/a > 1, and so ROC includes unit circle*

On taking $\mathcal{Z}$-transform of unit step signal we get,

$$U(z) = \mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)\, z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = \frac{1}{1-z^{-1}} = \frac{z}{z-1}\ ;\ \text{ROC } |z| > | \quad .....(2)$$

Let y(n) be step response. Now the step response is given by convolution of step input u(n) and impulse response h(n)

$$\therefore\ y(n) = u(n) * h(n)$$

On taking $\mathcal{Z}$-transform we get,

$$\mathcal{Z}\{y(n)\} = \mathcal{Z}\{u(n) * h(n)\}$$

$$\therefore\ Y(z) = U(z)\, H(z)$$

By convolution property
$\mathcal{Z}\{u(n) * h(n)\} = U(z)\, H(z)$

On substituting for U(z) and H(z) from equations (1) and (2) respectively we get,

$$Y(z) = U(z)\, H(z) = \left(\frac{z}{z-1}\right)\left(\frac{-1/a}{z-1/a}\right)$$

By partial fraction expansion we can write,

$$\frac{Y(z)}{z} = \frac{-1/a}{(z-1)(z-1/a)} = \frac{A}{z-1} + \frac{B}{z-1/a}$$

$$A = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1} = \left.\frac{-1/a}{z-1/a}\right|_{z=1} = \frac{-1/a}{1-1/a} = \frac{-1/a}{\frac{a-1}{a}} = \frac{-1}{a-1} = \frac{1}{1-a}$$

$$B = (z-1/a)\left.\frac{Y(z)}{z}\right|_{z=1/a} = \left.\frac{-1/a}{z-1}\right|_{z=1/a} = \frac{-1/a}{1/a-1} = \frac{-1/a}{\frac{1-a}{a}} = \frac{-1}{1-a}$$

$$\therefore\ \frac{Y(z)}{z} = \frac{1}{(1-a)}\,\frac{1}{(z-1)} - \frac{1}{(1-a)}\,\frac{1}{(z-1/a)}$$

$$\therefore\ Y(z) = \frac{1}{(1-a)}\,\frac{z}{(z-1)} - \frac{1}{(1-a)}\,\frac{z}{(z-1/a)}$$

$\mathcal{Z}\{-u(-n-1)\} = \dfrac{z}{z-1}$
$\mathcal{Z}\{-b^n\, u(-n-1)\} = \dfrac{z}{z-b}$

Note : *Since impulse response is anticausal, the step response is also anticausal.*

On taking inverse $\mathcal{Z}$-transform of Y(z) we get step response.

$$\therefore\ \text{Step response, } y(n) = -\frac{1}{1-a}\, u(-n-1) + \frac{1}{1-a}\left(\frac{1}{a}\right)^n u(-n-1) = \left[\left(\frac{1}{a}\right)^n - 1\right]\left(\frac{1}{1-a}\right) u(-n-1)$$

Example 7.24

Test the stability of the first order system governed by the equation, y(n) = x(n) + b y(n–1), where |b| < 1.

Solution

$\mathcal{Z}\{y(n)\} = Y(z)$; $\mathcal{Z}\{y(n-1)\} = z^{-1}Y(z)$; $\mathcal{Z}\{x(n)\} = X(z)$

Given that, y(n) = x(n) + b y(n – 1)

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = X(z) + b\, z^{-1}\, Y(z) \quad \Rightarrow \quad Y(z) - b\, z^{-1}\, Y(z) = X(z) \quad \Rightarrow \quad (1 - b\, z^{-1})\, Y(z) = X(z)$$

$$\therefore\ \frac{Y(z)}{X(z)} = \frac{1}{1 - b\, z^{-1}}$$

We know that, Y(z)/X(z) is equal to H(z).

$$\therefore\ H(z) = \frac{1}{1 - b\, z^{-1}} = \frac{1}{z^{-1}(z-b)} = \frac{z}{z-b}$$

On taking inverse $\mathcal{Z}$-transform of H(z) we get the impulse response h(n).

$\therefore$ Impulse response, $h(n) = b^n\, u(n)$

The condition to be satisfied for the stability of the system is, $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |b^n| = \sum_{n=0}^{\infty} |b|^n$$

Since $|b| < 1$, using the infinite geometric series sum formula we can write,

$$\sum_{n=0}^{\infty} |b|^n = \frac{1}{1-|b|}$$

$$\therefore \sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{1-|b|} = \text{constant}$$

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \quad \text{if, } 0 < |C| < 1$$

The term $1/(1-|b|)$ is less than infinity and so the system is stable.

Example 7.25

Using $\mathcal{Z}$-transform, find the autocorrelation of the causal sequence $x(n) = a^n\, u(n)$, $-1 < a < 1$.

Solution

Given that, $x(n) = a^n\, u(n)$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{a^n u(n)\} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$\therefore X(z^{-1}) = X(z)\Big|_{z=z^{-1}} = \frac{1}{1-az} = -\frac{1}{a}\,\frac{1}{z-1/a}$$

Let, $r_{xx}(m)$ be autocorrelation sequence

By correlation property of $\mathcal{Z}$-transform,

$$\mathcal{Z}\{r_{xx}(m)\} = X(z)\,X(z^{-1}) = \frac{z}{z-a} \times \frac{1}{-a}\,\frac{1}{z-1/a} = -\frac{1}{a}\,\frac{z}{(z-a)\,(z-1/a)}$$

Let, $\dfrac{z}{(z-a)\,(z-1/a)} = \dfrac{A}{z-a} + \dfrac{B}{z-1/a}$

$$A = \frac{z}{(z-a)\,(z-1/a)} \times (z-a)\Big|_{z=a} = \frac{z}{z-1/a}\Bigg|_{z=a} = \frac{a}{a-1/a} = \frac{a}{\frac{a^2-1}{a}} = \frac{a^2}{a^2-1}$$

$$B = \frac{z}{(z-a)\,(z-1/a)} \times (z-1/a)\Big|_{z=1/a} = \frac{z}{z-a}\Bigg|_{z=1/a} = \frac{1/a}{1/a-a} = \frac{1/a}{\frac{1-a^2}{a}} = \frac{1}{1-a^2} = \frac{-1}{a^2-1}$$

$$\therefore \mathcal{Z}\{r_{xx}(m)\} = -\frac{1}{a}\left(\frac{a^2}{a^2-1}\,\frac{1}{z-a} - \frac{1}{a^2-1}\,\frac{1}{z-1/a}\right) = -\frac{a}{a^2-1}\,\frac{1}{z-a} + \frac{1}{a(a^2-1)}\,\frac{1}{z-1/a}$$

$$\therefore r_{xx}(m) = \mathcal{Z}^{-1}\left\{-\frac{a}{a^2-1}\,\frac{1}{z-a} + \frac{1}{a(a^2-1)}\,\frac{1}{z-1/a}\right\} = -\frac{a}{a^2-1}\,\mathcal{Z}^{-1}\left\{z^{-1}\frac{1}{z-a}\right\} + \frac{1}{a(a^2-1)}\,\mathcal{Z}^{-1}\left\{z^{-1}\frac{z}{z-1/a}\right\}$$

$$= -\frac{a}{a^2-1}\,a^{(n-1)}\,u(n-1) + \frac{1}{a(a^2-1)}\left(\frac{1}{a}\right)^{n-1} u(n-1)$$

$$= \frac{1}{a^2-1}\left[\frac{1}{a}\left(\frac{1}{a}\right)^{n-1} - a\,(a)^{n-1}\right] u(n-1) = \frac{1}{a^2-1}\left[\left(\frac{1}{a}\right)^{n} - a^{n}\right] u(n-1)$$

$\boxed{z^{-1}\,z = z^0 = 1}$

If $\mathcal{Z}\{x(n)\} = X(z)$
then by shifting property
$\mathcal{Z}\{x(n-m)\} = z^{-m}\,X(z)$

7.9 Summary of Important Concepts

1. The Ƶ-transform provides a method for analysis of discrete time signals and systems in frequency domain.
2. The region of convergence (ROC) of X(z) is the set of all values of z, for which X(z) attains a finite value.
3. The zeros are defined as values of z at which the function X(z) becomes zero.
4. The poles are defined as values of z at which the function X(z) becomes infinite.
5. In a realizable system, the number of zeros will be less than or equal to number of poles.
6. The ROC of X(z) is a ring or disk in z-plane, with centre at origin.
7. If x(n) is finite duration right sided (causal) signal, then the ROC is entire z-plane except $z = 0$.
8. If x(n) is finite duration left sided (anticausal) signal, then the ROC is entire z-plane except $z = \infty$.
9. If x(n) is finite duration two sided (noncausal) signal, then the ROC is entire z-plane except $z = 0$ and $z = \infty$.
10. If x(n) is infinite duration right sided (causal) signal, then the ROC is exterior of a circle of radius r_1.
11. If x(n) is infinite duration left sided (anticausal) signal, then the ROC is interior of a circle of radius r_2.
12. If x(n) is infinite duration two sided (noncausal) signal, then the ROC is the region in between two circles of radius r_1 and r_2.
13. If X(z) is rational, (where X(z) is Ƶ-transform of x(n)), then the ROC does not include any poles of X(z).
14. If X(z) is rational, (where X(z) is Ƶ-transform of x(n)), and if x(n) is right sided, then ROC is exterior of a circle whose radius corresponds to pole with largest magnitude.
15. If X(z) is rational, (where X(z) is Ƶ-transform of x(n)), and if x(n) is left sided, then ROC is interior of a circle whose radius corresponds to pole with smallest magnitude.
16. If X(z) is rational, (where X(z) is Ƶ-transform of x(n)), and if x(n) is two sided, then ROC is region in between two circles whose radii corresponds to pole of causal part with largest magnitude and pole of anticausal part with smallest magnitude.
17. The inverse Ƶ-transform is the process of recovering the discrete time signal x(n) from its Ƶ-transform X(z).
18. The transfer function of an LTI discrete time system is defined as the ratio of Ƶ-transform of output and Ƶ-transform of input.
19. The transfer function of an LTI discrete time system is also given by Ƶ-transform of the impulse response.
20. The inverse Ƶ-transform of transfer function is the impulse response of the system.
21. The zero-input response $y_{zi}(n)$ is mainly due to initial output (or initial stored energy) in the system.
22. The zero-state response $y_{zs}(n)$ is the response of the system due to input signal and with zero initial output.
23. The total response y(n) is the response of the system due to input signal and initial output (or initial stored energy).
24. The convolution operation is performed to find the response y(n) of an LTI discrete time system from the input x(n) and impulse response h(n).
25. The deconvolution operation is performed to extract the input x(n) of an LTI system from the response y(n) of the system.
26. A point-s_1 on left half of s-plane (LHP), will map as a point-z_1 inside the unit circle in z-plane.
27. A point-s_1 on imaginary axis of s-plane, will map as a point-z_1 on the unit circle in z-plane.
28. A point-s_1 on the right half of s-plane (RHP), will map as a point-z_1 outside the unit circle in z-plane.
29. The mapping of s-plane to z-plane, using the transformation, $e^{sT} = z$ is not one-to-one.
30. The mapping of the continuous time frequency Ω to the discrete time frequency ω is many-to-one.

7.10 Short Questions and Answers

Infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C} \quad ; \text{ if, } 0 < |C| < 1$$

Q7.1 ***Find the Z-transform of $a^n u(n)$.***

By the definition of Z-transform,

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1-az^{-1}} = \frac{1}{1-a/z} = \frac{z}{z-a}$$

Q7.2 ***Find the Z-transform of $e^{-anT} u(n)$.***

By the definition of Z-transform,

$$\mathcal{Z}\{e^{-anT} u(n)\} = \sum_{n=0}^{\infty} e^{-anT} z^{-n} = \sum_{n=0}^{\infty} (e^{-aT} z^{-1})^n = \frac{1}{1-e^{-aT} z^{-1}} = \frac{1}{1-e^{-aT}/z} = \frac{z}{z-e^{-aT}}$$

Q7.3 ***Find the Z-transform of x(n) defined as,***

$$x(n) = b^n \quad ; \quad 0 \le n \le N-1$$
$$= 0 \quad ; \quad \text{otherwise}$$

Solution

By the definition of Z-transform,

Finite geometric series sum formula.

$$\sum_{n=0}^{N-1} c^n = \frac{1-c^N}{1-c}$$

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{N-1} b^n z^{-n}$$

$$= \sum_{n=0}^{N-1} (b z^{-1})^n = \frac{1-(bz^{-1})^N}{1-bz^{-1}} = \frac{1-b^N z^{-N}}{1-bz^{-1}} = \frac{z^{-N}(z^N - b^N)}{z^{-1}(z-b)} = \frac{z^{-N+1}(z^N-b^N)}{z-b}$$

Q7.4 ***Find the Z-transform of $x(n) = a^{n+1} u(n+1)$.***

Solution

Given that, $x(n) = a^{n+1} u(n+1) = a^{n+1} \quad ; \quad n \ge -1$

By the definition of Z-transform,

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=-1}^{+\infty} a^{n+1} z^{-n} = a^{-1+1} z + \sum_{n=0}^{+\infty} a^{n+1} z^{-n} = a^0 z + \sum_{n=0}^{+\infty} a^n\, a z^{-n}$$

$$= z + a \sum_{n=0}^{+\infty} \left(a z^{-1}\right)^n = z + a\frac{1}{1-a z^{-1}} = z + \frac{az}{z-a} = \frac{z(z-a)+az}{z-a} = \frac{z^2}{z-a}$$

Q7.5 ***Determine the inverse Z-transform of $X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$***

Solution

Given that, $X(z) = \log(1 + az^{-1}) \quad ; \quad |z| > |a|$

Let, $x(n) = \mathcal{Z}^{-1}\{X(z)\}$

Since ROC is exterior of a circle of radius (a), the x(n) should be a causal signal

By differentiation property of Z-transform we get,

$$\mathcal{Z}\{n x(n)\} = -z\, \frac{d}{dz}\, X(z)$$

$$= -z\, \frac{d}{dz}\left[\log\,(1+a z^{-1})\right] = -z\, \frac{1}{1+a z^{-1}}(-a z^{-2}) = \frac{a z^{-1}}{1+a z^{-1}}$$

$$= \frac{a z^{-1}}{z^{-1}(z+a)} = \frac{a}{z+a} = a z^{-1}\, \frac{z}{z-(-a)}$$

$$\therefore \; n x(n) = \mathcal{Z}^{-1}\left\{a z^{-1} \frac{z}{z-(-a)}\right\}$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by shifting property $\mathcal{Z}\{x(n-m)\} = z^{-m} X(z)$

$$= a(-a)^{n-1}\, u(n-1)$$

$$\therefore \quad x(n) = \frac{a}{n}\, (-a)^{n-1}\, u(n-1)$$

Q7.6 Determine x(0) if the Z-transform of x(n) is $X(z) = \dfrac{2z^2}{(z+3)(z-4)}$.

Solution

By initial value theorem of Z-transform,

$$x(0) = \underset{z\to\infty}{\text{Lt}}\ X(z) = \underset{z\to\infty}{\text{Lt}}\ \frac{2z^2}{(z+3)\ (z-4)}$$

$$= \underset{z\to\infty}{\text{Lt}}\ \frac{2z^2}{z^2\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \underset{z\to\infty}{\text{Lt}}\ \frac{2}{\left(1+\dfrac{3}{z}\right)\left(1-\dfrac{4}{z}\right)} = \frac{2}{(1+0)\ (1-0)} = 2$$

Q7.7 Determine the Z-transform of x(n) = (n – 3) u(n).

Solution

$$\boxed{\mathcal{Z}\{u(n)\} = \frac{z}{z-1}}$$

$$\boxed{\mathcal{Z}\{n\ x(n)\} = -z\frac{d}{dz}\ X(z)}$$

$$\boxed{d\frac{u}{v} = v\ du - u\ dv}$$

$$\mathcal{Z}\{x(n)\} = \mathcal{Z}\{(n-3)\ u(n)\} = \mathcal{Z}\{n\ u(n) - 3\ u(n)\}$$

$$= \mathcal{Z}\{n\ u(n)\} - 3\ \mathcal{Z}\{u(n)\}$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right) - 3\ \frac{z}{z-1} = -z\ \frac{z-1-z}{(z-1)^2} - \frac{3z}{z-1}$$

$$= \frac{z}{(z-1)^2} - \frac{3z}{z-1} = \frac{z-3z(z-1)}{(z-1)^2} = \frac{z-3z^2+3z}{(z-1)^2} = \frac{-3z^2+4z}{(z-1)^2} = \frac{z(4-3z)}{(z-1)^2}$$

Q7.8 Determine the transfer function of the LTI system defined by the equation,

y(n) – 0.5 y(n – 1) = x(n) + 0.4 x(n – 1)

Solution

Given that, $y(n) - 0.5\ y(n-1) = x(n) + 0.4\ x(n-1)$

On taking Z-transform we get,

$$Y(z) - 0.5\ z^{-1}Y(z) = X(z) + 0.4\ z^{-1}X(z) \quad \Rightarrow \quad Y(z)\left[1-0.5\ z^{-1}\right] = X(z)\left[1+0.4\ z^{-1}\right]$$

$\therefore$ Transfer function, $\dfrac{Y(z)}{X(z)} = \dfrac{1+0.4\ z^{-1}}{1-0.5\ z^{-1}}$

Q7.9 The transfer function of a system is given by, $H(z) = 1 - z^{-1}$. Find the response of the system for any input, x(n).

Solution

Given that, $H(z) = 1 - z^{-1}$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$\therefore$ Response in z - domain, $\quad Y(z) = H(z)\ X(z) = (1-z^{-1})\ X(z) = X(z) - z^{-1}\ X(z)$

$\therefore$ Response in time domain, $\quad y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{X(z) - z^{-1}\ X(z)\right\} = x(n) - x(n-1)$

Q7.10 An LTI system is governed by the equation, y(n) = –2 y(n – 2) – 0.5 y(n – 1) + 3 x(n – 1) + 5 x(n). Determine the transfer function of the system.

Solution

Given that, $y(n) = -2\ y(n-2) - 0.5\ y(n-1) + 3\ x(n-1) + 5\ x(n)$

On taking Z-transform of above equation we get,

$$Y(z) = -2\ z^{-2}\ Y(z) - 0.5z^{-1}\ Y(z) + 3z^{-1}\ X(z) + 5X(z)$$

$$Y(z) + 2\ z^{-2}\ Y(z) + 0.5z^{-1}\ Y(z) = 3z^{-1}\ X(z) + 5X(z)$$

$$Y(z)\left[1 + 2\ z^{-2} + 0.5z^{-1}\right] = \left[3z^{-1} + 5\right] X(z)$$

$\therefore$ Transfer function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{3z^{-1}+5}{1+0.5z^{-1}+2z^{-2}} = \dfrac{5z^2+3z}{z^2+0.5z+2}$

Q7.11 The transfer function of an LTI system is $H(z) = \dfrac{z-1}{(z-2)\,(z+3)}$***. Determine the impulse response.***

Solution

$$H(z) = \frac{z-1}{(z-2)\,(z+3)} = \frac{A}{z-2} + \frac{B}{z+3}$$

$$A = \frac{z-1}{(z-2)\,(z+3)} \times (z-2)\Big|_{z=2} = \frac{z-1}{z+3}\Bigg|_{z=2} = \frac{2-1}{2+3} = \frac{1}{5}$$

$$B = \frac{z-1}{(z-2)\,(z+3)} \times (z+3)\Big|_{z=-3} = \frac{z-1}{z-2}\Bigg|_{z=-3} = \frac{-3-1}{-3-2} = \frac{-4}{-5} = \frac{4}{5}$$

$$H(z) = \frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}$$

Impulse response, $h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{5}\,\frac{1}{z-2} + \frac{4}{5}\,\frac{1}{z+3}\right\}$

$$= \mathcal{Z}^{-1}\left\{\frac{1}{5}z^{-1}\,\frac{z}{z-2} + \frac{4}{5}z^{-1}\,\frac{z}{z-(-3)}\right\}$$

$$= \frac{1}{5}\,2^{(n-1)}\,u(n-1) + \frac{4}{5}\,(-3)^{(n-1)}\,u(n-1) = \frac{1}{5}\left[2^{(n-1)} + 4\,(-3)^{(n-1)}\right]u(n-1)$$

Q7.12 Determine the response of LTI system governed by the difference equation,y(n) – 0.5 y(n – 1) = x(n), for input x(n) = 5ⁿ u(n), and initial condition y(–1) = 2.

Solution

Given that, $x(n) = 5^n u(n)$; $\therefore X(z) = \mathcal{Z}\{u(n)\} = \dfrac{z}{z-5}$

Given that, $y(n) - 0.5\,y(n-1) = x(n)$,

On taking $\mathcal{Z}$-transform of above equation we get,

$$Y(z) - 0.5\left[z^{-1}Y(z) + y(-1)\right] = X(z)$$

$$Y(z) - 0.5\left[z^{-1}Y(z) + 2\right] = \frac{z}{z-5}$$

$$Y(z) - 0.5z^{-1}Y(z) - 1 = \frac{z}{z-5} \quad\Rightarrow\quad Y(z)\left[1 - \frac{0.5}{z}\right] = \frac{z}{z-5} + 1 \quad\Rightarrow\quad Y(z)\left[\frac{z-0.5}{z}\right] = \frac{z+z-5}{z-5}$$

$$\therefore Y(z) = \frac{z(2z-5)}{(z-0.5)\,(z-5)} \quad\Rightarrow\quad \frac{Y(z)}{z} = \frac{2z-5}{(z-0.5)\,(z-5)}$$

Let, $\dfrac{Y(z)}{z} = \dfrac{2z-5}{(z-0.5)\,(z-5)} = \dfrac{A}{z-0.5} + \dfrac{B}{z-5}$

$$A = \frac{2z-5}{(z-0.5)\,(z-5)} \times (z-0.5)\Big|_{z=0.5} = \frac{2\times 0.5 - 5}{0.5-5} = \frac{-4}{-4.5} = \frac{40}{45} = \frac{8}{9}$$

$$B = \frac{2z-5}{(z-0.5)\,(z-5)} \times (z-5)\Big|_{z=5} = \frac{2\times 5 - 5}{5-0.5} = \frac{5}{4.5} = \frac{50}{45} = \frac{10}{9}$$

$$\therefore \frac{Y(z)}{z} = \frac{8}{9}\,\frac{1}{z-0.5} + \frac{10}{9}\,\frac{1}{z-5} \quad\Rightarrow\quad Y(z) = \frac{8}{9}\,\frac{z}{z-0.5} + \frac{10}{9}\,\frac{z}{z-5}$$

$\therefore$ Response, $y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{\dfrac{8}{9}\,\dfrac{z}{z-0.5} + \dfrac{10}{9}\,\dfrac{z}{z-5}\right\}$

$$= \frac{8}{9}\,0.5^n\,u(n) + \frac{10}{9}\,5^n\,u(n) = \left[\frac{8}{9}\,0.5^n + \frac{10}{9}\,5^n\right]u(n)$$

Q7.13 ***A signal $x(t) = a^t$ is sampled at a frequency of 1/T Hz in the range $-\infty < t < 0$. Find the Z-transform of the sampled version of the signal.***

Solution

Given that, $x(t) = a^t$; $-\infty < t < 0$

The sampled version of the signal x(nT) is given by, $x(nT) = a^{nT}$; $-\infty < nT < 0$

Now the Z - transform of x(nT) is,

$$Z\{x(nT)\} = \sum_{n=-\infty}^{+\infty} x(nT)\, z^{-n} = \sum_{n=-\infty}^{0} a^{nT}\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a^{-T}z\right)^n = \frac{1}{1-a^{-T}z} = \frac{1}{1-z/a^T} = \frac{a^T}{a^T - z}$$

Q7.14 ***The transfer function of a system is given by, $H(z) = \dfrac{1}{1-0.5\,z^{-1}} + \dfrac{1}{1-2\,z^{-1}}$. Determine the stability and causality of the system for a) ROC : |z| > 2 ; b) ROC : |z| < 0.5.***

Solution

a) ROC is |z| > 2

When ROC is |z| > 2, the impulse response h(n) should be right sided signal.

$$\therefore \text{ Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(0.5^n + 2^n\right) u(n)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is right sided signal. Hence the system is causal.

b) ROC is |z| < 0.5

When ROC is |z| < 0.5, the impulse response h(n) should be left sided signal.

$$\therefore \text{ Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{1}{1-0.5z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = \left(-0.5^n - 2^n\right) u(-n-1)$$

1. The ROC does not include unit circle. Hence the system is unstable.
2. The impulse response is left sided sequence. Hence the system is anticausal.

Q7.15 ***Determine the stability and causality of the system described by the transfer function, $H(z) = \dfrac{1}{1-0.25\,z^{-1}} + \dfrac{1}{1-2\,z^{-1}}$ for ROC : 0.25 < |z| < 2.***

Solution

Given that, ROC is 0.25 < |z| < 2

When ROC is 0.25 < |z| < 2, the impulse response h(n) is two sided signal. Since |z| > 0.25, the term with pole z = 0.25 corresponds to right sided signal. Since |z| < 2, the term with pole z = 2 corresponds to left sided signal.

$$\therefore \text{ Impulse response, } h(n) = Z^{-1}\{H(z)\} = Z^{-1}\left\{\frac{1}{1-0.25z^{-1}} + \frac{1}{1-2z^{-1}}\right\} = 0.25^n\, u(n) - 2^n\, u(-n-1)$$

1. The ROC includes the unit circle. Hence the system is stable.
2. The impulse response is two sided noncausal signal. Hence the system is noncausal.

Q7.16 ***Using $\mathcal{Z}$-transform, determine the response of the LTI system with impulse response, $h(n) = \{1, \ -1, \ 1\}$, for an input $x(n) = \{-2, \ 3, \ 1\}$.***

Solution

Given that, $x(n) = \{-2, \ 3, \ 1\}$

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n} = \sum_{n=0}^{2} x(n)\, z^{-n} = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} = -2 + 3z^{-1} + z^{-2}$$

Given that, $h(n) = \{1, \ -1, \ 1\}$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} = \sum_{n=0}^{2} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\, H(z) = \left(-2 + 3z^{-1} + z^{-2}\right) \times \left(1 - z^{-1} + z^{-2}\right)$$

$$= -2 + 2z^{-1} - 2z^{-2} + 3z^{-1} - 3z^{-2} + 3z^{-3} + z^{-2} - z^{-3} + z^{-4}$$

$$= -2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4} \qquad .....(1)$$

By definition of Z - transform,

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n}$$

On expanding the above summation we get,

$$Y(z) = ...+ y(0) + y(1)\, z^{-1} + y(2)\, z^{-2} + y(3)\, z^{-3} + y(4)\, z^{-4} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$y(0) = -2 \quad ; \quad y(1) = 5 \quad ; \quad y(2) = -4 \quad ; \quad y(3) = 2 \quad ; \quad y(4) = 1$

$\therefore$ Response, $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$

Q7.17 ***Using $\mathcal{Z}$-transform, perform deconvolution of response $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$ and impulse response $h(n) = \{1, \ -1, \ 1\}$, to extract the input x(n).***

Solution

Given that, $y(n) = \{-2, \ 5, \ -4, \ 2, \ 1\}$

$$Y(z) = \mathcal{Z}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, z^{-n} = \sum_{n=0}^{4} y(n)\, z^{-n}$$

$$= y(0) + y(1)\, z^{-1} + y(2)\, z^{-2} + y(3)\, z^{-3} + y(4)\, z^{-4} = -2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4}$$

Given that, $h(n) = \{1, \ -1, \ 1\}$

$$H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n} = \sum_{n=0}^{2} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} = 1 - z^{-1} + z^{-2}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{-2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4}}{1 - z^{-1} + z^{-2}}$$

$$= -2 + 3z^{-1} + z^{-2} \qquad .....(1)$$

$$\begin{array}{r|l}
 & -2 + 3z^{-1} + z^{-2} \\
\hline
1 - z^{-1} + z^{-2} & -2 + 5z^{-1} - 4z^{-2} + 2z^{-3} + z^{-4} \\
 & \underset{(+)}{-}2 \underset{(-)}{+} 2z^{-1} \underset{(+)}{-} 2z^{-2} \\
\cline{2-2}
 & 3z^{-1} - 2z^{-2} + 2z^{-3} \\
 & \underset{(-)}{}3z^{-1} \underset{(+)}{-} 3z^{-2} \underset{(-)}{+} 3z^{-3} \\
\cline{2-2}
 & z^{-2} - z^{-3} + z^{-4} \\
 & \underset{(-)}{}z^{-2} \underset{(+)}{-} z^{-3} \underset{(-)}{+} z^{-4} \\
\cline{2-2}
 & 0
\end{array}$$

By definition of $\mathcal{Z}$ - transform,

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, z^{-n}$$

On expanding the above summation we get,

$$X(z) = ...+ x(0) + x(1)\ z^{-1} + x(2)\ z^{-2} + x(3)\ z^{-3} + ... \qquad .....(2)$$

On comparing equations (1) and (2) we get,

$$x(0) = -2 \quad ; \quad x(1) = 3 \quad ; \quad x(2) = 1$$

$\therefore$ Input, $x(n) = \{-2, \ 3, \ 1\}$

Q7.18 ***In an LTI system the impulse response h(n) = C^n for n $\leq$ 0. Determine the range of values of C, for which the system is stable.***

Solution

Given that, $h(n) = C^n$ for $n \leq 0$.

$$\therefore \sum_{n=-\infty}^{+\infty} h(n) = \sum_{n=-\infty}^{0} C^n = \sum_{n=0}^{\infty} C^{-n} = \sum_{n=0}^{+\infty} (C^{-1})^n$$

If, $0 < \left|C^{-1}\right| < 1$, then $\displaystyle\sum_{n=0}^{+\infty} (C^{-1})^n = \frac{1}{1 - C^{-1}}$

If, $\left|C^{-1}\right| > 1$, then $\displaystyle\sum_{n=0}^{+\infty} (C^{-1})^n = \infty$

$\therefore$ For stability, $\left|C^{-1}\right| < 1 \quad \Rightarrow \quad \dfrac{1}{C} < 1 \quad \Rightarrow \quad C > 1$

Q7.19 ***Using Z-transform, determine the response of the LTI system with impulse response $h(n) = 0.4^n\ u(n)$, for an input $x(n) = 0.2^n\ u(n)$.***

Solution

Given that, $x(n) = 0.2^n\ u(n)$.

$$\therefore X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{0.2^n\ u(n)\} = \frac{z}{z - 0.2}$$

Given that, $h(n) = 0.4^n\ u(n)$

$$\therefore H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\{0.4^n\ u(n)\} = \frac{z}{z - 0.4}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)\ H(z) = \frac{z}{z - 0.2} \times \frac{z}{z - 0.4} = \frac{z^2}{(z - 0.2)\ (z - 0.4)}$$

Let, $\dfrac{Y(z)}{z} = \dfrac{z}{(z - 0.2)\ (z - 0.4)} = \dfrac{A}{z - 0.2} + \dfrac{B}{z - 0.4}$

$$A = \frac{z}{(z - 0.2)\ (z - 0.4)} \times (z - 0.2)\Big|_{z=0.2} = \frac{0.2}{0.2 - 0.4} = \frac{0.2}{-0.2} = -1$$

$$B = \frac{z}{(z - 0.2)\ (z - 0.4)} \times (z - 0.4)\Big|_{z=0.4} = \frac{0.4}{0.4 - 0.2} = \frac{0.4}{0.2} = 2$$

$$\therefore \frac{Y(z)}{z} = \frac{-1}{z - 0.2} + \frac{2}{z - 0.4} \qquad \Rightarrow \qquad Y(z) = -\frac{z}{z - 0.2} + 2\frac{z}{z - 0.4}$$

Response, $y(n) = \mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{-\dfrac{z}{z - 0.2} + 2\dfrac{z}{z - 0.4}\right\}$

$$= -(0.2)^n\ u(n) + 2\ (0.4)^n\ u(n) = \left[2\ (0.4)^n - (0.2)^n\right] u(n)$$

Q7.20 ***Using Z-transform perform deconvolution of response, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$ and impulse response, $h(n) = 0.4^n\,u(n)$, to extract the input $x(n)$.***

Solution

Given that, $y(n) = 2\,(0.4)^n\,u(n) - (0.2)^n\,u(n)$

$$\therefore\; Y(z) = \mathcal{Z}\{y(n)\} = \mathcal{Z}\left\{2\,(0.4)^n u(n) - (0.2)^n\,u(n)\right\}$$

$$= \frac{2z}{z-0.4} - \frac{z}{z-0.2} = \frac{2z(z-0.2) - z(z-0.4)}{(z-0.4)\,(z-0.2)} = \frac{2z^2 - 0.4z - z^2 + 0.4z}{(z-0.4)\,(z-0.2)} = \frac{z^2}{(z-0.4)\,(z-0.2)}$$

Given that, $h(n) = 0.4^n\,u(n)$

$$\therefore\; H(z) = \mathcal{Z}\{h(n)\} = \mathcal{Z}\left\{0.4^n u(n)\right\} = \frac{z}{z-0.4}$$

We know that, $H(z) = \dfrac{Y(z)}{X(z)}$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = Y(z) \times \frac{1}{H(z)} = \frac{z^2}{(z-0.4)\,(z-0.2)} \times \frac{z-0.4}{z} = \frac{z}{z-0.2}$$

$$\therefore \text{Input, } x(n) = \mathcal{Z}^{-1}\{X(z)\} = \mathcal{Z}^{-1}\left\{\frac{z}{z-0.2}\right\} = 0.2^n\,u(n)$$

7.11 MATLAB Programs

Program 7.1

Write a MATLAB program to find one sided Z-transform of the following standard causal signals.

a) n   b) a^n c) na^n d) e^{-anT}

```
%Program to find the Z-transform of some standard signals

  clear all
  syms n T a real;      %Let n, T, a be real variable
  syms z complex;       %Let z be complex variable

 %(a)
  x = n;
  disp('(a) z-transform of "n" is');
  ztrans(x)

 %(b)
  x = a^n;
  disp('(b) z-transform of  "a^n" is');
  ztrans(x)

 %(c)
  x=n*(a^n);
  disp('(c) z-transform of  "n(a^n)" is');
  ztrans(x)

 %(d)
  x=exp(-a*n*T);
  disp('(d) z-transform of "exp(-a*n*T)" is');
  ztrans(x)
```

OUTPUT

```
(a) z-transform of "n" is
      ans =
            z/(z-1)^2
```

```
(b) z-transform of   "a^n" is
        ans =
                z/a/(z/a-1)
(c) z-transform of   "n(a^n)" is
        ans =
                z*a/(z-a)^2
(d) z-transform of "exp(-a*n*T)" is
        ans =
                z/exp(-a*T)/(z/exp(-a*T)-1)
```

Program 7.2

Write a MATLAB program to find Z-transform of the following causal signals.

a) 0.5^n b) $1+n(0.4)^{(n-1)}$

```
%********** program to determine z-transform of the given signals
clear all
syms n real; %Let n be real variable

%(a)
x1=0.5^n;
disp('(a) z-transform of "0.5^n" is');
X1=ztrans(x1)

%(b)
x2=1+n*(0.4^(n-1));
disp('(b) z-transform of "1+n*(0.4^(n-1))" is');
X2=ztrans(x2)
```

OUTPUT

```
(a) z-transform of "0.5^n" is
        X1 =
              2*z/(2*z-1)
(b) z-transform of "1+n*(0.4^(n-1))" is
        X2 =
               z/(z-1)+25*z/(5*z-2)^2
```

Program 7.3

Write a MATLAB program to find inverse Z-transform of the following z-domain signals.

a) $1/(1-1.5z^{-1}+0.5z^{-2})$ b) $1/((1+z^{-1})(1-z^{-1})^2)$

```
%***********Program to determine the inverse z-transform
syms n z

X=1/(1-1.5*(z^(-1))+0.5*(z^(-2)));
disp('Inverse z-transform of 1/(1-1.5z^-1+0.5z^-2)is');
x=iztrans(X,z,n);
simplify(x)

X=1/((1+(z^(-1)))*((1-(z^(-1))^2)));
disp('Inverse z-transform of 1/((1+z^-1)*(1-z^-1)^2))is');
x=iztrans(X,z,n);
simplify(x)
```

OUTPUT

```
Inverse z-transform of 1/(1-1.5z^-1+0.5z^-2) is
        ans =
                2-2^(-n)
Inverse z-transform of 1/((1+z^-1)*(1-z^-1)^2)) is
        ans =
                3/4*(-1)^n+1/2*(-1)^n*n+1/4
```

Program 7.4

Write a MATLAB program to perform convolution of signals, $x_1(n)= (0.4)^n u(n)$ and $x_2(n)=(0.5)^n u(n)$, using Z-transform, and then to perform deconvolution using the result of convolution to extract $x_1(n)$ and $x_2(n)$.

```
%*** Program to perform convolution and deconvolution using z-transform

clear all;
syms n z
x1n=0.4^n;
x2n=0.5^n;

X1z=ztrans(x1n);
X2z=ztrans(x2n);
X3z=X1z*X2z;                    %product of z-transform of inputs
con12=iztrans(X3z);
disp('Convolution of x1(n) and x2(n) is');
simplify(con12)                 % convolution output

decon_X1z=X3z/X1z;
decon_x1n=iztrans(decon_X1z);
disp('The signal x1(n) obtained by deconvolution is');
simplify(decon_x1n)

decon_X2z=X3z/X2z;
decon_x2n=iztrans(decon_X2z);
disp('The signal x2(n) obtained by deconvolution is');
simplify(decon_x2n)
```

OUTPUT

```
Convolution of x1(n) and x2(n) is
      ans =
            5*2^(-n)-4*2^n*5^(-n)
The signal x1(n) obtained by deconvolution is
      ans =
           2^(-n)
The signal x2(n) obtained by deconvolution is
      ans =
           2^n*5^(-n)
```

Program 7.5

Write a MATLAB program to find residues and poles of z-domain signal, $(3z^2+2z+1)/(z^2-3z+2)$

```
%*** Program to find partial fraction expansion of rational
% function of z

clear all
H=tf('z');
Ts=0.1;

b=[3 2 1 ];         %Numerator coefficients
a=[1 -3 2];         %Denominator coefficients

disp('The given transfer function is,');
H=tf([b], [a],Ts)

disp('The residues, poles and direct terms of given TF are,');
disp('(r - residue ; p - poles ; k - direct terms)');
[r,p,k]=residue(b,a)

disp('The num. and den. coefficients extracted from r,p,k,');
[b,a]=residue(r,p,k)
```

OUTPUT

```
The given transfer function is,
      Transfer function:
                        3 z^2 + 2 z + 1
                         ---------------
                        z^2 - 3 z + 2
Sampling time: 0.1

The residues, poles and direct terms of given TF are,
(r - residue ; p - poles ; k - direct terms)
      r =
          17
         -6
      p =
          2
          1
      k =
          3
The num. and den. coefficients extracted from r,p,k are,
      b =
            3     2     1
      a =
            1    -3     2
```

Program 7.6

Write a MATLAB program to find poles and zeros of z-domain signal, $(z^2+0.8z+0.8)/(z^2+0.49)$, and sketch the pole zero plot.

```
% Program to determine poles and zeros of rational function of Z and
% to plot the poles and zeros in z-plane

clear all
syms z

num_coeff=[1 0.8 0.8];        %find the factors of z^2+0.8z+0.8
disp('Roots of numerator polynomial z^2+0.8z+0.8 are zeros.');
zeros=roots(num_coeff)

den_coeff=[1 0 0.49];         %find the factors of z^2+0.49
disp('Roots of denominator polynomial z^2+0.49 are poles.');
poles=roots(den_coeff)

H=tf('z');
Ts=0.1;

H=tf([num_coeff],[den_coeff],Ts);
zgrid on;
pzmap(H);                      %Pole-zero plot
```

OUTPUT

```
Roots of numerator polynomial z^2+0.8z+0.8 are zeros.
      zeros =
            -0.4000 + 0.8000i
            -0.4000 - 0.8000i

Roots of denominator polynomial z^2+4z+13 are poles.
      poles =
            0 + 0.7000i
            0 - 0.7000i
```

The pole-zero plot is shown in fig P7.6.

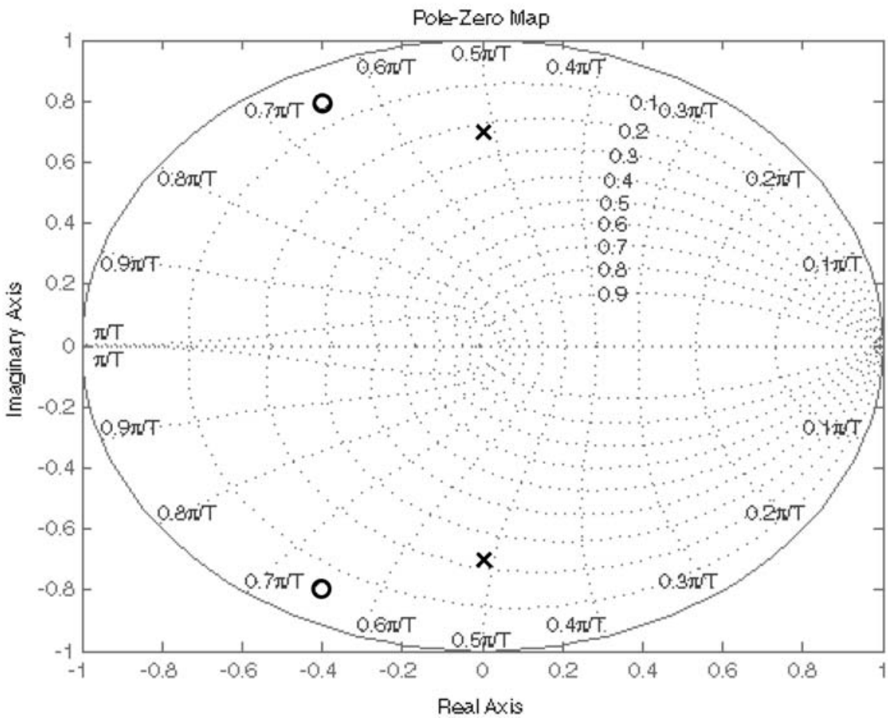


Fig P7.6 *: Pole-Zero plot of program 7.6.*

Program 7.7

Write a MATLAB program to compute and sketch the impulse response of discrete time system governed by transfer function, $H(z)=1/(1-0.8z^{-1}+0.16z^{2})$.

```
%******* Program to find impulse response of a discrete time system
clear all
syms z n

H=1/(1-0.8*(z^(-1))+0.16*(z^(-2)));
disp('Impulse response h(n) is');
h=iztrans(H);        %compute impulse response
simplify(h)

N=15;
b=[0 0 1];           %numerator coefficients
a=[1 -0.8 0.16];     %denominator coefficients
[H,n]=impz(b,a,N);   %compute N samples of impulse response

stem(n,H);           %sketch impulse response
xlabel('n');
ylabel('h(n)');
```

OUTPUT

```
Impulse response h(n) is
      ans =
            2^n*5^(-n)+2^n*5^(-n)*n
```

The sketch of impulse response is shown in fig P7.7.

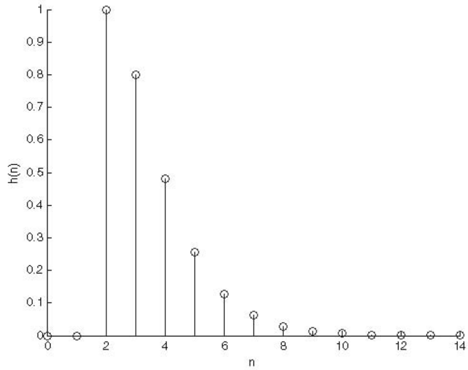


Fig P7.7 : Impulse response of program 7.7.

7.12 Exercises

I. Fill in the blanks with appropriate words

1. The ———— of X(z) is the set of all values of z, for which X(z) attains a finite value.
2. The transformation ——— maps the s-plane into z-plane.
3. The ——— of s-plane can be mapped into the ——— of the unit circle in z-plane.
4. The ratio of Ƶ-transform of output to Ƶ-transform of input is called _______ of the system.
5. In the mapping $z = e^{sT}$, the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
6. In impulse invariant mapping the _______ poles of s-plane are mapped into _______ of unit circle in z-plane.
7. In impulse invariant mapping the poles on the imaginary axis in s-plane are mapped on the _______ in z-plane.
8. In _______ transformation any strip of width $2\pi/T$ in s-plane is mapped into the entire z-plane.
9. The phenomena of high frequency components acquiring the identity of low frequency components is called _______.
10. For a causal LTI discrete time system the ROC should be _______ the circle of radius whose value corresponds to pole with _______ magnitude.
11. If X(z) is rational, then the ROC does not include _______ of X(z).
12. The sequences multiplied by u(–n –1) are _______ and defined for _______.
13. The inverse Ƶ-transform of transfer function is _______ of the system.
14. If Ƶ-transform of x(n) is X(z), then Ƶ-transform of x*(n) is _______.
15. The Ƶ-transform of a shifted signal, shifted by 'q' units of time is obtained by _______ to Ƶ-transform of unshifted signal.

Answers

1. region of convergence	6. right half, exterior	11. poles
2. $s = (1/T)\, ln\, z$	7. unit circle	12. anticausal sequences, $n \leq 0$
3. left half, interior	8. impulse invariant	13. impulse response
4. transfer function	9. aliasing	14. $X^*(z^*)$
5. left half, interior	10. outside, largest	15. multiplying z^q

II. State whether the following statements are True/False.

1. The Ƶ-transform exists only for those values of z for which X(z) is finite.
2. When the input is an impulse sampled signal, the z-domain transfer function can be directly obtained from s-domain transfer function.
3. The jΩ axis in s-plane maps into the unit circle of z-plane in the clockwise direction.
4. The left half of s-plane maps into the interior of the unit circle in z-plane.
5. The system is unstable if all the poles of transfer function lies inside the unit circle in z-plane.
6. The Ƶ-transform of impulse response gives the transfer function of LTI system.
7. If X(z) and H(z) are Ƶ-transform of input and impulse response respectively, then the response of LTI system is given by inverse Ƶ-transform of the product X(z) H(z).
8. For a stable LTI continuous time system the poles should lie on the right half of s-plane.
9. For a stable LTI discrete time system the poles should lie on the unit circle.
10. If Ƶ{x(n)} = X(z), then $\mathcal{Z}\left\{n^m x(n)\right\} = -z\left(\frac{d}{dz}\right)^m X(z)$.

Answers

1. True	4. True	7. True	10. False
2. True	5. False	8. False	
3. False	6. True	9. False	

III. Choose the right answer for the following questions

1. ***The impulse response,*** $h(n) = 1 \quad ; \; n = 0$, ***can be represented as,***
$\qquad = -(1-b)\, b^{n-1} \quad ; \; n \ge 1$

   a) $\delta(n)$ **b)** $u(n) - (1-b)\, b^{n-1}\, u(n-1)$
   c) $\delta(n) - (1-b)\, b^{n-1}\, u(n-1)$ **d)** $u(n) - (1-b)\, b^{n-1}\, u(n)$

2. ***The Ƶ-transform of*** $a^{-n}\, u(-n-1)$ ***is,***

   a) $\frac{-z}{z - 1/a}$ **b)** $\frac{z}{z - 1/a}$ **c)** $\frac{z}{z-a}$ **d)** $\frac{-z}{z-a}$

3. ***The ROC of the sequence x(n) = u(–n) is,***

   a) $|z| > 1$ **b)** $|z| < 1$ **c)** no ROC **d)** $-1 < |z| < 1$

4. ***The inverse Ƶ-transform of*** $\frac{3}{z-4}$, $|z| > 4$ ***is,***

   a) $3(4)^n\, u(n-1)$ **b)** $3(4)^{n-1}\, u(n)$ **c)** $3(4)^{n-1}\, u(n+1)$ **d)** $3(4)^{n-1}\, u(n-1)$

5. ***ROC of x(n) contains,***

   a) poles **b)** zeros **c)** no poles **d)** no zeros

6. ***The inverse Ƶ-transform of*** $X(z) = e^{a/z}$, $|z| > 0$ ***is,***

   a) $x(n) = \frac{-a^n}{n!}\, u(n)$ **b)** $x(n) = \frac{a^n}{n!}\, u(n)$ **c)** $x(n) = \frac{a^{n-1}}{n!}\, u(n-1)$ **d)** None of the above

7. ***The Ƶ-transform of*** $x(n) = [u(n) - u(n-3)]$, ***for ROC*** $|z| > 1$ ***is,***

   a) $X(z) = \frac{z - z^{-2}}{z-1}$ **b)** $X(z) = \frac{z^{-2}}{(z-1)^2}$ **c)** $X(z) = \frac{z - 4z^{-2} + 3z^{-3}}{(z-1)^2}$ **d)** $X(z) = \frac{z - z^{-2}}{z-1}$

8. ***The system function*** $H(z) = \dfrac{z^3 - 2z^2 + z}{z^2 + 0.25z + 0.125}$ ***is,***

a) causal **b)** noncausal **c)** unstable but causal **d)** can not be defined

9. ***If all the poles of the system function H(z) have magnitude smaller than 1, then the system will be,***

a) stable **b)** unstable **c)** BIBO stable **d)** a and c

10. ***If*** $x(n) = \{0.5,\ \ -0.25,\ \ 1\}$, ***then Z-transform of the signal is,***

a) $\dfrac{z^2}{0.5z^2 - 0.25z + 1}$ **b)** $\dfrac{z^2}{z^2 - 0.5z + 0.25}$ **c)** $\dfrac{0.5z^2 - 0.25z + 1}{z^2}$ **d)** $\dfrac{2z^2 + 4z + 1}{z^2}$

11. ***The ROC of the signal*** $x(n) = a^n$ ***for*** $-5 < n < 5$ ***is,***

a) Entire z-plane
b) Entire z-plane except z = 0 and z = ∞
c) Entire z-plane except z = 0
d) Entire z-plane except z = ∞

12. ***If Z-transform of x(n) is X(z) then Z-transform of x(–n) is,***

a) $-X(z)$ **b)** $X(-z)$ **c)** $-X(z^{-1})$ **d)** $X(z^{-1})$

13. ***The inverse Z-transform of X(z) can be defined as,***

a) $x(n) = \dfrac{1}{2\pi}\oint_c X(z)\ z^{n-1}\ dz$
b) $x(n) = \dfrac{1}{2j}\oint_c X(z)\ z^{n-1}\ dz$
c) $x(n) = \dfrac{1}{2\pi j}\oint_c X(z)\ z^{n-1}\ dz$
d) $x(n) = \dfrac{1}{2\pi j}\oint_c X(z)\ z^{-n}\ dz$

14. ***The Z-transform is a,***

a) finite series **b)** infinite power series **c)** geometric series **d)** both a and c

15. ***If the Z-transform of x(n) is X(z), then Z-transform of*** $(0.5)^n\ x(n)$ ***is,***

a) $X(0.5\ z)$ **b)** $X(0.5^{-1}\ z)$ **c)** $X(2^{-1}\ z)$ **d)** $X(2z)$

16. ***The Z-transform of correlation of the sequences x(n) and y(n) is,***

a) $X^*(z)\ Y^*(z^{-1})$ **b)** $X(z)\ Y(z^{-1})$ **c)** $X(z) * Y(z)$ **d)** $X(z^{-1})\ Y(z^{-1})$

17. ***The parseval's relation states that if*** $Z\{x_1(n)\} = X_1(z)$ ***and*** $Z\{x_2(n)\} = X_2(z)$ ***then*** $\sum\limits_{n=-\infty}^{+\infty} x_1(n)\ x_2^*(n)$ ***is,***

a) $\dfrac{1}{2\pi}\oint_c X_1(z)\ \ X_2^*\left(\dfrac{1}{z}\right) z^{-1}\ dz$
b) $\dfrac{1}{2\pi}\oint_c X_1(z)\ \ X_2\left(\dfrac{1}{z^*}\right) z^{-1}\ dz$
c) $\dfrac{1}{2\pi j}\oint_c X_1(z)\ \ X_2^*\left(\dfrac{1}{z^*}\right) z^{-1}\ dz$
d) $\dfrac{1}{2\pi}\oint_c X_1(z)\ \ X_2^*\left(\dfrac{1}{z^*}\right) z^{-1}\ dz$

18. ***For a stable LTI discrete time system poles should lie ——— and unit circle should be ——— .***

a) outside unit circle, included in ROC
b) inside unit circle, outside of ROC
c) inside unit circle, included in ROC
d) outside unit circle, outside of ROC

19. ***An LTI system with impulse response,*** $h(n) = (-a)^n\ u(n)$ ***and*** $-a < -1$ ***will be,***

a) stable system
b) unstable system
c) anticausal system
d) neither stable nor causal

20. ***If X(z) has a single pole on the unit circle, on negative real axis then, x(n) is,***

a) signed constant sequence
b) signed decaying sequence
c) signed growing sequence
d) constant sequence

21. The $\mathcal{Z}$-transform of $x(n) = -na^n\, u(-n-1)$ is,

a) $X(z) = \dfrac{az}{(z-a)^2}$ b) $\dfrac{az(z+a)}{(z-a)^3}$ c) $X(z) = \dfrac{az^{-1}}{(1-az^{-1})^2}$ d) both a and c

22. The ROC for $x(n) \overset{\mathcal{Z}}{\underset{\mathcal{Z}^{-1}}{\longleftrightarrow}} X(z)$ is R_1, then ROC of $a^n x(n) \overset{\mathcal{Z}}{\underset{\mathcal{Z}^{-1}}{\longleftrightarrow}} X\left(\frac{z}{a}\right)$ is,

a) $\dfrac{R_1}{a}$ b) aR_1 c) R_1 d) $\dfrac{1}{R_1}$

23. The $\mathcal{Z}$-transform of a ramp function $x(n) = n\,u(n)$ is,

a) $X(z) = \dfrac{z}{(z-1)^2}$; ROC is $|z| > 1$ b) $X(z) = \dfrac{-z}{(z-1)^2}$; ROC is $|z| > 1$

c) $X(z) = \dfrac{z}{(z-1)^2}$; ROC is $|z| < 1$ d) $X(z) = \dfrac{-z}{(z-1)^2}$; ROC is $|z| < 1$

24. By impulse invariant transformation, if $x(nT)$ is sampled version of $x(t)$, then $\mathcal{Z}\{x(nT)\}$ is,

a) $\mathcal{L}\{x(nT)\}\big|_{z=e^{sT}}$ b) $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{-sT}}$ c) $\mathcal{L}\{x(nT)\}\big|_{z=e^{-sT}}$ d) $\mathcal{L}^{-1}\{x(nT)\}\big|_{z=e^{sT}}$

25. The $\mathcal{Z}$-transform of $x(n) = \left[\sin\frac{\pi}{2}n\right] u(n)$ is,

a) $\dfrac{z}{z+1}$ b) $\dfrac{z^2}{z^2+1}$ c) $\dfrac{1}{z+1}$ d) $\dfrac{z}{z^2+1}$

Answers

1. c	6. b	11. b	16. b	21. d
2. a	7. d	12. d	17. c	22. a
3. b	8. b	13. c	18. c	23. a
4. d	9. a	14. b	19. b	24. a
5. c	10. c	15. b	20. a	25. d

IV. Answer the following questions

1. Define one sided and two-sided $\mathcal{Z}$-transform.
2. What is region of convergence (ROC)?
3. State the final value theorem with regard to $\mathcal{Z}$-transform.
4. State the initial value theorem with regard to $\mathcal{Z}$-transform.
5. Define $\mathcal{Z}$-transform of unit step signal.
6. What are the different methods available for inverse $\mathcal{Z}$-transform?
7. When the z-domain transfer function of the system can be directly obtained from s-domain transfer function?
8. Define the transfer function of an LTI system.
9. Write the transfer function of N^{th} order LTI system.
10. What is impulse invariant transformation?
11. How a-point in s-plane is mapped to z-plane in impulse invariant transformation?
12. Why an impulse invariant transformation is not considered to be one-to-one?
13. Give the importance of convolution and deconvolution operations using $\mathcal{Z}$-transform.
14. Give the conditions for stability of an LTI discrete time system in z-plane.

15. Explain when an LTI dicrete time system will be causal.
16. Define ROC for various finite and infinite discrete time signals.
17. Explain the shifting property of a discrete time signal defined in the range $0 < n < \infty$ with an example.
18. What are all the properties of ROC of a rational function of z?
19. List the various elements that can be used to realise the structure of an LTI discrete time system.
20. State and prove the linearity property of Z-transform.

V. Solve the Following Problems

E7.1 Determine the Z-transform and their ROC of the following discrete time signals.

a) $x(n) = \{\underset{\uparrow}{1}, \ 3, \ 5, \ 6\}$ **b)** $x(n) = \{3, \ 0, \ 9, \ 0, \ 27, \ \underset{\uparrow}{2}\}$

c) $x(n) = \{2, \ 1.0, \ 1, \ \underset{\uparrow}{2}, \ 5, \ 7, \ 2\}$ **d)** $x(n) = -0.4^n \ u(n-1)$

e) $x(n) = (0.1)^n \ u(n) + (0.3)^n \ u(-n-1)$ **f)** $x(n) = (0.4)^{|n|}$

E7.2 Find the one sided Z-transform of the following discrete time signals.

a) $x(n) = n^2 \ 2^n \ u(n)$ **b)** $x(n) = n(0.5)^{n+2}$ **c)** $x(n) = (0.5)^n \ [u(n) - u(n-2)]$

E7.3 Find the one sided Z-transform of the discrete signals generated by mathematically sampling the following continuous time signals.

a) $x(n) = 3t\,e^{-0.5t} \ u(t)$ **b)** $x(t) = 2\,t^3 \ u(t)$

E7.4 Find the time domain initial value x(0) and final value $x(\infty)$ of the following z-domain functions.

a) $X(z) = \dfrac{2}{(1+z^{-1})^2 \ (1-z^{-1})}$ **b)** $X(z) = \dfrac{z^2}{(z-1) \ (z-0.2)}$

E7.5 Determine the inverse Z-transform of the following functions using contour integral method.

a) $X(z) = \dfrac{(2z-1)z}{4(z-1)^3}$ **b)** $X(z) = \dfrac{z^2+z}{(z-2)^2}$ **c)** $X(z) = \dfrac{(1-e^{-a})z}{(z-1) \ (z-e^{-a})}$

E7.6 Determine the inverse Z-transform of the following functions by partial fraction method.

a) $X(z) = \dfrac{5z^2}{(z+1) \ (z+2)^2}$ **b)** $X(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$ **c)** $X(z) = \dfrac{z(z^2-1)}{(z^2+1)^2}$

E7.7 Determine the inverse Z-transform of the function, $X(z) = \dfrac{3 - \frac{5}{6} z^{-1}}{\left[1-\frac{1}{4} z^{-1}\right]\left[1-\frac{1}{3} z^{-1}\right]}$

a) ROC : $|z| > \frac{1}{3}$, **b)** ROC : $|z| < \frac{1}{4}$, **c)** ROC : $\frac{1}{4} < |z| < \frac{1}{3}$.

E7.8 Determine the inverse Z-transform for the following function using power series method.

$$X(z) = \frac{z}{2z^2 - 3z + 1}$$

a) ROC : $|z| < 0.5$, **b)** ROC : $|z| > 1$

E7.9 Determine the inverse Z-transform for the following functions using power series method.

a) $X(z) = \dfrac{z^2+z}{z^2-2z+1}$; ROC : $|z| > 1$ **b)** $X(z) = \dfrac{1-0.5\,z^{-1}}{1+0.5\,z^{-1}}$; ROC : $|z| > 0.5$

E7.10 Determine the transfer function and impulse response for the systems described by the following equations.

a) $y(n) - 2y(n-1) - 3y(n-2) = x(n-1)$

b) $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 2x(n)$

c) $y(n) = 0.2x(n) - 0.5x(n-1) + 0.6y(n-1) - 0.08y(n-2)$

d) $y(n) - \frac{1}{2}y(n-1) = x(n) + \frac{1}{3}x(n-1)$

E7.11 A discrete time LTI system is characterized by a transfer function, $H(z) = \dfrac{z(3z-4)}{\left(z-\frac{1}{2}\right)(z-3)}$. Specify the ROC of H(z) and determine h(n) for the system to be (i) stable (ii) causal.

E7.12 Determine the unit step response of a discrete time LTI system, whose input and output relation is described by the differential equation, $y(n) + 3\,y(n-1) = x(n)$, where the initial condition is $y(-1) = 1$.

E7.13 Determine the response of discrete time LTI system governed by the following differential equation,

$3\,y(n) - 4\,y(n-1) + y(n-2) = x(n)$; with initial conditions, $y(-2) = -2$ and $y(-1) = 1$ for the input

$x(n) = \left(\frac{1}{2}\right)^n u(n)$.

E7.14 An LTI system has the impulse response h(n) defined by $h(n) = x_1(n-1) * x_2(n)$. The $\mathcal{Z}$-transform of the two signals $x_1(n)$ and $x_2(n)$ are $X_1(z) = 1 - 3z^{-1}$ and $X_2(z) = 1 + 2z^{-2}$. Determine the output of the system for the input $\delta(n-1)$.

Answers

E7.1

a) $X(z) = 1 + \dfrac{3}{z} + \dfrac{5}{z^2} + \dfrac{6}{z^3}$

ROC is entire z-plane except at $z = 0$.

b) $X(z) = 3z^5 + 9z^3 + 27z + 2$

ROC is entire z-plane except at $z = \infty$.

c) $X(z) = 2z^3 + 10z^2 + z + 2 + \dfrac{5}{z} + \dfrac{7}{z^2} + \dfrac{2}{z^3}$

ROC is entire z-plane except at $z = 0$ and $z = \infty$.

d) $X(z) = \dfrac{-0.4}{z-0.4}$; ROC is exterior of the circle of radius 0.4 in z-plane.

e) $X(z) = \dfrac{-0.2z}{(z-0.1)(z-0.3)}$; ROC is $0.1 < |z| < 0.3$

f) $X(z) = \dfrac{-2.1z}{(z-0.4)(z-2.5)}$; ROC is $0.4 < |z| < 2.5$

E7.2

a) $X(z) = \dfrac{2z(z+2)}{(z-2)^3}$

b) $X(z) = \dfrac{0.5^3 z}{(z-0.5)^2}$

c) $X(z) = \dfrac{z^2 - 0.25}{z(z-0.5)}$

E7.3

a) $X(z) = \dfrac{3zT\,e^{-0.5T}}{(z-e^{-0.5T})^2}$

b) $X(z) = \dfrac{2T^3 z\left(z^2+4z+1\right)}{(z-1)^4}$

E7.4

a) Initial value, $x(0) = 2$

Final value, $x(\infty) = 0.5$

b) Initial value, $x(0) = 1$

Final value, $x(\infty) = 1.25$

E7.5	**a)** $x(n)=\left[0.125n^2+0.375n\right]u(n)$ **b)** $x(n)=(n+1)\,2^n\,u(n)+n\,2^{(n-1)}\,u(n-1)$ **c)** $(1-e^{-an})\,u(n)$
E7.6	**a)** $x(n)=5\left[(-2)^n-(-1)^n-n(-2)^n\right]u(n)$ **b)** $x(n)=\left[2-2^{n+1}+3n\,2^n\right]u(n)$ **c)** $x(n)=\dfrac{1}{2j}\left[(j)^n-(-j)^n\right]u(n)$
E7.7	**a)** $x(n)=\left[\left(\dfrac{1}{4}\right)^n+2\left(\dfrac{1}{3}\right)^n\right]u(n)$ **b)** $x(n)=-\left(\dfrac{1}{4}\right)^n u(-n-1)-2\left(\dfrac{1}{3}\right)^n u(-n-1)$ **c)** $x(n)=\left(\dfrac{1}{4}\right)^n u(n)-2\left(\dfrac{1}{3}\right)^n u(-n-1)$
E7.8	**a)** $x(n)=\dfrac{2^n-1}{2^n}\,u(n)$ **b)** $x(n)=1\ ;\ n=0$; $=2^{\lvert n\rvert+1}-1\ ;\ n\le -1$
E7.9	**a)** $x(n)=(2n+1)\,u(n)$ **b)** $x(n)=1\ ;\ n=0$; $=\dfrac{(-1)^n}{2^{n-1}}\ ;\ n\ge 1$
E7.10	**a)** $H(z)=\dfrac{z}{z^2-2z-3}\ ;\ h(n)=\dfrac{1}{4}\left[-(-1)^n\,u(n)+3^n\,u(n)\right]$ **b)** $H(z)=\dfrac{2z^2}{z^2-\frac{3}{4}z+\frac{1}{8}}\ ;\ h(n)=\left[4\left(\dfrac{1}{2}\right)^n-2\left(\dfrac{1}{4}\right)^n\right]u(n)$ **c)** $H(z)=\dfrac{0.2z^2-0.5z}{z^2-0.6z+0.08}\ ;\ h(n)=\left[2.3\,(0.2)^n-2.1\,(0.4)^n\right]u(n)$ **d)** $H(z)=\dfrac{z+\frac{1}{3}}{z-\frac{1}{2}}\ ;\ h(n)=\left(\dfrac{1}{2}\right)^n u(n)+\dfrac{1}{3}\left(\dfrac{1}{2}\right)^{n-1}u(n-1)$
E7.11	**i) Stable system** ROC : $0.5<\lvert z\rvert<3\ ;\ h(n)=\left(\dfrac{1}{2}\right)^n u(n)-2\,(3)^n\,u(-n-1)$ **ii) Causal system** ROC : $\lvert z\rvert>3\ ;\ h(n)=\left(\dfrac{1}{2}\right)^n u(n)+2\,(3)^n\,u(n)$
E7.12	$y(n)=\dfrac{1}{4}\left[1-9\,(-3)^n\right]u(n)$
E7.13	$y(n)=\left[10.5-3\left(\dfrac{1}{2}\right)^n-0.5\left(\dfrac{1}{3}\right)^n\right]u(n)$
E7.14	$y_1(n)=\delta(n-2)-3\delta(n-3)+2\delta(n-4)-6\delta(n-5)$ (or) $y_1(n)=\{\underset{\uparrow}{0},\ 0,\ 1,\ -3,\ 2,\ -6\}$

CHAPTER 8

Fourier Series and Fourier Transform of Discrete Time Signals

8.1 Introduction

A periodic discrete time signal with fundamental period N can be decomposed into N harmonically related frequency components. The summation of the frequency components gives the Fourier series representation of periodic discrete time signal, in which the discrete time signal is represented as a function of frequency, ω. The Fourier series of discrete time signal is called ***Discrete Time Fourier Series*** (***DTFS***). The frequency components are also called frequency spectrum of the discrete time signal.

The Fourier representation of periodic discrete time signals has been extended to nonperiodic signals by letting the fundamental period N to infinity, and this Fourier method of representing nonperiodic discrete time signals as a function of discrete time frequency, ω is called Fourier transform of discrete time signals or ***Discrete Time Fourier Transform*** (***DTFT***). The Fourier represention of discrete time signals is also known as frequency domain representation. In general the Fourier series representation can be obtained only for periodic discrete time signals, but the Fourier transform technique can be applied to both periodic and nonperiodic signals to obtain the frequency domain representation of the discrete time signals.

The Fourier representation of discrete time signals can be used to perform frequency domain analysis of discrete time signals, in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components. The graphical plots of magnitude and phase as a function of frequency are also drawn. The plot of magnitude versus frequency is called ***magnitude spectrum*** and the plot of phase versus frequency is called ***phase spectrum***. In general these plots are called ***frequency spectrum***.

8.2 Fourier Series of Discrete Time Signals (Discrete Time Fourier Series)

The Fourier series (or ***Discrete Time Fourier Series***, DTFS) of discrete time periodic signal x(n) with periodicity N is defined as,

$$\boxed{x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_0 kn} = \sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}} \qquad .....(8.1)$$

where, c_k = Fourier coefficients; ω_0 = Fundamental frequency of x(n)

$$\omega_k = \frac{2\pi k}{N} = k^{th} \text{ harmonic frequency of } x(n)$$

$$c_k e^{j\omega_k n} = k^{th} \text{ harmonic component of } x(n)$$

The Fourier coefficients, c_k for k = 0, 1, 2, ..., N–1 can be evaluated using equation (8.2).

$$\boxed{c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, ..., N-1} \qquad .....(8.2)$$

The ***Fourier coefficient c_k*** represents the amplitude and phase associated with the k^{th} frequency component. Hence we can say that the fourier coefficients provide the description of x(n) in the frequency domain.

Proof :

Consider the Fourier series representation of the discrete time signal x(n).

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

Let us replace k by p

$$\therefore\ x(n) = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}$$

Let us multiply the above equation by $e^{\frac{-j2\pi kn}{N}}$ on both sides.

$$x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

On evaluating the above equation for n = 0 to N – 1 and summing up the values we get,

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\sum_{p=0}^{N-1} c_p\, e^{\frac{j2\pi pn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

Let us interchange the order of summation in the right hand side of the above equation and rearrange as shown below.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} c_p \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}}$$

When p = k the right hand side of the above equation reduces to c_k N.

$$\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = c_k N$$

$$\therefore\ c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Note : The sum over one period of the values of a periodic complex exponential is zero, unless that complex exponential is a constant.

$$\therefore \sum_{n=0}^{N-1} e^{\frac{j2\pi(p-k)n}{N}} = N \; ; \; (p-k) = 0, \pm N, \pm 2N, ...$$
$$= 0 \; ; \; (p-k) \neq N$$

Difference Between Continuous Time and Discrete Time Fourier Series

1. The frequency range of continuous time signal is $-\infty$ to $+\infty$, and so it has infinite frequency spectrum.

2. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has finite frequency spectrum. A discrete time signal with fundamental period N will have N frequency components whose frequencies are,

$$\boxed{\omega_k = \frac{2\pi k}{N} \quad ; \text{ for } k = 0, 1, 2, ... , N-1}$$

8.2.1 Frequency Spectrum of Periodic Discrete Time Signals

Let x(n) be a periodic discrete time signal. Now, the Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}}$$

where, c_k is the Fourier coefficient of k^{th} harmonic component

The Fourier coefficient, c_k is a complex quantity and so it can be expressed in the polar form as shown below.

$$c_k = |c_k| \angle c_k \,; \qquad \text{for } k = 0, 1, 2, 3, \text{.......} N-1$$

where, $|c_k|$ = Magnitude of c_k ; $\angle c_k$ = Phase of c_k

The term, $|c_k|$ represents the magnitude of k^{th} harmonic component and the term $\angle c_k$ represents the phase of the k^{th} harmonic component.

The plot of harmonic magnitude / phase of a discrete time signal versus "k" (or harmonic frequency ω_k) is called ***Frequency spectrum***. The plot of harmonic magnitude versus "k" (or ω_k) is called ***magnitude spectrum*** and the plot of harmonic phase versus "k" (or ω_k) is called ***phase spectrum***.

The Fourier coefficients are periodic with period N.

$$\therefore c_{k+N} = c_k$$

Since Fourier coefficients are periodic, the frequency spectrum is also periodic, with period N.

Proof :

Consider the Fourier coefficient c_k of the discrete time signal x(n).

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Now, the Fourier coefficient c_{k+N} is given by,

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi\,(k+N)\,n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\left(\frac{-j2\pi kn}{N} + \frac{-j2\pi Nn}{N}\right)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi\,k\,n}{N}}\, e^{-j2\pi\,n} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi\,k\,n}{N}} = c_k$$

For a periodic discrete time signal with period N, there are N Fourier coefficients denoted as c_0, c_1, c_2, ... c_{N-1}, and so the N-number of Fourier coefficients can be expressed as a sequence consisting of N values.

Fourier coefficients,	c_k	=	$\{c_0, c_1, c_2, c_3, \text{.........} c_{N-1}\}$
Magnitude spectrum,	$\|c_k\|$	=	$\{\|c_0\|, \|c_1\|, \|c_2\|, \|c_3\|, \text{.........} \|c_{N-1}\|\}$
Phase spectrum,	$\angle c_k$	=	$\{\angle c_0, \angle c_1, \angle c_2, \angle c_3, \text{.........} \angle c_{N-1}\}$

8.2.2 Properties of Discrete Time Fourier Series

The properties of discrete time Fourier series coefficients are listed in table 8.1. The proof of these properties are left as exercise to the readers.

Table 8.1 : Properties of Discrete Time Fourier Series Coefficients

Note : *c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).*

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\,e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\,x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\,c_k$
Multiplication	$x(n)\;y(n)$	$\sum\limits_{m=0}^{N-1} c_m\,d_{k-m}$
Circular convolution	$\sum\limits_{m=0}^{N-1} x(m)\;y((n-m))_N$	$N\,c_K\,d_K$
Symmetry of real signals	x(n) is real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $Re\{c_k\} = Re\{c_{-k}\}$ $Im\{c_k\} = -Im\{c_{-k}\}$
Real and even	x(n) is real and even	c_k are real and even
Real and odd	x(n) is real and odd	c_k are imaginary and odd
Parseval's relation	Average power P of x(n) is defined as, $P = \frac{1}{N}\sum\limits_{n=0}^{N-1} \lvert x(n)\rvert^2$	Average power P in terms of Fourier series coefficients is, $P = \sum\limits_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N}\sum\limits_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum\limits_{k=0}^{N-1} \lvert c_k\rvert^2$	

Note : *The average power in the signal is the sum of the powers of the individual frequency components. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N – 1) is the distribution of power as a function of frequency and so it is called the* ***power density spectrum*** *(or)* ***power spectral density*** *of the periodic signal.*

Example 8.1

Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 2\cos\sqrt{3}\pi n$ b) $x(n) = 4\cos\frac{\pi n}{2}$ c) $x(n) = 3\,e^{\frac{j5\pi n}{2}}$

Solution

a) Given that, $x(n) = 2\cos\sqrt{3}\pi n$

Test for Periodicity

$$\text{Let, } x(n+N) = 2\cos\sqrt{3}\pi(n+N) = 2\cos\left(\sqrt{3}\pi n + \sqrt{3}\pi N\right)$$

For periodicity $\sqrt{3}\pi N$ should be equal to integral multiple of 2π.

Let, $\sqrt{3}\pi N = M \times 2\pi$; where M and N are integers. $\Rightarrow$ $N = \frac{2}{\sqrt{3}}M$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

Fourier Series

Here x(n) is nonperiodic signal and so Fourier series does not exists.

---

b) Given that, $x(n) = 4\cos\frac{\pi n}{2}$

Test for Periodicity

$$\text{Let, } x(n+N) = 4\cos\frac{\pi}{2}(n+N) = 4\cos\left(\frac{\pi n}{2} + \frac{\pi N}{2}\right)$$

For periodicity $\frac{\pi N}{2}$ should be integral multiple of 2π.

Let, $\frac{\pi N}{2} = 2\pi \times M$; where M and N are integers $\Rightarrow$ $N = 4M$

Here N is an integer for M = 1, 2, 3, ...

Let M = 1, $\therefore$ N = 4

Hence x(n) is periodic, with fundamental period N = 4, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$.

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, 3, \ldots.., N-1$$

$$\text{Here } N = 4 \text{ and } x(n) = 4\cos\frac{\pi n}{2}$$

$$\therefore c_k = \frac{1}{4}\sum_{n=0}^{3} 4\cos\frac{\pi n}{2}\,e^{\frac{-j2\pi kn}{4}} \quad ; \text{ for } k = 0, 1, 2, 3$$

$$= \frac{4}{4}\sum_{n=0}^{3}\cos\frac{\pi n}{2}\,e^{\frac{-j\pi kn}{2}} = \sum_{n=0}^{3}\cos\frac{\pi n}{2}\left(\cos\frac{\pi kn}{2} - j\sin\frac{\pi kn}{2}\right)$$

$\cos 0 = 1$;	$\cos\pi = -1$
$\cos\frac{\pi}{2} = 0$;	$\cos\frac{3\pi}{2} = 0$

$$= \cos 0\,(\cos 0 - j\sin 0) + \cos\frac{\pi}{2}\left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right)$$

$$+ \cos\pi\,(\cos\pi k - j\sin\pi k) + \cos\frac{3\pi}{2}\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

$$= 1 + 0 - (\cos\pi k - j\sin\pi k) + 0 = 1 - \cos\pi k + j\sin\pi k$$

When k = 0 ; $c_k = c_0 = 1 - \cos 0 + j\sin 0 \quad = 1 - 1 + j0 = 0$

When k = 1 ; $c_k = c_1 = 1 - \cos \pi + j\sin \pi \quad = 1 + 1 + j0 = 2$

When k = 2 ; $c_k = c_2 = 1 - \cos 2\pi + j\sin 2\pi = 1 - 1 + j0 = 0$

When k = 3 ; $c_k = c_3 = 1 - \cos 3\pi + j\sin 3\pi = 1 + 1 + j0 = 2$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}} = c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}}$$

$$= 0 + 2 e^{\frac{j\pi n}{2}} + 0 + 2 e^{\frac{j3\pi n}{2}} = 2 e^{\frac{j\pi n}{2}} + 2 e^{\frac{j3\pi n}{2}} = 2 e^{j\omega_0 n} + 2 e^{j3\omega_0 n} \quad ; \quad \text{where } \omega_0 = \frac{\pi}{2}$$

---

c) Given that, $x(n) = 3\, e^{\frac{j5\pi n}{2}}$

Test for Periodicity

$$\text{Let, } x(n + N) = 3\, e^{\frac{j5\pi(n+N)}{2}} = 3\, e^{\left(\frac{j5\pi n}{2} + \frac{j5\pi N}{2}\right)}$$

For periodicity $\frac{5\pi N}{2}$ should be integral multiple of 2π.

$$\text{Let, } \frac{5\pi N}{2} = 2\pi \times M \qquad \Rightarrow \qquad N = \frac{4}{5}M$$

Here N is integer for M = 5, 10, 15, ...

Let, M = 5, $\therefore$ N = 4

Here x(n) is periodic with fundamental period N = 4, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4} = \frac{\pi}{2}$

Fourier Series

The Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, 3, \ldots.., N-1$$

$$\text{Here } N = 4 \text{ and } x(n) = 3\, e^{\frac{j5\pi n}{2}}$$

$$\therefore c_k = \frac{1}{4}\sum_{n=0}^{3} 3\, e^{\frac{j5\pi n}{2}}\, e^{\frac{-j2\pi kn}{4}} \quad ; \quad \text{for } k = 0, 1, 2, 3$$

$$= \frac{3}{4}\sum_{n=0}^{3} e^{\frac{j\pi n(5-k)}{2}} = \frac{3}{4}\left[e^0 + e^{\frac{j\pi(5-k)}{2}} + e^{\frac{j2\pi(5-k)}{2}} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{3}{4}\left[1 + e^{\frac{j\pi(5-k)}{2}} + e^{j\pi(5-k)} + e^{\frac{j3\pi(5-k)}{2}}\right]$$

$$= \frac{3}{4}\left[1 + \cos\frac{\pi(5-k)}{2} + j\sin\frac{\pi(5-k)}{2} + \cos\pi(5-k) + j\sin\pi(5-k) + \cos\frac{3\pi(5-k)}{2} + j\sin\frac{3\pi(5-k)}{2}\right]$$

$$\text{When } k = 0;\ c_k = c_0 = \frac{3}{4}\left[1 + \cos\frac{5\pi}{2} + j\sin\frac{5\pi}{2} + \cos 5\pi + j\sin 5\pi + \cos\frac{15\pi}{2} + j\sin\frac{15\pi}{2}\right]$$

$$= \frac{3}{4}\left[1 + 0 + j - 1 + j0 + 0 - j\right] = 0$$

$$\text{When } k = 1;\ c_k = c_1 = \frac{3}{4}\left[1+\cos 2\pi + j\sin 2\pi + \cos 4\pi + j\sin 4\pi + \cos 6\pi + j\sin 6\pi\right]$$

$$= \frac{3}{4}\left[1 + 1 + j0 + 1 + j0 + 1 + j0\right] = 3$$

$$\text{When } k = 2;\ c_k = c_2 = \frac{3}{4}\left[1 + \cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2} + \cos 3\pi + j\sin 3\pi + \cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\right]$$

$$= \frac{3}{4}\left[1 + 0 - j - 1 + j0 + 0 + j\right] = 0$$

$$\text{When } k = 3;\ c_k = c_3 = \frac{3}{4}\left[1+\cos \pi + j\sin \pi + \cos 2\pi + j\sin 2\pi + \cos 3\pi + j\sin 3\pi\right]$$

$$= \frac{3}{4}\left[1 - 1 + j0 + 1 + j0 - 1 + j0\right] = 0$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{3} c_k\, e^{\frac{j2\pi kn}{4}} = \sum_{k=0}^{3} c_k\, e^{\frac{j\pi kn}{2}}$$

$$= c_0 + c_1 e^{\frac{j\pi n}{2}} + c_2 e^{j\pi n} + c_3 e^{\frac{j3\pi n}{2}} = 0 + 3e^{\frac{j\pi n}{2}} + 0 + 0 = 3e^{\frac{j\pi n}{2}} = 3e^{j\omega_0 n}$$

> *Note :* $x(n) = 3e^{\frac{j5\pi n}{2}} = 3e^{j\left(\frac{4\pi n}{2} + \frac{\pi n}{2}\right)} = 3e^{j2\pi n}\, e^{\frac{j\pi n}{2}} = 3e^{\frac{j\pi n}{2}} = 3e^{j\omega_0 n}$
>
> $\therefore$ *The given signal itself is in the Fourier series form.*

Example 8.2

Determine the Fourier series representation of the following discrete time signal and sketch the frequency spectrum.

$$x(n) = \{\ldots\ldots, 1, 2, -1, \underset{\uparrow}{1}, 2, -1, 1, 2, -1, \ldots\ldots\}$$

Solution

Given that, $x(n) = \{\ldots\ldots, 1, 2, -1, \underset{\uparrow}{1}, 2, -1, 1, 2, -1, \ldots\ldots\}$

Here x(n) is periodic with periodicity of N = 3, and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{3}$.

Let, x(n) = {1, 2, –1} (considering one period). Now, the Fourier coefficients c_k are given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} = \frac{1}{3}\sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[x(0) + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}}\right] = \frac{1}{3}\left[1 + 2e^{\frac{-j2\pi k}{3}} - e^{\frac{-j4\pi k}{3}}\right]$$

$$\text{When } k = 0;\ c_k = c_0 = \frac{1}{3}[1 + 2 - 1] = \frac{2}{3} = 0.667$$

$$\text{When } k = 1;\ c_k = c_1 = \frac{1}{3}\left[1 + 2e^{\frac{-j2\pi}{3}} - e^{\frac{-j4\pi}{3}}\right]$$

$$= \frac{1}{3}\left[1 + 2\cos\frac{2\pi}{3} - j2\sin\frac{2\pi}{3} - \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2\times\frac{1}{2} - j2\times\frac{\sqrt{3}}{2} + \frac{1}{2} - j\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{1}{2} - j\frac{3\sqrt{3}}{2}\right] = \frac{1}{6} - j\frac{\sqrt{3}}{2} = 0.1667 - j0.866$$

$$= 0.88\angle -1.38\ \text{rad} = 0.88\angle -0.44\pi = 0.88\, e^{-j0.44\pi}$$

When $k = 2;\ c_k = c_2 = \dfrac{1}{3}\left[1 + 2\,e^{\frac{-j4\pi}{3}} - e^{\frac{-j8\pi}{3}}\right]$

$$= \frac{1}{3}\left[1 + 2\cos\frac{4\pi}{3} - j2\sin\frac{4\pi}{3} - \cos\frac{8\pi}{3} + j\sin\frac{8\pi}{3}\right]$$

$$= \frac{1}{3}\left[1 - 2 \times \frac{1}{2} + j2 \times \frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{3}\left[\frac{1}{2} + j\frac{3\sqrt{3}}{2}\right] = \frac{1}{6} + j\frac{\sqrt{3}}{2} = 0.1667 + j0.866$$

$$= 0.88\ \angle 1.38 \text{ rad} = 0.88\ \angle 0.44\pi = 0.88\ e^{j0.44\pi}$$

The Fourier series representation of x(n) is,

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}} = \sum_{k=0}^{2} c_k\, e^{\frac{j2\pi kn}{3}}$$

$$= c_0 + c_1\, e^{\frac{j2\pi n}{3}} + c_2\, e^{\frac{j4\pi n}{3}}$$

$$= 0.667 + 0.88\, e^{-j0.44\pi}\, e^{\frac{j2\pi n}{3}} + 0.88\, e^{j0.44\pi}\, e^{\frac{j4\pi n}{3}}$$

$$= 0.667 + 0.88\, e^{-j0.44\pi}\, e^{j\omega_0 n} + 0.88\, e^{j0.44\pi}\, e^{j2\omega_0 n}$$

Frequency Spectrum

The frequency spectrum has two components : Magnitude spectrum and Phase spectrum.

The magnitude spectrum is obtained from magnitude of c_k and phase spectrum is obtained from phase of c_k.

Here, $c_k = \{c_0,\ \ c_1,\ \ c_2\} = \{0.667,\ \ 0.88\ \angle -0.44\pi,\ \ 0.88\ \angle 0.44\pi\}$

$\therefore$ Magnitude spectrum, $|c_k| = \{0.667,\ 0.88,\ 0.88\}$

Phase spectrum, $\angle c_k = \{0,\ -0.44\pi,\ 0.44\pi\}$

The sketch of magnitude and phase spectrum are shown in fig 1.

Here both the spectrum are periodic with period, N = 3.

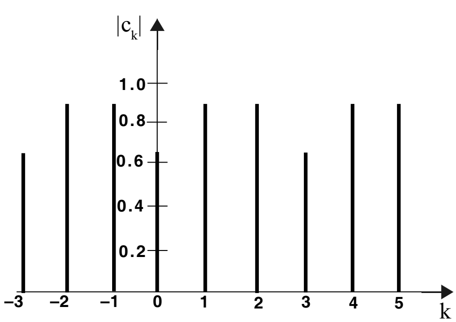


Fig 1.a : *Magnitude spectrum.*

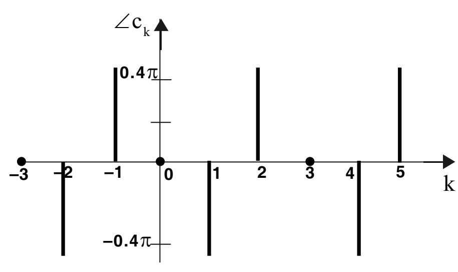


Fig 1.b : *Phase spectrum.*

Fig 1 : *Frequency spectrum.*

8.3 Fourier Transform of Discrete Time Signals (Discrete Time Fourier Transform)

8.3.1 Development of Discrete Time Fourier Transform From Discrete Time Fourier Series

Let $\tilde{x}(n)$ be a periodic sequence with period N. If the period N tends to infinity then the periodic sequence $\tilde{x}(n)$ will become a nonperiodic sequence x(n).

$$\therefore\ x(n) = \underset{N\to\infty}{Lt}\ \tilde{x}(n)$$

Let c_k be Fourier coefficients of $\tilde{x}(n)$.

$$\therefore c_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}} \qquad \Rightarrow \qquad Nc_k = \sum_{n=0}^{N-1}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N, the summation index in the above equation can be changed from to $n = -\left(\frac{N}{2}-1\right)$ to $+\frac{N}{2}$. (For odd values of N, the summation index is $n = -\frac{N}{2}$ to $+\frac{N}{2}$).

$$\therefore Nc_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{\frac{-j2\pi kn}{N}} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{-j\omega_k n} \qquad .....(8.3)$$

$$\text{where,}\quad \omega_k = \frac{2\pi k}{N}$$

Let us define Nc_k as a function of $e^{j\omega_k}$.

$$\therefore\ X(e^{j\omega_k}) = Nc_k \qquad .....(8.4)$$

Now, using equation (8.3), the equation (8.4) can be expressed as shown below.

$$X(e^{j\omega_k}) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\,e^{-j\omega_k n} \qquad .....(8.5)$$

Let, $N \to \infty$, in equation (8.5).

Now, $\tilde{x}(n) \to x(n)$, $\omega_k \to \omega$, and the summation index become $-\infty$ to $+\infty$.

Therefore, the equation (8.5) can be written as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} \qquad .....(8.6)$$

The equation (8.6) is called Fourier transform of x(n), which is used to represent nonperiodic discrete time signal (as a function of frequency,ω) in frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$ given below.

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k\,e^{\frac{j2\pi kn}{N}}$$

Let us multiply and divide the above equation by $N/2\pi$.

$$\tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}} = \frac{N}{2\pi} \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} Nc_k \, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N}$$

$$\boxed{\omega_k = \frac{2\pi k}{N}}$$

Using equation (8.4).

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega_k}) \, e^{j\omega_k n} \frac{2\pi}{N} \quad .....(8.7)$$

Let, $N \rightarrow \infty$, in equation (8.7).

Now, $\tilde{x}(n) \rightarrow x(n), \omega_k \rightarrow \omega, 2\pi/N \rightarrow d\omega$, and summation becomes integral with limits 0 to 2π.

Therefore, the equation (8.7) can be written as shown below.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) \, e^{j\omega n} \, d\omega \quad .....(8.8)$$

The equation (8.8) is called inverse Fourier transform of x(n), which is used to extract the discrete time signal from its frequency domain representation.

Since equation (8.6) extracts the frequency components of discrete time signal, the transformation using equation (8.6) is also called ***analysis*** of discrete time signal x(n). Since equation (8.8) integrates or combines the frequency components of discrete time signal, the inverse transformation using equation (8.8) is also called ***synthesis*** of discrete time signal x(n).

8.3.2 Definition of Discrete Time Fourier Transform

The Fourier transform (FT) of discrete-time signals is called ***Discrete Time Fourier Transform*** (i.e., DTFT). But for convenience the DTFT is also referred as FT in this book.

Let, $x(n)$ = Discrete time signal

$X(e^{j\omega})$ = Fourier transform of x(n)

The Fourier transform of a finite energy discrete time signal, x(n) is defined as,

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n) \, e^{-j\omega n}}$$

Symbolically the Fourier transform of x(n) is denoted as,

$\mathcal{F}\{x(n)\}$

where, $\mathcal{F}$ is the operator that represents Fourier transform.

$$\boxed{\therefore \; X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n) \, e^{-j\omega n}}$$

The Fourier transform of a signal is said to exist if it can be expressed in a valid functional form. Since the computation of Fourier transform involves summing infinite number of terms, the Fourier transform exists only for the signals that are absolutely summable, i.e., given a signal x(n), the $X(e^{j\omega})$ exists only when,

$$\sum_{n=-\infty}^{+\infty} |x(n)| < \infty$$

8.3.3 Frequency Spectrum of Discrete Time Signal

The Fourier transform $X(e^{j\omega})$ of a signal x(n) represents the frequency content of x(n). We can say that, by taking Fourier transform, the signal x(n) is decomposed into its frequency components. Hence $X(e^{j\omega})$ is also called ***frequency spectrum*** of discrete time signal or ***signal spectrum***.

Magnitude and Phase Spectrum

The $X(e^{j\omega})$ is a complex valued function of ω, and so it can be expressed in rectangular form as,

$$X(e^{j\omega}) = X_r(e^{j\omega}) + jX_i(e^{j\omega})$$

where, $X_r(e^{j\omega})$ = Real part of $X(e^{j\omega})$

$X_i(e^{j\omega})$ = Imaginary part of $X(e^{j\omega})$

The polar form of $X(e^{j\omega})$ is,

$$X(e^{j\omega}) = |X(e^{j\omega})| \angle X(e^{j\omega})$$

where, $|X(e^{j\omega})|$ = Magnitude spectrum

$\angle X(e^{j\omega})$ = Phase spectrum

The ***magnitude spectrum*** is defined as,

$$\boxed{\begin{aligned} |X(e^{j\omega})|^2 &= X(e^{j\omega})\, X^*(e^{j\omega}) \\ &= [X_r(e^{j\omega}) + jX_i(e^{j\omega})]\,[X_r(e^{j\omega}) - jX_i(e^{j\omega})] \\ &\text{where, } X^*(e^{j\omega}) \text{ is complex conjugate of } X(e^{j\omega}) \end{aligned}}$$

Alternatively, $|X(e^{j\omega})|^2 = X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})$

$$\boxed{\text{or } |X(e^{j\omega})| = \sqrt{X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})}}$$

The ***phase spectrum*** is defined as,

$$\boxed{\angle X(e^{j\omega}) = \mathrm{Arg}[X(e^{j\omega})] = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]}$$

8.3.4 Inverse Discrete Time Fourier Transform

Let, x(n) = Discrete time signal

$X(e^{j\omega})$ = Fourier transform of x(n)

The ***inverse discrete time Fourier transform*** of $X(e^{j\omega})$ is defined as,

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad ; \text{ for } n = -\infty \text{ to } +\infty} \qquad .....(8.9)$$

Symbolically the inverse Fourier transform can be expressed as, $\mathcal{F}^{-1}\{X(e^{j\omega})\}$, where, $\mathcal{F}^{-1}$ is the operator that represents the inverse Fourier transform.

$$\boxed{\therefore\ x(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \ ; \text{ for } n = -\infty \text{ to } +\infty}$$

Since $X(e^{j\omega})$ is periodic with period 2π, the limits of integral in the above definition of inverse Fourier transform can be either "$-\pi$ to $+\pi$", or "0 to 2π", or "any interval of 2π".

We also refer to x(n) and $X(e^{j\omega})$ as a Fourier transform pair and this relation is expressed as,

$$\boxed{x(n) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} X(e^{j\omega})}$$

Alternate Method for Inverse Fourier Transform

The integral solution of equation (8.9)for the inverse Fourier transform is useful for analytic purpose, but sometimes it will be difficult to evaluate for typical functional forms of $X(e^{j\omega})$. An alternate and more useful method of determining the values of x(n) follows directly from the definition of the Fourier transform.

Consider the definition of Fourier transform of x(n).

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Let us expand the above equation of $X(e^{j\omega})$ as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$= \;.....\; + x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)\, e^{0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + \;..... \qquad .....(8.10.1)$$

Let us express the given function of $X(e^{j\omega})$ as a power series of $e^{-j\omega}$ by long division as shown below.

$$X(e^{j\omega}) = \;.....\; + b_2\, e^{j2\omega} + b_1\, e^{j\omega} + a_0\, e^{0} + a_1\, e^{-j\omega} + a_2\, e^{-j2\omega} + \;..... \qquad .....(8.10.2)$$

On comparing the equations (8.10.1) and (8.10.2) we can say that the samples of signal x(n) are simply the coefficients of $e^{-j\omega n}$.

8.3.5 Comparison of Fourier Transform of Discrete and Continuous Time Signals

1. The Fourier transform of a continuous time signal consists of a spectrum with a frequency range $-\infty$ to $+\infty$. But the Fourier transform of a discrete time signal is unique in the frequency range $-\pi$ to $+\pi$ (or equivalently 0 to 2π). Also Fourier transform of discrete time signal is periodic with period 2π. Hence the frequency range for any discrete-time signal is limited to $-\pi$ to π (or 0 to 2π) and any frequency outside this interval has an equivalent frequency within this interval.

2. Since the continuous time signal is continuous in time the Fourier transform of continuous time signal involves integration but the Fourier transform of discrete time signal involves summation because the signal is discrete.

8.4 Properties of Discrete Time Fourier Transform

1. Linearity property

The linearity property of Fourier transform states that the Fourier transform of a linear weighted combination of two or more signals is equal to the similar linear weighted combination of the Fourier transform of the individual signals.

Let $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$ then by linearity property

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega}) \;;\; \text{where } a_1 \text{ and } a_2 \text{ are constants.}$$

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad \text{.....(8.11)}$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad \text{.....(8.12)}$$

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n)\, e^{-j\omega n} + a_2 x_2(n)]\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1 x_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{+\infty} a_2 x_2(n)\, e^{-j\omega n}$$

$$= a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}$$

$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Using equations (8.11) and (8.12)

2. Periodicity

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $X(e^{j\omega})$ is periodic with period 2π.

$$\therefore X(e^{j(\omega + 2\pi m)}) = X(e^{j\omega}) \;;\; \text{where m is an integer}$$

Proof :

$$X(e^{j(\omega + 2\pi m)}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega+2\pi m)n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi mn}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = X(e^{j\omega})$$

Since m and n are integers, $e^{-j2\pi mn} = 1$

3. Time shifting or Fourier transform of delayed signal

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(n-m)\} = e^{-j\omega m}\, X(e^{j\omega})$

Also $\mathcal{F}\{x(n+m)\} = e^{j\omega m}\, X(e^{j\omega})$

This relation means that if a signal is shifted in time domain by m samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $-\omega m$. This result can be explained if we recall that the frequency content of a signal depends only on its shape. Mathematically, we can say that delaying by m units in time domain is equivalent to multiplying the spectrum by $e^{-j\omega m}$ in the frequency domain.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{.....(8.13)}$$

$$\mathcal{F}\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, e^{-j\omega n}$$

Let, $n - m = p$, $\therefore n = p + m$
when $n \to -\infty$, $p \to -\infty$
when $n \to +\infty$, $p \to +\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega(m+p)}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega m}\ e^{-j\omega p}$$

$$= e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

Let, $p \to n$

$$= e^{-j\omega m}\, X(e^{j\omega})$$

Using equation (8.13)

4. *Time reversal*

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(-n)\} = X(e^{-j\omega})$

This means that if a signal is folded about the origin in time, its magnitude spectrum remains unchanged and the phase spectrum undergoes a change in sign (phase reversal).

Proof :

By the definition of Fourier transform,

$$\mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \quad \text{....(8.14)}$$

Let, $p = -n$
when $n \to -\infty$, $p \to +\infty$
when $n \to +\infty$, $p \to -\infty$

$$\mathcal{F}\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n)\, e^{-j\omega n}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{j\omega p}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, (e^{-j\omega})^{-p} \quad \text{.....(8.15)}$$

$$= X(e^{-j\omega})$$

The equation (8.15) is similar to the form of equation (8.14)

5. *Conjugation*

If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$

then $\mathcal{F}\{x^*(n)\} = X^*(e^{-j\omega})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$\mathcal{F}\{x^*(n)\} = \sum_{n=-\infty}^{+\infty} x^*(n)\, e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, (e^{-j\omega})^{-n}\right]^*$$

$$= \left[X(e^{-j\omega})\right]^*$$

$$= X^*(e^{-j\omega})$$

6. Frequency shifting

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = X(e^{j(\omega - \omega_0)})$

According to this property, multiplication of a sequence x(n) by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{j\omega})$ by ω_0.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{.....(8.16)}$$

$$\therefore \mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = \sum_{n=-\infty}^{+\infty} e^{j\omega_0 n}\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega - \omega_0)n} \qquad \text{.....(8.17)}$$

$$= X(e^{j(\omega - \omega_0)})$$

The equation (8.17) is similar to the form of equation (8.16)

7. Fourier transform of the product of two signals

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$

$\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

$$\text{Now, } \mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi} \int_{-\pi}^{-\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega - \lambda)})\, d\lambda \qquad \text{.....(8.18)}$$

The equation (8.18) is convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$

This relation is the dual of time domain convolution. In other words, the Fourier transform of the product of two discrete time signals is equivalent to the convolution of their Fourier transform. On the other hand, the Fourier transform of the convolution of two discrete time signals is equivalent to the product of their Fourier transform.

Proof :

Let, $x_2(n)\, x_1(n) = x_3(n)$

$$\text{Now, } \mathcal{F}\{x_2(n)\, x_1(n)\} = \mathcal{F}\{x_3(n)\} = \sum_{n=-\infty}^{+\infty} x_3(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x_2(n)\, x_1(n)\, e^{-j\omega n} \qquad \text{.....(8.19)}$$

By the definition of inverse Fourier transform we get,

$$x_1(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Let, $\omega = \lambda$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda \qquad \text{.....(8.20)}$$

On substituting for $x_1(n)$ from equation (8.20) in equation (8.19) we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n) \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda \right] e^{-j\omega n}$$

On interchanging the order of summation and integration in the above equation we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j(\omega-\lambda)n}\right] X_1(e^{j\lambda})\, d\lambda$$

The term in the paranthesis in the above equation is similar to the definition of fourier transform of $x_2(n)$ but at a frequency argument of $(\omega - \lambda)$

$$\therefore\ \mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2(e^{j(\omega-\lambda)})\, X_1(e^{j\lambda})\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$$

8. Differentiation in frequency domain

If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$

then $\mathcal{F}\{n\, x(n)\} = j\dfrac{d}{d\omega}X(e^{j\omega})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.21)$$

$$\mathcal{F}\{n\, x(n)\} = \sum_{n=-\infty}^{+\infty} n\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} n\, x(n)\, j(-j)\, e^{-j\omega n} \qquad \boxed{\text{Multiply by j and } -j}$$

$$= j\sum_{n=-\infty}^{+\infty} x(n)\, [(-jn)\, e^{-j\omega n}]$$

$$= j\sum_{n=-\infty}^{+\infty} x(n)\left[\frac{d}{d\omega}e^{-j\omega n}\right] \qquad \boxed{\frac{d}{d\omega}e^{-j\omega n} = -jn\, e^{-j\omega n}}$$

$$= j\frac{d}{d\omega}\left[\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\right] \qquad \boxed{\text{Interchanging summation and differentiation}}$$

$$= j\frac{d}{d\omega}X(e^{j\omega}) \qquad \boxed{\text{Using equation (8.21)}}$$

9. Convolution theorem

If $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$

and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

then $\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\, X_2(e^{j\omega})$

$$\text{where,}\quad x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad ....(8.22)$$

The Fourier transform of the convolution of $x_1(n)$ and $x_2(n)$ is equal to the product of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$. It means that if we convolve two signals in time domain, it is equivalent to multiplying their spectra in frequency domain.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(8.23)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(8.24)$$

$$\mathcal{F}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \big[x_1(n) * x_2(n)\big]\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] e^{-j\omega n} \qquad \boxed{\text{Using equation (8.22)}}$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, e^{-j\omega n}\, e^{-j\omega m}\, e^{j\omega m} \qquad \boxed{\text{Multiply by } e^{-j\omega m} \text{ and } e^{j\omega m}}$$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, e^{-j\omega (n-m)}$$

Let, $n - m = p$
when $n \to -\infty$, $p \to -\infty$
when $n \to +\infty$, $p \to +\infty$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x_2(p)\, e^{-j\omega p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right] \left[\sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}\right]$$

Let $m = n$, in first summation
Let $p = n$, in second summation

$$= X_1(e^{j\omega})\, X_2(e^{j\omega}) \qquad \boxed{\text{Using equations (8.23) and (8.24)}}$$

10. Correlation

If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$

then $\mathcal{F}\{r_{xy}(m)\} = X(e^{j\omega})\, Y(e^{-j\omega})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad .....(8.25)$$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.26)$$

$$Y(e^{j\omega}) = \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, e^{-j\omega n} \qquad .....(8.27)$$

$$\mathcal{F}\{r_{xy}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xy}(m)\, e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\right] e^{-j\omega m} \qquad \boxed{\text{Using equation (8.25)}}$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\, e^{-j\omega m}\, e^{-j\omega n}\, e^{j\omega n} \qquad \boxed{\text{Multiply by } e^{-j\omega n} \text{ and } e^{j\omega n}}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{m=-\infty}^{+\infty} y(n-m)\, e^{j\omega (n-m)}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{p=-\infty}^{+\infty} y(p)\, e^{j\omega p}$$

Let, $n - m = p$ $\quad \therefore m = n - p$
when $m \to -\infty$, $p \to +\infty$,
when $m \to +\infty$, $p \to -\infty$.

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\right] \left[\sum_{p=-\infty}^{+\infty} y(p)\, (e^{-j\omega})^{-p}\right] \qquad \boxed{\text{Using equations (8.26) and (8.27)}}$$

$$= X(e^{j\omega})\, Y(e^{-j\omega})$$

11. Parseval's relation

If $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

then the Parseval's relation states that,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega \qquad .....(8.28)$$

When $x_1(n) = x_2(n) = x(n)$, then Parseval's relation can be written as,

$$\sum_{n=-\infty}^{+\infty} \left|x(n)\right|^2 = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

The above equation is also called energy density spectrum of the signal x(n).

Proof :

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

Now, by definition of Fourier transform,

$$\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(8.29)$$

Now, by definition of inverse Fourier transform,

$$x_2(n) = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(8.30)$$

Consider left hand side of Parseval's relation [equation (8.28)],

$$\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega$$

In the above expression, Let us substitute for $X_1(e^{j\omega})$ from equation (8.29),

$$\therefore\ \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right] X_2^*(e^{j\omega})\; d\omega$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n) \left[\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2^*(e^{j\omega})\, e^{-j\omega n}\, d\omega\right] \qquad \boxed{\text{Interchanging summation and integration}}$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n) \left[\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\, d\omega\right]^*$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) \qquad \boxed{\text{Using equation (8.30)}}$$

Table 8.2 : Properties of Discrete Time Fourier Transform

Note : *$X(e^{j\omega}) = \mathcal{F}\{x(n)\}$; $X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\}$; $X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\}$; $Y(e^{j\omega}) = \mathcal{F}\{y(n)\}$*

Property	Discrete time signal	Fourier transform
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$
Periodicity	$x(n)$	$X(e^{j\omega + 2\pi m}) = X(e^{j\omega})$
Time shifting	$x(n - m)$	$e^{-j\omega m} X(e^{j\omega})$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
Frequency shifting	$e^{j\omega_0 n} x(n)$	$X(e^{j(\omega - \omega_0)})$
Multiplication	$x_1(n)\, x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$
Differentiation in frequency domain	$n\, x(n)$	$j\,\frac{dX(e^{j\omega})}{d\omega}$
Convolution	$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(e^{j\omega})\, X_2(e^{j\omega})$
Correlation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(e^{j\omega})\, Y(e^{-j\omega})$
Symmetry of real signals	$x(n)$ is real	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\}$ $\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert,\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
Symmetry of real and even signal	$x(n)$ is real and even	$X(e^{j\omega})$ is real and even
Symmetry of real and odd signal	$x(n)$ is real and odd	$X(e^{j\omega})$ is imaginary and odd
Parseval's relation	$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$
Parseval's relation	Energy in time domain, $E = \sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2$	Energy in frequency domain, $E = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$
	$\sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2\, d\omega$	

Note : *The term $\left|X(e^{j\omega})\right|^2$ represents the distribution of energy as a function of frequency and so it is called **energy density spectrum** or **energy spectral density**.*

8.5 Discrete Time Fourier Transform of Periodic Discrete Time Signals

The Fourier transform of any periodic discrete time signal can be obtained from the knowledge of Fourier transform of periodic discrete time signal $e^{j\omega_0 n}$, with period N.

In chapter-4, section 4.12, it is observed that the Fourier transform of continuous time periodic signal is a train of impulses. Similarly, the Fourier transform of discrete time periodic signal is also a train of impulses, but the impulse train should be periodic. Therefore, the Fourier transform of $e^{j\omega_0 n}$ will be in the form of periodic impulse train with period 2π as shown in equation (8.31).

$$\text{Let,} \quad g(n) = e^{j\omega_0 n}$$

$$\therefore G(e^{j\omega}) = \mathcal{F}\{g(n)\} = \mathcal{F}\{e^{j\omega_0 n}\} = \sum_{m=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi m) \qquad .....(8.31)$$

$$\text{where,} \quad \omega_0 = \frac{2\pi}{N} = \text{Fundamental frequency of } g(n).$$

In equation (8.31), $\delta(\omega)$ is an impulse function of ω and ω_0 lie in the range $-\pi$ to $+\pi$.

The equation (8.31) can be proved by taking inverse Fourier transform of $G(e^{j\omega})$ as shown below.

Proof :

$$G(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi m)$$

By the definition of inverse Fourier transform,

$$g(n) = \mathcal{F}^{-1}\{G(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{m=-\infty}^{+\infty} 2\pi\,\delta(\omega - \omega_0 - 2\pi m)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{+\pi} \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Note : Here the integral limit is $-\pi$ to $+\pi$, and in this range there is only one impulse located at ω_0.

Consider the Fourier series representation of periodic discrete time signal x(n), shown below.

$$x(n) = \sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}$$

$$\text{where,} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, ..., (N-1) \qquad .....(8.32)$$

$$\omega_k = \frac{2\pi k}{N}$$

On comparing g(n) and x(n), we can say that the Fourier transform of x(n) can be obtained from its Fourier series representation, as shown below.

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\left\{\sum_{k=0}^{N-1} c_k\, e^{j\omega_k n}\right\} = \sum_{k=-\infty}^{+\infty} c_k\, 2\pi\, \delta(\omega - \omega_k) \qquad .....(8.33)$$

The equation (8.33) can be used to compute Fourier transform of any periodic discrete time signal x(n), and the Fourier transform consists of train of impulses located at the harmonic frequencies of x(n).

Table - 8.3 : Some Common Discrete Time Fourier Transform Pairs

x(t)	x(n)	$X(e^{j\omega})$ with positive power of $e^{j\omega}$	$X(e^{j\omega})$ with negative power of $e^{j\omega}$
	$\delta(n)$	1	1
	$\delta(n-n_0)$	$\dfrac{1}{e^{j\omega n_0}}$	$e^{-j\omega n_0}$
	$u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-1}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$
	$a^n\,u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$	$\dfrac{1}{1-a e^{-j\omega}}$
	$n\,a^n\,u(n)$	$\dfrac{a e^{j\omega}}{(e^{j\omega}-a)^2}$	$\dfrac{a e^{-j\omega}}{(1-a\,e^{-j\omega})^2}$
	$n^2\,a^n\,u(n)$	$\dfrac{a e^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$	$\dfrac{a e^{-j\omega}\,(1+a e^{-j\omega})}{(1-a e^{-j\omega})^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)$	$\dfrac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$	$\dfrac{1}{1-e^{-j\omega}e^{-aT}}$
	1	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$	
	$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1-2a\cos\omega+a^2}$	
	$\sum_{m=-\infty}^{+\infty}\delta(n-mN)$	$\dfrac{2\pi}{N}\sum_{m=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi m}{N}\right)$	
$e^{j\Omega_0 t}$	$e^{j\Omega_0 nt}=e^{j\omega_0 n}$ where, $\omega_0=\Omega_0 T$	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi m)$	
$\sin\Omega_0 t$	$\sin\Omega_0 nT$ $=\sin\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\dfrac{\pi}{j}\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)-\delta(\omega+\omega_0-2\pi m)\right]$	
$\cos\Omega_0 t$	$\cos\Omega_0 nT$ $=\cos\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\pi\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)+\delta(\omega+\omega_0-2\pi m)\right]$	

8.6 Analysis of LTI Discrete Time System Using Discrete Time Fourier Transform

8.6.1 Transfer Function of LTI Discrete Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI discrete time system in frequency domain.

Let, x(n) = Input to the discrete time system

y(n) = Output of the discrete time system

$\therefore$ $X(e^{j\omega})$ = Fourier transform of x(n)

$Y(e^{j\omega})$ = Fourier transform of y(n)

$$\text{Now, Transfer function} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(8.34)$$

The transfer function of an LTI discrete time system in frequency domain can be obtained from the difference equation governing the input-output relation of the LTI discrete time system given below, (refer chapter-6, equation (6.17)).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

On taking Fourier transform of above equation and rearraging the resultant equation as a ratio of $Y(e^{j\omega})$ and $X(e^{j\omega})$, the transfer function of LTI discrete time system in frequency domain is obtained.

Impulse Response and Transfer Function

Let, x(n) = Input of an LTI discrete time system

y(n) = Output / Response of the LTI discrete time system for the input x(n)

h(n) = Impulse response (i.e., response for impulse input)

Now, the response y(n) of the discrete time system is given by convolution of input and impulse response, (Refer chapter-6, equation (6.33)).

$$\therefore\ y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(8.35)$$

Let, $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$; $\mathcal{F}\{x(n)\} = X(e^{j\omega})$; $\mathcal{F}\{h(n)\} = H(e^{j\omega})$

Now by convolution theorem of Fourier transform,

$$\mathcal{F}\{x(n) * h(n)\} = X(e^{j\omega})\, H(e^{j\omega}) \qquad .....(8.36)$$

Using equation (8.35), the equation (8.36) can be written as,

$$\mathcal{F}\{y(n)\} = X(e^{j\omega})\, H(e^{j\omega})$$

$$\therefore Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$$

$$\boxed{\therefore\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}} \qquad .....(8.37)$$

From equations (8.34) and (8.37) we can say that the ***transfer function*** of a discrete time system in frequency domain is also given by discrete time Fourier transform of impulse response.

8.6.2 Response of LTI Discrete Time System using Discrete Time Fourier Transform

Consider the transfer function of LTI discrete time system.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Now, response in frequency domain, $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$ .....(8.38)

On taking inverse Fourier transform of equation (8.38) we get,

$$y(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\, H(e^{j\omega})\} \qquad .....(8.39)$$

From the equation (8.39) we can say that the output y(n) is given by the inverse Fourier transform of the product of $X(e^{j\omega})$ and $H(e^{j\omega})$.

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (8.39) is the forced response or steady state response of discrete time system.

8.6.3 Frequency Response of LTI Discrete Time System

The output y(n) of LTI system is given by convolution of h(n) and x(n).

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m) \qquad .....(8.40)$$

Consider a special class of input (sinusoidal input),

$\boxed{Ae^{j\omega n} = A(\cos\omega n + j\sin\omega n)}$

$$x(n) = A\, e^{j\omega n} \; ; \quad -\infty < n < \infty \qquad .....(8.41)$$

where, A = Amplitude

ω = Arbitrary frequency in the interval –π to +π.

$$\therefore \; x(n-m) = Ae^{j\omega(n-m)} \qquad .....(8.42)$$

On substituting for x(n–m) from equation (8.42) in equation (8.40) we get,

$$y(n) = \sum_{m=-\infty}^{+\infty} h(m)\, A\, e^{j\omega(n-m)} = \sum_{m=-\infty}^{+\infty} h(m)\, A\, e^{j\omega n}\, e^{-j\omega m}$$

$$= A\, e^{j\omega n} \sum_{m=-\infty}^{+\infty} h(m)\, e^{-j\omega m} \qquad .....(8.43)$$

By the definition of Fourier transform,

$\boxed{\text{Replace n by m.}}$

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{m=-\infty}^{+\infty} h(m)\, e^{-j\omega m} \qquad .....(8.44)$$

Using equations (8.41) and (8.44), the equation (8.43) can be written as,

$$y(n) = x(n)\, H(e^{j\omega}) \qquad .....(8.45)$$

From equation (8.45) we can say that if a complex sinusoidal signal is given as input signal to an LTI system, then the output is a sinusoidal of the same frequency modified by $H(e^{j\omega})$. Hence $H(e^{j\omega})$ is called the ***frequency response*** of the system.

The $H(e^{j\omega})$ produces a change in the amplitude and phase of the input signal. An LTI system is characterized in the frequency domain by its frequency response. The function $H(e^{j\omega})$ is a complex quantity and so it can be expressed as magnitude function and phase function.

$$\therefore H(e^{j\omega}) = |H(e^{j\omega})| \angle H(e^{j\omega})$$

where, $|H(e^{j\omega})|$ = Magnitude function

$\angle H(e^{j\omega})$ = Phase function

The sketch of magnitude function and phase function with respect to ω will give the frequency response graphically.

Let, $H(e^{j\omega}) = H_r(e^{j\omega}) + jH_i(e^{j\omega})$

where, $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

$H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$

The ***magnitude function*** is defined as,

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = [H_r(e^{j\omega}) + jH_i(e^{j\omega})]\,[H_r(e^{j\omega}) - jH_i(e^{j\omega})]$$

where, $H^*(e^{j\omega})$ is complex conjugate of $H(e^{j\omega})$

$$\therefore\ |H(e^{j\omega})|^2 = H_r^2(e^{j\omega}) + H_i^2(e^{j\omega}) \quad \Rightarrow \quad |H(e^{j\omega})| = \sqrt{H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})}$$

The ***phase function*** is defined as,

$$\angle H(e^{j\omega}) = \mathrm{Arg}[H(e^{j\omega})] = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

The drawback in frequency response analysis using Fourier transform is that the frequency response is a continuous function of ω and so it cannot be processed by digital systems. This drawback is overcome in Discrete Fourier Transform (DFT) discussed in chapter-9.

From equation (8.37) we can say that the frequency response $H(e^{j\omega})$ of an LTI system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\text{i.e., Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(8.46)$$

Properties of Frequency Response

1. The frequency response is periodic function of ω with a period of 2π.
2. If h(n) is real, then the magnitude of $H(e^{j\omega})$ is symmetric and phase of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
3. If h(n) is complex, then the real part of $H(e^{j\omega})$ is symmetric and the imaginary part of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
4. The impulse response h(n) is discrete, whereas the frequency response $H(e^{j\omega})$ is continuous function of ω.

8.6.4 Frequency Response of First Order Discrete Time System

A first order discrete time system is characterized by the difference equation,

$$y(n) = x(n) + a\, y(n-1) \qquad .....(8.47)$$

On taking Fourier transform of equation(8.47) we get,

$$Y(e^{j\omega}) = X(e^{j\omega}) + a\, e^{-j\omega}\, Y(e^{j\omega}) \quad \Rightarrow \quad Y(e^{j\omega}) - a\, e^{-j\omega}\, Y(e^{j\omega}) = X(e^{j\omega})$$

$$\therefore Y(e^{j\omega})\,[1 - a\, e^{-j\omega}] = X(e^{j\omega}) \quad \Rightarrow \quad H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a\, e^{-j\omega}} \qquad .....(8.48)$$

The equation(8.48) is the frequency response of first order system. The frequency response can be expressed graphically as two functions: Magnitude function and Phase function.

The magnitude function of $H(e^{j\omega})$ is defined as,

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = \frac{1}{(1 - a\, e^{-j\omega})}\ \frac{1}{(1 - a\, e^{j\omega})} = \frac{1}{1 - a\, e^{j\omega} - a\, e^{-j\omega} + a^2\, e^{-j\omega}\, e^{j\omega}}$$

$$= \frac{1}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} = \frac{1}{1 - 2a\cos\omega + a^2} \qquad .....(8.49)$$

$$\therefore \left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

The phase function of $H(e^{j\omega})$ is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right];\ \text{where } H_r(e^{j\omega}) \text{ is real part and } H_i(e^{j\omega}) \text{ is imaginary part.}$$

To find the real part and imaginary part of $H(e^{j\omega})$, multiply the numerator and denominator of $H(e^{j\omega})$ [equation (8.48)], by the complex conjugate of the denominator as shown below.

$$H(e^{j\omega}) = \frac{1}{1 - a e^{-j\omega}} \times \frac{1 - a\, e^{+j\omega}}{1 - a\, e^{+j\omega}} = \frac{1 - a\, e^{j\omega}}{1 - 2a\cos\omega + a^2} = \frac{1 - a(\cos\omega + j\sin\omega)}{1 - 2a\cos\omega + a^2}$$

$$= \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} + j\,\frac{-a\sin\omega}{1 - 2a\cos\omega + a^2}$$

Using equation (8.49)
$e^{j\omega} = \cos\omega + j\sin\omega$

$$\therefore\ H_r(e^{j\omega}) = \frac{1 - a\cos\omega}{1 - 2a\cos\omega + a^2} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{-a\sin\omega}{1 - 2a\cos\omega + a^2}$$

$$\text{The phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-a\sin\omega}{1 - a\cos\omega}\right]$$

The Magnitude and Phase responses are calculated for a = 0.5, 0.8, –0.5 & –0.8 and tabulated in Table-8.4. Using the calculated values, the $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ are sketched graphically for a = 0.5, 0.8, –0.5 & –0.8 in fig 8.1, 8.2, 8.3 & 8.4 respectively. From the plots it is inferred that the first order system behaves as a lowpass filter when "a" is in the range of "$0 < a < 1$" and behaves as a highpass filter when "a" is in the range of "$-1 < a < 0$".

Table 8.4 : Frequency Response of First Order Discrete Time System

$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{1 - 2a\cos\omega + a^2}}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right)\right]\pi$$

ω	a = 0.5		a = 0.8		a = –0.5		a = –0.8	
	$\|H(e^{j\omega})\|$	$\angle H(e^{j\omega})$	$\|H(e^{j\omega})\|$	$\angle H(e^{j\omega})$	$\|H(e^{j\omega})\|$	$\angle H(e^{j\omega})$	$\|H(e^{j\omega})\|$	$\angle H(e^{j\omega})$
$\frac{-8\pi}{8} = -\pi$	0.667	0	0.556	0	2	0	5	0
$\frac{-7\pi}{8}$	0.678	0.04π	0.566	0.06π	1.751	-0.11π	2.486	-0.28π
$\frac{-6\pi}{8}$	0.715	0.08π	0.601	0.11π	1.357	-0.16π	1.402	-0.29π
$\frac{-5\pi}{8}$	0.783	0.12π	0.666	0.16π	1.074	-0.17π	0.986	-0.26π
$\frac{-4\pi}{8} = \frac{-\pi}{2}$	0.894	0.15π	0.781	0.21π	0.894	-0.15π	0.781	-0.21π
$\frac{-3\pi}{8}$	1.074	0.17π	0.986	0.26π	0.783	-0.12π	0.666	-0.16π
$\frac{-2\pi}{8}$	1.357	0.16π	1.402	0.29π	0.715	-0.08π	0.601	-0.11π
$\frac{-\pi}{8}$	1.751	0.11π	2.486	0.28π	0.678	-0.04π	0.566	-0.06π
0	2	0	5	0	0.667	0	0.556	0
$\frac{\pi}{8}$	1.751	-0.11π	2.486	-0.28π	0.678	0.04π	0.566	0.06π
$\frac{2\pi}{8}$	1.357	-0.16π	1.402	-0.29π	0.715	0.08π	0.601	0.11π
$\frac{3\pi}{8}$	1.074	-0.17π	0.986	-0.26π	0.783	0.12π	0.666	0.16π
$\frac{4\pi}{8} = \frac{\pi}{2}$	0.894	-0.15π	0.781	-0.21π	0.894	0.15π	0.781	0.21π
$\frac{5\pi}{8}$	0.783	-0.12π	0.666	-0.16π	1.074	0.17π	0.986	0.26π
$\frac{6\pi}{8}$	0.715	-0.08π	0.601	-0.11π	1.357	0.16π	1.402	0.29π
$\frac{7\pi}{8}$	0.678	-0.04π	0.566	-0.06π	1.751	0.11π	2.486	0.28π
$\frac{8\pi}{8} = \pi$	0.667	0	0.556	0	2	0	5	0

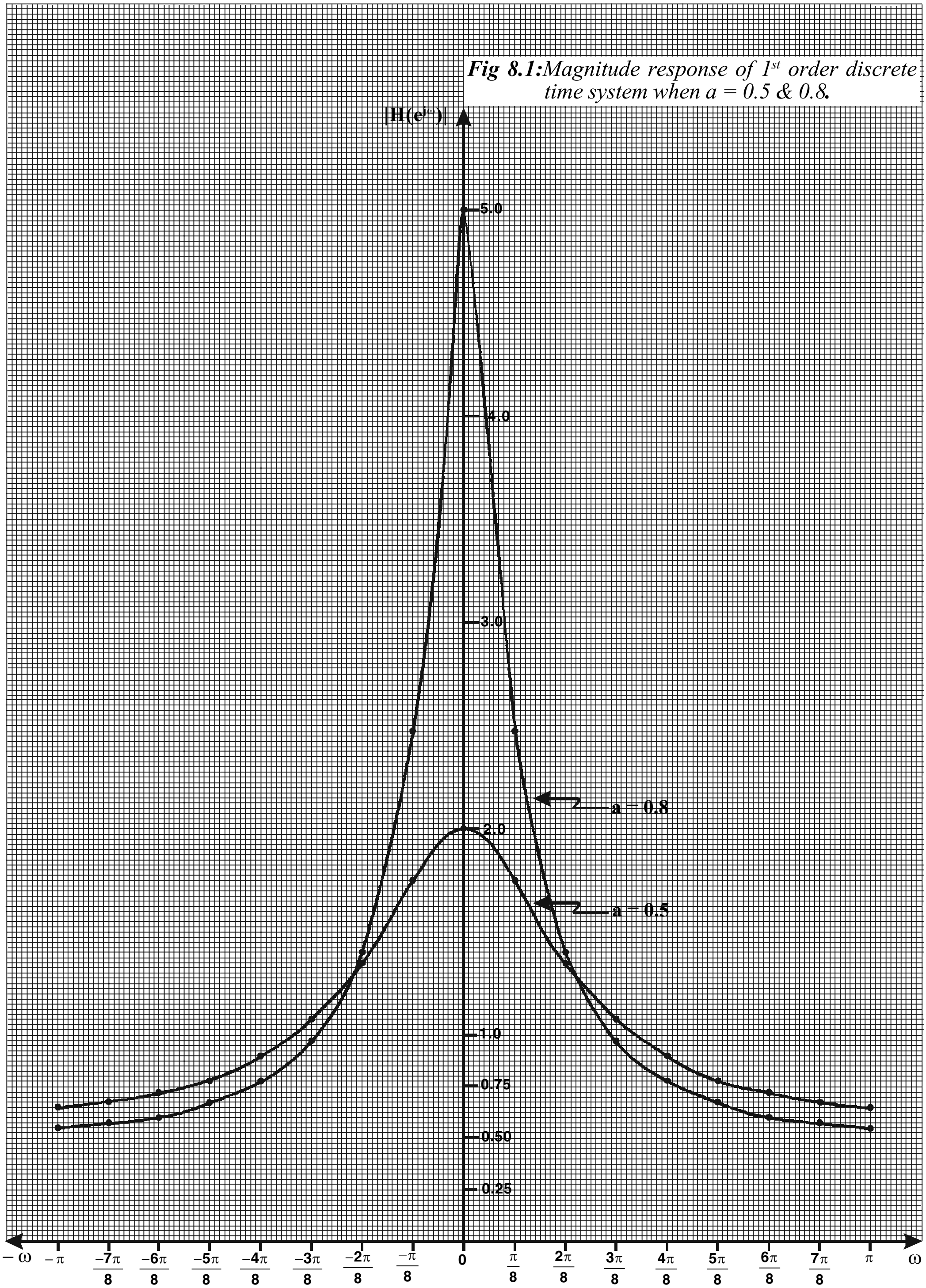


Fig 8.1: *Magnitude response of 1st order discrete time system when a = 0.5 & 0.8.*

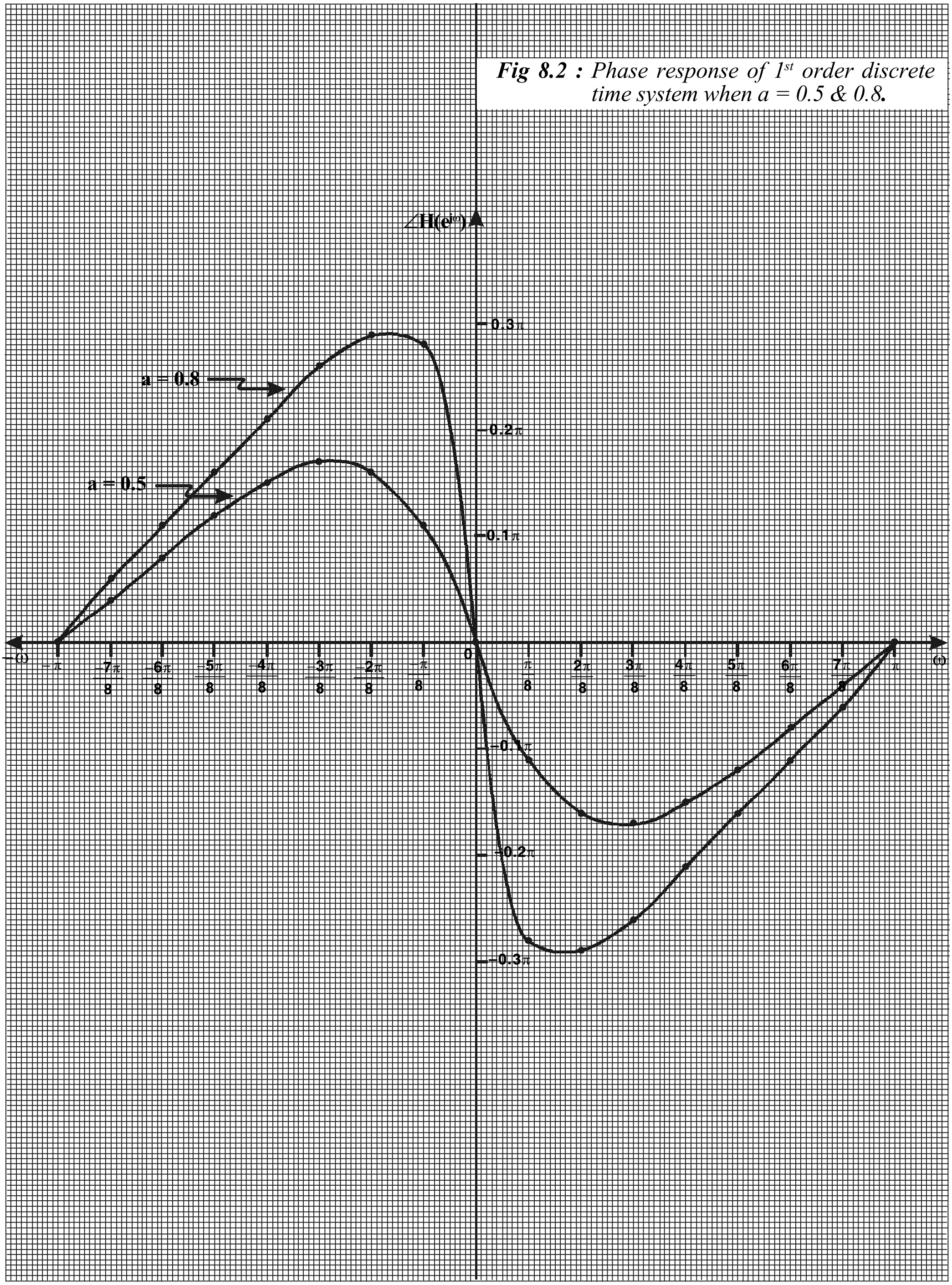


Fig 8.2 : *Phase response of 1st order discrete time system when a = 0.5 & 0.8.*

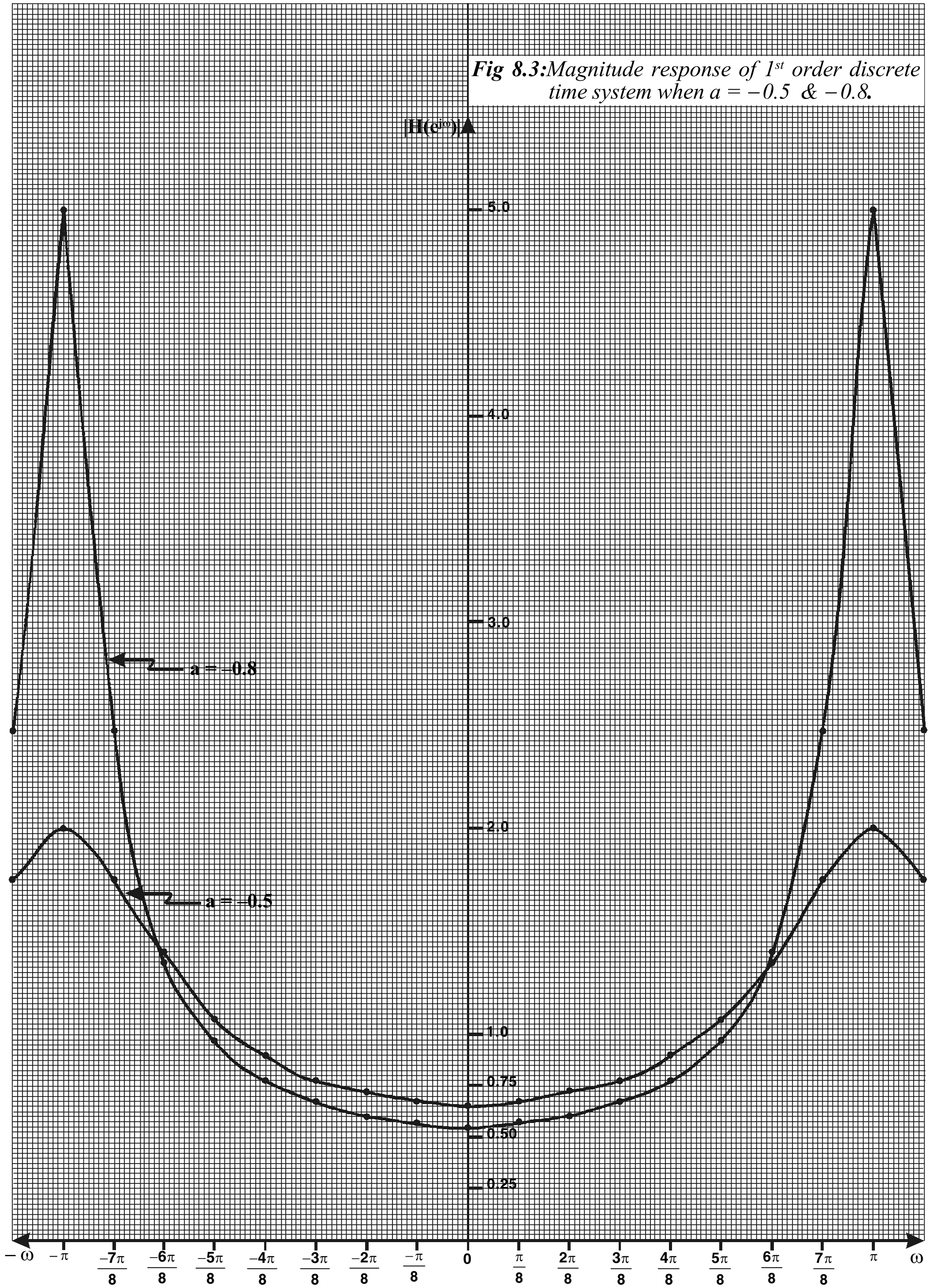


Fig 8.3: Magnitude response of 1st order discrete time system when $a = -0.5$ & -0.8.
$|H(e^{j\omega})|$
5.0
4.0
3.0
2.0
1.0
0.75
0.50
0.25
$a = -0.8$
$a = -0.5$
$-\omega$
$-\pi$
$\frac{-7\pi}{8}$
$\frac{-6\pi}{8}$
$\frac{-5\pi}{8}$
$\frac{-4\pi}{8}$
$\frac{-3\pi}{8}$
$\frac{-2\pi}{8}$
$\frac{-\pi}{8}$
0
$\frac{\pi}{8}$
$\frac{2\pi}{8}$
$\frac{3\pi}{8}$
$\frac{4\pi}{8}$
$\frac{5\pi}{8}$
$\frac{6\pi}{8}$
$\frac{7\pi}{8}$
π
ω

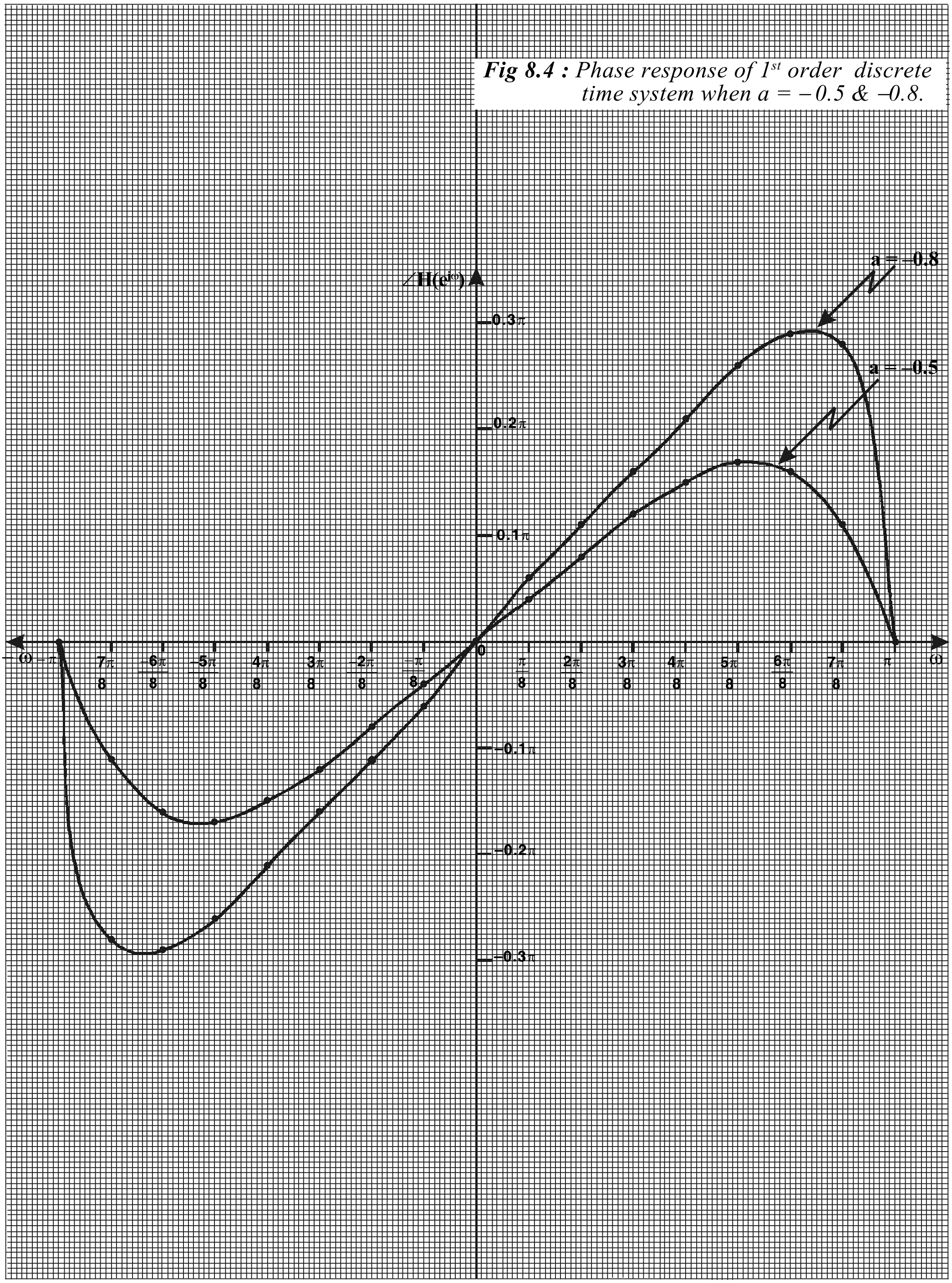


Fig 8.4 : *Phase response of 1st order discrete time system when a = – 0.5 & –0.8.*

8.6.5 Frequency Response of Second Order Discrete Time System

A second order discrete time system is characterized by the difference equation.

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

Let $a = -r\cos\omega_0$; $\alpha = -2r\cos\omega_0$; $\beta = r^2$

$$\therefore\ y(n) = -\alpha\, y(n-1) - \beta\, y(n-2) + x(n) + a\, x(n-1) \qquad .....(8.50)$$

On taking Fourier transform of the equation (8.50) we get,

$$Y(e^{j\omega}) = -\alpha\, e^{-j\omega}\, Y(e^{j\omega}) - \beta\, e^{-j2\omega}\, Y(e^{j\omega}) + X(e^{j\omega}) + a\, e^{-j\omega}\, X(e^{j\omega})$$
$$Y(e^{j\omega}) + \alpha\, e^{-j\omega}\, Y(e^{j\omega}) + \beta\, e^{-j2\omega}\, Y(e^{j\omega}) = X(e^{j\omega}) + a\, e^{-j\omega}\, X(e^{j\omega})$$
$$Y(e^{j\omega})\left[1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}\right] = X(e^{j\omega})\left[1 + a\, e^{-j\omega}\right]$$

$$\therefore\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + a\, e^{-j\omega}}{1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}} \qquad .....(8.51)$$

The equation (8.51) is the frequency response of second order system.The frequency response can be expressed graphically as two functions: Magnitude function and Phase function.

The magnitude function of $H(e^{j\omega})$ is defined as,

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = \frac{1 + a\, e^{-j\omega}}{1 + \alpha e^{-j\omega} + \beta\, e^{-j2\omega}}\ \frac{1 + a\, e^{+j\omega}}{1 + \alpha\, e^{+j\omega} + \beta\, e^{+j2\omega}}$$

$$= \frac{1 + a\, e^{j\omega} + a\, e^{-j\omega} + a^2}{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega} + \alpha\, e^{-j\omega} + \alpha^2 + \alpha\,\beta\, e^{j\omega} + \beta\, e^{-j2\omega} + \alpha\,\beta\, e^{-j\omega} + \beta^2}$$

$$= \frac{1 + a(e^{j\omega} + e^{-j\omega}) + a^2}{1 + \alpha^2 + \beta^2 + \alpha\beta(e^{j\omega} + e^{-j\omega}) + \beta(e^{j2\omega} + e^{-j2\omega}) + \alpha\,(e^{j\omega} + e^{-j\omega})}$$

$$= \frac{1 + 2a\cos\omega + a^2}{1 + \alpha^2 + \beta^2 + 2\alpha\beta\cos\omega + 2\,\beta\cos 2\omega + 2\alpha\cos\omega} \qquad .....(8.52)$$

$$\therefore \text{Magnitude function, } \left|H(e^{j\omega})\right| = \left[\frac{1 + a^2 + 2a\cos\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}\right]^{\frac{1}{2}}$$

The phase function of $H(e^{j\omega})$ is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right];$$ where $H_r(e^{j\omega})$ is real part and $H_i(e^{j\omega})$ is imaginary part.

To find the real part and imaginary part of $H(e^{j\omega})$, multiply the numerator and denominator of $H(e^{j\omega})$ [equation (8.51)], by the complex conjugate of the denominator as shown below.

$$\therefore\ H(\omega) = \frac{1 + a\, e^{-j\omega}}{1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}}\ \frac{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega}}{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega}}$$

$$= \frac{1 + \alpha\, e^{j\omega} + \beta\, e^{j2\omega} + a\, e^{-j\omega} + a\alpha + a\beta\, e^{j\omega}}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

Using equation (8.52)

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$H(e^{j\omega}) = \frac{1 + a\alpha + ae^{-j\omega} + (a\beta + \alpha)\, e^{j\omega} + \beta\, e^{j2\omega}}{1 + \alpha^2 + \beta^2 + 2\alpha\,(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$= \frac{1 + a\alpha + a(\cos\omega - j\sin\omega) + (a\beta + \alpha)\,(\cos\omega + j\sin\omega) + \beta(\cos 2\omega + j\sin 2\omega)}{1 + \alpha^2 + \beta^2 + 2\alpha\,(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\text{The real part, } H_r(e^{j\omega}) = \frac{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\text{The imaginary part, } H_i(e^{j\omega}) = \frac{(\alpha\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}$$

$$\therefore \text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{(\alpha\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}$$

The magnitude and phase response are calculated for r = 0.5 & 0.9 and $\omega_0 = \pi/4$, and tabulated in table 8.5. Using the calculated values, the $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$ are sketched graphically for r = 0.5 & 0.8 and $\omega_0 = \pi/4$ as shown in fig 8.5 . From the plots it can be infered that the second order system behaves as a resonant filter (or bandpass filter). The magnitude response shows a sharp peak close to the frequency $\omega = \omega_0 = \pi/4$, which is called resonant frequency.

Table 8.5 : Frequency Response of Second Order Discrete Time System

$$\left|H(e^{j\omega})\right| = \left(\frac{1 + a^2 + 2a\cos\omega}{1 + \alpha^2 + \beta^2 + 2\alpha(1+\beta)\cos\omega + 2\beta\cos 2\omega}\right)^{1/2}$$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{(\alpha\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}\right) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{(\alpha\beta + \alpha - a)\sin\omega + \beta\sin 2\omega}{1 + a\alpha + (a + a\beta + \alpha)\cos\omega + \beta\cos 2\omega}\right)\right]\pi$$

Case - i

$r = 0.5,\quad \omega_0 = \frac{\pi}{4}$ $\therefore\ a = -r\cos\omega_0 = -0.5\cos\frac{\pi}{4} = -0.3536$ $2a = -0.7072;\ a^2 = 0.125$ $\alpha = -2\,r\cos\omega_0 = -2 \times 0.5\cos\frac{\pi}{4} = -0.7071;\ \alpha^2 = 0.5$ $\beta = r^2 = 0.5^2 = 0.25;\ \beta^2 = 0.0625$	$\left\|H(e^{j\omega})\right\| = \left(\frac{1.125 - 0.7072\cos\omega}{1.5625 - 1.7678\cos\omega + 0.5\cos 2\omega}\right)^{1/2}$ $\angle H(e^{j\omega}) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{-0.5303\sin\omega + 0.25\sin 2\omega}{1.25 - 1.1491\cos\omega + 0.25\cos 2\omega}\right)\right]\pi$

Case - ii

$r = 0.9,\quad \omega_0 = \frac{\pi}{4}$ $\therefore\ a = -r\cos\omega_0 = -0.9\cos\frac{\pi}{4} = -0.6364$ $2a = -1.2728;\ a^2 = 0.405$ $\alpha = -2\,r\cos\omega_0 = -2 \times 0.9\cos\frac{\pi}{4} = -1.2728;\ \alpha^2 = 1.62$ $\beta = r^2 = 0.9^2 = 0.81;\ \beta^2 = 0.6561$	$\left\|H(e^{j\omega})\right\| = \left(\frac{1.405 - 1.2728\cos\omega}{3.2761 - 4.6075\cos\omega + 1.62\cos 2\omega}\right)^{1/2}$ $\angle H(e^{j\omega}) = \left[\frac{1}{\pi}\tan^{-1}\left(\frac{-1.6674\sin\omega + 0.81\sin 2\omega}{1.81 - 2.4247\cos\omega + 0.81\cos 2\omega}\right)\right]\pi$

Table 8.5 : Continued...

ω	r = 0.5		r = 0.9	
	$\lvert H(e^{j\omega})\rvert$	$\angle H(e^{j\omega})$	$\lvert H(e^{j\omega})\rvert$	$\angle H(e^{j\omega})$
$\frac{-8\pi}{8} = -\pi$	0.69	0	0.53	0
$\frac{-7\pi}{8}$	0.71	0.05π	0.55	0.06π
$\frac{-6\pi}{8}$	0.76	0.09π	0.59	0.1π
$\frac{-5\pi}{8}$	0.86	0.13π	0.7	0.17π
$\frac{-4\pi}{8} = \frac{-\pi}{2}$	1.03	0.16π	0.92	0.33π
$\frac{-3\pi}{8}$	1.27	0.15π	1.58	0.79π
$\frac{-2\pi}{8}$	1.41	0.09π	5.28	1.18π
$\frac{-\pi}{8}$	1.29	0.02π	1.18	0.15π
0	1.19	0	0.68	0
$\frac{\pi}{8}$	1.29	-0.02π	1.18	-0.15π
$\frac{2\pi}{8}$	1.41	-0.09π	5.28	-1.18π
$\frac{3\pi}{8}$	1.27	-0.15π	1.58	-0.79π
$\frac{4\pi}{8} = \frac{\pi}{2}$	1.03	-0.16π	0.92	-0.33π
$\frac{5\pi}{8}$	0.86	-0.13π	0.7	-0.17π
$\frac{6\pi}{8}$	0.76	-0.09π	0.59	-0.1π
$\frac{7\pi}{8}$	0.71	-0.05π	0.55	-0.06π
$\frac{8\pi}{8} = \pi$	0.69	0	0.53	0

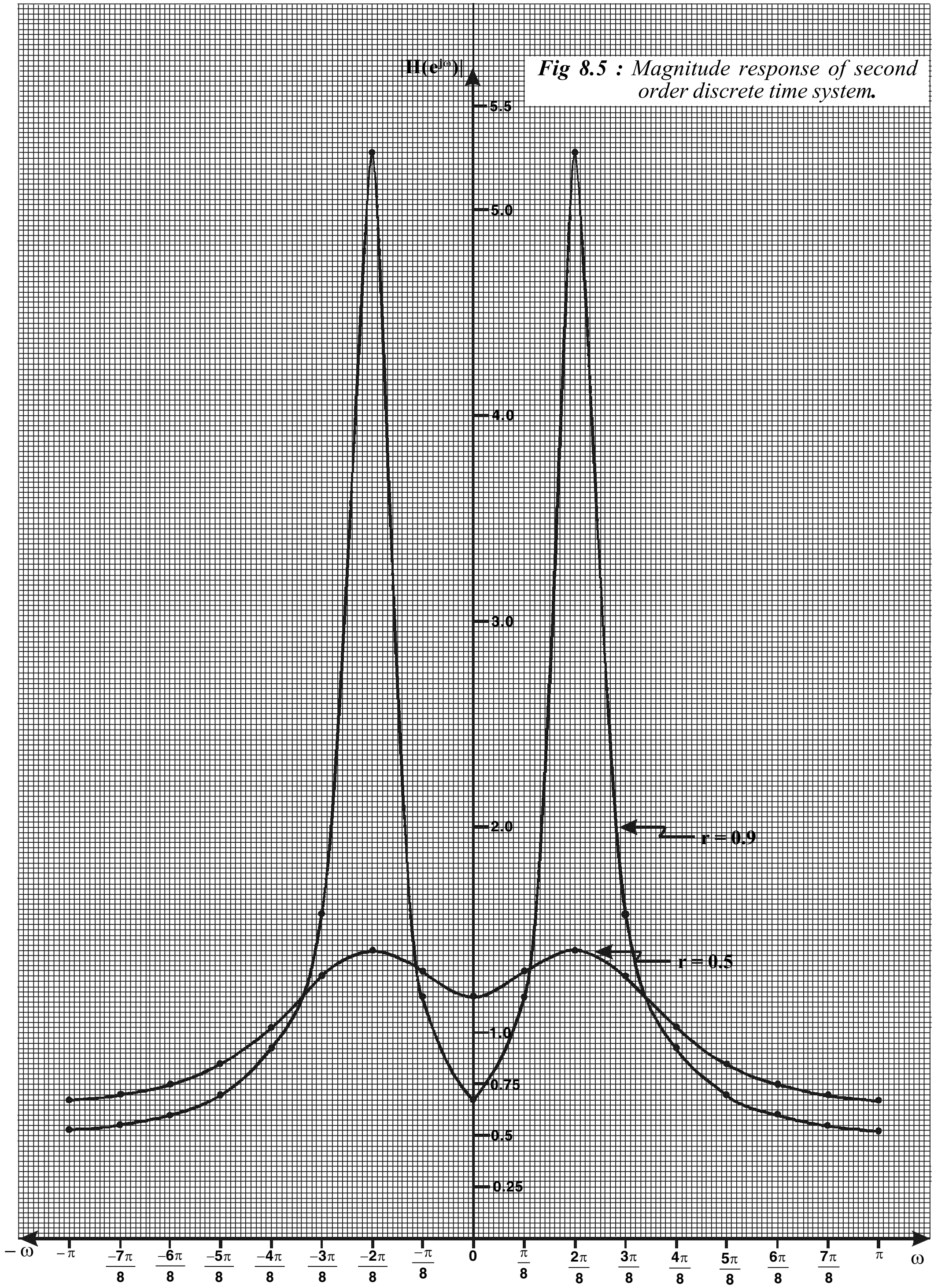


Fig 8.5 : *Magnitude response of second order discrete time system.*

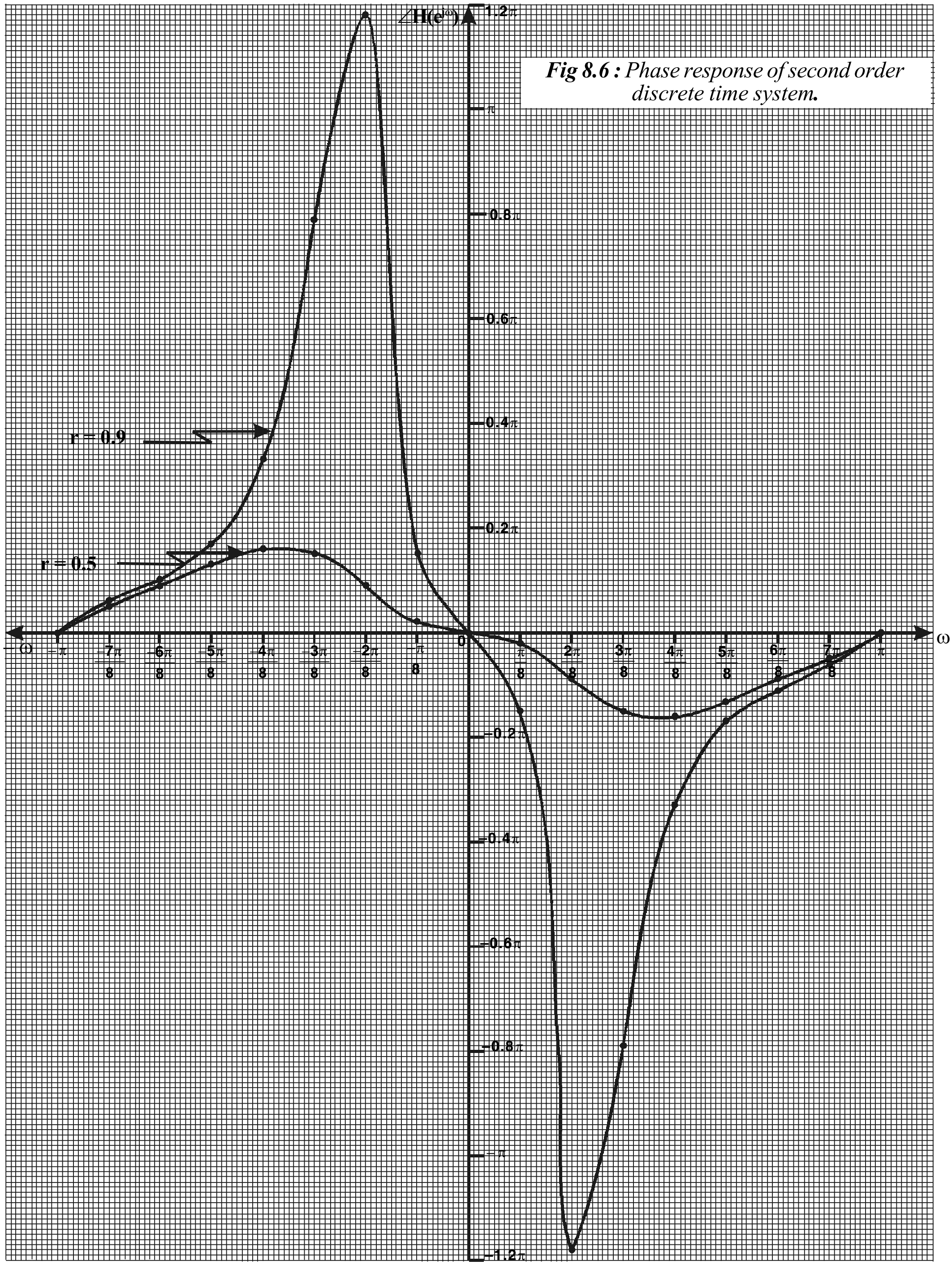


Fig 8.6 : *Phase response of second order discrete time system.*

8.7 Aliasing in Frequency Spectrum Due to Sampling

Let x(t) be an analog signal and X(jΩ) be Fourier transform of x(t).

Now by definition of continuous time inverse Fourier transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega \quad .....(8.53)$$

Let x(nT) be a discrete time signal obtained by sampling x(t) with sampling period, T.

$$\therefore x(nT) = x(t)\Big|_{t = nT}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega\Big|_{t = nT}$$

$\boxed{\text{Using equation (8.53)}}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{j\Omega nT}\, d\Omega$$

$\boxed{\text{Expressing the integration as summation of infinite number of integrals.}}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{\frac{(2m-1)\pi}{T}}^{\frac{(2m+1)\pi}{T}} X\left(j\left(\Omega+\tfrac{2\pi m}{T}\right)\right) e^{j\left(\Omega+\frac{2\pi m}{T}\right)nT}\, d\Omega$$

$\boxed{X(j\Omega) \text{ in the interval } \frac{(2m-1)\pi}{T} \text{ to } \frac{(2m+1)\pi}{T} \text{ is identical with } X(j\Omega) \text{ in the interval } -\frac{\pi}{T} \text{ to } +\frac{\pi}{T}}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\pi/T}^{+\pi/T} X\left(j\left(\Omega+\tfrac{2\pi m}{T}\right)\right) e^{j\Omega nT}\, e^{j2\pi mn} d\Omega$$

$\boxed{\text{Since m and n are integers } e^{j2\pi mn}=1}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\int_{-\pi/T}^{+\pi/T} X\left(j\left(\tfrac{\omega}{T}+\tfrac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega$$

$\boxed{\text{The relation between analog and digital frequency is } \Omega = \frac{\omega}{T}}$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{+\infty}\frac{1}{T}\int_{-\pi}^{\pi} X\left(j\left(\tfrac{\omega}{T}+\tfrac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\tfrac{\omega}{T}+\tfrac{2\pi m}{T}\right)\right) e^{j\omega n}\, d\omega \quad .....(8.54)$$

By the definition of inverse Fourier transform of a discrete time signal, the x(nT) can be written as,

$$x(nT) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad .....(8.55)$$

On comparing equations (8.54) and (8.55) we can write,

$$X(e^{j\omega}) = \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\tfrac{\omega}{T}+\tfrac{2\pi m}{T}\right)\right) \quad .....(8.56)$$

$$= \frac{1}{T}\sum_{m=-\infty}^{+\infty} X\left(j\left(\Omega+\tfrac{2\pi m}{T}\right)\right) \quad .....(8.57)$$

In equation (8.57) if X(jΩ) is the original spectrum of analog signal, then $X\left(j\left(\Omega+\frac{2\pi m}{T}\right)\right)$ is the frequency shifted version of X(jΩ), shifted by $\frac{2\pi m}{T}$. In equation (8.57) the term $\frac{1}{T}$ will scale the amplitude of the spectrum $X\left(j\left(\Omega+\frac{2\pi m}{T}\right)\right)$ by a factor $\frac{1}{T}$.

Therefore from equation (8.57) we can say that $X(e^{j\omega})$ is sum of frequency shifted and amplitude scaled version of $X(j\Omega)$. In general we can say that the *frequency spectrum of a discrete time signal obtained by sampling continuous time signal will be sum of frequency shifted and amplitude scaled spectrum of continuous time signal*. This concept is illustrated in fig 8.7.

The frequency Ω of a continuous time signal can be converted to frequency ω of a discrete time signal by choosing the transformation, $\omega = \Omega T$, where T is the sampling time, $1/T = F_s$ is the sampling cyclic frequency, and $2\pi F_s = \Omega_s$ is the radian sampling frequency. (Refer Chapter-7, section 7.7.3).

In this transformation, the radian frequency ω of sampled version of discrete time signal is unique in the interval $-\pi$ to $+\pi$, and the cyclic frequency f of sampled version of discrete time signal is unique in the interval $-1/2$ to $+1/2$.

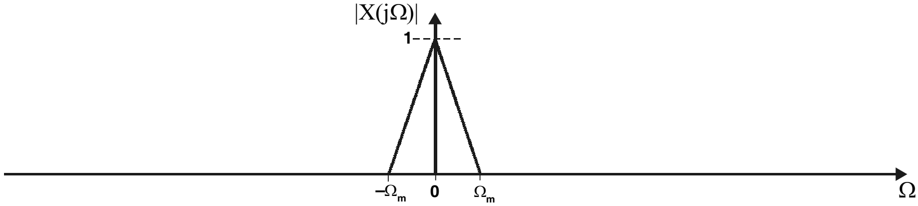


***Fig 8.7a :** Spectrum of a continuous time signal x(t), with maximum frequency Ω_m.*

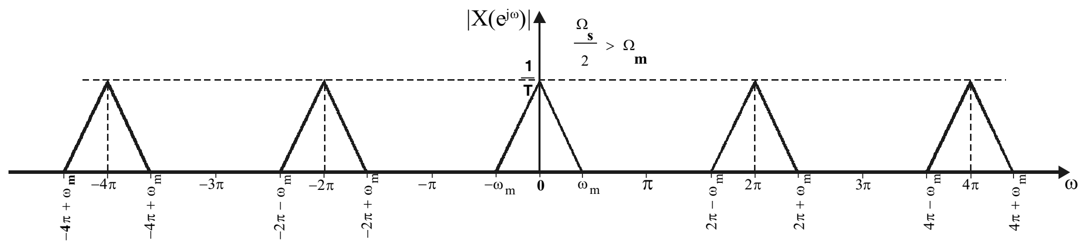


***Fig 8.7b :** Spectrum of sampled version of x(t), with $\Omega_s/2 > \Omega_m$.*

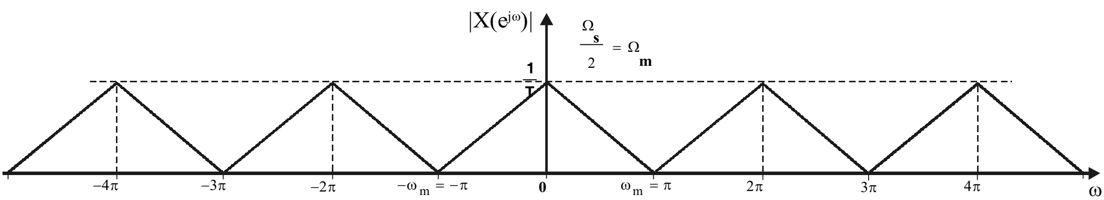


***Fig 8.7c :** Spectrum of sampled version of x(t), with $\Omega_s/2 = \Omega_m$.*

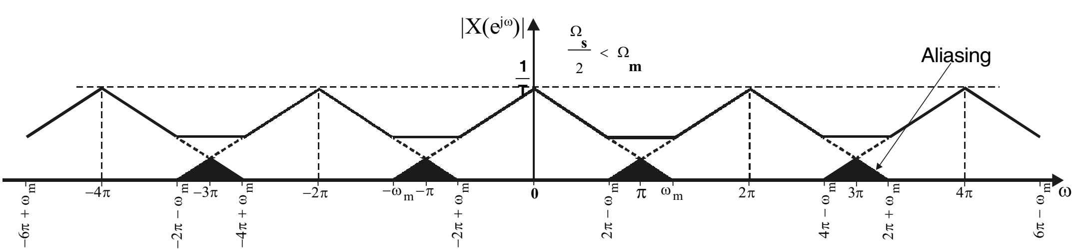


***Fig 8.7d :** Spectrum of sampled version of x(t), with $\Omega_s/2 < \Omega_m$.*

***Fig 8.7 :** Spectrum of a continuous time signal and its sampled version, sampled at various sampling rates.*

The maximum frequency in the spectrum shown in fig 8.7a is Ω_m. Let ω_m be the corresponding maximum frequency of the sampled version of the discrete time signal when the spectrum of fig 8.7a is sampled at a frequency of $\Omega_s/2$. If Ω_m is equal to $\Omega_s/2$, then the corresponding value of ω_m is given by,

$$\omega_m = \Omega_m T = \frac{\Omega_s}{2} T = \frac{2\pi F_s}{2} T = \frac{\pi}{T} T = \pi$$

From the above equation we can say that if Ω_m is less than $\Omega_s/2$, then corresponding ω_m will be less than π and if Ω_m is greater than $\Omega_s/2$ then corresponding ω_m will be greater than π. From fig 8.7b and fig 8.7c it is observed that, as long as Ω_m is less than $\Omega_s/2$, then corresponding ω_m is less than or equal to π, and so there is no overlapping of the components of frequency spectrum.

Form fig 8.7c it is observed, when Ω_m is greater than $\Omega_s/2$, then corresponding ω_m will be greater than π, and so the components of frequency spectrum overlaps. Due to overlap of frequency spectrum, the high frequency components get the identity of low frequency components. This phenomenon is called ***aliasing***. Due to aliasing the information shifts from one band of frequency to another band of frequency.

Therefore in order to avoid aliasing, $\Omega_s/2$, should be greater than or equal to Ω_m.

Since, $\Omega_m = 2\pi F_m$ and $\Omega_s = 2\pi F_s$, to avoid aliasing, $2\pi F_s/2 > 2\pi F_m$

$$\therefore F_s > 2F_m \qquad .....(8.58)$$

Therefore, *in order to avoid aliasing the sampling frequency F_s should be greater than twice the maximum frequency of continuous time signal F_m .*

8.7.1 Signal Reconstruction (Recovery of Continuous Time Signal)

In the above discussion it is observed that, if the sampling frequency $F_s > 2F_m$, then the spectrum $X(e^{j\omega})$ of the sampled continuous time signal will have aliased components of the spectrum $X(j\Omega)$ of original continuous time signal. The aliasing of spectral components prevents the recovery of original signal x(t) from the sampled signal x(n).

When the spectrum of sampled signal has no aliasing then it is possible to recover the original signal from the sampled signal. When there is no aliasing, the spectrum $X(e^{j\omega})$ can be passed through a low pass filter with cut-off frequency, ω_s/π. Now the equation of spectrum $X(e^{j\omega})$ [equation 8.57] can be written as shown below.

$$X(e^{j\omega}) = \frac{1}{T} X(j\Omega) \quad \Rightarrow \quad X(j\Omega) = T\, X(e^{j\omega}) \qquad .....(8.59)$$

On taking inverse Fourier transform of $X(j\Omega)$ we get x(t). Hence by definition of inverse Fourier transform of continuous time signal we get,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

Because $X(j\Omega)$ is zero outside the interval $-\pi/T$ to π/T

$$= \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T\, X(e^{j\omega})\, e^{j\Omega t}\, d\Omega$$

Substituting for $X(j\Omega)$ from equation (8.59).

$$= \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j\omega n}\, e^{j\Omega t}\, d\Omega$$

Using the definition of Fourier transform of discrete time signal.

$$x(t) = \frac{1}{2\pi}\int_{-\pi/T}^{+\pi/T} T \sum_{n=-\infty}^{+\infty} x(nT)\, e^{-j\Omega Tn}\, e^{j\Omega t}\, d\Omega = \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\int_{-\pi/T}^{+\pi/T} e^{j\Omega(t-nT)}\, d\Omega$$

$$= \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\left[\frac{e^{j\Omega(t-nT)}}{j(t-nT)}\right]_{-\pi/T}^{+\pi/T} = \frac{T}{2\pi}\sum_{n=-\infty}^{+\infty} x(nT)\left[\frac{e^{j(\pi/T)(t-nT)}}{j(t-nT)} - \frac{e^{j(-\pi/T)(t-nT)}}{j(t-nT)}\right]$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{1}{(\pi/T)(t-nT)}\left[\frac{e^{j(\pi/T)(t-nT)} - e^{-j(\pi/T)(t-nT)}}{2j}\right]$$

$$= \sum_{n=-\infty}^{+\infty} x(nT)\,\frac{\sin((\pi/T)(t-nT))}{(\pi/T)(t-nT)} \qquad .....(8.60)$$

The equation (8.60) can be used to reconstruct the original continuous time signal x(t) from its samples and the equation (8.60) is also called ***ideal interpolation formula***.

The concepts discussed above are summarized as sampling theorem given below.

Sampling Theorem : *A bandlimited continuous time signal with maximum frequency F_m hertz can be fully recovered from its samples provided that the sampling frequency F_s is greater than or equal to two times the maximum frequency F_m, (.i.e., $F_s \geq 2F_m$).*

8.7.2 Sampling of Bandpass Signal

A continuous time signal is called ***bandpass signal*** if its frequency spectrum lies in a narrow band of frequencies. Let the lower and upper value of this narrow band of frequency be F_1 and F_2 respectively. Now the ***bandwidth***, "B = $F_1 - F_2$". Let F_c be a frequency corresponding to centre of bandwidth. The frequency spectrum of some of the bandpass signals are shown in fig 8.8.

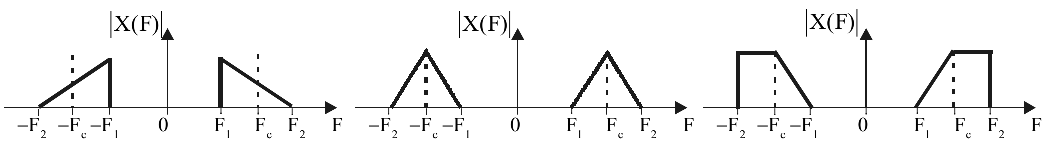


Fig 8.8 : *Sample frequency spectrum of continuous time bandpass signals* **.**

The maximum frequency in the bandpass signal is F_2. According to sampling theorem, to avoid aliasing the bandpass signal has to be sampled at a sampling frequency greater than $2F_2$. When F_2 happens to be a very high frequency, then sampling rate will be very high. In order to avoid high sampling rates the bandpass signals can be shifted in frequency to an equivalent lowpass signal and the equivalent lowpass signal can be sampled at a lower rate.

A bandpass signal can be shifted in frequency by an amount F_c to convert the signal to an equivalent lowpass signal, and when the upper cutoff frequency F_2 is an integer multiple of bandwidth B, then the equivalent lowpass signal can be sampled at a rate of 2B samples per second.When the upper cutoff frequency F_2 is not an integer multiple of bandwidth B, then the sampling rate has to be slighty increased and go upto 4B.

In general, the bandpass signals with a bandwidth of B Hz can be sampled at a rate of 2B to 4B Hz.

8.8 Relation Between Ƶ-Transform and Discrete Time Fourier Transform

The Ƶ-transform of a discrete time signal x(n) is defined as,

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n} \qquad ..... (8.61)$$

where, z is a complex variable (or number)

The Fourier transform of a discrete time signal x(n) is given by,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad ..... (8.62)$$

From equation (8.61) and (8.62) we can say that if we replace **z** by $\mathbf{e^{j\omega}}$ in the Ƶ-transform of x(n) we get Fourier transform of x(n).

The X(z) can be viewed as a unique representation of the signal x(n) in the complex z-plane. In z-plane, the point $z = e^{j\omega}$, represents a point with unit magnitude and having a phase of ω. The range of digital frequency ω is 0 to 2π. Hence we can say that, the points on unit circle in z-plane are given by $z = e^{j\omega}$, when ω is varied from 0 to 2π. Therefore the Fourier transform of a discrete time signal x(n) can be obtained by evaluating the Ƶ- transform on a circle of unit radius as shown in equation (8.63).

$$\therefore\ X(e^{j\omega}) = X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad .....(8.63)$$

It is important to note that X(z) exists for $z = e^{j\omega}$ if unit circle is included in ROC of X(z). Therefore the Fourier transform can be obtained from Ƶ-transform by evaluating X(z) at $z = e^{j\omega}$, if and only if ROC of X(z) includes the unit circle. Fourier transform of some of the common signals that can be obtained from Ƶ-transform are listed in table 8.6.

Table - 8.6 : Some Common Ƶ-transform and Fourier Transform Pairs

x(t)	**x(n)**	**X(z)**	**X(e^jω)**
	$\delta(n)$	1	1
	$a^n\, u(n)\ ;\quad \lvert a\rvert < 1$	$\dfrac{z}{z-a}$	$\dfrac{e^{j\omega}}{e^{j\omega} - a}$
	$n\, a^n\ u(n)\ ;\quad \lvert a\rvert < 1$	$\dfrac{az}{(z-a)^2}$	$\dfrac{a\, e^{j\omega}}{(e^{j\omega} - a)^2}$
	$n^2\, a^n\ u(n)\ ;\quad \lvert a\rvert < 1$	$\dfrac{az\,(z+a)}{(z-a)^3}$	$\dfrac{a\, e^{j\omega}\ (e^{j\omega} + a)}{(e^{j\omega} - a)^3}$
$e^{-at} u(t)$	$e^{-anT} u(nT)\ ;\quad \lvert e^{-aT}\rvert < 1$	$\dfrac{z}{z-e^{-aT}}$	$\dfrac{e^{j\omega}}{e^{j\omega} - e^{-aT}}$
$te^{-at} u(t)$	$nTe^{-anT} u(nT)\ ;\ \lvert e^{-aT}\rvert < 1$	$\dfrac{z\, T\, e^{-aT}}{(z-e^{-aT})^2}$	$\dfrac{e^{j\omega}\ T\, e^{-aT}}{(e^{j\omega} - e^{-aT})^2}$

Example 8.3

Find the Fourier transform of x(n), where $x(n) = 1 \; ; \; 0 \le n \le 4$
$= 0 \; ; \;$ otherwise

Solution

By the definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{4} x(n)\, e^{-j\omega n} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}$$

Using finite geometric series sum formula, $\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$

$$= \frac{1 - e^{\frac{-j5\omega}{2}}\, e^{\frac{-j5\omega}{2}}}{1 - e^{\frac{-j\omega}{2}}\, e^{\frac{-j\omega}{2}}} = \frac{\left(e^{\frac{j5\omega}{2}} - e^{\frac{-j5\omega}{2}}\right) e^{\frac{-j5\omega}{2}}}{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right) e^{\frac{-j\omega}{2}}}$$

$$= \left(\frac{2j \sin\frac{5\omega}{2}}{2j \sin\frac{\omega}{2}}\right) e^{\frac{-j5\omega}{2} + \frac{j\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}\, e^{\frac{-j4\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}\, e^{-j2\omega}$$

$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

Example 8.4

Determine the Fourier transform of the signal $x(n) = a^{|n|} \; ; \; -1 < a < 1$

Solution

The signal x(n) can be expressed as sum of two signals $x_1(n)$ and $x_2(n)$ as shown below.

$\therefore \; x(n) = x_1(n) + x_2(n)$

where, $x_1(n) = a^n \; ; \; n \ge 0$ and $x_2(n) = a^{-n} \; ; \; n < 0$
$= 0 \; ; \; n < 0 \qquad = 0 \; ; \; n \ge 0$

Let, $X_1(e^{j\omega})$ = Fourier transform of $x_1(n)$ and $X_2(e^{j\omega})$ = Fourier transform of $x_2(n)$.

By definition of Fourier transform,

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} a^n\, e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - a\, e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

$(a\, e^{j\omega})^0 = 1$

By definition of Fourier transform,

$$X_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} a^{-n}\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} (a\, e^{j\omega})^{-n} = \sum_{n=1}^{+\infty} (ae^{j\omega})^n = \sum_{n=0}^{+\infty} (ae^{j\omega})^n - 1$$

$$= \frac{1}{1 - a\, e^{j\omega}} - 1 = \frac{1 - 1 + ae^{j\omega}}{1 - a\, e^{j\omega}} = \frac{a\, e^{j\omega}}{1 - a\, e^{j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

Let $X(e^{j\omega})$ = Fourier transform of x(n).

By property of linearity,

$$X(e^{j\omega}) = X_1(e^{j\omega}) + X_2(e^{j\omega}) = \frac{1}{1 - a\, e^{-j\omega}} + \frac{a\, e^{j\omega}}{1 - a\, e^{j\omega}}$$

$$= \frac{1 - a\, e^{j\omega} + a\, e^{j\omega}(1 - a\, e^{-j\omega})}{(1 - a\, e^{-j\omega})\,(1 - a\, e^{j\omega})} = \frac{1 - a\, e^{j\omega} + a\, e^{j\omega} - a^2}{1 - a\, e^{-j\omega} - a\, e^{j\omega} + a^2}$$

$$= \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$

Example 8.5

Find $X(e^{j\omega})$, if $x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n)$

Solution

Given that, $x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] u(n)$; for all n

$$\therefore\ x(n) = \frac{1}{2}\left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n\right] = \frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n \quad ; \text{ for } n \geq 0$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} \left[\frac{1}{2}\left(\frac{1}{2}\right)^n + \frac{1}{2}\left(\frac{1}{4}\right)^n\right] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\omega n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\omega}\right)^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= \frac{1}{2}\,\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \frac{1}{2}\,\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$

$$= \frac{1}{2}\left[\frac{1-\frac{1}{4}e^{-j\omega} + 1 - \frac{1}{2}e^{-j\omega}}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\left[\frac{2-\frac{3}{4}e^{-j\omega}}{\left(1-\frac{1}{4}e^{-j\omega}-\frac{1}{2}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}\right)}\right] = \frac{1-0.375\,e^{-j\omega}}{1-0.75\,e^{-j\omega}+0.125\,e^{-j2\omega}}$$

Example 8.6

Compute the Fourier transform and sketch the magnitude and phase function of causal three sample sequence given by,

$$x(n) = \frac{1}{3}\ ;\ 0 \leq n \leq 2$$
$$= 0\ ;\ \text{else}$$

Solution

Let, $X(e^{j\omega})$ be Fourier transform of x(n).

Now by definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n}$$

$$= x(0)\,e^0 + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega} = \frac{1}{3} + \frac{1}{3}e^{-j\omega} + \frac{1}{3}e^{-j2\omega}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1}{3} + \frac{1}{3}(\cos\omega - j\sin\omega) + \frac{1}{3}(\cos 2\omega - j\sin 2\omega)$$

$$= \frac{1}{3}(1 + \cos\omega + \cos 2\omega) - j\frac{1}{3}(\sin\omega + \sin 2\omega)$$

The $X(e^{j\omega})$ is evaluated for various values of ω and tabulated in table 1. The magnitude and phase of $X(e^{j\omega})$ for various values of ω are also listed in table 1. Using the values listed in table 1, the magnitude and phase function are sketched as shown in fig 1 and fig 2 respectively.

Table 1 : Frequency Response of the System

ω	$X(e^{j\omega})$	$\|X(e^{j\omega})\|$	$\angle X(e^{j\omega})$ in rad
0	$1 + j0 \quad = 1\angle 0$	1	0
$\frac{\pi}{8}$	$0.877 - j0.363 \quad = 0.949\angle{-0.392} = 0.949\angle{-0.125\pi}$	0.949	-0.125π
$\frac{2\pi}{8}$	$0.569 - j0.569 \quad = 0.805\angle{-0.785} = 0.805\angle{-0.25\pi}$	0.805	-0.25π
$\frac{3\pi}{8}$	$0.225 - j0.544 \quad = 0.587\angle{-1.179} = 0.587\angle{-0.375\pi}$	0.587	-0.375π
$\frac{4\pi}{8} = \frac{\pi}{2}$	$0 - j0.333 \quad = 0.333\angle{-1.571} = 0.333\angle{-0.5\pi}$	0.333	-0.5π
$\frac{5\pi}{8}$	$-0.03 - j0.072 \quad = 0.078\angle{-1.966} = 0.078\angle{-0.625\pi}$	0.078	-0.625π
$\frac{6\pi}{8}$	$0.098 - j0.098 \quad = 0.139\angle{-0.785} = 0.139\angle{-0.25\pi}$	0.139	-0.25π
$\frac{7\pi}{8}$	$0.261 + j0.108 \quad = 0.282\angle 0.392 = 0.282\angle 0.125\pi$	0.282	0.125π
$\frac{8\pi}{8} = \pi$	$0.333 + j0 \quad = 0.333\angle 0 \quad = 0.333\angle 0$	0.333	0
$\frac{9\pi}{8}$	$0.261 - j0.108 \quad = 0.282\angle 0.392 = 0.282\angle{-0.125\pi}$	0.282	-0.125π
$\frac{10\pi}{8}$	$0.098 + j0.098 \quad = 0.139\angle 0.785 = 0.139\angle 0.25\pi$	0.139	0.25π
$\frac{11\pi}{8}$	$-0.03 + j0.072 \quad = 0.078\angle 1.966 = 0.078\angle 0.625\pi$	0.078	0.625π
$\frac{12\pi}{8} = \frac{3\pi}{2}$	$0 + j0.333 \quad = 0.333\angle 1.571 = 0.333\angle 0.5\pi$	0.333	0.5π
$\frac{13\pi}{8}$	$0.225 + j0.544 \quad = 0.589\angle 1.179 = 0.589\angle 0.375\pi$	0.589	0.375π
$\frac{14\pi}{8}$	$0.569 + j0.569 \quad = 0.805\angle 0.785 = 0.805\angle 0.25\pi$	0.805	0.25π
$\frac{15\pi}{8}$	$0.877 + j0.363 \quad = 0.949\angle 0.392 = 0.949\angle 0.125\pi$	0.949	0.125π
$\frac{16\pi}{8} = 2\pi$	$1 + j0 \quad = 1\angle 0$	1	0

Note : *The function $X(e^{j\omega})$ is calculated using complex mode of calculator. The magnitude and phase are calculated using rectangular to polar conversion technique.*

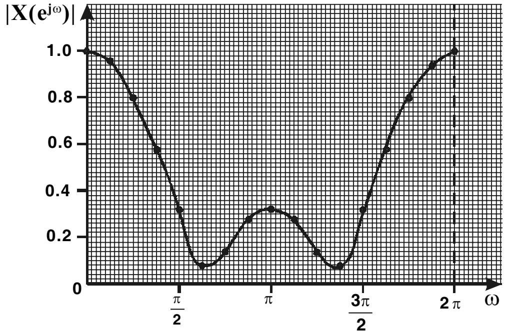


***Fig 1 :** Magnitude of $X(e^{j\omega})$.*

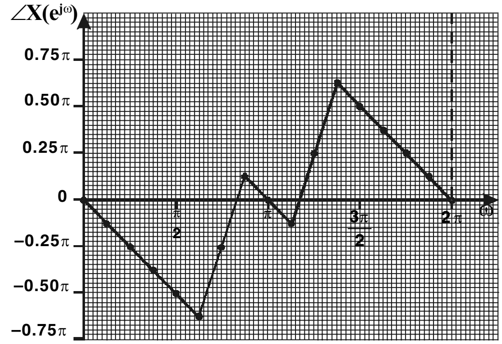


***Fig 2 :** Phase of $X(e^{j\omega})$.*

Example 8.7

Find the convolution of the sequences, $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

Solution

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=-1}^{+1} x_1(n) e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega}$$

Since, $x_1(n) = x_2(n)$, $\quad X_2(e^{j\omega}) = X_1(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega}$

Let, $x(n) = x_1(n) * x_2(n)$, and $X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\{x_1(n) * x_2(n)\}$

By convolution property of Fourier transform.

$$\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\, X_2(e^{j\omega})$$

$$\therefore\ X(e^{j\omega}) = X_1(e^{j\omega})\, X_2(e^{j\omega}) = (e^{j\omega} + 1 + e^{-j\omega})\,(e^{j\omega} + 1 + e^{-j\omega})$$

$$= e^{j2\omega} + e^{j\omega} + 1 + e^{j\omega} + 1 + e^{-j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

$$= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} \qquad .....(1)$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = ...\ x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)$$

$$+ x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + ... \qquad .....(2)$$

On comparing the coefficient of $e^{j\omega n}$ in the two equations of $X(e^{j\omega})$ [equations (1) and (2)] we get,

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

Example 8.8

If $H(e^{j\omega}) = \frac{1}{3}(1 + 2\cos\omega)$, find h(n).

Solution

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Given that, $H(e^{j\omega}) = \frac{1}{3}(1 + 2\cos\omega) = \frac{1}{3}(1 + e^{-j\omega} + e^{j\omega}) \qquad .....(1)$

Let, h(n) = Inverse Fourier transform of $H(e^{j\omega})$.

By definition of Fourier transform we get,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = ... + h(-2)\, e^{j2\omega} + h(-1)\, e^{j\omega} + h(0) + h(1)\, e^{-j\omega} + h(2)\, e^{-j2\omega} + ... \qquad .....(2)$$

On comparing the two expressions for $H(e^{j\omega})$, [equations (1) and (2)] we can say that the samples of h(n) are the coefficients of $e^{-j\omega n}$. Hence by inspection we can write,

$$h(-1) = \frac{1}{3} \;;\quad h(0) = \frac{1}{3} \;;\quad h(1) = \frac{1}{3} \;;\ \text{and}\quad h(n) = 0,\ \text{for } n < -1 \text{ and } n > 1$$

$$\therefore\ h(n) = \frac{1}{3} \quad ;\ \text{for } n = -1, 0, 1$$

$$= 0 \quad ;\ \text{for other } n$$

Example 8.9

Find the inverse Fourier transform of the frequency response of first order system, $H(e^{j\omega}) = (1 - a\,e^{-j\omega})^{-1}$.

Solution

Given that, $H(e^{j\omega}) = (1 - a\,e^{-j\omega})^{-1} = \dfrac{1}{1 - a\,e^{-j\omega}}$

Using Taylor series expansion, the above equation of $H(e^{j\omega})$ can be expanded as shown below.

$$H(e^{j\omega}) = 1 + a\,e^{-j\omega} + a^2\,e^{-j2\omega} + \ldots\ldots + a^k\,e^{-jk\omega} + \ldots\ldots \qquad \ldots..(1)$$

Let, h(n) = Inverse Fourier transform of $H(e^{j\omega})$.

By definition of Fourier transform we get,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n}$$

$$= \ldots.. + h(-2)\,e^{j2\omega} + h(-1)\,e^{j\omega} + h(0) + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + \ldots.. \qquad \ldots..(2)$$

On comparing the two expressions for $H(e^{j\omega})$ [equation (1) and (2)] we can say that the samples of h(n) are the coefficients of $e^{-j\omega n}$.

$$\therefore\ h(n) = \Big\{\underset{\uparrow}{1},\ a,\ a^2, \ldots..,\ a^k, \ldots..\Big\}$$

$$h(n) = \begin{cases} a^n & ;\ n \ge 0 \\ 0 & ;\ n < 0 \end{cases} \quad \Rightarrow \quad h(n) = a^n u(n)$$

Example 8.10

Determine the output sequence from the output spectrum $Y(e^{j\omega})$, where $Y(e^{j\omega}) = \dfrac{1}{3}\,\dfrac{e^{j\omega} + 1 + e^{-j\omega}}{1 - a\,e^{-j\omega}}$

Solution

The output sequence y(n) is obtained by taking inverse Fourier transform of $Y(e^{j\omega})$.

$$Y(e^{j\omega}) = \frac{1}{3}\,\frac{e^{j\omega} + 1 + e^{-j\omega}}{1 - a\,e^{-j\omega}} = \frac{1}{3}\left[\frac{e^{j\omega}}{1 - a\,e^{-j\omega}} + \frac{1}{1 - a\,e^{-j\omega}} + \frac{e^{-j\omega}}{1 - a\,e^{-j\omega}}\right]$$

$$Y(e^{j\omega}) = \frac{1}{3}\left[Y_1(e^{j\omega}) + Y_2(e^{j\omega}) + Y_3(e^{j\omega})\right]$$

$$\text{where, } Y_1(e^{j\omega}) = \frac{e^{j\omega}}{1 - a\,e^{-j\omega}} \;;\quad Y_2(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} \ \text{ and } Y_3(e^{j\omega}) = \frac{e^{-j\omega}}{1 - a\,e^{-j\omega}}$$

Let, $y_1(n) = \mathcal{F}^{-1}\{Y_1(e^{j\omega})\}$; $y_2(n) = \mathcal{F}^{-1}\{Y_2(e^{j\omega})\}$; $y_3(n) = \mathcal{F}^{-1}\{Y_3(e^{j\omega})\}$

By Taylor's series expansion we get,

$$Y_2(e^{j\omega}) = \frac{1}{1 - a\,e^{-j\omega}} = 1 + a\,e^{-j\omega} + a^2 e^{-j2\omega} + a^3\,e^{-j3\omega} + \ldots..$$

$u(n) = 1$ for $n \ge 0$ $= 0$ for $n < 0$

$$= \sum_{n=0}^{+\infty} a^n\,e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} a^n\,u(n)\,e^{-j\omega n} \qquad \ldots..(1)$$

By definition of Fourier transform we can write,

$$Y_2(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y_2(n)\, e^{-j\omega n} \qquad .....(2)$$

By comparing equations (1) and (2) we can write,

$$y_2(n) = a^n u(n)$$

Here, $Y_1(e^{j\omega}) = \dfrac{e^{j\omega}}{1 - a\,e^{-j\omega}} = e^{j\omega}\, Y_2(e^{j\omega})$

$\therefore\ y_1(n) = a^{(n+1)} u(n+1)$ — Using shifting property

Here, $Y_3(e^{j\omega}) = \dfrac{e^{-j\omega}}{1 - a\,e^{-j\omega}} = e^{-j\omega}\, Y_2(e^{j\omega})$

$\therefore\ y_3(n) = a^{(n-1)} u(n-1)$ — Using shifting property

Let, y(n) = Inverse Fourier transform of $Y(e^{j\omega})$.

$$\therefore\ y(n) = \mathcal{F}^{-1}\{Y(e^{j\omega})\} = \mathcal{F}^{-1}\left\{\frac{1}{3}\left[Y_1(e^{j\omega}) + Y_2(e^{j\omega}) + Y_3(e^{j\omega})\right]\right\}$$

$$= \frac{1}{3}\left[\mathcal{F}^{-1}\{Y_1(\omega)\} + \mathcal{F}^{-1}\{Y_2(\omega)\} + \mathcal{F}^{-1}\{Y_3(\omega)\}\right]$$

$$= \frac{1}{3}\left[y_1(n) + y_2(n) + y_3(n)\right]$$

$$= \frac{1}{3}\left[a^{(n+1)} u(n+1) + a^n u(n) + a^{(n-1)} u(n-1)\right]$$

Example 8.11

If $X(e^{j\omega}) = e^{-j1.5\omega}$; $|\,\omega\,| \le 1$

$= 0$; $1 \le \omega \le \pi$, Find x(n) and plot.

Solution

The x(n) is obtained by taking inverse Fourier transform of $X(e^{j\omega})$.

By definition of inverse Fourier transform,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{+1} e^{-j1.5\omega}\ e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{+1} e^{j\omega(n-1.5)}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-1.5)}}{j(n-1.5)}\right]_{-1}^{1} = \frac{1}{j2\pi(n-1.5)}\left[e^{j(n-1.5)} - e^{-j(n-1.5)}\right]$$

$$= \frac{1}{\pi(n-1.5)}\left[\frac{e^{j(n-1.5)} - e^{-j(n-1.5)}}{2j}\right] = \frac{1}{\pi(n-1.5)}\ \sin(n-1.5)$$

$$= \frac{\sin(n-1.5)}{\pi(n-1.5)} \quad ; \quad \text{for all } n$$

The signal x(n) is an infinite duration signal and can be evaluated for all integer values of n in the range $n = -\infty$ to $+\infty$. Here x(n) is evaluated for n = –3 to + 6 and plotted.

$$x(n) = \frac{\sin(n-1.5)}{\pi(n-1.5)}$$

Note : *Evaluate sin(n – 1.5) by keeping calculator in radians mode*

$n = -3\,;\; x(n) = \dfrac{\sin(-3 - 1.5)}{\pi(-3 - 1.5)} = -0.069$

$n = -2\,;\; x(n) = \dfrac{\sin(-2 - 1.5)}{\pi(-2 - 1.5)} = -0.032$

$n = -1\,;\; x(n) = \dfrac{\sin(-1 - 1.5)}{\pi(-1 - 1.5)} = 0.076$

$n = 0\,;\; x(n) = \dfrac{\sin(0 - 1.5)}{\pi(0 - 1.5)} = 0.212$

$n = 1\,;\; x(n) = \dfrac{\sin(1 - 1.5)}{\pi(1 - 1.5)} = 0.305$

$n = 2\,;\; x(n) = \dfrac{\sin(2 - 1.5)}{\pi(2 - 1.5)} = 0.305$

$n = 3\,;\; x(n) = \dfrac{\sin(3 - 1.5)}{\pi(3 - 1.5)} = 0.212$

$n = 4\,;\; x(n) = \dfrac{\sin(4 - 1.5)}{\pi(4 - 1.5)} = 0.076$

$n = 5\,;\; x(n) = \dfrac{\sin(5 - 1.5)}{\pi(5 - 1.5)} = -0.032$

$n = 6\,;\; x(n) = \dfrac{\sin(6 - 1.5)}{\pi(6 - 1.5)} = -0.069$

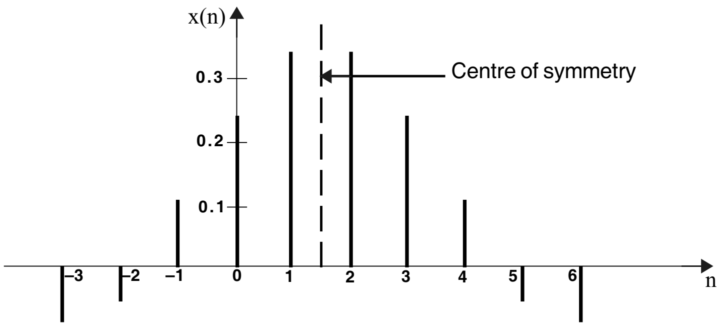


***Fig 1 :** Graphical representation of x(n).*

Here x(n) is a symmetrical signal with centre of symmetry lies between n = 1 and n = 2.

Example 8.12

Find x(n), if $X(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

Solution

Given that, $X(e^{j\omega}) = \dfrac{1}{1 - \frac{1}{2}e^{-j\omega}}$

By Taylor's series expansion we can write,

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}} = 1 + \frac{1}{2}e^{-j\omega} + \left(\frac{1}{2}e^{-j\omega}\right)^2 + \left(\frac{1}{2}e^{-j\omega}\right)^3 + \,.....$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{2}e^{-j\omega}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n e^{-j\omega n} \qquad ..... (1)$$

By definition of Fourier transform we can write,

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\, e^{-j\omega n} \;;\; \text{for } n \geq 0 \qquad ..... (2)$$

On comparing equations (1) and (2) we get,

$$x(n) = \left(\frac{1}{2}\right)^n u(n)\,;\; \text{for all } n \quad \left[\text{or} \quad x(n) = \left(\frac{1}{2}\right)^n \;;\; \text{for } n \geq 0\right]$$

Example 8.13

If $H(e^{j\omega}) = 1 \;;\quad \omega \le \omega_0$

$\quad = 0 \;;\quad |\omega_0| < \omega \le \pi$, Find the impulse response h(n).

Solution

The impulse response h(n) can be obtained by taking inverse Fourier transform of $H(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{j2\pi n}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] = \frac{1}{\pi n}\left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] = \frac{\sin \omega_0 n}{\pi n} \quad \text{except when } n = 0$$

L' Hospitals rule

$$\underset{\theta \to 0}{\mathrm{Lt}} \frac{\sin A\theta}{\theta} = A$$

When n = 0; h(n) can be evaluated using L' Hospitals rule.

$$\text{When } n = 0;\; h(n) = \underset{n \to 0}{\mathrm{Lt}} \frac{\sin\omega_0 n}{\pi n} = \frac{1}{\pi}\underset{n \to 0}{\mathrm{Lt}} \frac{\sin\omega_0 n}{n} = \frac{1}{\pi}\omega_0 = \frac{\omega_0}{\pi}$$

$$\therefore \text{ Impulse response, } h(n) = \frac{\omega_0}{\pi}, \quad \text{when } n = 0$$

$$= \frac{\sin \omega_0 n}{\pi n}, \text{ when } n \ne 0$$

Example 8.14

Find the transfer function of the second order recursive filter in frequency domain whose impulse response is $h(n) = r^n \cos(\omega_0 n)\, u(n)$ for all n.

Solution

The transfer function of a system is the Fourier transform of impulse response.

By definition of Fourier transform,

$u(n) = 1$ for $n \ge 0$
$\quad = 0$ for $n < 0$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{n=0}^{+\infty} r^n \cos\omega_0 n\;\; e^{-j\omega n}$$

$$= \sum_{n=0}^{+\infty} r^n \left[\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}\right] e^{-j\omega n} = \frac{1}{2}\sum_{n=0}^{+\infty}\left[r^n e^{j\omega_0 n} e^{-j\omega n} + r^n e^{-j\omega_0 n} e^{-j\omega n}\right]$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\left[r\, e^{j\omega_0} e^{-j\omega}\right]^n + \frac{1}{2}\sum_{n=0}^{\infty}\left[r\, e^{-j\omega_0} e^{-j\omega}\right]^n$$

For |r| < 1, we can apply the infinite geometric series sum formula to give,

$$H(e^{j\omega}) = \frac{1}{2}\,\frac{1}{1 - r\,e^{j\omega_0} e^{-j\omega}} + \frac{1}{2}\,\frac{1}{1 - r\,e^{-j\omega_0} e^{-j\omega}} = \frac{1}{2}\left[\frac{1 - r\,e^{-j\omega_0} e^{-j\omega} + 1 - r\,e^{j\omega_0} e^{-j\omega}}{\left(1 - r\,e^{j\omega_0} e^{-j\omega}\right)\left(1 - r\,e^{-j\omega_0} e^{-j\omega}\right)}\right]$$

$$= \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\left(e^{-j\omega_0} + e^{j\omega_0}\right)}{1 - r\,e^{-j\omega_0} e^{-j\omega} - r\,e^{j\omega_0} e^{-j\omega} + r^2 e^{-j2\omega}} = \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right)}{1 - r\,e^{-j\omega}\left(e^{j\omega_0} + e^{-j\omega_0}\right) + r^2 e^{-j2\omega}}$$

$$= \frac{1}{2}\,\frac{2 - r\,e^{-j\omega}\, 2\cos\omega_0}{1 - r\,e^{-j\omega}\, 2\cos\omega_0 + r^2 e^{-j2\omega}} = \frac{1 - r\cos\omega_0\, e^{-j\omega}}{1 - 2r\cos\omega_0\, e^{-j\omega} + r^2 e^{-j2\omega}}$$

Example 8.15

Find the output spectrum of an LTI system, if input $x(n) = \frac{1}{3}$; $-1 \le n \le 1$
$= 0$; else

and the impulse response $h(n) = a^n$; $n \ge 1$
$= 0$; else

Solution

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \frac{1}{3}(1 + 2\cos\omega)$$ Refer example 8.8

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \frac{1}{1 - a\,e^{-j\omega}}$$ Refer example 8.4

The output spectrum $Y(e^{j\omega})$ is given by,

$$Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega}) = \frac{1}{3}(1 + 2\cos\omega) \times \frac{1}{1 - a\,e^{-j\omega}} = \frac{1 + 2\cos\omega}{3(1 - a\,e^{-j\omega})}$$

Example 8.16

The impulse response of an LTI system is $h(n) = \{1, 2, 1, -1\}$. Find the response of the system for the input $x(n) = \{1, 2, 3, 1\}$

Solution

The response y(n) of the system is given by convolution of x(n) and h(n).

$$\therefore\ y(n) = x(n) * h(n) \qquad ..... (1)$$

By convolution theorem of Fourier transform we get,

$$\mathcal{F}\{x(n) * h(n)\} = X(e^{j\omega})\,H(e^{j\omega}) \qquad ..... (2)$$

From equations (1) and (2) we can write,

$$\mathcal{F}\{y(n)\} = X(e^{j\omega})\,H(e^{j\omega})$$

Let $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$; $\quad \therefore\ Y(e^{j\omega}) = X(e^{j\omega})\,H(e^{j\omega})$

$$\therefore\ y(n) = \mathcal{F}^{-1}\{Y(e^{j\omega})\} = \mathcal{F}^{-1}\{X(e^{j\omega})\,H(e^{j\omega})\}$$

By definition of Fourier transform, we can write

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{3} x(n)\,e^{-j\omega n}$$

$$= x(0)\,e^{0} + x(1)\,e^{-j\omega} + x(2)\,e^{-j2\omega} + x(3)\,e^{-j3\omega}$$

$$= 1 + 2\,e^{-j\omega} + 3\,e^{-j2\omega} + e^{-j3\omega}$$

By definition of Fourier transform we can write,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n} = \sum_{n=0}^{3} h(n)\,e^{-j\omega n}$$

$$= h(0)\,e^{0} + h(1)\,e^{-j\omega} + h(2)\,e^{-j2\omega} + h(3)\,e^{-j3\omega}$$

$$= 1 + 2e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}$$

$$X(e^{j\omega})\,H(e^{j\omega}) = (1 + 2\,e^{-j\omega} + 3\,e^{-j2\omega} + e^{-j3\omega})\,(1 + 2\,e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega})$$

$$= 1 + 2\,e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}$$
$$+ 2\,e^{-j\omega} + 4\,e^{-j2\omega} + 2\,e^{-j3\omega} - 2\,e^{-j4\omega}$$
$$+ 3\,e^{-j2\omega} + 6\,e^{-j3\omega} + 3\,e^{-j4\omega} - 3\,e^{-j5\omega}$$
$$+ e^{-j3\omega} + 2\,e^{-j4\omega} + e^{-j5\omega} - e^{-j6\omega}$$

$$\therefore\; Y(e^{j\omega}) = 1 + 4\,e^{-j\omega} + 8\,e^{-j2\omega} + 8\,e^{-j3\omega} + 3\,e^{-j4\omega} - 2\,e^{-j5\omega} - e^{-j6\omega} \quad \text{..... (3)}$$

By definition of Fourier transform we get,

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y(n)\, e^{-j\omega n}$$

$$= \text{.....}\; y(0)\,e^{0} + y(1)\,e^{-j\omega} + y(2)\,e^{-j2\omega} + y(3)\,e^{-j3\omega} + y(4)\,e^{-j4\omega} + y(5)\,e^{-j5\omega} + y(6)\,e^{-j6\omega} + \text{.....} \quad \text{..... (4)}$$

On comparing equations (3) and (4) we get,

$$y(n) = \{\underset{\uparrow}{1}, 4, 8, 8, 3, -2, -1\}$$

Example 8.17

Determine the impulse response and frequency response of the LTI system defined by, $y(n) = x(n) + b\,y(n-1)$.

Solution

a) Impulse Response

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z), where, $H(z) = \dfrac{Y(z)}{X(z)}$.

Given that, $y(n) = x(n) + b\,y(n-1)$. .....(1)

On taking $\mathcal{Z}$-transform of equation (1) we get,

$$Y(z) = X(z) + b\,z^{-1}\,Y(z) \quad \Rightarrow \quad Y(z) - b\,z^{-1}\,Y(z) = X(z) \quad \Rightarrow \quad Y(z)\,(1 - b\,z^{-1}) = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}} \quad \text{.....(2)}$$

On taking inverse $\mathcal{Z}$-transform of equation (2) we get,

$$h(n) = \mathcal{Z}^{-1}\{H(z)\} = b^n\,u(n)$$

The impulse response, $h(n) = b^n\,u(n)$, for all n.

b) Frequency Response

The frequency response $H(e^{j\omega})$ is obtained by evaluating H(z) when $z = e^{j\omega}$.

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = H(z)\big|_{z = e^{j\omega}} = \left.\frac{1}{1 - bz^{-1}}\right|_{z = e^{j\omega}} = \frac{1}{1 - be^{-j\omega}}$$

The magnitude function of $H(e^{j\omega})$ is defined as,

$$\left|H(e^{j\omega})\right| = \sqrt{H(e^{j\omega})\,H^*(e^{j\omega})}\,,\ \text{where } H^*(e^{j\omega}) = \text{Conjugate of } H(e^{j\omega}).$$

$$\therefore \text{ Magnitude function, } \left|H(e^{j\omega})\right| = \left[\frac{1}{1 - b\,e^{-j\omega}} \times \frac{1}{1 - b\,e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1}{1 - b\,e^{j\omega} - b\,e^{-j\omega} + b^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 + b^2 - b\left(e^{j\omega} + e^{-j\omega}\right)}\right]^{\frac{1}{2}} \quad \frac{1}{\left(1 + b^2 - 2b\,\cos\omega\right)^{\frac{1}{2}}} \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\text{The phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

where, $H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$ and $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

To separate the real parts and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore\ H(e^{j\omega}) = \frac{1}{1-be^{-j\omega}} \times \frac{1-b\,e^{j\omega}}{1-b\,e^{j\omega}} = \frac{1-b\,e^{j\omega}}{1-b\,e^{j\omega}-b\,e^{-j\omega}+b^2}$$

$$= \frac{1-b(\cos\omega + j\sin\omega)}{1+b^2-b\left(e^{j\omega}+e^{-j\omega}\right)} = \frac{1-b\cos\omega - jb\sin\omega}{1+b^2-2b\cos\omega}$$

$$= \frac{1-b\cos\omega}{1+b^2-2b\cos\omega} - j\frac{b\sin\omega}{1+b^2-2b\cos\omega}$$

$$\therefore\ H_i(e^{j\omega}) = \frac{-b\sin\omega}{1+b^2-2b\cos\omega} \quad \text{and} \quad H_r(e^{j\omega}) = \frac{1-b\cos\omega}{1+b^2-2b\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-b\sin\omega}{1-b\cos\omega}\right]$$

Example 8.18

The impulse response of an LTI system is given by $h(n) = 0.6^n\,u(n)$. Find the frequency response.

Solution

The frequency response $H(e^{j\omega})$ is obtained by taking Fourier transform of the impulse response $h(n)$.

Given that, impulse response, $h(n) = 0.6^n\,u(n)$ for all n.

On taking Fourier transform we get,

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} 0.6^n\,u(n)\,e^{-j\omega n} = \sum_{n=0}^{\infty} 0.6^n\,e^{-j\omega n} = \sum_{n=0}^{\infty}\left(0.6\,e^{-j\omega}\right)^n$$

$$= \frac{1}{1-0.6\,e^{-j\omega}}$$

$u(n) = 1;\ n \ge 0$
$= 0;\ n < 0$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $|C| < 1$

Here $H(e^{j\omega})$ is a complex function of ω. To separate real and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore\ H(e^{j\omega}) = \frac{1}{1-0.6\,e^{-j\omega}} \times \frac{1-0.6\,e^{j\omega}}{1-0.6\,e^{j\omega}}$$

$$= \frac{1-0.6\,e^{j\omega}}{1-0.6\,e^{j\omega}-0.6\,e^{-j\omega}+0.36} = \frac{1-0.6(\cos\omega + j\sin\omega)}{1-0.6\left(e^{j\omega}+e^{-j\omega}\right)+0.36}$$

$$= \frac{1-0.6\cos\omega - j0.6\sin\omega}{1.36-1.2\cos\omega} = \frac{1-0.6\cos\omega}{1.36-1.2\cos\omega} - j\left(\frac{0.6\sin\omega}{1.36-1.2\cos\omega}\right)$$

The magnitude function of $H(e^{j\omega})$ is defined as, $\left|H(e^{j\omega})\right| = \left[H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})\right]^{\frac{1}{2}}$

$$\therefore\ \text{Magnitude function, } \left|H(e^{j\omega})\right| = \left[\left(\frac{1-0.6\cos\omega}{1.36-1.2\cos\omega}\right)^2 + \left(\frac{-0.6\sin\omega}{1.36-1.2\cos\omega}\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{(1-0.6\cos\omega)^2 + (-0.6\sin\omega)^2}{(1.36-1.2\cos\omega)^2}\right)\right]^{\frac{1}{2}}$$

$$= \frac{\left(1+0.36\cos^2\omega - 1.2\cos\omega + 0.36\sin^2\omega\right)^{\frac{1}{2}}}{1.36-1.2\cos\omega}$$

$$\therefore \text{ Magnitude function, } \left|H(e^{j\omega})\right| = \frac{\left(1 + 0.36(\sin^2\omega + \cos^2\omega) - 1.2\cos\omega\right)^{\frac{1}{2}}}{1.36 - 1.2\cos\omega}$$

$$= \frac{\left(1 + 0.36 - 1.2\cos\omega\right)^{\frac{1}{2}}}{1.36 - 1.2\cos\omega} = \frac{1}{\left(1.36 - 1.2\cos\omega\right)^{\frac{1}{2}}}$$

The phase function is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-0.6\sin\omega}{1 - 0.6\cos\omega}\right]$$

Example 8.19

A system has impulse response h(n) given by, $h(n) = -0.25\,\delta(n+1) + 0.5\,\delta(n) - 0.25\,\delta(n-1)$.

a) Is the system BIBO stable? b) Is the system causal? Justify your answer. c) Find the frequency response.

Solution

We know that, $\delta(n) = 1$; when $n = 0$
$\qquad\qquad\qquad\quad = 0$; when $n \neq 0$

Let us evaluate h(n) for different values of n.

When $n = -2$; $h(n) = h(-2) = -0.25\,\delta(-2+1) + 0.5\,\delta(-2) - 0.25\,\delta(-2-1)$
$\qquad\qquad\qquad\qquad\qquad = -0.25\,\delta(-1) + 0.5\,\delta(-2) - 0.25\,\delta(-3) = 0 + 0 + 0 = 0$

When $n = -1$; $h(n) = h(-1) = -0.25\,\delta(0) + 0.5\,\delta(-1) - 0.25\,\delta(-2) = -0.25 + 0 + 0 = -0.25$

When $n = 0$; $h(n) = h(0) = -0.25\,\delta(1) + 0.5\,\delta(0) - 0.25\,\delta(-1) = 0 + 0.5 + 0 = 0.5$

When $n = 1$; $h(n) = h(1) = -0.25\,\delta(2) + 0.5\,\delta(1) - 0.25\,\delta(0) = 0 + 0 - 0.25 = -0.25$

When $n = 2$; $h(n) = h(2) = -0.25\,\delta(3) + 0.5\,\delta(2) - 0.25\,\delta(1) = 0 + 0 + 0 = 0$

From the above analysis, we can infer that $h(n) = 0$ for $n < -1$ & $n > 1$, and

$h(-1) = -0.25,\; h(0) = 0.5,\; h(1) = -0.25$

$\therefore$ Impulse response, $h(n) = \{-0.25, \underset{\uparrow}{0.5}, -0.25\}$

a) Check for Stability

For stability of a system, $\displaystyle\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$

$$\sum_{n=-\infty}^{+\infty} |h(n)| = |h(-1)| + |h(0)| + |h(1)| = 0.25 + 0.5 + 0.25 = 1$$

Since $\displaystyle\sum_{n=-\infty}^{+\infty} |h(n)| < \infty$, the system is BIBO stable.

b) Check for Causality

In a causal system the present output should depend only on present and past inputs or outputs, and should not depend on future inputs or outputs. In the given system the response h(n) at any value of n depends on the future input $\delta(n+1)$. Hence the system is noncausal.

c) Frequency Response

The frequency response, $H(e^{j\omega})$ is given by the Fourier transform of h(n).

By definition of Fourier transform,

The frequency response, $$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = h(-1)\, e^{j\omega} + h(0) + h(1)\, e^{-j\omega}$$

$$= -0.25e^{j\omega} + 0.5 - 0.25\, e^{-j\omega} = 0.5 - 0.25(e^{j\omega} + e^{-j\omega})$$

$$= 0.5 - 0.25(2\cos\omega) = 0.5(1 - \cos\omega)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

The frequency response is purely real function of ω.

$\therefore$ Magnitude function, $\left|H(e^{j\omega})\right| = 0.5(1-\cos\omega)$

Phase function, $\angle H(e^{j\omega}) = 0$

Example 8.20

A causal system is represented by the following difference equation.

$$y(n) + \frac{1}{4}\, y(n-1) = x(n) + \frac{1}{2}\, x(n-1)$$

Find the system transfer function H(z), the impulse response and frequency response of the system.

Solution

a) System Transfer Function

The system transfer function, $H(z) = \dfrac{Y(z)}{X(z)}$

Given that, $y(n) + \dfrac{1}{4}\, y(n-1) = x(n) + \dfrac{1}{2}\, x(n-1)$

Let, $\mathcal{Z}\{y(n)\} = Y(z)$, $\quad\therefore\ \mathcal{Z}\{y(n-1)\} = z^{-1}\,Y(z)$

Let, $\mathcal{Z}\{x(n)\} = X(z)$, $\quad\therefore\ \mathcal{Z}\{x(n-1)\} = z^{-1}\,X(z)$

On taking $\mathcal{Z}$-transform of the difference equation governing the system we get,

$$Y(z) + \frac{1}{4}\, z^{-1}\, Y(z) = X(z) + \frac{1}{2}\, z^{-1}\, X(z)$$

$$Y(z)\left(1 + \frac{1}{4}\, z^{-1}\right) = X(z)\left(1 + \frac{1}{2}\, z^{-1}\right)$$

$$\therefore \text{ System transfer function, } H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{2}\, z^{-1}}{1 + \frac{1}{4}\, z^{-1}}$$

b) Impulse Response

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z).

$$H(z) = \frac{1 + \frac{1}{2}\, z^{-1}}{1 + \frac{1}{4}\, z^{-1}} = \frac{z^{-1}\left(z + \frac{1}{2}\right)}{z^{-1}\left(z + \frac{1}{4}\right)} = \frac{z + \frac{1}{2}}{z + \frac{1}{4}}$$

$$= 1 + \frac{\frac{1}{4}}{z + \frac{1}{4}} = 1 + \frac{1}{4} z^{-1}\, \frac{z}{z - \left(-\frac{1}{4}\right)}$$

On dividing numerator by denominator we get,

$$\begin{array}{r|l} & \ \ 1 \\ \hline z + 1/4 & z + 1/2 \\ & z + 1/4 \\ \hline & \quad 1/4 \end{array}$$

$$\therefore \frac{z + \frac{1}{2}}{z + \frac{1}{4}} = 1 + \frac{\frac{1}{4}}{z + \frac{1}{4}}$$

We know that, $\mathcal{Z}\{\delta(n)\} = 1\,;\ \mathcal{Z}\left\{\dfrac{z}{z-a}\right\} = a^n\,u(n)$
If $\mathcal{Z}\left\{\dfrac{z}{z-a}\right\} = a^n\,u(n)$ then by time shifting property, $\mathcal{Z}\left\{z^{-1}\dfrac{z}{z-a}\right\} = a^{(n-1)}\,u(n-1)$

On taking inverse $\mathcal{Z}$-transform of H(z) we get,

$$h(n) = \delta(n) + \frac{1}{4}\left(-\frac{1}{4}\right)^{(n-1)} = \delta(n) - \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right)^{(n-1)} u(n-1) = \delta(n) - \left(-\frac{1}{4}\right)^{n} u(n-1)$$

c) Frequency Response

The frequency response $H(e^{j\omega})$ is the Fourier transform of h(n), or $H(e^{j\omega})$ is obtained by evaluating H(z) at $z = e^{j\omega}$, or $H(e^{j\omega})$ is given by $\dfrac{Y(e^{j\omega})}{X(e^{j\omega})}$.

Method 1

By definition of Fourier transform,

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\,e^{-j\omega n} = \sum_{n=-\infty}^{+\infty}\left(\delta(n) - \left(-\frac{1}{4}\right)^n u(n-1)\right)e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty}\delta(n)\,e^{-j\omega n} - \sum_{n=-\infty}^{+\infty}\left(-\frac{1}{4}\right)^n u(n-1)\,e^{-j\omega n}$$

$$= 1 - \sum_{n=1}^{+\infty}\left(-\frac{1}{4}\right)^n e^{-j\omega n} = 1 - \left(\sum_{n=0}^{+\infty}\left(-\frac{1}{4}\right)^n e^{-j\omega n} - 1\right)$$

$$= 1 - \sum_{n=0}^{+\infty}\left(\left(-\frac{1}{4}\right)e^{-j\omega}\right)^n + 1$$

$\because\ \delta(n) = 1\,;\ n = 0$ $= 0\,;\ n \neq 0$
$u(n-1) = 1\,;\ n \geq 1$ $= 0\,;\ n < 1$

By infinite geometric series sum formula,

$$H(e^{j\omega}) = 2 - \frac{1}{1 - \left(-\frac{1}{4}\right)e^{-j\omega}} = 2 - \frac{1}{1 + \frac{1}{4}\,e^{-j\omega}} = \frac{2\left(1 + \frac{1}{4}\,e^{-j\omega}\right) - 1}{1 + \frac{1}{4}\,e^{-j\omega}}$$

$$= \frac{2 + \frac{1}{2}\,e^{-j\omega} - 1}{1 + \frac{1}{4}\,e^{-j\omega}} = \frac{1 + \frac{1}{2}\,e^{-j\omega}}{1 + \frac{1}{4}\,e^{-j\omega}}$$

Method 2

The frequency response, $H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \left.\dfrac{1 + \frac{1}{2}\,z^{-1}}{1 + \frac{1}{4}\,z^{-1}}\right|_{z=e^{j\omega}} = \dfrac{1 + \frac{1}{2}\,e^{-j\omega}}{1 + \frac{1}{4}\,e^{-j\omega}}$

Method 3

Given that, $y(n) + \dfrac{1}{4}\,y(n-1) = x(n) + \dfrac{1}{2}\,x(n-1)$

On taking Fourier transform,

$$Y(e^{j\omega}) + \frac{1}{4}\,e^{-j\omega}\,Y(e^{j\omega}) = X(e^{j\omega}) + \frac{1}{2}\,e^{-j\omega}\,X(e^{j\omega}) \quad\Rightarrow\quad Y(e^{j\omega})\left[1 + \frac{1}{4}\,e^{-j\omega}\right] = X(e^{j\omega})\left[1 + \frac{1}{2}\,e^{-j\omega}\right]$$

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + \frac{1}{2}\,e^{-j\omega}}{1 + \frac{1}{4}\,e^{-j\omega}}$$

Magnitude and Phase Function

Magnitude function, $\left|H(e^{j\omega})\right| = \left[H(e^{j\omega})\,H^*(e^{j\omega})\right]^{\frac{1}{2}}$, where $H^*(e^{j\omega})$ is conjugate of $H(e^{j\omega})$

$$= \left[\frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}} \times \frac{1+\frac{1}{2}e^{j\omega}}{1+\frac{1}{4}e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1+\frac{1}{2}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{4}}{1+\frac{1}{4}e^{j\omega}+\frac{1}{4}e^{-j\omega}+\frac{1}{16}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1+\frac{1}{2}(e^{j\omega}+e^{-j\omega})+\frac{1}{4}}{1+\frac{1}{4}(e^{j\omega}+e^{-j\omega})+\frac{1}{16}}\right]^{\frac{1}{2}} = \left[\frac{\frac{5}{4}+\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}\right]^{\frac{1}{2}} \qquad \boxed{\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}}$$

The phase function of $H(e^{j\omega})$ is defined as, $\angle H(e^{j\omega}) = \tan^{-1}\dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})}$

where, $H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$ and $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$.

In order to separate the real part and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator of $H(e^{j\omega})$ by the conjugate of denominator of $H(e^{j\omega})$.

$$\therefore\ H(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1+\frac{1}{4}e^{-j\omega}} \times \frac{1+\frac{1}{4}e^{j\omega}}{1+\frac{1}{4}e^{j\omega}} = \frac{1+\frac{1}{4}e^{j\omega}+\frac{1}{2}e^{-j\omega}+\frac{1}{8}}{1+\frac{1}{4}e^{j\omega}+\frac{1}{4}e^{-j\omega}+\frac{1}{16}}$$

$$= \frac{\frac{9}{8}+\frac{1}{4}(\cos\omega+j\sin\omega)+\frac{1}{2}(\cos\omega-j\sin\omega)}{\frac{17}{16}+\frac{1}{4}(e^{j\omega}+e^{-j\omega})} \qquad \boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{\frac{9}{8}+\frac{1}{4}\cos\omega+\frac{1}{2}\cos\omega+j\frac{1}{4}\sin\omega-j\frac{1}{2}\sin\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$= \frac{\frac{9}{8}+\frac{3}{4}\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega} + \frac{j\left(-\frac{1}{4}\sin\omega\right)}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{\frac{9}{8}+\frac{3}{4}\cos\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{-\frac{1}{4}\sin\omega}{\frac{17}{16}+\frac{1}{2}\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\frac{-\frac{1}{4}\sin\omega}{\frac{9}{8}+\frac{3}{4}\cos\omega}\right] = \tan^{-1}\left[\frac{-2\sin\omega}{9+6\cos\omega}\right]$$

Example 8.21

Find the frequency response of the LTI system, governed by the difference equation,

$$y(n) - a_1 y(n-1) - a_2 y(n-2) = x(n)$$

Solution

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$, $\therefore\ \mathcal{F}\{y(n-k)\} = e^{-j\omega k}\,Y(e^{j\omega})$

Given that, $y(n) - a_1\, y(n-1) - a_2\, y(n-2) = x(n)$

On taking Fourier transform we get,

$$Y(e^{j\omega}) - a_1 e^{-j\omega} Y(e^{j\omega}) - a_2 e^{-j2\omega} Y(e^{j\omega}) = X(e^{j\omega}) \quad \Rightarrow \quad (1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega}) Y(e^{j\omega}) = X(e^{j\omega})$$

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega}}$$

The magnitude function of $H(e^{j\omega})$ is defined as,

$$|H(e^{j\omega})| = [H(e^{j\omega})\ H^*(e^{j\omega})]^{1/2} \quad ; \text{ where } H^*(e^{j\omega}) \text{ is conjugate of } H(e^{j\omega})$$

$$\therefore |H(e^{j\omega})| = \left[\frac{1}{1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega}} \times \frac{1}{1 - a_1 e^{j\omega} - a_2 e^{j2\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 - a_1 e^{j\omega} - a_2 e^{j2\omega} - a_1 e^{-j\omega} + a_1^2 + a_1 a_2 e^{j\omega} - a_2 e^{-j2\omega} + a_1 a_2 e^{-j\omega} + a_2^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 - a_1 (e^{j\omega} + e^{-j\omega}) - a_2 (e^{j2\omega} + e^{-j2\omega}) + a_1^2 + a_1 a_2 (e^{j\omega} + e^{-j\omega}) + a_2^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 - 2a_1 \cos\omega - 2a_2 \cos 2\omega + a_1^2 + a_2^2 + 2a_1 a_2 \cos\omega}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}\right]^{\frac{1}{2}} \qquad .....(1)$$

The Phase function of $H(e^{j\omega})$ is defined as, $\angle H(e^{j\omega}) = \tan^{-1} \dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})}$

where, $H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$ and $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

To separate the real and imaginary parts, multiply the numerator and denominator of $H(e^{j\omega})$ by the conjugate of the denominator of $H(e^{j\omega})$.

$$\therefore H(e^{j\omega}) = \frac{1}{1 - a_1 e^{-j\omega} - a_2 e^{-j2\omega}} \times \frac{1 - a_1 e^{j\omega} - a_2 e^{j2\omega}}{1 - a_1 e^{j\omega} - a_2 e^{j2\omega}} \qquad .....(2)$$

Using equation (1), the equation (2) can be written as,

$$H(e^{j\omega}) = \frac{1 - a_1 e^{j\omega} - a_2 e^{j2\omega}}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

$$= \frac{1 - a_1(\cos\omega + j\sin\omega) - a_2(\cos 2\omega + j\sin 2\omega)}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

$$= \frac{1 - a_1 \cos\omega - a_2 \cos 2\omega}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

$$+ j\frac{-a_1 \sin\omega - a_2 \sin 2\omega}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

$$\therefore H_r(e^{j\omega}) = \frac{1 - a_1 \cos\omega - a_2 \cos 2\omega}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

$$H_i(\omega) = \frac{-a_1 \sin\omega - a_2 \sin 2\omega}{1 + a_1^2 + a_2^2 + 2a_1(a_2 - 1)\cos\omega - 2a_2 \cos 2\omega}$$

The phase function, $\angle H(e^{j\omega}) = \tan^{-1} \dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\dfrac{-a_1 \sin\omega - a_2 \sin 2\omega}{1 - a_1 \cos\omega - a_2 \cos 2\omega}\right]$

Example 8.22

The impulse response of an LTI system is given by $h(n) = r^n \cos(\omega_0 n)\, u(n)$. Find the frequency response of the system.

Solution

The frequency response $H(e^{j\omega})$ is obtained by taking Fourier transform of h(n). For the solution of $H(e^{j\omega})$ refer example 8.14.

$$\text{Frequency response, } H(e^{j\omega}) = \frac{1 - r\cos\omega_0\, e^{-j\omega}}{1 - 2r\cos\omega_0\, e^{-j\omega} + r^2 e^{-j2\omega}}$$

Let, $-r\cos\omega_0 = a$; $-2r\cos\omega_0 = \alpha$; and $r^2 = \beta$.

$$\therefore\ H(e^{j\omega}) = \frac{1 + a\, e^{-j\omega}}{1 + \alpha\, e^{-j\omega} + \beta\, e^{-j2\omega}}$$

The function $H(e^{j\omega})$ is same as frequency response of standard second order system. Hence refer section 8.6.5.

Example 8.23

An LTI system is described by the difference equation, $y(n) = ay(n-1) + bx(n)$. Find the impulse response, magnitude function and phase function. Solve b, if $|H(e^{j\omega})| = 1$. Sketch the magnitude and phase response for a = 0.9

Solution

a) To Find Impulse Response

Let, $\mathcal{Z}\{x(n)\} = X(z)$, $\mathcal{Z}\{y(n)\} = Y(z)$, $\quad \therefore \mathcal{Z}\{y(n-1)\} = z^{-1}Y(z)$.

Given that, $y(n) = ay(n-1) + bx(n)$.

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = az^{-1}Y(z) + b\,X(z) \quad\Rightarrow\quad Y(z) - az^{-1}Y(z) = b\,X(z) \quad\Rightarrow\quad (1 - az^{-1})\,Y(z) = b\,X(z)$$

$$\therefore\ H(z) = \frac{Y(z)}{X(z)} = \frac{b}{1 - az^{-1}} = \frac{b}{z^{-1}(z - a)} = \frac{bz}{z - a}$$

The impulse response is obtained by taking inverse $\mathcal{Z}$-transform of H(z).

$$\therefore\ \text{Impulse response, } h(n) = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{bz}{z - a}\right\} = b\,\mathcal{Z}^{-1}\left\{\frac{z}{z - a}\right\} = b\,a^n\,u(n); \quad \text{for all } n$$

$$\text{or}\quad h(n) = b\,a^n; \quad \text{for } n \geq 0$$

b) To Find Frequency Response

The frequency response $H(e^{j\omega})$ is obtained by evaluating H(z) at, $z = e^{j\omega}$.

$$\therefore\ \text{Frequency response, } H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}} = \frac{b}{1 - az^{-1}}\Bigg|_{z = e^{j\omega}} = \frac{b}{1 - a\,e^{-j\omega}}$$

$$\text{Magnitude function, } |H(e^{j\omega})| = \left[H(e^{j\omega}) \times H^*(e^{j\omega})\right]^{\frac{1}{2}} = \left[\frac{b}{1 - a\,e^{-j\omega}} \times \frac{b}{1 - a\,e^{j\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{b^2}{1 - ae^{j\omega} - ae^{-j\omega} + a^2}\right]^{\frac{1}{2}} = \left[\frac{b^2}{1 + a^2 - a\left(e^{j\omega} + e^{-j\omega}\right)}\right]^{\frac{1}{2}}$$

$$= \left[\frac{b^2}{1 + a^2 - 2a\cos\omega}\right]^{\frac{1}{2}} = \frac{b}{\sqrt{1 + a^2 - 2a\cos\omega}}$$

The phase function is defined as,

$$\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]; \quad \text{where, } H_i(e^{j\omega}) \text{ and } H_r(e^{j\omega}) \text{ are imaginary and real parts of } H(e^{j\omega}).$$

To separate real and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator of $H(e^{j\omega})$ by the complex conjugate of the denominator.

$$\therefore H(e^{j\omega}) = \frac{b}{1 - a\,e^{-j\omega}} \times \frac{1 - a\,e^{j\omega}}{1 - a\,e^{j\omega}} = \frac{b - ab\,e^{j\omega}}{1 - a\,e^{j\omega} - a\,e^{-j\omega} + a^2} = \frac{b - ab(\cos\omega + j\sin\omega)}{1 + a^2 - a(e^{j\omega} + e^{-j\omega})}$$

$$= \frac{b - ab\cos\omega - jab\sin\omega}{1 + a^2 - 2a\cos\omega} = \frac{b - ab\cos\omega}{1 + a^2 - 2a\cos\omega} + j\frac{-ab\sin\omega}{1 + a^2 - 2a\cos\omega}$$

$$\therefore H_r(e^{j\omega}) = \frac{b - ab\cos\omega}{1 + a^2 - 2a\cos\omega} \quad \text{and} \quad H_i(e^{j\omega}) = \frac{-ab\sin\omega}{1 + a^2 - 2a\cos\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-ab\sin\omega}{b - ab\cos\omega}\right] = \tan^{-1}\left[\frac{-a\sin\omega}{1 - a\cos\omega}\right]$$

c) To Evaluate b and Sketch Frequency Response

Given that, $|H(e^{j\omega})| = 1$

$$\therefore \frac{b}{\sqrt{1 + a^2 - 2a\cos\omega}} = 1 \quad \text{or} \quad b = \sqrt{1 + a^2 - 2a\cos\omega}$$

$$\text{When } a = 0.9,\ \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-a\sin\omega}{1 - a\cos\omega}\right) = \tan^{-1}\left(\frac{-0.9\sin\omega}{1 - 0.9\cos\omega}\right)$$

The phase function is periodic in the range $-\pi$ to $+\pi$. Hence the phase function is evaluated for various values of ω in the range $-\pi$ to $+\pi$.

$$\text{When } \omega = -\pi;\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin(-\pi)}{1 - 0.9\cos(-\pi)} = 0$$

$$\text{When } \omega = \frac{-3\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{-3\pi}{4}\right)}{1 - 0.9\cos\left(\frac{-3\pi}{4}\right)} = 0.37 \text{ rad} = \frac{0.37}{\pi} \times \pi = 0.12\pi \text{ rad}$$

$$\text{When } \omega = \frac{-\pi}{2};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{-\pi}{2}\right)}{1 - 0.9\cos\left(\frac{-\pi}{2}\right)} = 0.73 \text{ rad} = \frac{0.73}{\pi} \times \pi = 0.23\pi \text{ rad}$$

$$\text{When } \omega = \frac{-\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{-\pi}{4}\right)}{1 - 0.9\cos\left(\frac{-\pi}{4}\right)} = 1.05 \text{ rad} = \frac{1.05}{\pi} \times \pi = 0.33\pi \text{ rad}$$

$$\text{When } \omega = 0;\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin(0)}{1 - 0.9\cos(0)} = 0$$

$$\text{When } \omega = \frac{\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{\pi}{4}\right)}{1 - 0.9\cos\left(\frac{\pi}{4}\right)} = -1.05 \text{ rad} = \frac{-1.05}{\pi} \times \pi = -0.33\pi \text{ rad}$$

$$\text{When } \omega = \frac{\pi}{2};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{\pi}{2}\right)}{1 - 0.9\cos\left(\frac{\pi}{2}\right)} = -0.73 \text{ rad} = \frac{-0.73}{\pi} \times \pi = -0.23\pi \text{ rad}$$

$$\text{When } \omega = \frac{3\pi}{4};\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\left(\frac{3\pi}{4}\right)}{1 - 0.9\cos\left(\frac{3\pi}{4}\right)} = -0.37 \text{ rad} = \frac{-0.37}{\pi} \times \pi = -0.12\pi \text{ rad}$$

$$\text{When } \omega = \pi;\ \angle H(e^{j\omega}) = \tan^{-1}\frac{-0.9\sin\pi}{1 - 0.9\cos\pi} = 0$$

The phase function of fig 2 is sketched using the above calculated values. The magnitude function is a straight line, passing through "1" as shown in fig 1.

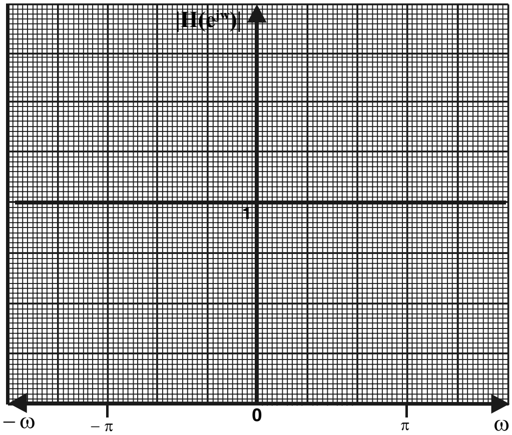


Fig 1 : Magnitude function.

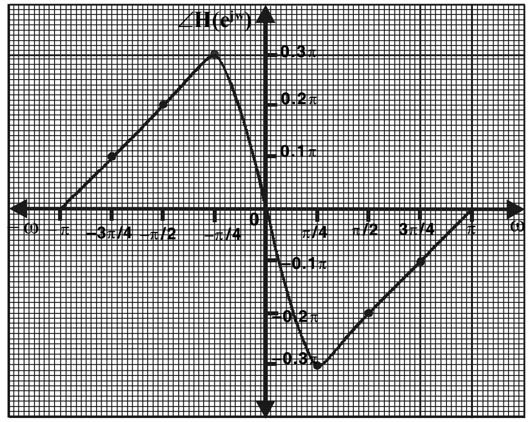


Fig 2 : Phase function.

Example 8.24

Determine the frequency response of an LTI system governed by the difference equation,

$$y(n) = x(n) + 0.81\,x(n-1) + 0.81\,x(n-2) - 0.45\,y(n-2)$$

Solution

Let, $\mathcal{F}\{y(n)\} = Y(e^{j\omega}) \quad \therefore \; \mathcal{F}\{y(n-k)\} = e^{-j\omega k}\,Y(e^{j\omega})$

Let, $\mathcal{F}\{x(n)\} = X(e^{j\omega}) \quad \therefore \; \mathcal{F}\{x(n-k)\} = e^{-j\omega k}\,X(e^{j\omega})$

Given that, $y(n) = x(n) + 0.81\,x(n-1) + 0.81\,x(n-2) - 0.45\,y(n-2)$

On taking Fourier transform we get,

$$Y(e^{j\omega}) = X(e^{j\omega}) + 0.81\,e^{-j\omega}\,X(e^{j\omega}) + 0.81\,e^{-j2\omega}\,X(e^{j\omega}) - 0.45\,e^{-j2\omega}\,Y(e^{j\omega})$$

$$Y(e^{j\omega}) + 0.45e^{-j2\omega}Y(e^{j\omega}) = X(e^{j\omega}) + 0.81\,e^{-j\omega}\,X(e^{j\omega}) + 0.81\,e^{-j2\omega}\,X(e^{j\omega})$$

$$(1 + 0.45\,e^{-j2\omega})\,Y(e^{j\omega}) = (1 + 0.81e^{-j\omega} + 0.81e^{-j2\omega})\,X(e^{j\omega})$$

$$\therefore \quad \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1 + 0.81\,e^{-j\omega} + 0.81\,e^{-j2\omega}}{1 + 0.45\,e^{-j2\omega}}$$

The frequency response, $H(e^{j\omega}) = \dfrac{Y(e^{j\omega})}{X(e^{j\omega})} = \dfrac{1 + 0.81\,e^{-j\omega} + 0.81\,e^{-j2\omega}}{1 + 0.45\,e^{-j2\omega}}$

Magnitude function, $|H(e^{j\omega})| = \left[H(e^{j\omega})\,H^*(e^{j\omega})\right]^{\frac{1}{2}}$

$$= \left[\frac{1 + 0.81\,e^{-j\omega} + 0.81\,e^{-j2\omega}}{1 + 0.45\,e^{-j2\omega}} \times \frac{1 + 0.81\,e^{j\omega} + 0.81\,e^{j2\omega}}{1 + 0.45\,e^{j2\omega}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{\begin{array}{r}1 + 0.81\,e^{j\omega} + 0.81\,e^{j2\omega} + 0.81\,e^{-j\omega} + 0.81^2 + 0.81^2\,e^{j\omega} \\ + 0.81\,e^{-j2\omega} + 0.81^2\,e^{-j\omega} + 0.81^2\end{array}}{1 + 0.45\,e^{j2\omega} + 0.45\,e^{-j2\omega} + 0.45^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2.31 + 0.81(e^{j\omega} + e^{-j\omega}) + 0.66(e^{j\omega} + e^{-j\omega}) + 0.81(e^{j2\omega} + e^{-j2\omega})}{1.2 + 0.45(e^{j2\omega} + e^{-j2\omega})}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2.31 + 1.62\cos\omega + 1.32\cos\omega + 1.62\cos 2\omega}{1.2 + 0.9\cos 2\omega}\right]^{\frac{1}{2}}$$

$$\therefore\ |H(e^{j\omega})| = \left[\frac{2.31 + 2.94\cos\omega + 1.62\cos 2\omega}{1.2 + 0.9\cos 2\omega}\right]^{\frac{1}{2}} \quad .....(1)$$

Phase function, $\angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$; where, $H_i(e^{j\omega})$ = Imaginary part and $H_r(e^{j\omega})$ = Real part

To separate real part and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator of $H(e^{j\omega})$ by the complex conjugate of $H(e^{j\omega})$.

$$\therefore\ H(e^{j\omega}) = \frac{1 + 0.81e^{-j\omega} + 0.81e^{-j2\omega}}{1 + 0.45e^{-j2\omega}} \times \frac{1 + 0.45e^{j2\omega}}{1 + 0.45e^{j2\omega}} \quad ..... (2)$$

$$= \frac{(1 + 0.81e^{-j\omega} + 0.81e^{-j2\omega})(1 + 0.45e^{j2\omega})}{1.2 + 0.9\cos 2\omega}$$

Using equation (1)

$$= \frac{1 + 0.45e^{j2\omega} + 0.81e^{-j\omega} + 0.36e^{j\omega} + 0.81e^{-j2\omega} + 0.36}{1.2 + 0.9\cos 2\omega}$$

$$= \frac{\begin{array}{c}1.36 + 0.45(\cos 2\omega + j\sin 2\omega) + 0.81(\cos\omega - j\sin\omega) \\ + 0.36(\cos\omega + j\sin\omega) + 0.81(\cos 2\omega - j\sin 2\omega)\end{array}}{1.2 + 0.9\cos 2\omega}$$

$$\therefore\ H_r(e^{j\omega}) = \frac{1.36 + 0.45\cos 2\omega + 0.81\cos\omega + 0.36\cos\omega + 0.81\cos 2\omega}{1.2 + 0.9\cos 2\omega}$$

$$= \frac{1.36 + 1.17\cos\omega + 1.26\cos 2\omega}{1.2 + 0.9\cos 2\omega}$$

$$H_i(e^{j\omega}) = \frac{0.45\sin 2\omega - 0.81\sin\omega + 0.36\sin\omega - 0.81\sin 2\omega}{1.2 + 0.9\cos 2\omega}$$

$$= \frac{-0.45\sin\omega - 0.36\sin 2\omega}{1.2 + 0.9\cos 2\omega}$$

$$\text{Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-0.45\sin\omega - 0.36\sin 2\omega}{1.36 + 1.17\cos\omega + 1.26\cos 2\omega}\right]$$

Example 8.25

The impulse response of system is $h(n) = 1 \ ; \ 0 \le n \le (N-1)$
$= 0 \ ;$ otherwise

Find the transfer function and frequency response.

Solution

The transfer function H(z) is obtained by taking $\mathcal{Z}$-transform of the impulse response,

Using finite geometric series sum formula
$$\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$$

$$\therefore\ \text{Transfer function, } H(z) = \mathcal{Z}\{h(n)\} = \sum_{n=0}^{\infty} h(n)\, z^{-n} = \sum_{n=0}^{N-1} z^{-n} = \frac{1 - \left(z^{-1}\right)^N}{1 - z^{-1}} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

The frequency response $H(e^{j\omega})$ is obtained by evaluating H(z) at $z = e^{j\omega}$.

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \left.\frac{1-z^{-N}}{1-z^{-1}}\right|_{z=e^{j\omega}} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$\text{Magnitude function, } |H(e^{j\omega})| = \left[H(e^{j\omega})\,H^*(e^{j\omega})\right]^{\frac{1}{2}}$$

$$= \left[\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \times \frac{1-e^{j\omega N}}{1-e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1-e^{j\omega N}-e^{-j\omega N}+1}{1-e^{j\omega}-e^{-j\omega}+1}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2-(e^{j\omega N}+e^{-j\omega N})}{2-(e^{j\omega}+e^{-j\omega})}\right]^{\frac{1}{2}} = \left[\frac{2-2\cos\omega N}{2-2\cos\omega}\right]^{\frac{1}{2}} = \left[\frac{1-\cos\omega N}{1-\cos\omega}\right]^{\frac{1}{2}}$$

In order to determine the phase function, the real and imaginary part of $H(e^{j\omega})$ has to be separated.

$$\therefore H(e^{j\omega}) = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \times \frac{1-e^{j\omega}}{1-e^{j\omega}}$$

$$= \frac{1-e^{j\omega}-e^{-j\omega N}+e^{-j\omega N}e^{j\omega}}{1-e^{j\omega}-e^{-j\omega}+1} = \frac{1-e^{j\omega}-e^{-j\omega N}+e^{-j\omega(N-1)}}{2-(e^{j\omega}+e^{-j\omega})}$$

$$= \frac{1-(\cos\omega+j\sin\omega)-(\cos\omega N-j\sin\omega N)+((\cos\omega(N-1)-j\sin\omega(N-1))}{2-2\cos\omega}$$

$$\text{Real part,} \quad H_r(e^{j\omega}) = \frac{1-\cos\omega-\cos\omega N+\cos\omega(N-1)}{2-2\cos\omega}$$

$$\text{Imaginary part,} \quad H_i(e^{j\omega}) = \frac{-\sin\omega+\sin\omega N-\sin\omega(N-1)}{2-2\cos\omega}$$

$$\therefore \text{ Phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right] = \tan^{-1}\left[\frac{-\sin\omega+\sin\omega N-\sin\omega(N-1)}{1-\cos\omega-\cos\omega N+\cos\omega(N-1)}\right]$$

Example 8.26

Consider the analog signal, $x_a(t) = 2\cos 2000\pi t + 5\sin 4000\pi t + 12\cos 12000\pi t$.

a) Determine the Nyquist sampling rate.

b) If the analog signal is sampled at $F_s = 5000$ Hz, determine the discrete time signal obtained by sampling.

Solution

a) To Find Nyquist Sampling Rate

The given analog signal can be written as shown below.

$$x_a(t) = 2\cos 2000\pi t + 5\sin 4000\pi t - 12\cos 12000\pi t = 2\cos 2\pi F_1 t + 5\sin 2\pi F_2 t - 12\cos 2\pi F_3 t$$

$$\text{where, } 2\pi F_1 = 2000\pi \Rightarrow F_1 = 1000\text{Hz}$$

$$2\pi F_2 = 4000\pi \Rightarrow F_2 = 2000\text{Hz}$$

$$2\pi F_3 = 12000\pi \Rightarrow F_3 = 6000\text{Hz}$$

The maximum analog frequency in the given signal, F_{max} is 6000 Hz. The Nyquist sampling rate is twice that of this maximum analog frequency.

$$\therefore \text{ Nyquist sampling rate, } F_s = 2F_{max} = 2 \times 6000 = 12000 \text{ Hz}$$

In order to avoid aliasing the sampling frequency, F_s should be greater than or equal to Nyquist rate.

b) To Determine the Discrete Time Signal Sampled at 5000 Hz

Let $x_a(nT)$ be the discrete time signal obtained by sampling the given analog signal.

$$\therefore\ x_a(nT) = x_a(t)\Big|_{t=nT} = x_a(t)\Big|_{t=\frac{n}{F_s}} = 2\cos\frac{2000\pi n}{F_s} + 5\sin\frac{4000\pi n}{F_s} + 12\cos\frac{12000\pi n}{F_s}$$

$$= 2\cos\frac{2000\pi n}{5000} + 5\sin\frac{4000\pi n}{5000} + 12\cos\frac{12000\pi n}{5000}$$

$$= 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\frac{12\pi n}{5} = 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\left(\frac{2\pi n}{5}+\frac{10\pi n}{5}\right)$$

$$= 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\left(\frac{2\pi n}{5}+2\pi n\right) = 2\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5} + 12\cos\frac{2\pi n}{5}$$

$$= 14\cos\frac{2\pi n}{5} + 5\sin\frac{4\pi n}{5}$$

Comment : *When sampled at 5000 Hz, the component $12\cos 12000\pi t$ is an alias of the component $2\cos 2000\pi t$.*

8.9 Summary of Important Concepts

1. A periodic discrete time signal with a fundamental period N can be decomposed into N harmonically related frequency components.
2. The Fourier series representation can be obtained only for periodic discrete time signals.
3. The Fourier transform technique can be applied to both periodic and nonperiodic discrete time signals.
4. The Fourier coefficients of periodic discrete time signal with period N is also periodic with period N.
5. The Fourier coefficient c_k represents the amplitude and phase associated with the k^{th} frequency component.
6. The frequency range of discrete time signal is 0 to 2π (or $-\pi$ to $+\pi$) and so it has finite frequency spectrum.
7. The plot of harmonic magnitude / phase of a discrete time signal versus "k" (or harmonic frequency ω_k) is called Frequency spectrum.
8. The plot of harmonic magnitude versus "k" (or ω_k) is called magnitude spectrum.
9. The plot of harmonic phase versus "k" (or ω_k) is called phase spectrum.
10. The sequence $|c_k|^2$ for k = 0, 1, 2,....., (N − 1) is called the power density spectrum (or) power spectral density of the periodic signal.
11. The Fourier transform is also called analysis of discrete time signal x(n).
12. The inverse Fourier transform is also called synthesis of discrete time signal x(n).
13. The Fourier transform exists only for the discrete time signals that are absolutely summable,
14. The Fourier transform of a signal is also called signal spectrum.
15. The Fourier transform of a discrete time signal is periodic with period 2π.
16. The Fourier transform of any periodic discrete time signal consists of train of impulses located at harmonic frequencies of the signal..
17. The ratio of Fourier transform of output and input of an LTI discrete time system is called transfer function of the LTI discrete time system in frequency domain.
18. The frequency domain transfer function is also given by Fourier transform of impulse response.
19. The Fourier transform of impulse response is called frequency response of the system.
20 The frequency response of discrete time system is periodic continuous function of ω with period 2π.
21. The first order discrete time system behaves as either lowpass filter or highpass filter .
22. The second order system behaves as a resonant filter (or bandpass filter).

23. The frequency spectrum of a discrete time signal obtained by sampling continuous time signal will be sum of frequency shifted and amplitude scaled spectrum of continuous time signal.
24. The frequency Ω of a continuous time signal can be converted to frequency ω of a discrete time signal by choosing the transformation, $\omega = \Omega T$, where T is the sampling time.
25. The overlap of frequency spectrum is called aliasing.
26. Due to aliasing the information shifts from one band of frequency to another band of frequency.
27. In order to avoid aliasing, the sampling frequency F_s should be greater than or equal to twice the maximum frequency F_m of continuous time signal .
28. When the spectrum of sampled signal has no aliasing then it is possible to recover the original signal from the sampled signal.
29. The bandpass signals with a bandwidth of B Hz can be sampled at a rate of 2B to 4B Hz.
30. The Fourier transform of a discrete time signal can be obtained by evaluating the $\mathcal{Z}$- transform on a circle of unit radius provided the ROC of $\mathcal{Z}$- transform includes unit circle.

8.10 Short Questions and Answers

Q8.1 ***Find Fourier coefficients of $x(n)$, where $x(n) = \sum_{k=-\infty}^{+\infty} \delta(n-3k)$.***

Solution

Given signal is a periodic impulse signal with impulses located at n = 3k, for integer values of k.

Let, one period of the given signal be $x_1(n)$.

Now, $x_1(n) = \{1, 0, 0\}$, with period N = 3, and with fundamental frequency, $\omega_0 = 2\pi/3$.

The Fourier coefficient c_k is given by,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-jk2\pi n}{N}} = \frac{1}{3}\sum_{n=0}^{2} x_1(n)\, e^{\frac{-jk2\pi n}{3}} = \frac{1}{3}\,x(0) + 0 + 0 = \frac{1}{3}\ ;\ \text{ for all k.}$$

Q8.2 ***Determine the discrete time Fourier series of $x(n) = \cos^2\left(\frac{\pi}{6}n\right)$.***

Solution

Given that, $x(n) = \cos^2\left(\frac{\pi}{6}n\right)$. Let us check, whether the given signal is periodic.

$$x(n+N) = \cos^2\frac{\pi}{6}(n+N) = \left(\cos\left(\frac{\pi n}{6} + \frac{\pi N}{6}\right)\right)^2$$

Since $\cos(\theta + 2\pi M) = \cos\theta$, For periodicity, $\frac{\pi N}{6}$ should be an integral multiple of 2π.

Let, $\frac{\pi N}{6} = M \times 2\pi$, where M and N are integers. $\Rightarrow$ $N = 12\,M$, Let $M = 1$, $\therefore\ N = 12$.

$\therefore$ x(n) is periodic with fundamental period, $N = 12$ and fundamental frequency, $\omega_0 = \frac{2\pi}{N} = \frac{2\pi}{12} = \frac{\pi}{6}$

The Fourier series of x(n) can be obtained from Euler's formula as shown below.

$$x(n) = \cos^2\left(\frac{\pi}{6}n\right) = \left[\cos\left(\frac{\pi n}{6}\right)\right]^2 = \left[\frac{e^{\frac{j\pi n}{6}} + e^{-\frac{j\pi n}{6}}}{2}\right]^2 = \left[\frac{e^{\frac{j\pi n}{6}}}{2} + \frac{e^{-\frac{j\pi n}{6}}}{2}\right]^2$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{4}e^{\frac{j2\pi n}{6}} + \frac{1}{4}e^{\frac{-j2\pi n}{6}} + \frac{1}{2} = \frac{1}{4}e^{\frac{-j2\pi n}{6}} + \frac{1}{2} + \frac{1}{4}e^{\frac{j2\pi n}{6}}$$

$$= \frac{1}{4}e^{-j2\omega_0 n} + \frac{1}{2} + \frac{1}{4}e^{j2\omega_0 n}\quad ;\ \text{where } \omega_0 = \frac{\pi}{6}$$

Q8.3 ***Find the Fourier transform of $x(n) = \{ 2, 1, 2 \}$.***

Solution

By definition of Fourier transform,

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n} = x(0)\, e^{0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega}$$

$$= 2 + e^{-j\omega} + 2\, e^{-j2\omega} = 2\, e^{-j\omega}\,(e^{j\omega} + e^{-j\omega}) + e^{-j\omega}$$

$$= 4\cos\omega\, e^{-j\omega} + e^{-j\omega} = (1 + 4\cos\omega)\, e^{-j\omega}$$

Q8.4 ***Determine the Fourier transform of $x(n) = u(n) - u(n-N)$.***

Solution

$x(n)$ can be expressed as, $x(n) = 1$; for $n = 0$ to $N-1$.

By definition of Fourier transform,

Using finite geometric series sum formula

$$\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{N-1} 1 \times e^{-j\omega n} = \sum_{n=0}^{N-1} \left(e^{-j\omega}\right)^n = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}}$$

$$= \frac{1 - \left(e^{\frac{-j\omega N}{2}}\, e^{\frac{-j\omega N}{2}}\right)}{1 - \left(e^{\frac{-j\omega}{2}}\, e^{\frac{-j\omega}{2}}\right)} = \frac{e^{\frac{-j\omega N}{2}}\left[e^{\frac{j\omega N}{2}} - e^{\frac{-j\omega N}{2}}\right]}{e^{\frac{-j\omega}{2}}\left[e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right]} = e^{-j\omega\left(\frac{N}{2} - \frac{1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right] = e^{-j\omega\left(\frac{N-1}{2}\right)}\left[\frac{\sin\frac{\omega N}{2}}{\sin\frac{\omega}{2}}\right]$$

Q8.5 ***Find the Fourier transform of , $x(n) = -a^n\, u(-n-1)$, where $|a| < 1$.***

Solution

Using finite geometric series sum formula

$$\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$$

By definition of Fourier transform,

$$\boxed{\text{when } n = 0;\ a^{-n}\, e^{j\omega n} = 1}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{-1} -a^n\, e^{-j\omega n} = \sum_{n=1}^{\infty} -a^{-n}\, e^{j\omega n} = 1 - \sum_{n=0}^{\infty} a^{-n}\, e^{j\omega n} = 1 - \sum_{n=0}^{\infty} \left(a^{-1} e^{j\omega}\right)^n$$

$$= 1 - \frac{1}{1 - a^{-1} e^{j\omega}} = 1 - \frac{a}{a - e^{j\omega}} = \frac{a - e^{j\omega} - a}{a - e^{j\omega}} = \frac{-e^{j\omega}}{a - e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega} - a}$$

Q8.6 ***Find the discrete time Fourier transform of the signal , $x(n) = (0.2)^n\, u(n) + (0.2)^{-n}\, u(-n-1)$.***

Solution

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} (0.2)^n\, u(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} (0.2)^{-n}\, u(-n-1)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=-\infty}^{-1} (0.2\, e^{j\omega})^{-n} = \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=1}^{\infty} (0.2\, e^{j\omega})^n$$

$$\boxed{\text{when } n = 0;\ (0.2\, e^{j\omega})^n = 1}$$

$$= \sum_{n=0}^{\infty} (0.2\, e^{-j\omega})^n + \sum_{n=0}^{\infty} (0.2\, e^{j\omega})^n - 1 = \frac{1}{1 - 0.2\, e^{-j\omega}} + \frac{1}{1 - 0.2\, e^{j\omega}} - 1$$

Using infinite geometric series sum formula

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$$

when $|C| < 1$

$$= \frac{1 - 0.2\, e^{j\omega} + 1 - 0.2\, e^{-j\omega} - (1 - 0.2\, e^{-j\omega})(1 - 0.2\, e^{j\omega})}{(1 - 0.2\, e^{-j\omega})(1 - 0.2\, e^{j\omega})}$$

$$= \frac{1 - 0.2\, e^{j\omega} + 1 - 0.2\, e^{-j\omega} - (1 - 0.2\, e^{j\omega} - 0.2\, e^{-j\omega} + 0.04)}{1 - 0.2\, e^{j\omega} - 0.2\, e^{-j\omega} + 0.04}$$

$$= \frac{1 - 0.04}{1 - 0.2\,(e^{j\omega} + e^{-j\omega}) + 0.04} = \frac{0.96}{1.04 - 0.4\cos\omega}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

Q8.7 ***Determine the energy density spectrum of a discrete time signal, $x(n) = a^n\,u(n)$ for $-1 < a < 1$.***

Solution

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)\,e^{-j\omega n} = \sum_{n=0}^{\infty} a^n\,e^{-j\omega n} = \sum_{n=0}^{\infty}\left(a\,e^{-j\omega}\right)^n = \frac{1}{1-a\,e^{-j\omega}}$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$ when $|C| < 1$

Now the energy density spectrum is,

$$\left|X(e^{j\omega})\right|^2 = X(e^{j\omega})\,X^*(e^{j\omega}) = \frac{1}{(1-a\,e^{-j\omega})} \times \frac{1}{(1-a\,e^{j\omega})}$$

$$= \frac{1}{1-a\,e^{j\omega} - a\,e^{-j\omega} + a^2} = \frac{1}{1-a\,(e^{j\omega}+e^{-j\omega}) + a^2} = \frac{1}{1-2a\cos\omega + a^2}$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$

Q8.8 ***Find the inverse Fourier transform of the rectangular pulse spectrum defined as,***

$$X(e^{j\omega}) = 1 \;;\; |\omega| \le W$$
$$= 0 \;;\; W \le |\omega| \le \pi$$

Solution

By definition inverse Fourier transform,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-W}^{W} = \frac{1}{2\pi}\left[\frac{e^{jWn}}{jn} - \frac{e^{-jWn}}{jn}\right] = \frac{1}{\pi n}\left[\frac{e^{jWn}-e^{-jWn}}{2j}\right]$$

$$= \frac{\sin Wn}{\pi n} = \frac{W}{\pi}\,\frac{\sin Wn}{Wn} = \frac{W}{\pi}\,\mathrm{sinc}\,Wn$$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

$\frac{\sin\theta}{\theta} = \mathrm{sinc}\,\theta$

Q8.9 ***Determine the inverse Fourier transform of $X(e^{j\omega}) = 2\pi\,\delta(\omega-\omega_0)$, $|\omega_0| \le \pi$.***

Solution

The inverse Fourier transform of $X(e^{j\omega})$ is,

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\,e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\;\delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega-\omega_0)\,e^{j\omega n}\,d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Note : *Here the integral limit is $-\pi$ to $+\pi$, and in this range there is only one impulse located at ω_0.*

Q8.10 ***A causal discrete time LTI system has a system function $H(z) = \frac{1-2a\,z^{-1}}{2b+z^{-1}}$. Here 'a' is real and $|a| < 1$. Find the value of 'b' so that the frequency response $H(e^{j\omega})$ of the system satisfies the condition $|X(e^{j\omega})| = 1$ for all ω.***

Solution

Given that, $H(z) = \frac{1-2a\,z^{-1}}{2b+z^{-1}} = \frac{1-2a\,z^{-1}}{2b+z^{-1}}$

The frequency response of the system can be obtained by putting, $z = e^{j\omega}$ in H(z).

$$\therefore H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{1-2a\,e^{-j\omega}}{2b+e^{-j\omega}}$$

Here, $\left|H(e^{j\omega})\right| = 1$; $\quad\therefore \left|\frac{1-2a\,e^{-j\omega}}{2b+e^{-j\omega}}\right| = 1 \quad\Rightarrow\quad \left|1-2a\,e^{-j\omega}\right| = \left|2b+e^{-j\omega}\right|$

$$\therefore \quad |1 - 2a\cos\omega + j\,2a\sin\omega| = |2b + \cos\omega - j\sin\omega|$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$(1 - 2a\cos\omega)^2 + (2a\sin\omega)^2 = (2b + \cos\omega)^2 + (\sin\omega)^2$$

$$1 + 4a^2\cos^2\omega - 4a\cos\omega + 4a^2\sin^2\omega = 4b^2 + 4b\cos\omega + \cos^2\omega + \sin^2\omega$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

$$1 + 4a^2 - 4a\cos\omega = 4b^2 + 4b\cos\omega + 1$$

The above equation is true, when $b = -a$.

Hence to satisfy the condition $|H(e^{j\omega})| = 1$ for all ω, $b = -a$.

Q8.11 ***Determine the sampling period for the signal $X(j\Omega) = U(j\Omega + j\Omega_0) - U(j\Omega - j\Omega_0)$, to sample without aliasing.***

Solution

The frequency spectrum of the given signal can be plotted as shown in fig Q8.11.

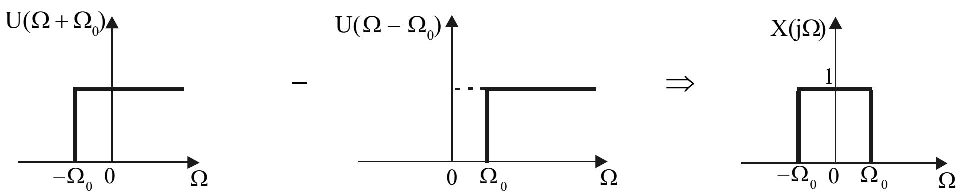


Fig Q8.11.

From the frequency spectrum of fig Q8.11, it is observed that the maximum frequency, Ω_{max} is,

$$\Omega_{max} = \Omega_0\,; \qquad \therefore\ 2\pi\,F_{max} = \Omega_0 \qquad \Rightarrow \qquad F_{max} = \frac{\Omega_0}{2\pi}$$

$\therefore$ Sampling frequency, $F_s \ge 2\,F_{max} \quad \Rightarrow \quad$ Sampling period, $T \le \dfrac{1}{F_s}$

$\therefore$ Minimum sampling period, $T = \dfrac{1}{F_s} = \dfrac{1}{2F_{max}} = \dfrac{\pi}{\Omega_0}$

$\therefore$ In order to avoid aliasing the sampling period, T should be less than $\dfrac{\pi}{\Omega_0}\left(\text{i.e., } T < \dfrac{\pi}{\Omega_0}\right)$.

Q8.12 ***Determine the Nyquist sampling frequency and Nyquist interval for the signal, $x(t) = \left[\dfrac{\sin 200\,\pi\,t}{\pi\,t}\right]^2$.***

Solution

$$x(t) = \left[\frac{\sin 200\pi t}{\pi t}\right]^2 = \frac{1}{\pi^2 t^2}\sin^2(200\pi t)$$

$$= \frac{1}{\pi^2 t^2}\,\frac{[1 - \cos 2(200\pi t)]}{2}$$

$$\boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$= \frac{1}{2\pi^2 t^2}\,[1 - \cos 400\pi t] = \frac{1}{2\pi^2 t^2} - \frac{\cos 400\pi t}{2\pi^2 t^2}$$

From the above analysis it is observed that, the maximum frequency in the signal $F_{max} = 200$ Hz.

$\therefore$ Nyquist rate $= 2\,F_{max} = 2 \times 200 = 400$ Hz

$$\text{Nyquist interval} = \frac{1}{\text{Nyquist rate}} = \frac{1}{400} = 2.5\text{ ms}$$

Q8.13 ***A signal x(t) whose spectrum is shown in fig Q8.13.1 is sampled at a rate of 300 samples / sec. Sketch the spectrum of the sampled discrete time signal.***

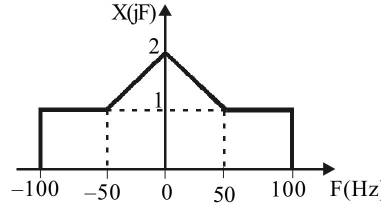


Fig Q8.13.1.

Solution

From the spectrum shown in fig Q3.18.1 it is observed that the maximum frequency, F_m in the signal is 100 Hz. Given that, Sampling frequency, F_s is 300 Hz, which is greater than 2 F_m , and so the signal is sampled without aliasing.

Frequency "f" of sampled discrete time signal corresponding to any frequency "F" of continuous time signal is given by, $f = F / F_s$.

The magnitude of the spectrum of discrete time signal will be scaled by 1/T, where $T = 1/ F_s$. The frequency spectrum of a discrete time signal will be periodic with periodicity of – 0.5 to + 0.5. (Refer Chapter-6, Section 6.3). Therefore the frequency spectrum of sampled discrete time signal will be as shown in fig Q3.13.2.

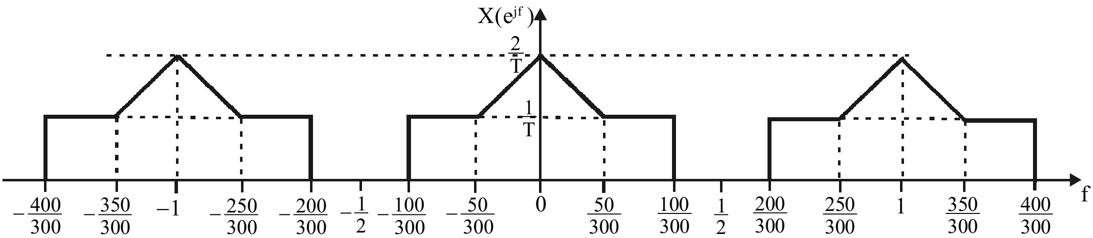


Fig Q8.13.2.

Q8.14 ***If the spectrum shown in fig Q8.13.1 is sampled at a rate of 100 samples / sec. Sketch the spectrum of the sampled discrete time signal.***

Since the sampling frequency is less than 2 F_m , the spectrum of the sampled signal will have aliasing as shown in fig Q8.14.1.

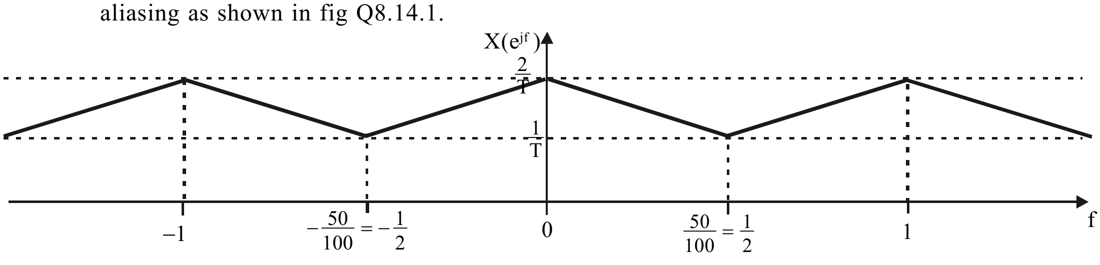


Fig Q8.14.1.

Q8.15 ***Consider the sampling of the bandpass signal whose frequency spectrum is shown in fig Q8.15. Determine the minimum sampling rate F_s to avoid aliasing.***

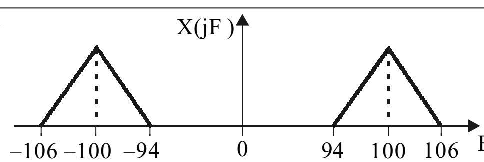


Fig Q8.15.

Solution

The given signal is a bandpass signal. The bandwidth, B = 106 – 94 = 12 Hz.

Here the upper cutoff frequency (106 Hz) is not an integer multiple of bandwidth, B. Hence the minimum sampling rate should be 4B, in order to avoid aliasing.

$$\therefore \text{Minimum sampling rate} = 4 \times B = 4 \times 12 = 48\text{Hz}$$

8.11 MATLAB Programs

Program 8.1

Write a MATLAB program to find Fourier coefficients of the discrete time signal x(n)={1,2,-1}, and sketch the magnitude and phase spectrum.

```
% Program to find Fourier coefficients of x(n)={1,2,-1}
% and to sketch the magnitude and phase spectrum

clear all
N=3; i=sqrt(-1);
x0=1; x1=2; x2=-1;
Ck=[];
for k=0:1:11
C=(1/N)*(x0+(x1*(exp(-i*2*pi*k/N)))+(x2*(exp(-i*4*pi*k/N))));
Ck=[Ck,C];
end

k = 0:1:11;
Ck                      %print the Fourier coefficients Ck
Mag_of_Ck = abs(Ck)     %evaluate and print the magnitude of Fourier
                        %coefficients
Pha_of_Ck = angle(Ck)   %evaluate and print the phase of Fourier
                        %coefficients

subplot(2,1,1), stem(k,Mag_of_Ck);
xlabel('k'), ylabel('Magnitude of Ck');
subplot(2,1,2), stem(k,Pha_of_Ck);
xlabel('k'), ylabel('Phase of Ck in rad.');
```

OUTPUT

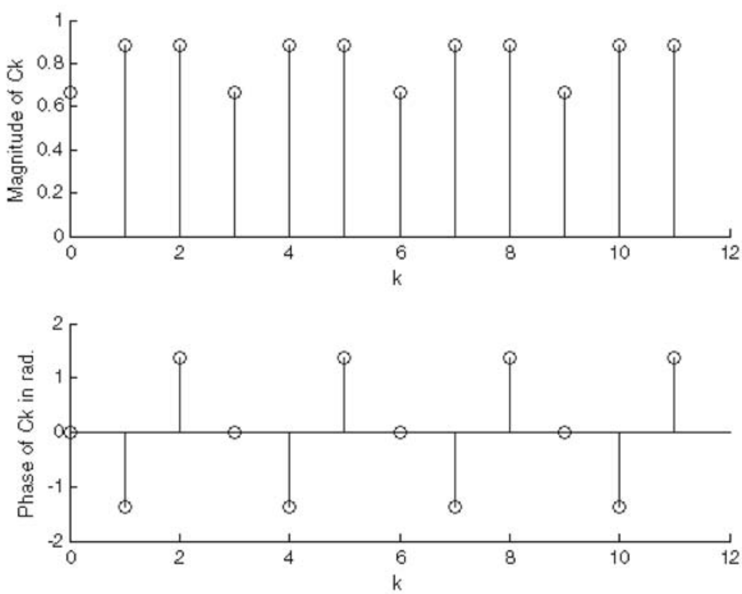


Fig P8.1 *: Magnitude and phase spectrum of program 8.1.*

```
Ck =
   Columns 1 through 7
    0.6667                 0.1667 - 0.8660i    0.1667 + 0.8660i    0.6667
    0.1667 - 0.8660i    0.1667 + 0.8660i    0.6667

   Columns 8 through 12
    0.1667 - 0.8660i    0.1667 + 0.8660i    0.6667                 0.1667 - 0.8660i
    0.1667 + 0.8660i


Mag_of_Ck =
     0.6667    0.8819    0.8819    0.6667    0.8819    0.8819    0.6667    0.8819
     0.8819    0.6667    0.8819    0.8819


Pha_of_Ck =
          0   -1.3807    1.3807         0   -1.3807    1.3807         0   -1.3807
     1.3807         0   -1.3807    1.3807
```

The magnitude and phase spectrum of program 8.1 are shown in fig P8.1.

Program 8.2

Write a MATLAB program to sketch the magnitude and phase spectrum of discrete time systems represented by the following transfer functions.

a) $H(e^{j\omega})=(1-e^{-j3\omega})/3(1-e^{-j\omega})$ b) $H(e^{j\omega})=2e^{-j\omega/2}\cos(\omega/2)$

c) $H(e^{j\omega})=2e^{-j\omega/2}\sin(\omega/2)$

```
% Program to sketch the magnitude and phase spectrum
% of the given discrete time systems

clear all

MagH1=[]; MagH2=[]; MagH3=[]; PhaH1=[]; PhaH2=[]; PhaH3=[]; w1=[];

for w=-2*pi:0.01:2*pi
H1=(1/3)*(1-exp(-3*i*w))/(1-exp(-i*w));
H2=2*(exp(-i*w/2))*(cos(w/2));
H3=2*(exp(-i*w/2))*(sin(w/2));

H1_M=abs(H1);   H2_M=abs(H2);    H3_M=abs(H3);
H1_P=angle(H1); H2_P=angle(H2);  H3_P=angle(H3);

MagH1=[MagH1,H1_M];    %store the magnitude as an array
MagH2=[MagH2,H2_M];
MagH3=[MagH3,H3_M];

PhaH1=[PhaH1,H1_P];    %store the phase as an array
PhaH2=[PhaH2,H2_P];
PhaH3=[PhaH3,H3_P];

w1=[w1,w];                 %store the frequency as an array
end

subplot(3,2,1),plot(w1,MagH1);
xlabel('w in rad.'),ylabel('Mag. of H1');
subplot(3,2,2),plot(w1,PhaH1);
xlabel('w in rad.'),ylabel('Pha. of H1');

subplot(3,2,3),plot(w1,MagH2);
xlabel('w in rad.'),ylabel('Mag. of H2');
```

```
subplot(3,2,4),plot(w1,PhaH2);
xlabel('w in rad.'),ylabel('Pha. of H2');

subplot(3,2,5),plot(w1,MagH3);
xlabel('w in rad.'),ylabel('Mag. of H3');
subplot(3,2,6),plot(w1,PhaH3);
xlabel('w in rad.'),ylabel('Pha. of H3');
```

OUTPUT

The magnitude and phase spectrum of program 8.2 are shown in fig P8.2.

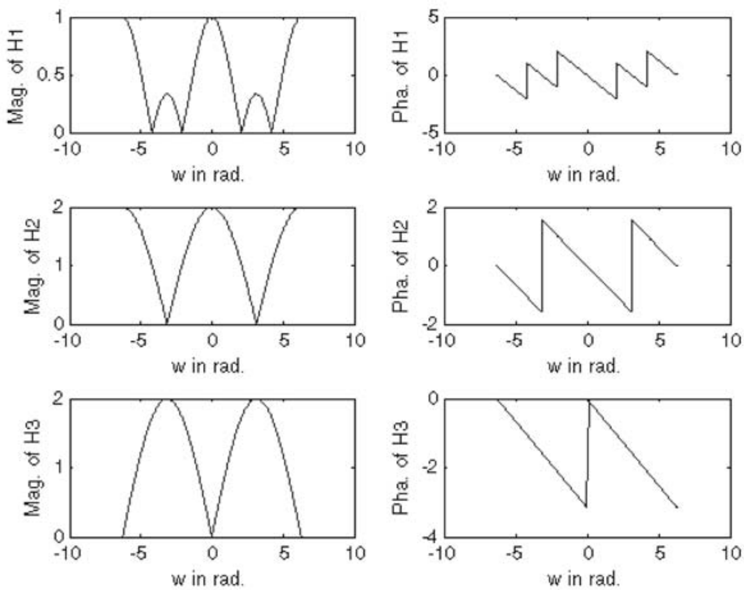


***Fig P8.2** : Magnitude and phase spectrum of program 8.2.*

Program 8.3

Write a MATLAB program to sketch the frequency response of the first order discrete time system governed by the transfer function,

$H(e^{j\omega})=1/(1-ae^{-j\omega})$ for a=0.5 and a=-0.5.

```
% Program to sketch frequency response of first order discrete
% time system

clear all

j=sqrt(-1);w=[];Mag_H1=[];Pha_H1=[];Mag_H2=[];Pha_H2=[];

for w1=-pi:0.01:pi
    H1 = 1/(1-0.5*exp(-j*w1));
    H2 = 1/(1+0.5*exp(-j*w1));
    H1_M = abs(H1);
    H2_M = abs(H2);
```

```
        H1_P  =  angle(H1);
        H2_P  =  angle(H2);
        Mag_H1=[Mag_H1,  H1_M];
        Mag_H2=[Mag_H2,  H2_M];
        Pha_H1=[Pha_H1,H1_P];
        Pha_H2=[Pha_H2,H2_P];
        w=[w,w1];
end

subplot(2,2,1),plot(w,Mag_H1);
xlabel('w in rad.'),ylabel('Magnitude of H1(jw)');
subplot(2,2,2),plot(w,Mag_H2);
xlabel('w in rad.'),ylabel('Magnitude of H2(jw)');
subplot(2,2,3),plot(w,Pha_H1);
xlabel('w in rad.'),ylabel('Phase of H1(jw) in rad.');
subplot(2,2,4),plot(w,Pha_H2);
xlabel('w in rad.'),ylabel('Phase of H2(jw) in rad.');
```

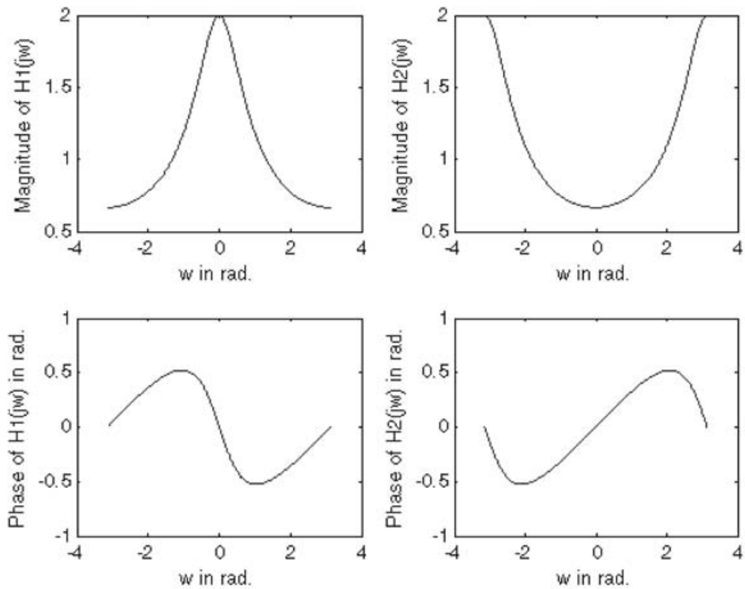


Fig P8.3 *: Magnitude and phase spectrum of first order discrete time system.*

OUTPUT

The frequency response consists of two parts : Magnitude spectrum and Phase spectrum. The magnitude and phase spectrum of first order discrete time system for a=0.5 and for a=-0.5 are shown in fig P8.3.

Program 8.4

Write a MATLAB program to sketch the frequency response of the second order discrete time system governed by the transfer function,

$H(e^{j\omega})=(1+ae^{-j\omega})/(1+\alpha e^{-j\omega}+\beta e^{-j2\omega})$

where, $a=-r\cos\omega_0$; $\alpha=2a$; $\beta=r^2$; $r=0.9$; $\omega_0=\pi/2$.

```
% Program to sketch frequency response of second order
% discrete time system

clear all

j=sqrt(-1);w=[];Mag_H=[];Pha_H=[];
r=0.9;  wo=pi/2;
a=(-1*r*cos(wo));
alpha=2*a;
Beta=r^2;

for w1=-pi:0.01:pi
    Num_of_H=(1+a*exp(-j*w1));
    Den_of_H=(1+((alpha)*exp(-j*w1))+((Beta)*exp(-j*2*w1)));
    H=Num_of_H / Den_of_H;
    H_M=abs(H);
    H_P=angle(H);
    Mag_H=[Mag_H,H_M];
    Pha_H=[Pha_H,H_P];
    w=[w,w1];
end
subplot(2,1,1),plot(w,Mag_H);
xlabel('w in radians'),ylabel('Magnitude of H(jw)');
subplot(2,1,2),plot(w,Pha_H);
xlabel('w in radians'),ylabel('Phase of H(jw)');
```

OUTPUT

The frequency response consists of two parts : Magnitude spectrum and Phase spectrum. The magnitude and phase spectrum of the given second order discrete time system are shown in fig P8.4.

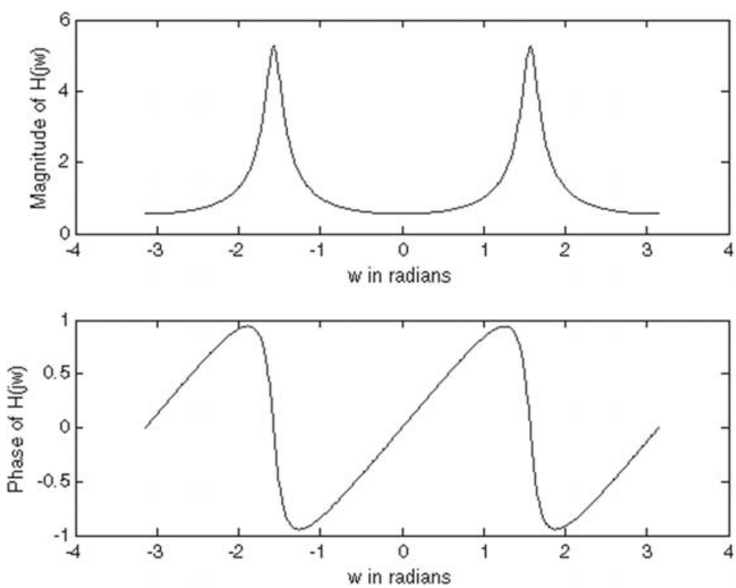


Fig P8.4 *: Magnitude and phase spectrum of second order discrete time system.*

8.12 Exercises

I. Fill in the blanks with appropriate words

1. The Fourier transform of continuous time signal involves integration, whereas the Fourier transform of discrete time signal involves _______.
2. In Fourier transform of a real signal, the magnitude function is symmetric and phase function is _______.
3. The _______ operation of x(n) with h(n) is equal to the product $X(e^{j\omega})\ H(e^{j\omega})$.
4. The Fourier transform of product of two time domain signals is equivalent to _______ of their Fourier transforms.
5. The Fourier transform of the impulse response of an LTI system is called _______.
6. The Fourier transform of the discrete signal can be obtained by evaluating the $\mathcal{Z}$-transform along _______.
7. A second order LTI system will behave as a _______ filter.
8. A first order LTI system will behave as a _______ filter.
9. A bandlimited signal with maximum frequency F_m can be fully recovered from its samples if sampled at a frequency greater than or equal to _______.
10. The sampling rate for a bandpass signal with bandwidth "B" is _______.

Answers

1. summation	4. convolution	7. bandpass	9. $2\ F_m$
2. antisymmetric	5. frequency response	8. lowpass or highpass	10. 2B to 4B
3. convolution	6. unit circle		

II. State whether the following statements are True/False

1. The discrete time Fourier series exists only for periodic discrete time signal.
2. The convergence of the discrete time Fourier series is exact at every point.
3. The Fourier coefficients of a discrete time signal is periodic.
4. The Fourier transform exists only for signals that are absolutely summable.
5. The Fourier transform of discrete signal is a discrete function of ω.
6. Fourier transform of an even signal is purely real and Fourier transform of an odd signal is purely imaginary.
7. The frequency response is periodic with a periodicity of 2π.
8. When the impulse response is complex, the real part of frequency response is symmetric and imaginary part is antisymmetric.
9. Convolving two signals in time domain is equivalent to multiplying their spectra in frequency domain.
10. Multiplication of a sequence x(n) by $e^{-j\omega_0 n}$ is same as frequency translation of the spectrum $X(e^{j\omega})$ by ω_0.
11. Impulse response h(n) is discrete, whereas frequency response $H(e^{j\omega})$ is continuous function of ω.
12. The first order system can be designed to behave as either low pass or high pass filter.
13. The spectrum of sampled version of a discrete time signal is sum of frequency shifted and amplitude scaled version of original spectrum of continuous time signal, $X(j\Omega)$.
14. If a discrete time signal is shifted in time by 'n_0' samples, then its magnitude spectrum shifts by ωn_0 .
15. The Fourier transform can be obtained from $\mathcal{Z}$-transform only if ROC of X(z) includes unit circle.

Answers

1. True	4. True	7. True	10. False	13. True
2. True	5. False	8. True	11. True	14. False
3. True	6. True	9. True	12. True	15. True

III. Choose the right answer for the following questions

1. *The Fourier coefficients of x(n) is,* $c_k = \{3,\ \ 2+j,\ \ 1,\ \ 2-j\}$. *The value of x(7) is,*

   a) 1 **b)** 0 **c)** $2-j$ **d)** $2+j$

2. *For a periodic discrete time signal x(n), the Fourier coefficient* $c_1 = -1 + j4.5$. *The value of* c_{1+N} *will be,*

   a) $-1-j\,4.5$ **b)** -1 **c)** $j4.5$ **d)** $-1+j\,4.5$

3. *The Fourier coefficients of x(n) is* c_k, *then Fourier coefficients of* $x^*(n)$ *is,*

   a) c^*_k **b)** c^*_{-k} **c)** c_{-k} **d)** c_k

4. *The average power of x(n) in terms of Fourier series coefficient* c_k *is,*

   a) $\sum\limits_{k=0}^{\infty} |c_k|^2$ **b)** $\frac{1}{N}\sum\limits_{k=0}^{\infty} |c_k|^2$ **c)** $\frac{1}{N}\sum\limits_{k=0}^{N-1} |c_k|^2$ **d)** $\sum\limits_{k=0}^{N-1} |c_k|^2$

5. *The Fourier transform of x(n) = 1, for all 'n' is,*

   a) $2\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$ **b)** $\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$ **c)** $2\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-m)$ **d)** $2\pi\sum\limits_{m=-\infty}^{+\infty}\delta(\omega-\pi m)$

6. *If* $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, *then* $\mathcal{F}\{x(n-3)\}$ *will be,*

   a) $e^{-j3\omega}\,X(e^{-j\omega})$ **b)** $e^{j3\omega}\,X(e^{-j\omega})$ **c)** $e^{-j3\omega}\,X(e^{j\omega})$ **d)** $e^{j3\omega}\,X(e^{j\omega})$

7. *If a signal is folded about the origin in time then its,*

   a) magnitude spectrum undergoes change in sign **b)** phase spectrum undergoes change in sign

   c) magnitude remains unchanged **d)** both c and b

8. *The Fourier transform of correlation sequence of two discrete time signals* $x_1(n)$ *and* $x_2(n)$ *is given by,*

   a) $X_1(e^{j\omega})\,X_2(e^{j\omega})$ **b)** $X_1(e^{j\omega})\,X_2(e^{-j\omega})$ **c)** $X_1(e^{-j\omega})\,X_2(e^{-j\omega})$ **d)** none of the above

9. *If h(n) is real, then magnitude of* $H(e^{j\omega})$ *is ______ and phase of* $H(e^{j\omega})$ *is _____.*

   a) symmetric, antisymmetric **b)** antisymmetric, symmetric

   c) symmetric, symmetric **d)** antisymmetric, antisymmetric

10. *The second order LTI discrete time system behaves as,*

    a) low pass filter **b)** high pass filter **c)** resonant filter **d)** all pass filter

11. *The ideal interpolation formula is used to,*

    a) obtain frequency spectrum of discrete time signal **b)** sample continuous time signal

    c) reconstruct original continuous time signal **d)** remove aliasing

12. *If* $X(j\Omega)$ *is frequency spectrum of a continuous time signal then, the frequency spectrum of sampled version of the signal* $X(e^{j\omega})$ *is, (where* $\omega = \Omega T$*),*

    a) $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty}X\left(j\left(\frac{\omega}{T}+\frac{2\pi m}{T}\right)\right)$ **b)** $\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}X\left(j\frac{\omega}{T}\right)e^{j\omega nT}\,d\omega$ **c)** $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty}X\left(j\left(\omega T+\frac{2\pi m}{T}\right)\right)$ **d)** $\frac{1}{T}\sum\limits_{m=-\infty}^{+\infty}X\left(j\left(\frac{m\omega}{T}\right)\right)$

13. A bandlimited continuous time signal with maximum frequency F_m, sampled at a frequency F_s, can be fully recovered from its samples, provided that,

a) $F_s \geq 2F_m$ **b)** $F_s = 2F_m$ **c)** $F_m \geq 2F_s$ **d)** $F_s = F_m$

14. If Z-transform of x(n) includes unit circle in its ROC, then the Fourier transform of x(n) can be expressed as,

a) $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right|_{z=e^{-j\omega}}$ **b)** $\left.\sum_{n=0}^{\infty} x(n)\, z^{-jn}\right|_{z=e^{-\omega}}$ **c)** $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{n}\right|_{z=\omega}$ **d)** $\left.\sum_{n=-\infty}^{\infty} x(n)\, z^{-n}\right|_{z=e^{j\omega}}$

15. Let x(n) is real and $x(n) = x_e(n) + x_o(n)$. If $A(e^{j\omega})$ is Fourier transform of $x_e(n)$ and if $B(e^{j\omega})$ is Fourier transform of $x_o(n)$, then Fourier transform of x(n) is,

a) $A(e^{j\omega}) + B(e^{j\omega})$ **b)** $A(e^{-j\omega}) + j\,B(e^{-j\omega})$ **c)** $A(e^{j\omega}) - jB(e^{j\omega})$ **d)** $A(e^{-j\omega}) - j\,B(e^{-j\omega})$

16. If a continuous time signal x(t) has a nyquist rate of Ω_0, then nyquist rate for the continuous time signal $x^2(t)$ is,

a) $\frac{\Omega_0}{2}$ **b)** $2\Omega_0$ **c)** $\frac{\Omega_0}{4}$ **d)** Ω_0

17. If the bandwidth of a bandpass signal x(t) is 2F, then the minimum sampling rate for bandpass signal must be,

a) 2F samples/sec **b)** 4F samples/sec **c)** $\frac{F}{2}$ samples/sec **d)** $\frac{F}{4}$ samples/sec

18. If $X(e^{j\omega}) = e^{-j\omega}$ for $-\pi \leq \omega \leq \pi$, then the discrete time signal x(n) is,

a) $\frac{\sin 2\pi(n-1)}{2\pi(n-1)}$ **b)** $\sin\pi(n-1)$ **c)** $\frac{\sin\pi(n-1)}{\pi(n-1)}$ **d)** $\frac{\sin\pi(2n-1)}{\pi(2n-1)}$

19. The discrete time Fourier transform of the signal, $x(n) = 0.5^{(n-1)}\, u(n-1)$ is,

a) $\frac{e^{-j\omega}}{1-0.5e^{-j\omega}}$ **b)** $e^{-j\omega}\left(1-0.5e^{-j\omega}\right)$ **c)** $\frac{0.5e^{-j\omega}}{1-0.5e^{-j\omega}}$ **d)** $\frac{0.5e^{j\omega}}{1-0.5e^{-j\omega}}$

20. The Fourier transform of, $x(n) = (0.8)^n$; $n = 0, \pm 1, \pm 2, \ldots\ldots$ is,

a) does not exist **b)** $\frac{1}{1-0.8e^{-j\omega}}$ **c)** $\frac{0.8}{1-0.8e^{-j\omega}}$ **d)** $\frac{0.8e^{-j\omega}}{1-0.8e^{-j\omega}}$

Answer

1. c	5. a	9. a	13. a	17. b
2. d	6. c	10. c	14. d	18. c
3. b	7. d	11. c	15. a	19. a
4. d	8. b	12. a	16. b	20. a

IV. Answer the following questions

1. Define Fourier series of a periodic discrete time signal.
2. Define Fourier coefficients of a periodic discrete time signal.
3. Write any two properties of Fourier series coefficients of discrete time signal.
4. Define the frequency spectrum of a periodic discrete time signal in terms of Fourier series coefficients.
5. Write the differences between Fourier series of a discrete time signal and continuous time signal.
6. Define Fourier transform of a discrete time signal.
7. State and prove any two properities of Fourier transform.
8. State and prove the time delay property of Fourier transform.
9. Give the significance of Parseval's relation.
10. Define inverse Fourier transform.
11. Write the differences between Fourier transform of discrete time signal and continuous time signal.
12. Define the frequency spectrum of a discrete time signal in terms Fourier transform.
13. Write a short note on Fourier transform of periodic discrete time signal.
14. Write the properities of frequency response of an LTI system.
15. What is frequency response of an LTI system?
16. What is the relation between Fourier transform and $\mathcal{Z}$-transform?
17. What is aliasing of frequency spectrum?
18. Explain how a band limited signal can be sampled without aliasing?
19. What is ideal interpolation formula? What is its significance?
20. Write a short note on sampling of bandpass signals.

V. Solve the Following Problems

E8.1 Determine the Fourier series representation of the following discrete time signals.

a) $x(n) = 3\cos\sqrt{5}\,\pi n$ **b)** $x(n) = \{.....1,\ 2,\ 3,\ 4,\ \underset{\uparrow}{1},\ 2,\ 3,\ 4,\ 1,\ 2,\ 3,\ 4,.....\}$

c) $x(n) = 7\,e^{\frac{j3\pi n}{2}}$ **d)** $x(n) = 8\sin\frac{2\pi n}{3}$ **e)** $x(n) = \cos\frac{\pi n}{3} + \sin\frac{\pi n}{5}$

E8.2 Determine the Fourier transform of the following signals.

a) $x(n) = 3\cos\frac{2\pi}{5}n$ **b)** $x(n) = \{\underset{\uparrow}{-2},\ 1,\ 4,\ -4\}$

c) $x(n) = (-1)^n \quad ; \quad 0 \le n \le 5$
$\quad\ \ = 0 \quad ; \quad \text{otherwise}$

d) $x(n) = 0.4\left[\left(\frac{1}{0.2}\right)^n - \left(\frac{1}{0.4}\right)^n\right]u(n)$

E8.3 Determine the convolution of the following sequences, using Fourier transform.

a) $x_1(n) = \{1,\ \underset{\uparrow}{-1},\ 1\},\ x_2(n) = \{-1,\ \underset{\uparrow}{1},\ -1\}$ **b)** $x_1(n) = \{\underset{\uparrow}{1},\ -2,\ 0\},\ x_2(n) = \{\underset{\uparrow}{-3},\ 1,\ -1\}$

E8.4 Determine the inverse Fourier transform of the following functions of ω.

a) $X(e^{j\omega}) = j\omega$ **b)** $X(e^{j\omega}) = \dfrac{1}{\left(1 - ae^{-j\omega}\right)^2}\ ; \quad |a| < 1$

c) $Y(e^{j\omega}) = \dfrac{1 + 0.25e^{-j\omega}}{1 - 0.25e^{-j\omega}}$ **d)** $H(e^{j\omega}) = \dfrac{1}{\left(1 - \frac{1}{2}e^{-j\omega}\right)\left(1 - \frac{1}{4}e^{-j\omega}\right)}$

E8.5 **a)** A causal discrete time system is described by the equation, $y(n)-\frac{7}{12}y(n-1)+\frac{7}{12}y(n-2)=x(n)$, where x(n) and y(n) are input and output of the system. Find the impulse response h(n), frequency response $H(e^{j\omega})$, magnitude function and phase function of the system.

b) Consider an LTI system described by, $y(n)-\frac{1}{5}y(n-1)=x(n)+\frac{1}{5}x(n-1)$

(i) Determine the frequency response $H(e^{j\omega})$ and impulse response h(n) of the system.

(ii) Determine the response y(n) for the input $x(n)=\cos\frac{\pi n}{2}$.

E8.6 A discrete LTI system is described by a difference equation, y(n) = x(n) + x(n – 1). Determine the frequency response $H(e^{j\omega})$, impulse response h(n). Sketch the magnitude function and phase function.

E8.7 Sketch the magnitude and phase function of the discrete time LTI system described by the equation y(n) = x(n) – x(n – 1).

E8.8 The impulse response of a system is, $h(n)=\frac{1}{0.3}\delta(n+2)+\frac{1}{0.1}\delta(n+1)+\frac{1}{0.3}\delta(n)+\frac{1}{0.1}\delta(n-1)$

(i) Is the system BIBO stable, (ii) Is the system causal, (iii) Find the frequency response.

E8.9 The impulse response of an LTI system is $h(n)=\{-2, \ -1, \ 3, \ -2\}$. Find the response of the system for the input $x(n)=\{2, \ 3, \ 4, \ 1\}$, using convolution property of Fourier transform.

E8.10 A causal system is represented by the following difference equation,

$$y(n)-0.25y(n-1) = x(n)-0.5x(n-1).$$

Find the system transfer function H(z), impulse response and frequency response of the system. Also determine the magnitude and phase function.

Answers

E8.1

a) x(n) is nonperiodic.

b) $x(n)=\frac{5}{2}+\frac{1}{\sqrt{2}}e^{\frac{j3\pi}{4}}e^{j\omega_0 n}-\frac{1}{2}e^{j2\omega_0 n}+\frac{1}{\sqrt{2}}e^{-\frac{j3\pi}{4}}e^{j3\omega_0 n}$; $\omega_0=\frac{\pi}{2}$

c) $x(n)=7\,e^{j3\omega_0 n}$; $\omega_0=\frac{\pi}{2}$

d) $x(n)=-4j\,e^{j\omega_0 n}+4\,e^{j2\omega_0 n}$; $\omega_0=\frac{2\pi}{3}$

e) $x(n)=\frac{1}{2}e^{-j5\omega_0 n}+j\frac{1}{2}e^{-j3\omega_0 n}-j\frac{1}{2}e^{j3\omega_0 n}+\frac{1}{2}e^{j5\omega_0 n}$; $\omega_0=\frac{\pi}{15}$

$\left(\text{or } x(n)=-j\frac{1}{2}e^{j3\omega_0 n}+\frac{1}{2}e^{j5\omega_0 n}+\frac{1}{2}e^{j25\omega_0 n}+j\frac{1}{2}e^{j27\omega_0 n}\right)$

E8.2

a) $X(e^{j\omega})=3\pi\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\frac{2\pi}{5}-2\pi m)+\delta(\omega+\frac{2\pi}{5}-2\pi m)\right]$

b) $X(e^{j\omega})=-2+e^{-j\omega}+4e^{-j2\omega}-4e^{-j3\omega}$

c) $X(e^{j\omega})=e^{-\frac{j5\omega}{2}}\dfrac{\cos 3\omega}{\cos\frac{\omega}{2}}$

d) $X(e^{j\omega})=\dfrac{e^{-j\omega}}{1-7.5e^{-j\omega}+12.5e^{-j2\omega}}$

E8.3

a) $x(n)=\{-1, \ 2, \ \underset{\uparrow}{-3}, \ 2, \ -1\}$

b) $x(n)=\{\underset{\uparrow}{-3}, \ 7, \ -3, \ 2\}$

E8.4

a) $x(n)=\frac{(-1)^n}{n}$; for $n\neq 0$

$= \infty$; for $n=0$

b) $x(n)=(n+1)\,a^n u(n)$; $|a|<1$

c) $y(n)=(0.25)^n\left[u(n)+u(n-1)\right]$

d) $h(n)=\left[2\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\right]u(n)$

E8.5

a) $h(n)=\left[4\left(\frac{1}{3}\right)^n-3\left(\frac{1}{4}\right)^n\right]u(n)$; $H(e^{j\omega})=\dfrac{1}{\left(1-\frac{1}{3}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$

$$\left|H(e^{j\omega})\right| = \left[\frac{1}{0.667 + 0.167\cos 2\omega - 1.264\cos\omega}\right]; \quad \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-7\,\sin\omega + \sin 2\omega}{12 - 7\,\cos\omega + \cos 2\omega}\right)$$

b) (i) $H(e^{j\omega}) = \left[\frac{1}{1 - \frac{1}{5}e^{-j\omega}}\right] + \frac{1}{5}\left(\frac{e^{-j\omega}}{1 - \frac{1}{5}e^{-j\omega}}\right)$; $\quad h(n) = \left(\frac{1}{5}\right)^n u(n) + \frac{1}{5}\left(\frac{1}{5}\right)^{n-1} u(n-1)$

(ii) $y(n) = \cos\left(\frac{\pi n}{2} - 2\tan^{-1}\frac{1}{5}\right)$

E8.6 $H(e^{j\omega}) = 2\,e^{\frac{-j\omega}{2}}\cos\left(\frac{\omega}{2}\right)$; $\quad h(n) = \delta(n) + \delta(n-1)$

$\left|H(e^{j\omega})\right| = 2\,\cos\left(\frac{\omega}{2}\right)$; $\quad \angle H(e^{j\omega}) = -\frac{\omega}{2}$

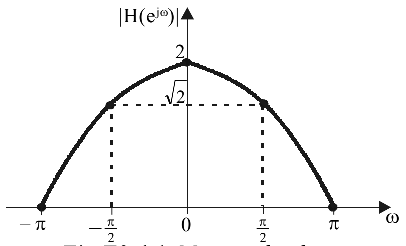


Fig E8.6.1 Magnitude plot.

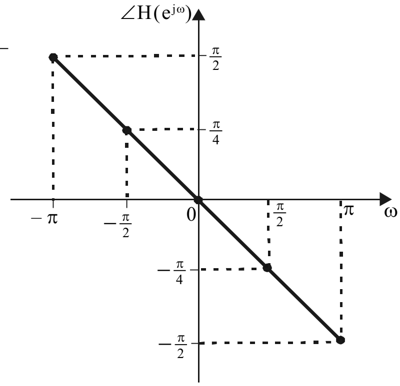


Fig E8.6.2 Phase plot.

E8.7 $H(e^{j\omega}) = 2\,e^{\frac{j(\pi-\omega)}{2}}\sin\left(\frac{\omega}{2}\right)$; $\quad \left|H(e^{j\omega})\right| = 2\,\left|\sin\,\frac{\omega}{2}\right|$

$\angle H(e^{j\omega}) = \frac{\pi}{2} - \frac{\omega}{2}$; $\omega \geq 0$

$= -\frac{\pi}{2} - \frac{\omega}{2}$; $\omega \leq 0$

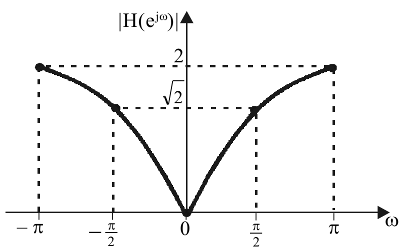


Fig E8.7.1 Magnitude plot.

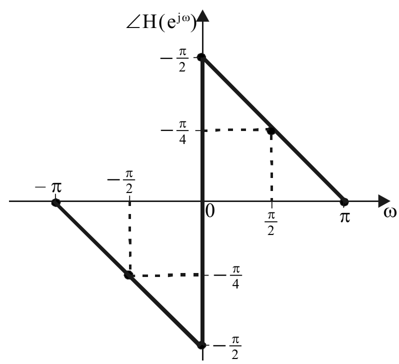


Fig E8.7.2 Phase plot.

E8.8 $h(n) = \left\{\frac{1}{0.3}, \quad \frac{1}{0.1}, \quad \underset{\uparrow}{\frac{1}{0.3}}, \quad \frac{1}{0.1}\right\}$; **(i)** The system is stable ; **(ii)** The system is noncausal.

(iii) $H(e^{j\omega}) = \frac{1}{0.3}\left[1 + 6\cos\omega + \cos 2\omega + j\sin 2\omega\right]$

E8.9 $y(n) = \{\underset{\uparrow}{-4}, \quad -8, \quad -5, \quad -1, \quad 5, \quad -5, \quad -2\}$

E8.10 $H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.25z^{-1}}$; $\quad H(e^{j\omega}) = \frac{1 - 0.5e^{-j\omega}}{1 - 0.25e^{-j\omega}}$; $\quad h(n) = \delta(n) + 0.25\,(0.25)^{n-1}\,u(n-1)$

$\left|H(e^{j\omega})\right| = \sqrt{\frac{1.25 - \cos\omega}{1.0625 - 0.5\cos\omega}}$; $\quad \angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-0.75\,\sin\omega}{1.125 - 0.75\cos\omega}\right)$

CHAPTER 9

Discrete Fourier Transform(DFT) and Fast Fourier Transform(FFT)

9.1 Introduction

The discrete time Fourier transform (DTFT) discussed in chapter-8, provides a method to represent a discrete time signal in frequency domain and to perform frequency analysis of discrete time signal.

The drawback in DTFT is that the frequency domain representation of a discrete time signal obtained using DTFT will be a continuous function of ω and so it cannot be processed by digital system. The discrete Fourier transform (DFT) has been developed to convert a continuous function of ω to a discrete function of ω, so that frequency analysis of discrete time signals can be performed on a digital system.

Basically, the DFT of a discrete time signal is obtained by sampling the DTFT of the signal at uniform frequency intervals and the number of samples should be sufficient to avoid aliasing of frequency spectrum. The samples of DTFT are represented as a function of integer k, and so the DFT is a sequence of complex numbers represented as X(k) for k = 0,1,2,3,etc.,.

Since X(k) is a sequence consisting complex numbers, the magnitude and phase of each sample can be computed and listed as magnitude sequence and phase sequence respectively. The graphical plots of magnitude and phase as a function of k are also drawn.

The plot of magnitude versus k is called ***magnitude spectrum*** and the plot of phase versus k is called ***phase spectrum***. In general these plots are called ***frequency spectrum***.

The drawback in ***DFT*** is that, the computation of each sample of DFT involves a large number of calculations and when large number of samples are required, the number of calculations will further increase. In order to overcome this drawback, a number of methods or algorithms have been developed to reduce the number of calculations. The various methods developed to compute DFT with reduced number of calculations are collectively called ***Fast Fourier Transform*** (FFT).

9.2 Discrete Fourier Transform (DFT) of Discrete Time Signal

9.2.1 Development of DFT from DTFT

The frequency domain representation of a discrete time signal obtained using discrete time Fourier transform (DTFT) will be a continuous and periodic function of ω, with periodicity of 2π. In order to obtain discrete function of ω, the DTFT can be sampled at sufficient number of frequency intervals.

Let $X(e^{j\omega})$ be discrete time Fourier transform of the discrete time signal x(n). The discrete Fourier transform (DFT) of x(n) is obtained by sampling one period of the discrete time Fourier transform $X(e^{j\omega})$ at a finite number of frequency points.

The frequency domain sampling is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$. The sampling frequency points are denoted as ω_k and they are given by,

$$\omega_k = \frac{2\pi k}{N} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1$$

Now, the DFT is a sequence consisting of N-samples of DTFT. Let the samples are denoted by X(k) for k = 0, 1, 2, ... N–1. Therefore, the sampling of $X(e^{j\omega})$ is mathematically expressed as,

$$X(k) = X(e^{j\omega})\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \ldots\ldots(9.1)$$

The DFT sequence starts at k = 0, corresponding to ω = 0 but does not include k = N, corresponding to ω = 2π, (since the sample at ω = 0 is same as the sample at ω = 2π). Generally, the DFT is defined along with number of samples and is called ***N-point DFT***. The number of samples N for a finite duration sequence x(n) of length L should be such that, N ≥ L, in order to avoid aliasing of frequency spectrum.

The sampling of Fourier transform of a sequence to get DFT is shown in example 9.1. To calculate DFT of a sequence it is not necessary to compute Fourier transform, since the DFT can be directly computed using the definition of DFT as given by equation (9.2).

9.2.2 Definition of Discrete Fourier Transform (DFT)

Let, x(n) = Discrete time signal of length L

X(k) = DFT of x(n)

Now, the ***N-point DFT*** of x(n), where N ≥ L, is defined as,

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0,1,2,\ldots\ldots, N-1} \qquad \ldots\ldots(9.2)$$

Symbolically the N-point DFT of x(n) can be expressed as,

$\mathcal{DFT}\{x(n)\}$

where, $\mathcal{DFT}$ is the operator that represents discrete Fourier transform.

$$\boxed{\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, N-1}$$

Since X(k) is a sequence consisting of N-complex numbers for k = 0, 1, 2, ... N–1, the DFT of x(n) can be expressed as a sequence as shown below.

$$X(k) = \{X(0),\ X(1),\ X(2), \ldots\ldots\ldots\ldots\ X(N-1)\}$$

9.2.3 Frequency Spectrum Using DFT

The X(k) is a discrete function of discrete time frequency ω, and so it is also called discrete frequency spectrum (or signal spectrum) of the discrete time signal x(n).

The X(k) is a complex valued function of k and so it can be expressed in rectangular form as,

$X(k) = X_r(k) + jX_i(k)$

where, $X_r(k)$ = Real part of X(k)

$X_i(k)$ = Imaginary part of X(k)

Now the Magnitude function (or Magnitude spectrum) $|X(k)|$ is defined as,

$$|X(k)|^2 = X(k)\,X^*(k) \quad \text{or} \quad |X(k)|^2 = X_r^2(K) + X_i^2(k)$$

$$\therefore \ |X(k)| = \sqrt{X(k)\,X^*(k)} \quad \text{or} \quad \sqrt{X_r^2(k) + X_i^2(k)}$$

The Phase function (or Phase spectrum) $\angle X(k)$ is defined as,

$$\angle X(k) = \mathrm{Arg}[X(k)] = \tan^{-1}\left[\frac{X_i(k)}{X_r(k)}\right]$$

Since $X(k)$ is a sequence consisting of N-complex numbers for k = 0, 1, 2, ... N–1, the magnitude and phase spectrum of $X(k)$ can be expressed as a sequence as shown below.

$$\text{Magnitude sequence, } |X(k)| = \{|X(0)|,\ |X(1)|,\ |X(2)|, \ldots\ldots\ldots\ldots\ldots\ |X(N-1)|\}$$

$$\text{Phase sequence, } \angle X(k) = \{\angle X(0),\ \angle X(1),\ \angle X(2), \ldots\ldots\ldots\ldots\ldots\ \angle X(N-1)\}$$

The magnitude and phase sequence can be sketched graphically as a function of k.

The plot of samples of magnitude sequence versus k is called ***magnitude spectrum*** and the plot of samples of phase sequence versus k is called ***phase spectrum***. In general these plots are called ***frequency spectrum***.

9.2.4 Inverse DFT

Let, $x(n)$ = Discrete time signal

$X(k)$ = N-point DFT of $x(n)$

The inverse discrete Fourier transform of the sequence $X(k)$ of length N is defined as,

$$\boxed{x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \ ; \quad \text{for } n = 0, 1, \ldots.., N-1} \qquad \ldots..(9.3)$$

Symbolically the inverse discrete Fourier transform of x(n) can be expressed as,

$$\mathcal{DFT}^{-1}\{X(k)\}$$

where, $\mathcal{DFT}^{-1}$ is the operator that represents inverse discrete Fourier transform.

$$\boxed{\mathcal{DFT}^{-1}\{X(k)\} = x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \ ; \quad \text{for } n = 0, 1, \ldots\ldots, N-1}$$

We also refer to x(n) and X(k) as a DFT pair and this relation is expressed as,

$$\boxed{x(n) \ \underset{\mathcal{DFT}^{-1}}{\overset{\mathcal{DFT}}{\rightleftarrows}} \ X(k)}$$

9.3 Properties of DFT

1. Linearity

The linearity property of DFT states that the DFT of a linear weighted combination of two or more signals is equal to similar linear weighted combination of the DFT of individual signals.

Let, $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$ then by linearity property,

$$\mathcal{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(k) + a_2 X_2(k)$$, where a_1 and a_2 are constants.

Proof :

By definition of discrete Fourier transform,

$$X_1(k) = \mathcal{DFT}\{x_1(n)\} = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(9.4)$$

$$X_2(k) = \mathcal{DFT}\{x_2(n)\} = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(9.5)$$

$$\mathcal{DFT}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1} \left[a_1 x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 x_2(n)]\, e^{\frac{-j2\pi kn}{N}}\right]$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{\frac{-j2\pi kn}{N}} + a_2 \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}}$$

$$= a_1 X_1(k) + a_2 X_2(k) \qquad \boxed{\text{Using equations (9.4) and (9.5)}}$$

2. Periodicity

If a sequence x(n) is periodic with periodicity of N samples then N-point DFT, X(k) is also periodic with periodicity of N samples.

Hence, if x(n) and X(k) are N point DFT pair then,

$$x(n + N) = x(n) \quad ; \quad \text{for all } n$$

$$X(k + N) = X(k) \quad ; \quad \text{for all } k$$

Proof :

By definition of DFT, the $(k + N)^{th}$ coefficient of X(k) is given by,

$$X(k + N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(k+N)}{N}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n k}{N}}\, e^{\frac{-j2\pi n N}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n k}{N}}\, e^{-j2\pi n} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n k}{N}} \qquad \boxed{e^{-j2\pi n} = 1}$$

$$= X(k) \qquad \boxed{\text{Using definition of DFT}}$$

3. Circular time shift

The circular time shift property of DFT says that if a discrete time signal is circularly shifted in time by m units then its DFT is multiplied by $e^{\frac{-j2\pi km}{N}}$.

i.e., if $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x((n-m))_N\} = X(k)\,e^{-j\frac{2\pi km}{N}}$

Proof :

$$\mathcal{DFT}\{x((n-m))_N\} = \sum_{n=0}^{N-1} x((n-m))_N\, e^{\frac{-j2\pi kn}{N}} = \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi k(p+m)}{N}}$$

Let, $p = n - m$, $\therefore n = p + m$

$$= \sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}\, e^{\frac{-j2\pi km}{N}}$$

$$= \left[\sum_{p=0}^{N-1} x(p)\, e^{\frac{-j2\pi kp}{N}}\right] e^{\frac{-j2\pi km}{N}}$$

$$= X(k)\, e^{\frac{-j2\pi km}{N}}$$

Using definition of DFT

4. Time reversal

The time reversal property of DFT says that, reversing the N-point sequence in time is equivalent to reversing the DFT sequence.

i.e., if $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x\,(N-n)\} = X(N-k)$.

Proof :

$$\mathcal{DFT}\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}}$$

Let, $m = N - n$, $\therefore n = N - m$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi kN}{N}}\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi k}$$

Since k is an integer, $e^{-j2\pi k} = 1$.

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi m}$$

Since m is an integer, $e^{-j2\pi m} = 1$.

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{\frac{-j2\pi mN}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$= X(N-k)$$

Using definition of DFT

5. Conjugation

Let x(n) be a complex N-point discrete sequence and $x^*(n)$ be its conjugate sequence.

Now if, $\mathcal{DFT}\{x(n)\} = X(k)$, then $\mathcal{DFT}\{x^*(n)\} = X^*(N-k)$.

Proof :

$$\mathcal{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{\frac{-j2\pi kn}{N}} = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{-j2\pi}\right]^* = \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi kn}{N}}\, e^{\frac{-j2\pi N}{N}}\right]^*$$

$e^{-j2\pi} = 1$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi k(N-n)}{N}}\right]^* = [X(N-k)]^* = X^*(N-k)$$

Using definition of DFT

6. Circular frequency shift

The circular frequency shift property of DFT says that if a discrete time signal is multiplied by $e^{\frac{j2\pi mn}{N}}$ its DFT is circularly shifted by m units.

$$\text{i.e., if}\quad \mathcal{DFT}\{x(n)\} = X(k) \quad \text{then} \quad \mathcal{DFT}\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = X((k-m))_N$$

Proof :

$$\mathcal{DFT}\left\{x(n)\, e^{\frac{j2\pi mn}{N}}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{j2\pi mn}{N}}\, e^{\frac{-j2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi(k-m)n}{N}}$$

$$= X(k-m)_N \qquad \boxed{\text{Using definition of DFT}}$$

7. Multiplication

The multiplication property of DFT says that, the DFT of product of two discrete time sequences is equivalent to circular convolution of the DFTs of the individual sequences scaled by a factor 1/N.

$$\text{i.e., if}\quad \mathcal{DFT}\{x(n)\} = X(k),\ \text{then}\ \mathcal{DFT}\{x_1(n)\, x_2(n)\} = \frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$$

Proof :

By definition of inverse DFT,
$$x_1(n) = \frac{1}{N}\sum_{K=0}^{N-1} X_1(k)\, e^{\frac{j2\pi kn}{N}} = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}} \qquad \boxed{\text{Let } k = m} \quad .....(9.6)$$

By definition of DFT,

$$\mathcal{DFT}\{x_1(n)\, x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)\, x_2(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{\frac{j2\pi mn}{N}}\right] x_2(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \boxed{\text{Using equation (9.6)}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi kn}{N}}\, e^{\frac{j2\pi mn}{N}}\right] \qquad \boxed{\text{Rearranging the order of summation}}$$

$$\boxed{\text{Using definition of DFT}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\left[\sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi(-(m-k))n}{N}}\right] = \frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\ X_2((m-k))_N$$

$$= \frac{1}{N}\left[X_1(k) \circledast X_2(k)\right] \qquad \boxed{\text{Using definition of Circular convolution}}$$

8. Circular convolution

The convolution property of DFT says that, the DFT of circular convolution of two sequences is equivalent to product of their individual DFTs.

Let $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$, then by convolution property,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k)$$

Proof :

Let, $x_1(n)$ and $x_2(n)$ be N-point sequences. Now by definition of DFT,

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi mk}{N}} \quad ; \quad k = 0, 1, 2, ... N-1 \qquad \boxed{\text{Let } n = m} \qquad .....(9.7)$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi pk}{N}} \quad ; \quad k = 0, 1, 2, ... N-1 \qquad \boxed{\text{Let } n = p} \qquad .....(9.8)$$

Consider the product $X_1(k)\, X_2(k)$. The inverse DFT of the product is given by,

$$\mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m)\, e^{\frac{-j2\pi mk}{N}}\right]\left[\sum_{p=0}^{N-1} x_2(p)\, e^{\frac{-j2\pi pk}{N}}\right] e^{\frac{j2\pi nk}{N}} \qquad \boxed{\text{Using equations (9.7) and (9.8)}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} \qquad .....(9.9)$$

Let, $n - m - p = qN$, where q is an integer.

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m-p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1} \left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N \qquad \boxed{\text{Since q is an integer, } e^{j2\pi q} = 1} \qquad .....(9.10)$$

Since, $n - m - p = qN$, $p = n - m - qN$

$$\therefore \sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n - m - qN) = \sum_{m=0}^{N-1} x_2(n-m, \bmod N) = \sum_{m=0}^{N-1} x_2((n-m))_N \qquad .....(9.11)$$

Using equations (9.10) and (9.11), the equation (9.9) can be written as shown below.

$$\mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2((n-m))_N\, N = \sum_{m=0}^{N-1} x_1(m)\, x_2((n-m))_N$$

$$= x_1(n) \circledast x_2(n) \qquad \boxed{\text{Using definition of Circular convolution}}$$

$$\therefore X_1(k)\, X_2(k) = \mathcal{DFT}\{x_1(n) \circledast x_2(n)\}$$

9. Circular correlation

The circular correlation of two sequences x(n) and y(n) is defined as,

$$\bar{r}_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N$$

Let, $\mathcal{DFT}\{x(n)\} = X(k)$ and $\mathcal{DFT}\{y(n)\} = Y(k)$, then by correlation property,

$$\mathcal{DFT}\{\bar{r}_{xy}(m)\} = \mathcal{DFT}\left\{\sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N\right\} = X(k)\, Y^*(k)$$

Proof :

Let, x(n) and y(n) be N-point sequences. Now by definition of DFT,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi mk}{N}} \quad ; \quad k = 0, 1, 2, ... N-1 \qquad \boxed{\text{Let } n = m} \qquad ....(9.12)$$

$$Y(k) = \sum_{n=0}^{N-1} y(n)\, e^{\frac{-j2\pi nk}{N}} = \sum_{p=0}^{N-1} y(p)\, e^{\frac{-j2\pi pk}{N}} \quad ; \quad k = 0, 1, 2, ... N-1 \qquad \boxed{\text{Let } n = p} \qquad .....(9.13)$$

Consider the product $X(k)Y^*(k)$. The inverse DFT of the product is given by,

$$\mathcal{DFT}^{-1}\{X(k)\,Y^*(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\,Y^*(k)\,e^{\frac{j2\pi nk}{N}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x(m)\,e^{\frac{-j2\pi mk}{N}}\right]\left[\sum_{p=0}^{N-1} y(p)\;e^{\frac{-j2\pi pk}{N}}\right]^* e^{\frac{j2\pi nk}{N}}$$ Using equations (9.12) and (9.13)

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m)\sum_{p=0}^{N-1} y^*(p)\sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m+p)}{N}} \quad .....(9.14)$$

Let, $n-m+p = qN$, where q is an integer.

Since q is an integer, $e^{j2\pi q} = 1$

$$\therefore \sum_{k=0}^{N-1} e^{\frac{j2\pi k(n-m+p)}{N}} = \sum_{k=0}^{N-1} e^{\frac{j2\pi kqN}{N}} = \sum_{k=0}^{N-1}\left(e^{j2\pi q}\right)^k = \sum_{k=0}^{N-1} 1^k = N \quad .....(9.15)$$

Since, $n-m+p = qN$, $p = n-m+qN$

$$\therefore \sum_{p=0}^{N-1} y^*(p) = \sum_{m=0}^{N-1} y^*(n-m+qN) = \sum_{m=0}^{N-1} y^*(n-m,\ \text{mod } N) = \sum_{m=0}^{N-1} y^*((n-m))_N \quad .....(9.16)$$

Using equations (9.15) and (9.16), the equation (9.14) can be written as shown below.

$$\mathcal{DFT}^{-1}\{X(k)\,Y^*(k)\} = \frac{1}{N}\sum_{m=0}^{N-1} x(m)\sum_{m=0}^{N-1} y^*((n-m))_N\,N$$

$$= \sum_{m=0}^{N-1} x(m)\,y^*((n-m))_N = \bar{r}_{xy}(m)$$ Using definition of circular convolution

$$\therefore X(k)\,Y^*(k) = \mathcal{DFT}\left\{\bar{r}_{xy}(m)\right\}$$

10. Parseval's relation

Let $\mathcal{DFT}\{x_1(n)\} = X_1(k)$ and $\mathcal{DFT}\{x_2(n)\} = X_2(k)$, then by Parseval's relation,

$$\sum_{n=0}^{N-1} x_1(n)\,x_2^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k)$$

Proof :

Let, $x_1(n)$ and $x_2(n)$ be N-point sequences.

Now by definition of DFT, $X_1(k) = \sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi nk}{N}}$ .....(9.17)

Now by definition of inverse DFT, $x_2(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\,e^{\frac{j2\pi nk}{N}}$ .....(9.18)

Consider the right hand side term of parseval's relation.

$$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\,X_2^*(k) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)\,e^{\frac{-j2\pi nk}{N}}\right]X_2^*(k)$$ Using equation (9.17)

$$= \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2^*(k)\,e^{\frac{-j2\pi nk}{N}}\right] = \sum_{n=0}^{N-1} x_1(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} X_2(k)\,e^{\frac{j2\pi nk}{N}}\right]^*$$

$$= \sum_{n=0}^{N-1} x_1(n)\,x_2^*(n)$$ Using equation (9.18)

Table 9.1 : Properties of Discrete Fourier Transform (DFT)

Note : $X(k) = \mathcal{DFT}\{x(n)\}$; $X_1(k) = \mathcal{DFT}\{x_1(n)\}$; $X_2(k) = \mathcal{DFT}\{x_2(n)\}$; $Y(k) = \mathcal{DFT}\{y(n)\}$

Property	Discrete time signal	Discrete Fourier Transform
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2\,X_2(k)$
Periodicity	$x(n + N) = x(n)$	$X(k + N) = X(k)$
Circular time shift	$x((n - m))_N$	$X(k)\,e^{\frac{-j2\pi k m}{N}}$
Time reversal	$x(N - n)$	$X(N - k)$
Conjugation	$x^*(n)$	$X^*(N - k)$
Circular frequency shift	$x(n)\,e^{\frac{j2\pi m n}{N}}$	$X((k - m))_N$
Multiplication	$x_1(n)\,x_2(n)$	$\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$
Circular convolution	$x_1(n) \circledast x_2(n) = \sum\limits_{m=0}^{N-1} x_1(m)\,x_2((n - m))_N$	$X_1(k)\,X_2(k)$
Circular correlation	$\bar{r}_{xy}(m) = \sum\limits_{n=0}^{N-1} x(n)\,y^*((n - m))_N$	$X(k)\,Y^*(k)$
Symmetry of real signals	$x(n)$ is real	$X(k) = X^*(N - k)$ $X_r(k) = X_r(N - k)$ $X_i(k) = -X_i(N - k)$ $\lvert X(k)\rvert = \lvert X(N - k)\rvert$ $\angle X(k) = -\angle X(N - k)$
Symmetry of real and even signal	$x(n)$ is real and even $x(n) = x(N - n)$	$X(k) = X_r(k)$ and $X_i(k) = 0$
Symmetry of real and odd signal	$x(n)$ is real and odd $x(n) = -x(N - n)$	$X(k) = jX_i(k)$ and $X_r(k) = 0$
Parseval's relation	$\sum\limits_{n=0}^{N-1} x_1(n)\,x_2^*(n)$	$\frac{1}{N}\sum\limits_{k=0}^{N-1} X_1(k)\,X_2^*(k)$

9.4 Relation Between DFT and $\mathcal{Z}$-Transform

The $\mathcal{Z}$-transform of N-point sequence x(n) is given by,

$$\mathcal{Z}\{x(n)\} = X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

Let us evaluate X(z) at N equally spaced points on unit circle, i.e., at $z = e^{\frac{j2\pi k}{N}}$

> ***Note :*** Since, $\left| e^{\frac{j2\pi k}{N}} \right| = 1$ *and* $\angle e^{\frac{j2\pi k}{N}} = \frac{2\pi k}{N}$,
>
> *the term,* $z = e^{\frac{j2\pi k}{N}}$, *for* $k = 0,1,2,3.....N-1$
>
> *represents N equally spaced points on unit circle in z-plane.*

$$\therefore\ X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n) z^{-n}\Big|_{z=e^{\frac{j2\pi k}{N}}} = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(9.19)$$

By the definition of N-point DFT we get,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \qquad .....(9.20)$$

From equations (9.19) and (9.20) we can say that,

$$X(k) = X(z)\Big|_{z=e^{\frac{j2\pi k}{N}}} \qquad .....(9.21)$$

From equation (9.21) we can conclude that the N-point DFT of a finite duration sequence can be obtained from the $\mathcal{Z}$-transform of the sequence, by evaluating the $\mathcal{Z}$-transform of the sequence at N equally spaced points around the unit circle. Since the evaluation is performed on unit circle the ROC of X(z) should include unit circle.

9.5 Analysis of LTI Discrete Time Systems Using DFT

In chapter-8 section-8.6, it is shown that Fourier transform is an useful tool for the analysis of discrete time systems in frequency domain. But the drawback in Fourier transform is that it is a continuous function of ω and so it will not be useful for digital processing of signals and systems. Hence DFT is proposed, therefore the analysis of discrete time systems in frequency domain can be conveniently performed using DFT for digital processing of signals and systems.

Discrete Frequency Spectrum

In general the DFT of a signal gives the discrete frequency spectrum of a signal.

Let x(n) and X(k) be a DFT pair.

Now, X(k) = Discrete frequency spectrum of discrete time signal.

|X(k)| = Magnitude spectrum of discrete time signal.

∠X(k) = Phase spectrum of discrete time signal.

In particular, the DFT of impulse response, h(n) of a discrete time system gives discrete frequency response or frequency spectrum of the discrete time system.

Let h(n) and H(k) be a DFT pair.

Now, H(k) = Discrete frequency spectrum of discrete time system.

|H(k)| = Magnitude spectrum of discrete time system.

∠H(k) = Phase spectrum of discrete time system.

Response of LTI Discrete Time System Using DFT

The response of an LTI discrete time system is given by linear convolution of input and impulse response of the system.

Let, x(n) = Input to an LTI system

h(n) = Impulse response of the LTI system

Now, the response or output of the system y(n) is given by,

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$\text{where,}\ \ x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(9.22)$$

The DFT supports only circular convolution and so, the linear convolution of equation (9.22) has to be computed via circular convolution. If x(n) is N_1-point sequence and h(n) is N_2-point sequence then linear convolution x(n) and h(n) will generate y(n) of size $N_1 + N_2 - 1$. Therefore in order to perform linear convolution via circular convolution the x(n) and h(n) should be converted to $N_1 + N_2 - 1$ point sequences by appending zeros. Now the circular convolution of $N_1 + N_2 - 1$ point sequences x(n) and h(n) will give same result as that of linear convolution.

Let, x(n) be N_1-point sequence and h(n) be N_2-point sequence. Let us convert x(n) and h(n) to N_1+N_2-1 point sequences.

Let, Y(k) = $N_1 + N_2 - 1$ point DFT of y(n)

X(k) = $N_1 + N_2 - 1$ point DFT of x(n)

H(k) = $N_1 + N_2 - 1$ point DFT of h(n)}

Now by circular convolution theorem of DFT,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\, H(k)$$

On taking inverse DFT of the above equation we get,

$$x(n) \circledast h(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\}$$

Since, $x(n) \circledast h(n) = y(n)$, the above equation can be written as,

$$y(n) = \mathcal{DFT}^{-1}\{X(k)\, H(k)\} \qquad .....(9.23)$$

From the equation (9.23) we can say that the output y(n) is given by the inverse DFT of the product of X(k) and H(k). Hence to determine the response of an LTI discrete time system, first find $N_1 + N_2 - 1$ point DFT of input x(n) to get X(k) and $N_1 + N_2 - 1$ point DFT of impulse response h(n) to get H(k), then take inverse DFT of the product X(k) H(k).

Example 9.1

Compute 4-point DFT and 8-point DFT of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \quad ; \quad 0 \le n \le 2$$
$$= 0 \quad ; \quad \text{else}$$

Show that DFT coefficients are samples of Fourier transform of x(n), (Refer example 8.6 for Fourier transform).

Solution

By the definition of N-point DFT, the k^{th} complex coefficient of X(k), for $0 \le k \le N-1$, is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

a) 4-point DFT ($\therefore$ N = 4)

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{2}} = x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k}$$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= \frac{1}{3} + \frac{1}{3} e^{\frac{-j\pi k}{2}} + \frac{1}{3} e^{-j\pi k} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos\pi k - j\sin\pi k\right]$$

For 4-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3.

$$\text{When } k = 0\,;\; X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$$
$$= \frac{1}{3}(1 + 1 + 1) = 1 = 1\angle 0$$

$$\text{When } k = 1\,;\; X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$$
$$= \frac{1}{3}(1 + 0 - j - 1 - j0) = -j\frac{1}{3} = \frac{1}{3}\angle -\pi/2 = 0.333\angle -0.5\pi$$

$$\text{When } k = 2\,;\; X(2) = \frac{1}{3}[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi]$$
$$= \frac{1}{3}(1 - 1 - j0 + 1 - j0) = \frac{1}{3} = 0.333\angle 0$$

$$\text{When } k = 3\,;\; X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi\right]$$
$$= \frac{1}{3}(1 + 0 + j - 1 - j0) = j\frac{1}{3} = \frac{1}{3}\angle\pi/2 = 0.333\angle 0.5\pi$$

$\therefore$ The 4-point DFT sequence X(k) is given by,

$$X(k) = \{\, 1\angle 0,\;\; 0.333\angle -0.5\pi,\;\; 0.333\angle 0,\;\; 0.333\angle 0.5\pi \,\}$$

$$\therefore \text{ Magnitude Function, } |X(k)| = \{\, 1,\;\; 0.333,\;\; 0.333,\;\; 0.333 \,\}$$

$$\text{Phase Function, } \angle X(k) = \{0,\;\; -0.5\pi,\;\; 0,\;\; 0.5\pi\,\}$$

Phase angles are in radians

b) 8-point DFT ($\therefore$ N = 8)

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j2\pi kn}{8}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{4}} = x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{4}} + x(2)\, e^{\frac{-j\pi k}{2}} \qquad \boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1}{3} + \frac{1}{3} e^{\frac{-j\pi k}{4}} + \frac{1}{3} e^{\frac{-j\pi k}{2}} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{4} - j\sin\frac{\pi k}{4} + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right]$$

For 8-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3, 4, 5, 6, 7.

When k = 0 ; $X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$

$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = 1 = 1\angle 0$

When k = 1 ; $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$

$= 0.333\ (1 + 0.707 - j0.707 + 0 - j1)$

$= 0.568 - j0.568 = 0.803\angle -0.785 = 0.803\angle -0.25\pi$

When k = 2 ; $X(2) = \frac{1}{3}\left[1 + \cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4} + \cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right]$

$= 0.333\ (1 + 0 - j1 - 1 - j0)$

$= -j0.333 = 0.333\angle -\pi/2 = 0.333\angle -0.5\pi$

When k = 3 ; $X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 0.333\ (1 - 0.707 - j0.707 + 0 + j1)$

$= 0.098 + j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$

When k = 4 ; $X(4) = \frac{1}{3}\left[1 + \cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4} + \cos\frac{4\pi}{2} - j\sin\frac{4\pi}{2}\right]$

$= 0.333\ (1 - 1 - j0 + 1 - j0) = 0.333 = 0.333\angle 0$

When k = 5 ; $X(5) = \frac{1}{3}\left[1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}\right]$

$= 0.333\ (1 - 0.707 + j0.707 + 0 - j1)$

$= 0.098 - j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$

When k = 6 ; $X(6) = \frac{1}{3}\left[1 + \cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4} + \cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right]$

$= 0.333\ (1 + 0 + j1 - 1 - j0)$

$= j0.333 = 0.333\angle \pi/2 = 0.333\angle 0.5\pi$

When k = 7 ; $X(7) = \frac{1}{3}\left[1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}\right]$

$= 0.333\ (1 + 0.707 + j0.707 + 0 + j1)$

$= 0.568 + j0.568 = 0.803\angle 0.785 = 0.803\angle 0.25\pi$

$\therefore$ The 8-point DFT sequence X(k) is given by,

Phase angles are in radians

$X(k) = \{1\angle 0,\ 0.803\angle -0.25\pi,\ 0.333\angle -0.5\pi,\ 0.139\angle -0.25\pi,\ 0.333\angle 0,\ 0.139\angle 0.25\pi,\ 0.333\angle 0.5\pi,\ 0.803\angle 0.25\pi\}$

$\therefore$ Magnitude Function, $|X(k)| = \{1,\ 0.803,\ 0.333,\ 0.139,\ 0.333,\ 0.139,\ 0.333,\ 0.803\}$

Phase Function, $\angle X(k) = \{0,\ -0.25\pi,\ -0.5\pi,\ -0.25\pi,\ 0,\ 0.25\pi,\ 0.5\pi,\ 0.25\pi\}$

The Magnitude spectrum of X(k) are shown in fig 1, 2 & 3 for N = 4, N = 8, & N = 16 respectively. The curve shown in dotted line is the sketch of Magnitude function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the magnitude of DFT coefficients are samples of Magnitude function of $X(e^{j\omega})$, (Refer example 8.6 for the Magnitude function of $X(e^{j\omega})$).

The Phase spectrum of X(k) are shown in fig 4, 5 & 6 for N = 4, N = 8, & N = 16 respectively. The curve shown in dotted line is the sketch of Phase function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the phase of the DFT coefficients are samples of Phase function of $X(e^{j\omega})$, (Refer example 8.6 for the Phase function of $X(e^{j\omega})$).

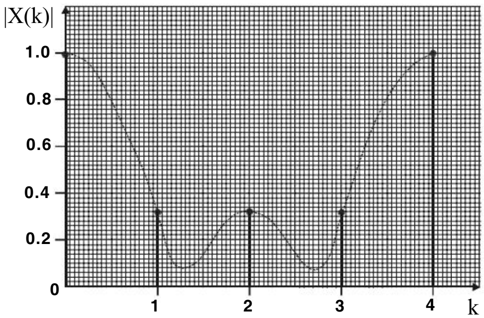


***Fig 1 :** Magnitude spectrum of X(k) for N=4.*

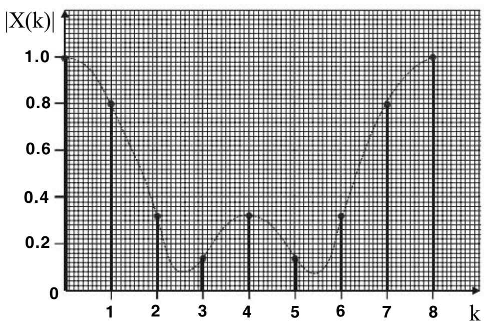


***Fig 2 :** Magnitude spectrum of X(k) for N=8.*

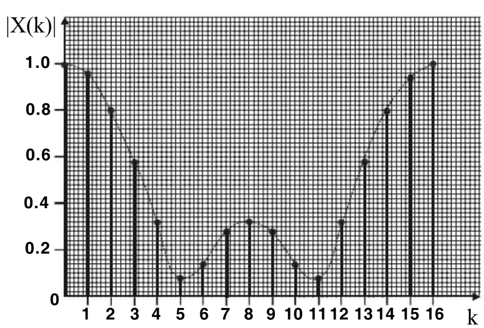


***Fig 3 :** Magnitude spectrum of X(k) for N=16.*

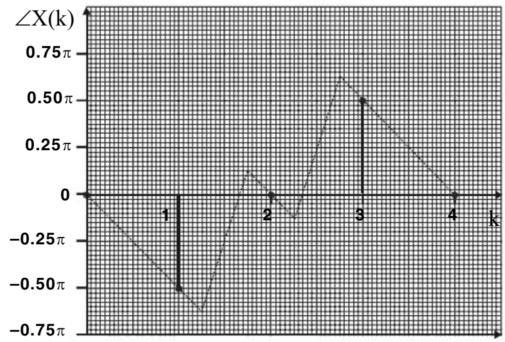


***Fig 4 :** Phase spectrum of X(k) for N=4.*

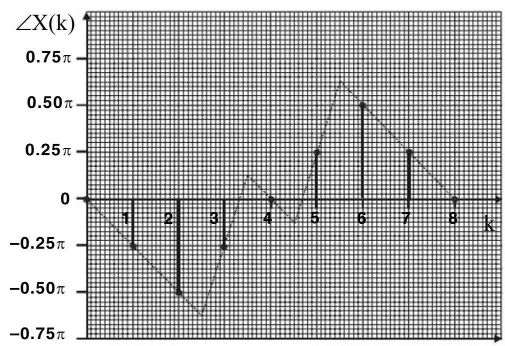


***Fig 5 :** Phase spectrum of X(k) for N=8.*

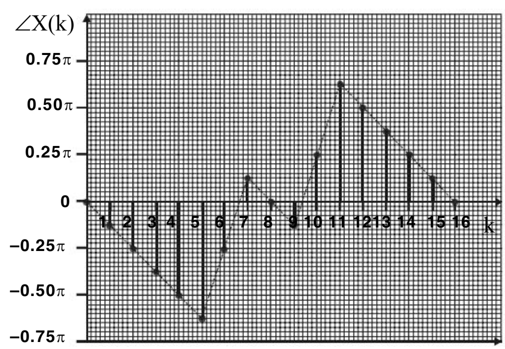


***Fig 6 :** Phase spectrum of X(k) for N=16.*

Example 9.2

Compute the DFT of the sequence, $x(n) = \{0, 1, 2, 3\}$. Sketch the magnitude and phase spectrum.

Solution

By the definition of DFT, the 4-point DFT is given by,

$\boxed{\text{Here, } x(n) \text{ is 4-point sequence}, \therefore \text{compute 4-point DFT}}$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

$$= x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi k}{2}} + x(2)\,e^{-j\pi k} + x(3)\,e^{\frac{-j3\pi k}{2}} = 0 + e^{\frac{-j\pi k}{2}} + 2\,e^{-j\pi k} + 3\,e^{\frac{-j3\pi k}{2}}$$

$$= \left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) + 2(\cos \pi k - j\sin \pi k) + 3\left(\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right)$$

When $k = 0$; $X(0) = (\cos 0 - j\sin 0) + 2(\cos 0 - j\sin 0) + 3(\cos 0 - j\sin 0)$

$$= (1 - j0) + 2(1 - j0) + 3(1 - j0) = 6 = 6\angle 0$$

When $k = 1$; $X(1) = \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 2(\cos\pi - j\sin\pi) + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$

$$= (0 - j1) + 2(-1 - j0) + 3(0 + j1) = -2 + 2j$$

$$= 2.8\angle 135^\circ = 2.8\angle 135^\circ \times \frac{\pi}{180^\circ} = 2.8\angle 0.75\pi$$

When $k = 2$; $X(2) = \left(\cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right) + 2(\cos 2\pi - j\sin 2\pi) + 3\left(\cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right)$

$$= (-1 - j0) + 2(1 - j0) + 3(-1 + j0) = -2$$

$$= 2\angle 180^\circ = 2\angle 180^\circ \times \frac{\pi}{180^\circ} = 2\angle\pi$$

When $k = 3$; $X(3) = \left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 2(\cos 3\pi - j\sin 3\pi) + 3\left(\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right)$

$$= (0 + j1) + 2(-1 - j0) + 3(0 - j1) = -2 - j2$$

$$= 2.8\angle -135^\circ = 2.8\angle -135^\circ \times \frac{\pi}{180^\circ} = 2.8\angle -0.75\pi$$

$\therefore X(k) = \{6\angle 0,\ \ 2.8\angle 0.75\pi,\ \ 2\angle\pi,\ \ 2.8\angle -0.75\pi\}$

Magnitude Spectrum, $|X(k)| = \{6,\ 2.8,\ 2,\ 2.8\}$

$\boxed{\text{Phase angles are in radians}}$

Phase Spectrum, $\angle X(k) = \{0,\ 0.75\pi,\ \pi,\ -0.75\pi\}$

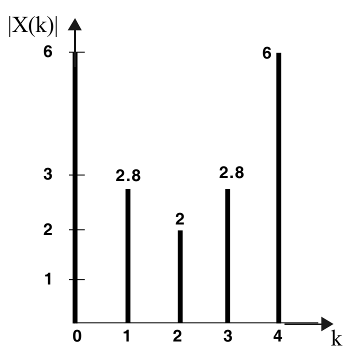


Fig 1 : Magnitude Spectrum.

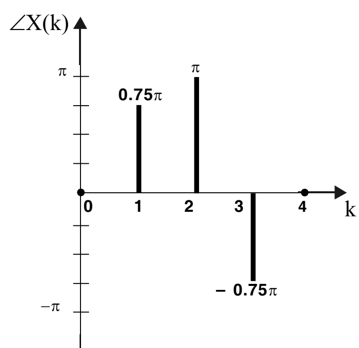


Fig 2 : Phase Spectrum.

Example 9.3

Compute circular convolution of the following two sequences using DFT.

$x_1(n) = \{ \underset{\uparrow}{2}, 1, 2, 1 \}$ and $x_2(n) = \{ \underset{\uparrow}{1}, 2, 3, 4 \}$

Solution

Given that, $x_1(n) = \{ 2, 1, 2, 1 \}$. The 4-point DFT of $x_1(n)$ is,

$$\mathcal{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{3} x_1(n)\, e^{\frac{-j2\pi nk}{4}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_1(0)\, e^0 + x_1(1)\, e^{\frac{-j2\pi k}{4}} + x_1(2)\, e^{\frac{-j2\pi 2k}{4}} + x_1(3)\, e^{\frac{-j2\pi 3k}{4}} = 2 + e^{\frac{-j\pi k}{2}} + 2\, e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_1(0) = 2 + 1 + 2 + 1 = 6$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 1$; $X_1(1) = 2 + e^{\frac{-j\pi}{2}} + 2\, e^{-j\pi} + e^{\frac{-j3\pi}{2}} = 2 - j - 2 + j = 0$

When $k = 2$; $X_1(2) = 2 + e^{-j\pi} + 2\, e^{-j2\pi} + e^{-j3\pi} = 2 - 1 + 2 - 1 = 2$

When $k = 3$; $X_1(3) = 2 + e^{\frac{-j3\pi}{2}} + 2\, e^{-j3\pi} + e^{\frac{-j9\pi}{2}} = 2 + j - 2 - j = 0$

Given that, $x_2(n) = \{ 1, 2, 3, 4 \}$. The 4-point DFT of $x_2(n)$ is,

$$\mathcal{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{3} x_2(n)\, e^{\frac{-j2\pi nk}{4}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_2(0)\, e^0 + x_2(1)\, e^{\frac{-j\pi k}{2}} + x_2(2)\, e^{-j\pi k} + x_2(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 1 + 2\, e^{\frac{-j\pi k}{2}} + 3\, e^{-j\pi k} + 4\, e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_2(0) = 1 + 2 + 3 + 4 = 10$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 1$; $X_2(1) = 1 + 2\, e^{\frac{-j\pi}{2}} + 3\, e^{-j\pi} + 4\, e^{\frac{-j3\pi}{2}} = 1 - 2j - 3 + 4j = -2 + j2$

When $k = 2$; $X_2(2) = 1 + 2\, e^{-j\pi} + 3\, e^{-j2\pi} + 4\, e^{-j3\pi} = 1 - 2 + 3 - 4 = -2$

When $k = 3$; $X_2(3) = 1 + 2\, e^{\frac{-j3\pi}{2}} + 3\, e^{-j3\pi} + 4\, e^{\frac{-j9\pi}{2}} = 1 + 2j - 3 - 4j = -2 - j2$

$$X_1(k) = \begin{cases} 6 \; ; \; k = 0 \\ 0 \; ; \; k = 1 \\ 2 \; ; \; k = 2 \\ 0 \; ; \; k = 3 \end{cases} \qquad X_2(k) = \begin{cases} 10 & ; \; k = 0 \\ -2 + j2 & ; \; k = 1 \\ -2 & ; \; k = 2 \\ -2 - j2 & ; \; k = 3 \end{cases}$$

Let $X_3(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$$\therefore X_3(k) = X_1(k)\, X_2(k)$$

When $k = 0$; $X_3(0) = X_1(0) \times X_2(0) = 6 \times 10 = 60$

When $k = 1$; $X_3(1) = X_1(1) \times X_2(1) = 0 \times (-2 + j2) = 0$

When $k = 2$; $X_3(2) = X_1(2) \times X_2(2) = 2 \times (-2) = -4$

When $k = 3$; $X_3(3) = X_1(3) \times X_2(3) = 0 \times (-2 - j2) = 0$

$$\therefore X_3(k) = \{60, \; 0, \; -4, \; 0\}$$

By circular convolution theorem of DFT we get,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\, X_2(k) \qquad \Rightarrow \qquad x_1(n) \circledast x_2(n) = \mathcal{DFT}^{-1}\{X_1(k)\, X_2(k)\} = \mathcal{DFT}^{-1}\{X_3(k)\}$$

Let $x_3(n)$ be the sequence obtained by taking inverse DFT of $X_3(k)$.

$$\mathcal{DFT}^{-1}\{X_3(k)\} = x_3(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_3(k)\, e^{\frac{j2\pi nk}{4}} \qquad ; \qquad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[X_3(0)\, e^0 + X_3(2)\, e^{j\pi n}\right] = \frac{1}{4}\left[60 - 4\, e^{j\pi n}\right]$$

When $n = 0$; $\quad x_3(0) = \frac{1}{4}(60 - 4) = \frac{56}{4} = 14$

When $n = 1$; $\quad x_3(1) = \frac{1}{4}(60 - 4e^{j\pi}) = \frac{64}{4} = 16$

When $n = 2$; $\quad x_3(2) = \frac{1}{4}(60 - 4e^{j2\pi}) = \frac{56}{4} = 14$

When $n = 3$; $\quad x_3(3) = \frac{1}{4}(60 - 4e^{j3\pi}) = \frac{64}{4} = 16$

$$\therefore\ x_1(n) \circledast x_2(n) = x_3(n) = \{14,\ 16,\ 14,\ 16\}$$

Example 9.4

Compute linear and circular convolution of the following two sequences using DFT.

$$x(n) = \{\underset{\uparrow}{1},\ 0.5\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{0.5},\ 1\}$$

Solution

Linear Convolution by DFT

The linear convolution of x(n) and h(n) will produce a 3 sample sequence. To avoid time aliasing we convert the 2 sample input sequences into 3-sample sequences by padding with zeros.

$$\therefore\ x(n) = \{\underset{\uparrow}{1},\ 0.5,\ 0\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{0.5},\ 1,\ 0\}$$

By the definition of N-point DFT, the three point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{\frac{-j2\pi kn}{3}} = x(0)\, e^0 + x(1)\, e^{\frac{-j2\pi k}{3}} + x(2)\, e^{\frac{-j4\pi k}{3}} = 1 + 0.5\ e^{\frac{-j2\pi k}{3}}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

When $k = 0$; $\quad X(0) = 1 + 0.5 = 1.5$

When $k = 1$; $\quad X(1) = 1 + 0.5\, e^{\frac{-j2\pi}{3}} = 1 + 0.5(-0.5 - j0.866) = 0.75 - j0.433$

When $k = 2$; $\quad X(2) = 1 + 0.5\, e^{\frac{-j4\pi}{3}} = 1 + 0.5(-0.5 + j0.866) = 0.75 + j0.433$

By the definition of N-point DFT, the three point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{2} h(n)\, e^{\frac{-j2\pi kn}{3}} = h(0)\, e^0 + h(1)\, e^{\frac{-j2\pi k}{3}} + h(2)\, e^{\frac{-j4\pi k}{3}} = 0.5 + e^{\frac{-j2\pi k}{3}}$$

When $k = 0$; $\quad H(0) = 0.5 + 1 = 1.5$

When $k = 1$; $\quad H(1) = 0.5 + e^{\frac{-j2\pi}{3}} = 0.5 - 0.5 - j0.866 = -j0.866$

When $k = 2$; $\quad H(2) = 0.5 + e^{\frac{-j4\pi}{3}} = 0.5 - 0.5 + j0.866 = j0.866$

Let, Y(k) = X(k) H(k) ; for k = 0, 1, 2

When k = 0 ; $Y(0) = X(0)\ H(0) = 1.5 \times 1.5 = 2.25$

When k = 1 ; $Y(1) = X(1)\ H(1) = (0.75 - j0.433) \times (-j0.866) = -0.375 - j0.6495$

When k = 2 ; $Y(2) = X(2)\ H(2) = (0.75 + j0.433) \times (j0.866) = -0.375 + j0.6495$

$$\therefore\ Y(k) = \{\underset{\uparrow}{2.25},\quad -0.375 - j0.6495,\quad -0.375 + j0.6495\}$$

The sequence y(n) is obtained from inverse DFT of Y(k). By definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

$$y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k)\, e^{\frac{j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[Y(0)\, e^{0} + Y(1)\, e^{\frac{j2\pi n}{3}} + Y(2)\, e^{\frac{j4\pi n}{3}}\right] \quad ; \quad \text{for } n = 0, 1, 2$$

When n = 0; $y(0) = \frac{1}{3}\big[Y(0) + Y(1) + Y(2)\big]$

$$= \frac{1}{3}\big[2.25 - 0.375 - j0.6495 - 0.375 + j0.6495\big] = \frac{1}{3}\big[1.5\big] = 0.5$$

When n = 1; $y(1) = \frac{1}{3}\left[Y(0) + Y(1)\, e^{\frac{j2\pi}{3}} + Y(2)\, e^{\frac{j4\pi}{3}}\right]$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= \frac{1}{3}\left[\begin{array}{l} 2.25 + (-0.375 - j0.6495)\,(-0.5 + j0.866) \\ \quad + (-0.375 + j0.6495)\quad(-0.5 - j0.866) \end{array}\right]$$

$$= \frac{1}{3}\big[2.25 + 0.75 + j0 + 0.75 + j0\big] = 1.25$$

When n = 2; $y(2) = \frac{1}{3}\left[Y(0) + Y(1)\, e^{\frac{j4\pi}{3}} + Y(2)\, e^{\frac{j8\pi}{3}}\right]$

$$= \frac{1}{3}\left[\begin{array}{l} 2.25 + (-0.375 - j0.6495)\,(-0.5 - j0.866) \\ \quad + (-0.375 + j0.6495)\,(-0.5 + j0.866) \end{array}\right]$$

$$= \frac{1}{3}\big[2.25 - 0.375 + j6495 - 0.375 - j0.649\big] = \frac{1}{3}\big[1.5\big] = 0.5$$

$$\therefore\ x(n) * h(n) = y(n) = \{\underset{\uparrow}{0.5},\ 1.25,\ 0.5\}$$

Circular Convolution by DFT

The given sequences are 2-point sequences. Hence 2-point DFT of the sequences are obtained as follows.

The 2-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{1} x(n)\, e^{\frac{-j2\pi kn}{2}} = x(0)\, e^{0} + x(1)\ e^{-j\pi k} = 1 + 0.5\, e^{-j\pi k} \quad ; \quad \text{for } k = 0, 1$$

When k = 0; $X(0) = 1 + 0.5 = 1.5$

When k = 1; $X(1) = 1 + 0.5\, e^{-j\pi} = 1 - 0.5 = 0.5$

$$\therefore\ X(k) = \{\underset{\uparrow}{1.5},\ 0.5\}$$

The 2-point DFT of h(n) is given by,

$$H(k) = \sum_{n=0}^{1} h(n)\, e^{\frac{-j2\pi kn}{2}} = h(0)\, e^{0} + h(1)\, e^{-j\pi k} = 0.5 + e^{-j\pi k} \quad ; \quad \text{for } k = 0,\ 1$$

When $k = 0$; $\quad H(0) = 0.5 + 1 = 1.5$

When $k = 1$; $\quad H(1) = 0.5 + e^{-j\pi} = 0.5 - 1 = -0.5$

$$\therefore \quad H(k) = \{\underset{\uparrow}{1.5},\ -0.5\}$$

Let the product of X(k) and H(k) be equal to Y(k).

$$\therefore\ Y(k) = X(k)\, H(k) = \begin{cases} 1.5 \times 1.5 & = 2.25 \quad ;\ k = 0 \\ 0.5 \times (-0.5) & = -0.25 \quad ;\ k = 1 \end{cases}$$

The sequence y(n) is obtained from inverse DFT of Y(k). By the definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0,\ 1,\ 2,\ \ldots..,\ N-1$$

Here, $N = 2$

$$\therefore\ y(n) = \frac{1}{2} \sum_{k=0}^{1} Y(k)\, e^{\frac{j2\pi kn}{2}} = \frac{1}{2}\left(Y(0) + Y(1)\, e^{j\pi n}\right) = \frac{1}{2}\left(2.25 - 0.25\, e^{j\pi n}\right)$$

When $n = 0$; $\quad y(0) = \frac{1}{2}(2.25 - 0.25) = 1$

When $n = 1$; $\quad y(1) = \frac{1}{2}(2.25 - 0.25\, e^{j\pi}) = \frac{1}{2}(2.25 + 0.25) = 1.25$

$$\therefore\ x(n) \circledast h(n) = y(n) = \{\underset{\uparrow}{1},\ 1.25\}$$

9.6 Fast Fourier Transform (FFT)

The **F**ast **F**ourier **T**ransform (**FFT**) is a method (or algorithm) for computing the discrete Fourier transform (DFT) with reduced number of calculations. The computational efficiency is achieved if we adopt a divide and conquer approach. This approach is based on the decomposition of an N-point DFT into successively smaller DFTs. This basic approach leads to a family of an efficient computational algorithms known collectively as FFT algorithms.

Radix-r FFT

In an N-point sequence, if N can be expressed as $N = r^m$, then the sequence can be decimated into r-point sequences. For each r-point sequence, r-point DFT can be computed. From the results of r-point DFT, the r^2-point DFTs are computed. From the results of r^2-point DFTs, the r^3-point DFTs are computed and so on, until we get r^m point DFT. This FFT algorithm is called radix-**r** FFT. In computing N-point DFT by this method the number of stages of computation will be **m** times.

Radix-2 FFT

For radix-2 FFT, the value of N should be such that, $N = 2^m$, so that the N-point sequence is decimated into 2-point sequences and the 2-point DFT for each decimated sequence is computed. From the results of 2-point DFTs, the 4-point DFTs can be computed. From the results of 4-point DFTs, the 8-point DFTs can be computed and so on, until we get N-point DFT.

Number of Calculations in N-point DFT

Let, X(k) be N-point DFT of an L-point discrete time sequence x(n), where $N \geq L$. Now, the N-point DFT is a sequence consisting of N-complex numbers. Each complex number of the sequence is calculated using the following equation, (equation 9.2).

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \quad \text{for } k = 0,1,2,\ldots\ldots, N-1$$

$$= x(0)\, e^{0} + x(1)\, e^{\frac{-j2\pi k}{N}} + x(2)\, e^{\frac{-j4\pi k}{N}} + x(3)\, e^{\frac{-j6\pi k}{N}} + \ldots\ldots\ldots\ldots + x(N-1)\, e^{\frac{-j2(N-1)\pi k}{N}}$$

The computation of above equation for one value of k involves N number of complex multiplications and N−1 number of complex additions.Therefore, the computation of all the N values of the sequence X(k) involves $N\times N = N^2$ complex multiplications and $N\times(N-1)$ complex additions. Hence, in direct computation of N-point DFT, the total number of complex additions are N(N–1) and total number of complex multiplications are N^2.

Number of Calculations in Radix-2 FFT

In radix-2 FFT, $N = 2^m$, and so there will be m-stages of computations, where $m = \log_2 N$, with each stage having N/2 butterflies. (Refer section 9.7.2 and 9.8.2). Each butterfly involve one complex multiplication and two complex additions. The number of computations in each stage are $1\times N/2 = N/2$ complex multiplications and $2\times N/2 = N$ complex additions.Therefore, the total number of computations for m-stages will be $(N/2)\times m = (N/2)\log_2 N$ complex multiplications and $N\times m = N\log_2 N$ complex additions. Hence, in radix-2 FFT, the total number of complex additions are reduced to $N\log_2 N$ and total number of complex multiplications are reduced to $(N/2)\log_2 N$.

The table 9.2 presents a comparison of the number of complex multiplications and additions in radix-2 FFT and in direct computation of DFT. From the table it can be observed that for large values of N, the percentage reduction in calculations is also very large.

$\boxed{\log_2 2^m = m}$ $\boxed{\log_y x = \frac{\log_{10} x}{\log_{10} y}}$

Table 9.2 : Comparison of Number of Computation in Direct DFT and FFT

Number of points N	Direct Computation		Radix-2 FFT	
	Complex additions N(N–1)	Complex Multiplications N^2	Complex additions $N\log_2 N$	Complex Multiplications $(N/2)\log_2 N$
$4\ (=2^2)$	12	16	$4\times\log_2 2^2 = 8$	$\frac{4}{2}\times\log_2 2^2 = 4$
$8\ (=2^3)$	56	64	$8\times\log_2 2^3 = 24$	$\frac{8}{2}\times\log_2 2^3 = 12$
$16\ (=2^4)$	240	256	$16\times\log_2 2^4 = 64$	$\frac{16}{2}\times\log_2 2^4 = 32$
$32\ (=2^5)$	992	1,024	$32\times\log_2 2^5 = 160$	$\frac{32}{2}\times\log_2 2^5 = 80$
$64\ (=2^6)$	4,032	4,096	$64\times\log_2 2^6 = 384$	$\frac{64}{2}\times\log_2 2^6 = 192$
$128\ (=2^7)$	16,256	16,384	$128\times\log_2 2^7 = 896$	$\frac{128}{2}\times\log_2 2^7 = 448$

Phase or Twiddle Factor

By the definition of DFT, the N-point DFT is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \quad ; \text{ for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \ldots\ldots(9.24)$$

To simplify the notation it is desirable to define the complex valued phase factor W_N (also called as twiddle factor) which is an Nth root of unity as,

$$W_N = e^{\frac{-j2\pi}{N}}$$

Here, W represents a complex number $1\angle-2\pi$. Hence the phase or argument of W is -2π. Therefore, when a number is multiplied by W, only its phase changes by -2π but magnitude remains same.

$$\therefore\ W = e^{-j2\pi}$$

The phase value -2π of W can be multiplied by any integer and it is represented as prefix in W. For example multiplying -2π by k can be represented as W^k.

$$\therefore\ e^{-j2\pi \times k} \quad \Rightarrow \quad W^k$$

The phase value -2π of W can be divided by any integer and it is represented as suffix in W. For example dividing -2π by N can be represented as W_N.

$$\therefore\ e^{-j2\pi \div N} = e^{-j2\pi \times \frac{1}{N}} \quad \Rightarrow \quad W_N$$

$$\therefore\ e^{\frac{-j2\pi nk}{N}} = \left(e^{-j2\pi}\right)^{\frac{nk}{N}} = W_N^{nk} \qquad \ldots\ldots(9.25)$$

Using equation (9.25) the equation (9.24) can be written as,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad ; \text{ for } k = 0, 1, 2, \ldots\ldots, N-1 \qquad \ldots\ldots(9.26)$$

The equation (9.26) is the definition of N-point DFT using phase factor, and this equation is popularly used in FFT.

9.7 Decimation in Time (DIT) Radix-2 FFT

The N-point DFT of a sequence x(n) converts the time domain N-point sequence x(n) to a frequency domain N-point sequence X(k). In **D**ecimation **I**n **T**ime (DIT) algorithm, the time domain sequence x(n) is decimated and smaller point DFTs are performed. The results of smaller point DFTs are combined to get the result of N-point DFT.

In DIT radix-2 FFT, the time domain sequence is decimated into 2-point sequences. For each two point sequence, the two point DFT is computed. The results of 2-point DFTs are used to compute 4-point DFTs. Two numbers of 2-point DFTs are combined to get an 4-point DFT. The results of 4-point DFTs are used to compute 8-point DFTs. Two numbers of 4-point DFTs are combined to get an 8-point DFT. This process is continued until we get N-point DFT.

In general we can say that, in decimation in time algorithm, the N-point DFT can be realized from two numbers of N/2 point DFTs, the N/2 point DFT can be realized from two numbers of N/4 points DFTs, and so on.

Let x(n) be N-sample sequence. We can decimate x(n) into two sequences of N/2 samples. Let the two sequences be $f_1(n)$ and $f_2(n)$. Let $f_1(n)$ consists of even numbered samples of x(n) and $f_2(n)$ consists of odd numbered samples of x(n).

$$\therefore\ f_1(n) = x(2n) \quad ; \text{for } n = 0,\ 1,\ 2,\ 3\ .....,\ \frac{N}{2}-1$$
$$f_2(n) = x(2n+1) \quad ; \text{for } n = 0,\ 1,\ 2,\ 3\ .....,\ \frac{N}{2}-1$$

Let, X(k) = N-point DFT of x(n)

$F_1(k)$ = N/2 point DFT of $f_1(n)$

$F_2(k)$ = N/2 point DFT of $f_2(n)$

By definition of DFT the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by,

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn}$$
$$F_2(k) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn}$$

Now, N-point DFT X(k), in terms of N/2 point DFTs $F_1(k)$ and $F_2(k)$ is given by,

$$X(k) = F_1(k) + W_N^k\ F_2(k), \quad \text{where, } k = 0,\ 1,\ 2,\ .....,\ N-1 \qquad .....(9.27)$$

The proof of equation (9.27) is given below.

Proof :

By definition of DFT, the N-point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$
$$= \sum_{n=\text{even}} x(n)\, W_N^{kn} + \sum_{n=\text{odd}} x(n)\, W_N^{kn}\ ;\ k = 0,\ 1,\ 2\ .....,N-1$$
$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_N^{k(2n)} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_N^{k(2n+1)} \qquad .....(9.27)$$

when n → 2n, even numbered samples of x(n) are selected.
when n → 2n +1, odd numbered samples of x(n) are selected.

The phase factors in equation (9.28) can be modified as shown below.

$$W_N^{k(2n)} = \left(e^{-j2\pi}\right)^{\frac{k(2n)}{N}} = \left(e^{-j2\pi}\right)^{\frac{kn}{N/2}} = W_{N/2}^{kn} \qquad .....(9.29)$$
$$W_N^{k(2n+1)} = \left(e^{-j2\pi}\right)^{\frac{k(2n+1)}{N}} = \left(e^{-j2\pi}\right)^{\frac{k2n}{N}} \left(e^{-j2\pi}\right)^{\frac{k}{N}} = \left(e^{-j2\pi}\right)^{\frac{kn}{N/2}} \left(e^{-j2\pi}\right)^{\frac{k}{N}} = W_{N/2}^{kn}\, W_N^k \qquad .....(9.30)$$

Using equations (9.29) and (9.30), the equation (9.28) can be written as,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(2n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1)\, W_{N/2}^{kn}\ W_N^k$$
$$= \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn} \qquad .....(9.31)$$

$x(2n) = f_1(n)$ and $x(2n+1) = f_2(n)$

By definition of DFT the N/2 point DFT of $f_1(n)$ and $f_2(n)$ are given by.

$$F_1(k) = \sum_{n=0}^{\frac{N}{2}-1} f_1(n)\, W_{N/2}^{kn} \quad \text{and} \quad F_2(n) = \sum_{n=0}^{\frac{N}{2}-1} f_2(n)\, W_{N/2}^{kn} \qquad .....(9.32)$$

Using equation (9.32) in equation(9.31) we get,

$$X(k) = F_1(k) + W_N^k\, F_2(k)\,, \qquad \text{where } k = 0, 1, 2, ..., N-1$$

Having performed the decimation in time once, we can repeat the process for each of the sequences $f_1(n)$ and $f_2(n)$. Thus $f_1(n)$ would result in the two N/4 point sequences and $f_2(n)$ would result in another two N/4 point sequences.

Let the decimated N/4 point sequences of $f_1(n)$ be $v_{11}(n)$ and $v_{12}(n)$.

$$\therefore\ v_{11}(n) = f_1(2n) \qquad ;\ \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

$$v_{12}(n) = f_1(2n+1) \qquad ;\ \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

Let the decimated N/4 point sequences of $f_2(n)$ be $v_{21}(n)$ and $v_{22}(n)$.

$$\therefore\ v_{21}(n) = f_2(2n) \qquad ;\ \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

$$v_{22}(n) = f_2(2n+1) \qquad ;\ \text{for } n = 0, 1, 2, ..., \frac{N}{4}-1$$

Let, $V_{11}(k)$ = N/4 point DFT of $v_{11}(n)$; $\quad V_{21}(k)$ = N/4 point DFT of $v_{21}(n)$

$V_{12}(k)$ = N/4 point DFT of $v_{12}(n)$; $\quad V_{22}(k)$ = N/4 point DFT of $v_{22}(n)$

Then like earlier analysis we can show that,

$$F_1(k) = V_{11}(k) + W_{N/2}^k\, V_{12}(k)\ ;\ \text{for } k = 0, 1, 2, ..., \frac{N}{2}-1 \qquad .....(9.33)$$

$$F_2(k) = V_{21}(k) + W_{N/2}^k\, V_{22}(k)\ ;\ \text{for } k = 0, 1, 2, ..., \frac{N}{2}-1 \qquad .....(9.34)$$

Hence the N/2 point DFTs are obtained from the results of N/4 point DFTs.

The decimation of the data sequence can be repeated again and again until the resulting sequences are reduced to 2-point sequences.

9.7.1 8-Point DFT Using Radix-2 DIT FFT

The computation of 8-point DFT using radix-2 FFT, involves three stages of computation. Here $N = 8 = 2^3$, therefore $r = 2$ and $m = 3$. The given 8-point sequence is decimated to 2-point sequences. For each 2-point sequence, the 2-point DFT is computed. From the result of 2-point DFT, the 4-point DFT can be computed. From the result of 4-point DFT, the 8-point DFT can be computed.

Let the given sequence be x(0), x(1), x(2), x(3), x(4),x(5), x(6), x(7), which consists of 8 samples. The 8-samples should be decimated into sequences of 2-samples. Before decimation they are arranged in bit reversed order, as shown in table 9.3.

The x(n) in bit reversed order is decimated into 4 numbers of 2-point sequences as shown below.

Sequence-1 : {x(0), x(4)}

Sequence-2 : {x(2), x(6)}

Sequence-3 : {x(1), x(5)}

Sequence-4 : {x(3), x(7)}

Using the decimated sequences as input the 8-point DFT is computed.The fig 9.1 shows the three stages of computation of an 8-point DFT.

Table 9.3

Normal order		**Bit reversed order**	
x(0)	x(000)	x(0)	x(000)
x(1)	x(001)	x(4)	x(100)
x(2)	x(010)	x(2)	x(010)
x(3)	x(011)	x(6)	x(110)
x(4)	x(100)	x(1)	x(001)
x(5)	x(101)	x(5)	x(101)
x(6)	x(110)	x(3)	x(011)
x(7)	x(111)	x(7)	x(111)

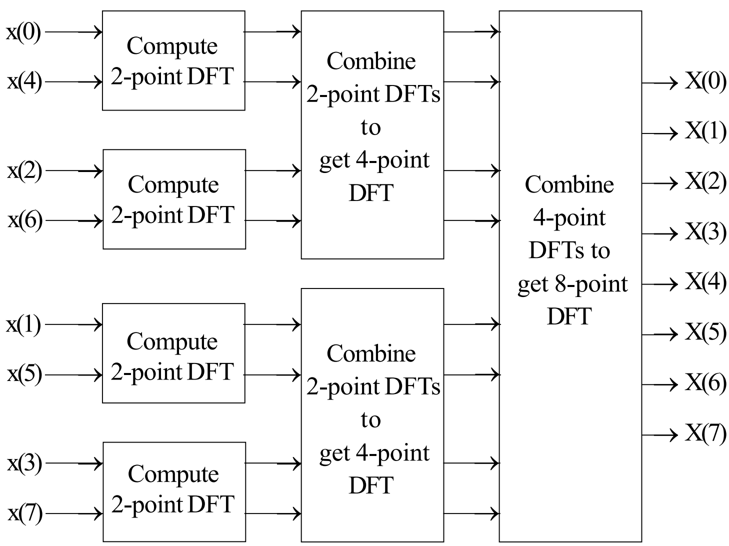


Fig 9.1. *Three stages of computation in 8-point DFT.*

Let us examine the 8-point DFT of an 8-point sequence in detail. The 8-point sequence is decimated into 4-point sequences and 2-point sequences as shown below.

Let x(n) = 8-point sequence

$f_1(n)$, $f_2(n)$ = 4-point sequences obtained from x(n)

$v_{11}(n)$, $v_{12}(n)$ = 2-point sequences obtained from $f_1(n)$

$v_{21}(n)$, $v_{22}(n)$ = 2-point sequences obtained from $f_2(n)$.

The relations between the samples of various sequences are given below

$v_{11}(0) = f_1(0) = x(0)$	$v_{21}(0) = f_2(0) = x(1)$
$v_{11}(1) = f_1(2) = x(4)$	$v_{21}(1) = f_2(2) = x(5)$
$v_{12}(0) = f_1(1) = x(2)$	$v_{22}(0) = f_2(1) = x(3)$
$v_{12}(1) = f_1(3) = x(6)$	$v_{22}(1) = f_2(3) = x(7)$

First Stage Computation

In the first stage of computation the two point DFTs of the 2-point sequences are computed.

Let, $V_{11}(k) = \mathcal{DFT}\{v_{11}(n)\}$.

Using equation (9.26), the 2-point DFT of $v_{11}(n)$ is given by,

$$V_{11}(k) = \sum_{n=0,1} v_{11}(n)\, W_2^{nk} \quad ; \quad \text{for } k = 0, 1$$

When $k = 0$; $V_{11}(k) = V_{11}(0) = v_{11}(0)\, W_2^0 + v_{11}(1)\, W_2^0 = v_{11}(0) + v_{11}(1) = x(0) + x(4)$

When $k = 1$; $V_{11}(k) = V_{11}(1) = v_{11}(0)\, W_2^0 + v_{11}(1)\, W_2^1 = v_{11}(0) - W_2^0 v_{11}(1) = x(0) - W_2^0 x(4)$

$$\boxed{W_2^0 = e^{j2\pi \times \frac{0}{2}} = e^0 = 1} \quad \boxed{W_2^1 = e^{-j2\pi \times \frac{1}{2}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1 \times W_2^0 = -W_2^0}$$

Let, $V_{12}(k) = \mathcal{DFT}\{v_{12}(n)\}$.

Using equation (9.26), the 2-point DFT of $v_{12}(n)$ is given by,

$$V_{12}(k) = \sum_{n=0,1} v_{12}(n)\, W_2^{nk} \quad ; \quad \text{for } k = 0, 1$$

When $k = 0$; $V_{12}(k) = V_{12}(0) = v_{12}(0)\, W_2^0 + v_{12}(1)\, W_2^0 = v_{12}(0) + v_{12}(1) = x(2) + x(6)$

When $k = 1$; $V_{12}(k) = V_{12}(1) = v_{12}(0)\, W_2^0 + v_{12}(1)\, W_2^1 = v_{12}(0) - W_2^0 v_{12}(1) = x(2) - W_2^0 x(6)$

Let, $V_{21}(k) = \mathcal{DFT}\{v_{21}(n)\}$.

Using equation (9.26), the 2-point DFT of $v_{21}(n)$ is given by,

$$V_{21}(k) = \sum_{n=0,1} v_{21}(n)\, W_2^{nk} \quad ; \quad \text{for } k = 0, 1$$

When $k = 0$; $V_{21}(k) = V_{21}(0) = v_{21}(0)\, W_2^0 + v_{21}(1)\, W_2^0 = v_{21}(0) + v_{21}(1) = x(1) + x(5)$

When $k = 1$; $V_{21}(k) = V_{21}(1) = v_{21}(0)\, W_2^0 + v_{21}(1)\, W_2^1 = v_{21}(0) - W_2^0 v_{21}(1) = x(1) - W_2^0 x(5)$

Let, $V_{22}(k) = \mathcal{DFT}\{v_{22}(n)\}$.

Using equation (9.26), the 2-point DFT of $v_{22}(n)$ is given by,

$$V_{22}(k) = \sum_{n=0,1} v_{22}(n)\, W_2^{nk} \quad ; \quad \text{for } k = 0, 1.$$

When $k = 0$; $V_{22}(k) = V_{22}(0) = v_{22}(0)\, W_2^0 + v_{22}(1)\, W_2^0 = v_{22}(0) + v_{22}(1) = x(3) + x(7)$

When $k = 1$; $V_{22}(k) = V_{22}(1) = v_{22}(0)\, W_2^0 + v_{22}(1)\, W_2^1 = v_{22}(0) - W_2^0 v_{22}(1) = x(3) - W_2^0 x(7)$

Second Stage Computation

In the second stage of computations the 4-point DFTs are computed using the results of first stage as input. Let, $F_1(k) = \mathcal{DFT}\{f_1(n)\}$. The 4-point DFT of $f_1(n)$ can be computed using equation (9.33).

$$\therefore\ F_1(k) = V_{11}(k) + W_4^k\, V_{12}(k)\ ;\ \text{for } k = 0, 1, 2, 3.$$

$V_{11}(k)$ and $V_{12}(k)$ are periodic with periodicity of 2 samples.
$\therefore\ V_{11}(k+2) = V_{11}(k)$
$V_{12}(k+2) = V_{12}(k)$

$$\text{When } k = 0;\ F_1(k) = F_1(0) = V_{11}(0) + W_4^0\, V_{12}(0)$$
$$\text{When } k = 1;\ F_1(k) = F_1(1) = V_{11}(1) + W_4^1\, V_{12}(1)$$
$$\text{When } k = 2;\ F_1(k) = F_1(2) = V_{11}(2) + W_4^2\, V_{12}(2) = V_{11}(0) - W_4^0\, V_{12}(0)$$
$$\text{When } k = 3;\ F_1(k) = F_1(3) = V_{11}(3) + W_4^3\, V_{12}(3) = V_{11}(1) - W_4^1\, V_{12}(1)$$

$$\boxed{W_4^2 = e^{-j2\pi \times \frac{2}{4}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1 = -1 \times W_4^0 = -W_4^0}$$

$$\boxed{W_4^3 = e^{-j2\pi \times \frac{3}{4}} = e^{-j2\pi \times \frac{2}{4}}\, e^{-j2\pi \times \frac{1}{4}} = e^{-j\pi}\, e^{-j2\pi \times \frac{1}{4}} = (\cos\pi - j\sin\pi)\, W_4^1 = -1 \times W_4^1 = -W_4^1}$$

Let, $F_2(k) = \mathcal{DFT}\{f_2(n)\}$. The 4-point DFT of $f_2(n)$ can be computed using equation (9.34).

$$\therefore\ F_2(k) = V_{21}(k) + W_4^k\, V_{22}(k)\ ;\ \text{for } k = 0, 1, 2, 3.$$

$V_{21}(k)$ and $V_{22}(k)$ are periodic with periodicity of 2 samples.
$\therefore\ V_{21}(k+2) = V_{21}(k)$
$V_{22}(k+2) = V_{22}(k)$

$$\text{When } k = 0;\ F_2(k) = F_2(0) = V_{21}(0) + W_4^0\, V_{22}(0)$$
$$\text{When } k = 1;\ F_2(k) = F_2(1) = V_{21}(1) + W_4^1\, V_{22}(1)$$
$$\text{When } k = 2;\ F_2(k) = F_2(2) = V_{21}(2) + W_4^2\, V_{22}(2) = V_{21}(0) - W_4^0\, V_{22}(0)$$
$$\text{When } k = 3;\ F_2(k) = F_2(3) = V_{21}(3) + W_4^3\, V_{22}(3) = V_{21}(1) - W_4^1\, V_{22}(1)$$

Third Stage Computation

In the third stage of computations the 8-point DFTs are computed using the results of second stage as inputs.

Let, $X(k) = \mathcal{DFT}\{X(n)\}$. The 8-point DFT of x(n) can be computed using equation (9.27).

$$\therefore\ X(k) = F_1(k) + W_8^k\, F_2(k)\ ;\ \text{for } k = 0, 1, 2, 3, 4, 5, 6, 7$$

$F_1(k)$ and $F_2(k)$ are periodic with periodicity of 4 samples.
$\therefore\ F_1(k+4) = F_1(k)$
$F_2(k+4) = F_2(k)$

$$\text{When } k = 0;\ X(k) = X(0) = F_1(0) + W_8^0\, F_2(0)$$
$$\text{When } k = 1;\ X(k) = X(1) = F_1(1) + W_8^1\, F_2(1)$$
$$\text{When } k = 2;\ X(k) = X(2) = F_1(2) + W_8^2\, F_2(2)$$
$$\text{When } k = 3;\ X(k) = X(3) = F_1(3) + W_8^3\, F_2(3)$$
$$\text{When } k = 4;\ X(k) = X(4) = F_1(4) + W_8^4\, F_2(4) = F_1(0) - W_8^0\, F_2(0)$$
$$\text{When } k = 5;\ X(k) = X(5) = F_1(5) + W_8^5\, F_2(5) = F_1(1) - W_8^1\, F_2(1)$$
$$\text{When } k = 6;\ X(k) = X(6) = F_1(6) + W_8^6\, F_2(6) = F_1(2) - W_8^2\, F_2(2)$$
$$\text{When } k = 7;\ X(k) = X(7) = F_1(7) + W_8^7\, F_2(7) = F_1(3) - W_8^3\, F_2(3)$$

$$W_8^4 = e^{-j2\pi \times \frac{4}{8}} = e^{-j\pi} = (\cos\pi - j\sin\pi) = -1$$

$$\boxed{W_8^4 = W_8^4 \times W_8^0 = -W_8^0} \quad \boxed{W_8^5 = W_8^4 \times W_8^1 = -W_8^1} \quad \boxed{W_8^6 = W_8^4 \times W_8^2 = -W_8^2} \quad \boxed{W_8^7 = W_8^4 \times W_8^3 = -W_8^3}$$

9.7.2 Flow Graph for 8-Point DIT Radix-2 FFT

If we observe the basic computation performed at every stage, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The complex number "b" is multiplied by a phase factor "W_N^k".
3. The product "bW_N^k" is added to complex number "a" to form new complex number "A".
4. The product "bW_N^k" is subtracted from complex number "a" to form new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 9.2. (For detailed discussion on signal flow graph, refer chapter-6, Section 6.6.2).

The signal flow graph is also called butterfly diagram since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8-point DFT via radix-2 DIT FFT can be arrived as shown below.

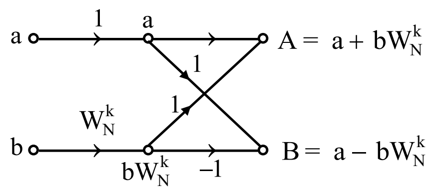


***Fig 9.2:** Basic butterfly or flow graph of DIT radix-2 FFT.*

The sequence is arranged in bit reversed order and then decimated into two sample sequences as shown below.

x(0)	x(2)	x(1)	x(3)
x(4)	x(6)	x(5)	x(7)

Flow Graph or (Butterfly Diagram) for First Stage of Computation

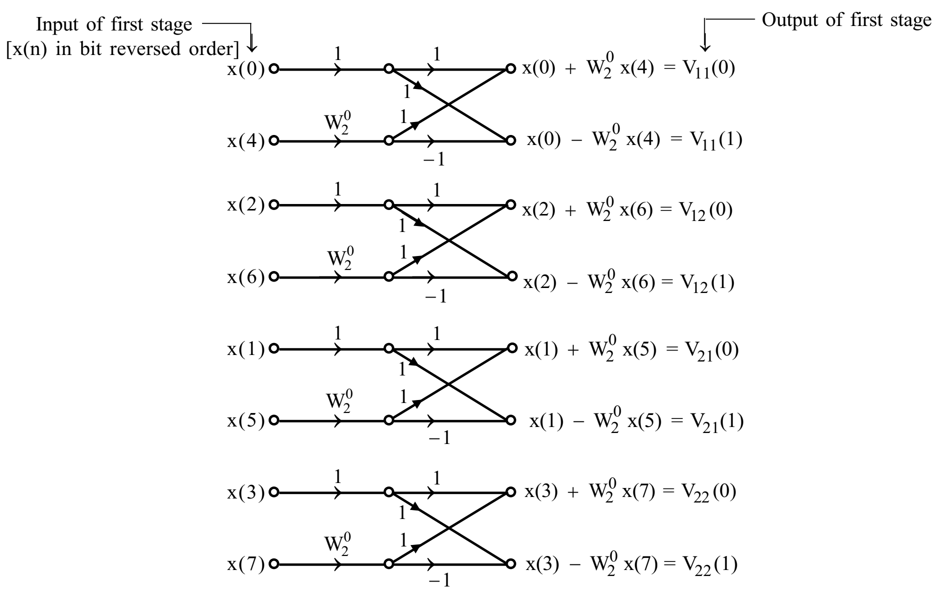


***Fig 9.3 :** First stage of flow graph (or butterfly diagram) for 8-point DFT via DIT.*

Flow Graph (or Butterfly Diagram) for Second Stage of Computation

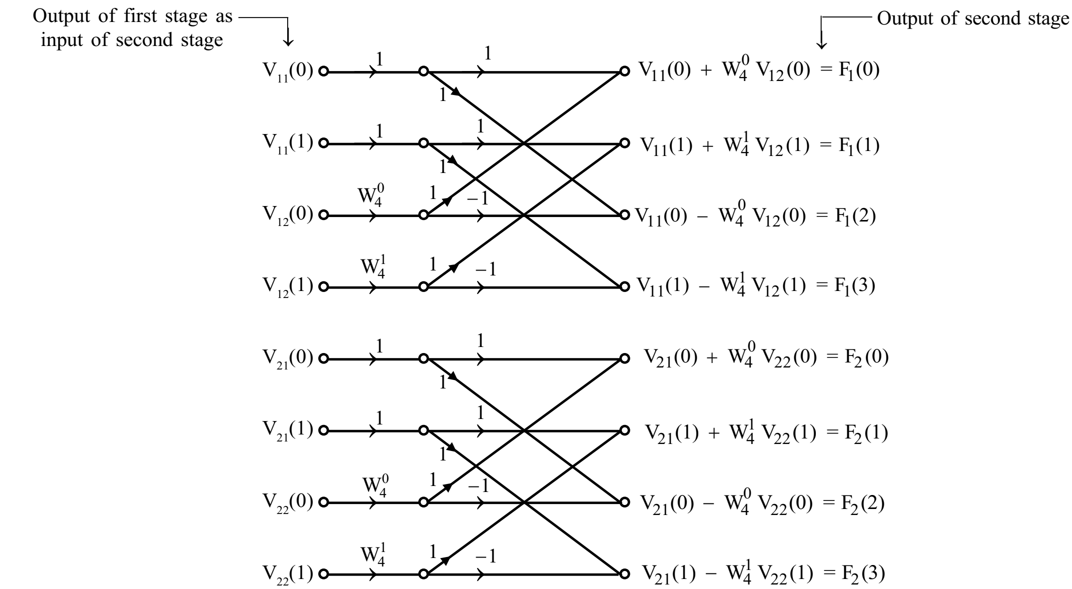


Fig : 9.4 : *Second stage of flow graph (or butterfly diagram) for 8-point DFT via DIT.*

Flow Graph (or Butterfly Diagram) for Third Stage of Computation

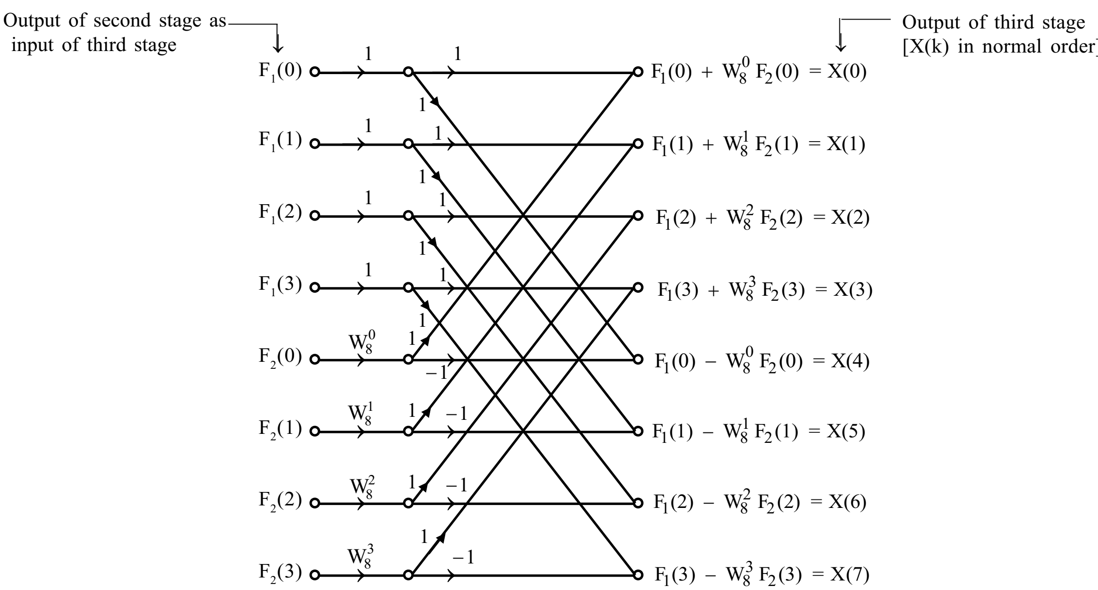


Fig 9.5 : *Third stage of flow graph (or butterfly diagram) for 8-point DFT via DIT.*

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

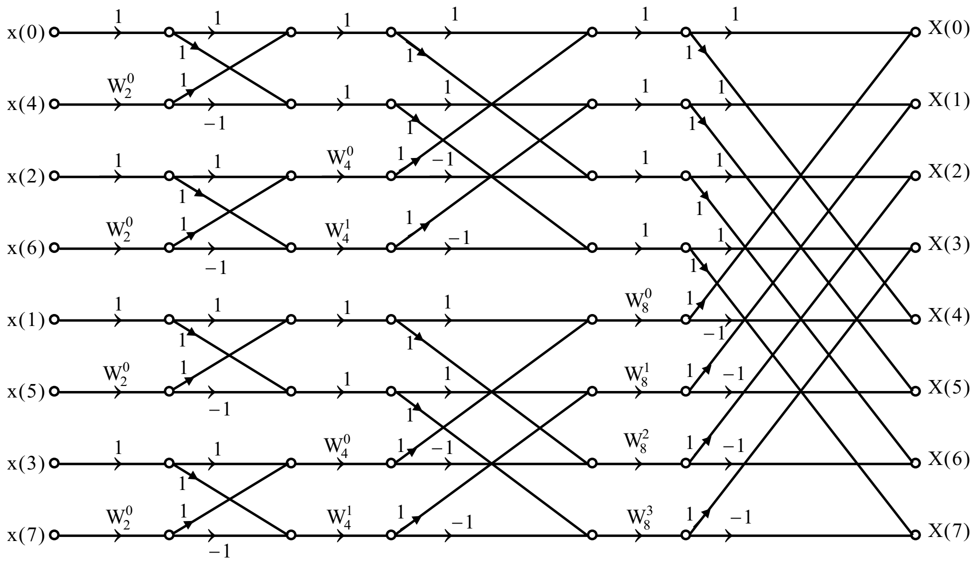


Fig 9.6 : The flow graph (or butterfly diagram) for 8-point DIT radix-2 FFT.

9.8 Decimation in Frequency (DIF) Radix-2 FFT

In decimation in frequency algorithm the frequency domain sequence X(k) is decimated, (but in decimation in time algorithm, the time domain sequence x(n) is decimated).

In this algorithm, the N-point time domain sequence is converted to two numbers of N/2 point sequences. Then each N/2 point sequence is converted to two numbers of N/4 point sequences. Thus we get 4 numbers of N/4 point sequences. This process is continued until we get N/2 numbers of 2-point sequences. Finally the 2-point DFT of each 2-point sequence is computed. The 2-point DFTs of N/2 numbers of 2-point sequences will give N samples, which is the N-point DFT of the time domain sequence.

Here the equations for forming N/2 point sequences, N/4 point sequences, etc., are obtained by decimation of frequency domain sequences. Hence this method is called DIF. For example the N-point frequency domain sequence X(k) can be decimated to two numbers of N/2 point frequency domain sequences $G_1(k)$ and $G_2(k)$. The $G_1(k)$ and $G_2(k)$ defines new time domain sequences $g_1(n)$ and $g_2(n)$ respectively, whose samples are obtained from x(n).

It can be shown that the N-point DFT of x(n) can be realized from two numbers of N/2 point DFTs. The N/2 point DFTs can be realized from two numbers of N/4 point DFTs and so on. The decimation continues upto 2-point DFTs.

Let x(n) and X(k) be N-point DFT pair.

Let $G_1(k)$ and $G_2(k)$ be two numbers of N/2 point sequences obtained by the decimation of X(k).

Let $G_1(k)$ be N/2 point DFT of $g_1(n)$, and $G_2(k)$ be N/2 point DFT of $g_2(n)$.

Now, the N-point DFT X(k) can be obtained from the two numbers of N/2 point DFTs $G_1(k)$ and $G_2(k)$, as shown below.

$$X(k)\Big|_{k\,=\,\text{even}} = G_1(k)$$

$$X(k)\Big|_{k\,=\,\text{odd}} = G_2(k)$$

Proof :

By definition of DFT, the N-point DFT of x(n) is,

$$W_N^{\frac{kN}{2}} = e^{-j\frac{2\pi}{N}\frac{kN}{2}} = e^{-j\pi k} = \left(e^{-j\pi}\right)^k = (-1)^k$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\,W_N^{kn} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=\frac{N}{2}}^{N-1} x(n)\,W_N^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{k\left(n+\frac{N}{2}\right)} = \sum_{n=0}^{\frac{N}{2}-1} x(n)\,W_N^{kn} + \sum_{n=0}^{\frac{N}{2}-1} x\left(n+\frac{N}{2}\right) W_N^{kn}\,W_N^{\frac{kN}{2}}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n)\,W_N^{kn} + (-1)^k\,x\left(n+\frac{N}{2}\right) W_N^{kn}\right] = \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^k\,x\left(n+\frac{N}{2}\right)\right] W_N^{kn}$$

Let us split X(k) into even and odd numbered samples.

$$X(k)\Big|_{k\,=\,\text{even}} = X(2k) \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{2k}\,x\left(n+\frac{N}{2}\right)\right] W_N^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + x\left(n+\frac{N}{2}\right)\right] W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\,W_{N/2}^{kn} = G_1(k)$$

$$g_1(n) = x(n) + x\left(n+\frac{N}{2}\right); \text{ for } n = 0, 1, 2, \ldots \frac{N}{2}-1$$

$G_1(k)$ is $\frac{N}{2}$ point DFT of $g_1(n)$

$$\therefore G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\,W_{N/2}^{kn} \text{ ; for } k = 0, 1, 2, \ldots \frac{N}{2}-1$$

$$X(k)\Big|_{k\,=\,\text{odd}} = X(2k+1) \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, \frac{N}{2}-1$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) + (-1)^{(2k+1)}\,x\left(n+\frac{N}{2}\right)\right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{2kn}\,W_N^{n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1}\left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^{n}\,W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\,W_{N/2}^{kn} = G_2(k)$$

$$g_2(n) = \left(x(n) - x\left(n+\frac{N}{2}\right)\right) W_N^{n}; \text{ for } n = 0, 1, 2, \ldots\ldots, \frac{N}{2}-1$$

$G_2(k)$ is $\frac{N}{2}$ point DFT of $g_1(n)$

$$\therefore G_2(k) = \sum_{n=0}^{\frac{N}{2}-1} g_2(n)\,W_{N/2}^{kn} \text{ ; for } k = 0, 1, 2, \ldots\ldots, \frac{N}{2}-1$$

In the next stage of decimation the N/2 point frequency domain sequence $G_1(k)$ is decimated into two numbers of N/4 point sequences $D_{11}(k)$ and $D_{12}(k)$, and $G_2(k)$ is decimated into two numbers of N/4 point sequences $D_{21}(k)$ and $D_{22}(k)$.

Let $D_{11}(k)$ and $D_{12}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_1(k)$.

Let $D_{11}(k)$ be N/4 point DFT of $d_{11}(n)$, and $D_{12}(k)$ be N/4 point DFT of $d_{12}(n)$.

Let $D_{21}(k)$ and $D_{22}(k)$ be two numbers of N/4 point sequences obtained by the decimation of $G_2(k)$.

Let $D_{21}(k)$ be N/4 point DFT of $d_{21}(n)$, and $D_{22}(k)$ be N/4 point DFT of $d_{22}(n)$.

Now, N/2 point DFTs can be obtained from two numbers of N/4 point DFTs as shown below.

$$G_1(k)\Big|_{k\,=\,\text{even}} = D_{11}(k)$$
$$G_1(k)\Big|_{k\,=\,\text{odd}} = D_{12}(k)$$
$$G_2(k)\Big|_{k\,=\,\text{even}} = D_{21}(k)$$
$$G_2(k)\Big|_{k\,=\,\text{odd}} = D_{22}(k)$$

Proof :

By definition of DFT, the N/2 point DFT of $G_1(k)$ is,

$$G_1(k) = \sum_{n=0}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{n=\frac{N}{4}}^{\frac{N}{2}-1} g_1(n)\, W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{n=0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{k\left(n+\frac{N}{4}\right)} = \sum_{n=0}^{\frac{N}{4}-1} g_1(n)\, W_{N/2}^{kn} + \sum_{0}^{\frac{N}{4}-1} g_1\left(n+\frac{N}{4}\right) W_{N/2}^{kn}\, W_{N/2}^{k\frac{N}{4}}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + W_{N/2}^{\frac{kn}{4}}\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^k\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{kn}$$

$$W_{N/2}^{\frac{kN}{4}} = e^{\frac{-j2\pi}{N/2}\frac{kN}{4}} = \left(e^{-j\pi}\right)^k = (-1)^k$$

Let us split $G_1(k)$ into even and odd numbered samples.

$$G_1(k)\Big|_{k\,=\,\text{even}} = G_1(2k) \quad ; \quad \text{for } k = 0, 1, 2, \ldots\ldots, \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^{2k}\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{2kn}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + g_1\left(n+\frac{N}{4}\right)\right] W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)\, W_{N/4}^{kn} = D_{11}(k)$$

$$d_{11}(n) = g_1(n) + g_1\left(n+\frac{N}{4}\right)$$

$D_{11}(k)$ is $\frac{N}{4}$ point DFT of $d_{11}(n)$

$$\therefore D_{11}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{11}(n)\, W_{N/4}^{kn}$$

$$G_1(k)\Big|_{k\,=\,\text{odd}} = G_1(2k+1) \quad ; \text{for } k = 0, 1, 2, \ldots\ldots, \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) + (-1)^{(2k+1)}\, g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{(2k+1)n}$$

$$= \sum_{n=0}^{\frac{N}{4}-1} \left[g_1(n) - g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}\, W_{N/4}^{kn} = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)\, W_{N/4}^{kn} = D_{12}(k)$$

$$d_{12}(n) = \left[g_1(n) + g_1\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}$$

$D_{12}(k)$ is $\frac{N}{4}$ point DFT of $d_{12}(n)$

$$\therefore D_{12}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{12}(n)\, W_{N/4}^{kn}$$

Similarly the N/2 point sequence $G_2(k)$ can be decimated into two numbers of N/4 point sequences.

$$G_2(k)\big|_{k\,=\,even} = G_2(2k) \; ; \quad for \;\; k = 0, 1, 2, \ldots.., \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)\, W_{N/4}^{kn} = D_{21}(k)$$

$$G_2(k)\big|_{k\,=\,odd} = G_2(2k+1) \quad ; for \; k = 0, 1, 2, \ldots.., \frac{N}{4}-1$$

$$= \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)\, W_{N/4}^{kn} = D_{22}(k)$$

$$d_{21}(n) = g_2(n) + g_2\left(n+\frac{N}{4}\right)$$

$D_{21}(k)$ is $\frac{N}{4}$ point DFT of $d_{21}(n)$

$$\therefore D_{21}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{21}(n)\, W_{N/4}^{kn}$$

$$d_{22}(n) = \left[g_2(n) + g_2\left(n+\frac{N}{4}\right)\right] W_{N/2}^{n}$$

$D_{22}(k)$ is $\frac{N}{4}$ point DFT of $d_{22}(n)$

$$\therefore D_{22}(k) = \sum_{n=0}^{\frac{N}{4}-1} d_{22}(n)\, W_{N/4}^{kn}$$

The decimation of the frequency domain sequence can be continued until the resulting sequence are reduced to 2-point sequences. The entire process of decimation involves, m stages of decimation where $m = \log_2 N$. The computation of the N-point DFT via the decimation in frequency FFT algorithm requires $(N/2)\log_2 N$ complex multiplications and $N \log_2 N$ complex additions. (i.e., the total number of computations remains same in both DIT and DIF).

9.8.1 8-point DFT Using Radix-2 DIF FFT

The DIF computations for an eight sequence is discussed in detail in this section. Let x(n) be an 8-point sequence. Therefore N = 8. The samples of x(n) are,

$$x(0),\; x(1),\; x(2),\; x(3),\; x(4),\; x(5),\; x(6),\; x(7)$$

First Stage Computation

In the first stage of computation, two numbers of 4-point sequences $g_1(n)$ and $g_2(n)$ are obtained from x(n) as shown below.

$$g_1(n) = \left[x(n) + x\left(n+\frac{N}{2}\right)\right] = [x(n) + x(n+4)] \quad ; for \; n = 0, 1, 2, 3$$

When n = 0; $g_1(n) = g_1(0) = x(0) + x(4)$

When n = 1; $g_1(n) = g_1(1) = x(1) + x(5)$

When n = 2; $g_1(n) = g_1(2) = x(2) + x(6)$

When n = 3; $g_1(n) = g_1(3) = x(3) + x(7)$

$$g_2(n) = \left[x(n) - x\left(n+\frac{N}{2}\right)\right] W_N^n = [x(n) - x(n+4)]\, W_8^n \;\; ; for \; n = 0, 1, 2, 3$$

When n = 0; $g_2(n) = g_2(0) = [x(0) - x(4)]\, W_8^0$

When n = 1; $g_2(n) = g_2(1) = [x(1) - x(5)]\, W_8^1$

When n = 2; $g_2(n) = g_2(2) = [x(2) - x(6)]\, W_8^2$

When n = 3; $g_2(n) = g_2(3) = [x(3) - x(7)]\, W_8^3$

Second Stage Computation

In the second stage of computation, 2 numbers of 2-point sequences $d_{11}(n)$ and $d_{12}(n)$ are generated from the samples of $g_1(n)$, and another 2 numbers of 2-point sequences $d_{21}(n)$ and $d_{22}(n)$ are generated from the samples of $g_2(n)$, as shown below.

$$d_{11}(n) = g_1(n) + g_1(n+N/4) = g_1(n) + g_1(n+2) \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{11}(n) = d_{11}(0) = g_1(0) + g_1(2)$

When $n = 1$; $d_{11}(n) = d_{11}(1) = g_1(1) + g_1(3)$

$$d_{12}(n) = \left[g_1(n) - g_1(n+N/4)\right] W_{N/2}^{n} = \left[g_1(n) - g_1(n-2)\right] W_4^{n} \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{12}(n) = d_{12}(0) = [g_1(0) - g_1(2)]\, W_4^0$

When $n = 1$; $d_{12}(n) = d_{12}(1) = [g_1(1) - g_1(3)]\, W_4^1$

$$d_{21}(n) = g_2(n) + g_2(n+N/4) = g_2(n) + g_2(n+2) \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{21}(n) = d_{21}(0) = [g_2(0) + g_2(2)]$

When $n = 1$; $d_{21}(n) = d_{21}(1) = [g_2(1) + g_2(3)]$

$$d_{22}(n) = \left[g_2(n) - g_2(n+N/4)\right] W_{N/2}^{n} = \left[g_2(n) - g_2(n+2)\right] W_4^{n} \; ; \text{ for } n = 0, 1$$

When $n = 0$; $d_{22}(n) = d_{22}(0) = [g_2(0) - g_2(2)]\, W_4^0$

When $n = 1$; $d_{22}(n) = d_{22}(1) = [g_2(1) - g_2(3)]\, W_4^1$

Third Stage Computation

In the third stage of computation, 2-point DFTs of the 2-point sequences $d_{11}(n)$, $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed.

The 2-point DFT of the 2-point sequence $d_{11}(n)$ is computed as shown below.

$$\mathcal{DFT}\{d_{11}(n)\} = D_{11}(k) = \sum_{n=0}^{1} d_{11}(n)\, W_2^{kn} \; ; \text{ for } k = 0, 1$$

When $k = 0$; $D_{11}(0) = \sum_{n=0}^{1} d_{11}(n)\, W_2^0 = d_{11}(0) + d_{11}(1)$ $\boxed{W_2^0 = 1}$

When $k = 1$; $D_{11}(1) = \sum_{n=0}^{1} d_{11}(n)\, W_2^n = d_{11}(0)\, W_2^0 + d_{11}(1)\, W_2^1$ $\boxed{W_2^1 = -1 = -1 \times W_2^1}$

$$= d_{11}(0)\, W_2^0 + d_{11}(1)\, W_2^1\, W_2^0 = \left[d_{11}(0) - d_{11}(1)\right] W_2^0$$

Similarly the 2-point DFTs of the 2-point sequences $d_{12}(n)$, $d_{21}(n)$ and $d_{22}(n)$ are computed and the results are given below.

$$D_{12}(0) = d_{12}(0) + d_{12}(1)$$
$$D_{12}(1) = \left[d_{12}(0) - d_{12}(1)\right] W_2^0$$

$$D_{21}(0) = d_{21}(0) + d_{21}(1)$$
$$D_{21}(1) = \left[d_{21}(0) - d_{21}(1)\right] W_2^0$$

$$D_{22}(0) = d_{22}(0) + d_{22}(1)$$
$$D_{22}(1) = \left[d_{22}(0) - d_{22}(1)\right] W_2^0$$

Combining the Three Stages of Computation

The final output $D_{ij}(k)$ gives the X(k). The relation can be obtained as shown below.

$X(2k) = G_1(k)\ ;\ k = 0,1,2,3$ $\therefore\ X(0) = G_1(0)$ $X(2) = G_1(1)$ $X(4) = G_1(2)$ $X(6) = G_1(3)$	$X(2k+1) = G_2(k)\ ;\ k = 0, 1, 2, 3$ $\therefore\ X(1) = G_2(0)$ $X(3) = G_2(1)$ $X(5) = G_2(2)$ $X(7) = G_2(3)$
$G_1(2k) = D_{11}(k)\ ;\ k = 0, 1$ $\therefore\ G_1(0) = D_{11}(0)$ $G_1(2) = D_{11}(1)$	$G_1(2k+1) = D_{12}(k)\ ;\ k = 0, 1$ $\therefore\ G_1(1) = D_{12}(0)$ $G_1(3) = D_{11}(1)$
$G_2(2k) = D_{21}(k)\ ;\ k = 0, 1$ $\therefore\ G_2(0) = D_{21}(0)$ $G_2(2) = D_{21}(1)$	$G_2(2k+1) = D_{22}(k)\ ;\ k = 0, 1$ $\therefore\ G_2(1) = D_{22}(0)$ $G_2(3) = D_{22}(1)$

From above relations we get,

$$X(0) = G_1(0) = D_{11}(0)$$
$$X(4) = G_1(2) = D_{11}(1)$$
$$X(2) = G_1(1) = D_{12}(0)$$
$$X(6) = G_1(3) = D_{12}(1)$$
$$X(1) = G_2(0) = D_{21}(0)$$
$$X(5) = G_2(2) = D_{21}(1)$$
$$X(3) = G_2(1) = D_{22}(0)$$
$$X(7) = G_2(3) = D_{22}(1)$$

From the above we observe that the output is in bit reversed order. In radix-2 DIF FFT, the input is in normal order the output will be in bit reversed order.

9.8.2 Flow Graph for 8-point Radix-2 DIF FFT

If we observe the basic computation performed at every stage, we can arrive at the following conclusion.

1. In each computation two complex numbers "a" and "b" are considered.
2. The sum of the two complex number is computed which forms a new complex number "A".
3. Then subtract complex number "b" from "a" to get the term "a-b". The difference term "a-b" is multiplied with the phase factor or twiddle factor "W_N^k" to form a new complex number "B".

The above basic computation can be expressed by a signal flow graph shown in Fig 9.7. (For detailed discussion on signal flow graph, refer chapter-6, Section 6.6.2).

The signal flow graph is also called butterfly diagram since it resembles a butterfly. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process. The butterfly diagram used to compute the 8-point DFT via radix-2 DIF FFT can be arrived as shown below.

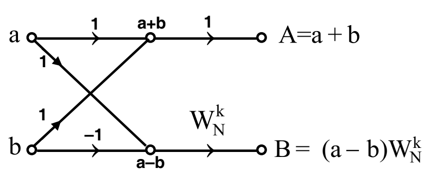


***Fig 9.7** : Basic butterfly or flow graph of DIF - FFT.*

Flow Graph (or Butterfly Diagram) for First Stage of Computation

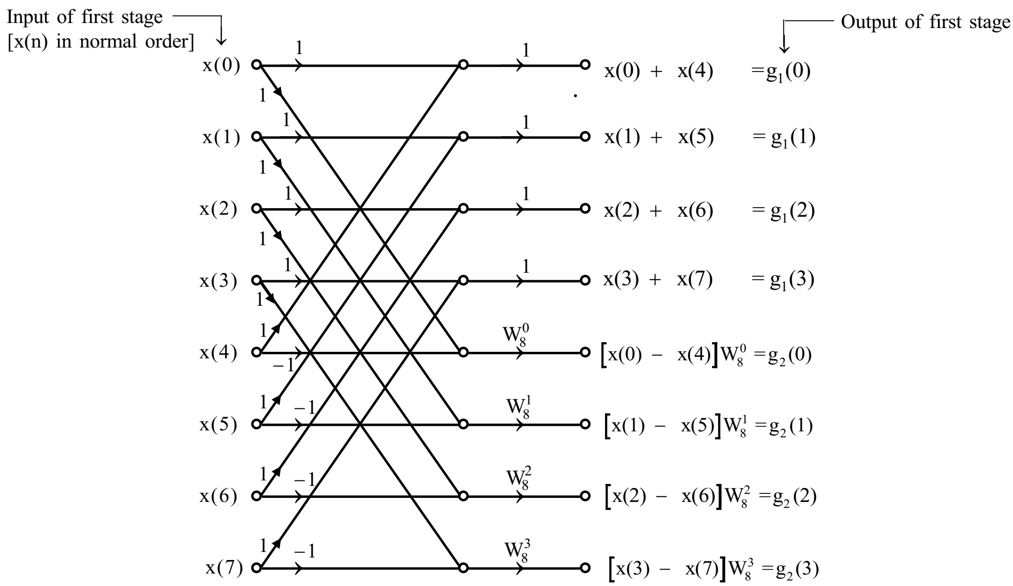


Fig 9.8 : First stage of flow graph (or butterfly diagram) for 8-point DFT via DIF.

Flowgraph or Butterfly Diagram for Second stage of computation

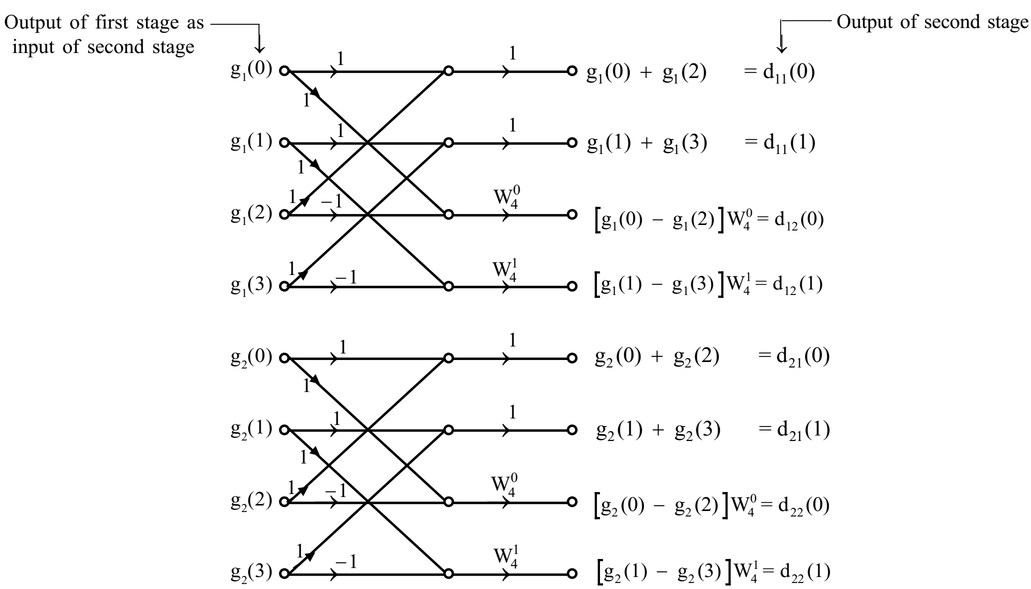


Fig 9.9: Second stage of flow graph (or butterfly diagram) for 8-point DFT via DIF.

Flow Graph(or Butterfly Diagram) for Third Stage of Computation

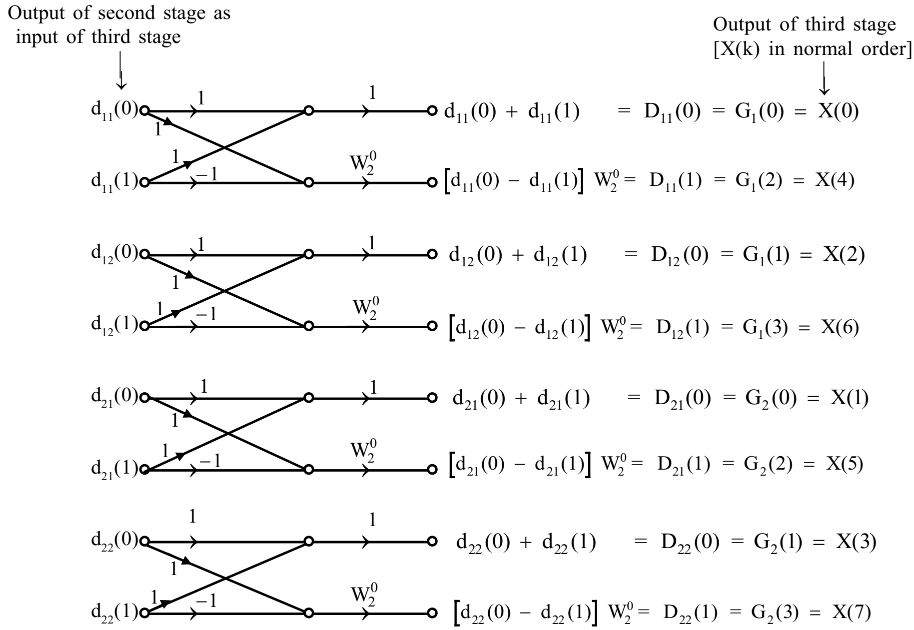


Fig 9.10 :*Third stage of flow graph (or butterfly diagram) for 8-point DFT via DIF.*

The Combined Flow Graph (or Butterfly Diagram) of All the Three Stages of Computation

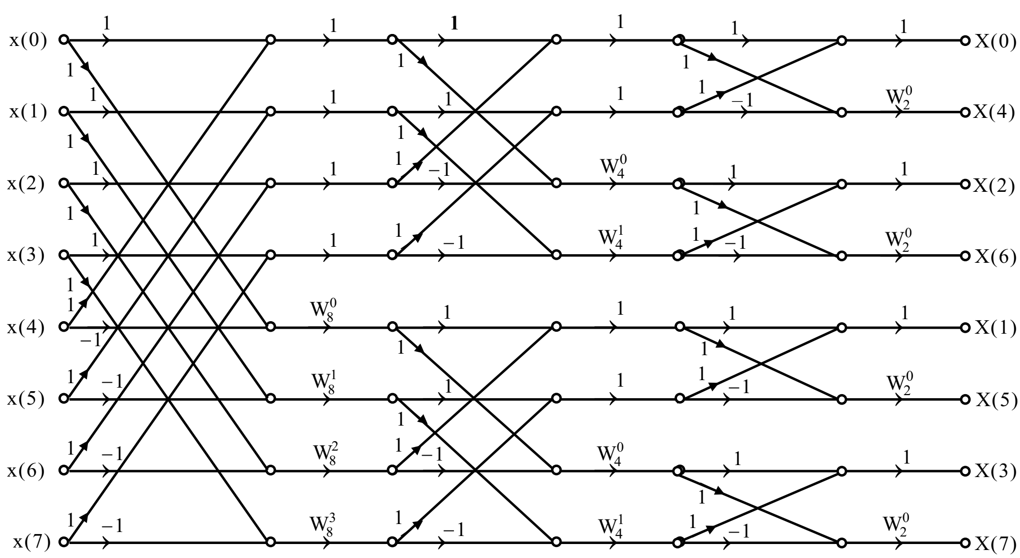


Fig 9.11 : *The flow graph (or butterfly diagram) for 8-point DIF radix-2 FFT.*

9.8.3 Comparison of DIT and DIF Radix-2 FFT

Differences in DIT and DIF

- In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
- In DIT the input should be in bit-reversed order and the output will be in normal order. For DIF the reverse is true, i.e., input is normal order, while output is bit reversed.
- Considering the butterfly diagram, in DIT the complex multiplication takes place before the add-subtract operation, whereas in DIF the complex multiplication takes place after the add-subtract operation.

Similarities in DIT and DIF

- For both the algorithms the value of N should be such that, $N = 2^m$, and there will be m stages of butterfly computations.
- Both algorithms involve same number of operations. The total number of complex additions are $N\log_2 N$ and total number of complex multiplications are $(N/2)\log_2 N$.
- Both algorithms require bit reversal at some place during computation.

9.9 Computation of Inverse DFT Using FFT

Let x(n) and X(k) be N-point DFT pair.

Now by the definition of inverse DFT,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi nk}{N}} \quad ; \qquad \text{for } n = 0, 1, 2, \ldots.., N-1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(e^{\frac{-j2\pi nk}{N}}\right)^* = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\left(W_N^{nk}\right)^* = \frac{1}{N}\left[\sum_{k=0}^{N-1} X(k)\left(W_N^{nk}\right)^*\right] \quad .....(9.35)$$

In equation (9.35), the expression inside the bracket is similar to that of DFT computation of a sequence, with following differences.

1. The summation index is k instead of n.
2. The input sequence is X(k) instead of x(n).
3. The phase factors are conjugate of the phase factor used for DFT.

Hence, in order to compute inverse DFT of X(k), the FFT algorithm can be used by taking the conjugate of phase factors. Also from equation (9.35) it is observed that the output of FFT computation should be divided by N to get x(n).

The following procedure can be followed to compute inverse DFT using FFT algorithm.

1. Take N-point frequency domain sequence X(k) as input sequence.
2. Compute FFT by using conjugate of phase factors.
3. Divide the output sequence obtained in FFT computation by N, to get the sequence x(n).

Thus a single FFT algorithm can be used for evaluation of both DFT and inverse DFT.

Example 9.5

An 8-point sequence is given by x(n) = {2, 2, 2, 2, 1, 1, 1, 1}. Compute 8-point DFT of x(n) by a) radix-2 DIT-FFT and b) radix-2 DIF-FFT. Also sketch the magnitude and phase spectrum.

Solution

a) 8-point DFT by Radix-2 DIT-FFT

The given sequence is first arranged in the bit reversed order.

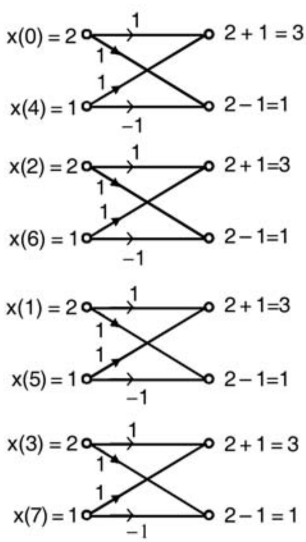
The sequence x(n) in normal order	The sequence x(n) in bit reversed order
x(0) = 2	x(0) = 2
x(1) = 2	x(4) = 1
x(2) = 2	x(2) = 2
x(3) = 2	x(6) = 1
x(4) = 1	x(1) = 2
x(5) = 1	x(5) = 1
x(6) = 1	x(3) = 2
x(7) = 1	x(7) = 1

For 8-point DFT by radix-2 FFT we require 3 stages of computation with 4-butterfly computations in each stage. The sequence rearranged in the bit reversed order forms the input to the first stage. For other stages of computation the output of previous stage will be the input for current stage.

First stage computation

The input sequence to first stage computation = { 2, 1, 2, 1, 2, 1, 2, 1 }

The butterfly computations of first stage are shown in fig 1.



The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Fig 1 : *Butterfly diagram for first stage of radix-2 DIT FFT.*

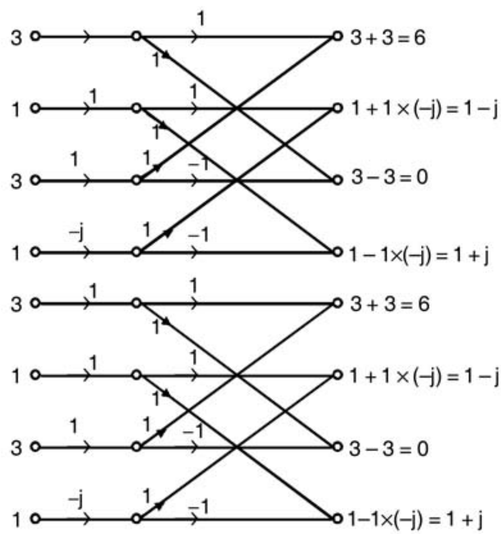
The output sequence of first stage of computation = { 3, 1, 3, 1, 3, 1, 3, 1 }

Second stage computation

The input sequence to second stage computation $= \{3, 1, 3, 1, 3, 1, 3, 1\}$

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown fig 2.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

***Fig 2 :** Butterfly diagram for second stage of radix-2 DIT FFT.*

$$\left.\begin{array}{r}\text{The output sequence of second} \\ \text{stage of computation}\end{array}\right\} = \{ 6,\ 1-j,\ 0,\ 1+j,\ 6,\ 1-j,\ 0,\ 1+j \}$$

Third stage computation

The input sequence to third stage computation $= \{ 6,\ 1-j,\ 0,\ 1+j,\ 6,\ 1-j,\ 0,\ 1+j \}$

The phase factors involved in third stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

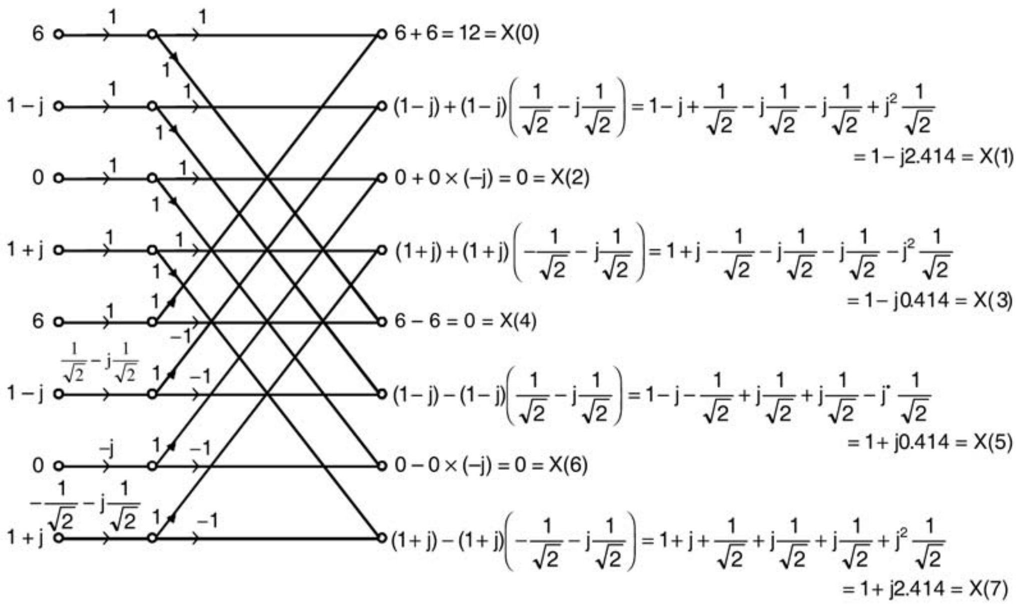
The butterfly computations of third stage are shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$



$$\left.\begin{array}{r}\text{The output sequence of third}\\ \text{stage of computation}\end{array}\right\} = \{12,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

The output sequence of third stage of computation is the 8-point DFT of the given sequence in normal order.

$$\boxed{\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \{12,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}}$$

b) 8-point DFT by Radix-2 DIF-FFT

For 8-point DFT by radix-2 FFT we require 3-stages of computation with 4-butterfly computation in each stage. The given sequence is the input to first stage. For other stages of computations, the output of previous stage will be the input for current stage.

First stage computation

The input sequence for first stage of computation $= \{2,\ 2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1\}$

The phase factors involved in first stage computation are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computations of first stage are shown in fig 4.

$$W_8^0 = e^{-j2\pi\times\frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi\times\frac{1}{8}} = e^{-j\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi\times\frac{2}{8}} = e^{-j\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi\times\frac{3}{8}} = e^{-j\frac{3\pi}{4}} = \cos\left(-\frac{3\pi}{4}\right) + j\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

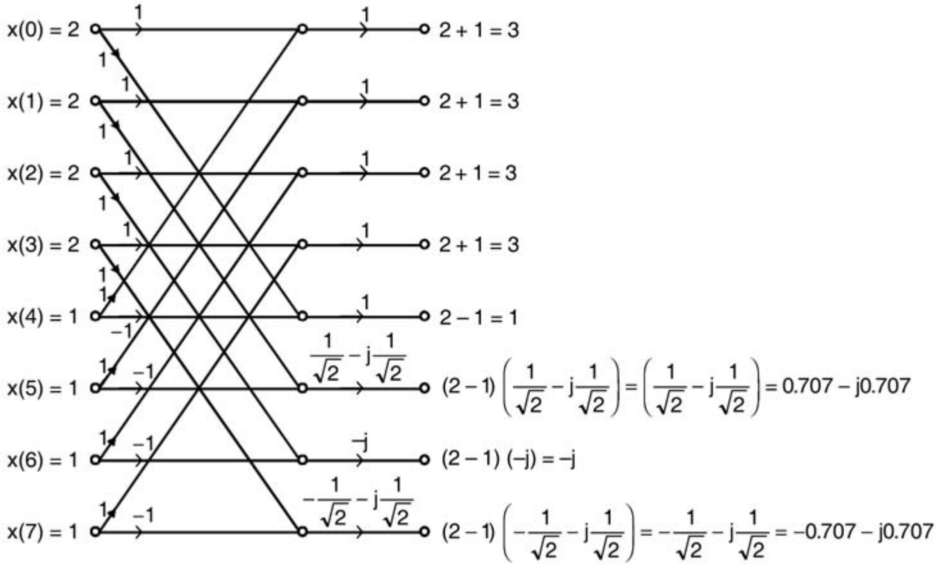


***Fig 4 :** Butterfly diagram for first stage of radix-2 DIF FFT.*

$$\left.\begin{array}{r}\text{The output sequence of first}\\ \text{stage of computation}\end{array}\right\} = \{3,\ 3,\ 3,\ 3,\ 1,\ 0.707 - j0.707,\ -j,\ -0.707 - j0.707\}$$

Second stage computation

$$\left.\begin{array}{r}\text{The input sequence for second}\\ \text{stage of computation}\end{array}\right\} = \{3,\ 3,\ 3,\ 3,\ 1,\ 0.707 - j0.707,\ -j,\ -0.707 - j0.707\}$$

The phase factors involved in second stage computation are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 5.

$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

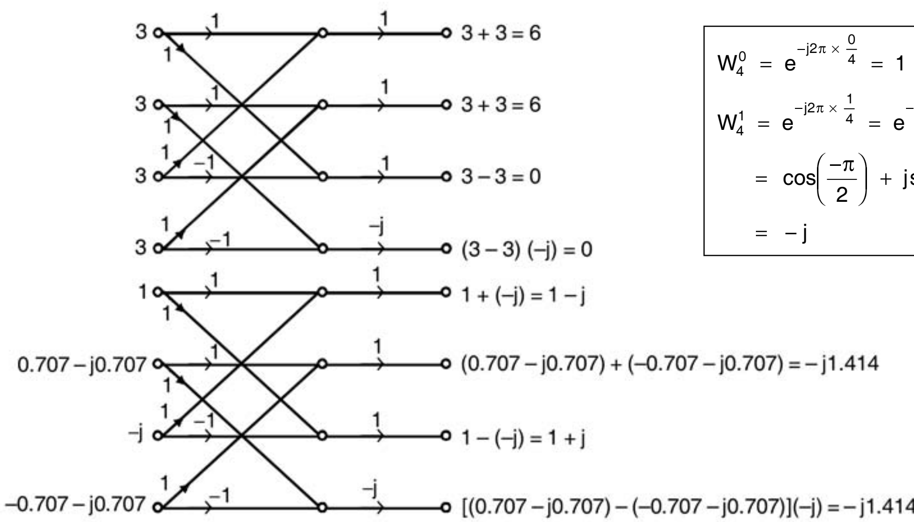


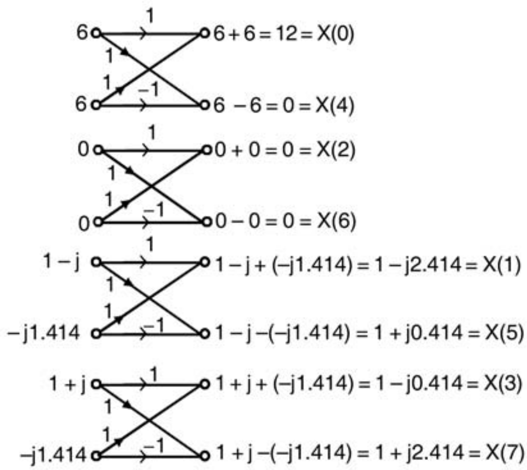
***Fig 5 :** Butterfly diagram for second stage of radix-2 DIF FFT.*

$$\left.\begin{array}{r}\text{The output sequence of second}\\ \text{stage of computation}\end{array}\right\} = \{\,6,\ \ 6,\ \ 0,\ \ 0,\ \ 1-j,\ \ -j1.414,\ \ 1+j,\ \ -j1.414\,\}$$

Third stage computation

The input sequence to third stage of computation $= \{\,6,\ \ 6,\ \ 0,\ \ 0,\ \ 1-j,\ \ -j1.414,\ \ 1+j,\ \ -j1.414\,\}$

The butterfly computations of third stage are shown in fig 6.



The phase factor involved in third stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

Fig 6 : *Butterfly diagram for third stage of radix-2 DIF FFT.*

The output sequence of third stage of computation $= \{\,12,\ 0,\ 0,\ 0,\ 1-j2.414,\ 1+j0.414,\ 1-j0.414,\ 1+j2.414\,\}$
The output sequence of third stage of computation is the 8-point DFT of the given sequence in bit reversed order.

In DIF-FFT algorithm the input to first stage is in normal order and the output of third stage will be in the bit reversed order. Hence the actual result is obtained by arranging the output sequence of third stage in normal order as shown below.

The sequence X(k) in bit reversed order	**The sequence X(k) in normal order**
$X(0) = 12$	$X(0) = 12$
$X(4) = 0$	$X(1) = 1 - j2.414$
$X(2) = 0$	$X(2) = 0$
$X(6) = 0$	$X(3) = 1 - j0.414$
$X(1) = 1 - j2.414$	$X(4) = 0$
$X(5) = 1 + j0.414$	$X(5) = 1 + j0.414$
$X(3) = 1 - j0.414$	$X(6) = 0$
$X(7) = 1 + j2.414$	$X(7) = 1 + j2.414$

$$\boxed{\therefore\ \mathcal{DFT}\{x(n)\} = X(k) = \{\,12,\ \ 1-j2.414,\ \ 0,\ \ 1-j0.414,\ \ 0,\ \ 1+j0.414,\ \ 0,\ \ 1+j2.414\,\}}$$

Magnitude and phase specturm

Each element of the sequence X(k) is a complex number and they are expressed in rectangular coordinates. If they are converted to polar coordinates then the magnitude and phase of each element can be obtained.

***Note :** The rectangular to polar conversion can be obtained by using* R $\rightarrow$ P *conversion in calculator.*

$$X(k) = \{12,\ 1-j2.414,\ 0,\ 1-j0.414,\ 0,\ 1+j0.414,\ 0,\ 1+j2.414\}$$

$$= \{12\angle 0°,\ 2.61\angle -67°,\ 0\angle 0°,\ 1.08\angle -22°,\ 0\angle 0°,\ 1.08\angle 22°,\ 0\angle 0°,\ 2.61\angle 67°\}$$

$$= \left\{12\angle 0,\ 2.61\angle -67°\times\frac{\pi}{180°},\ 0\angle 0,\ 1.08\angle -22°\times\frac{\pi}{180°},\ 0\angle 0,\ 1.08\angle 22°\times\frac{\pi}{180°},\ 0\angle 0,\ 2.61\angle 67°\times\frac{\pi}{180°}\right\}$$

$$= \{12\angle 0,\ 2.61\angle -0.37\pi,\ 0\angle 0,\ 1.08\angle -0.12\pi,\ 0\angle 0,\ 1.08\angle 0.12\pi,\ 0\angle 0,\ 2.61\angle 0.37\pi\}$$

$$\therefore |X(k)| = \{12,\ 2.61,\ 0,\ 1.08,\ 0,\ 1.08,\ 0,\ 2.61\}$$

$$\angle X(k) = \{0,\ -0.37\pi,\ 0,\ -0.12\pi,\ 0,\ 0.12\pi,\ 0,\ 0.37\pi\}$$

The magnitude specturm is the plot of the magnitude of each sample of X(k) as a function of k as shown in fig 7. The phase spectrum is the plot of phase of X(k) as a function of k as shown in fig 8.

When N-point DFT is performed on a sequence x(n) then the DFT sequence X(k) will have a periodicity of N. Hence in this example the magnitude and phase specturm will have a periodicity of 8 as shown in fig 7 and fig 8.

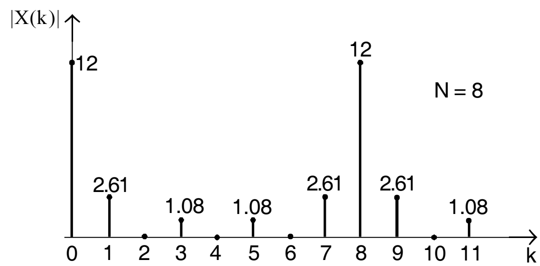


***Fig 7 :** Magnitude spectrum.*

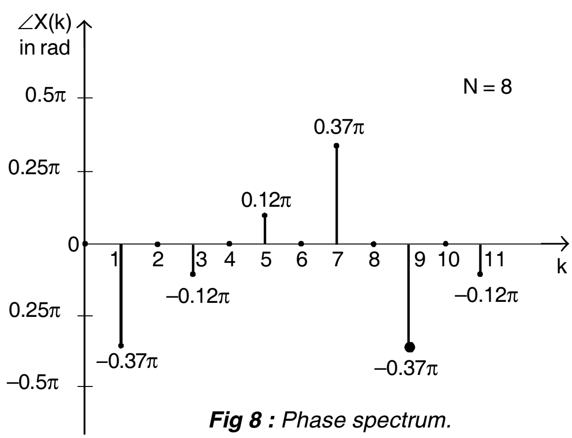


***Fig 8 :** Phase spectrum.*

Example 9.6

In an LTI system the input x(n) = {1, 1, 1} and the impulse response h(n) = {–1, –1}. Determine the response of the LTI system by radix-2 DIT FFT.

Solution

The response y(n) of LTI system is given by linear convolution of input x(n) and impulse response h(n).

$$\therefore \text{ Response or Output, } y(n) = x(n) * h(n)$$

The DFT (or FFT) supports only circular convolution. Hence to get the result of linear convolution from circular convolution, the sequences x(n) and h(n) should be converted to the size of y(n) by appending with zeros and circular convolution of x(n) and h(n) is performed.

The length of x(n) is 3 and h(n) is 2. Hence the length of y(n) is 3 + 2 – 1 = 4. Therefore given sequences x(n) and h(n) are converted to 4 point sequences by appending zeros.

$$\therefore x(n) = \{1, 1, 1, 0\} \text{ and } h(n) = \{-1, -1, 0, 0\}$$

Now the response y(n) is given by, $y(n) = x(n) \circledast h(n)$.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$, $\mathcal{DFT}\{h(n)\} = H(k)$, $\mathcal{DFT}\{y(n)\} = Y(k)$.

By convolution theorem of DFT we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\, H(k)$$

$$\therefore y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \mathcal{DFT}^{-1}\{X(k)\, H(k)\}$$

The various steps in computing y(n) are,

Step - 1 : Determine X(k) using radix-2 DIT algorithm.

Step - 2 : Determine H(k) using radix-2 DIT algorithm.

Step - 3 : Determine the product X(k)H(k).

Step - 4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1: To determine X(k)

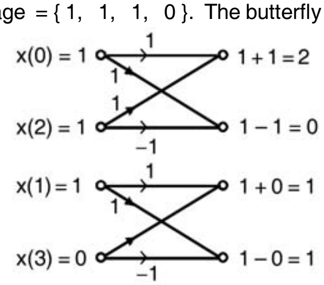
Since x(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The given sequence x(n), is first arranged in bit reversed order as shown in table.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

x(n) Normal order	x(n) Bit reversed order
x(0) = 1	x(0) = 1
x(1) = 1	x(2) = 1
x(2) = 1	x(1) = 1
x(3) = 0	x(3) = 0

First stage computation

Input sequence to first stage = { 1, 1, 1, 0 }. The butterfly computations of first stage are shown in fig 1.



The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

***Fig 1 :** Butterfly diagram for first stage of radix-2 DIT FFT of X(k).*

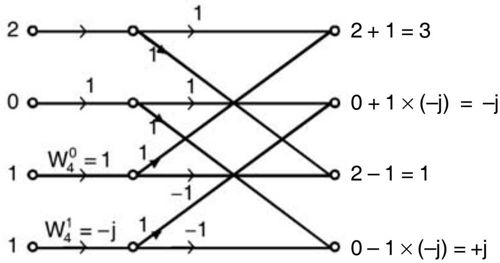
Output sequence of first stage of computation = { 2, 0, 1, 1 }

Second stage computation

Input sequence to second stage computation = { 2, 0, 1, 1 }

The phase factors involved in second stage computation are W_4^0 and W_4^1 .

The butterfly computations of second stage are shown in fig 2.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

***Fig 2 :** Butterfly diagram for second stage of radix-2 DIT FFT of X(k).*

Output sequence of second stage computation = { 3, –j, 1, +j }

The output sequence of second stage of computation is the 4-point DFT of x(n).

$$\boxed{\therefore X(k) = \mathcal{DFT}\{x(n)\} = \{3,\ -j,\ 1,\ j\}}$$

Step - 2: To determine H(k)

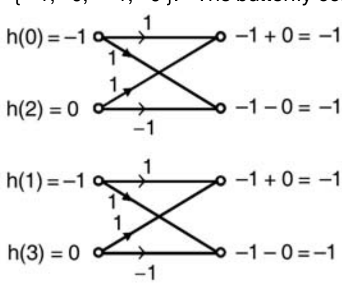
Since h(n) is a 4-point sequence, we have to compute 4-point DFT. The 4-point DFT by radix-2 FFT consists of two stages of computations with 2-butterflies in each stage. The sequence h(n) is first arranged in bit reversed order as shown below.

The sequence in bit reversed order forms the input sequence to first stage computation.

h(n) Normal order	h(n) Bit reversed order
h(0) = –1	h(0) = –1
h(1) = –1	h(2) = 0
h(2) = 0	h(1) = –1
h(3) = 0	h(3) = 0

First stage computation

Input sequence of first stage = { –1, 0, –1, 0 }. The butterfly computations of first stage are shown in fig 3.



The phase factor involved in first stage of computation is W_2^0 . Since, $W_2^0 = 1$, it is not considered for computation.

***Fig 3 :** Butterfly diagram for first stage of radix-2 DIT FFT of H(k).*

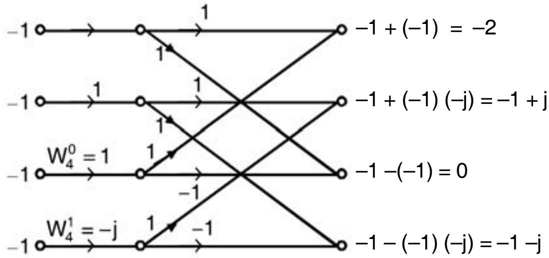
Output sequence of first stage computation = { –1, –1, –1, –1 }

Second stage computation

Input sequence to second stage computation = { –1, –1, –1, –1 }

The phase factors involved are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 4.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

***Fig 4 :** Butterfly diagram for second stage of radix-2 DIT FFT of H(k).*

Output sequence of second stage computation = { –2, –1 + j, 0, –1 –j }

The output sequence of second stage computation is the 4-point DFT of h(n).

$$\boxed{\therefore H(k) = \mathcal{DFT}\{h(n)\} = \{-2,\ -1+j,\ 0,\ -1-j\}}$$

Step 3 : To determine the product X(k)H(k)

Let the product, $X(k)H(k) = Y(k)$; for k = 0, 1, 2, 3.

when k = 0; $Y(0) = X(0) \times H(0) = 3 \times (-2) = -6$

when k = 1; $Y(1) = X(1) \times H(1) = (-j) \times (-1 + j) = 1 + j$

when k = 2; $Y(2) = X(2) \times H(2) = 1 \times 0 = 0$

when k = 3; $Y(3) = X(3) \times H(3) = j \times (-1-j) = 1 - j$

$$\boxed{\therefore Y(k) = \{-6,\ 1+j,\ 0,\ 1-j\}}$$

Step - 4: To determine inverse DFT of Y(k)

The 4-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 4.

Y(k) = { –6, 1 + j, 0, 1 – j }

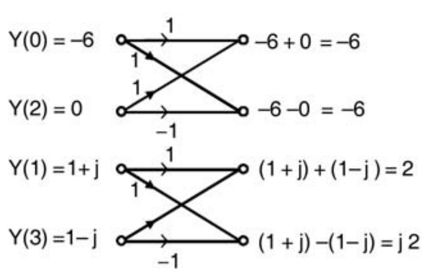
The 4-point inverse DFT of Y(k) using radix-2 DIT FFT involves two stages of computations with 2-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in the table.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

Y(k) Normal order	Y(k) Bit reversed order
Y(0) = –6	Y(0) = –6
Y(1) = 1+ j	Y(2) = 0
Y(2) = 0	Y(1) = 1 + j
Y(3) = 1 – j	Y(3) = 1 – j

First stage computation

Input sequence to first stage = { –6, 0, 1+j, 1–j }. The butterfly computations of first stage are shown in fig 5.



The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

***Fig 5 :** Butterfly diagram for first stage of inverse DFT of Y(K).*

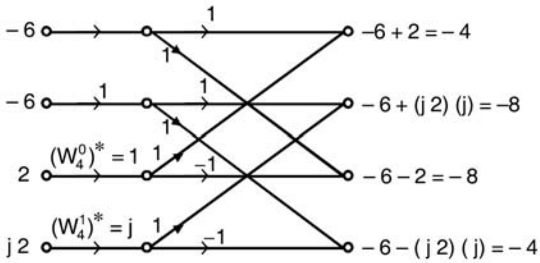
The output sequence of first stage computation = { – 6, – 6, 2, j 2 }

Second stage computation

Input sequence to second stage computation = { – 6, – 6, 2, j 2 }

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$.

The butterfly computation of second stage is shown in fig 6.



$$(W_4^0)^* = e^{j2\pi \times \frac{0}{4}} = 1$$

$$(W_4^1)^* = e^{j2\pi \times \frac{1}{4}} = e^{j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right)$$

$$= j$$

***Fig 4 :** Butterfly diagram for second stage of inverse DFT of Y(k).*

The output sequence of second stage computation = { – 4, – 8, – 8, – 4 }

The sequence y(n) is obtained by dividing each sample of output sequence of second stage by 4.

$\boxed{\therefore \text{ The response of the LTI system, } y(n) = \{-1, -2, -2, -1\}}$

Example 9.7

Determine the response of LTI system when the input sequence x(n) = {–1, 1, 2, 1, –1} by radix 2 DIT FFT. The impulse response of the system is h(n) = {–1, 1, –1, 1}.

Solution

The response of an LTI system is given by linear convolution of input sequence x(n) and impulse response h(n).

$\therefore$ Response or output, $y(n) = x(n) * h(n)$.

The DFT (or FFT) supports only circular convolution. Hence to get the result of linear convolution from circular convolution, the sequence x(n) and h(n) should be converted to the size of y(n), by appending with zeros, and then circular convolution of x(n) and h(n) is performed.

The length of x(n) = 5, and h(n) = 4. Hence the length of y(n) is 5 + 4 – 1 = 8.

Therefore x(n) and h(n) are converted into 8-point sequence by appending zeros.

$\therefore$ x(n) = { –1, 1, 2, 1, –1, 0, 0, 0 } and h(n) = { –1, 1, –1, 1, 0, 0, 0, 0 }

Now, the response y(n) is given by, $y(n) = x(n) \circledast h(n)$.

Let, $\mathcal{DFT}\{x(n)\} = X(k)$, $\mathcal{DFT}\{h(n)\} = H(k)$, $\mathcal{DFT}\{y(n)\} = Y(k)$.

By convolution theorem of DFT we get,

$$\mathcal{DFT}\{x(n) \circledast h(n)\} = X(k)\,H(k)$$

$$\therefore y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \mathcal{DFT}^{-1}\{X(k)\,H(k)\}$$

The various steps in computing y(n) are,

Step - 1 : Determine X(k) using radix-2 DIT algorithm.

Step - 2 : Determine H(k) using radix-2 DIT algorithm.

Step - 3 : Determine the product X(k)H(k).

Step - 4 : Take inverse DFT of the product X(k)H(k) using radix-2 DIT algorithm.

Step-1 : To determine X(k)

Since x(n) is an 8 point sequence, we have to compute 8-point DFT.

The 8-point DFT by radix-2 FFT algorithm consists of 3 stages of computations with 4 butterflies in each stage.

The given sequence x(n) is arranged in bit reversed order as shown in the following table.

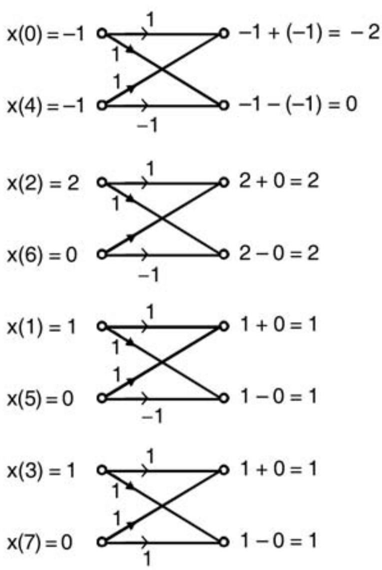
x(n) Normal order	x(n) Bit reversed order
x(0) = –1	x(0) = –1
x(1) = 1	x(4) = –1
x(2) = 2	x(2) = 2
x(3) = 1	x(6) = 0
x(4) = –1	x(1) = 1
x(5) = 0	x(5) = 0
x(6) = 0	x(3) = 1
x(7) = 0	x(7) = 0

The sequence arranged in bit-reversed order forms the input sequence to the first stage computation.

First stage computation

Input sequence to first stage = { –1, –1, 2, 0, 1, 0, 1, 0 }.

The butterfly computation of first stage is shown in fig 1.



The phase factor involved in first stage of computation is W_2^0. Since, $W_2^0 = 1$, it is not considered for computation.

***Fig 1 :** Butterfly diagram for first stage of radix-2 DIT FFT of X(k).*

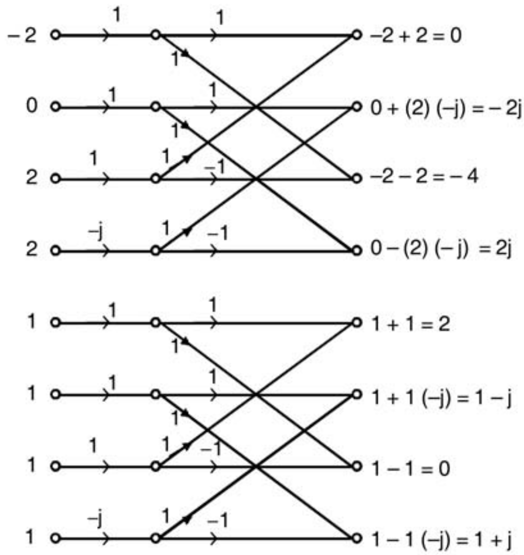
Output sequence of first stage of computation = { –2, 0, 2, 2, 1, 1, 1, 1 }

Second stage computation

The input sequence to second stage of computation = { –2, 0, 2, 2, 1, 1, 1, 1 }

Phase factors involved in second stage are W_4^0 and W_4^1 .

The butterfly computation of second stage is shown in fig 2.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

***Fig 2 :** Butterfly diagram for second stage of radix-2 DIT FFT of X(k).*

Output sequence of second stage of computation = { 0, –2j, –4, 2j, 2, 1 – j, 0, 1 + j }

Third stage computation

Input sequence to third stage computation = { 0, –2j, –4, 2j, 2, 1 – j, 0, 1 + j }.

Phase factors involved are W_8^0, W_8^1, W_8^2 and W_8^3.

The butterfly computation of third stage is shown in fig 3.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$\left.\begin{array}{r}\text{Output sequence of third}\\ \text{stage of computation}\end{array}\right\} = \left\{2,\ -j\left(2+\sqrt{2}\right),\ -4,\ j\left(2-\sqrt{2}\right),\ -2,\ j\left(-2+\sqrt{2}\right),\ -4,\ j\left(2+\sqrt{2}\right)\right\}$$

$$\boxed{\therefore\ \text{DFT}\ \{x(n)\} = X(k) = \left\{2,\ -j\left(2+\sqrt{2}\right),\ -4,\ j\left(2-\sqrt{2}\right),\ -2,\ j\left(-2+\sqrt{2}\right),\ -4,\ j\left(2+\sqrt{2}\right)\right\}}$$

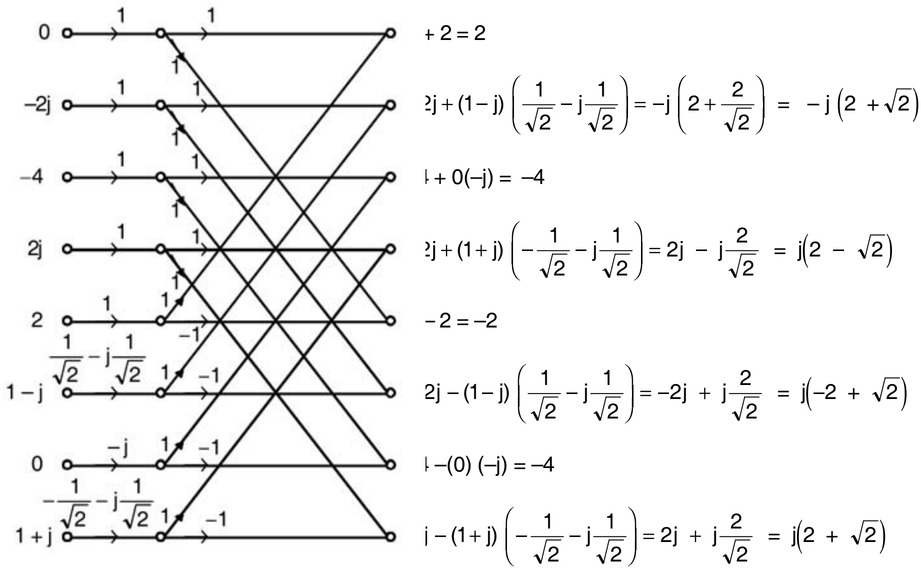


Fig 3 : *Butterfly diagram for third stage of radix-2 DIT FFT of X(k).*

Step 2 : To determine H(k)

Since h(n) is an 8-point sequence, we have to compute 8-point DFT. The 8-point DFT by radix-2 FFT consists of three stages of computations with four butterflies in each stage.

The sequence h(n) is first arranged in bit reversed order as shown in the following table .

h(n) Normal order	h(n) Bit reversed order
h(0) = –1	h(0) = –1
h(1) = 1	h(4) = 0
h(2) = –1	h(2) = –1
h(3) = 1	h(6) = 0
h(4) = 0	h(1) = 1
h(5) = 0	h(5) = 0
h(6) = 0	h(3) = 1
h(7) = 0	h(7) = 0

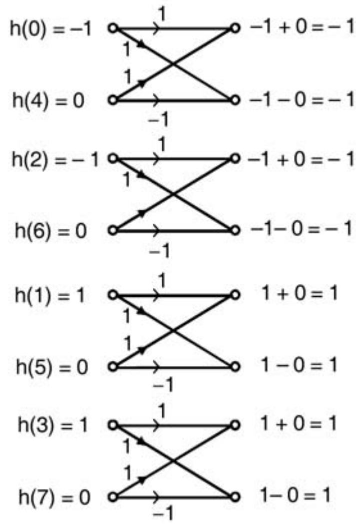
The sequence arranged in bit reversed order forms the input sequence to the first stage.

First stage computation

Input sequence to first stage computation = { –1, 0, –1, 0, 1, 0, 1, 0 }

The butterfly computations of first stage is shown in fig 4.

Outtput sequence of first stage of computation = { –1, –1, –1, –1, 1, 1, 1, 1 }



The phase factor involved in first stage of computation is W_2^0 *. Since,* $W_2^0 = 1$*, it is not considered for computation.*

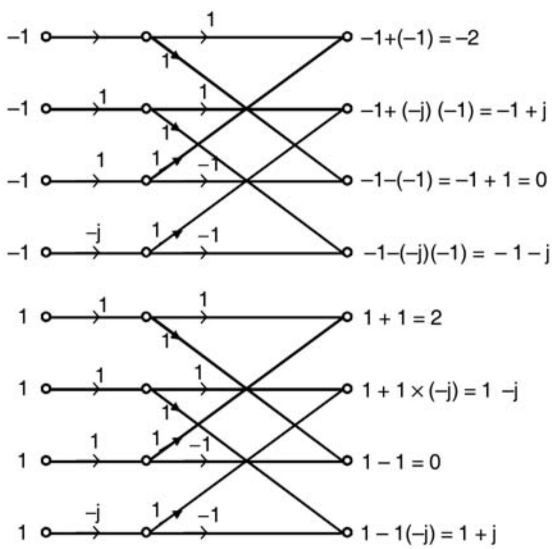
***Fig 4 :** Butterfly diagram for first stage of radix-2 DIT FFT of H(k).*

Second stage computation

Input sequence to second stage of computation = { –1, –1, –1, –1, 1, 1, 1, 1 }

Phase factors involved in second stage are W_4^0 and W_4^1.

The butterfly computations of second stage are shown in fig 5.



$$W_4^0 = e^{-j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{-j2\pi \times \frac{1}{4}} = e^{-j \times \frac{\pi}{2}}$$

$$= \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right)$$

$$= -j$$

***Fig 5 :** Butterfly diagram for second stage of radix-2 DIT FFT of H(k).*

$$\left.\begin{array}{r}\text{Output sequence of second} \\ \text{stage of computation}\end{array}\right\} = \{-2,\ -1+j,\ 0,\ -1-j,\ 2,\ 1-j,\ 0,\ 1+j\}$$

Third stage computation

Input sequence to third stage computation $= \{-2,\ -1+j,\ 0,\ -1-j,\ 2,\ 1-j,\ 0,\ 1+j\}$

Phase factors involved in third stage computations are $W_8^0,\ W_8^1,\ W_8^2,$ and W_8^3.

The butterfly computations of third stage are shown in fig 6.

$$W_8^0 = e^{-j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{-j \times \frac{\pi}{4}} = \cos\left(\frac{-\pi}{4}\right) + j\sin\left(\frac{-\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{-j2\pi \times \frac{2}{8}} = e^{-j \times \frac{\pi}{2}} = \cos\left(\frac{-\pi}{2}\right) + j\sin\left(\frac{-\pi}{2}\right) = -j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}} = e^{-j \times \frac{3\pi}{4}} = \cos\left(\frac{-3\pi}{4}\right) + j\sin\left(\frac{-3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

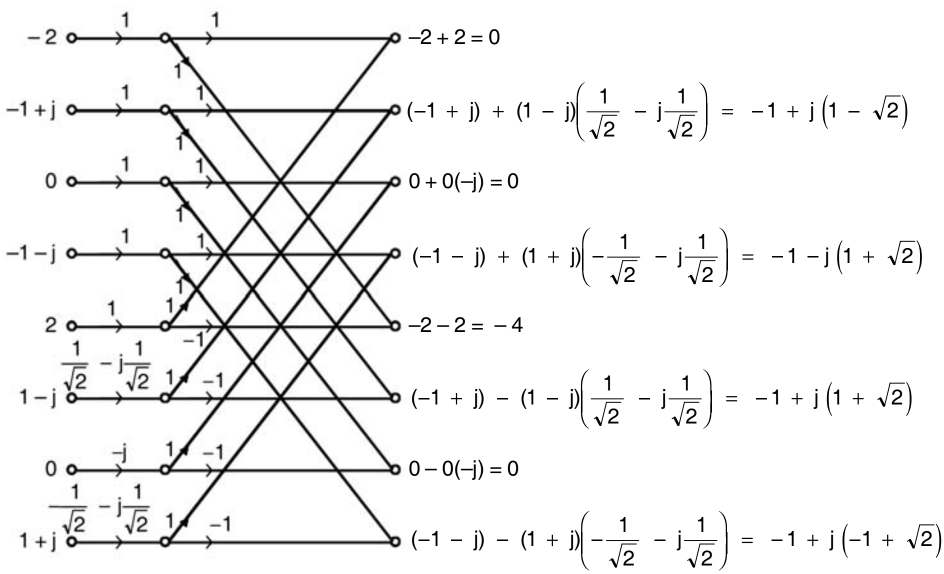


***Fig 6 :** Butterfly diagram for third stage of radix-2 DIT FFT of H(k).*

$$\left.\begin{matrix}\text{Output sequence of third} \\ \text{stage computation}\end{matrix}\right\} = \left\{0,\ -1+j\left(1-\sqrt{2}\right),\ 0,\ -1-j\left(1+\sqrt{2}\right),\ -4,\ -1+j\left(1+\sqrt{2}\right),\ 0,\ -1+j\left(-1+\sqrt{2}\right)\right\}$$

The output sequence of third stage computation is the 8-point DFT of h(n).

$$\boxed{\therefore\ \mathcal{DFT}\{h(n)\} = H(k) = \left\{0,\ -1+j\left(1-\sqrt{2}\right),\ 0,\ -1-j\left(1+\sqrt{2}\right),\ -4,\ -1+j\left(1+\sqrt{2}\right),\ 0,\ -1+j\left(-1+\sqrt{2}\right)\right\}}$$

Step 3 : To determine the product X(k)H(k)

Let the product of $X(k)H(k) = Y(k)$; for $k = 0, 1, 2, 3, 4, 5, 6, 7$

$$\therefore Y(k) = X(k)H(k)$$

when $k = 0$; $Y(0) = X(0)\,H(0) = 2 \times 0 = 0$

when $k = 1$; $Y(1) = X(1)\,H(1) = -j\left(2+\sqrt{2}\right) \times -1 + j(1-\sqrt{2}) = -\sqrt{2}+j(2+\sqrt{2})$

when $k = 2$; $Y(2) = X(2)\,H(2) = -4 \times 0 = 0$

when $k = 3$; $Y(3) = X(3)\,H(3) = j\left(2-\sqrt{2}\right) \times -1 - j(1+\sqrt{2}) = \sqrt{2}-j(2-\sqrt{2})$

when $k = 4$; $Y(4) = X(4)\,H(4) = -2 \times -4 = 8$

when $k = 5$; $Y(5) = X(5)\,H(5) = j\left(-2+\sqrt{2}\right) \times \left(-1 + j(1+\sqrt{2})\right) = \sqrt{2} + j(2-\sqrt{2})$

when $k = 6$; $Y(6) = X(6)\,H(6) = -4 \times 0 = 0$

when $k = 7$; $Y(7) = X(7)\,H(7) = j\left(2+\sqrt{2}\right) \times \left(-1 + j(-1+\sqrt{2})\right) = -\sqrt{2} - j(2+\sqrt{2})$

$$\boxed{\therefore Y(k) = \left\{0,\ -\sqrt{2}+j(2+\sqrt{2}),\ 0,\ \sqrt{2}-j(2-\sqrt{2}),\ 8,\ \sqrt{2}+j(2-\sqrt{2}),\ 0,\ -\sqrt{2}-j(2+\sqrt{2})\right\}}$$

Step - 4: To determine inverse DFT of Y(k)

The 8-point inverse DFT of Y(k) can be computed using radix-2 DIT FFT by taking conjugate of the phase factors and then dividing the output sequence of FFT by 8.

The 8-point inverse DFT of Y(k) using radix-2 DIT FFT involves three stages of computations with 4-butterflies in each stage. The sequence Y(k) is arranged in bit reversed order as shown in the following table.

The sequence arranged in bit reversed order forms the input sequence to first stage computation.

Y(k) Normal order	Y(k) Bit reversed order
$Y(0) = 0$	$Y(0) = 0$
$Y(1) = -\sqrt{2}+j(2+\sqrt{2})$	$Y(4) = 8$
$Y(2) = 0$	$Y(2) = 0$
$Y(3) = \sqrt{2}-j(2-\sqrt{2})$	$Y(6) = 0$
$Y(4) = 8$	$Y(1) = -\sqrt{2}+j(2+\sqrt{2})$
$Y(5) = \sqrt{2}+j(2-\sqrt{2})$	$Y(5) = \sqrt{2}+j(2-\sqrt{2})$
$Y(6) = 0$	$Y(3) = \sqrt{2}-j(2-\sqrt{2})$
$Y(7) = -\sqrt{2}-j(2+\sqrt{2})$	$Y(7) = -\sqrt{2}-j(2+\sqrt{2})$

First stage computation

Input sequence of first stage $= \left\{0,\ 8,\ 0,\ 0,\ -\sqrt{2}+j(2+\sqrt{2}),\ \sqrt{2}+j(2-\sqrt{2}),\ \sqrt{2}-j(2-\sqrt{2}),\ -\sqrt{2}-j(2+\sqrt{2})\right\}$

The butterfly computations of first stage are shown in fig 7.

Output sequence of first stage $= \left\{8,\ -8,\ 0,\ 0,\ j4,\ -2\sqrt{2}+j2\sqrt{2},\ -j4,\ 2\sqrt{2}+j2\sqrt{2}\right\}$

The phase factor involved in first stage of computation is $(W_2^0)^*$. Since, $(W_2^0)^* = 1$, it is not considered for computation.

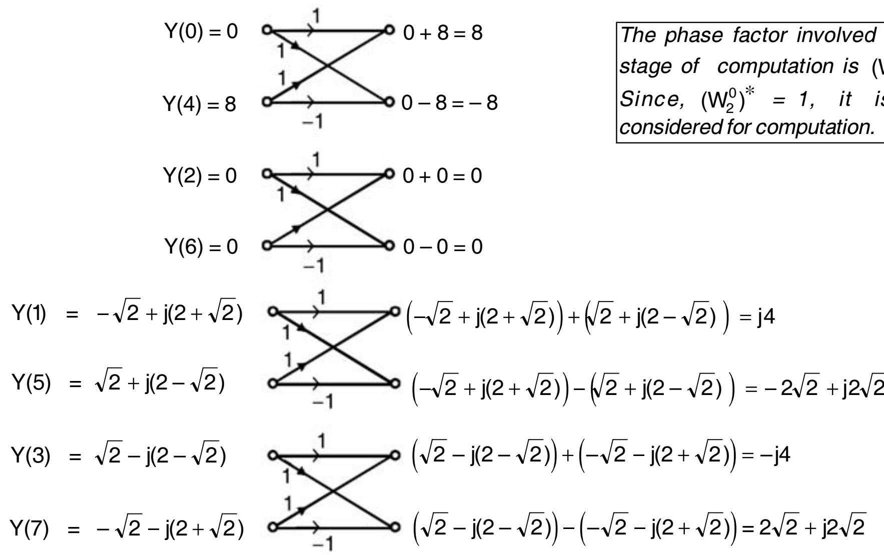


***Fig 7 :** Butterfly diagram for first stage of inverse DFT of Y(K).*

Second stage computation

Input sequence of second stage $= \left\{8,\ -8,\ 0,\ 0,\ j4,\ -2\sqrt{2}+j2\sqrt{2},\ -j4,\ 2\sqrt{2}+j2\sqrt{2}\right\}$

The butterfly computation of second stage is shown in fig 8.

The phase factors involved are $(W_4^0)^*$ and $(W_4^1)^*$.

$$(W_4^0)^* = e^{j2\pi \times \frac{0}{4}} = 1$$

$$(W_4^1)^* = e^{j2\pi \times \frac{1}{4}} = e^{j \times \frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$$

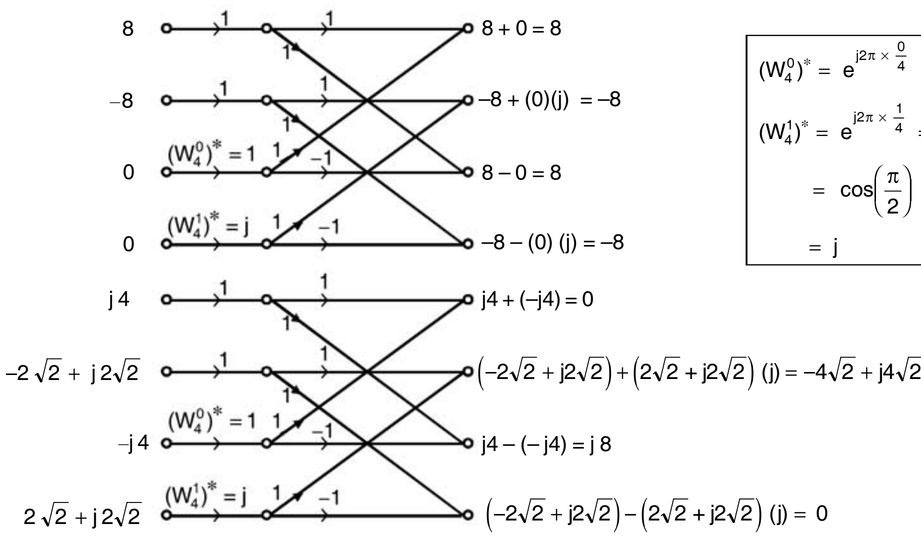


***Fig 8 :** Butterfly diagram for second stage of inverse DFT of Y(k).*

Output sequence of second stage computation $= \left\{8,\ -8,\ 8,\ -8,\ 0,\ -4\sqrt{2}+j4\sqrt{2},\ j8,\ 0\right\}$

Third Stage Computation

Input sequence of third stage computation $= \left\{8,\ -8,\ 8,\ -8,\ 0,\ -4\sqrt{2}+j4\sqrt{2},\ \ j8,\ 0\right\}$

The butterfly computation of third stage is shown in fig 9.

The phase factors involved are $(W_8^0)^*$, $(W_8^1)^*$, $(W_8^2)^*$ and $(W_8^3)^*$.

$$(W_8^0)^* = e^{j2\pi \times \frac{0}{8}} = 1$$

$$(W_8^1)^* = e^{j2\pi \times \frac{1}{8}} = e^{j \times \frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

$$(W_8^2)^* = e^{j2\pi \times \frac{2}{8}} = e^{j \times \frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + j\sin\left(\frac{\pi}{2}\right) = j$$

$$(W_8^3)^* = e^{j2\pi \times \frac{3}{8}} = e^{j \times \frac{3\pi}{4}} = \cos\left(\frac{3\pi}{4}\right) + j\sin\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

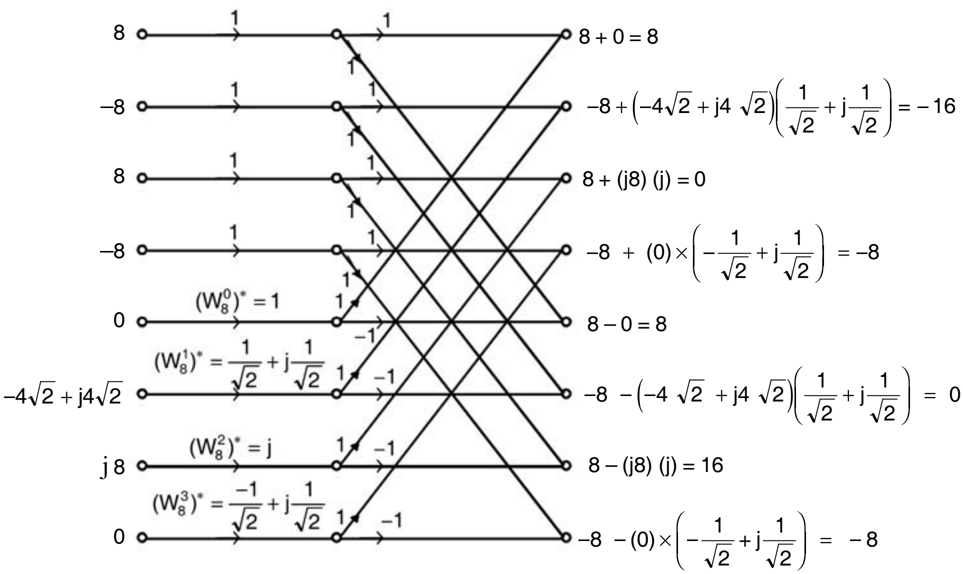


***Fig 9 :** Butterfly diagram for third stage of inverse DFT of Y(k).*

Output sequence of third stage computation = { 8, –16, 0, –8, 8, 0, 16, –8 }

The sequence y(n) is obtained by dividing each sample of output sequence of third stage by 8.

$\therefore$ The response of the LTI system, y(n) = { 1, –2, 0, –1, 1, 0, 2, –1 }

9.10 Summary of Important Concepts

1. The drawback in DTFT is that the frequency domain representation of a discrete time signal obtained using DTFT will be a continuous function of ω.
2. The DFT has been developed to convert a continuous function of ω to a discrete function of ω.
3. The DFT of a discrete time signal can be obtained by sampling the DTFT of the signal.
4. The sampling of the DTFT is conventionally performed at N equally spaced frequency points in the period, $0 \le \omega \le 2\pi$.
5. DFT sequence starts at $k = 0$, corresponding to $\omega = 0$ but does not include $k = N$, corresponding to $\omega = 2\pi$.
6. The DFT is defined along with number of samples and is called N-point DFT.
7. The number of samples N for a finite duration sequence x(n) of length L should be such that, $N \ge L$, in order to avoid aliasing of frequency spectrum.
8. The X(k) is also called discrete frequency spectrum (or signal spectrum) of the discrete time signal x(n).
9. The plot of samples of magnitude sequence versus k is called magnitude spectrum.
10. The plot of samples of phase sequence versus k is called phase spectrum.
11. The DFT sequence X(k) is periodic with periodicity of N samples.
12. The DFT of circular convolution of two sequences is equivalent to product of their individual DFTs.
13. The N-point DFT of a finite duration sequence can be obtained from the $\mathcal{Z}$-transform of the sequence, by evaluating the $\mathcal{Z}$-transform at N equally spaced points around the unit circle.
14. The DFT supports only circular convolution and so, the linear convolution using DFT has to be computed via circular convolution.
15. The FFT is a method (or algorithm) for computing the DFT with reduced number of calculations.
16. In N-point DFT by radix-r FFT, the number of stages of computation will be "m" times, where $m = \log_r N$.
17. In direct computation of N-point DFT, the total number of complex additions are $N(N-1)$ and total number of complex multiplications are N^2.
18. In computation of N-point DFT via radix-2 FFT, the total number of complex additions are $N\log_2 N$ and total number of complex multiplications are $(N/2)\log_2 N$.
19. The complex valued phase factor or twiddle factor W_N is defined as, $W_N = e^{\frac{-j2\pi}{N}}$
20. The term W in phase factor represents a complex number $1\angle{-2\pi}$.
21. The multiplication of the phase value -2π of W by k can be represented as W^k.
22. The division of the phase value -2π of W by N can be represented as W_N.
23. In DIT the time domain sequence is decimated, whereas in DIF the frequency domain sequence is decimated.
24. In radix-2 FFT algorithm, the N-point DFT can be realized from two numbers of N/2 point DFTs, the N/2 point DFT can be realized from two numbers of N/4 points DFTs, and so on.
25. In radix-2 FFT, N/2 butterflies per stage are required to represent the computational process.
26. In radix-2 DIT FFT, the input should be in bit reversed order and the output will be in normal order.
27. In radix-2 DIF FFT, the input should be in normal order and the output will be in bit reversed order.
28. In butterfly computation of DIT, the complex multiplication takes place before the add-subtract operation.
29. In butterfly computation of DIF, the complex multiplication takes place after the add-subtract operation.
30. In FFT, the phase factor for computing inverse DFT will be conjugate of phase factors for computing DFT.

9.11 Short Questions and Answers

Q9.1 Calculate the DFT of the sequence, x(n) = {1, 1, –2, –2}.

Solution

The N-point DFT of x(n) is given by,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \quad ; \text{ for } k = 0, 1, 2, ...N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\therefore X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi nk}{4}} = x(0)e^{0} + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= 1 + e^{-j\frac{\pi k}{2}} - 2e^{-j\pi k} - 2e^{-j\frac{3\pi k}{2}} \quad ; \text{ for } k = 0, 1, 2, 3$$

Q9.2 Find the DFT of the sequence x(n) = {1, 1, 0, 0}. Also find magnitude and phase sequence.

Solution

The N-point DFT of x(n) is given by,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} \; ; \text{ for } k = 0, 1, 2, ..., N-1$$

Since x(n) is a 4-point sequence, we can take 4-point DFT.

$$\therefore X(k) = \sum_{n=0}^{3} x(n)e^{-j\frac{2\pi nk}{4}} = x(0)e^{0} + x(1)e^{-j\frac{\pi k}{2}} + x(2)e^{-j\pi k} + x(3)e^{-j\frac{3\pi k}{2}}$$

$$= 1 + e^{-j\frac{\pi k}{2}} + 0 + 0 = e^{-j\frac{\pi k}{4}}\left(e^{j\frac{\pi k}{4}} + e^{-j\frac{\pi k}{4}}\right)$$

$$= e^{-j\frac{\pi k}{4}}\, 2\cos\left(\frac{\pi k}{4}\right) = 2\cos\left(\frac{\pi k}{4}\right)e^{-j\frac{\pi k}{4}} \; ; \text{ for } k = 0, 1, 2, 3$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\therefore |X(k)| = 2\cos\left(\frac{\pi k}{4}\right) \text{ and } \angle X(k) = -\frac{\pi k}{4} \; ; \text{ for } k = 0, 1, 2, 3$$

Q9.3 Compute the DFT of the sequence x(n) = (–1)ⁿ for the period N = 16.

Solution

Given that, $x(n) = (-1)^n = \{...... 1, -1, 1, -1, 1, -1, ...\}$. On evaluating the sequence for all values of n, it can be observed that x(n) is periodic with periodicity of 2 samples. The DFT of x(n) has to be computed for the period N = 16. Let us consider the 16-sample of the infinite sequence from n = 0 to n = 15.

The 16-point DFT of x(n) is given by,

$$\boxed{\sum_{n=0}^{N-1} C^n = \frac{1-C^N}{1-C}}$$

$$\boxed{e^{j\theta} \times e^{-j\theta} = 1}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$X(k) = \sum_{n=0}^{15} x(n)e^{-j\frac{2\pi nk}{16}} = \sum_{n=0}^{15} (-1)^n \times e^{-j\frac{\pi nk}{8}} = \sum_{n=0}^{15}\left(-e^{-j\frac{\pi k}{8}}\right)^n$$

$$= \frac{1-\left(-e^{-j\frac{\pi k}{8}}\right)^{16}}{1-\left(-e^{-j\frac{\pi k}{8}}\right)} = \frac{1 - e^{-j\frac{\pi k 16}{8}}}{1 + e^{-j\frac{\pi k}{8}}} = \frac{1 - e^{-j2\pi k}}{1 + e^{-j\frac{\pi k}{8}}} = \frac{e^{-j\pi k}\left(e^{j\pi k} - e^{-j\pi k}\right)}{e^{-j\frac{\pi k}{16}}\left(e^{j\frac{\pi k}{16}} + e^{-j\frac{\pi k}{16}}\right)}$$

$$= e^{-j\pi k + j\frac{\pi k}{16}} \frac{\sin(\pi k)}{j\cos\left(\frac{\pi k}{16}\right)} = -j\frac{\sin(\pi k)}{\cos\left(\frac{\pi k}{16}\right)} e^{-j\frac{15\pi k}{16}} = 1\angle -\frac{\pi}{2}\ \frac{\sin(\pi k)}{\cos\left(\frac{\pi k}{16}\right)} e^{-j\frac{15\pi k}{16}}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= e^{-j\frac{\pi}{2}} \frac{\sin \pi k}{\cos\frac{\pi k}{16}} e^{-j\frac{15\pi k}{16}} = \frac{\sin \pi k}{\cos\frac{\pi k}{16}} e^{-\frac{23\pi k}{16}} \quad ; \quad \text{for } k = 0, 1, 2, ...15$$

Q9.4 ***Find the inverse DFT of Y(k) = {1, 0, 1, 0}.***

Solution

The inverse DFT of the sequence Y(k) of length 4 is given by,

$$\mathcal{DFT}^{-1}\{Y(k)\} = y(n) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\frac{2\pi kn}{4}} \ ; \text{for } n = 0, 1, 2, 3$$

$$\therefore\ y(n) = \frac{1}{4}\left[Y(0)e^{0} + Y(1)e^{j\frac{\pi n}{2}} + Y(2)e^{j\pi n} + Y(3)e^{j\frac{3\pi n}{2}}\right]$$

$$= \frac{1}{4}\left[1 + 0 + e^{j\pi n} + 0\right] = \frac{1}{4}\left[1 + \cos\pi n + j\sin\pi n\right] \ ; \quad \text{for } n = 0, 1, 2, 3$$

When $n = 0$; $y(0) = 1/4\ (1 + \cos 0 + j \sin 0) = 0.5$

When $n = 1$; $y(1) = 1/4\ (1 + \cos \pi + j \sin \pi) = 0$

When $n = 2$; $y(2) = 1/4\ (1 + \cos 2\pi + j \sin 2\pi) = 0.5$

When $n = 3$; $y(3) = 1/4\ (1 + \cos 3\pi + j \sin 3\pi) = 0$

$\therefore\ y(n) = \{0.5, 0, 0.5, 0\}$

Q9.5 ***Calculate the percentage saving in calculations in a 512-point radix-2 FFT, when compared to direct DFT.***

Solution

Direct computation of DFT

Number of complex additions $= N(N-1) = 512 \times (512 - 1) = 2{,}61{,}632$

Number of complex multiplications $= N^2 = 512^2 = 2{,}62{,}144$

Radix-2 FFT

Number of complex additions $= N\log_2 N = 512 \times \log_2 512$

$$= 512 \times \log_2 2^9 = 512 \times 9 = 4{,}608$$

Number of complex multiplications $= \frac{N}{2}\log_2 N = \frac{512}{2} \times \log_2 512$

$$= \frac{512}{2} \times \log_2 2^9 = \frac{512}{2} \times 9 = 2304$$

Percentage Saving

$$\text{Percentage saving in additions} = 100 - \frac{\text{Number of additions in radix-2 FFT}}{\text{Number of additions in direct DFT}} \times 100$$

$$= 100 - \frac{4{,}608}{2{,}61{,}632} \times 100 = 98.2\%$$

$$\text{Percentage saving in multiplications} = 100 - \frac{\text{Number of multiplications in radix-2 FFT}}{\text{Number of multiplications in direct DFT}} \times 100$$

$$= 100 - \frac{2{,}304}{2{,}62{,}144} \times 100 = 99.1\%$$

Q9.6 Arrange the 8-point sequence, x(n) = {1, 2, 3, 4, –1, –2, –3, –4} in bit reversed order.

The x(n) in normal order = {1, 2, 3, 4, –1, –2, –3, –4}

The x(n) in bit reversed order = {1, –1, 3, –3, 2, –2, 4, –4}

Q9.7 Compare the DIT and DIF radix-2 FFT.

DIT raidx-2 FFT	DIF radix-2 FFT
1. The time domain sequence is decimated.	1. The frequency domain sequence is decimated.
2. The input should be in bit reversed order, the output will be in normal order.	2. The input should be in normal order, the output will be in bit reversed order.
3. In each stage of computations, the phase factors are multiplied before add and subtract operations.	3. In each stage of computations, the phase factors are multiplied after add and subtract operations.
4. The value of N should be expressed such that $N = 2^m$ and this algorithm consists of m stages of computations.	4. The value of N should be expressed such that, $N = 2^m$ and this algorithm consists of m stages of computations.
5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.	5. Total number of arithmetic operations are $N\log_2 N$ complex additions and $(N/2)\log_2 N$ complex multiplications.

Q9.8 What are direct or slow convolution and fast convolution?

The response of an LTI system is given by convolution of input and impulse response.

The computation of the response of the LTI system by convolution sum formula is called slow convolution because it involves very large number of calculations.

The number of calculations in DFT computations can be reduced to a very large extent by FFT algorithms. Hence computation of the response of the LTI system by FFT algorithm is called fast convolution.

Q9.9 Why FFT is needed?

The FFT is needed to compute DFT with reduced number of calculations. The DFT is required for spectrum analysis and filtering operations on the signals using digital computers.

Q9.10 What is bin spacing?

Solution

The N-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

where, $W_N^{nk} = \left(e^{-j2\pi}\right)^{\frac{nk}{N}}$ is the phase factor or twiddle factor.

The phase factors are equally spaced around the unit circle at frequency increments of F_s/N where F_s is the sampling frequency of the time domain signal. This frequency increment or resolution is called bin spacing. (The X(k) consists of N-numbers of frequency samples whose discrete frequency locations are given by $f_k = kF_s/N$, for k = 0, 1, 2, ... N–1).

9.12 MATLAB Programs

Program 9.1

Write a MATLAB program to perform circular convolution of the discrete time sequences x1(n)={2,1,2,1} and x1(n)={1,2,3,4} using DFT.

```
% Program to perform Circular Convolution via DFT

clear all
clc

N = 4;            % declare the value of N
x1 = [2,1,2,1]; % declare the input sequences
x2 = [1,2,3,4];

disp('The 4-point DFT of  x1(n) is,');
X1 = fft(x1,N)   % compute 4-point DFT of x1(n)

disp('The 4-point DFT of  x2(n) is,');
X2 = fft(x2,N)   % compute 4-point DFT of x2(n)

disp('The product of DFTs is,');
X1X2 = X1.*X2    % product of DFTs

disp('Circular convolution of x1(n) and x2(n) is,');
X3 = ifft(X1X2) % perform IDFT to get result of circular convolution
```

OUTPUT

```
The 4-point DFT of  x1(n) is,
X1 =
     6     0     2     0

The 4-point DFT of  x2(n) is,
X2 =
  10.0000             -2.0000 + 2.0000i   -2.0000             -2.0000 - 2.0000i

The product of DFTs is,
X1X2 =
    60     0    -4     0

Circular convolution of x1(n) and x2(n) is,
X3 =
    14    16    14    16
```

(Note : Verify the above result with example 9.3 in chapter-9)

Program 9.2

Write a MATLAB program to perform 16-point DFT of the discrete time sequence x(n)={1/3,1/3,1/3} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum

clear all
clc

N = 16;                % specify the length of the DFT
j=sqrt(-1);

xn = zeros (1,N);     % initialize input sequence as zeros
```

```
xn(1) = 1/3;          %let given sequence be first three samples
xn(2) = 1/3;
xn(3) = 1/3;
Xk = zeros (1,N);     %initialize output sequence as zeros

for k = 0:1:N-1       % compute DFT
      for n = 0:1:N-1
       Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
     end
end

disp ('The DFT sequence is,'); Xk
disp ('The Magnitude sequence is,');MagXk = abs(Xk)
disp ('The Phase sequence is,');PhaXk = angle(Xk)

Wk=0:1:N-1;           %specify a discrete frequency vector

subplot(2,1,1)
stem(Wk,MagXk);
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,1,2)
stem(Wk,PhaXk);
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
The DFT sequence is,
Xk =
   Columns 1 through 7
    1.0000                0.8770 - 0.3633i    0.5690 - 0.5690i    0.2252 - 0.5437i
    0 - 0.3333i          -0.0299 - 0.0723i    0.0976 + 0.0976i

   Columns 8 through 14
    0.2611 + 0.1081i    0.3333 + 0.0000i    0.2611 - 0.1081i    0.0976 - 0.0976i   -
    0.0299 + 0.0723i   -0.0000 + 0.3333i    0.2252 + 0.5437i

   Columns 15 through 16
    0.5690 + 0.5690i    0.8770 + 0.3633i


The Magnitude sequence is,
MagXk =
   Columns 1 through 12
     1.0000     0.9493     0.8047     0.5885     0.3333     0.0782     0.1381     0.2826
     0.3333     0.2826     0.1381     0.0782

   Columns 13 through 16
     0.3333     0.5885     0.8047     0.9493


The Phase sequence is,
PhaXk =
   Columns 1 through 12
          0    -0.3927    -0.7854    -1.1781    -1.5708    -1.9635     0.7854     0.3927
     0.0000    -0.3927    -0.7854     1.9635

   Columns 13 through 16
     1.5708     1.1781     0.7854     0.3927
```

Note : Verify the above results with example 8.6 and example 9.1.

The magnitude and phase spectrum of program 9.2 are shown in fig P9.2.

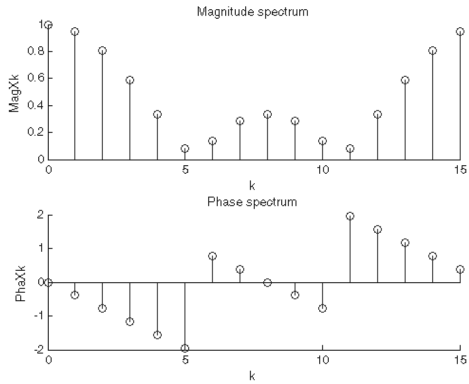


Fig P9.2 *: Magnitude and phase spectrum of program 9.2.*

Program 9.3

Write a MATLAB program to perform 8-point DFT of the discrete time sequence x(n)={2,2,2,2,1,1,1,1} and sketch the magnitude and phase spectrum.

```
% program to find DFT and frequency spectrum

clear all
clc
N = 8;                    % specify the length of the DFT
j=sqrt(-1);
xn = [2,2,2,2,1,1,1,1]; % input sequence
Xk = zeros (1,N);         % initialize output sequence as zeros

for k = 0:1:N-1           % compute DFT
     for n = 0:1:N-1
      Xk(k+1) = Xk(k+1)+xn(n+1)*exp(-j*2*pi*k*n/N);
     end
end

disp ('The DFT sequence is,'); Xk
disp ('The Magnitude sequence is,');MagXk = abs(Xk)
disp ('The Phase sequence is,');PhaXk = angle(Xk)

Wk=0:1:N-1;               % specify a discrete frequency vector
subplot(2,1,1)
stem(Wk,MagXk);
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')
subplot(2,1,2)
stem(Wk,PhaXk);
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
The DFT sequence is,
Xk =
  Columns 1 through 7
  12.0000              1.0000 - 2.4142i   -0.0000 - 0.0000i    1.0000 - 0.4142i
  0 - 0.0000i          1.0000 + 0.4142i   -0.0000 - 0.0000i
  Column 8
   1.0000 + 2.4142i
```

```
The Magnitude sequence is,
MagXk =
   12.0000    2.6131    0.0000    1.0824    0.0000    1.0824    0.0000    2.6131

The Phase sequence is,
PhaXk =
         0   -1.1781   -2.1426   -0.3927   -1.5708    0.3927   -2.7220    1.1781
```

Note : Verify the above results with example 9.5.

The magnitude and phase spectrum of program 9.3 are shown in fig P9.3.

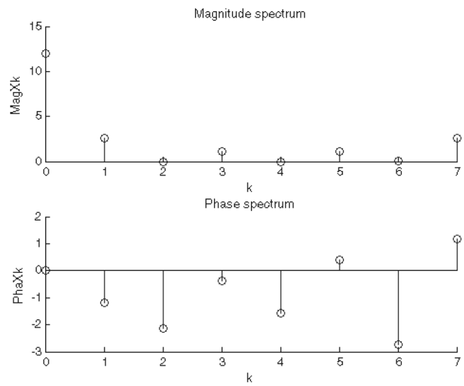


***Fig P9.3** : Magnitude and phase spectrum of program 9.3.*

Program 9.4

Write a MATLAB program to perform inverse DFT. Take the frequency domain output sequence of program 9.3 as input.

```
% program to compute N-point inverse DFT

clear all
clc
N = 8;                  % declare the length of the inverse DFT
j=sqrt(-1);
                        % Xk is input sequence
Xk = [12, 1-j*2.4142,0,1-j*0.4142,0,1+j*0.4142,0,1+j*2.4142];
xn = zeros (1,N); %initialize output sequence as zeros

for n= 0:1:N-1          % compute inverse DFT
    for k = 0:1:N-1
        xn(n+1) = xn(n+1)+(Xk(k+1)*exp(j*2*pi*n*k/N))/N;
    end
end
disp('The inverse DFT sequence is,' ); xn
```

OUTPUT

```
The inverse DFT sequence is,
xn =
  Columns 1 through 7
   2.0000               2.0000 - 0.0000i   2.0000 - 0.0000i   2.0000 - 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 - 0.0000i
  Column 8
   1.0000 + 0.0000i
```

Program 9.5

Write a MATLAB program to perform 4-point DFT of the discrete time sequence x(n)={1,1,2,3} using function FFT and sketch the magnitude and phase spectrum.

Also perform inverse DFT on the frequency domain sequence using function IFFT, to extract the time domain sequence.

```
% program to demonstrate DFT and inverse DFT Computation using FFT

clear all
clc

N = 4;                   % specify the value of N
xn = [1,1,2,3];          % input Sequence

disp('DFT of the sequence xn is, ')
Xk = fft(xn,N)           % compute N-point DFT of input

disp('The magnitude sequence is, ')
MagXk = abs(Xk)          % compute magnitude spectrum

disp('The phase sequence is, ')
PhaXk = angle(Xk)        % compute phase spectrum

disp('inverse DFT of the sequence Xk is, ')
Xn = ifft(Xk)            % compute inverse DFT

n = 0:1:N-1;             % declare a discrete time vector
Wk = 0:1:N-1;            % declare a discrete frequency vector

subplot(2,2,1)           % Plot the input sequence
stem(n,xn)
title(' Input sequence')
xlabel('n'); ylabel('xn')

subplot(2,2,2)
stem(n,Xn)
title('inverse DFT sequence')  % Plot the inverse DFT sequence
xlabel('n'); ylabel('Xn')

subplot(2,2,3)           % Plot the magnitude spectrum
stem(Wk,MagXk)
title('Magnitude spectrum')
xlabel('k'); ylabel('MagXk')

subplot(2,2,4)           % Plot the frequency spectrum
stem(Wk,PhaXk)
title('Phase spectrum')
xlabel('k'); ylabel('PhaXk')
```

OUTPUT

```
DFT of the sequence xn is,
Xk =
   7.0000            -1.0000 + 2.0000i   -1.0000            -1.0000 - 2.0000i

The magnitude sequence is,
MagXk =
    7.0000     2.2361     1.0000     2.2361
```

```
The phase sequence is,
PhaXk =
          0     2.0344     3.1416    -2.0344

inverse DFT of the sequence Xk is,
Xn =
     1     1     2     3
```

The input sequence, inverse DFT sequence, magnitude spectrum, and phase spectrum of program 9.5 are shown in fig P9.5.

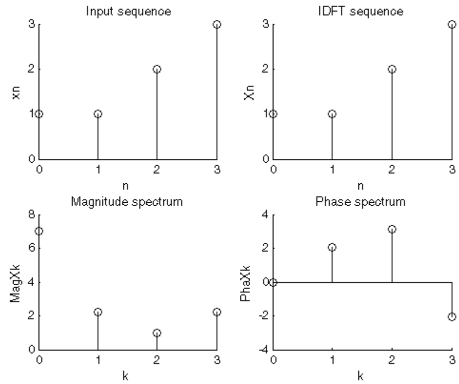


Fig P9.5 : Iuput sequence, Magnitude spectrum and phase spectrum of program 9.5.

9.13 Exercises

I. Fill in the blanks with appropriate words.

1. In an N-point DFT of a finite duration sequence x(n) of length L, the value of N should be such that_______.
2. The N-point DFT of a L-point sequence will have a periodicity of _______.
3. The convolution property of DFT says that DFT{x(n)$\circledast$ h(n)} = _______.
4. The N-point DFT of a sequence is given by $\mathcal{Z}$-transform of the sequence at N equally spaced points around the _______ in z-plane.
5. The convolution by FFT is called _______.
6. The convolution using convolution sum formula is called _______.
7. Appending zeros to a sequence in order to increase its length is called _______.
8. In DFT computation using radix-2 FFT, the value of N should be such that _______.
9. The number of complex additions and multiplications in radix-2 FFT are _______ and _______ respectively.
10. The number of complex additions and multiplications in direct DFT are _______ and _______ respectively.
11. In 8-point DFT by radix-2 FFT there are _______ stages of computations with _______ butterflies per stage.
12. In _______ butterfly diagram the _______ is multiplied after add-subtract operations.

Answers

1. $N \geq L$	4. unit circle	7. zero padding	10. $N(N-1)$, N^2
2. N-samples	5. fast convolution	8. $N = 2^m$	11. four, four
3. X(k) H(k)	6. slow convolution	9. $N\log_2 N$, $(N/2)\log_2 N$	12. DIF, phase factor

II. State whether the following statements are True/False.

1. The DFT of a sequence is a continuous function of ω.
2. The DFT of a signal can be obtained by sampling one period of Fourier Transform of the signal.
3. In sampling $X(e^{j\omega})$, the value of sample of $\omega = 0$ is same as the value of sample at $\omega=2\pi$.
4. The DFT of even sequence is purely imaginary and DFT of odd sequence is purely real.
5. In a DFT of real sequence, the real component is even and imaginary component is odd.
6. The multiplication of the DFTs of the two sequences is equal to the DFT of the linear convolution of two sequences.

7. The DFT supports only circular convolution.
8. In FFT algorithm the N-point DFT is decomposed into successively smaller DFTs.
9. In N-point DFT using radix-2 FFT, the decimation is performed m times, where $m=\log_2 N$.
10. Both DIT and DIF algorithms involves same number of computations.
11. Bit reversing is required for both DIT and DIF algorithm.

Answers

1. False	3. True	5. True	7. True	9. True	11. True
2. True	4. False	6. False	8. True	10. True	

III. Choose the right answer for the following questions

1. In N-point DFT of L-point sequence, the value of N to avoid aliasing in frequency spectrum is,

a) $N \neq L$ **b)** $N \leq L$ c) $N \geq L$ **d)** $N = L$

2. The inverse DFT of x(n) can be expressed as,

a) $x(n) = \frac{1}{N}\sum_{k=0}^{N} X(k)\, e^{-\frac{j2\pi kn}{N}}$ **b)** $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$

c) $x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\, e^{-\frac{j2\pi kn}{N}}$ **d)** $x(n) = N\sum_{n=0}^{N-1} X(k)\, e^{-\frac{j2\pi kn}{N}}$

3. If DFT {x(n)} = X(k), then DFT {x(n + m)$_N$}

a) $X(k)\, e^{\frac{-j2\pi km}{N}}$ **b)** $X(k)\, e^{\frac{-j2\pi k}{mN}}$ **c)** $X(k)\, e^{\frac{j2\pi km}{N}}$ **d)** $X(k)\, e^{\frac{j2\pi k}{mN}}$

4. The DFT of product of two discrete time sequences $x_1(n)$ and $x_2(n)$ is equivalent to,

a) $\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$ **b)** $\frac{1}{N}\left[X_1(k)\, X_2(k)\right]$ **c)** $\frac{1}{N}\left[X_1(k) \circledast X_2^*(k)\right]$ **d)** $X_1(k) \circledast X_2(k)$

5. By correlation property, the DFT of circular correlation of two sequences x(n) and y(n) is,

a) $X(k)\, Y^*(k)$ **b)** $X(k) \circledast Y(k)$ **c)** $X(k) \circledast Y^*(k)$ **d)** $X(k)\, Y(k)$

6. The N-point DFT of a finite duration sequence can be obtained as,

a) $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi n}{N}}}$ **b)** $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi k}{N}}}$

c) $X(k) = X(z)\Big|_{z = e^{-\frac{j2\pi kn}{N}}}$ **d)** $X(k) = X(z)\Big|_{z = e^{\frac{j2\pi kn}{N}}}$

7. In an N-point sequence, if N = 16, the total number of complex additions and multiplications using Radix-2 FFT are,

a) 64 and 80 **b)** 80 and 64 **c)** 64 and 32 **d)** 24 and 12

8. The complex valued phase factor/twiddle factor, W_N can be represented as,

a) $e^{-j2\pi N}$ **b)** $e^{-\frac{j2\pi}{N}}$ **c)** $e^{-j2\pi}$ **d)** $e^{-j2\pi kN}$

9. The phase factors are multiplied before the add and subtract operations in,

a) DIT radix-2 FFT **b)** DIF radix-2 FFT **c)** inverse DFT **d)** both a and c

10. If X(k) consists of N-number of frequency samples, then its discrete frequency locatioins are given by,

a) $f_k = \frac{kF_s}{N}$ **b)** $f_k = \frac{F_s}{N}$ **c)** $f_k = \frac{kN}{F_s}$ **d)** $f_k = N$

Answers

1. c	3. c	5. a	7. c	9. a
2. b	4. a	6. b	8. b	10. a

IV. Answer the Following questions

1. Define DFT of a discrete time sequence.
2. Define inverse DFT.
3. What is the relation between DTFT and DFT?
4. What is the drawback in Fourier transform and how it is overcome?
5. List any four properties of DFT.
6. State and prove the shifting property of DFT.
7. What is FFT?
8. What is radix-2 FFT?
9. How many multiplications and additions are involved in radix-2 FFT?
10. What is DIT radix-2 FFT?
11. What is phase factor or twiddle factor?
12. Draw and explain the basic butterfly diagram or flow graph of DIT radix-2 FFT.
13. What are the phase factors involved in the third stage of computation in the 8-point DIT radix-2 FFT?
14. What is DIF radix-2 FFT?
15. Draw and explain the basic butterfly diagram or flow graph of DIF radix-2 FFT.
16. What are the phase factors involved in first stage of computation in 8-point DIF radix-2 FFT?
17. How will you compute inverse DFT using radix-2 FFT algorithm?
18. What is magnitude and phase spectrum?

V. Solve the Following Problems

E9.1 Compute 4-point DFT and 8-point DFT of causal sequence given by, $x(n) = 0.5 \;;\; 0 \le n \le 3$
$\qquad = 0 \;;\; \text{else}$

E9.2 Compute DFT of the sequence, $x(n) = \{0, -1, 2, 1\}$. Sketch the magnitude and phase spectrum.

E9.3 Compute DFT of the sequence, $x(n) = \{1, 1, 1, 2, 2\}$. Sketch the magnitude and phase spectrum.

E9.4 Compute circular convolution of the following sequences using DFT.

$$x_1(n) = \{\underset{\uparrow}{-1}, 2, -2, 1\} \text{ and } x_2(n) = \{\underset{\uparrow}{1}, -2, -1, 2\}.$$

E9.5 Compute linear and circular convolution of the following sequences using DFT.
$x(n) = \{1, 0.2, -1\}$, $\quad h(n) = \{1, -1, 0.2\}$.

E9.6 Compute 8-point DFT of the discrete time signal, $x(n) = \{1, 3, 1, 2, 1, 3, 1, 2\}$,
a) using radix-2 DIT-FFT and b) using radix-2 DIF-FFT.
Also sketch the magnitude and phase spectrum.

E9.7 In an LTI system the input, $x(n) = \{-1, 2, -1\}$ and the impulse response, $h(n) = \{2, -1\}$. Determine the response of LTI system by radix-2 DIT-FFT.

E9.8 Compute the DFT and plot the magnitude and phase spectrum of the discrete time sequence, $x(n) = \{2, 0, 0, 1\}$, and verify the result using the inverse DFT.

E9.9 Determine the response of LTI system when the input sequence $x(n) = \{2, -1, 1, 0, 1\}$ by radix 2 DIT-FFT. The impulse response of the system is $h(n) = \{1, -1, -1, 1\}$.

Answers

E9.1 **4-point DFT:** $X(k) = \{2\angle 0,\ 0,\ 0,\ 0\}$

8-point DFT: $X(k) = \{2\angle 0,\ 1.306\angle -0.375\pi,\ 0,\ 0.54\angle -0.125\pi,\ 0,\ 0.54\angle 0.125\pi,\ 0,\ 1.306\angle 0.375\pi\}$

E9.2 $X(k) = \{0,\ 2.828\angle 0.75\pi,\ 0,\ 2.828\angle -0.75\pi\}$

$|X(k)| = \{0,\ 2.828,\ 0,\ 2.828\}$; $\angle X(k) = \{0,\ 0.75\pi,\ 0,\ -0.75\pi\}$

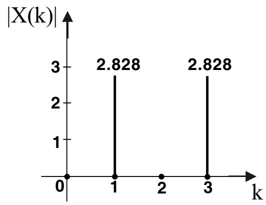


Fig E9.2.1 Magnitude spectrum.

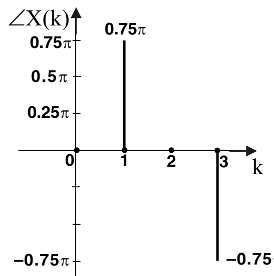


Fig E9.2.2 Phase spectrum.

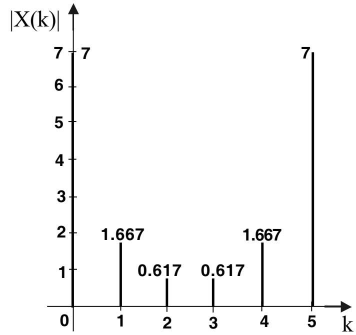


Fig E9.3.1 Magnitude spectrum.

E9.3 $X(k) = \{7\angle 0,\ 1.667\angle 0.6\pi,\ 0.617\angle -0.8\pi,\ 0.617\angle 0.8\pi,\ 1.667\angle -0.6\pi\}$

$|X(k)| = \{7,\ 1.667,\ 0.617,\ 0.617,\ 1.667\}$; $\angle X(k) = \{0,\ 0.6\pi,\ -0.8\pi,\ 0.8\pi,\ -0.6\pi\}$

E9.4 $x_1(n) \circledast x_2(n) = \{3,\ -1,\ -3,\ 1\}$

E9.5 $x(n) * h(n) = \{1,\ -0.8,\ -1,\ 1.04,\ -0.2\}$

$x(n) \circledast h(n) = \{2.04,\ -1,\ -1\}$

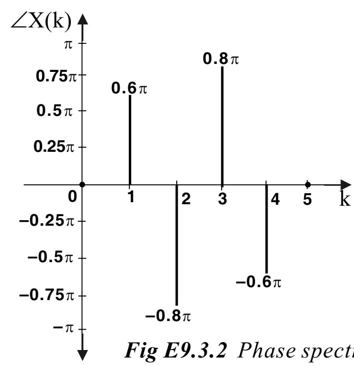


Fig E9.3.2 Phase spectrum.

E9.6 $X(k) = \{14,\ 0,\ -2j,\ 0,\ -6,\ 0,\ 2j,\ 0\}$

$= \{14\angle 0,\ 0,\ 2\angle -0.5\pi,\ 0,\ 6\angle \pi,\ 0,\ 2\angle 0.5\pi,\ 0\}$

$|X(k)| = \{14,\ 0,\ 2,\ 0,\ 6,\ 0,\ 2,\ 0\}$

$\angle X(k) = \{0,\ 0,\ -0.5\pi,\ 0,\ \pi,\ 0,\ 0.5\pi,\ 0\}$

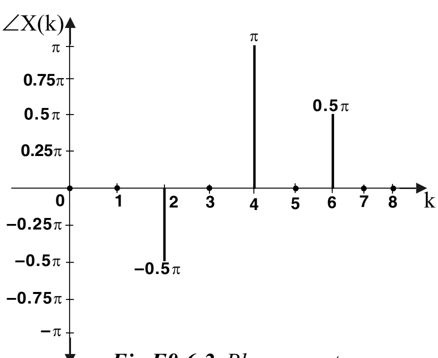


Fig E9.6.2 Phase spectrum.

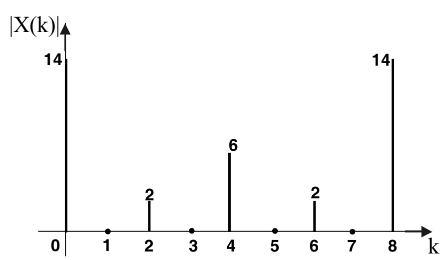


Fig E9.6.1 Magnitude spectrum.

E9.7 $y(n) = \{-2,\ 5,\ -4,\ 1\}$

E9.8 $X(k) = \{3\angle 0,\ 2.236\angle 0.463,\ 1\angle 0,\ 2.236\angle -0.463\}$

E9.9 $y(n) = \{2,\ -3,\ 0,\ 2,\ -1,\ 0,\ -1,\ 1\}$

CHAPTER 10

Structures for Realization of IIR and FIR Systems

10.1 Introduction

A discrete time system is a system that accepts a discrete time signal as input and processes it, and delivers the processed discrete time signal as output. Mathematically, a discrete time system is represented by a difference equation. Physically, a discrete time system is realized or implemented either as a digital hardware (like special purpose Microprocessor / Microcontroller) or as a software running on a digital hardware (like PC-Personal Computer).

The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay. Also the calculations are performed either by using fixed point arithmetic or floating point arithmetic. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation. These issues are addressed in structures for realization of discrete time systems.

From the implementation point of view, the discrete time systems are basically classified as IIR and FIR systems. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.

10.2 Discrete Time IIR and FIR Systems

A discrete time system is usually designed for a specified frequency response, $H(e^{j\omega})$. Now, the impulse response, h(n) of the system is given by inverse Fourier transform of the frequency response, $H(e^{j\omega})$. The impulse response, h(n) will be a sequence with infinite samples.

When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called ***IIR (Infinite Impulse Response) system***. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called ***FIR (Finite Impulse Response) system***.

In the design of IIR systems, the infinite samples of impulse response cannot be handled in digital domain. Therefore, the frequency response of IIR system will be transferred to a corresponding frequency response of a continuous time system, and a continuous time system is designed, then the continuous time system is transformed to discrete time system.

10.2.1 Discrete Time IIR System

Let, $H(e^{j\omega})$ = Frequency response of discrete time IIR system

H(s) = Transfer function of continuous time system

H(z) = Transfer function of discrete time system

The frequency response of discrete time system is transferred to a corresponding frequency response of continuous time system. Using this frequency response of continuous time system the transfer function of continuous time system is designed. Then the transfer function of continuous time system, H(s) is transformed to transfer function of discrete time system, H(z).

The general form of H(z) is,

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}}$$

Let, X(z) = Input of the discrete time system in z-domain

Y(z) = Output of the discrete time system in z-domain

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}} \qquad \ldots\ldots(10.1)$$

On cross multiplying the equation (10.1) we get,

$$\left[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots\ldots + a_N z^{-N}\right] Y(z) = \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots\ldots + b_M z^{-M}\right] X(z)$$

$$Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + \ldots\ldots + a_N z^{-N} Y(z)$$
$$= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots + b_M z^{-M} X(z)$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

If $\mathcal{Z}\{x(n)\} = X(z)$ then,
$\mathcal{Z}\{x(n-k)\} = z^{-k} X(z)$

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \ldots\ldots + a_N y(n-N)$$
$$= b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots\ldots + b_M x(n-M)$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \ldots\ldots - a_N y(n-N)$$
$$+ b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots\ldots + b_M x(n-M)$$

$$\therefore y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m) \qquad \ldots\ldots(10.2)$$

The equation (10.1) is the transfer function of discrete time IIR system and the equation (10.2) is the time domain equation governing discrete time IIR system. From equation (10.2), it is observed that the output at any time n depends on past outputs and so the IIR systems are recursive systems.

10.2.2 Discrete Time FIR system

Let, $H(e^{j\omega})$ = Frequency response of discrete time FIR system

h(n) = Impulse response of discrete time FIR system

Here, the impulse response is obtained by inverse Fourier transform of the frequency response of discrete time system. The impulse response will have infinite samples. Let us choose N-samples of h(n) for n = 0 to N–1. (or for n = –(N–1)/2 to + (N–1)/2).

Let the samples of h(n) be, $b_0, b_1, b_2, \ldots\ldots b_{N-1}$ for n = 0, 1, 2, ... N–1 respectively.

$$\therefore\ h(n) = \{\underset{\uparrow}{b_0},\ b_1,\ b_2,\ \ldots\ldots\ b_{N-1}\}$$

On taking $\mathcal{Z}$-transform of h(n) we get,

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

Let, X(z) = Input of the discrete time system in z-domain

Y(z) = Output of the discrete time system in z-domain

$$\therefore H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)} \qquad .....(10.3)$$

On cross multiplying the equation (10.3) we get,

$$\begin{aligned} Y(z) &= \left[b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}\right]X(z) \\ &= b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + ... + b_{N-1} z^{-(N-1)} X(z) \end{aligned}$$

On taking inverse $\mathcal{Z}$-transform of the above equation we get,

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_{N-1} x(n-(N-1))$$

$$\therefore y(n) = \sum_{m=0}^{N-1} b_m x(n-m) \qquad .....(10.4)$$

The equation (10.3) is the transfer function of discrete time FIR system and the equation (10.4) is the time domain equation governing discrete time FIR system. From equation (10.4), it is observed that the output at any time n does not depend on past outputs and so the FIR systems are nonrecursive systems.

10. 3 Structures for Realization of IIR Systems

In general, the time domain representation of an N^{th} order IIR system is,

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)$$

and the z-domain representation of an N^{th} order IIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

The above two representations of IIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). Also, in the above representations the value of M gives the number of zeros and the value of N gives the number of poles of the IIR system.

The computations in the above equation can be arranged into various equivalent sets of difference equations, with each set of equations defining a computational procedure or algorithm for implementing the system. The main advantage of rearranging the sets of difference equations is to reduce the computational complexity, memory requirements and finite-word-length effects in computations.

For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred as realization of system or equivalently as a structure for realizing system. (For the block diagram representation of discrete system refer chapter-6, section 6.6.2). Some of the block diagram representation of the system gives a direct relation between the time domain equation and the z-domain equation.

The different types of structures for realizing the IIR systems are,

1. Direct form-I structure
2. Direct form-II structure
3. Cascade form structure
4. Parallel form structure

10.3.1 Direct Form-I Structure of IIR System

Consider the difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - \ldots\ldots - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots\ldots + b_M\, x(n-M)$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \ldots\ldots - a_N z^{-N} Y(z)$$
$$+ b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots\ldots + b_M z^{-M} X(z) \qquad \ldots\ldots(10.5)$$

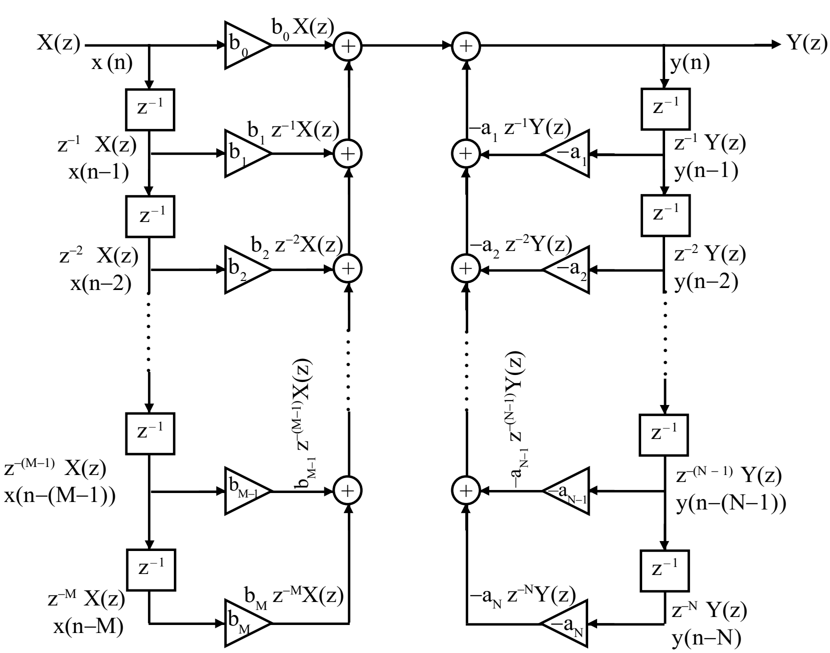


***Fig 10.1 :** Direct form-I structure of IIR system.*

The equation of Y(z) [equation (10.5)] can be directly represented by a block diagram as shown in fig 10.1 and this structure is called direct form-I structure. The direct form-I structure provides a direct relation between time domain and z-domain equations. The direct form-I structure uses separate delays (z^{-1}) for input and output samples. Hence for realizing direct form-I structure more memory is required.

From the direct form-I structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. Also this structure involves M+N delays and so M+N memory locations are required to store the delayed signals.

When the number of delays in a structure is equal to the order of the system, the structure is called ***canonic structure***. In direct form-I structure the number of delays is not equal to order of the system and so direct form-I structure is noncanonic structure.

10.3.2 Direct Form-II Structure of IIR System

An alternative structure called direct form-II structure can be realized which uses less number of delay elements than the direct form-I structure.

Consider the general difference equation governing an IIR system.

$$y(n) = -\sum_{m=1}^{N} a_m \, y(n-m) + \sum_{m=0}^{M} b_m \, x(n-m)$$

$$\begin{aligned} y(n) = & -a_1 \, y(n-1) - a_2 \, y(n-2) - .....- a_N \, y(n-N) \\ & + b_0 \, x(n) + b_1 \, x(n-1) + b_2 \, x(n-2) + ... + b_M \, x(n-M) \end{aligned}$$

On taking $\mathcal{Z}$-transform of the above equation we get,

$$\begin{aligned} Y(z) = & -a_1 \, z^{-1} \, Y(z) - a_2 \, z^{-2} \, Y(z) - ..... - a_N \, z^{-N} \, Y(z) \\ & + b_0 \, X(z) + b_1 \, z^{-1} \, X(z) + b_2 \, z^{-2} \, X(z) + ... + b_M \, z^{-M} \, X(z) \end{aligned}$$

$$\begin{aligned} & Y(z) + a_1 \, z^{-1} \, Y(z) + a_2 \, z^{-2} \, Y(z) + ... + a_N \, z^{-N} \, Y(z) \\ & \quad = b_0 \, X(z) + b_1 \, z^{-1} \, X(z) + b_2 \, z^{-2} \, X(z) + ... + b_M \, z^{-M} \, X(z) \end{aligned}$$

$$\begin{aligned} & Y(z) \left[1 + a_1 \, z^{-1} + a_2 \, z^{-2} + ... + a_N \, z^{-N}\right] \\ & \quad = X(z) \left[b_0 + b_1 \, z^{-1} + b_2 \, z^{-2} + ... + b_M \, z^{-M}\right] \end{aligned}$$

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 \, z^{-1} + b_2 \, z^{-2} + ... + b_M \, z^{-M}}{1 + a_1 \, z^{-1} + a_2 \, z^{-2} + ... + a_N \, z^{-N}}$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)} \times \dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 \, z^{-1} + a_2 \, z^{-2} + ... + a_N \, z^{-N}} \qquad .....(10.6)$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 \, z^{-1} + b_2 \, z^{-2} + ... + b_M \, z^{-M} \qquad .....(10.7)$$

On cross multiplying equation (10.6) we get,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + ... + a_N z^{-N} W(z) = X(z)$$

$$\therefore W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) - ..... - a_N z^{-N} W(z) \qquad ..... (10.8)$$

On cross multiplying equation (10.7) we get,

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + ... + b_M z^{-M} W(z) \qquad ..... (10.9)$$

The equations (10.8) and (10.9) represent the IIR system in z-domain and can be realized by a direct structure called direct form-II structure as shown in fig 10.2. In direct form-II structure the number of delays is equal to order of the system and so the direct form-II structure is canonic structure.

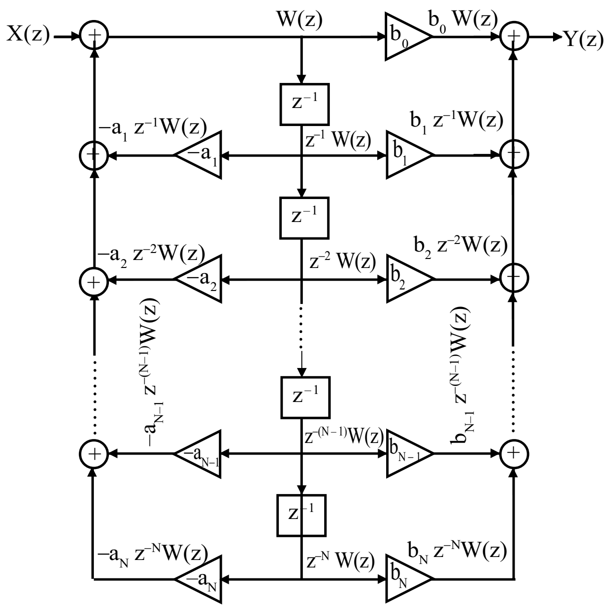


Fig 10.2 : *Direct form-II structure of IIR system for N = M.*

From the direct form-II structure it is observed that the realization of an N^{th} order discrete time system with M number of zeros and N number of poles, involves M+N+1 number of multiplications and M+N number of additions. In a realizable system, $N \geq M$, and so the number of delays in direct form-II structure will be equal to N. Hence, when a system is realized using direct form-II structure, N memory locations are required to store the delayed signals.

Conversion of Direct Form-I Structure to Direct Form-II Structure

The direct form-I structure can be converted to direct form-II structure by considering the direct form-I structure as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig 10.3. By linearity property the order of cascading can be interchanged as shown in fig 10.4 and fig 10.5.

In fig 10.5 we can observe that the input to the delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delay elements in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same. Therefore instead of having separate delays for $\mathcal{H}_1$ and $\mathcal{H}_2$, a single set of delays can be used. Hence the delays can be merged to combine the cascaded systems to a single system and the resultant structure will be direct form-II structure as that of fig 10.2.

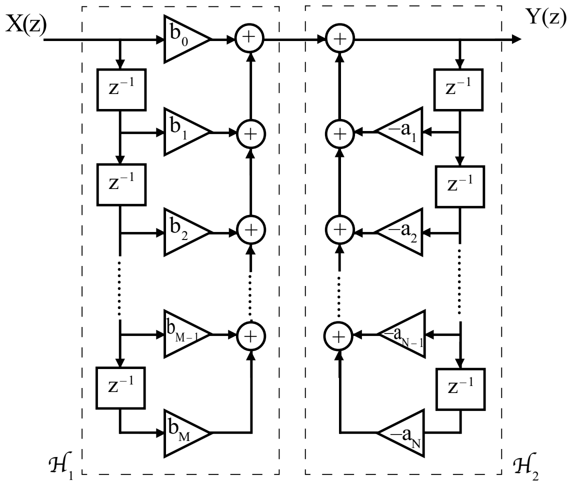


Fig 10.3 : *Direct form-I structure as cascade of two systems.*

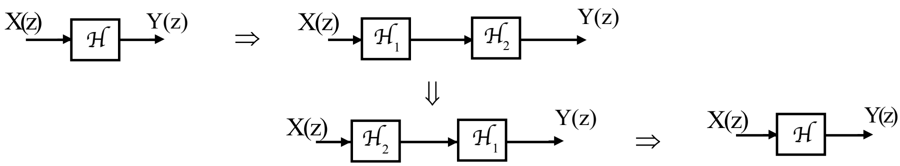


Fig 10.4 : *Conversion of Direct form-I structure to Direct form-II structure.*

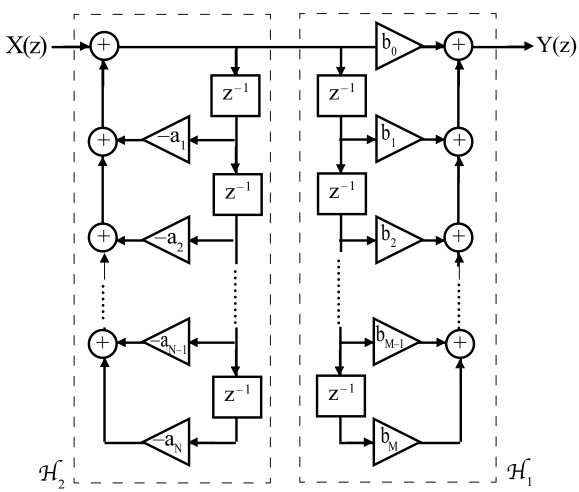


Fig 10.5 : *Direct form-I structure after interchanging the order of cascading.*

10.3.3 Cascade Form Realization of IIR System

The transfer function H(z) can be expressed as a product of a number of second order or first order sections, as shown in equation (10.10).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \;..... \; H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad .....(10.10)$$

$$\text{where,} \quad H_i(z) = \frac{c_{0i} + c_{1i}\,z^{-1} + c_{2i}\,z^{-2}}{d_{0i} + d_{1i}\,z^{-1} + d_{2i}\,z^{-2}} \qquad \boxed{\text{Second order section}}$$

$$\text{or,} \quad H_i(z) = \frac{c_{0i} + c_{1i}\,z^{-1}}{d_{0i} + d_{1i}\,z^{-1}} \qquad \boxed{\text{First order section}}$$

The individual second order or first order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by cascading the individual sections as shown in fig 10.6. The number of calculations and the memory requirement depends on the realization of individual sections.

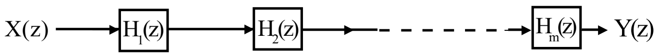


Fig 10.6 : *Cascade form realization of IIR system.*

The difficulty in cascade structure are,

1. Decision of pairing poles and zeros.
2. Deciding the order of cascading the first and second order sections.
3. Scaling multipliers should be provided between individual sections to prevent the system variables from becoming too large or too small.

10.3.4 Parallel Form Realization of IIR System

The transfer function H(z) of a discrete time system can be expressed as a sum of first and second order sections, using partial fraction expansion technique as shown in equation (10.11).

$$H(z) = \frac{Y(z)}{X(z)} = C + H_1(z) + H_2(z) + \;..... + H_m(z) \qquad .....(10.11)$$

$$= C + \sum_{i=1}^{m} H_i(z)$$

$$\text{where,} \quad H_i(z) = \frac{c_{0i} + c_{1i}\,z^{-1}}{d_{0i} + d_{1i}\,z^{-1} + d_{2i}\,z^{-2}} \qquad \boxed{\text{Second order section}}$$

$$\text{or} \quad H_i(z) = \frac{c_{0i}}{d_{0i} + d_{1i}\,z^{-1}} \qquad \boxed{\text{First order section}}$$

The individual first and second order sections can be realized either in direct form-I or direct form-II structures. The overall system is obtained by connecting the individual sections in parallel as shown in fig 10.7.The number of calculations and the memory requirement depends on the realization of individual sections.

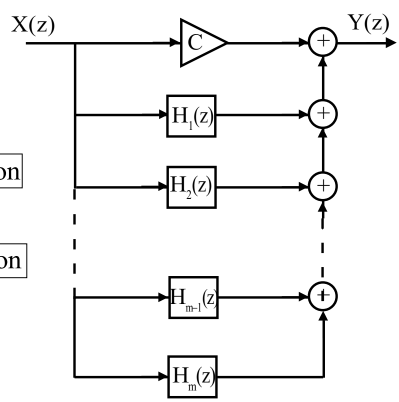


Fig 10.7 : *Parallel form realization of IIR system.*

Example 10.1

Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2).$$

Solution

Direct Form-I

Given that,

$$y(n) = -\frac{3}{8}y(n-1) + \frac{3}{32}y(n-2) + \frac{1}{64}y(n-3) + x(n) + 3x(n-1) + 2x(n-2) \quad .....(1)$$

On taking $\mathcal{Z}$-transform of equation(1) we get,

$$Y(z) = -\frac{3}{8}z^{-1}Y(z) + \frac{3}{32}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z) \quad .....(2)$$

The direct form-I structure can be obtained from equation (2), as shown in fig 1.

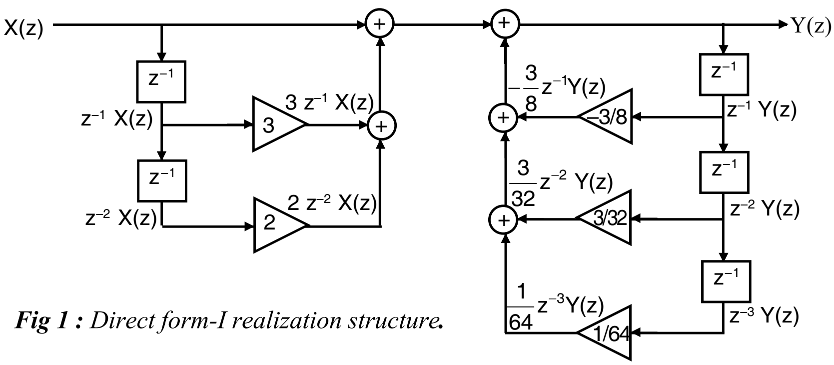


Fig 1 : Direct form-I realization structure.

Direct Form-II

Consider equation (2).

$$Y(z) = -\frac{3}{8}z^{-1}Y(z) + \frac{3}{32}z^{-2}Y(z) + \frac{1}{64}z^{-3}Y(z) + X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$

$$Y(z) + \frac{3}{8}z^{-1}Y(z) - \frac{3}{32}z^{-2}Y(z) - \frac{1}{64}z^{-3}Y(z) = X(z) + 3z^{-1}X(z) + 2z^{-2}X(z)$$

$$Y(z)\left[1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right] = X(z)\left[1 + 3z^{-1} + 2z^{-2}\right]$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \quad .....(3)$$

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)}\,\dfrac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \quad .....(4)$$

$$\frac{Y(z)}{W(z)} = 1 + 3z^{-1} + 2z^{-2} \quad .....(5)$$

On cross multiplying equation (4) we get,

$$W(z)\left(1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}\right) = X(z)$$

$$\text{or} \quad W(z) = X(z) - \frac{3}{8}z^{-1}\,W(z) + \frac{3}{32}z^{-2}\,W(z) + \frac{1}{64}z^{-3}\,W(z) \qquad .....(6)$$

On cross multiplying equation (5) we get,

$$Y(z) = W(z) + 3z^{-1}\,W(z) + 2z^{-2}\,W(z) \qquad .....(7)$$

The equations (6) and (7) can be realized by a direct form-II structure as shown in fig 2.

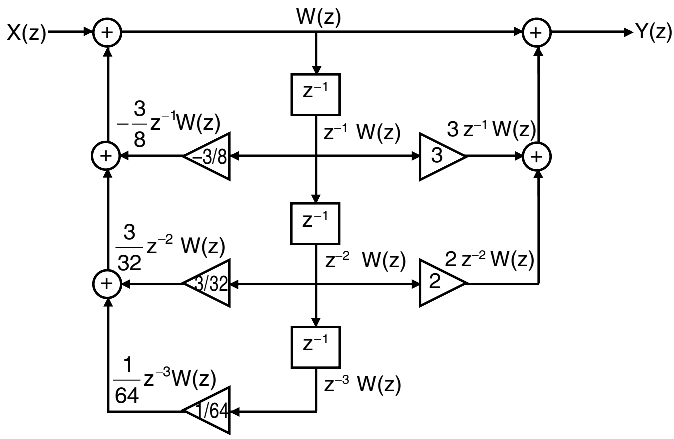


Fig 2: *Direct form-II realization structure.*

Cascade Form

Consider equation (3).

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} \qquad .....(8)$$

The numerator and denominator polynomials should be expressed in the factored form.

Consider the numerator polynomial of equation (8).

$$1 + 3z^{-1} + 2z^{-2} = z^{-2}\left(\frac{1}{z^{-2}} + \frac{3}{z^{-1}} + 2\right) = \frac{1}{z^2}\,(z^2 + 3z + 2)$$

$$= \frac{1}{z^2}\,(z + 1)\,(z + 2) = \frac{(z + 1)}{z}\,\frac{(z + 2)}{z}$$

$$= \left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right) \qquad .....(9)$$

Consider the denominator polynomial of equation (8)

$$1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} = z^{-3}\left(\frac{1}{z^{-3}} + \frac{3}{8}\,\frac{1}{z^{-2}} - \frac{3}{32}\,\frac{1}{z^{-1}} - \frac{1}{64}\right)$$

$$= \frac{1}{z^3}\left(z^3 + \frac{3}{8}z^2 - \frac{3}{32}z - \frac{1}{64}\right) \qquad .....(10)$$

$$= \frac{1}{z^3}\left(z + \frac{1}{8}\right)\left(z^2 + \frac{2}{8}z - \frac{8}{64}\right)$$

–1/8	1	3/8	–3/32	–1/64
	↓	–1/8	–2/64	+1/64
	1	2/8	–8/64	0

z = –1/8 is one of the root of the equation (10).

$$\therefore 1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3} = \frac{1}{z^3}\left(z + \frac{1}{8}\right)\left(z^2 + \frac{1}{4}z - \frac{1}{8}\right)$$

$$= \frac{1}{z^3}\left(z + \frac{1}{8}\right)\left(z + \frac{1}{2}\right)\left(z - \frac{1}{4}\right)$$

$$= \frac{\left(z + \frac{1}{8}\right)}{z}\,\frac{\left(z + \frac{1}{2}\right)}{z}\,\frac{\left(z - \frac{1}{4}\right)}{z}$$

$$= \left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right) \qquad .....(11)$$

From equations(8), (9) and (11) we can write,

$$H(z) = \frac{1 + 3z^{-1} + 2z^{-2}}{1 + \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} - \frac{1}{64}z^{-3}} = \frac{(1 + z^{-1})(1 + 2z^{-1})}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \qquad .....(12)$$

Since there are three first order factors in the denominator of equation (12), H(z) can be expressed as a product of 3 sections as shown in equation (13).

$$\text{Let,}\quad H(z) = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}} \times \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}} \times \frac{1}{1 - \frac{1}{4}z^{-1}} = H_1(z) \times H_2(z) \times H_3(z) \qquad .....(13)$$

$$\text{where,}\quad H_1(z) = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}}\ ;\quad H_2(z) = \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}}\quad \text{and}\quad H_3(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

The transfer function $H_1(z)$ can be realized in direct form-II structure as shown in fig 3.

$$\text{Let,}\ H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{W_1(z)}{X(z)}\,\frac{Y_1(z)}{W_1(z)} = \frac{1 + z^{-1}}{1 + \frac{1}{8}z^{-1}}$$

$$\text{where,}\ \frac{W_1(z)}{X(z)} = \frac{1}{1 + \frac{1}{8}z^{-1}}\ \text{and}\ \frac{Y_1(z)}{W_1(z)} = 1 + z^{-1}$$

$$\therefore\ W_1(z) = X(z) - \frac{1}{8}\,z^{-1}\,W_1(z)$$

$$Y_1(z) = W_1(z) + z^{-1}\,W_1(z)$$

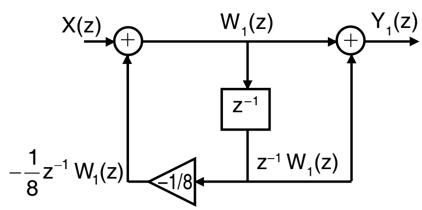


Fig 3 : *Direct form-II structure of* $H_1(z)$.

The transfer function $H_2(z)$ can be realized in direct form-II structure as shown in fig 4.

$$\text{Let,}\ H_2(z) = \frac{Y_2(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\,\frac{Y_2(z)}{W_2(z)} = \frac{1 + 2z^{-1}}{1 + \frac{1}{2}z^{-1}}$$

$$\text{where,}\ \frac{W_2(z)}{Y_1(z)} = \frac{1}{1 + \frac{1}{2}z^{-1}}\ \text{and}\ \frac{Y_2(z)}{W_2(z)} = 1 + 2z^{-1}$$

$$\therefore\ W_2(z) = Y_1(z) - \frac{1}{2}z^{-1}\,W_2(z)$$

$$Y_2(z) = W_2(z) + 2z^{-1}\,W_2(z)$$

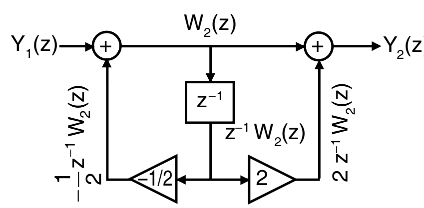


Fig 4 : *Direct form-II structure of* $H_2(z)$.

The transfer function $H_3(z)$ can be realized in direct form-II structure as shown in fig 5.

$$\text{Let,}\ H_3(z) = \frac{Y(z)}{Y_2(z)} = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

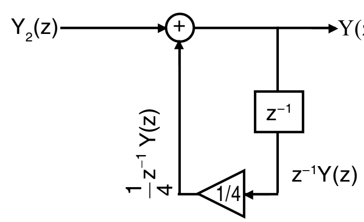


Fig 5 : *Direct form-II structure of* $H_3(z)$.

$$\therefore\ Y(z) - \frac{1}{4}z^{-1}Y(z) = Y_2(z)$$

$$Y(z) = Y_2(z) + \frac{1}{4}z^{-1}Y(z)$$

The cascade structure of the given system is obtained by connecting the individual sections shown in fig 3, fig 4 and fig 5 in cascade as shown in fig 6.

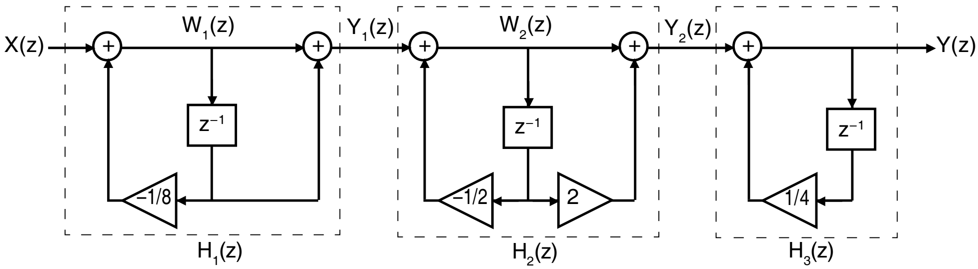


***Fig 6 :** Cascade realization of the system.*

Parallel Form

Consider the equation (12).

$$H(z) = \frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)}$$

By partial fraction expansion,

$$H(z) = \frac{A}{1 + \frac{1}{8}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}} + \frac{C}{1 - \frac{1}{4}z^{-1}}$$

$$A = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1 + \frac{1}{8}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{4}z^{-1}\right)} \times \left(1 + \frac{1}{8}z^{-1}\right)\right|_{z^{-1} = -8} = \frac{(1 - 8)\,(1 - 16)}{(1 - 4)\,(1 + 2)} = -\frac{35}{3}$$

$$B = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \times \left(1+\frac{1}{2}z^{-1}\right)\right|_{z^{-1} = -2} = \frac{(1-2)\,(1-4)}{\left(1-\frac{1}{4}\right)\left(1+\frac{1}{2}\right)} = \frac{(-1)\times(-3)}{\frac{3}{4}\times\frac{3}{2}} = \frac{8}{3}$$

$$C = \left.\frac{\left(1 + z^{-1}\right)\left(1 + 2z^{-1}\right)}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} \times \left(1-\frac{1}{4}z^{-1}\right)\right|_{z^{-1} = 4} = \frac{(1+4)\,(1+8)}{\left(1+\frac{1}{2}\right)(1+2)} = \frac{5\times 9}{\frac{3}{2}\times 3} = 10$$

$$\therefore\ H(z) = \frac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}} + \frac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}} + \frac{10}{1 - \frac{1}{4}z^{-1}} = H_1(z) + H_2(z) + H_3(z)$$

where, $H_1(z) = \dfrac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}}$; $\quad H_2(z) = \dfrac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}}$; $\quad H_3(z) = \dfrac{10}{1 - \frac{1}{4}z^{-1}}$

Let, $H(z) = \dfrac{Y(z)}{X(z)}$; $H_1(z) = \dfrac{Y_1(z)}{X(z)}$; $H_2(z) = \dfrac{Y_2(z)}{X(z)}$; $H_3(z) = \dfrac{Y_3(z)}{X(z)}$

$$\therefore H(z) = H_1(z) + H_2(z) + H_3(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)} + \frac{Y_3(z)}{X(z)}$$

$$\therefore Y(z) = Y_1(z) + Y_2(z) + Y_3(z)$$

The transfer function $H_1(z)$ can be realized in direct form-I structure as shown in fig 7.

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{-\frac{35}{3}}{1 + \frac{1}{8}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_1(z) = -\frac{1}{8}z^{-1}\,Y_1(z) - \frac{35}{3}\,X(z)$$

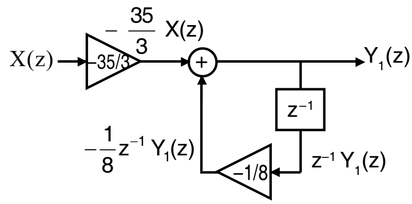


***Fig 7 :** Direct form-I structure of $H_1(z)$.*

The transfer function $H_2(z)$ can be realized in direct form-I structure as shown in fig 8.

$$\text{Let, } H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{\frac{8}{3}}{1 + \frac{1}{2}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_2(z) = -\frac{1}{2}z^{-1}\,Y_2(z) + \frac{8}{3}\,X(z)$$

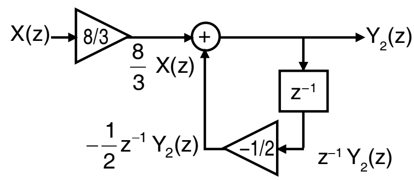


***Fig 8 :** Direct form-I structure of $H_2(z)$.*

The transfer function $H_3(z)$ can be realized in direct form-I structure as shown in fig 9.

$$\text{Let, } H_3(z) = \frac{Y_3(z)}{X(z)} = \frac{10}{1 - \frac{1}{4}z^{-1}}$$

On cross multiplying and rearranging we get,

$$Y_3(z) = \frac{1}{4}z^{-1}\,Y_3(z) + 10\,X(z)$$

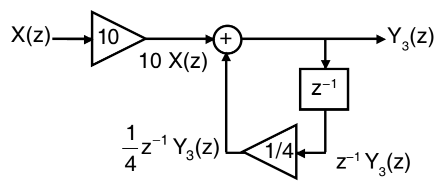


***Fig 9 :** Direct form-I structure of $H_3(z)$.*

The overall structure is obtained by connecting the individual sections shown in fig 7, fig 8 and fig 9 in parallel as shown in fig 10.

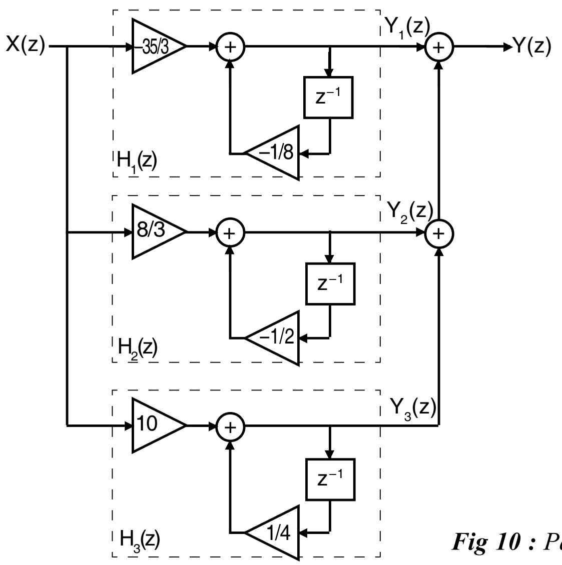


***Fig 10 :** Parallel form realization.*

Example 10.2

Find the direct form-I and direct form-II realizations of a discrete time system represented by transfer function,

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

Solution

Direct Form-I

Let, $H(z) = \frac{Y(z)}{X(z)}$; where, $Y(z)$ = Output and $X(z)$ = Input.

$$\therefore \frac{Y(z)}{X(z)} = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)} = \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - z^2 + \frac{1}{2}z - \frac{1}{4}z^2 + \frac{1}{4}z - \frac{1}{8}}$$

$$= \frac{8z^3 - 4z^2 + 11z - 2}{z^3 - \frac{5}{4}z^2 + \frac{3}{4}z - \frac{1}{8}} = \frac{z^3\left(8 - 4z^{-1} + 11z^{-2} - 2z^{-3}\right)}{z^3\left(1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}\right)}$$

$$= \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}} \quad .....(1)$$

On cross multiplying equation (1) we get,

$$Y(z) - \frac{5}{4}z^{-1}Y(z) + \frac{3}{4}z^{-2}Y(z) - \frac{1}{8}z^{-3}Y(z) = 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z)$$

$$\therefore Y(z) = 8X(z) - 4z^{-1}X(z) + 11z^{-2}X(z) - 2z^{-3}X(z) + \frac{5}{4}z^{-1}Y(z) - \frac{3}{4}z^{-2}Y(z) + \frac{1}{8}z^{-3}Y(z) \quad .....(2)$$

The direct form-I structure can be obtained from equation (2) as shown in fig 1.

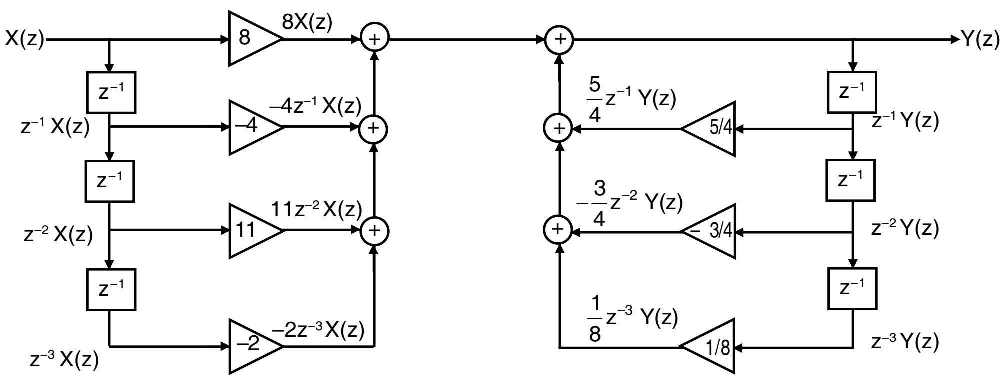


***Fig 1 :** Direct form-I realization.*

Direct Form-II

From equation (1) we get,

$$\frac{Y(z)}{X(z)} = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

Let, $\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \frac{Y(z)}{W(z)}$

where, $$\frac{W(z)}{X(z)} = \frac{1}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}} \quad .....(3)$$

$$\frac{Y(z)}{W(z)} = 8 - 4z^{-1} + 11z^{-2} - 2z^{-3} \quad .....(4)$$

On cross multiplying equation (3) we get,

$$W(z) - \frac{5}{4}z^{-1}W(z) + \frac{3}{4}z^{-2}W(z) - \frac{1}{8}z^{-3}W(z) = X(z)$$

$$\therefore\ W(z) = X(z) + \frac{5}{4}z^{-1}W(z) - \frac{3}{4}z^{-2}W(z) + \frac{1}{8}z^{-3}W(z) \quad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 8W(z) - 4z^{-1}W(z) + 11z^{-2}W(z) - 2z^{-3}W(z) \quad .....(6)$$

The equations (5) and (6) can be realized by a direct form-II Structure as shown in fig 2.

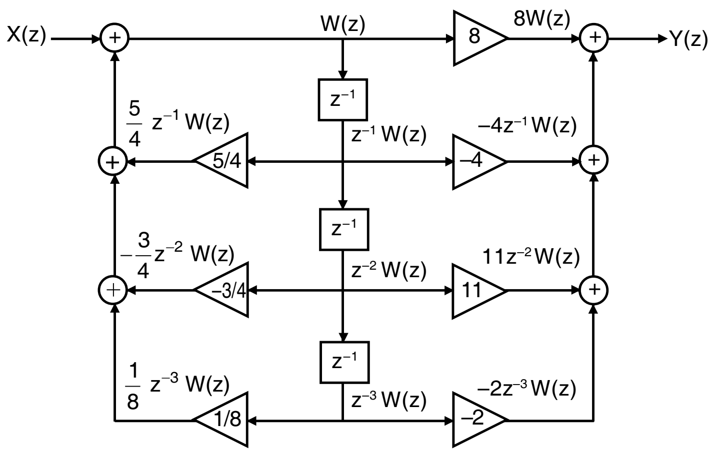


***Fig 2 :** Direct form-II realization.*

Example 10.3

Find the digital network in direct form-I and II for the system described by the difference equation,

$$y(n) = x(n) + 0.5\,x(n-1) + 0.4\,x(n-2) - 0.6\,y(n-1) - 0.7\,y(n-2).$$

Solution

Given that, $y(n) = x(n) + 0.5\,x(n-1) + 0.4\,x(n-2) - 0.6\,y(n-1) - 0.7\,y(n-2)$

On taking $\mathcal{Z}$-transform we get,

$$Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z) - 0.6z^{-1}Y(z) - 0.7z^{-2}\,Y(z) \quad .....(1)$$

The direct form-I digital network can be realized using equation (1) as shown in fig 1.

On rearranging equation (1) we get,

$$Y(z) + 0.6z^{-1}Y(z) + 0.7z^{-2}Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z)$$

$$(1+0.6z^{-1} + 0.7z^{-2})\,Y(z) = (1 + 0.5z^{-1} + 0.4z^{-2})\,X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1} + 0.4z^{-2}}{1 + 0.6z^{-1} + 0.7z^{-2}} \quad .....(2)$$

The equation (2) is the transfer function of the system.

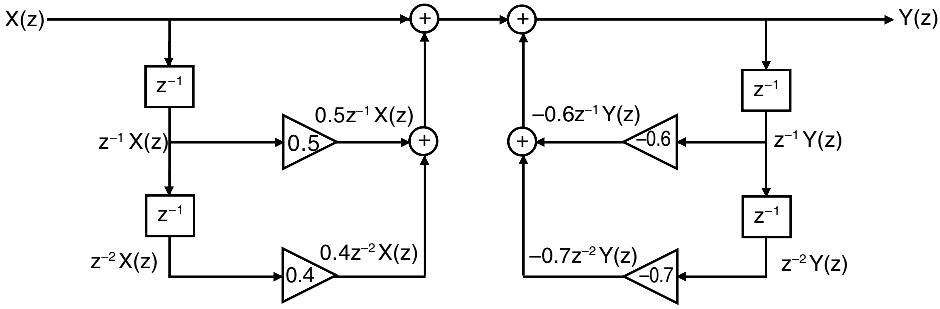


***Fig 1 :** Direct form-I digital network.*

Let, $\dfrac{Y(z)}{X(z)} = \dfrac{W(z)}{X(z)} \dfrac{Y(z)}{W(z)}$

where, $\dfrac{W(z)}{X(z)} = \dfrac{1}{1 + 0.6z^{-1} + 0.7z^{-2}}$ ..... (3)

$\dfrac{Y(z)}{W(z)} = 1 + 0.5z^{-1} + 0.4z^{-2}$ ..... (4)

On cross multiplying equation (3) we get,

$W(z) + 0.6z^{-1}W(z) + 0.7z^{-2}W(z) = X(z)$

$\therefore\; W(z) = X(z) - 0.6z^{-1}W(z) - 0.7z^{-2}W(z)$..... (5)

On cross multiplying equation (4) we get,

$Y(z) = W(z) + 0.5z^{-1}W(z) + 0.4z^{-2}W(z)$ ..... (6)

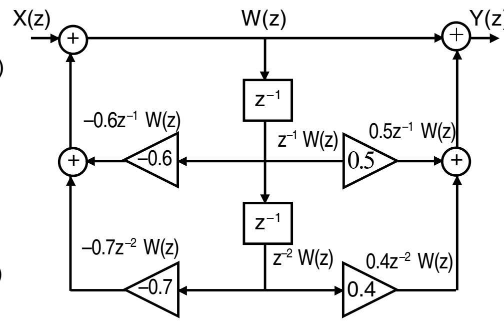


***Fig 2 :** Direct form-II digital network.*

The direct form-II digital network is realized using equations (5) and (6) as shown in fig 2.

Example 10.4

Realize the digital network described by H(z) in two ways. $H(z) = \dfrac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$

Solution

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$

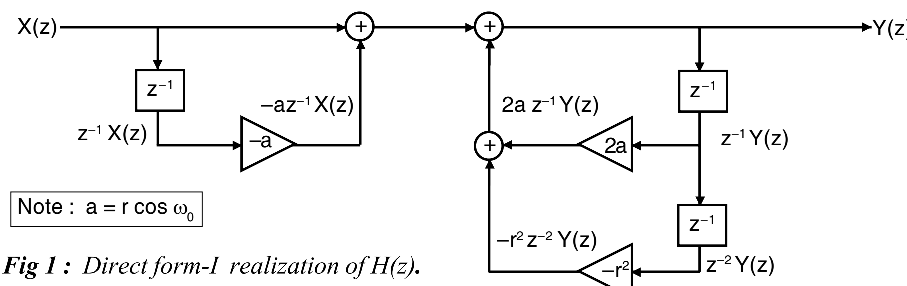
On cross multiplying we get,

$Y(z) - 2r\cos\omega_0 z^{-1}Y(z) + r^2z^{-2}Y(z) = X(z) - r\cos\omega_0 z^{-1}X(z)$

$\therefore\; Y(z) = X(z) - r\cos\omega_0 z^{-1}X(z) + 2r\cos\omega_0 z^{-1}Y(z) - r^2z^{-2}Y(z)$

Let, $r\cos\omega_0 = a$. $\therefore\; Y(z) = X(z) - az^{-1}X(z) + 2az^{-1}Y(z) - r^2z^{-2}Y(z)$ ..... (1)

The equation (1) can be used to construct direct form-I structure of H(z) as shown in fig 1.



Note : $a = r\cos\omega_0$

***Fig 1 :** Direct form-I realization of H(z).*

Consider the direct form-I structure as cascade of two systems $H_1(z)$ and $H_2(z)$ as shown in fig 2.

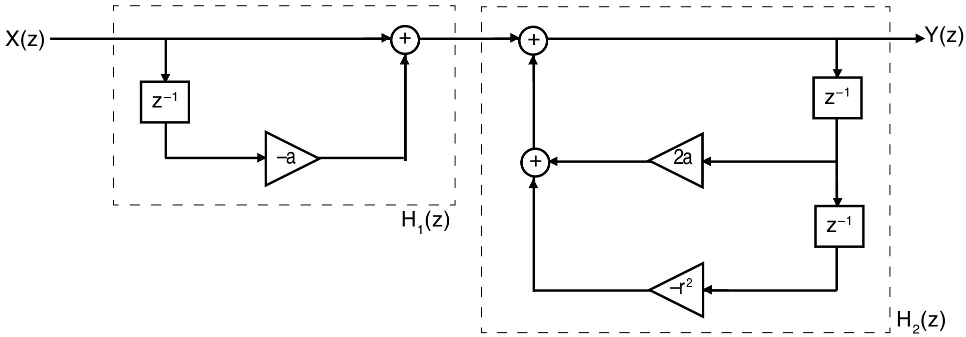


***Fig 2 :** Direct form-I structure as cascade of two systems.*

In an LT1 system, by linearity property, the order of cascading can be changed. Hence the systems $H_1(z)$ and $H_2(z)$ are interchanged and the fig 2 is redrawn as shown in fig 3.

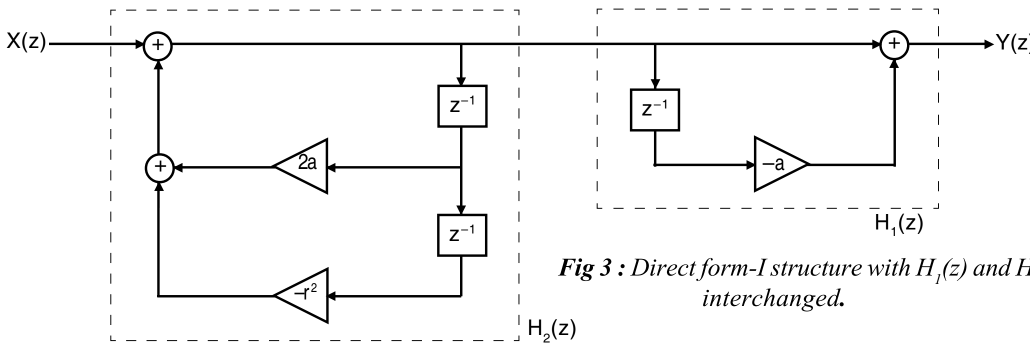
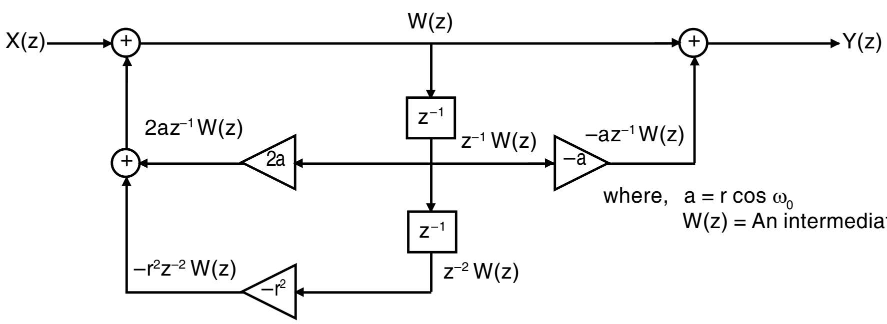


***Fig 3 :** Direct form-I structure with $H_1(z)$ and H_2 (z) interchanged.*

Since the input to delay elements in both the systems $H_1(z)$ and $H_2(z)$ are same, the outputs will also be same. Hence the delays can be combined and the resultant structure is direct form-II structure, which is shown in fig 4.



where, $a = r\cos\omega_0$
W(z) = An intermediate signal

***Fig 4 :** Direct form-II structure of H(z).*

Example 10.5

Realize the given system in cascade and parallel forms.

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

Solution

Cascade Form

Let us realize the system as cascade of two second order systems.

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \times \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

where, $H_1(z) = \dfrac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}}$; $H_2(z) = \dfrac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$

Let, $$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - z^{-1}\,Y_1(z) + \frac{1}{4}z^{-2}\,Y_1(z) = X(z)$$

$$\therefore Y_1(z) = X(z) + z^{-1}\,Y_1(z) - \frac{1}{4}z^{-2}\,Y_1(z) \qquad .....(2)$$

The equation (2) can be realized in direct form-II structure as shown in fig 1.

Let, $$H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + \frac{1}{2}z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $$\frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\,\frac{Y(z)}{W_2(z)}$$

where, $$\frac{W_2(z)}{Y_1(z)} = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 1 + \frac{1}{2}z^{-1} \qquad .....(4)$$

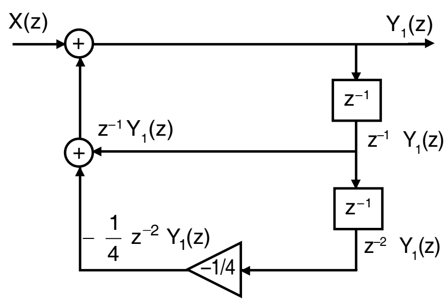


***Fig 1 :** Direct form-II structure of system $H_1(z)$.*

On cross multiplying equation (3) we get

$$W_2(z) - z^{-1}\,W_2(z) + \frac{1}{2}z^{-2}\,W_2(z) = Y_1(z)$$

$$\therefore\; W_2(z) = Y_1(z) + z^{-1}\,W_2(z) - \frac{1}{2}z^{-2}\,W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = W_2(z) + \frac{1}{2}z^{-1}\,W_2(z) \qquad .....(6)$$

Using equations (5) and (6) the system $H_2(z)$ can be realized in direct form-II structure as shown in fig 2.

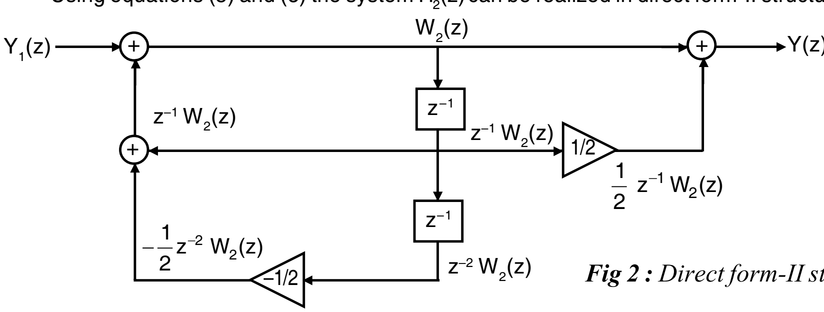


***Fig 2 :** Direct form-II structure of system $H_2(z)$.*

Cascade structure of H(z) is obtained by connecting structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

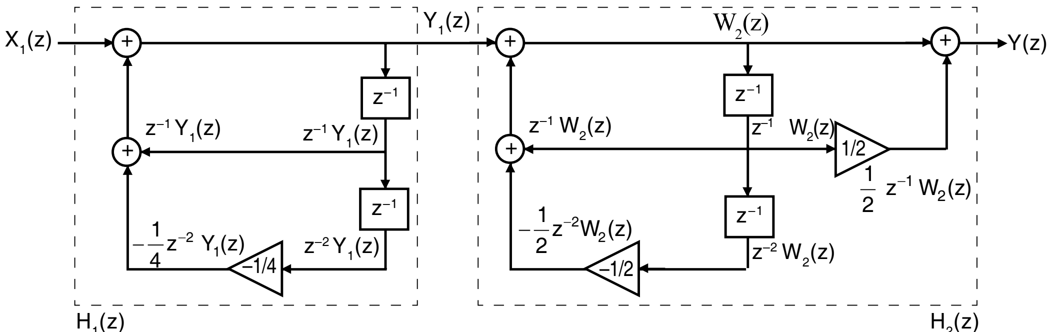


Fig 3 : *Cascade structure of H(z).*

Parallel Realization

$$\text{Given that, } H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

By partial fraction expansion we can write,

$$H(z) = \frac{1 + \frac{1}{2}z^{-1}}{\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{A + Bz^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{C + Dz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \quad .....(7)$$

On cross multiplying equation (7) we get,

$$1 + \frac{1}{2}z^{-1} = (A + Bz^{-1})\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right) + (C + Dz^{-1})\left(1 - z^{-1} + \frac{1}{4}z^{-2}\right)$$

$$1 + \frac{1}{2}z^{-1} = A - Az^{-1} + \frac{1}{2}Az^{-2} + Bz^{-1} - Bz^{-2} + \frac{1}{2}Bz^{-3}$$
$$+ C - Cz^{-1} + \frac{1}{4}Cz^{-2} + Dz^{-1} - Dz^{-2} + \frac{1}{4}Dz^{-3}$$

$$1 + \frac{1}{2}z^{-1} = (A + C) + (-A + B - C + D)\,z^{-1} + \left(\frac{1}{2}A - B + \frac{1}{4}C - D\right)z^{-2}$$
$$+ \left(\frac{1}{2}B + \frac{1}{4}D\right)z^{-3} \quad .....(8)$$

On equating the constants in equation (8) we get,

$$A + C = 1 \quad \Rightarrow \quad C = 1 - A$$

On equating the coefficients of z^{-3} in equation (8) we get,

$$\frac{1}{2}B + \frac{1}{4}D = 0 \qquad \therefore \ \frac{1}{4}D = -\frac{1}{2}B \qquad \Rightarrow \qquad D = -2B$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-A + B - C + D = \frac{1}{2}$$

On substituting $C = 1 - A$ and $D = -2B$ in the above equation we get,

$$-A + B - (1 - A) + (-2B) = \frac{1}{2} \quad \Rightarrow \quad -B = \frac{1}{2} + 1 \quad \Rightarrow \quad B = \frac{-3}{2}$$

$$\therefore \quad D = -2B = -2 \times \left(\frac{-3}{2}\right) = 3$$

On equating the coefficients of z^{-2} in equation (8) we get,

$$\frac{1}{2}A - B + \frac{1}{4}C - D = 0$$

On substituting $B = -3/2$, $C = 1 - A$, and $D = 3$ in the above equation we get,

$$\frac{1}{2}A - \left(-\frac{3}{2}\right) + \frac{1}{4}(1 - A) - 3 = 0 \quad \Rightarrow \quad \frac{1}{2}A - \frac{1}{4}A = -\frac{3}{2} - \frac{1}{4} + 3$$

$$\therefore \quad \frac{2A - A}{4} = \frac{-6 - 1 + 12}{4} \quad \Rightarrow \quad \frac{A}{4} = \frac{5}{4} \quad \Rightarrow \quad A = 5$$

$$\therefore C = 1 - A = 1 - 5 = -4$$

$$\therefore H(z) = \frac{A + Bz^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{C + Dz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\text{Let, } H(z) = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}} + \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = H_1(z) + H_2(z)$$

$$\text{where, } H_1(z) = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}}$$

$$H_2(z) = \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)}\ ; \quad H_1(z) = \frac{Y_1(z)}{X(z)}; \quad H_2(z) = \frac{Y_2(z)}{X(z)}$$

$$\therefore H(z) = H_1(z) + H_2(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)}$$

$$\therefore Y(z) = Y_1(z) + Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{5 - \frac{3}{2}z^{-1}}{1 - z^{-1} + \frac{1}{4}z^{-2}}$$

Let, $\frac{Y_1(z)}{X(z)} = \frac{W_1(z)}{X(z)} \frac{Y_1(z)}{W_1(z)}$

where, $$\frac{W_1(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{4}z^{-2}} \quad .....(9)$$

$$\frac{Y_1(z)}{W_1(z)} = 5 - \frac{3}{2}z^{-1} \quad .....(10)$$

On cross multiplying equation (9) we get,

$$W_1(z) - z^{-1} W_1(z) + \frac{1}{4}z^{-2} W_1(z) = X(z)$$

$$\therefore W_1(z) = X(z) + z^{-1} W_1(z) - \frac{1}{4}z^{-2} W_1(z) \quad .....(11)$$

On cross multiplying equation (10) we get,

$$Y_1(z) = 5W_1(z) - \frac{3}{2}z^{-1} W_1(z) \quad .....(12)$$

The direct form-II structure of system $H_1(z)$ can be realized using equations (11) and (12) as shown in fig 4.

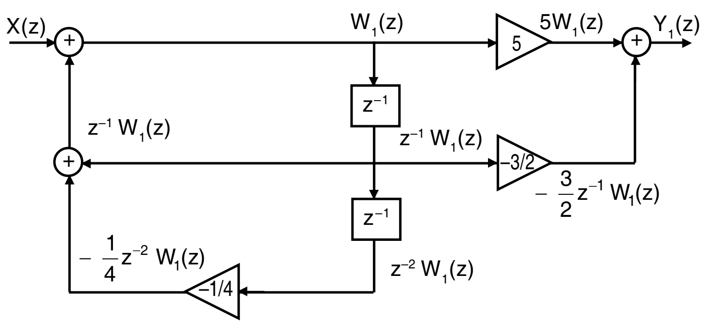


***Fig 4 :** Direct form-II structure of $H_1(z)$.*

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{-4 + 3z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $\frac{Y_2(z)}{X(z)} = \frac{W_2(z)}{X(z)} \frac{Y_2(z)}{W_2(z)}$

where, $$\frac{W_2(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{2}z^{-2}} \quad .....(13)$$

$$\frac{Y_2(z)}{W_2(z)} = -4 + 3z^{-1} \quad .....(14)$$

On cross multiplying the equation (13) we get,

$$W_2(z) - z^{-1} W_2(z) + \frac{1}{2}z^{-2} W_2(z) = X(z)$$

$$\therefore W_2(z) = X(z) + z^{-1} W_2(z) - \frac{1}{2}z^{-2} W_2(z) \quad .....(15)$$

On cross multiplying equation (14) we get,

$$Y_2(z) = -4 W_2(z) + 3z^{-1} W_2(z) \quad ..... (16)$$

The direct form-II structure of system $H_2(z)$ can be realized using equations (15) and (16) as shown in fig 5.

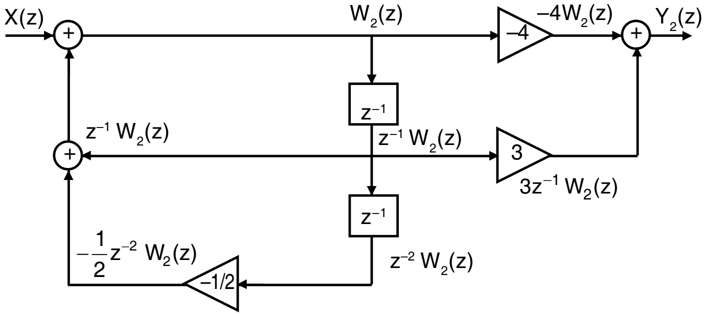


***Fig 5 :** Direct form-II structure of $H_2(z)$.*

The parallel form structure of H(z) is obtained by connecting the direct form-II structure of $H_1(z)$ and $H_2(z)$ in parallel as shown in fig 6.

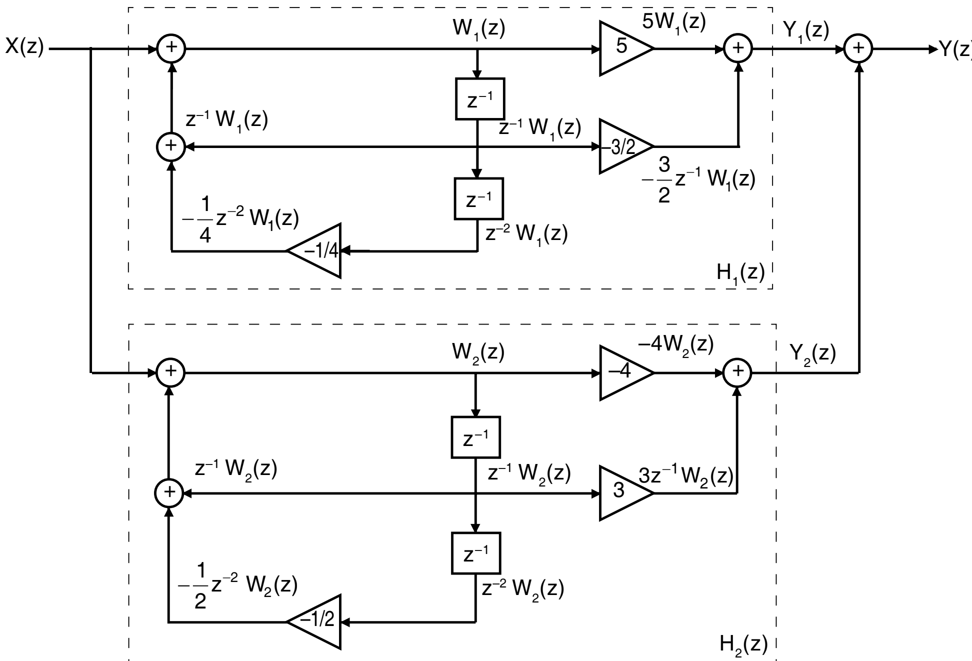


***Fig 6 :** Parallel form realization of system H(z).*

Example 10.6

Obtain the cascade realization of the system, $H(z) = \dfrac{2 + z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$

Solution

Given that, $H(z) = \dfrac{2 + z^{-1} + z^{-2}}{\left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + \frac{1}{2}z^{-1}\right)}$

On examining the roots of numerator polynomial it is found that the roots are complex conjugate. Hence H(z) can be realized on cascade of one first order and one second order system.

$$\therefore\ H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{2+z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)} = \frac{1}{1-\frac{1}{2}z^{-1}} \times \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

$$\text{where, } H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad \text{and} \quad H_2(z) = \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

$$\text{Let, } H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad .....(1)$$

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{2}z^{-1}\,Y_1(z) = X(z)\ ; \quad \therefore Y_1(z) = X(z) + \frac{1}{2}z^{-1}\,Y_1(z) \qquad .....(2)$$

The direct form-II structure of $H_1(z)$ can be obtained from equation (2) as shown in fig 1.

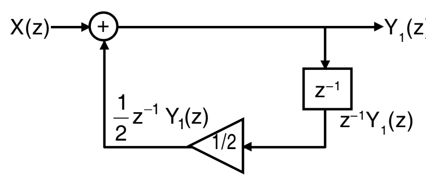


Fig 1 : *Direct form-II structure of $H_1(z)$.*

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{2+z^{-1}+z^{-2}}{1+z^{-1}+\frac{1}{4}z^{-2}}$$

$$\text{Let, } \frac{Y(z)}{Y_1(z)} = \frac{W_2(z)}{Y_1(z)}\ \frac{Y(z)}{W_2(z)}$$

$$\text{where, } \frac{W_2(z)}{Y_1(z)} = \frac{1}{1+z^{-1}+\frac{1}{4}z^{-2}} \qquad .....(3)$$

$$\frac{Y(z)}{W_2(z)} = 2+z^{-1}+z^{-2} \qquad .....(4)$$

On cross multiplying equation (3) we get,

$$W_2(z) + z^{-1}\,W_2(z) + \frac{1}{4}z^{-2}\,W_2(z) = Y_1(z)$$

$$\therefore\ W_2(z) = Y_1(z) - z^{-1}\,W_2(z) - \frac{1}{4}z^{-2}\,W_2(z) \qquad .....(5)$$

On cross multiplying equation (4) we get,

$$Y(z) = 2W_2(z) + z^{-1}\,W_2(z) + z^{-2}\,W_2(z) \qquad ..... (6)$$

The direct form-II structure of $H_2(z)$ can be obtained using equations (5) and (6) as shown in fig 2.

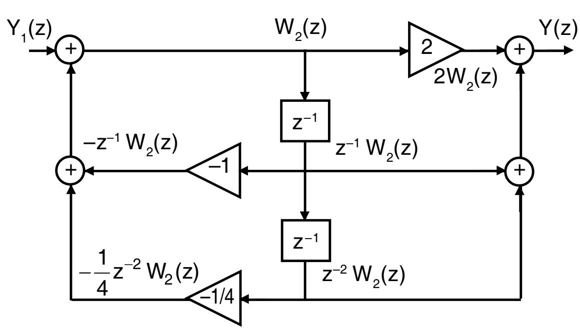


Fig 2 : *Direct form-II structure of $H_2(z)$.*

The cascade realization of H(z) is obtained by connecting the direct form-II structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

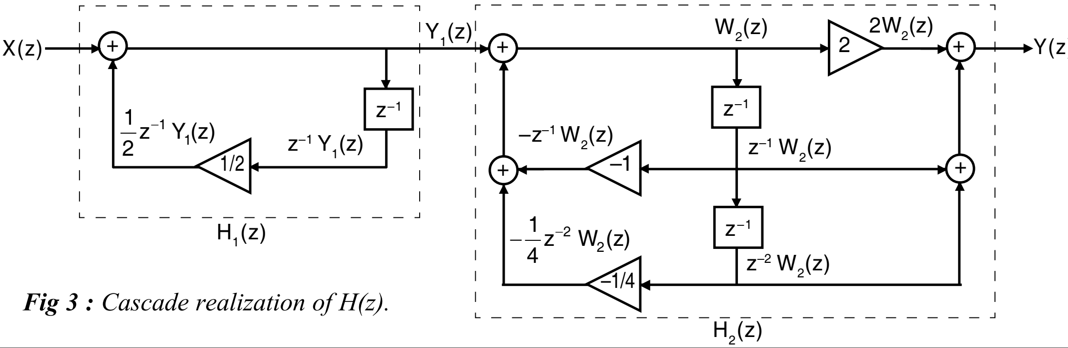


Fig 3 : *Cascade realization of H(z).*

Example 10.7

The transfer function of a system is given by, $H(z) = \dfrac{\left(1+z^{-1}\right)^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$

Realize the system in cascade and parallel structures.

Solution

Cascade Realization

Given that $H(z) = \dfrac{\left(1+z^{-1}\right)^3}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-z^{-1}+\frac{1}{2}z^{-2}\right)}$

On examining the roots of the quadratic factor in the denominator it is observed that the roots are complex conjugate. Hence the system has to be realized as cascading of one first order section and one second order section.

$$\therefore\ H(z) = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \times \frac{\left(1+z^{-1}\right)^2}{1-z^{-1}+\frac{1}{2}z^{-2}} = \frac{1+z^{-1}}{1-\frac{1}{4}z^{-1}} \times \frac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

Let, $H(z) = H_1(z) \times H_2(z)$

where, $H_1(z) = \dfrac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$ and $H_2(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-z^{-1}+\frac{1}{2}z^{-2}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$ ..... (1)

On cross multiplying equation (1) we get,

$$Y_1(z) - \frac{1}{4}z^{-1}Y_1(z) = X(z) + z^{-1}X(z)$$

$$\therefore\ Y_1(z) = X(z) + z^{-1}X(z) + \frac{1}{4}z^{-1}Y_1(z) \quad \text{..... (2)}$$

The direct form-I structure of $H_1(z)$ can be drawn using equation (2) as shown in fig 1.

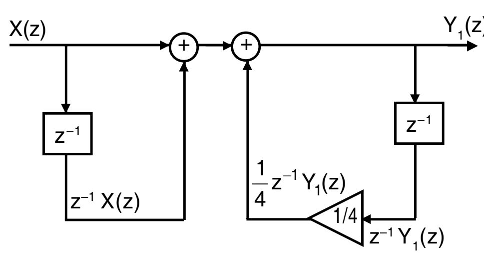


Fig 1 : *Direct form-I realization of $H_1(z)$.*

$$\text{Let, } H_2(z) = \frac{Y(z)}{Y_1(z)} = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \qquad .....(3)$$

On cross multiplying equation (3) we get

$$Y(z) - z^{-1}Y(z) + \frac{1}{2}z^{-2}Y(z) = Y_1(z) + 2z^{-1}Y_1(z) + z^{-2}Y_1(z)$$

$$\therefore\ Y(z) = Y_1(z) + 2z^{-1}Y_1(z) + z^{-2}Y_1(z) + z^{-1}Y(z) - \frac{1}{2}z^{-2}Y(z) \qquad .....(4)$$

The direct form-I structure of $H_2(z)$ can be drawn using equation (4) as shown in fig 2.

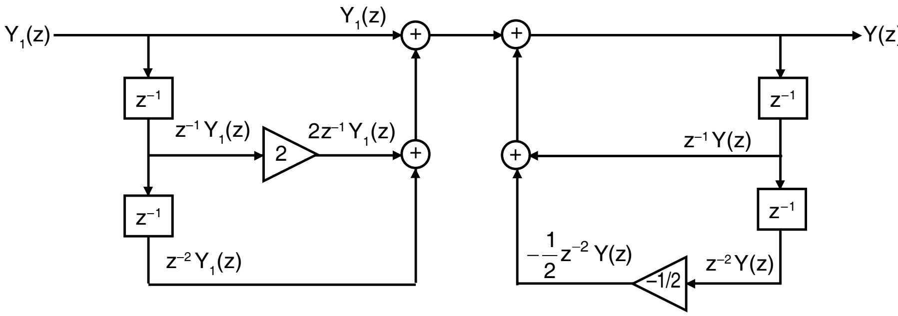


Fig 2 : The direct form-I structure of $H_2(z)$.

The cascade realization of H(z) is obtained by connecting the direct form - I structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

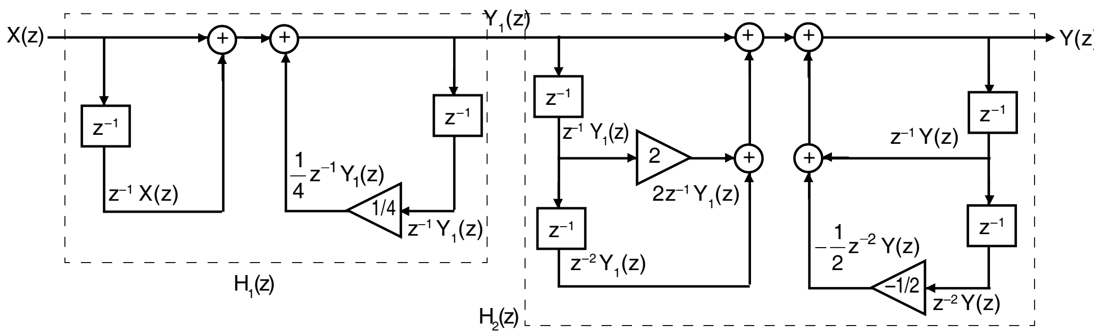


Fig 3 : Cascade realization of H(z).

Parallel Realization

$$\text{Given that, } H(z) = \frac{(1 + z^{-1})^3}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{(1 + z^{-1})(1 + z^{-1})^2}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}$$

$$= \frac{(1 + z^{-1})(1 + 2z^{-1} + z^{-2})}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{1 + 2z^{-1} + z^{-2} + z^{-1} + 2z^{-2} + z^{-3}}{1 - z^{-1} + \frac{1}{2}z^{-2} - \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$= \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$= 1 + \frac{\frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3}}{1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3}}$$

$$\begin{array}{r|l} & \quad 1 \\ \hline 1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3} & 1 + 3z^{-1} + 3z^{-2} + 2z^{-3} \\ & 1 - \frac{5}{4}z^{-1} + \frac{3}{4}z^{-2} - \frac{1}{8}z^{-3} \\ & (-) \quad (+) \qquad (-) \qquad (+) \\ \hline & \frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3} \end{array}$$

$$\therefore\ H(z) = 1 + \frac{\frac{17}{4}z^{-1} + \frac{9}{4}z^{-2} + \frac{9}{8}z^{-3}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = 1 + z^{-1}\left[\frac{\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)}\right] \quad .....(5)$$

By partial fraction expansion we can write,

$$\frac{\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right)} = \frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B + Cz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \quad .....\ (6)$$

On cross multiplying equation (6) we get,

$$\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2} = A\left(1 - z^{-1} + \frac{1}{2}z^{-2}\right) + (B + Cz^{-1})\left(1 - \frac{1}{4}z^{-1}\right) \quad .....(7)$$

$$\frac{17}{4} + \frac{9}{4}z^{-1} + \frac{9}{8}z^{-2} = A - Az^{-1} + \frac{1}{2}Az^{-2} + B - \frac{1}{4}Bz^{-1} + Cz^{-1} - \frac{1}{4}z^{-2} \quad .....(8)$$

The residue A can be solved by putting, $z^{-1} = 4$, in equation (7) as shown below.

$$\frac{17}{4} + \frac{9}{4}\times 4 + \frac{9}{8}\times 4^2 = A\left(1 - 4 + \frac{1}{2}\times 4^2\right) \quad\Rightarrow\quad \frac{17}{4} + 9 + 18 = A(1 - 4 + 8)$$

$$\therefore\quad \frac{17 + 36 + 72}{4} = 5A \quad\Rightarrow\quad \frac{125}{4} = 5A$$

$$\therefore\ A = \frac{125}{4}\times\frac{1}{5} = \frac{25}{A}$$

On equating the constants in equation (8) we get,

$$A + B = \frac{17}{4} \quad\Rightarrow\quad B = \frac{17}{4} - A = \frac{17}{4} - \frac{25}{4} = -\frac{8}{4} = -2$$

On equating the coefficients of z^{-1} in equation (8) we get,

$$-A - \frac{1}{4}B + C = \frac{9}{4} \qquad\Rightarrow\quad C = \frac{9}{4} + A + \frac{1}{4}B = \frac{9}{4} + \frac{25}{4} - \frac{2}{4} = \frac{32}{4} = 8$$

From equations (5) and (6) we can write,

$$H(z) = 1 + z^{-1}\left[\frac{A}{1 - \frac{1}{4}z^{-1}} + \frac{B + Cz^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}}\right]$$

$$\therefore\ H(z) = 1 + \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $$H(z) = 1 + \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} + \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}} = 1 + H_1(z) + H_2(z)$$

where, $$H_1(z) = \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\ ; \qquad H_2(z) = \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

Let, $H(z) = \frac{Y(z)}{X(z)}$; $H_1(z) = \frac{Y_1(z)}{X(z)}$; $H_2(z) = \frac{Y_2(z)}{X(z)}$

$$\therefore\ H(z) = 1 + H_1(z) + H_2(z) \quad \Rightarrow \quad \frac{Y(z)}{X(z)} = 1 + \frac{Y_1(z)}{X(z)} + \frac{Y_2(z)}{X(z)}$$

$$\therefore Y(z) = X(z) + Y_1(z) + Y_2(z)$$

Realization of $H_1(z)$

$$H_1(z) = \frac{Y_1(z)}{X(z)} = \frac{\frac{25}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}$$

On cross multiplying the above equation we get,

$$Y_1(z) - \frac{1}{4}z^{-1}\,Y_1(z) = \frac{25}{4}z^{-1}\,X(z)$$

$$\therefore\ Y_1(z) = \frac{25}{4}z^{-1}\,X(z) + \frac{1}{4}z^{-1}\,Y_1(z) \qquad \text{..... (9)}$$

Using equation (9) the direct form-I structure of $H_1(z)$ is drawn as shown in fig 4.

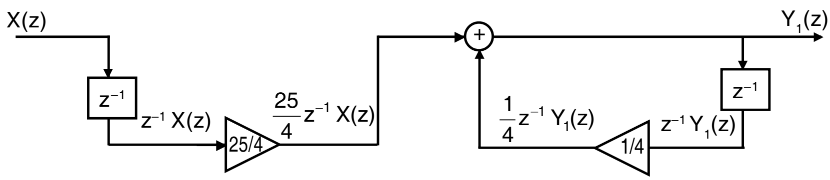


***Fig 4 :** Direct form-I structure of $H_1(z)$.*

Realization of $H_2(z)$

$$H_2(z) = \frac{Y_2(z)}{X(z)} = \frac{-2z^{-1} + 8z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}}$$

On cross multiplying the above equation we get,

$$Y_2(z) - z^{-1}\,Y_2(z) + \frac{1}{2}z^{-2}\,Y_2(z) = -2z^{-1}\,X(z) + 8z^{-2}\,X(z)$$

$$\therefore\ Y_2(z) = -2z^{-1}\,X(z) + 8z^{-2}\,X(z) + z^{-1}\,Y_2(z) - \frac{1}{2}z^{-2}\,Y_2(z) \qquad \text{.....(10)}$$

Using equation (10) the direct form-I structure of $H_2(z)$ is drawn as shown in fig 5.

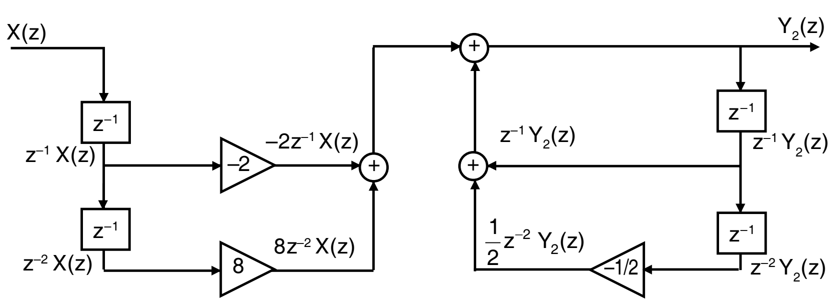


***Fig 5 :** Direct form-I structure of $H_2(z)$.*

Parallel Structure

The parallel structure of H(z) is obtained by connecting the direct form-I structure of $H_1(z)$ and $H_2(z)$ as shown in fig 6.

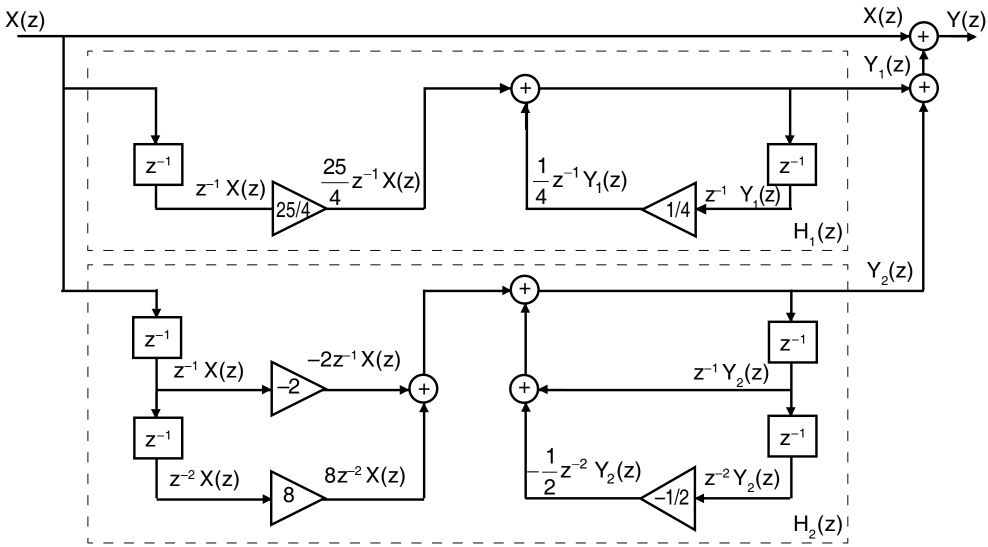


***Fig 6 :** The parallel structure of H(z).*

Example 10.8

An LTI System is described by the equation, $y(n) + y(n-1) - \frac{1}{4}y(n-2) = x(n)$.
Determine the cascade realization structure of the system.

Solution

Given that, $y(n) + y(n-1) - \frac{1}{4}y(n-2) = x(n)$

On taking Z-transform we get,

$$Y(z) + z^{-1}Y(z) - \frac{1}{4}z^{-2}Y(z) = X(z)$$

$$\left(1 + z^{-1} - \frac{1}{4}z^{-2}\right)Y(z) = X(z)$$

$$\therefore \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} - \frac{1}{4}z^{-2}}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + z^{-1} - \frac{1}{4}z^{-2}} = \frac{1}{z^{-2}\left(z^2 + z - \frac{1}{4}\right)}$$

$$= \frac{1}{z^{-2}(z - 0.207)(z + 1.207)} = \frac{1}{(1 - 0.207z^{-1})(1 + 1.207z^{-1})}$$

$$z^2 + z - \frac{1}{4} = 0$$

$$z = \frac{-1 \pm \sqrt{1 + 4x\frac{1}{4}}}{2}$$

$$= \frac{-1 \pm \sqrt{2}}{2}$$

$$= +0.207 \text{ or } -1.207$$

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = \frac{1}{1 - 0.207z^{-1}}$; $\qquad H_2(z) = \frac{1}{1 + 1.207z^{-1}}$

Let, $H_1(z) = \dfrac{Y_1(z)}{X(z)} = \dfrac{1}{1 - 0.207z^{-1}}$ .....(1)

On cross multiplying equation (1) we get,

$Y_1(z) - 0.207z^{-1} Y_1(z) = X(z)$

$\therefore\ Y_1(z) = X(z) + 0.207z^{-1} Y_1(z)$ ..... (2)

The direct form-I structure of $H_1(z)$ is obtained using equation (2) as shown in fig 1.

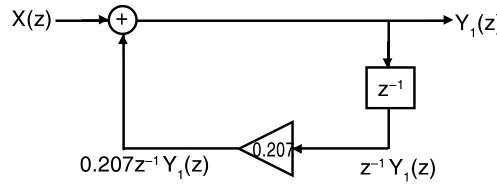


***Fig 1 :** Direct form-I structure of $H_1(z)$.*

Let, $H_2(z) = \dfrac{Y(z)}{Y_1(z)} = \dfrac{1}{1 + 1.207z^{-1}}$ ..... (3)

On cross multiplying equation (3) we get,

$Y(z) + 1.207\ z^{-1}\ Y(z) = Y_1(z)$

$Y(z) = Y_1(z) - 1.207\ z^{-1}\ Y(z)$ ..... (4)

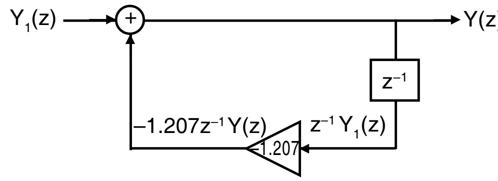


***Fig 2 :** Direct form-I structure of $H_2(z)$.*

The direct form-I structure of $H_2(z)$ is obtained using equation (4) as shown in fig 2.The cascade structure is obtained by connecting the direct form structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 3.

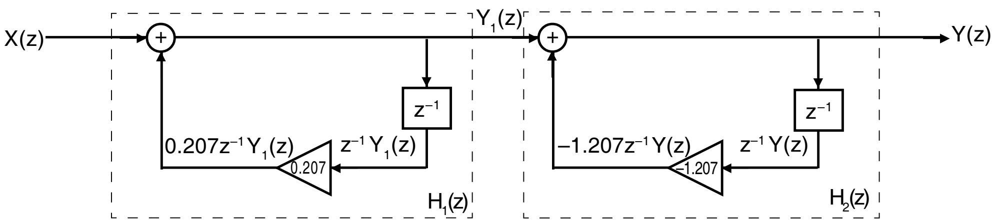


***Fig 3 :** Cascade structure.*

10.4 Structures for Realization of FIR Systems

In general, the time domain representation of an Nth order FIR system is,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m) = b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots .. + b_{N-1}\, x(n-(N-1))$$

and the z-domain representation of a FIR system is,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots .. + b_{N-1} z^{-(N-1)}$$

The above two representations of FIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence y(n) from the input sequence x(n). These equations can be used to construct the block diagram of the system using delays, adders and multipliers. This block diagram is referred to as realization of the system or equivalently as a structure for realizing the system, (For block diagram representation of discrete system refer chapter - 6, section 6.6.2). Some of the block diagram representation of the system gives a direct relation between time domain equation and z-domain equation.

The different types of structures for realizing FIR systems are,

1. Direct form realization
2. Cascade realization
3. Linear phase realization

10.4.1 Direct Form Realization of FIR System

Consider the difference equation governing a FIR system,

$$y(n) = \sum_{m=0}^{N-1} b_m\, x(n-m)$$

$$= b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + ... + b_{N-1}\, x(n-(N-1))$$

If $\mathcal{Z}\{x(n)\} = X(z)$ then,
$\mathcal{Z}\{x(n-k)\} = z^{-k}\, X(z)$

On taking $\mathcal{Z}$-transform of the above equation we get,.

$$\therefore\; Y(z) = b_0\, X(z) + b_1\, z^{-1}\, X(z) + b_2\, z^{-2}\, X(z) + b_3\, z^{-3}\, X(z) + ... + b_{N-2}\, z^{-(N-2)}\, X(z) + b_{N-1}\, z^{-(N-1)}\, X(z) \qquad ..... (10.12)$$

The equation of Y(z) [equation (10.12)] can be directly represented by a block diagram as shown in fig 10.8 and this structure is called direct form structure. The direct form structure provides a direct relation between time domain and z-domain equations.

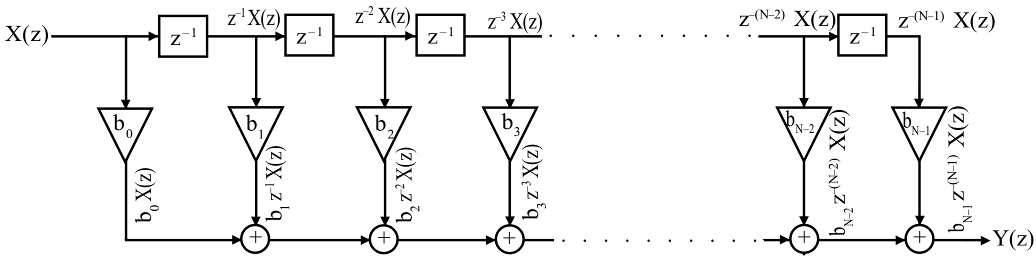


***Fig 10.8 :** Direct form structure of FIR system.*

From the direct form structure it is observed that the realization of an N^{th} order FIR discrete time system involves N number of multiplications and N−1 number of additions. Also the structure involves N−1 delays and so N−1 memory locations are required to store the delayed signals.

10.4.2 Cascade Form Realization of FIR System

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1\, z^{-1} + b_2\, z^{-2} + ... + b_{N-1}\, z^{-(N-1)}$$

The transfer function of FIR system is $(N-1)^{th}$ order polynomial in z. This polynomial can be factorized into first and second order factors and the transfer function can be expressed as a product of first and second order factors or sections as shown in equation (10.13).

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) ..... H_m(z) = \prod_{i=1}^{m} H_i(z) \qquad .....(10.13)$$

where, $H_i(z) = c_{0i} + c_{1i}\, z^{-1} + c_{2i}\, z^{-2}$ Second order section

or, $H_i(z) = c_{0i} + c_{1i}\, z^{-1}$ First order section

The individual second order or first order sections can be realized either in direct form structure or linear phase structure. The overall system is obtained by cascading the individual sections as shown in fig 10.9. The number of calculations and the memory requirement depends on the realization of individual sections.

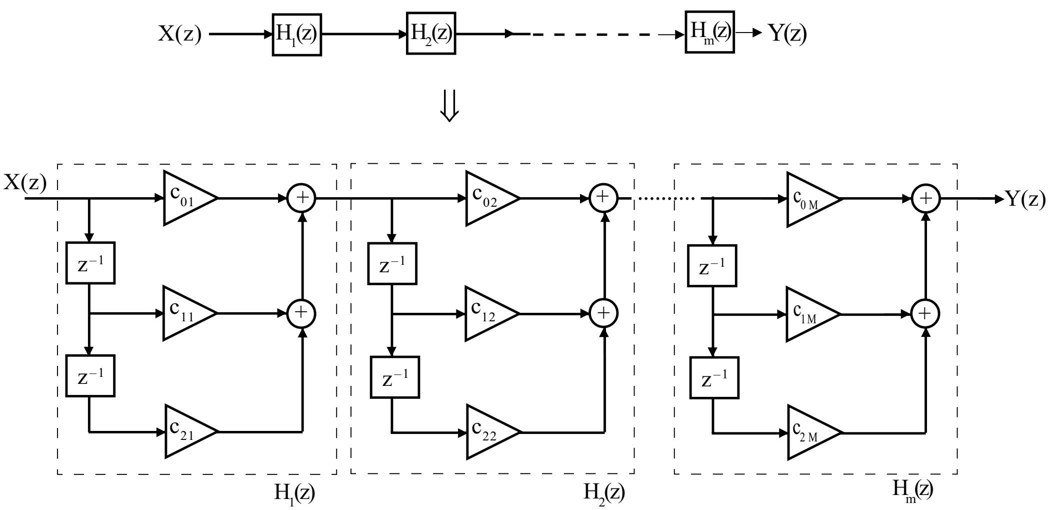


***Fig 10.9 :** Cascade structure of FIR system.*

10.4.3 Linear Phase Realization of FIR System

Consider the impulse response, h(n) of FIR system,

$$h(n) = \{\underset{\uparrow}{b_0},\ b_1,\ b_2,\ \ldots\ldots\ldots\ldots\ b_{N-1}\}$$

In FIR system, for linear phase response the impulse response should be symmetrical.

The condition for symmetry is,

$$h(n) = h(N-1-n)$$

Proof :

Let, N = 7, $\therefore$ h(n) = h(6–n)

n = 0, 1, 2, 3, 4, 5, 6

When n = 0; h(0) = h(6)

When n = 1; h(1) = h(5)

When n = 2; h(2) = h(4)

When n = 3; h(3) = h(3)

Let, N = 8, $\therefore$ h(n) = h(7–n)

n = 0, 1, 2, 3, 4, 5, 6, 7

When n = 0; h(0) = h(7)

When n = 1; h(1) = h(6)

When n = 2; h(2) = h(5)

When n = 3; h(3) = h(4)

When the impulse response is symmetric, the samples of impulse response will satisfy the condition,

$$b_n = b_{N-1-n}$$

By using the above symmetry condition it is possible to reduce the number of multipliers required for the realization of FIR system. Hence, the linear phase realization is also called ***realization with minimum number of multipliers.***

Consider the transfer function of a FIR system,

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots.. + b_{N-1} z^{-(N-1)}$$

The linear phase realization of the FIR system using the above equation for even and odd values of N are discussed below.

Case i : When N is even

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_{N-1} z^{-(N-1)}$$

$$= \sum_{m=0}^{N-1} b_m z^{-m} = \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=\frac{N}{2}}^{N-1} b_m z^{-m}$$

Let, p = N–1–m, $\quad \therefore$ m = N–1–p

When $m = \frac{N}{2}$; $p = N-1-\frac{N}{2} = \frac{N}{2}-1$

When m = N–1; p = N–1–(N–1) = 0

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{p=0}^{\frac{N}{2}-1} b_{N-1-p} z^{-(N-1-p)}$$

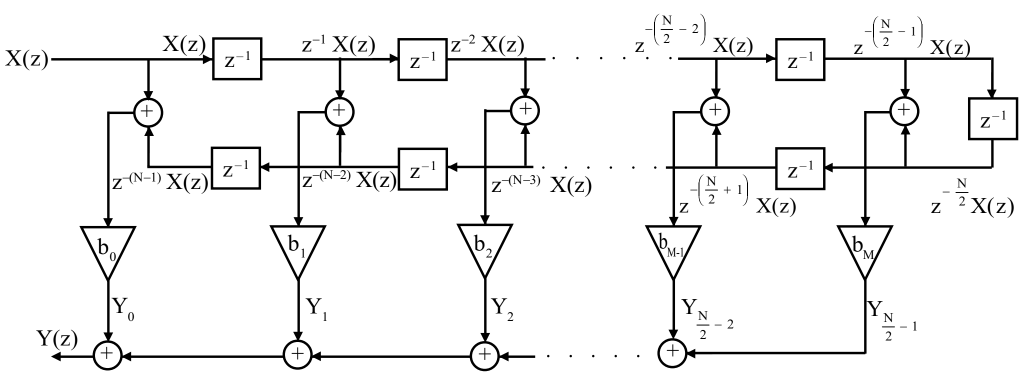
$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_{N-1-m} z^{-(N-1-m)}$$

Let p = m

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-m} + \sum_{m=0}^{\frac{N}{2}-1} b_m z^{-(N-1-m)}$$

$\because b_m = b_{N-1-m}$

$$= \sum_{m=0}^{\frac{N}{2}-1} b_m \left[z^{-m} + z^{-(N-1-m)}\right]$$



where $M = \frac{N}{2}-1$; $Y_0 = b_0\left[X(z) + z^{-(N-1)} X(z)\right]$; $\quad Y_{\frac{N}{2}-2} = b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)} X(z) + z^{-\left(\frac{N}{2}+1\right)} X(z)\right]$

$Y_1 = b_1\left[z^{-1} X(z) + z^{-(N-2)} X(z)\right]$; $\quad Y_{\frac{N}{2}-1} = b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)} X(z) + z^{-\frac{N}{2}} X(z)\right]$

***Fig 10.10 :** Direct form realization of a linear phase FIR system when N is even.*

$$\therefore\ Y(z) = b_0\left[X(z) + z^{-(N-1)}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] + \ldots..$$

$$+\ b_{\frac{N}{2}-2}\left[z^{-\left(\frac{N}{2}-2\right)}X(z) + z^{-\left(\frac{N}{2}+1\right)}X(z)\right] + b_{\frac{N}{2}-1}\left[z^{-\left(\frac{N}{2}-1\right)}X(z) + z^{-\frac{N}{2}}X(z)\right]$$

When N is even, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 10.10. From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications and N–1 number of additions. Also the structure involves N–1 delays and so N–1 memory locations are required to store the delayed signals.

Case ii : When N is odd

$$H(z) = \frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots.. + b_{N-1} z^{-(N-1)} = \sum_{m=0}^{N-1} b_m z^{-m}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=\frac{N+1}{2}}^{N-1} b_m z^{-m}$$

Let, p = N–1–m, $\therefore$ m = N–1–p
When m $= \frac{N+1}{2}$; p $= N-1-\frac{N+1}{2} = \frac{N-3}{2}$
When m = N–1 ; p = N–1 – (N–1) = 0

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{p=0}^{\frac{N-3}{2}} b_{N-1-p} z^{-(N-1-p)}$$

$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_{N-1-m} z^{-(N-1-m)} \qquad \boxed{\text{Let } p = m}$$

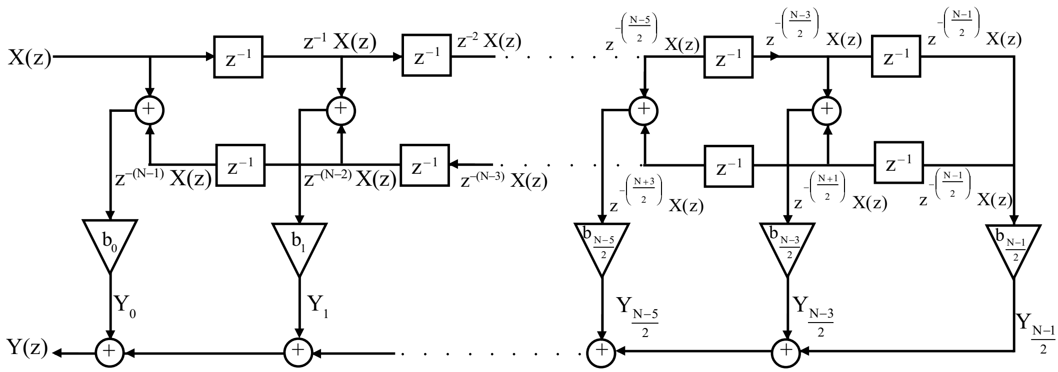
$$= \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-m} + b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m z^{-(N-1-m)} \qquad \boxed{\because b_m = b_{N-1-m}}$$

$$= b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} + \sum_{m=0}^{\frac{N-3}{2}} b_m\left[z^{-m} + z^{-(N-1-m)}\right]$$

$$\therefore\ Y(z) = b_{\frac{N-1}{2}} z^{-\left(\frac{N-1}{2}\right)} X(z) + b_0\left[X(z) + z^{-(N-1)}X(z)\right] + b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right] +$$

$$\ldots.. + b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right] + b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

When N is odd, the above equation can be used to construct the direct form structure of linear phase FIR system with minimum number of multipliers, as shown in fig 10.11.



$$Y_0 = b_0\left[X(z) + z^{-(N-1)}X(z)\right]; \quad Y_{\frac{N-5}{2}} = b_{\frac{N-5}{2}}\left[z^{-\left(\frac{N-5}{2}\right)}X(z) + z^{-\left(\frac{N+3}{2}\right)}X(z)\right]; \quad Y_{\frac{N-1}{2}} = b_{\frac{N-1}{2}}\, z^{-\left(\frac{N-1}{2}\right)}X(z)$$

$$Y_1 = b_1\left[z^{-1}X(z) + z^{-(N-2)}X(z)\right]; \quad Y_{\frac{N-3}{2}} = b_{\frac{N-3}{2}}\left[z^{-\left(\frac{N-3}{2}\right)}X(z) + z^{-\left(\frac{N+1}{2}\right)}X(z)\right]$$

***Fig 10.11 :** Direct form realization of a linear phase FIR system when N is odd.*

From the direct form linear phase structure it is observed that the realization of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications and N–1 number of additions. Also the structure involves N–1 delays and so N–1 memory locations are required to store the delayed signals.

Example 10.9

Draw the direct form structure of the FIR system described by the transfer function

$$H(z) = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5}$$

Solution

$$\text{Let,}\quad H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{4}z^{-3} + \frac{1}{2}z^{-4} + \frac{1}{8}z^{-5}$$

$$\therefore\ Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{4}z^{-3}X(z) + \frac{1}{2}z^{-4}X(z) + \frac{1}{8}z^{-5}X(z) \quad .....(1)$$

The direct form structure of FIR system can be obtained directly from equation (1).

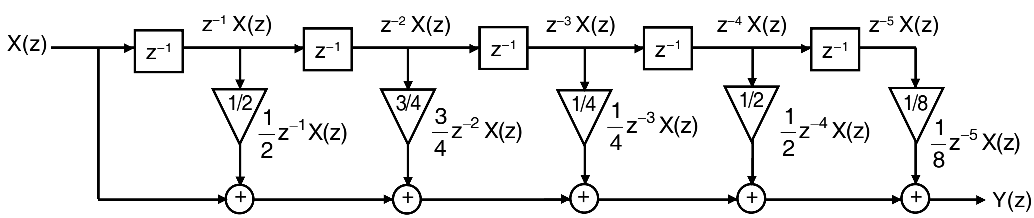


***Fig 1 :** Direct form structure of H(z).*

Example 10.10

Realize the following system with minimum number of multipliers.

a) $H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$

b) $H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$

c) $H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)$

Solution

a) Given that, $\mathbf{H(z) = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}}$ ..... (1)

By the definition of $\mathcal{Z}$-transform we get,

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} = h(0) + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + \ldots\ldots \quad \text{..... (2)}$$

On comparing equations (1) and (2) we get,

$$\text{Impulse response, } h(n) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}\right\}$$

Here h(n) satisfies the condition h(n) = h(N – 1 – n) and so impulse response is symmetrical. Hence the system has linear phase and can be realized with minimum number of multipliers.

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \frac{1}{4} + \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{2}z^{-3} + \frac{1}{4}z^{-4}$$

$$\therefore Y(z) = \frac{1}{4}X(z) + \frac{1}{2}z^{-1}X(z) + \frac{3}{4}z^{-2}X(z) + \frac{1}{2}z^{-3}X(z) + \frac{1}{4}z^{-4}X(z)$$

$$= \frac{1}{4}\left[X(z) + z^{-4}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-3}X(z)\right] + \frac{3}{4}z^{-2}X(z) \quad \text{.....(3)}$$

The direct form structure of linear phase FIR system is constructed using equation (3) as shown in fig 1.

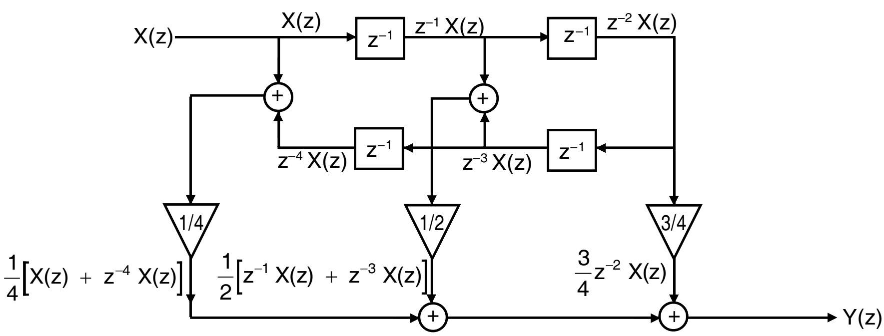


Fig 1 : Linear phase realization of H(z).

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b) Given that, $\mathbf{H(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}}$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + z^{-3}$$

$$\therefore Y(z) = X(z) + \frac{1}{2}z^{-1}X(z) + \frac{1}{2}z^{-2}X(z) + z^{-3}X(z)$$

$$= \left[X(z) + z^{-3}X(z)\right] + \frac{1}{2}\left[z^{-1}X(z) + z^{-2}X(z)\right] \quad \text{.....(4)}$$

The direct form realization of H(z) with minimum number of multipliers (i.e., linear phase realization) is obtained using equation (4) as shown in fig 2.

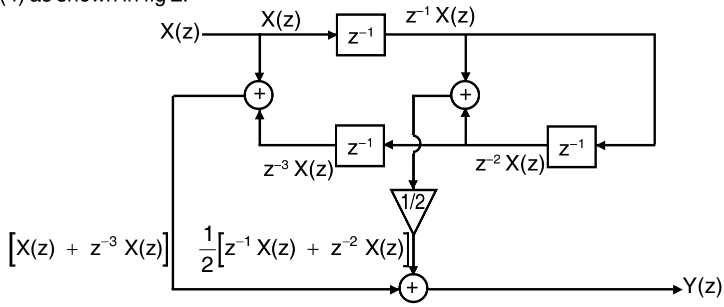


Fig 2 : Linear phase realization of H(z).

c) Given that, $\mathbf{H(z) = \left(1 + \frac{1}{2}z^{-1} + z^{-2}\right)\left(1 + \frac{1}{4}z^{-1} + z^{-2}\right)}$

The given system can be realized as cascade of two second order systems. Each system can be realized with minimum number of multipliers.

Let, $H(z) = H_1(z)\,H_2(z)$

where, $H_1(z) = 1 + \frac{1}{2}z^{-1} + z^{-2}$; $H_2(z) = 1 + \frac{1}{4}z^{-1} + z^{-2}$

Let, $H_1(z) = \frac{Y_1(z)}{X(z)} = 1 + \frac{1}{2}z^{-1} + z^{-2}$

$$\therefore\ Y_1(z) = X(z) + \frac{1}{2}z^{-1} X(z) + z^{-2} X(z)$$

$$= \left[X(z) + z^{-2} X(z)\right] + \frac{1}{2}z^{-1} X(z) \qquad .....(5)$$

The linear phase realization structure of $H_1(z)$ is obtained using equation (5) as shown in fig 3.

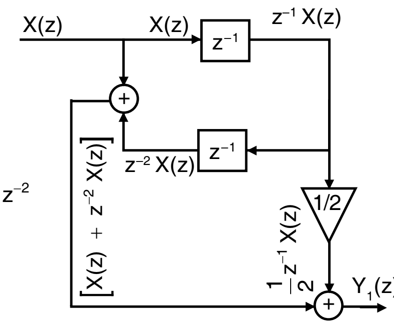


Fig 3 : Linear phase realization of $H_1(z)$.

Let, $H_2(z) = \frac{Y(z)}{Y_1(z)} = 1 + \frac{1}{4}z^{-1} + z^{-2}$

$$\therefore\ Y(z) = Y_1(z) + \frac{1}{4}z^{-1} Y_1(z) + z^{-2} Y_1(z)$$

$$= \left[Y_1(z) + z^{-2} Y_1(z)\right] + \frac{1}{4}z^{-1} Y_1(z) \qquad .....(6)$$

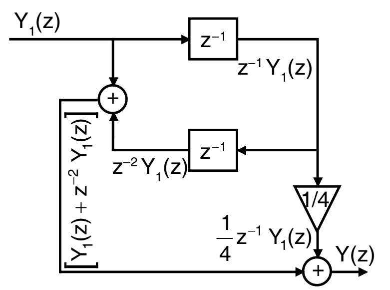


Fig 4 : Linear phase realization of $H_2(z)$.

The linear phase realization structure of $H_2(z)$ is obtained using equation (6) as shown in fig 4. The linear phase structure of H(z) is obtained by connecting the linear phase realization structures of $H_1(z)$ and $H_2(z)$ in cascade as shown in fig 5.

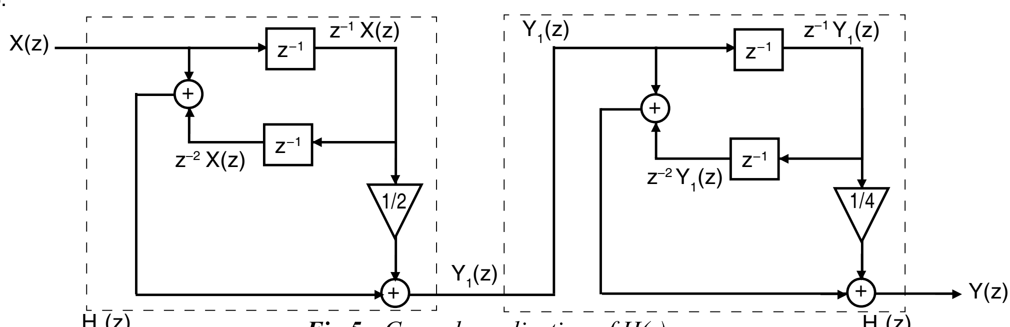


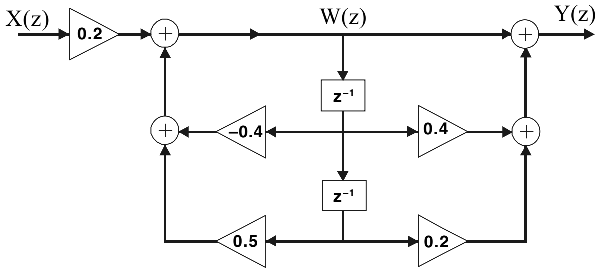
Fig 5 : Cascade realization of H(z).

10.5 Summary of Important Concepts

1. Mathematically, a discrete time system is represented by a difference equation.
2. Physically, a discrete time system is realized or implemented either as a digital hardware or as a software running on a digital hardware.
3. The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay.
4. The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithmetic used for computation.
5. The various structures proposed for IIR and FIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.
6. When a discrete time system is designed by considering all the infinite samples of the impulse response, then the system is called IIR (Infinite Impulse Response) system.
7. When a discrete time system is designed by choosing only finite samples (usually N-samples) of the impulse response, then the system is called FIR (Finite Impulse Response) system.
8. The IIR systems are recursive systems, whereas the FIR systems are nonrecursive systems.
9. The direct form-I structure of IIR system offers a direct relation between time domain and z-domain equations.
10. Since separate delays are employed for input and output samples, realizing IIR system using direct form-I structure require more memory.
11. The direct form-I and II structure realization of an N^{th} order IIR discrete time system involves M+N+1 number of multiplications and M+N number of additions.
12. The direct form-I structure realization of an N^{th} order IIR discrete time system involves M+N delays and so M+N memory locations are required to store the delayed signals.
13. In a realizable N^{th} order IIR discrete time system, the direct form-II structure realization involves N delays and so N memory locations are required to store the delayed signals.
14. In canonic structure, the number of delays will be equal to the order of the system.
15. The direct form-II structure of IIR system is canonic whereas the direct form-I structure is noncanonic.
16. In cascade realization of IIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form-I or II structure and then connected in cascade.
17. In parallel realization of IIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form-I or II structure and then connected in parallel.
18. In cascade and parallel realization of IIR systems, the number of calculations and the memory requirement depends on the realization of individual sections.
19. Direct form structure of FIR system provides a direct relation between time domain and z-domain equations.
20. The realization of an N^{th} order FIR discrete time system using direct form structure and linear phase structure involves N number of multiplications and N−1 number of additions.
21. The realization of an N^{th} order FIR discrete time system using direct form structure involves N−1 delays and so N−1 memory locations are required to store the delayed signals.
22. The condition for symmetry of impulse response of FIR system is, $h(n) = h(N-1-n)$.
23. The linear phase realization is also called realization with minimum number of multipliers.
24. In cascade realization of FIR system, the N^{th} order transfer function is divided into first and second order sections and they are realized in direct form or linear phase structure and then connected in cascade.
25. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for even values of N involves N/2 number of multiplications, and N−1 number of additions.
26. The direct form linear phase realization structure of an N^{th} order FIR discrete time system for odd values of N involves (N+1)/2 number of multiplications, and N−1 number of additions.

10. 6 Short Questions and Answers

Q10.1 Obtain the transfer function for the following structure.



Solution

The following z-domain equations can be obtained from the given direct form-II structure.

$$W(z) = -0.4z^{-1}W(z) + 0.5z^{-2}W(z) + 0.2\,X(z)$$

$$\therefore W(z) + 0.4z^{-1}W(z) - 0.5z^{-2}W(z) = 0.2\,X(z) \quad \Rightarrow \quad \frac{W(z)}{X(z)} = \frac{0.2}{1+0.4\,z^{-1} - 0.5\,z^{-2}}$$

$$Y(z) = W(z) + 0.4\,z^{-1}W(z) + 0.2\,z^{-2}W(z) \quad \Rightarrow \quad \frac{Y(z)}{W(z)} = 1 + 0.4z^{-1} + 0.2z^{-2}$$

The given direct form-II digital network can be realized by the transfer function,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)} = \frac{0.2\,(1 + 0.4z^{-1} + 0.2z^{-2})}{1 + 0.4z^{-1} - 0.5z^{-2}}$$

Q10.2 Realize the following FIR system with minimum number of multipliers.
h(n) = { –0.5, 0.8, –0.5 }

Solution

Given that, h(n) = {–0.5, 0.8, –0.5 }

On taking Z- transform,

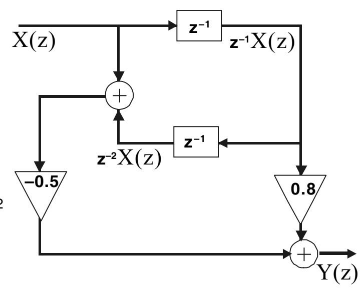
$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-1} = \sum_{n=0}^{2} h(n)z^{-1}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} = -0.5 + 0.8\,z^{-1} - 0.5\,z^{-2}$$

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = -0.5 + 0.8\,z^{-1} - 0.5z^{-2}$$

$$\therefore Y(z) = -0.5\,X(z) + 0.8z^{-1}\,X(z) - 0.5z^{-2}\,X(z)$$

$$= -0.5\,[\,X(z) + z^{-2}\,X(z)\,] + 0.8z^{-1}\,X(z)$$



FigQ10.2: *Linear phase realization.*

The linear phase structure is drawn using the above equation as shown in fig Q10.2.

Q10.3 The transfer function of an IIR system has 'Z' number of zeros and 'P' number of poles. How many number of additions, multiplications and memory locations are required to realize the system in direct form-I and direct form-II.

The realization of IIR system with Z zeros and P poles in direct form-I and II structure, involves Z+P number of additions and Z+P+1 number of multiplications. The direct form-I structure requires Z+P memory locations whereas the direct form-II structure requires only P number of memory locations.

Q10.4 What are the factors that influence the choice of structure for realization of an LTI system?

The factors that influence the choice of realization structure are computational complexity, memory requirements, finite word length effects, parallel processing and pipelining of computations.

Q10.5 ***Draw the direct form-I structure of second order IIR system with equal number of poles and zeros.***

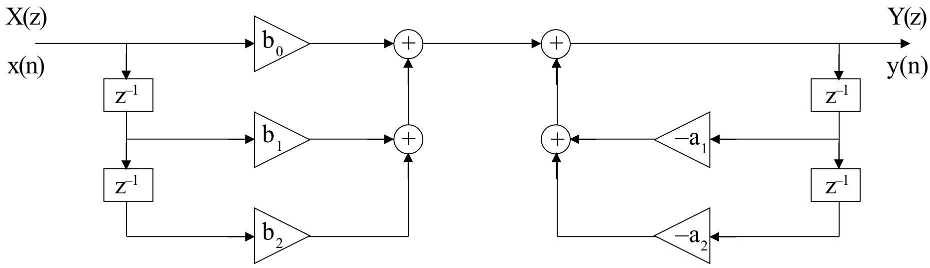


Fig Q10.5 : *Direct form-I structure of second order IIR system.*

Q10.6 ***An LTI system is described by the difference equation, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$. Realize it in direct form-I structure and convert to direct form-II structure.***

Solution

Given that, $y(n) = a_1 y(n-1) + x(n) + b_1 x(n-1)$. Using the given equation the direct form-I structure is drawn as shown in fig Q10.6a.

Direct form-I structure can be considered as cascade of two systems $\mathcal{H}_1$ and $\mathcal{H}_2$ as shown in fig Q10.6b. By linearity property, order of cascading can be changed as shown in fig Q10.6c.

In fig Q10.6c, we can observe that the input to the delay in $\mathcal{H}_1$ and $\mathcal{H}_2$ are same and so the output of delays will be same. Hence the delays can be combined to get direct form-II structure as shown in fig Q10.6d.

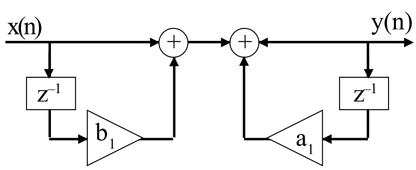


Fig Q10.6a*: Direct form-I structure.*

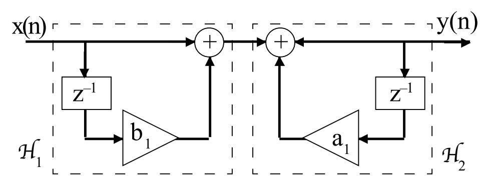


Fig Q10.6b*: Direct form-I structure as cascade of two systems.*

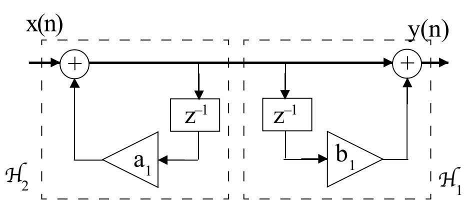


Fig Q10.6c*: Direct form-I structure after interchanging the order of cascading.*

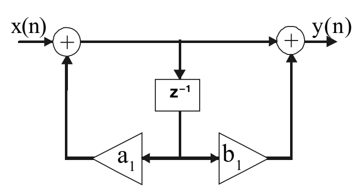


Fig Q10.6d*: Direct form-II structure.*

Q10.7 ***What is the advantage in cascade and parallel realization of IIR systems ?***

In digital implementation of LTI system the coefficients of the difference equation governing the system are quantized. While quantizing the coefficients the value of poles may change. This will end up in a frequency response different to that of desired frequency response.

These effects can be avoided or minimized, if the LTI system is realized in cascade or parallel structure. [i.e, The sensitivity of frequency response characteristics to quantization of the coefficients is minimized]

Q10.8 Compare the direct form-I and II structures of an IIR systems, with M zeros and N poles.

Direct form-I	*Direct form-II*
1. Separate delay for input and output.	1. Same delay for input and output.
2. M + N + 1 multiplications are involved.	2. M + N + 1 multiplications are involved.
3. M + N additions are involved.	3. M + N additions are involved.
4. M + N delays are involved.	4. N delays are involved.
5. M + N memory locations are required.	5. N memory locations are required.
6. Noncanonical structure.	6. Canonical structure.

Q10.9 Compare the direct form and linear phase structures of an N^{th} order FIR system.

Direct form	*Linear phase*
1. Impulse response need not be symmetric.	1. Impulse response should be symmetric.
2. N multiplications are involved.	2. N/2 or (N+1)/2 multiplications are involved.
3. N−1 additions and delays are involved.	3. N−1 additions and delays are involved.
4. N−1 memory locations are required.	4. N−1 memory locations are required.

Q10.10 What is the advantage in linear phase realization of FIR systems ?

The advantage in the linear phase realization structure is that it involves minimum number of multiplications. In linear phase realization of N^{th} order FIR system, the number of multiplications for even values of N will be N/2 and for odd values of N will be (N+1)/2, whereas the direct form realization involves N multiplications.

10. 7 Exercises

I. Fill in the blanks with appropriate words.

1. In IIR systems, the ___________ structure will give direct relation between time domain and z-domain.
2. When number of delays is equal to order of the system, the structure is called ___________.
3. The direct form realization of IIR system with M zeros and N poles involves ___________ multiplications.
4. The direct form-II realization of N^{th} order IIR system requires ___________ delays and memory locations.
5. The direct form realization of N^{th} order FIR system involves ___________ additions.
6. ___________ realization is called realization with minimum number of multipliers.

Answers

1. direct form-I	3. M+N+1	5. N−1
2. canonic structure	4. N	6. Linear phase

II. State whether the following statements are True/False.

1. The direct form-I structure of IIR system employs same delay for input and output samples.
2. In direct form-II realization of IIR system, N memory locations are required to store delayed signals.
3. In parallel or cascade realization, the memory requirement depends on realization of individual sections.
4. Scaling multipliers has to be provided between individual sections of cascade structure.
5. The linear phase realization of N^{th} order FIR system for odd values of N involves N/2 multiplications.
6. For linear phase realization of FIR system, the impulse response should be symmetric.

Answers

1. False 2. True 3. True 4. True 5. False 6. True

III. Choose the right answer for the following questions

1. ***The factor that influence the choice of realization of strurcture is,***

a) memory requirements b) computational complexity
c) parallel processing and pipelining d) all the above.

2. ***The structure that uses separate delays for input and output samples is,***

a) direct form-II b) direct form-I c) cascade form d) parallel form

3. ***The linear phase realization sturcture is used to represent,***

a) FIR systems b) IIR systems
c) both FIR and IIR systems d) all discrete time systems

4. ***The effect of quantization of coefficients on the frequency response is minimized in,***

a) cascade realization b) parallel realization
c) direct form structure d) both a and b

5. ***The direct form-I and II structures of IIR system will be identical in,***

a) all pole system b) all zero system
c) both a and b d) first order and second order systems.

6. ***The condition for symmetry of impulse response of FIR system is,***

a) h(n) = h(N–1) b) h(n) = h(N+1) c) h(n) = h(N–n) d) h(n) = h(N–1–n)

7. ***The linear phase realization is used in FIR systems, mainly to minimize,***

a) multipliers b) memory c) delays d) adders

8. ***Which one of the following FIR system has linear phase response?***

a) y(n) = 0.4 x(n) + 0.1 x(n–1) + 0.5 x(n–2) b) y(n) = 0.3 x(n) + x(n–1) + 3.0 x(n–2)
c) y(n) = 0.5 x(n) + 0.7 x(n–1) d) y(n) = 0.6x(n) + 0.6 x(n–1)

9. ***The quantization error increases, when the order of the system 'N' increases in case of,***

a) direct form realization b) cascade or parallel form realization
c) all IIR systems d) all FIR systems

10. ***The number of memory locations required to realize the system,*** $H(z) = \dfrac{1+z^{-2}+2z^{-3}}{1+z^{-2}+z^{-4}}$ ***is,***

a) 8 b) 7 c) 2 d) 10

11. ***Number of multipliers and adders required for direct form realization of N^{th} order FIR system are,***

a) N, N+1 b) N, N–1 c) N+1, N, N d) N–1, N+1

12. ***The realization of linear phase FIR system for odd values of 'N' needs,***

a) $\dfrac{N}{2}$ multipliers b) $\dfrac{N+1}{2}$ multipliers c) N–1 multipliers d) N multipliers

Answers

1. d	3. a	5. c	7. a	9. a	11. b
2. b	4. d	6. d	8. d	10. b	12. b

IV. Answer the following questions

1. What are the various issues that are addressed by realization structures?
2. What are the basic elements used to construct the realization structures of discrete time system?
3. List the different types of structures for realization of IIR systems.
4. Draw the direct form-I structure of an N^{th} order IIR system with equal number of poles and zeros.
5. Draw the direct form-II structure of an N^{th} order IIR system with equal number of poles and zeros.
6. Explain the conversion of direct form-I structure to direct form-II structure with an example.
7. What are the difficulties in cascade realization?
8. Explain the realization of cascade structure of an IIR system.
9. Explain the realization of parallel structure of an IIR system.
10. What are the different types of structure for realization of FIR systems?
11. Draw the direct form structure of an N^{th} order FIR system.
12. What is the necessary condition for Linear phase realization of FIR system?
13. Draw the linear phase realization structure of an Nth order FIR system when 'N' is even.
14. Draw the linear phase realization structure of an Nth order FIR system when 'N' is odd.
15. Explain the realization of cascade structure of a FIR system.

V. Solve the following problems

E10.1 Obtain the direct form-I, direct form-II, cascade and parallel form realizations of the LTI system governed by the equation,

$$y(n) = -\frac{5}{4}\,y(n-1) + \frac{1}{8}\,y(n-2) + \frac{1}{16}\,y(n-3) + x(n) + 5x(n-1) + 6x(n-2)$$

E10.2 Realize the direct form-I and direct form-II of the IIR system represented by the transfer function,

$$H(z) = \frac{2(z+2)}{(z-0.1)\,(z+0.5)\,(z+0.4)}$$

E10.3 Determine the direct form-I, II, cascade and parallel realization of the following LTI system.

$$H(z) = \frac{(z^3 - 2z^2 + 2z - 1)}{(z-0.5)\,(z^2 + z - 0.5)}$$

E10.4 Realize the cascade and parallel structures of the system governed by the difference equation,

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + \frac{1}{2}x(n-1)$$

E10.5 Draw the direct form structure of the FIR systems described by the following equations,

a) $y(n) = x(n) + \frac{1}{3}x(n-1) + \frac{1}{4}\,x(n-2) + \frac{1}{5}\,x(n-3) + \frac{2}{7}\,x(n-4)$

b) $y(n) = 0.35\,x(n) + 0.3\,x(n-1) + 0.125x(n-2) - 0.25x(n-3)$
$\qquad - 0.35x(n-4) - 0.3x(n-5) - 0.125x(n-6)$

E10.6 Realize the following FIR systems with minimum number of multipliers.

a) $H(z) = 0.2 + 0.5z^{-1} + 0.3z^{-2} + 0.5z^{-3} + 0.2z^{-4}$

b) $H(z) = \left(1 + \frac{1}{8}\,z^{-1} + z^{-2}\right)\left(2 - \frac{1}{9}\,z^{-1} + 2z^{-2}\right)$

c) $y(n) = -\frac{1}{2}\,x(n) + \frac{3}{5}\,x(n-1) + \frac{3}{8}\,x(n-2) + \frac{3}{5}\,x(n-3) - \frac{1}{2}x(n-4)$

Answers

E10. 1

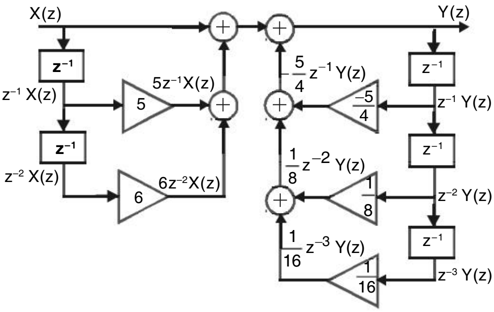


Fig E10.1.1 : *Direct form-I structure.*

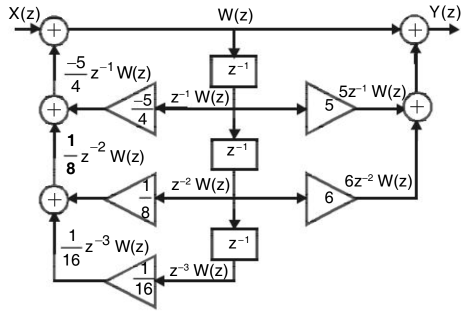


Fig E10.1.2 : *Direct form-II structure.*

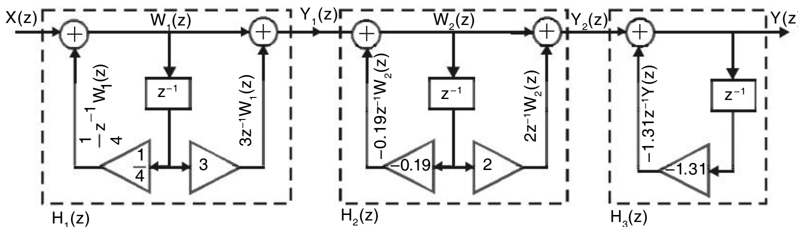


Fig E10.1.3 : *Cascade structure.*

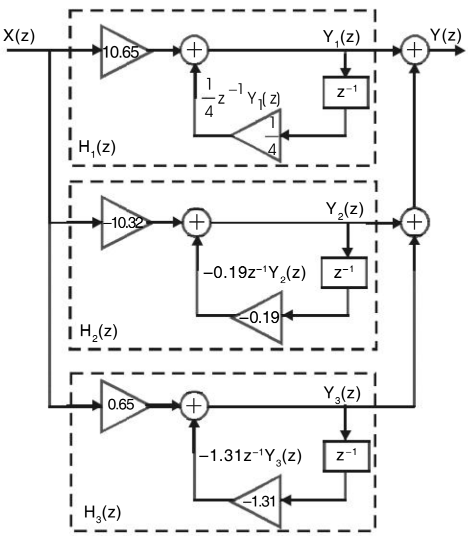


Fig E10.1.4 : *Parallel structure.*

E10. 2

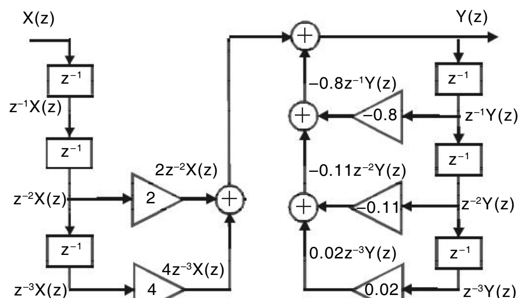


Fig E10.2.1 : *Direct form-I structure.*

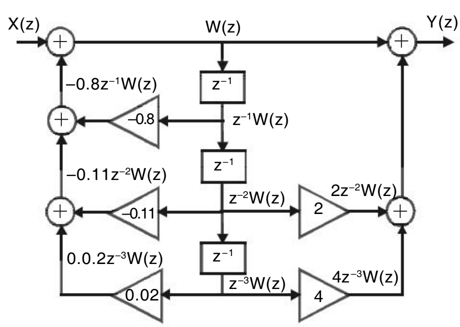


Fig E10.2.2 : *Direct form-II structure.*

E10.3

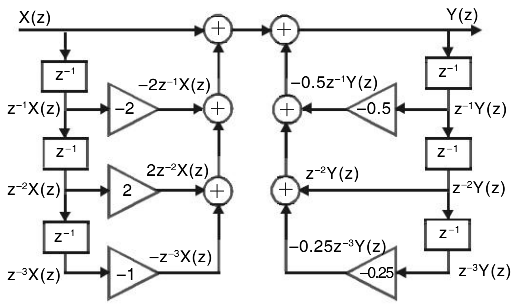


Fig E10.3.1 : *Direct form-I structure.*

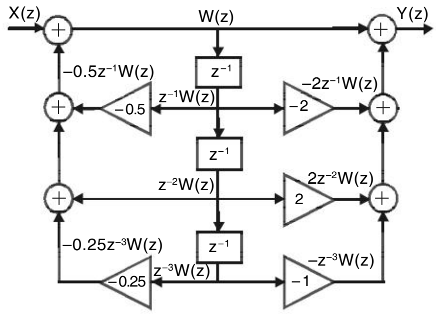


Fig E10.3.2 : *Direct form-II structure.*

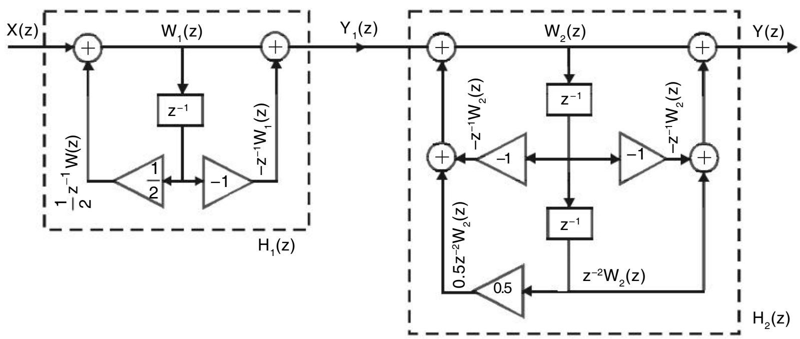


Fig E10.3.3 : *Cascade structure.*

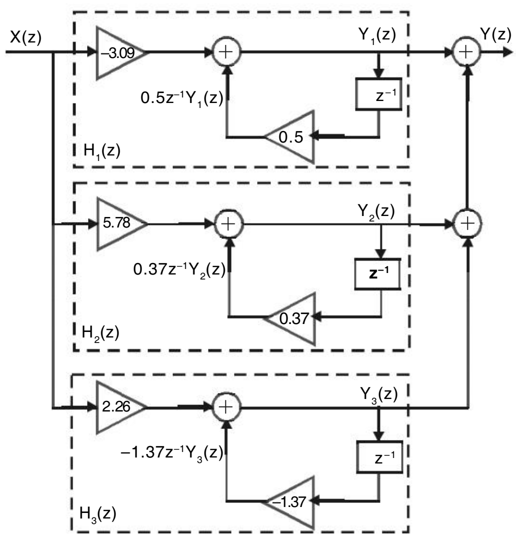


Fig E10.3.4 : *Parallel structure.*

E10.4

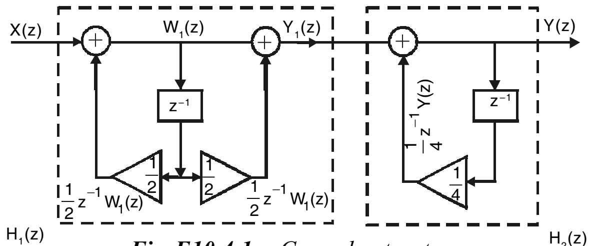


Fig E10.4.1 : Cascade structure.

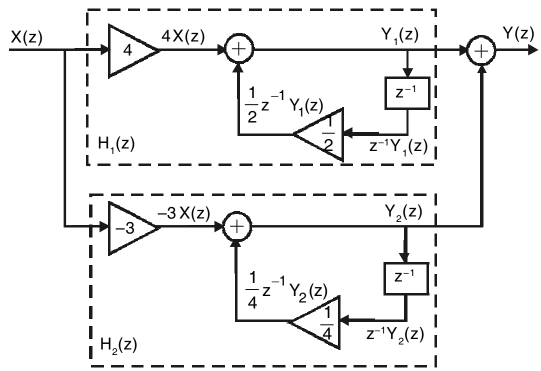


Fig E10.4.2 : Parallel structure.

E10.5 a)

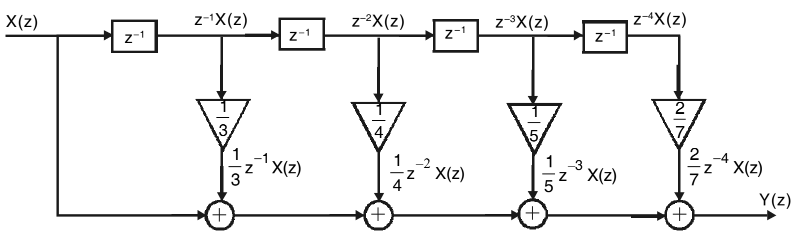


Fig E10.5a : Direct form structure.

E10.5 b)

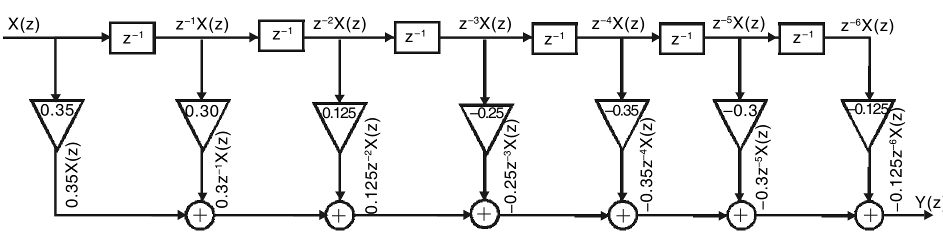


Fig E10.5b : Direct form structure.

E10.6 a)

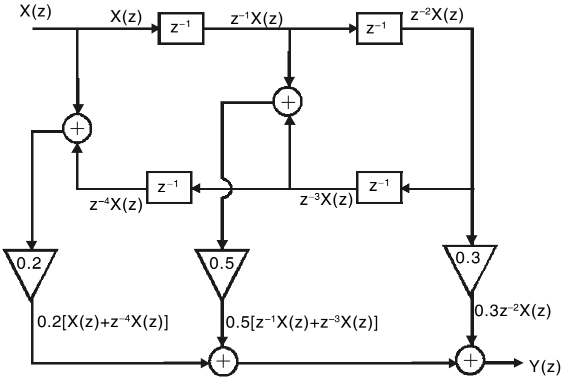


Fig E10.6a : Linear phase structure.

E10.6 b)

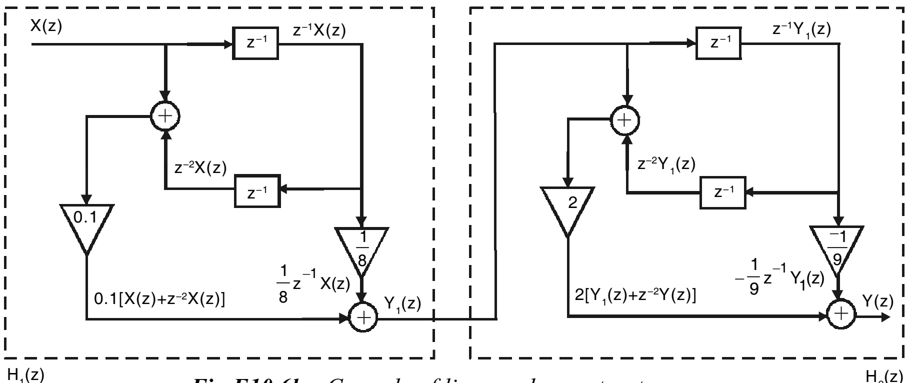


Fig E10.6b : Cascade of linear phase structure.

E10.6 c)

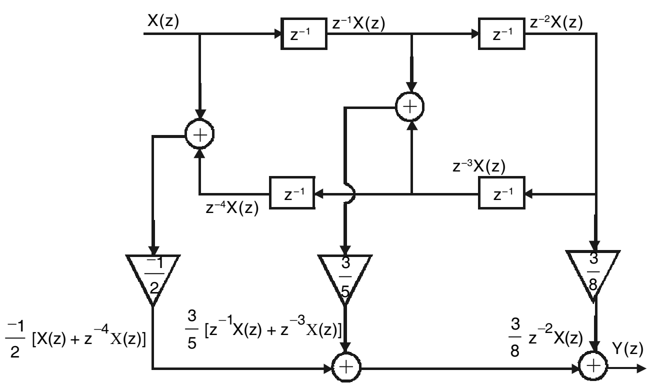


Fig E10.6c : Linear phase structure.

CHAPTER 11

State Space Analysis of Discrete Time Systems

11.1 Introduction

The state space analysis of continuous time system is presented in chapter-5. The concepts introduced in chapter-5 are extended to analysis of discrete time systems in this chapter. In state space analysis a discrete time system is represented by a state model, in which the state or condition of a system at any discrete time n is represented by a set of N state variables.

The state space model is the best choice for analysis of discrete time system using digital systems or computers.The state variable analysis can be applied for any type of systems and the analysis can be carried with initial conditions, and with multiple inputs and outputs.

11.2 State Model of Discrete Time systems

State and State Variables

The ***state*** of a discrete time system refers to the condition of the system at any discrete time instant, which is described using a minimum set of variables called state variables. The knowledge of these variables at n = 0 together with knowledge of inputs for n > 0, completely describes the behaviour of the system for $n \geq 0$. The ***state variables*** are a set of variables that completely describe the state or condition of a system at any discrete time instant, n.

State Equations

Consider a discrete time system with M-inputs, N-state variables, and P-outputs.

Let, $q_1(n)$, $q_2(n)$, $q_3(n)$, ..., $q_N(n)$ be N-state variables of the discrete time system,

$x_1(n)$, $x_2(n)$, $x_3(n)$, ..., $x_M(n)$ be M-inputs of the discrete time system.

$y_1(n)$, $y_2(n)$, $y_3(n)$, ..., $y_P(n)$ be P-outputs of the discrete time system.

Now, the discrete time system can be represented by the block diagram shown in fig 11.1.

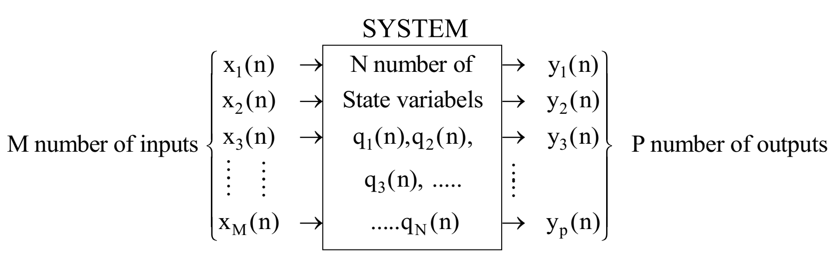


Fig 11.1 : *State space representation of a discrete time system.*

In state space representation,

-state variables $q_1(n), q_2(n), q_3(n), \ldots\ldots q_N(n)$ represents the present state of the system.

-state variables $q_1(n+1), q_2(n+1), q_3(n+1), \ldots\ldots q_N(n+1)$ represents the next state.

The state equations can be formed by taking next state of state variables as function of present state of state variables and present inputs. Therefore, the next state of the N-state variables can be expressed as shown below.

$$
\begin{aligned}
q_1(n+1) &= F\left[q_1(n),\ q_2(n),\ q_3(n) \ldots\ldots q_N(n),\ x_1(n),\ x_2(n),\ x_3(n) \ldots\ldots x_M(n)\right] \\
q_2(n+1) &= F\left[q_1(n),\ q_2(n),\ q_3(n) \ldots\ldots q_N(n),\ x_1(n),\ x_2(n),\ x_3(n) \ldots\ldots x_M(n)\right] \\
q_3(n+1) &= F\left[q_1(n),\ q_2(n),\ q_3(n) \ldots\ldots q_N(n),\ x_1(n),\ x_2(n),\ x_3(n) \ldots\ldots x_M(n)\right] \\
&\vdots \\
q_N(n+1) &= F\left[q_1(n),\ q_2(n),\ q_3(n) \ldots\ldots q_N(n),\ x_1(n),\ x_2(n),\ x_3(n) \ldots\ldots x_M(n)\right]
\end{aligned}
$$

Mathematically, the above functional form of the next state of the N-state variables can be expressed as first order difference equations shown below.

$$
\begin{aligned}
q_1(n+1) = {} & a_{11}\, q_1(n) + a_{12}\, q_2(n) + a_{13}\, q_3(n) + \ldots\ldots + a_{1N}\, q_N(n) \\
& + b_{11}\, x_1(n) + b_{12}\, x_2(n) + b_{13}\, x_3(n) + \ldots\ldots + b_{1M}\, x_M(n) \\
q_2(n+1) = {} & a_{21}\, q_1(n) + a_{22}\, q_2(n) + a_{23}\, q_3(n) + \ldots\ldots + a_{2N}\, q_N(n) \\
& + b_{21}\, x_1(n) + b_{22}\, x_2(n) + b_{23}\, x_3(n) + \ldots\ldots + b_{2M}\, x_M(n) \\
q_3(n+1) = {} & a_{31}\, q_1(n) + a_{32}\, q_2(n) + a_{33}\, q_3(n) + \ldots\ldots + a_{3N}\, q_N(n) \\
& + b_{31}\, x_1(n) + b_{32}\, x_2(n) + b_{33}\, x_3(n) + \ldots\ldots + b_{3M}\, x_M(n) \\
& \vdots \\
q_N(n+1) = {} & a_{N1}\, q_1(n) + a_{N2}\, q_2(n) + a_{N3}\, q_3(n) + \ldots\ldots + a_{NN}\, q_N(n) \\
& + b_{N1}\, x_1(n) + b_{N2}\, x_2(n) + b_{N3}\, x_3(n) + \ldots\ldots + b_{NM}\, x_M(n)
\end{aligned}
$$

The above equations of the next state of the N-state variables are called ***state equations*** and can be expressed in the matrix form as shown in the matrix equation (11.1).

$$
\underbrace{\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \\ \vdots \end{bmatrix}}_{\mathbf{Q}(n+1)}
=
\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13}\ldots \\ a_{21} & a_{22} & a_{23}\ldots \\ a_{31} & a_{32} & a_{33}\ldots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{A}}
\underbrace{\begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \\ \vdots \end{bmatrix}}_{\mathbf{Q}(n)}
+
\underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13}\ldots \\ b_{21} & b_{22} & b_{23}\ldots \\ b_{31} & b_{32} & b_{33}\ldots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{B}}
\underbrace{\begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ \vdots \end{bmatrix}}_{\mathbf{X}(n)}
\qquad \ldots.(11.1)
$$

where, $\mathbf{A}$ = System Matrix ; $\mathbf{B}$ = Input Matrix

$\mathbf{Q}(n)$ = State Vector ; $\mathbf{X}(n)$ = Input Vector

Therefore, the Matrix form of state equation can be written as shown in equation (11.2).

$$
\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n) \qquad \ldots.(11.2)
$$

Output Equations

The output equations can be formed by taking the outputs as function of state variables and inputs. Therefore the P-outputs can be expressed as shown below.

$$y_1(n) = F\left[q_1(n),\ q_2(n),\ q_3(n)\ .....\ q_N(n),\ x_1(n),\ x_2(n),\ x_3(n)\ .....\ x_M(n)\right]$$
$$y_2(n) = F\left[q_1(n),\ q_2(n),\ q_3(n)\ .....\ q_N(n),\ x_1(n),\ x_2(n),\ x_3(n)\ .....\ x_M(n)\right]$$
$$y_3(n) = F\left[q_1(n),\ q_2(n),\ q_3(n)\ .....\ q_N(n),\ x_1(n),\ x_2(n),\ x_3(n)\ .....\ x_M(n)\right]$$
$$\vdots$$
$$y_P(n) = F\left[q_1(n),\ q_2(n),\ q_3(n)\ .....\ q_N(n),\ x_1(n),\ x_2(n),\ x_3(n)\ .....\ x_M(n)\right]$$

Mathematically, the above functional form of the outputs can be expressed as algebriac equations shown below.

$$\begin{aligned}
y_1(n) &= c_{11}\,q_1(n) + c_{12}\,q_2(n) + c_{13}\,q_3(n) + ...+ c_{1N}\,q_N(n) \\
&\qquad + d_{11}\,x_1(n) + d_{12}\,x_2(n) + d_{13}\,x_3(n) + ... + d_{1M}\,x_M(n) \\
y_2(n) &= c_{21}\,q_1(n) + c_{22}\,q_2(n) + c_{23}\,q_3(n) + ...+ c_{2N}\,q_N(n) \\
&\qquad + d_{21}\,x_1(n) + d_{22}\,x_2(n) + d_{23}\,x_3(n) + ... + d_{2M}\,x_M(n) \\
y_3(n) &= c_{31}\,q_1(n) + c_{32}\,q_2(n) + c_{33}\,q_3(n) + ... + c_{3N}\,q_N(n) \\
&\qquad + d_{31}\,x_1(n) + d_{32}\,x_2(n) + d_{33}\,x_3(n) + ... + d_{3M}\,x_M(n) \\
&\ \vdots \\
y_P(n) &= c_{P1}\,q_1(n) + c_{P2}\,q_2(n) + c_{P3}\,q_3(n) + ... + c_{PN}\,q_N(n) \\
&\qquad + d_{P1}\,x_1(n) + d_{P2}\,x_2(n) + d_{P3}\,x_3(n) + ... + d_{PM}\,x_M(n)
\end{aligned}$$

The above equations are called ***output equations*** and can be expressed in the matrix form as shown in equation (11.3).

$$\underbrace{\begin{bmatrix} y_1(n) \\ y_2(n) \\ y_3(n) \\ \vdots \end{bmatrix}}_{\mathbf{Y}(n)} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13}\cdots \\ c_{21} & c_{22} & c_{23}\cdots \\ c_{31} & c_{32} & c_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \\ \vdots \end{bmatrix}}_{\mathbf{Q}(n)} + \underbrace{\begin{bmatrix} d_{11} & d_{12} & d_{13}\cdots \\ d_{21} & d_{22} & d_{23}\cdots \\ d_{31} & d_{32} & d_{33}\cdots \\ \vdots & \vdots & \vdots \end{bmatrix}}_{\mathbf{D}} \underbrace{\begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \\ \vdots \end{bmatrix}}_{\mathbf{X}(n)} \qquad ....(11.3)$$

where, $\mathbf{C}$ = Output Matrix ; $\mathbf{D}$ = Transmission Matrix

$\mathbf{Q}(n)$ = State Vector; $\mathbf{X}(n)$ = Input Vector

Therefore the Matrix form of output equation can be written as shown in equation (11.4).

$$\mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n) \qquad ....(11.4)$$

State Model

The ***state model*** of a discrete time system is given by state equations and output equations.

$$\left.\begin{aligned} &\text{State equations} \quad : && \mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n) \\ &\text{Output equations} : && \mathbf{Y}(n) \quad\ \ = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n) \end{aligned}\right\}\ \text{State model of discrete time system}$$

11.3 State Model of a Discrete Time System From Direct Form-II Structure

Consider the equation of 3rd order LTI system,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3)$$

The direct form-II structure of the system described by the above equation is shown in fig 11.2.

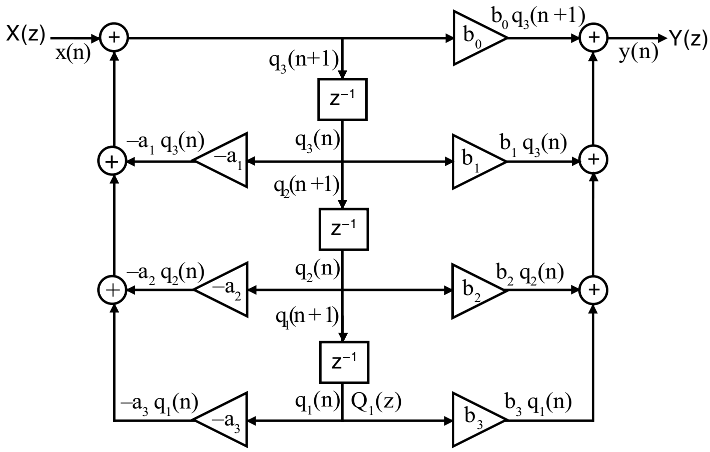


***Fig 11.2 :** Direct form-II structure of a discrete time system.*

The choice of number of state variables is equal to number of delay units. The direct form-II structure has one input x(n), one output y(n) and three delay units, and so let us choose three state variables. Let, $q_1(n)$, $q_2(n)$ and $q_3(n)$ be the three state variables. Assign state variables at the output of every delay unit. Hence the $(n+1)^{th}$ value of state variables will be available at the input of delay unit.

State Equations

The state equations are formed by equating the sum of incoming signals of delay unit to $(n+1)^{th}$ value of state variables, as shown below.

$$\left.\begin{aligned} q_1(n+1) &= q_2(n) \\ q_2(n+1) &= q_3(n) \\ q_3(n+1) &= -a_3\, q_1(n) - a_2\, q_2(n) - a_1\, q_3(n) + x(n) \end{aligned}\right\} \text{State equations}$$

$$\Downarrow$$

$$\left.\begin{aligned} q_1(n+1) &= 0 \times q_1(n) + 1 \times q_2(n) + 0 \times q_3(n) + 0 \times x(n) \\ q_2(n+1) &= 0 \times q_1(n) + 0 \times q_2(n) + 1 \times q_3(n) + 0 \times x(n) \\ q_3(n+1) &= -a_3 \times q_1(n) - a_2 \times q_2(n) - a_1 \times q_3(n) + 1 \times x(n) \end{aligned}\right\} \text{State equations}$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} \left[x(n)\right]$$

Output Equations

The output equation y(n) is formed by equating the incoming signals of output node / summing point to y(n) as shown below.

$$y(n) = b_0\, q_3(n+1) + b_3\, q_1(n) + b_2\, q_2(n) + b_1\, q_3(n)$$

On substituting for $q_3(n+1)$ from state equation we get,

$$y(n) = b_0\left[-a_3\, q_1(n) - a_2\, q_2(n) - a_1\, q_3(n) + x(n)\right] + b_3\, q_1(n) + b_2\, q_2(n) + b_1\, q_3(n)$$

$$= (b_3 - b_0\, a_3)\, q_1(n) + (b_2 - b_0\, a_2)\, q_2(n) + (b_1 - b_0 a_1)\, q_3(n) + b_0\, x(n)$$

On arranging the output equation in the matrix form we get,

$$y(n) = \underbrace{\begin{bmatrix} b_3 - b_0\, a_3 & b_2 - b_0\, a_2 & b_1 - b_0\, a_1 \end{bmatrix}}_{\mathbf{C}} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \underbrace{\begin{bmatrix} b_0 \end{bmatrix}}_{\mathbf{D}}\begin{bmatrix} x(n) \end{bmatrix}$$

State Model

State model of a discrete time system is given by state equations and output equations.

State equations : $\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$

$$\Downarrow$$

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} x(n) \end{bmatrix}$$

Output equations : $\mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)$

$$\Downarrow$$

$$y(n) = \begin{bmatrix} b_3 - b_0\, a_3 & b_2 - b_0\, a_2 & b_1 - b_0\, a_1 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} b_0 \end{bmatrix}\begin{bmatrix} x(n) \end{bmatrix}$$

11. 4 Transfer Function of a Discrete Time System From State Model

Consider the state equation,

$$\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$$

On taking $\mathcal{Z}$-transform of above equation with zero initial condition we get,

$$z\mathbf{Q}(z) = \mathbf{A}\,\mathbf{Q}(z) + \mathbf{B}\,\mathbf{X}(z)$$

$$z\mathbf{Q}(z) - \mathbf{A}\,\mathbf{Q}(z) = \mathbf{B}\,\mathbf{X}(z)$$

$(z\mathbf{I} - \mathbf{A})\,\mathbf{Q}(z) = \mathbf{B}\,\mathbf{X}(z)$, where $\mathbf{I}$ is the unit matrix.

On premultiplying the above equation by $(z\mathbf{I} - \mathbf{A})^{-1}$, we get,

$$\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\, \mathbf{B}\,\mathbf{X}(z) \qquad .....(11.5)$$

Consider the output equation,

$$\mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)$$

On taking $\mathcal{Z}$-transform of above equation we get,

$$\mathbf{Y}(z) = \mathbf{C}\,\mathbf{Q}(z) + \mathbf{D}\,\mathbf{X}(z) \qquad \text{.....(11.6)}$$

On substituting for $\mathbf{Q}(z)$ from equation (11.5) in (11.6) we get,

$$\mathbf{Y}(z) = \mathbf{C}\,(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z) + \mathbf{D}\,\mathbf{X}(z)$$

$$\therefore\ \mathbf{Y}(z) = [\,\mathbf{C}\,(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}\,]\ \mathbf{X}(z)$$

For single input and single output system,

$$\boxed{\frac{Y(z)}{X(z)} = \mathbf{C}\left(z\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}} \qquad \text{.....(11.7)}$$

The equation (11.7) is the ***transfer function*** of discrete time system.

11. 5 Solution of State Equations and Response of Discrete Time System

Solution of State Equations in time domain

Consider the state equations of discrete time system.

$$\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$$

when $n = 0$; $\mathbf{Q}(1) = \mathbf{A}\,\mathbf{Q}(0) + \mathbf{B}\,\mathbf{X}(0)$ .....(11.8)

where is $\mathbf{Q}(0)$ initial condition state vector

when $n = 1$;

$$\begin{aligned}\mathbf{Q}(2) &= \mathbf{A}\,\mathbf{Q}(1) + \mathbf{B}\,\mathbf{X}(1)\\ &= \mathbf{A}\,[\,\mathbf{A}\,\mathbf{Q}(0) + \mathbf{B}\,\mathbf{X}(0)\,] + \mathbf{B}\,\mathbf{X}(1)\\ &= \mathbf{A}^2\,\mathbf{Q}(0) + \mathbf{A}\,\mathbf{B}\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{X}(1) \qquad \text{.....(11.9)}\end{aligned}$$

Substituting for $\mathbf{Q}(1)$ from equation (11.8)

when $n = 2$;

$$\begin{aligned}\mathbf{Q}(3) &= \mathbf{A}\,\mathbf{Q}(2) + \mathbf{B}\,\mathbf{X}(2)\\ &= \mathbf{A}\,[\,\mathbf{A}^2\,\mathbf{Q}(0) + \mathbf{A}\,\mathbf{B}\,\mathbf{X}(0) + \mathbf{B}\,\mathbf{X}(1)\,] + \mathbf{B}\,\mathbf{X}(2)\\ &= \mathbf{A}^3\,\mathbf{Q}(0) + \mathbf{A}^2\,\mathbf{B}\,\mathbf{X}(0) + \mathbf{A}\,\mathbf{B}\,\mathbf{X}(1) + \mathbf{B}\,\mathbf{X}(2)\end{aligned}$$

Substituting for $\mathbf{Q}(2)$ from equation (11.9)

Hence, in general the solution of state equation for any value of n is given by,

$$\boxed{Q(n) = \underbrace{\mathbf{A}^{n}\,\mathbf{Q}(0)}_{\text{Solution due to initial condition}} + \underbrace{\mathbf{A}^{n-1}\,\mathbf{B}\,X(0) + \mathbf{A}^{n-2}\,\mathbf{B}\,X(1) + \mathbf{A}^{n-3}\,\mathbf{B}\,\mathbf{X}(2) + \ldots + \mathbf{A}\,\mathbf{B}\,\mathbf{X}(n-2) + \mathbf{B}\,\mathbf{X}(n-1)}_{\text{Solution due to input}}} \qquad \text{.....(11.10)}$$

The equation (11.10) is the solution of state equations of discrete time system in time domain.

Here, the matrix $\mathbf{A}^n$ is called ***state transition matrix***.

Solution of State Equations Using Ƶ-Transform

Consider the state equations,

$$\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$$

$$\boxed{\mathcal{Z}\,\{x(n+1)\} = z\,X(z) - z\,X(0)}$$

On taking Ƶ-transform of above equation, with initial conditions state vector as $\mathbf{Q}(0)$.

$$z\,\mathbf{Q}(z) - z\,\mathbf{Q}(0) = \mathbf{A}\,\mathbf{Q}(z) + \mathbf{B}\,\mathbf{X}(z)$$

$$z\,\mathbf{Q}(z) - \mathbf{A}\,\mathbf{Q}(Z) = \mathbf{B}\,\mathbf{X}(z) + z\,\mathbf{Q}(0)$$

$$[z\,\mathbf{I} - \mathbf{A}]\,\mathbf{Q}(z) = z\,\mathbf{Q}(0) + \mathbf{B}\,\mathbf{X}(z), \quad \text{where } \mathbf{I} \text{ is the unit matrix}$$

On premultiplying the above equation by $(z\,\mathbf{I} - \mathbf{A})^{-1}$ we get,

$$\boxed{\mathbf{Q}(z) = \underbrace{(z\mathbf{I} - \mathbf{A})^{-1}\,z\,\mathbf{Q}(0)}_{\substack{\text{Solution due to}\\ \text{initial condition}}} + \underbrace{(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\;\mathbf{X}(z)}_{\substack{\text{Solution due to}\\ \text{input } x(n)}}} \qquad \text{......(11.11)}$$

The equation (11.11) is the solution of state equations in z-domain. The solution of state equations in time domain is given by inverse Ƶ-transform of $\mathbf{Q}(z)$.

$$\therefore\ \mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{\mathbf{Q}(z)\right\}$$

$$= \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,z\,\mathbf{Q}(0) + (z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\right\}$$

$$= \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,z\right\}\mathbf{Q}(0) + \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\right\}$$

$$\boxed{\text{State equations, } \mathbf{Q}(n) = \underbrace{\mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,z\right\}\mathbf{Q}(0)}_{\substack{\text{Solution due to}\\ \text{initial condition}}} + \underbrace{\mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\right\}}_{\substack{\text{Solution due to}\\ \text{input } x(n)}}} \qquad \text{.....(11.12)}$$

The equation (11.12) is the time domain solution of state equations of discrete time system, which is obtained via Ƶ-transform.

On comparing equations (11.10) and (11.12) we get,

$$\boxed{\text{State Transition Matrix, } \mathbf{A}^n = \mathcal{Z}^{-1}\left\{(z\mathbf{I} - \mathbf{A})^{-1}\,z\right\}} \qquad \text{......(11.13)}$$

Response of Discrete Time System

The response, $\mathbf{Y}(n)$ of discrete time system can be computed by substituting the solution of state equations $\mathbf{Q}(n)$ from equation (11.10) or (11.12) in the output equation shown below.

$$\boxed{\therefore\ \text{Response of discrete time system, } \mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)}$$

11. 6 Solved Problems in State Space Analysis of Discrete Time System

Example 11.1

Determine the state model of the system governed by the equation.

$$y(n) = -2\,y(n-1) + 3\,y(n-2) + 0.5\,y(n-3) + 2\,x(n) + 1.5\,x(n-1) + 2.5\,x(n-2) + 4\,x(n-3)$$

Solution

Given that,

$$y(n) = -2\,y(n-1) + 3\,y(n-2) + 0.5\,y(n-3) + 2\,x(n) + 1.5\,x(n-1) + 2.5\,x(n-2) + 4\,x(n-3)$$

The direct form-II structure of the system described by the above equation is shown in fig 1.

The choice of number of state variables is equal to number of delay units. The direct form-II structure has one input x(n), one output y(n) and three delay units, and so let us choose three state variables. Let, $q_1(n)$, $q_2(n)$ and $q_3(n)$ be the three state variables. Assign state variables at the output of every delay unit. Hence the $(n+1)^{th}$ value of state variable will be available at the input of delay unit.

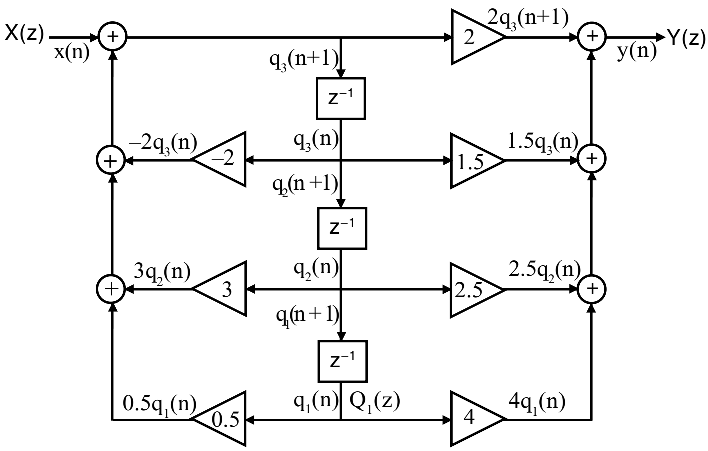


***Fig 1 :** Direct form-II structure.*

State Equations

The state equations are formed by equating the sum of incoming signals of delay unit to $(n+1)^{th}$ value of state variable, as shown below.

$$q_1(n+1) = q_2(n)$$
$$q_2(n+1) = q_3(n)$$
$$q_3(n+1) = 0.5\,q_1(n) + 3\,q_2(n) - 2\,q_3(n) + x(n)$$

On arranging the state equations in the matrix form we get,

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 3 & -2 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [x(n)]$$

Output Equations

Output equation y(n) is formed by equating incoming signals of output node / summing point to y(n) as shown below.

$$y(n) = 2\,q_3(n+1) + 4\,q_1(n) + 2.5\,q_2(n) + 1.5\,q_3(n)$$

On substituting for $q_3(n+1)$ from state equation we get,

$$y(n) = 2\big[0.5\,q_1(n) + 3\,q_2(n) - 2\,q_3(n) + x(n)\big] + 4\,q_1(n) + 2.5\,q_2(n) + 1.5\,q_3(n)$$

$$= q_1(n) + 6\,q_2(n) - 4\,q_3(n) + 2\,x(n) + 4\,q_1(n) + 2.5\,q_2(n) + 1.5\,q_3(n)$$

$$\therefore\ y(n) = 5\,q_1(n) + 8.5\,q_2(n) - 2.5\,q_3(n) + 2\,x(n)$$

On arranging the output equation in the matrix form we get,

$$y(n) = \begin{bmatrix} 5 & 8.5 & -2.5 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + [2]\,[x(n)]$$

State Model

State model of a discrete time system is given by state equations and output equations.

State equations : $\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$

$$\Downarrow$$

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.5 & 3 & -2 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [x(n)]$$

Output equation : $\mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)$

$$\Downarrow$$

$$y(n) = \begin{bmatrix} 5 & 8.5 & -2.5 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + [2]\,[x(n)]$$

Example 11.2

The transfer function of a system is given by, $H(z) = \dfrac{1.5 + 2z^{-1} + 3z^{-2} + 2z^{-3}}{1 + 3z^{-1} + 2z^{-2} + 4z^{-3}}$

Determine the state model of the system.

Solution

Let, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1.5 + 2z^{-1} + 3z^{-2} + 2z^{-3}}{1 + 3z^{-1} + 2z^{-2} + 4z^{-3}}$

On cross multiplication we get,

$$Y(z) + 3z^{-1}\,Y(z) + 2z^{-2}\,Y(z) + 4z^{-3}\,Y(z) = 1.5\,X(z) + 2z^{-1}\,X(z) + 3z^{-2}\,X(z) + 2z^{-3}\,X(z)$$

$$\therefore\ Y(z) = -3z^{-1}\,Y(z) - 2z^{-2}\,Y(z) - 4z^{-3}\,Y(z) + 1.5\,X(z) + 2z^{-1}\,X(z) + 3z^{-2}\,X(z) + 2z^{-3}\,X(z)$$

The direct form-II structure of the system described by the above equation is shown in fig 1.

The choice of number of state variables is equal to number of delay units. The direct form-II structure has one input x(n), one output y(n) and three delay units, and so let us choose three state variables. Let, $q_1(n)$, $q_2(n)$ and $q_3(n)$ be the three state variables. Assign state variables at the output of every delay unit. Hence the $(n + 1)^{th}$ value of state variable will be available at the input of delay unit.

State Equations

The state equations are formed by equating the sum of incoming signals of delay unit to $(n+1)^{th}$ value of state variable, as shown below.

$$q_1(n+1) = q_2(n)$$

$$q_2(n+1) = q_3(n)$$

$$q_3(n+1) = -4\,q_1(n) - 2\,q_2(n) - 3\,q_3(n) + x(n)$$

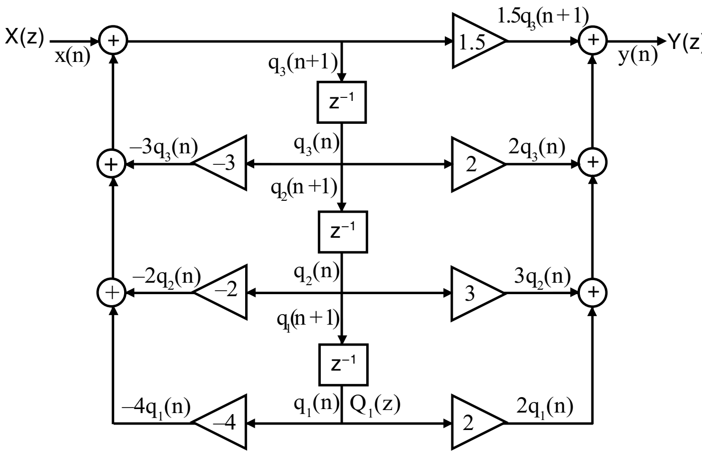


Fig 1 : *Direct form-II structure.*

On arranging the state equations in the matrix form, we get

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & -3 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [x(n)]$$

Output Equations

The output equation y(n) is formed by equating the incoming signals of output node to y(n) as shown below.

$$y(n) = 1.5\, q_3(n+1) + 2\, q_1(n) + 3\, q_2(n) + 2\, q_3(n)$$

On substituting for $q_3(n+1)$ from state equation we get,

$$y(n) = 1.5\left[-4\, q_1(n) - 2\, q_2(n) - 3\, q_3(n) + x(n)\right] + 2\, q_1(n) + 3\, q_2(n) + 2\, q_3(n)$$

$$y(n) = -6\, q_1(n) - 3\, q_2(n) - 4.5\, q_3(n) + 1.5\, x(n) + 2\, q_1(n) + 3\, q_2(n) + 2\, q_3(n)$$

$$y(n) = -4\, q_1(n) - 2.5\, q_3(n) + 1.5\, x(n)$$

On arranging the above equations in the matrix form we get,

$$y(n) = \begin{bmatrix} -4 & 0 & -2.5 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + [1.5]\,[x(n)]$$

State Model

State model of a discrete time system is given by state equations and output equations.

State equations : $\mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n) + \mathbf{B}\,\mathbf{X}(n)$

$$\Downarrow$$

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -2 & -3 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [x(n)]$$

Output equation : $\mathbf{Y}(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)$

$$\Downarrow$$

$$y(n) = \begin{bmatrix} -4 & 0 & -2.5 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + [1.5]\,[x(n)]$$

Example 11.3

The state space representation of a discrete time system is given by,

$$\mathbf{A} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \; ; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; ; \quad \mathbf{C} = \begin{bmatrix} 1 & 3 \end{bmatrix} \; ; \quad \mathbf{D} = [3]$$

Derive the transfer function of the system.

Solution

The transfer function of the discrete time system is given by,

$$\frac{Y(z)}{X(z)} = \mathbf{C}\,(z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$z\mathbf{I} - \mathbf{A} = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} z-2 & 1 \\ -3 & z-1 \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\therefore \; \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix} z-2 & 1 \\ -3 & z-1 \end{vmatrix}} \begin{bmatrix} z-1 & -1 \\ 3 & z-2 \end{bmatrix}$$

$$= \frac{1}{(z-2)\,(z-1) - (-3) \times 1} \begin{bmatrix} z-1 & -1 \\ 3 & z-2 \end{bmatrix}$$

$$= \frac{1}{z^2 - 3z + 5} \begin{bmatrix} z-1 & -1 \\ 3 & z-2 \end{bmatrix} = \begin{bmatrix} \dfrac{z-1}{z^2 - 3z + 5} & \dfrac{-1}{z^2 - 3z + 5} \\ \dfrac{3}{z^2 - 3z + 5} & \dfrac{z-2}{z^2 - 3z + 5} \end{bmatrix}$$

$$\therefore \; \frac{Y(z)}{X(z)} = \mathbf{C}\,(z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} \dfrac{z-1}{z^2 - 3z + 5} & \dfrac{-1}{z^2 - 3z + 5} \\ \dfrac{3}{z^2 - 3z + 5} & \dfrac{z-2}{z^2 - 3z + 5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3]$$

$$= \begin{bmatrix} \dfrac{z - 1 + 9}{z^2 - 3z + 5} & \dfrac{-1 + 3(z-2)}{z^2 - 3z + 5} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + [3]$$

$$= \begin{bmatrix} \dfrac{z - 1 + 9}{z^2 - 3z + 5} + \dfrac{2(-1 + 3(z-2))}{z^2 - 3z + 5} \end{bmatrix} + [3]$$

$$= \frac{z - 1 + 9 + 2(-1 + 3(z-2)) + 3(z^2 - 3z + 5)}{z^2 - 3z + 5}$$

$$= \frac{z - 1 + 9 - 2 + 6z - 12 + 3z^2 - 9z + 15}{z^2 - 3z + 5} = \frac{3z^2 - 2z + 9}{z^2 - 3z + 5}$$

Example 11.4

The state model of a discrete time system is given by,

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \; ; \quad \mathbf{C} = \begin{bmatrix} 1 & 3 \end{bmatrix}; \quad \mathbf{D} = 3$$

Find the response of the discrete time system for unit step input. Assume zero initial condition.

Solution

The response, $\mathbf{Y}(n) = \mathbf{CQ}(n) + \mathbf{D\,X}(n)$

$$\mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1} z\,\mathbf{Q}(0) + (z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\right\}$$

Here $\mathbf{Q}(0) = 0$, (because initial conditions are zero).

$$\therefore\ \mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\right\}$$

$$z\mathbf{I} - \mathbf{A} = z\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix} = \begin{bmatrix}z & 0\\ 0 & z\end{bmatrix} - \begin{bmatrix}2 & 0\\ 3 & 1\end{bmatrix} = \begin{bmatrix}z-2 & 0\\ -3 & z-1\end{bmatrix}$$

$$(z\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\begin{vmatrix}z-2 & 0\\ -3 & z-1\end{vmatrix}}\begin{bmatrix}z-1 & 0\\ 3 & z-2\end{bmatrix}$$

$$= \frac{1}{(z-2)\,(z-1)}\begin{bmatrix}z-1 & 0\\ 3 & z-2\end{bmatrix}$$

$$= \begin{bmatrix}\dfrac{1}{z-2} & 0\\ \dfrac{3}{(z-1)\,(z-2)} & \dfrac{1}{z-1}\end{bmatrix}$$

> Let, **P** be a square matrix.
>
> Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$
>
> If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.
>
> $$\mathbf{P} = \begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}$$
>
> $$\therefore\ \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{vmatrix}} \times \begin{bmatrix}p_{22} & -p_{12}\\ -p_{21} & p_{11}\end{bmatrix}$$

Given that, $x(n) = u(n)$,

$$\therefore\ X(z) = \mathcal{Z}\{x(n)\} = \mathcal{Z}\{u(n)\} = \frac{z}{z-1}$$

Now, $\mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\ \mathbf{X}(z)$

$$\therefore \mathbf{Q}(z) = (z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\ \mathbf{X}(z)$$

$$= \begin{bmatrix}\dfrac{1}{z-2} & 0\\ \dfrac{3}{(z-1)\,(z-2)} & \dfrac{1}{z-1}\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}\begin{bmatrix}\dfrac{z}{z-1}\end{bmatrix} = \begin{bmatrix}\dfrac{1}{z-2} + 0\\ \dfrac{3}{(z-1)\,(z-2)} + \dfrac{2}{z-1}\end{bmatrix}\begin{bmatrix}\dfrac{z}{z-1}\end{bmatrix}$$

$$= \begin{bmatrix}\dfrac{1}{z-2}\\ \dfrac{3 + 2\,(z-2)}{(z-1)\,(z-2)}\end{bmatrix}\begin{bmatrix}\dfrac{z}{z-1}\end{bmatrix} = \begin{bmatrix}\dfrac{1}{z-2}\\ \dfrac{2z-1}{(z-1)\,(z-2)}\end{bmatrix}\begin{bmatrix}\dfrac{z}{z-1}\end{bmatrix} = \begin{bmatrix}\dfrac{z}{(z-1)\,(z-2)}\\ \dfrac{z(2z-1)}{(z-1)^2\,(z-2)}\end{bmatrix}$$

$$\mathbf{Q}(n) = \mathcal{Z}^{-1}\{\mathbf{Q}(z)\} = \mathcal{Z}^{-1}\begin{bmatrix}\dfrac{z}{(z-1)\,(z-2)}\\ \dfrac{z(2z-1)}{(z-1)^2\,(z-2)}\end{bmatrix} = \begin{bmatrix}\mathcal{Z}^{-1}\left\{\dfrac{z}{(z-1)\,(z-2)}\right\}\\ \mathcal{Z}^{-1}\left\{\dfrac{z(2z-1)}{(z-1)^2\,(z-2)}\right\}\end{bmatrix}$$

> Let, $\dfrac{1}{(z-1)\,(z-2)} = \dfrac{k_1}{z-1} + \dfrac{k_2}{z-2}$
>
> where, $k_1 = \dfrac{1}{(z-1)\,(z-2)} \times (z-1)\Big|_{z=1} = \dfrac{1}{1-2} = -1$
>
> $k_2 = \dfrac{1}{(z-1)\,(z-2)} \times (z-2)\Big|_{z=2} = \dfrac{1}{2-1} = 1$
>
> $\therefore \dfrac{1}{(z-1)\,(z-2)} = \dfrac{-1}{z-1} + \dfrac{1}{z-2} \quad \Rightarrow \quad \dfrac{z}{(z-1)\,(z-2)} = \dfrac{-z}{z-1} + \dfrac{z}{z-2}$

Let, $\dfrac{2z-1}{(z-1)^2(z-2)} = \dfrac{k_a}{(z-1)^2} + \dfrac{k_b}{(z-1)} + \dfrac{k_c}{(z-2)}$

$$\text{where, } k_a = \left.\frac{2z - 1}{(z-1)^2\ (z-2)} \times (z-1)^2\right|_{z=1} = \frac{2-1}{1-2} = -1$$

$$k_b = \left[\frac{d}{dz}\left(\frac{2z-1}{(z-1)^2\ (z-2)} \times (z-1)^2\right)\right]_{z=1} = \left[\frac{d}{dz}\left(\frac{2z-1}{z-2}\right)\right]_{z=1}$$

$$= \left.\frac{(z-2)\,2 - (2z-1) \times 1}{(z-2)^2}\right|_{z=1} = \frac{-1 \times 2 - 1}{(-1)^2} = -3$$

$$k_c = \left.\frac{2z - 1}{(z-1)^2\ (z-2)} \times (z-2)\right|_{z=2} = \frac{2 \times 2 - 1}{(2-1)^2} = 3$$

$$\therefore \quad \frac{2z-1}{(z-1)^2\ (z-2)} = \frac{-1}{(z-1)^2} + \frac{-3}{z-1} + \frac{3}{z-2}$$

$$\therefore \quad \frac{z(2z-1)}{(z-1)^2(z-2)} = \frac{-z}{(z-1)^2} + \frac{-3z}{z-1} + \frac{3z}{z-2}$$

$$\mathbf{Q}(n) = \begin{bmatrix} \mathcal{Z}^{-1}\left\{\dfrac{z}{(z-1)\ (z-2)}\right\} \\ \mathcal{Z}^{-1}\left\{\dfrac{z(2z-1)}{(z-1)^2\ (z-2)}\right\} \end{bmatrix} = \begin{bmatrix} \mathcal{Z}^{-1}\left\{\dfrac{-z}{z-1} + \dfrac{z}{z-2}\right\} \\ \mathcal{Z}^{-1}\left\{\dfrac{-z}{(z-1)^2} - 3\,\dfrac{z}{z-1} + 3\,\dfrac{z}{z-2}\right\} \end{bmatrix}$$

$$= \begin{bmatrix} -u(n) + 2^n\ u(n) \\ -nu(n) - 3u(n) + 3(2)^n u(n) \end{bmatrix} = \begin{bmatrix} (2^n - 1)\ u(n) \\ [3(2^n-1) - n]\ u(n) \end{bmatrix}$$

Now, the response of the discrete time system is given by,

Response, $y(n) = \mathbf{C}\,\mathbf{Q}(n) + \mathbf{D}\,\mathbf{X}(n)$

$$= \begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} (2^n - 1)\ u(n) \\ \left[3(2^n-1) - n\right] u(n) \end{bmatrix} + \begin{bmatrix} 3 \end{bmatrix}\begin{bmatrix} u(n) \end{bmatrix}$$

$$= (2^n - 1)\ u(n) + 3\left[3(2^n-1) - n\right] u(n) + 3u(n)$$

$$= \left[(2^n-1) + 9\,(2^n-1) - 3n + 3\right] u(n)$$

$$= \left[10(2^n-1) - 3(n-1)\right] u(n)$$

Example 11.5

If system matrix, $\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$. Find the state transition matrix of discrete time system.

Solution

The state transition matrix, $A^n = \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - \mathbf{A})^{-1}\, z\right\}$

$$zI - A = z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} z+3 & 0 \\ 0 & z+2 \end{bmatrix}$$

$$(zI - A)^{-1} = \frac{1}{\begin{vmatrix} z+3 & 0 \\ 0 & z+2 \end{vmatrix}} \begin{bmatrix} z+2 & 0 \\ 0 & z+3 \end{bmatrix}$$

$$= \frac{1}{(z+3)(z+2)} \begin{bmatrix} z+2 & 0 \\ 0 & z+3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{z+3} & 0 \\ 0 & \frac{1}{z+2} \end{bmatrix}$$

Let, **P** be a square matrix.

Now, $\mathbf{P}^{-1} = \dfrac{\text{Transpose of Cofactor Matrix of } \mathbf{P}}{\text{Determinant of } \mathbf{P}}$

If, **P** is a square matrix of size 2 × 2, then its cofactor matrix is obtained by interchanging the elements of main diagonal and changing the sign of other two elements as shown in the following example.

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

$$\therefore \mathbf{P}^{-1} = \frac{1}{\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix}} \times \begin{bmatrix} p_{22} & -p_{12} \\ -p_{21} & p_{11} \end{bmatrix}$$

$$A^n = \mathcal{Z}^{-1}\left\{(z\,\mathbf{I} - A)^{-1} z\right\}$$

$$= \mathcal{Z}^{-1}\left\{ \begin{bmatrix} \frac{1}{z+3} & 0 \\ 0 & \frac{1}{z+2} \end{bmatrix} z \right\}$$

$$= \begin{bmatrix} \mathcal{Z}^{-1}\left\{\frac{z}{z+3}\right\} & 0 \\ 0 & \mathcal{Z}^{-1}\left\{\frac{z}{z+2}\right\} \end{bmatrix} = \begin{bmatrix} (-3)^n u(n) & 0 \\ 0 & (-2)^n u(n) \end{bmatrix}$$

11.7 Summary of Important Concepts

1. The state of a discrete time system refers to the condition of discrete time system at any discrete time instant.
2. The state variables of a discrete time system are a set of variables that completely describe the state of discrete time system at any discrete time instant.
3. The state equations of a discrete time system are a set of N-number of first order difference equations.
4. The state equations of a discrete time system are formed by taking next state of state variables as function of present state of state variables and present inputs.
5. The output equations of a discrete time system are a set of P-number of algebraic equations formed by taking outputs as function of state variables and inputs.
6. The state model of a discrete time system is given by state equations and output equations.
7. The transfer function of a discrete time system can be obtained from its state model using the equation $\mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B} + \mathbf{D}$.
8. The state transition matrix of a discrete time system is, $\mathbf{A}^n = \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1} z\}$.
9. The solution of state equations of a discrete time system due to initial condition (and with no input) is, $\mathbf{Q}(n) = \mathbf{A}^n\,\mathbf{Q}(0)$.
10. The solution of state equations of a discrete time system due to input (and with zero initial conditions) is, $\mathbf{Q}(n) = \mathcal{Z}^{-1}\{(z\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{B}\,\mathbf{X}(z)\}$.

11. 8 Short Questions And Answers

Q11.1 What are the advantages of state space analysis of discrete time system?

1. The state space analysis is applicable to any type of systems. They can be used for modelling and analysis of linear & non-linear systems, time invariant & time variant systems and multiple input & multiple output systems.
2. The state space analysis can be performed with initial conditions.
3. The variables that are used to represent the system can be any variable in the system.
4. Using this analysis the internal states of the system at any time instant can be predicted.

Q11.2 ***What is state vector of a discrete time system?***

The state vector of a discrete time system is a N×1 column matrix (or vector) whose elements are state variables of the system, (where N is the order of the system). It is denoted by **Q**(n).

Q11.3 ***What is state space of a discrete time system?***

The set of all possible values which the state vector of a discrete time system **Q**(n) can have (or assume) at any discrete time n forms the state space of the discrete time system.

Q11.4 ***What is input and output space of a discrete time system?***

The set of all possible values which the input vector of a discrete time system **X**(n) can have (or assume) at any discrete time n forms the input space of the discrete time system.

The set of all possible values which the output vector of a discrete time system **Y**(n) can have (or assume) at any discrete time n forms the output space of the discrete time system.

Q11.5 ***Determine system matrix of the system governed by the difference equation, $y(n-2)+2y(n-1)+4y(n)=0$.***

Solution

Given that, $y(n-2) + 2\,y(n-1) + 4\,y(n) = 0$

The given system is second order system and so let us choose two state variables $q_1(n)$ and $q_2(n)$. Let us equate the state variables to system variable y(n) and its delayed versions y(n–1) and y(n–2).

$$\therefore\ q_1(n) = y(n-2) \quad\Rightarrow\quad q_1(n+1) = y(n-1)$$

$$q_2(n) = y(n-1) \quad\Rightarrow\quad q_2(n+1) = y(n) \quad \text{and} \quad q_2(n) = q_1(n+1) \qquad .....(1)$$

On substituting the state variables and their shifted versions in the given system equation we get,

$$q_1(n) + 2\,q_1(n+1) + 4\,q_2(n+1) = 0 \quad\Rightarrow\quad 4\,q_2(n+1) = -q_1(n) - 2\,q_1(n+1)$$

$$\therefore\ q_2(n+1) = -\frac{1}{4}q_1(n) - \frac{2}{4}\,q_1(n+1) \quad\Rightarrow\quad q_2(n+1) = -\frac{1}{4}q_1(n) - \frac{1}{2}\,q_2(n) \qquad \boxed{\text{Using equation (1)}}$$

Now the state equations are,

$$q_1(n+1) = q_2(n)$$

$$q_2(n+1) = -\frac{1}{4}q_1(n) - \frac{1}{2}\,q_2(n) \quad\Rightarrow\quad \begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} \Rightarrow \mathbf{Q}(n+1) = \mathbf{A}\,\mathbf{Q}(n)$$

Therefore, the system matrix, $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{2} \end{bmatrix}$

Q11.6 ***Determine state model of the system shown in fiq Q11.6.1.***

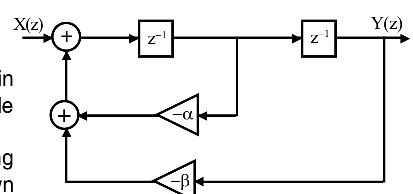


Fig Q11.6.1.

Solution

Let us assign state variables at the output of delay units as shown in figQ11.6.2. Hence the $(n+1)^{th}$ value of state variables will be available at the input of delay units.

The state equations are formed by equating the sum of incoming signals of delay unit to $(n + 1)^{th}$ value of state variables as shown below.

$$q_1(n+1) = q_2(n)$$

$$q_2(n+1) = -\beta q_1(n) - \alpha\, q_2(n) + x(n)$$

Here the output equation is, $y(n) = q_1(n)$.

Now the state model in matrix form is,

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\beta & -\alpha \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x(n)$$

$$y(n) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix}$$

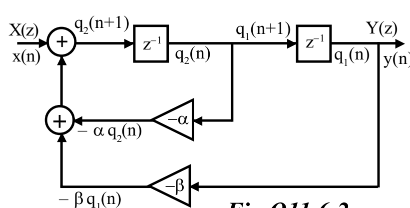


Fig Q11.6.2.

Q11.7 ***Determine state model of the system shown in fiq Q11.7.1.***

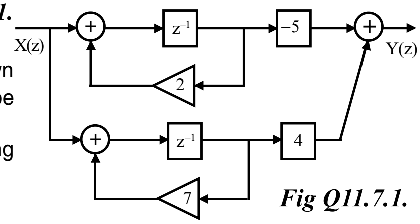


Fig Q11.7.1.

Solution

Let us assign state variables at the output of delay units as shown in figQ11.7.2. Hence the (n + 1)th value of state variables will be available at the input of delay units.

The state equations are formed by equating the sum of incoming signals of delay unit to (n + 1)th value of state variables.

$$q_1(n+1) \;=\; 7\,q_1(n) + x(n)$$
$$q_2(n+1) \;=\; 2\,q_2(n) + x(n)$$

Here the output equation is, $y(n) = 4q_1(n)–5q_2(n)$.

Now the state model in matrix form is,

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x(n)$$

$$y(n) \;=\; \begin{bmatrix} 4 & -5 \end{bmatrix} \begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix}$$

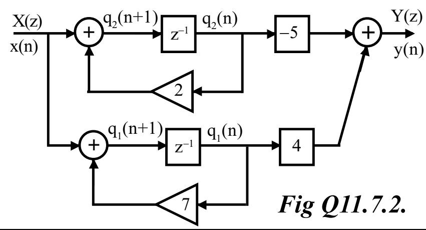


Fig Q11.7.2.

Q11.8 ***Determine the transfer function of the system described by the following state model.***

$$q(n+1) \;=\; -2q(n) + 0.5\,x(n) \;\; ; \qquad y(n) \;=\; 0.7\,q(n)$$

Solution

Given that, $q(n+1) = -2q(n) + 0.5x(n)$. On taking $\mathcal{Z}$-transform we get,

$$zQ(z) \;=\; -2Q(z) + 0.5X(z) \quad \Rightarrow \quad zQ(z) + 2Q(z) = 0.5X(z) \quad \Rightarrow \quad Q(z) = \frac{0.5}{z+2}\,X(z) \qquad .....(1)$$

Given that, $y(n) = 0.7q(n)$. On taking $\mathcal{Z}$-transform we get, $Y(z) \;=\; 0.7\,Q(z)$ .....(2)

From equations (1) and (2) we can write,

$$\text{Transfer function,} \quad \frac{Y(z)}{X(z)} = 0.7 \times \frac{0.5}{z+2} = \frac{0.35}{z+2}$$

Q11.9 ***The output equation of a discrete time system is, y(n) = [1 1] Q(n). If the state transition matrix,*** $A^n = \begin{bmatrix} (-4)^n u(n) & (3)^n u(n) \\ (-2)^n u(n) & 0 \end{bmatrix}$ ***and the initial condition vector,*** $Q(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$***, Find the zero-input response of the system.***

Solution

We know that, $\mathbf{Q}(n) = A^n\ \mathbf{Q}(0)$.

$$\therefore \text{Zero - input response, } y_{zi}(n) \;=\; \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{Q}(n) = \begin{bmatrix} 1 & 1 \end{bmatrix}\ \mathbf{A}^n\ \mathbf{Q}(0)$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} (-4)^n u(n) & (-3)^n u(n) \\ (2)^n u(n) & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (-4)^n u(n) + (2)^n u(n) & (-3)^n u(n) \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= 2(-4)^n u(n) + 2(2)^n u(n) + (-3)^n u(n) = \Big[\, 2\ \big((-4)^n + (2)^n\big) + (-3)^n \Big]\ u(n)$$

Q11.10 ***The output equation of a system is, y(n) = [1 2] Q(n) + [2] X(n).***

If the state vector, $Q(n) = \begin{bmatrix} (-2)^n u(n) \\ (-1)^n u(n) \end{bmatrix}$***.Find the zero-state response of the system for unit step input.***

Solution

$$\text{Zero - state response, } y_{zs}(n) \;=\; \begin{bmatrix} 1 & 2 \end{bmatrix} \mathbf{Q}(n) \;+\; \begin{bmatrix} 2 \end{bmatrix} \mathbf{X}(n) \;=\; \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} (-2)^n u(n) \\ (-1)^n u(n) \end{bmatrix} + \begin{bmatrix} 2 \end{bmatrix}\ u(n)$$

$$= (-2)^n u(n) + 2(-1)^n u(n) + 2 \times u(n) \;=\; \Big[(-2)^n + 2(-1)^n + 2\Big] u(n)$$

11.9 MATLAB Programs

Program 11.1

Write a MATLAB program to find the state model of the discrete time system governed by the transfer function,

H(s)= $(1.5z^3+2z^2+3z+2)/(z^3+3z^2+2z+4)$.

```
%Program to determine the state model from the transfer function

clear all
H=tf('z');
Ts = 0.1;
disp('Enter numerator coefficients of given transfer function');
b=input('');
disp('Enter denominator coefficients of given transfer function');
a=input('');
disp('The given transfer function is,');
H=tf([b], [a], Ts)                %display the given transfer function
[A,B,C,D]=tf2ss(b,a)              %compute matrices A,B,C,D of state model
```

OUTPUT

```
Enter numerator coefficients of given transfer function
      [1.5  2  3  2]

Enter denominator coefficients of given transfer function
      [1  3  2  4]

The given transfer function is,
      Transfer function:
                    1.5 z^3 + 2 z^2 + 3 z + 2
                    -------------------------
                      z^3 + 3 z^2 + 2 z + 4
Sampling time: 0.1
A =
   -3    -2    -4
    1     0     0
    0     1     0
B =
    1
    0
    0
C =
   -2.5000         0   -4.0000
D =
    1.5000
```

Program 11.2

Write a MATLAB program to determine the transfer function of a system by getting the state model from the user through keyboard.

```
%Program to determine the transfer function of Nth order system.
clear all
syms z
Ts=0.1;

disp('Enter elements of the system matrix A of size NxN');
disp('A='); A=input('');
disp('Enter elements of the input matrix B of size Nx1');
disp('B='); B=input('');
```

```
disp('Enter elements of the output matrix C of size 1xN');
disp('C='); C=input('');
disp('Enter elements of the transmission matrix D of size 1x1');
disp('D='); D=input('');
disp('Coefficients of numerator(b) and denominator(a) are,');
[b,a]=ss2tf(A,B,C,D)
disp('The transfer function of the state model is,');
tf(b,a,Ts)
```

OUTPUT

```
Enter elements of the system matrix A of size NxN
 A=
    [0 1 0; 0 0 1; -4 -2 -3]
Enter elements of the input matrix B of size Nx1
 B=
    [0; 0; 1]
Enter elements of the output matrix C of size 1xN
 C=
    [-4 0 -2.5]
Enter elements of the transmission matrix D of size 1x1
 D=
    [1.5]

Coefficients of numerator(b) and denominator(a) are,
b =
    1.5000    2.0000    3.0000    2.0000
a =
    1.0000    3.0000    2.0000    4.0000

The transfer function of the state model is,

      Transfer function:
                           1.5 z^3 + 2 z^2 + 3 z + 2
                           -------------------------
                              z^3 + 3 z^2 + 2 z + 4
Sampling time: 0.1
```

11.10 Exercises

I. Fill in the blanks with appropriate words

1. A set of variables that completely describes a discrete time system at any time instant are called ————.
2. The ————— of a discrete time system consists of the state equations and output equations.
3. The pictorial representation of the state model of a discrete time system is called ——————— .
4. The number of state variables in a state diagram of a discrete time system is equal to number of ————.
5. In vector notation, the state equation of a discrete time system is ——————.
6. In state model of a discrete time system, the number of ________ will be equal to order of the system.
7. In the matrix form of state equation of a discrete time system, **A** represents —————.
8. In the matrix form of output equation of a discrete time system, **C** represents —————.
9. The state transition matrix of a discrete time system, $\mathbf{A}^n$ =_____________.
10. In vector notation, the output equation of a discrete time system is ——————.

Answers

1. state variables 2. state model 3. state diagram 4. delay units 5. $\mathbf{Q}(n+1)=\mathbf{A}\,\mathbf{Q}(n)+\mathbf{B}\,\mathbf{X}(n)$
6. state variables 7. system matrix 8. output matrix 9. $\mathcal{Z}^{-1}\{(z\mathbf{I}-\mathbf{A})^{-1}z\}$ 10. $\mathbf{Y}(n)=\mathbf{C}\,\mathbf{Q}(n)+\mathbf{D}\,\mathbf{X}(n)$

II. State whether the following statements are True/False

1. The state space model of discrete time systems are quite analogous to state space model of continuous time systems.
2. The state space analysis of discrete time system permits multiple inputs and outputs.
3. The state model of a discrete time system is nonunique but the transfer function is unique.
4. The state equation of a discrete time system is a set of N numbers of first order differential equations.
5. In state space model of a discrete time system the value of state variables at $(k+1)^{th}$ time instant are functions of state variables and inputs at k^{th} instant of time.
6. In state space model of a discrete time system the value of output at $(k+1)^{th}$ time instant are functions of state variables and inputs at k^{th} instant of time.

Answers

1. True 2. True 3. True 4. False 5. True 6. False

III. Choose the right answer for the following questions

1. In the state equation of a discrete time system, $Q(n+1) = A\,Q(n) + B\,X(n)$, the input matrix is,

a) **B** **b)** **X**(n) **c)** **A** **d)** **Q**(n)

2. In the output equation of a discrete time system, $Y(n) = C\,Q(n) + D\,X(n)$, the output matrix is,

a) **D** **b)** **X**(n) **c)** **Q**(n) **d)** **C**

3. A discrete time system is described by four number of first order difference equations. The number of delay units present in the direct form-II structure is,

a) 2 **b)** 4 **c)** 1 **d)** 8

4. The number of state variables in the discrete time system governed by the difference equation, $y(n-3) + 2\,y(n) = 3\,x(n-2) + x(n)$ is,

a) 1 **b)** 2 **c)** 5 **d)** 3

5. The transfer function of a discrete time system is described by, $\dfrac{Y(z)}{X(z)} = \dfrac{z^4 + 4z + 3}{z^5 + 4z^3 + 1}$. The number of state variables in the state model of the system is,

a) 1 **b)** 5 **c)** 4 **d)** 2

6. The solution of state equation of a discrete time system with zero initial conditions is given by,

a) $\mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B}\ \mathbf{X}(z)\right\}$ **b)** $\mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{[z\mathbf{I}-\mathbf{A}]^{-1}\,z\,\mathbf{B}\ \mathbf{X}(z)\right\}$

c) $\mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{\mathbf{C}\,[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}\right\}$ **d)** $\mathbf{Q}(n) = \mathcal{Z}^{-1}\left\{[z\mathbf{I}-\mathbf{A}]^{-1}\,z\right\}\mathbf{Q}(0)$

7. If there is no input, the solution of state equation of a discrete time system is,

a) $\mathbf{Q}(n) = \mathbf{A}^n$ **b)** $\mathbf{Q}(n) = \mathbf{A}^n\,\mathbf{X}(0)$ **c)** $\mathbf{Q}(n) = \mathbf{A}^n\,\mathbf{Q}(0)$ **d)** $\mathbf{Q}(n) = \mathbf{A}^{-n}\,\mathbf{Q}(0)$

8. If A, B, C and D are the matrices that describe the state equation of a discrete time system, then the transfer function of the discrete time system is given by,

a) $\mathbf{A}[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}$ **b)** $\mathbf{C}[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}$ **c)** $[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}$ **d)** $[z\mathbf{I}-\mathbf{A}]^{-1}\,\mathbf{C} + \mathbf{D}$

9. The transfer function of the system having the following state variable description is,

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix};\quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix};\quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

a) $\dfrac{2}{(z-1)^2}$ **b)** $\dfrac{1}{z+1}$ **c)** $\dfrac{1}{z-1}$ **d)** $\dfrac{1}{(z+1)^2}$

10. The system matrix of a discrete time system is given by, $A=\begin{bmatrix}7 & 0\\ 0 & 3\end{bmatrix}$. ***The state transition matrix*** A^n ***is,***

a) $\begin{bmatrix}3^n u(n) & 0\\ 0 & 7^n u(n)\end{bmatrix}$ **b)** $\begin{bmatrix}0 & 3^n u(n)\\ 7^n u(n) & 0\end{bmatrix}$ **c)** $\begin{bmatrix}7^n u(n) & 0\\ 0 & 3^n u(n)\end{bmatrix}$ **d)** $\begin{bmatrix}7^n u(n) & 0\\ 3^n u(n) & 0\end{bmatrix}$

Answers

1. a	2. d	3. b	4. d	5. b
6. a	7. c	8.b	9. c	10. c

IV. Answer the following questions

1. Define state and state variables of a discrete time system.
2. What is state model of a discrete time system ?
3. Write the state model of N^{th} order discrete time system.
4. Write the state equations of N^{th} order discrete time system.
5. Write the output equations of N^{th} order discrete time system.
6. Explain the formation of state model of a discrete time system from the direct form-II structure.
7. How will you determine the transfer function of a discrete time system from its state model?.
8. Write the expression to compute the state transition matrix of a discrete time system via Ƶ-transform.
9. Write the expression for time domain solution of state equations of a discrete time system in terms of state transition matrix and using Ƶ-transform.

V. Solve the following problems

E11.1 Determine the state model of the discrete time systems governed by the equations,

a) $0.2\,y(n-3) - 0.4\,y(n-2) + 0.1\,y(n-1) + y(n) = 2\,x(n) + 2.5\,x(n-1) + 1.5\,x(n-2) + 3\,x(n-3)$

b) $y(n-2) + 5\,y(n-1) + 2\,y(n) = 3\,x(n-2) + 4\,x(n-1) + x(n)$

c) $3\,y(n-2) + 7\,y(n-1) + y(n) = 6\,x(n-2) + 3\,x(n-1) + 3\,x(n)$

E11.2 Compute the state transition matrix of discrete time system $\mathbf{A}^n$ for the following system matrices.

a) $\mathbf{A}=\begin{bmatrix}0 & 1\\ -5 & 6\end{bmatrix}$ **b)** $\mathbf{A}=\begin{bmatrix}-1 & 0\\ 0 & -3\end{bmatrix}$ **c)** $\mathbf{A}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$ **d)** $\mathbf{A}=\begin{bmatrix}-2 & 0\\ 0 & -1\end{bmatrix}$

E11.3 Obtain the state model of the discrete time systems whose transfer functions are given below.

a) $H(z)=\dfrac{\frac{1}{2}}{\frac{1}{2}z^{-2}+\frac{3}{2}z^{-1}+1}$ **b)** $H(z)=\dfrac{\frac{1}{4}z^{-1}+\frac{3}{4}}{\frac{1}{4}z^{-2}+\frac{3}{4}z^{-1}+1}$

c) $H(z)=\dfrac{z^2+4z+3}{z^2+9z+20}$ **d)** $H(z)=\dfrac{z^2+6z+8}{(z+3)\,(z^2+2z+2)}$

E11.4 Determine the transfer function of the digital system having the following state variable description.

a) $\mathbf{A}=\begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}1\\ 0\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}3 & 1\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}2\end{bmatrix}$

b) $\mathbf{A}=\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}0\\ 2\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}3 & 0\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}4\end{bmatrix}$

c) $\mathbf{A}=\begin{bmatrix}-1 & 2\\ 2 & -3\end{bmatrix}$; $\mathbf{B}=\begin{bmatrix}2\\ -1\end{bmatrix}$; $\mathbf{C}=\begin{bmatrix}1 & 4\end{bmatrix}$; $\mathbf{D}=\begin{bmatrix}1\end{bmatrix}$

E11.5 Compute the solution of state equations of the discrete time systems represented by the following state model.

a) $$\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\end{bmatrix}=\begin{bmatrix}3 & 1\\ 0 & -4\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix} \quad ; \quad \begin{bmatrix}q_1(0)\\ q_2(0)\end{bmatrix}=\begin{bmatrix}0\\ 2\end{bmatrix}$$

b) $$\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 2 & -5\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix} \quad ; \quad \begin{bmatrix}q_1(0)\\ q_2(0)\end{bmatrix}=\begin{bmatrix}-0.5\\ 1\end{bmatrix}$$

E11.6 Find the response of the discrete time system represented by the following state model. Assume zero initial conditions.

$$\mathbf{A}=\begin{bmatrix}-1 & -1\\ 0 & 2\end{bmatrix} \; ; \; \mathbf{B}=\begin{bmatrix}3\\ 5\end{bmatrix} \; ; \; \mathbf{C}=\begin{bmatrix}2 & -0.5\end{bmatrix} \; ; \; \mathbf{D}=\begin{bmatrix}1\end{bmatrix} \; ; \quad \text{Input, } x(n) = 2\,n\,u(n)$$

Answers

E11. 1 a) State equation

$$\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\\ q_3(n+1)\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -0.2 & 0.4 & -0.1\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\\ q_3(n)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

Output equation

$$y(n) = \begin{bmatrix}2.6 & 2.3 & 2.3\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\\ q_3(n)\end{bmatrix}+\begin{bmatrix}2\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

E11. 1 b) State equation

$$\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.5 & -2.5\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}0\\ 0.5\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

Output equation

$$y(n)=\begin{bmatrix}2.5 & 1.5\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}0.5\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

E11. 1 c) State equation

$$\begin{bmatrix}q_1(n+1)\\ q_2(n+1)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -3 & -7\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

Output equation

$$y(n)=\begin{bmatrix}-3 & -18\end{bmatrix}\begin{bmatrix}q_1(n)\\ q_2(n)\end{bmatrix}+\begin{bmatrix}3\end{bmatrix}\begin{bmatrix}x(n)\end{bmatrix}$$

E11. 2 **a)** $\mathbf{A}^n = \frac{1}{4}u(n)\begin{bmatrix} 5-5^n & -1+5^n \\ 5-5^{n+1} & -1+5^{n+1} \end{bmatrix}$ **b)** $\mathbf{A}^n = (-1)^n u(n)\begin{bmatrix} 1 & 0 \\ 0 & 3^n \end{bmatrix}$

c) $\mathbf{A}^n = \frac{(-1)^n}{2}u(n)\begin{bmatrix} 1+3^n & 1-3^n \\ 1-3^n & 1+3^n \end{bmatrix}$ **d)** $\mathbf{A}^n = (-1)^n u(n)\begin{bmatrix} 2^n & 0 \\ 0 & 1 \end{bmatrix}$

E11. 3 **a)** State equation

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}[x(n)]$$

Output equation

$$y(n) = \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \end{bmatrix}[x(n)]$$

E11. 3 **b)** State equation

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{3}{4} \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[x(n)]$$

Output equation

$$y(n) = \begin{bmatrix} -\frac{3}{16} & -\frac{5}{16} \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \end{bmatrix}[x(n)]$$

E11. 3 **c)** State equation

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -20 & -9 \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[x(n)]$$

Output equation

$$y(n) = \begin{bmatrix} -17 & -5 \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \end{bmatrix} + [x(n)]$$

E11. 3 **d)** State equation

$$\begin{bmatrix} q_1(n+1) \\ q_2(n+1) \\ q_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -8 & -5 \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}[x(n)]$$

Output equation

$$y(n) = \begin{bmatrix} 8 & 6 & 1 \end{bmatrix}\begin{bmatrix} q_1(n) \\ q_2(n) \\ q_3(n) \end{bmatrix}$$

E11. 4 **a)** $\frac{Y(z)}{X(z)} = \frac{2z^2 - z + 3}{(z-1)^2}$ **b)** $\frac{Y(z)}{X(z)} = \frac{4z^2 - 4z + 10}{z^2 - z + 1}$ **c)** $\frac{Y(z)}{X(z)} = \frac{z^2 + 2z + 15}{z^2 + 4z - 1}$

E11. 5 **a)** $\mathbf{Q}(n) = \frac{2}{7}u(n)\begin{bmatrix} 3^n - (-4)^n \\ 7(-4)^n \end{bmatrix}$ **b)** $\mathbf{Q}(n) = \frac{(-1)^n}{4}u(n)\begin{bmatrix} 2 \\ 5^{n+1} - 1 \end{bmatrix}$

E11. 6 $y(n) = \left[7 + 23n + \frac{7}{3}(-1)^n - \frac{35}{3}\,2^n\right]u(n)$

APPENDIX 1

Important Mathamatical Relations

A1.1 Trigonometric Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1, \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \pm \sin A \sin B$$

$$2\sin A \sin B = \cos(A - B) - \cos(A + B)$$

$$2\sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2\cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos(\theta \pm 90^\circ) = \pm\sin\theta$$

$$\sin(\theta \pm 90^\circ) = \pm\cos\theta$$

$$\tan(\theta \pm 90^\circ) = -\cot\theta$$

$$\cos(\theta \pm 180^\circ) = -\cos\theta$$

$$\sin(\theta \pm 180^\circ) = -\sin\theta$$

$$\tan(\theta \pm 180^\circ) = \tan\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}, \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

A1.2 Complex Variables

A complex number 'z' may be represented as,

$$z = x + jy = r\angle\theta = re^{j\theta} = r(\cos\theta + j\sin\theta)$$

$$\text{Where, } x = \operatorname{Re}(z) = r\cos\theta, \qquad y = \operatorname{Im}(z) = r\sin\theta$$

$$r = |z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$j = \sqrt{-1}, \qquad \frac{1}{j} = -j, \qquad j^2 = -1$$

The conjugate of the complex number 'z' may be represented as,

$$z^* = x - jy = r\angle-\theta = re^{-j\theta} = r(\cos\theta - j\sin\theta)$$

Demovier's theorem : $\left(e^{j\theta}\right)^n = e^{jn\theta} = \cos n\theta + j\sin n\theta$

Let, z_1 and z_2 be two complex numbers defined as, $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$.
Now, $z_1 = z_2$ only if $x_1 = x_2$ and $y_1 = y_2$.

$$z_1 \pm z_2 = (x_1 + x_2) \pm j(y_1 + y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \quad \text{or} \quad z_1 z_2 = r_1 r_2 \, e^{j(\theta_1 + \theta_2)} = r_1 r_2 \angle(\theta_1 + \theta_2)$$

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)}{(x_2 + jy_2)} \times \frac{(x_2 - jy_2)}{(x_2 - jy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}$$

$$\text{or} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)} = \frac{r_1}{r_2} \angle(\theta_1 - \theta_2)$$

The following relations hold good for a complex number 'z'.

$$\sqrt{z} = \sqrt{x + jy} = \sqrt{re^{j\theta}} = \sqrt{r}\, e^{j\frac{\theta}{2}} = \sqrt{r} \angle \tfrac{\theta}{2}$$

$$z^n = (x + jy)^n = r^n e^{jn\theta} = r^n \angle n\theta \quad \text{where, n is an integer.}$$

$$z^{1/n} = (x + jy)^{1/n} = r^{1/n} e^{j\theta/n} = r^{1/n} \angle\left(\tfrac{\theta}{n} + \tfrac{2\pi k}{n}\right) \quad \text{for, } k = 0, 1, 2, \dots\dots\dots n - 1$$

$$\text{In}\, z = \text{In}(re^{j\theta}) = \text{In}\, r + \text{In}\, e^{j\theta} = \text{In}\, r + j(\theta + 2k\pi) \quad \text{where, k is an integer.}$$

A1.3 Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}$$

$$\text{cosech}\, x = \frac{1}{\sinh x}, \qquad \text{sech}\, x = \frac{1}{\cosh x}$$

$$\sin jx = j \sinh x, \qquad \cos j\, x = \cosh x$$

$$\sinh jx = j \sin x, \qquad \cosh jx = \cos x$$

$$\sinh (x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh (x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\sinh (x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$$

$$\cosh (x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$$

$$\tanh (x \pm jy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm j \frac{\sin 2y}{\cosh 2x + \cos 2y}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\text{sech}^2 x + \tanh^2 x = 1$$

$$\sin (x \pm iy) = \sin x \cosh y \pm j \cos x \sinh y$$

$$\cos (x \pm jy) = \cos x \cosh y \mp j \sin x \sinh y$$

A1.4 Derivatives

Let, $U = U(x)$, $V = V(x)$, and $a = \text{constant}$.

$$\frac{d}{dx}(aU) = a\frac{dU}{dx}$$

$$\frac{d}{dx}(UV) = U\frac{dV}{dx} + V\frac{dU}{dx}$$

$$\frac{d}{dx}\left[\frac{U}{V}\right] = \frac{V\frac{dU}{dx} - U\frac{dV}{dx}}{V^2}$$

$$\frac{d}{dx}\left(aU^n\right) = naU^{n-1}$$

$$\frac{d}{dx}\log_a U = \frac{\log_a e}{U}\frac{dU}{dx}$$

$$\frac{d}{dx}\text{In } U = \frac{1}{U}\frac{dU}{dx}$$

$$\frac{d}{dx}a^U = a^U \text{ In } a\frac{dU}{dx}$$

$$\frac{d}{dx}e^U = e^U\frac{dU}{dx}$$

$$\frac{d}{dx}U^V = VU^{V-1}\frac{dU}{dx} + U^V \text{ In } U\frac{dV}{dx}$$

$$\frac{d}{dx}\sin U = \cos U\frac{dU}{dx}$$

$$\frac{d}{dx}\cos U = -\sin U\frac{dU}{dx}$$

$$\frac{d}{dx}\tan U = \sec^2 U\frac{dU}{dx}$$

A1.5 Indefinite Integrals

Let, $U = U(x)$, $V = V(x)$, and $a = \text{constant}$.

$$\int a\, dx = ax + C$$

$$\int U V = U \int V - \int \left[\int V \, dU \right] \quad \text{or} \quad \int U \, dV = U V - \int V \, dU$$

$$\int U^n \, dU = \frac{U^{n+1}}{n+1} + C, \qquad n \neq -1$$

$$\int \frac{1}{U} \, dU = \text{In}\, U + C$$

$$\int a^U \, dU = \frac{a^U}{\text{In}\, a} + C, \qquad a > 0 \text{ and } a \neq 1$$

$$\int e^U \, dU = e^U + C$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C$$

$$\int x e^{ax} \, dx = \frac{e^{ax}}{a^2} (ax - 1) + C$$

$$\int x^2 e^{ax} \, dx = \frac{e^{ax}}{a^3} (a^2 x^2 - 2ax + 2) + C$$

$$\int \text{In}\, x \, dx = x \, \text{In}\, x - x + C$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C$$

$$\int \tan ax \, dx = \frac{1}{a} \text{In}\,(\sec ax) + C = -\frac{1}{a} \text{In}\,(\cos ax) + C$$

$$\int \sec ax \, dx = \frac{1}{a} \text{In}\,(\sec ax + \tan ax) + C$$

APPENDIX 2

MATLAB Commands and Functions

	Operators and Special Characters
+	Plus; addition operator.
-	Minus; subtraction operator.
*	Scalar and matrix multiplication operator.
.*	Array multiplication operator.
^	Scalar and matrix exponentiation operator.
.^	Array exponentiation operator.
\	Left-division operator.
/	Right-division operator.
.\	Array left-division operator.
./	Array right-division operator.
:	Colon; generates regularly spaced elements and represents an entire row/column.
( )	Parentheses; encloses function arguments and array indices; overrides precedence.
[ ]	Brackets; enclosures array elements.
.	Decimal point.
...	Ellipsis; line-continuation operator.
,	Comma; separates statements and elements in a row.
;	Semicolon; separates columns and suppresses display.
%	Percent sign; designates a comment and specifies formatting.
_	Quote sign and transpose operator.
._	Nonconjugated transpose operator.
=	Assignment (replacement) operator.

	Logical and Relational Operators
==	Relational operator : equal to.
~=	Relational operator : not equal to.
<	Relational operator : less than.
<=	Relational operator : less than or equal to.
>	Relational operator : greater than.
>=	Relational operator : greater than or equal to.
&	Logical operator : AND.
|	Logical operator : OR.
~	Logical operator : NOT.
xor	Logical operator : EXCLUSIVE OR.

Special Variables and Constants	
`ans`	Most recent answer.
`eps`	Accuracy of floating-point precision.
`i,j`	The imaginary unit ; $\sqrt{-1}$.
`Inf`	Infinity.
`NaN`	Undefined numerical result (not a number).
`pi`	The number π .

Commands for Managing a Session	
`clc`	Clears Command window.
`clear`	Removes variables from memory.
`exist`	Checks for existence of file or variable.
`global`	Declares variables to be global.
`help`	Searches for a help topic.
`lookfor`	Searches help entries for a keyword.
`quit`	Stops MATLAB.
`who`	Lists current variables.
`whos`	Lists current variables (long display).

Input/Output Commands	
`disp`	Displays contents of an array or string.
`fscanf`	Read formatted data from a file.
`format`	Controls screen-display format.
`fprintf`	Performs formatted writes to screen or file.
`input`	Displays prompts and waits for input.
`;`	Suppresses screen printing.

Format Codes for fprintf and fscanf	
`%s`	Format as a string.
`%d`	Format as an integer.
`%f`	Format as a floating point value.
`%e`	Format as a floating point value in scientific notation.
`%g`	Format in the most compact form : %f or %e.
`\n`	Insert a new line in the output string.
`\t`	Insert a tab in the output string.

Array Commands	
`cat`	Concatenates arrays.
`find`	Finds indices of nonzero elements.
`length`	Computes number of elements.
`linspace`	Creates regularly spaced vector.
`logspace`	Creates logarithmically spaced vector.
`max`	Returns largest element.
`min`	Returns smallest element.
`prod`	Product of each column.
`reshape`	Change size
`size`	Computes array size.
`sort`	Sorts each column.
`sum`	Sums each column.

Special Matrices	
`eye`	Creates an identity matrix.
`ones`	Creates an array of ones.
`zeros`	Creates an array of zeros.

Program Flow Control	
`break`	Terminates execution of a loop.
`case`	Provides alternate execution paths within switch structure.
`else`	Delineates alternate block of statements.
`elseif`	Conditionally executes statements.
`end`	Terminates for, while, and if statements.
`error`	Displays error messages.
`for`	Repeats statements a specific number of times
`if`	Executes statements conditionally.
`otherwise`	Default part of switch statement.
`return`	Return to the invoking function.
`switch`	Directs program execution by comparing point with case expressions.
`warning`	Display a warning message.
`while`	Repeats statements an indefinite number of times.

Basic xy Plotting Commands	
`axis`	Sets axis limits.
`fplot`	Intelligent plotting of functions.
`grid`	Displays gridlines.
`plot`	Generates xy plot.
`print`	Prints plot or saves plot to a file
`title`	Puts text at top of plot.
`xlabel`	Adds text label to x-axis.
`ylabel`	Adds text label to y-axis.

Plot Enhancement Commands	
`axes`	Creates axes objects.
`close`	Closes the current plot.
`close`	Closes all plots.
`figure`	Opens a new figure window.
`gtext`	Enables label placement by mouse.
`hold`	Freezes current plot.
`legend`	Legend placement by mouse.
`refresh`	Redraws current figure window.
`set`	Specifies properties of objects such as axes.
`subplot`	Creates plots in subwindows.
`text`	Places string in figure.

Specialized Plot Commands	
`bar`	Creates bar chart.
`loglog`	Creates log-log plot.
`polar`	Creates polar plot.
`semilogx`	Creates semilog plot (logarithmic abscissa).
`semilogy`	Creates semilog plot (logarithmic ordinate).
`stairs`	Creates stairs plot.
`stem`	Creates stem plot.

Convolution Functions	
`conv(x,h)`	Returns convolution of x and h.
`deconv(y,x)`	Returns deconvolution of y and x.

Logical Functions	
`any`	True if any elements are nonzero.
`all`	True if all elements are nonzero.
`find`	Finds indices of nonzero elements.
`finite`	True if elements are finite.
`isnan`	True if elements are undefined.
`isinf`	True if elements are infinite.
`isempty`	True if matrix is empty.
`isreal`	True if all elements are real.

Exponential and Logarithmic Functions	
`exp(x)`	Exponential; e^x.
`log(x)`	Natural logarithm; ln(x).
`log10(x)`	Common (base 10) logarithm; $\log(x) = \log_{10}(x)$.
`sqrt(x)`	Square root of x; $\sqrt{x}$.

Trigonometric Functions	
`acos(x)`	Inverse cosine; $\cos^{-1}(x)$.
`acot(x)`	Inverse cotangent; $\cot^{-1}(x)$.
`acsc(x)`	Inverse cosecant; $\mathrm{cosec}^{-1}(x)$.
`asec(x)`	Inverse secant; $\sec^{-1}(x)$.
`asin(x)`	Inverse sine; $\sin^{-1}(x)$.
`atan(x)`	Inverse tangent; $\tan^{-1}(x)$.
`atan2(y,x)`	Four-quadrant inverse tangent.
`cos(x)`	Cosine; cos(x).
`cot(x)`	Cotangent; cot(x).
`csc(x)`	Cosecant; cosec(x).
`sec(x)`	Secant; sec(x).
`sin(x)`	Sine; sin(x).
`tan(x)`	Tangent; tan(x).

Hyperbolic Functions	
`acosh(x)`	Inverse hyperbolic cosine; $\cosh^{-1}(x)$.
`acoth(x)`	Inverse hyperbolic cotangent; $\coth^{-1}(x)$.
`acsch(x)`	Inverse hyperbolic cosecant; $\mathrm{cosech}^{-1}(x)$.
`asech(x)`	Inverse hyperbolic secant; $\mathrm{sech}^{-1}(x)$.
`asinh(x)`	Inverse hyperbolic sine; $\sinh^{-1}(x)$.
`atanh(x)`	Inverse hyperbolic tangent; $\tanh^{-1}(x)$.
`cosh(x)`	Hyperbolic cosine; cosh(x).
`coth(x)`	Hyperbolic cotangent; cosh(x)/sinh(x).
`csch(x)`	Hyperbolic cosecant; 1/sinh(x).
`sech(x)`	Hyperbolic secant; 1/cosh(x).
`sinh(x)`	Hyperbolic sine; sinh(x).
`tanh(x)`	Hyperbolic tangent; sinh(x)/cosh(x).

Complex Functions	
`abs(x)`	Absolute value; $\|x\|$.
`angle(x)`	Angle of a complex number x.
`conj(x)`	Complex conjugate of x.
`imag(x)`	Imaginary part of a complex number x.
`real(x)`	Real part of a complex number x.

State Space Functions	
`ss2tf`	Computes transfer function from state model.
`tf2ss`	Computes state model from transfer function.

Transform Functions	
`fft`	Computes DFT via FFT.
`fourier`	Returns the Fourier transform.
`ifft`	Computes inverse DFT via FFT.
`ifourier`	Returns the inverse Fourier transform.
`ilaplace`	Returns the inverse Laplace transform.
`iztrans`	Returns the inverse $\mathcal{Z}$-transform.
`laplace`	Returns the Laplace transform.
`ztrans`	Returns the $\mathcal{Z}$-transform.

APPENDIX 3

Summary of Various Standard Transform Pairs

Table - A3.1: Standard Continuous Time Fourier Tranform Pairs

x(t)	X(jΩ)
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\Omega t_0}$
A where, A is constant	$2\pi A\,\delta(\Omega)$
$u(t)$	$\pi\delta(\Omega) + \dfrac{1}{j\Omega}$
$\mathrm{sgn}(t)$	$\dfrac{2}{j\Omega}$
$t\,u(t)$	$\dfrac{1}{(j\Omega)^2}$
$\dfrac{t^{n-1}}{(n-1)!}\,u(t)$ where, $n = 1, 2, 3, ...$	$\dfrac{1}{(j\Omega)^n}$
$t^n\,u(t)$ where, $n = 1, 2, 3, ...$	$\dfrac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\dfrac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\dfrac{1}{(j\Omega + a)^2}$
$Ae^{-a\lvert t\rvert}$	$\dfrac{2Aa}{a^2 + \Omega^2}$
$Ae^{j\Omega_0 t}$	$2\pi A\;\delta(\Omega - \Omega_0)$
$\sin\Omega_0 t$	$\dfrac{\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$
$\cos\Omega_0 t$	$\pi[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)]$

Table - A3.2: Standard Laplace Transform Pairs

Note : σ = Real part of s

x(t)	X(s)	ROC
$\delta(t)$	1	Entire s-plane
$u(t)$	$\frac{1}{s}$	$\sigma > 0$
$t\,u(t)$	$\frac{1}{s^2}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
$t^n u(t)$ where, $n = 1, 2, 3 .....$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\sigma > -a$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{(s+a)^n}$	$\sigma > -a$
$t^n e^{-at}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$\sigma > 0$
$\cos\Omega_0 t\,u(t)$	$\frac{s}{s^2+\Omega_0^2}$	$\sigma > 0$
$\sin h\,\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2-\Omega_0^2}$	$\sigma > \Omega_0$
$\cos h\,\Omega_0 t\,u(t)$	$\frac{s}{s^2-\Omega_0^2}$	$\sigma > \Omega_0$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$\sigma > -a$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$\sigma > -a$

Table - A3.3 : Standard Fourier Tranform Pairs of Causal Signals via Laplace Transform

x(t) for t = 0 to ∞	X(s)	X(jΩ) $\left[X(j\Omega) = X(s)\big\|_{s=j\Omega}\right]$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{1}{j\Omega}$
$t\,u(t)$	$\frac{1}{s^2}$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{s^n}$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{s^{n+1}}$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\frac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\frac{1}{(j\Omega+a)^2}$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2+\Omega_0^2} = \frac{\Omega_0}{\Omega_0^2-\Omega^2}$
$\cos\Omega_0 t\;u(t)$	$\frac{s}{s^2+\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2+\Omega_0^2} = \frac{j\Omega}{\Omega_0^2-\Omega^2}$
$\sinh\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2-\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2-\Omega_0^2} = \frac{-\Omega_0}{\Omega^2+\Omega_0^2}$
$\cosh\Omega_0 t\,u(t))$	$\frac{s}{s^2-\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2-\Omega_0^2} = \frac{-j\Omega}{\Omega^2+\Omega_0^2}$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega+a)^2+\Omega_0^2}$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$\frac{j\Omega+a}{(j\Omega+a)^2+\Omega_0^2}$

Table - A3.4: Standard $\mathcal{Z}$-transform Pairs

x(t)	x(n)	X(z) With positive power of z	X(z) With negative power of z	ROC
	$\delta(n)$	1	1	Entire z-plane
	u(n) or 1	$\frac{z}{z-1}$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
	$a^n u(n)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
	$n a^n u(n)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
	$n^2 a^n u(n)$	$\frac{az(z+a)}{(z-a)^3}$	$\frac{az^{-1}(1+az^{-1})}{(1-az^{-1})^3}$	$\|z\| > \|a\|$
	$-a^n u(-n-1)$	$\frac{z}{z-a}$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
	$-na^n u(-n-1)$	$\frac{az}{(z-a)^2}$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$t\,u(t)$	$nT\,u(nT)$	$\frac{Tz}{(z-1)^2}$	$\frac{Tz^{-1}}{(1-z^{-1})^2}$	$\|z\| > 1$
$t^2 u(t)$	$(nT)^2 u(nT)$	$\frac{T^2 z(z+1)}{(z-1)^3}$	$\frac{T^2 z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$	$\|z\| > 1$
$e^{-at} u(t)$	$e^{-anT} u(nT)$	$\frac{z}{z-e^{-aT}}$	$\frac{1}{1-e^{-aT}z^{-1}}$	$\|z\| > \|e^{-aT}\|$
$te^{-at} u(t)$	$nTe^{-anT} u(nT)$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$	$\frac{z^{-1}Te^{-aT}}{(1-e^{-aT}z^{-1})^2}$	$\|z\| > \|e^{-aT}\|$
$\sin\Omega_0 t\, u(t)$	$\sin\Omega_0 nT\; u(nT)$ $= \sin\omega n\; u(nT)$ where, $\omega = \Omega_0 T$	$\frac{z\sin\omega}{z^2-2z\cos\omega+1}$	$\frac{z^{-1}\sin\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\|z\| > 1$
$\cos\Omega_0 t\, u(t)$	$\cos\Omega_0 nT\, u(nT)$ $= \cos\omega n\; u(nT)$ where, $\omega = \Omega_0 T$	$\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$	$\frac{1-z^{-1}\cos\omega}{1-2z^{-1}\cos\omega+z^{-2}}$	$\|z\| > 1$

Note : *1. The sequences multiplied by u(n) are causal sequences (defined for $n \geq 0$).*

2. The sequences multiplied by u(–n – 1) are anticausal sequences (defined for $n \leq 0$).

Table - A3.5 : Standard Discrete Time Fourier Transform Pairs

x(t)	x(n)	$X(e^{j\omega})$ with positive power of $e^{j\omega}$	$X(e^{j\omega})$ with negative power of $e^{j\omega}$
	$\delta(n)$	1	1
	$\delta(n-n_0)$	$\dfrac{1}{e^{j\omega n_0}}$	$e^{-j\omega n_0}$
	$u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-1}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$
	$a^n\,u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$	$\dfrac{1}{1-a e^{-j\omega}}$
	$n\,a^n\,u(n)$	$\dfrac{a e^{j\omega}}{(e^{j\omega}-a)^2}$	$\dfrac{a e^{-j\omega}}{(1-a e^{-j\omega})^2}$
	$n^2\,a^n\,u(n)$	$\dfrac{a e^{j\omega}(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$	$\dfrac{a e^{-j\omega}(1+a e^{-j\omega})}{(1-a e^{-j\omega})^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)$	$\dfrac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$	$\dfrac{1}{1-e^{-j\omega}e^{-aT}}$
	1	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-2\pi m)$	
	$a^{\lvert n\rvert}$	$\dfrac{1-a^2}{1-2a\cos\omega+a^2}$	
	$\sum_{m=-\infty}^{+\infty}\delta(n-mN)$	$\dfrac{2\pi}{N}\sum_{m=-\infty}^{+\infty}\delta\left(\omega-\frac{2\pi m}{N}\right)$	
$e^{j\Omega_0 t}$	$e^{j\Omega_0 nt}=e^{j\omega_0 n}$ where, $\omega_0=\Omega_0 T$	$2\pi\sum_{m=-\infty}^{+\infty}\delta(\omega-\omega_0-2\pi m)$	
$\sin\Omega_0 t$	$\sin\Omega_0 nT$ $=\sin\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\dfrac{\pi}{j}\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)-\delta(\omega+\omega_0-2\pi m)\right]$	
$\cos\Omega_0 t$	$\cos\Omega_0 nT$ $=\cos\omega_0 n$ where, $\omega_0=\Omega_0 T$	$\pi\sum_{m=-\infty}^{+\infty}\left[\delta(\omega-\omega_0-2\pi m)+\delta(\omega+\omega_0-2\pi m)\right]$	

Table - A3.6 : Standard Discrete Time Fourier Transform Pairs of Causal Signals via $\mathcal{Z}$-Transform

x(t)	x(n)	X(z)	X(e^jω)
	$\delta(n)$	1	1
	$a^n\, u(n)\,; \quad \lvert a\rvert < 1$	$\dfrac{z}{z-a}$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$
	$n\, a^n\, u(n)\,; \quad \lvert a\rvert < 1$	$\dfrac{az}{(z-a)^2}$	$\dfrac{a\,e^{j\omega}}{(e^{j\omega}-a)^2}$
	$n^2\, a^n\, u(n)\,; \quad \lvert a\rvert < 1$	$\dfrac{az\,(z+a)}{(z-a)^3}$	$\dfrac{a\,e^{j\omega}\,(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$
$e^{-at}u(t)$	$e^{-anT}u(nT)\,; \quad \lvert e^{-aT}\rvert < 1$	$\dfrac{z}{z-e^{-aT}}$	$\dfrac{e^{j\omega}}{e^{j\omega}-e^{-aT}}$
$te^{-at}u(t)$	$nTe^{-anT}u(nT)\,; \quad \lvert e^{-aT}\rvert < 1$	$\dfrac{z\,T\,e^{-aT}}{(z-e^{-aT})^2}$	$\dfrac{e^{j\omega}\,T\,e^{-aT}}{(e^{j\omega}-e^{-aT})^2}$

APPENDIX 4

Summary of Properties of Various Transforms

Table - A4.1 : Properties of Laplace Transform

Note : $\mathcal{L}\{x(t)\} = X(s);\quad \mathcal{L}\{x_1(t)\} = X_1(s);\quad \mathcal{L}\{x_2(t)\} = X_2(s)$

Property	Time domain signal	s-domain signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1 x_1(t) \pm a_2 x_2(t)$	$a_1 X_1(s) \pm a_2 X_2(s)$
Time differentiation	$\frac{d}{dt} x(t)$	$s\,X(s) - x(0)$
	$\frac{d^n}{dt^n} x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$s^n X(s) - \sum_{K=1}^{n} s^{n-K} \left. \frac{d^{(K-1)} x(t)}{dt^{K-1}} \right\|_{t=0}$
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$
	$\int \ldots\ldots \int x(t)\,(dt)^n$ where $n = 1, 2, 3 \ldots\ldots$	$\frac{X(s)}{s^n} + \sum_{K=1}^{n} \frac{1}{s^{n-K+1}} \left[\int \ldots \int x(t)\,(dt)^k\right]_{t=0}$
Frequency shifting	$e^{\pm at}\,x(t)$	$X(s \mp a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm \alpha s}\,X(s)$
Frequency differentiation	$t\,x(t)$	$-\frac{dX(s)}{ds}$
	$t^n\,x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$(-1)^n \frac{d^n}{ds^n} X(s)$
Frequency integration	$\frac{1}{t}\,x(t)$	$\int_s^{\infty} X(s)\,ds$
Time scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{s}{a}\right)$
Periodicity	$x(t + nT)$	$\frac{1}{1 - e^{-sT}} \int_0^T x(t)\,e^{-st}\,dt$
Initial value theorem	$\underset{t \to 0}{Lt}\ x(t) = x(0)$	$\underset{s \to \infty}{Lt}\ s\,X(s)$
Final value theorem	$\underset{t \to \infty}{Lt}\ x(t) = x(\infty)$	$\underset{s \to 0}{Lt}\ s\,X(s)$
Convolution theorem	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\,x_2(t - \lambda)\,d\lambda$	$X_1(s)\,X_2(s)$

Table - A4.2 : Properties of Exponential Fourier Series Coefficients of Continuous Time Signals

***Note :** c_n and d_n are exponential form of Fourier series coefficients of x(t) and y(t) respectively.*

Property	Continuous time periodic signal	Fourier series coefficients
Linearity	$A\,x(t) + B\,y(t)$	$A\,c_n + B\,d_n$
Time shifting	$x(t - t_0)$	$c_n\, e^{-jn\Omega_0 t_0}$
Frequency shifting	$e^{-jm\Omega_0 t}\, x(t)$	c_{n-m}
Conjugation	$x^*(t)$	c^*_{-n}
Time reversal	$x(-t)$	c_{-n}
Time scaling	$x(\alpha t)\ ;\ \alpha > 0$ (x(t) is periodic with period T/α)	c_n (No change in Fourier coefficient)
Multiplication	$x(t)\ y(t)$	$\sum_{m=-\infty}^{+\infty} c_m\, d_{n-m}$
Differentiation	$\frac{d}{dt}x(t)$	$jn\Omega_0\, c_n$
Integration	$\int_{-\infty}^{t} x(t)\, dt$ (Finite valued and periodic only if $a_0 = 0$)	$\frac{1}{jn\Omega_0}\, c_n$
Periodic convolution	$\int_T x(\tau)\ y(t-\tau)\ d\tau$	$T\, c_n\, d_n$
Symmetry of real signals	x(t) is real	$c_n = c^*_{-n}$ $\lvert c_n\rvert = \lvert c_{-n}\rvert\ ;\ \angle c_n = -\angle c_{-n}$ $\mathrm{Re}\{c_n\} = \mathrm{Re}\{c_{-n}\}$ $\mathrm{Im}\{c_n\} = -\mathrm{Im}\{c_{-n}\}$
Real and Even	x(t) is real and even	c_n are real and even
Real and Odd	x(t) is real and odd	c_n are imaginary and odd

Table - A4.3 : Properties of Continuous Time Fourier Transform

Note : $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Property	Time domain function	Frequency domain function
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0}\ X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t}\ x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt} x(t)$	$j\Omega\ X(j\Omega)$
Time integration	$\int_{-\infty}^{t} x(\tau)\ d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi X(0)\ \delta(\Omega)$
Frequency differentiation	$t\ x(t)$	$j \frac{d}{d\Omega} X(j\Omega)$
Time convolution	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\ x_2(t-\tau) d\tau$	$X_1(j\Omega)\ X_2(j\Omega)$
Frequency convolution (or Multiplication)	$x_1(t)\ x_2(t)$	$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\ X_2(j(\Omega-\lambda))\ d\lambda$
Symmetry of real signals	x(t) is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega)\rvert = \lvert X(-j\Omega)\rvert$ $\angle X(j\Omega) = -\angle X(-j\Omega)$ $\text{Re}\{X(j\Omega)\} = \text{Re}\{X(-j\Omega)\}$ $\text{Im}\{X(j\Omega)\} = -\text{Im}\{X(-j\Omega)\}$
Real and Even	x(t) is real and even	X(jΩ) are real and even
Real and Odd	x(t) is real and odd	X(jΩ) are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e., $X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under frequency domain signal	$\int_{-\infty}^{+\infty} X(j\Omega)\ d\Omega = 2\pi\ x(0)$	
Area under time domain signal	$\int_{-\infty}^{+\infty} x(t)\ dt = X(0)$	

Table - A4.4 : Properties of $\mathcal{Z}$-Transform

Note : $X(z) = \mathcal{Z}\{x(n)\}$; $X_1(z) = \mathcal{Z}\{x_1(n)\}$; $X_2(z) = \mathcal{Z}\{x_2(n)\}$; $Y(z) = \mathcal{Z}\{y(n)\}$

Property		Discrete sequence	$\mathcal{Z}$-transform
Linearity		$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Shifting ($m \ge 0$)	$x(n)$ for $n \ge 0$	$x(n - m)$	$z^{-m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$
		$x(n + m)$	$z^{m} X(z) - \sum_{i=0}^{m-1} x(i)\, z^{m-i}$
	$x(n)$ for all n	$x(n - m)$	$z^{-m} X(z)$
		$x(n + m)$	$z^{m} X(z)$
Multiplication by n^m (or differentiation in z-domain)		$n^m x(n)$	$\left(-z \dfrac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)		$a^n x(n)$	$X(a^{-1} z)$
Time reversal		$x(-n)$	$X(z^{-1})$
Conjugation		$x^*(n)$	$X^*(z^*)$
Convolution		$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n - m)$	$X_1(z)\, X_2(z)$
Correlation		$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\, y(n - m)$	$X(z)\, Y(z^{-1})$
Initial value		$x(0) = \underset{z\to\infty}{\text{Lt}}\ X(z)$	
Final value		$x(\infty) = \underset{z\to 1}{\text{Lt}}\ (1 - z^{-1})\, X(z)$ $= \underset{z\to 1}{\text{Lt}}\ \dfrac{(z-1)}{z} X(z)$ if $X(z)$ is analytic for $\lvert z\rvert > 1$	
Complex convolution theorem		$x_1(n)\, x_2(n)$	$\dfrac{1}{2\pi j} \oint_C X_1(v)\, X_2\!\left(\dfrac{z}{v}\right) v^{-1}\, dv$

Table - A4.5 : Properties Fourier Series Coefficients of Discrete Time Signals

Note : c_k are Fourier series coefficients of x(n) and d_k are Fourier series coefficients of y(n).

Property	Discrete time periodic signal	Fourier series coefficients
Linearity	$A\,x(n) + B\,y(n)$	$A\,c_k + B\,d_k$
Time shifting	$x(n-m)$	$c_k\,e^{\frac{j2\pi km}{N}}$
Frequency shifting	$e^{\frac{j2\pi nm}{N}}\,x(n)$	c_{k-m}
Conjugation	$x^*(n)$	c^*_{-k}
Time reversal	$x(-n)$	c_{-k}
Time scaling	$x(\frac{n}{m})$; for n multiple of m (periodic with period mN)	$\frac{1}{m}\,c_k$
Multiplication	$x(n)\,y(n)$	$\sum_{m=0}^{N-1} c_m\,d_{k-m}$
Circular convolution	$\sum_{m=0}^{N-1} x(m)\,y((n-m))_N$	$N\,c_K\,d_K$
Symmetry of real signals	$x(n)$ – real	$c_k = c^*_{-k}$ $\lvert c_k\rvert = \lvert c_{-k}\rvert$ $\angle c_k = -\angle c_{-k}$ $Re\{c_k\} = Re\{c_{-k}\}$ $Im\{c_k\} = -Im\{c_{-k}\}$
Real and Even	x(n) is real and even	c_k are real and even
Real and Odd	x(n) is real and odd	c_k are imaginary and odd

Table - A4.6 : Properties of Discrete Time Fourier Transform

Note : $X(e^{j\omega}) = \mathcal{F}\{x(n)\}$; $X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\}$; $X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\}$; $Y(e^{j\omega}) = \mathcal{F}\{y(n)\}$

Property	Discrete time signal	Fourier transform
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$
Periodicity	$x(n)$	$X(e^{j\omega + 2\pi m}) = X(e^{j\omega})$
Time shifting	$x(n - m)$	$e^{-j\omega m} X(e^{j\omega})$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
Frequency shifting	$e^{j\omega_0 n} x(n)$	$X(e^{j(\omega - \omega_0)})$
Multiplication	$x_1(n)\, x_2(n)$	$\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$
Differentiation in frequency domain	$n\, x(n)$	$j\,\dfrac{dX(e^{j\omega})}{d\omega}$
Convolution	$x_1(n) * x_2(n) = \displaystyle\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(e^{j\omega})\, X_2(e^{j\omega})$
Correlation	$\gamma_{xy}(m) = \displaystyle\sum_{m=-\infty}^{+\infty} x(n)\, y(n-m)$	$X(e^{j\omega})\, Y(e^{-j\omega})$
Symmetry of real signals	$x(n)$ is real	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $Re\{X(e^{j\omega})\} = Re\{X(e^{-j\omega})\}$ $Im\{X(e^{j\omega})\} = -Im\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$, $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
Symmetry of real and even signals	$x(n)$ is real and even	$X(e^{j\omega})$ is real and even
Symmetry of real and odd signals	$x(n)$ is real and odd	$X(e^{j\omega})$ is imaginary and odd

Table - A4.7 : Properties of Discrete Fourier Transform (DFT)

Note : $X(k) = \mathcal{DFT}\{x(n)\}$; $X_1(k) = \mathcal{DFT}\{x_1(n)\}$; $X_2(k) = \mathcal{DFT}\{x_2(n)\}$; $Y(k) = \mathcal{DFT}\{y(n)\}$

Property	Discrete time signal	Discrete Fourier Transform
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(k) + a_2\,X_2(k)$
Periodicity	$x(n + N)$	$X(k + N) = X(k)$
Circular time shift	$x((n - m))_N$	$X(k)\,e^{\frac{-j2\pi km}{N}}$
Time reversal	$x(N - n)$	$X(N - k)$
Conjugation	$x^*(n)$	$X^*(N - k)$
Circular frequency shift	$x(n)\,e^{\frac{j2\pi mn}{N}}$	$X((k - m))_N$
Multiplication	$x_1(n)\,x_2(n)$	$\frac{1}{N}\left[X_1(k) \circledast X_2(k)\right]$
Circular convolution	$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\,x_2((n - m))_N$	$X_1(k)\,X_2(k)$
Circular correlation	$\gamma_{xy}(m) = \sum_{n=0}^{N-1} x(n)\,y^*((n - m))_N$	$X(k)\,Y^*(k)$
Symmetry of real signals	x(n) is real	$X(k) = X^*(N-k)$ $X_r(k) = X_r(N-k)$ $X_i(k) = -X_i(N-k)$ $\lvert X(k)\rvert = \lvert X(N-k)\rvert$ $\angle X(k) = -\angle X(N-k)$
Symmetry of real and even signals	x(n) is real and even $x(n) = x(N - n)$	$X(k) = X_r(k)$ and $X_i(k) = 0$
Symmetry of real and odd signals	x(n) is real and odd $x(n) = -x(N - n)$	$X(k) = jX_i(k)$ and $X_r(k) = 0$

Table - A4.8 : Parseval's Relation in Various Transforms

Parseval's relation in continuous time fourier series	Average power, P of x(t) is defined as, $P = \frac{1}{T}\int_T \lvert x(t)\rvert^2\ dt$	The average power, P in terms of Fourier series coefficients is, $P = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$
	$\frac{1}{T}\int_T \lvert x(t)\rvert^2\ dt = \sum_{n=-\infty}^{+\infty} \lvert c_n\rvert^2$	
Parseval's relation in continuous time fourier transform	The energy in time domain is, $E = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2\, dt$	The energy in frequency domain is, $E = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\, d\Omega$
	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\ d\Omega$	
Parseval's relation in Z-transform	$\sum_{n=-\infty}^{+\infty} x_1(n)\ x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\ X_2^*\left(\frac{1}{z^*}\right) z^{-1}\ dz$	
Parseval's relation in discrete time fourier series	Average power P of x(n) is defined as, $P = \frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	The Average power P in terms of Fourier series coefficients is, $P = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$
	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2 = \sum_{k=0}^{N-1} \lvert c_k\rvert^2$	
Parseval's relation in discrete time fourier transform	Energy E in time domain, $E = \sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2$	Energy E in frequency domain, $E = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \lvert X(e^{j\omega})\rvert^2\ d\omega$
	$\sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2\ d\omega$	
Parseval's relation in discrete fourier transform (DFT)	$\sum_{n=0}^{N-1} x_1(n)\ x_2^*(n)$	$\frac{1}{N}\sum_{k=0}^{N-1} X_1(k)\ X_2^*(k)$
	$\sum_{n=-\infty}^{+\infty} x_1(n)\ x_2^*(n)$	$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega}) X_2^*(e^{j\omega})\ d\omega$

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