Digital Signal Analysis And Processing

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3. Analysis of LTI system in frequency domain (6 Hrs)

3.1 Frequency response of LTI system, response to complex exponential

3.2 Linear constant coefficient difference equations and corresponding system function

3.3 Relationship of frequency response to pole-zero of system

3.4 Linear phase of LTI system and its relationship to causality

3.1 Frequency response of LTI system, response to complex exponential

- An LTI system can be completely characterized in the time domain by its impulse response $h(n)$, with the output $y(n)$ due to a given input $\mathrm{x}(n)$ specified through the convolution sum

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- Since the frequency response and impulse response are directly related through the Fourier transfrom, the frequency response, assuming it exists (i.e., converges), provides an equally complete characterization of LTI system.
- Since the z-transform is a generalization of the Fourier transform, the Y(z), the z-transform of the output of the LTI system, is related to $X(z)$, the z-transform of the input, and $H(z)$, the z-trasform of the system impulse response, by

$$Y(z) = H(z)X(z)$$

- With an appropriate region of convergence. $H(z)$ is refered to as the system function.
- Both frequecny response and the system function are extremely useful in the anlaysis and representation of LTI systems, because we can readily infer many properties of the system response from them.

Frequency Response of LTI systems

The Fourier transforms of the system input and output are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

Where $X(e^{j\omega})$ and $\mathrm{Y}(e^{j\omega})$ are the Fourier transforms of the system input and output respectively.

With the frequency response expressed in polar form, the magnitude and phase of the Fourier transforms of the system input and output are related by

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|,$$

$$\measuredangle Y(e^{j\omega}) = \measuredangle H(e^{j\omega}) + \measuredangle X(e^{j\omega})$$

$|H(e^{j\omega})|$ is referred to as the magnitude response or the gain of the system, and $\measuredangle H(e^{j\omega})$ is referred to as the phase response or phase shift of the system.

The magnitude and phase effects can be either desirable, if the input signal is modified in a useful way, or undesirable, if the input is changed in a deleterious manner.

Frequency Response of the Ideal Lowpass Filter

The ideal low pass filter is defined as the discrete time linear time invariant system whose frequency response is

$$H_{lp}\left(e^{j\omega}\right) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi, \end{cases}$$

And $H_{lp}\left(e^{j\omega}\right)$ is also periodic with period 2π.

The ideal low pass filter selects the low frequency components of the signal and rejects the high frequency components. The impulse response $h_{lp}[n]$ can be found using the Fourier transform synthesis equation

$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{1}{2\pi j n}\left[e^{j\omega n}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi j n}\left(e^{j\omega_c n} - e^{-j\omega_c n}\right)$$

$$= \frac{\sin(\omega_c n)}{\pi n}, \qquad -\infty < n < \infty$$

Analogously, the ideal highpass filter is defined as

$$H_{hp}\left(e^{j\omega}\right) = \begin{cases} 0, & |\omega| < \omega_{c,} \\ 1, & \omega_c < |\omega| \leq \pi, \end{cases}$$

And since $H_{hp}\left(e^{j\omega}\right) = 1 - H_{lp}\left(e^{j\omega}\right)$, its impulse response is

$$h_{hp}[n] = \delta[n] - h_{hp}[n] = \delta[n] - \frac{\sin(\omega_c n)}{\pi n}$$

The ideal highpass filter passes the frequency band $\omega_c < \omega \leq \pi$ undistorted and rejects frequencies below ω_c

The ideal lowpass filters are noncausal, and their impulse responses extends from $-\infty\ to + \infty$. Therefore, it is not possible to compute the output of either the ideal lowpass or the ideal high pass filter either recursively or nonrecursively; i.e., the system are not computationally realizable.

The phase response of the ideal low pass filter is specified to be zero. If it were nit zero, the low frequency band selected by the filter would also have phase distortion.

Phase Distortion and Delay

To understand the effect of the phase of a linear system, consider the ideal delay system. The impulse response is

$$h_{id}[n] = \delta[n - n_d]$$

Where n_d is a fixed integer

The frequency response of the ideal delay is therefore

$$H_{id}\left(e^{j\omega}\right) = e^{-j\omega n_d}$$

The magintude and phase are

$$\left|H_{id}(e^{j\omega})\right| = 1,$$

$$\measuredangle H_{id}\left(e^{j\omega}\right) = -\omega n_d, \qquad |\omega| < \pi$$

With periodicity 2π

Delay only shift the sequence in time.

An ideal low pass filter with linear phase is defined as

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| \leq \pi, \end{cases}$$

Its impulse response is

$$h_{lp}[n] = \frac{\sin(\omega_c n - n_d)}{\pi(n - n_d)}, \qquad -\infty < n < \infty$$

No matter how large we make n_d, the ideal lowpass filter is always noncausal.

3.2 Linear constant coefficient difference equations and corresponding system function

An important subclass of linear time-invariant systems consists of those systems for which the input $x(n)$ and the output $y(n)$ satisfy an N^{th} order linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

If the system is causal, the difference equation can be used to compute the output recursively. If the auxiliary conditions correspond to initial rest, the system will be causal, linear and time invariant.

The properties and characteritics of LTI sys tms for which the input and output satisfy a linear constant-coefficience difference equation are best developed through the z-transform.

Applying the z-transform to both sides of above equation and using the linearity property and time shifting property, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$or, \qquad (\sum_{k=0}^{N} a_k z^{-k}) Y(z) = (\sum_{k=0}^{M} b_k z^{-k}) X(z)$$

And

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

In factored form as

$$H(z) = (\frac{b_0}{a_0}) \frac{\prod_{k=1}^{M}(1 - c_k z^{-1})}{\prod_{k=1}^{N}(1 - d_k z^{-1})}$$

Each of the factors $(1 - c_k z^{-1})$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$.

Similarly, each of the factors $(1 - d_k z^{-1})$ in the denominator contributes a zero at $z = 0$ and pole at $z = d_k$.

Example 1: Find the difference equation that satisfied by the input and output of the linear time-invariant system

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})}$$

Solution:

$$H(z) = \frac{(1+z^{-1})^2}{(1-\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} = \frac{1+2z^{-1}+z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}} = \frac{Y(z)}{X(z)}$$

Thus,

$$\left(1+\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}\right)\mathrm{Y(z)} = (1+2z^{-1}+z^{-2})X(z)$$

And the difference equation is

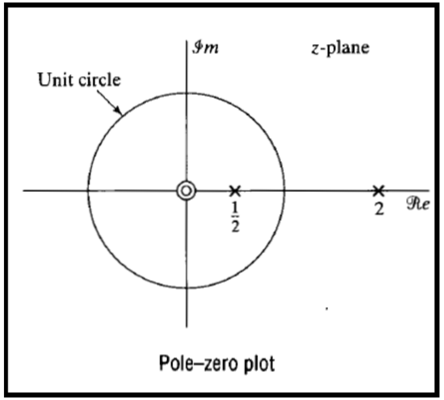
$$y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] + 2x\{n-1] + x[n-2]$$

Example 2: Consider the LTI system with input and output related through the difference equation

$$y[n] - \frac{5}{2}y[n-1] + y[n-2] = x[n]$$

$H(z)$ is calculated as

$$H(z) = \frac{1}{1 - \frac{5}{2}z^{-1} + z^{-2}} = \frac{1}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}$$



There are three possible choices for the ROC as the pole-zero plot

a. If the system is causal, then the ROC is outside the outermost ploe, i.e.,$|z| > 2$. In this case the system will not be stable, since the ROC does not include the unit circle.
b. If we assume the system is stable, then the ROC will be $\frac{1}{2} < z < 2$.
c. If the ROC is $|z| < \frac{1}{2}$, the system will be neither stable nor causal.

- ✓ Causality and stablility are not necessary compatible requirements
- ✓ A LTI system whose input and output satisfy a difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- ✓ To be causal and stable, the ROC of the corresponding system function must be outside the outermost pole and include the unit circle.
- ✓ Clearly, this requires that all poles of the system function be inside the unit circle.

Example 3: Find the system function, $H(z)$ and unit sample response $h(n)$ of the difference equation $y(n) = \frac{1}{2}y(n-1) + 2x(n)$, where $y(n)$ and $x(n)$ are the input and output of the system, respectively.
solution:

$$y(n) - \frac{1}{2}y(n-1) = 2x(n)$$

Taking z-transform

$$\left(1 - \frac{1}{2}z^{-1}\right)Y(z) = 2X(z)$$

Thus

$$H(z) = \frac{2}{1 - \frac{1}{2}z^{-1}}$$

This system has a pole at $z = \frac{1}{2}$ and zero at $z = 0$.
Taking inverse z-transform we get the unit sample response of the system, i.e.,

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Example 4: A causal discrete time LTI system is described by

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$$

Where $x(n)$ and $y(n)$ are the input and output of the system, respectively.

a. Determine the system function $H(z)$.
b. Find the impulse response $h(n)$ of the system.
c. Find the step response $s(n)$ of the system.

Solution:

a. Taking the z-transform we get

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

Thus

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{z^2}{(z - \frac{1}{2})(z - \frac{1}{4})}$$

Here the system has two zeros at $z = 0$ and pole at $z = \frac{1}{4}$ & $\frac{1}{2}$.

For the system to be causal and stable the ROC should be $|z| > \frac{1}{2}$

b. Using partial-fraction expansion, we have

$$\frac{H(z)}{z} = \frac{z}{(z - \frac{1}{2})(z - \frac{1}{4})} = \frac{A_1}{z - \frac{1}{2}} + \frac{A_2}{z - \frac{1}{4}}$$

Equating the numerator,

$$z = A_1\left(z - \frac{1}{4}\right) + A_2\left(z - \frac{1}{2}\right)$$

Comparing coefficient we get,

$$A_1 + A_2 = 1$$

$$-\frac{1}{4}A_1 - \frac{1}{2}A_2 = 0 \rightarrow A_1 = -2A_2$$

Thus $A_1 = 2$ & $A_2 = -1$

And

$$= 2\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}} H(z) = 2\frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{4}}, \qquad |z| > \frac{1}{2}$$

Taking inverse z-transform, we get

$$h(n) = \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right]u(n)$$

c. $x(n) = u(n)$ and $X(z) = \frac{1}{1-z^{-1}} = \frac{z}{z-1}, \qquad |z| > 1$

$$Y(z) = X(z)H(z) = \frac{z^3}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})}$$

Using partial fraction expansion

$$\frac{Y(z)}{z} = \frac{z^2}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{A_1}{z-1} + \frac{A_2}{z-\frac{1}{2}} + \frac{A_3}{1-\frac{1}{4}}$$

Solving we get, $A_1 = \frac{8}{3}, A_2 = -2, \& \ A_3 = \frac{1}{3}$

$$Y(z) = \frac{8}{3}\frac{1}{1-z^{-1}} - 2\frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{3}\frac{1}{1-\frac{1}{4}z^{-1}}, \qquad |z| > 1$$

Taking inverse z-transform we obtain

$$y(n) = s(n) = \left[\frac{8}{3} - 2\left(\frac{1}{2}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n\right]u(n)$$

❖ Difference Equation Representation of the Accumulator

The linear constant-coefficient difference equations is the accumulator system defined by

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

We can write the output for $n - 1$ as

$$y(n-1) = \sum_{k=-\infty}^{n-1} x(k)$$

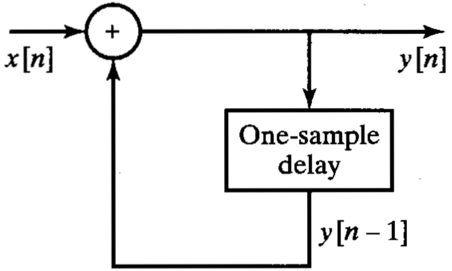
By separating the term $x(n)$ from the sum

$$y(n) = x(n) + \sum_{k=-\infty}^{n-1} x(k)$$

$$= x(n) + y(n-1)$$

Thus

$$y(n) - y(n-1) = x(n)$$



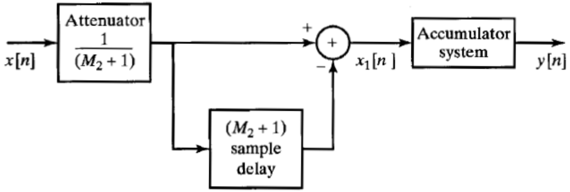
The difference equation gives us a better understanding of how we could implement the accumulator system.
From above equation for each value of n, we add the current input value $x(n)$ to the previously accumulated sum $y(n-1)$.
This interpretation of the accumulator is represented in the block diagram.
This equation and the block diagram are reffered to as a recursive representation of the system, since each value is computed using previously computed values.

Difference Equation Representation of the Moving-Average System

Consider the moving-average system with $M_1 = 0$ so that the system is causal. In this case the impuse response is

$$h(n) = \frac{1}{M_2 + 1}[u(n) - u(n - M_2 - 1)],$$

From which it follows that



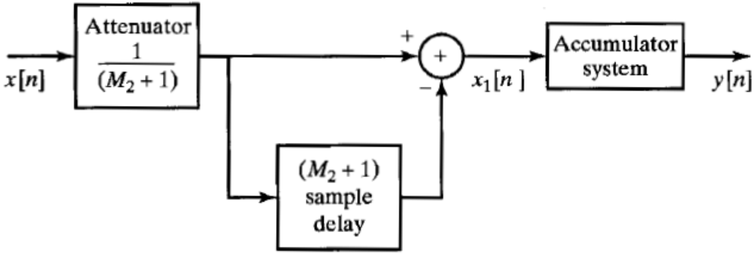
$$y(n) = \frac{1}{M_2 + 1}\sum_{k=0}^{M_2} x(n - k)$$

Also, the impusle response can be expressed as

$$h(n) = \frac{1}{M_2 + 1}[\delta(n) - \delta(n - M_2 - 1)] * u(n)$$

Which suggests that the causal moving-average system can be represented as the cascade system as in figure.

We can obtain a difference equation for this block diagram by noting first that



$x_1(n) = \frac{1}{M_2+1}[(x(n) - x(n - M_2 - 1)]$

And the accumulator satisfies the difference equation

$$y(n) - y(n-1) = x_1(n)$$

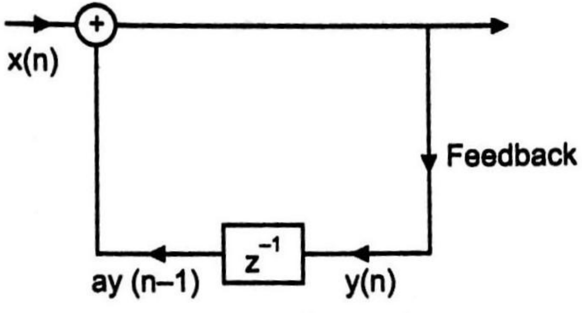
So that

$$y(n) - y(n-1) = \frac{1}{M_2+1}[(x(n) - x(n - M_2 - 1)]$$

Two different difference equation represent the moving-average system.

An unlimited number of distinct difference equations can be used to represent a given linear time-invariant input-output relation.

Let us consider a recursive system shown in figure.



The equation of output is,

$$y(n) = x(n) + ay(n-1)$$

This is first order difference equation. Hence, 'a' is a constant coefficient. Now, output $y(n)$ is calculated by putting different values of n in equation.

$y(0) = x(0) + ay(-1)$

$y(1) = x(1) + ay(0) = x(1) + a[x(0) + ay(-1)] = a^2y(-1) + ax(0) + x(1)$

$y(2) = x(2) + ay(1) = x(2) + a[a^2y(-1) + ax(0) + x(1)]$

$\quad = a^3y(-1) + a^2x(0) + ax(1) + x(2)$

Similarly,

$$y(n) = x(n) + ay(n-1) = a^{n+1}y(-1) + a^n x(1) + \cdots + ax(n-1) + x(n)$$

Or,

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k), \qquad n \geq 0$$

The first part, which contains the term $y(-1)$, is a result of the initial condition $y(-1)$ of the system. The second part is the response of the system to the input signal $x(n)$.

If the system is initially relaxed at time $n = 0$, then its memory (i.e., the output of the delay) should be zero. Hence $y(-1) = 0$. Thus a recursive system is relaxed if it starts with zero initial conditons. Because the memory of the system describes, in some sence, its “state”, we say that the system is at zero state and its corresponding output is called the zero-state response or forced response, and is denoted by $y_{zs}(n)$ and given by

$$y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k), \qquad n \geq 0$$

This is the convolution sumation involving the input signal convolved with the impluse response

$$h(n) = a^n u(n)$$

This is the impulse response of the relaxed recursive system described by the first order difference equation which is a linear time invariant IIR system.

Now, suppose that the system is initially nonrelaxed [i.e., $y(-1) \neq 0$] and the input $x(n) = 0$ for all n. Then the output of the system with zero input is called zero-input response or natural response and is denoted by $y_{zi}(n)$. Thus for $x(n) = 0$ for $-\infty < n < \infty$, we obtain

$$y_{zi}(n) = a^{n+1}y(-1), \qquad n \geq 0$$

A recursive system with nonzero initial condition is nonrelaxed in the sense that it can produce an output without being excited. Note that the zero-input response is due to the memory of the system.

The zero-input response is obtained by setting the input signal to zero, making it independent of the input. It depends only on the nature of the system and the initial conditon. Thus the zero-input response is a characteristic of the sysem itself, and it is known as the natural or free response of the system.
On the other hand, the zero-state response depends on the nature of the system and the input of the signal. Since this output is a response forced upon it by the input signal, it is usually called the forced response of the system.
In general, the total response of the system can be expresssed as

$$y(n) = y_{zi}(n) + y_{zs}(n)$$

The properties of linearity, time invariance, and stability in the context of recursive systems described by linear constant-coefficient difference equations can be restate as

❖ A system is linear if it satisfies the following three requirements:

1. The response is equal to the sum of the zero-input and zero-state responses. [i.e., $y(n) = y_{zi}(n) + y_{zs}(n)$].
2. The principle of superposition applies to the zero-state response (zero-state linear).
3. The principle of superposition applies to the zero-input response (zero-input linear).

A system that does not satisfy all three separete requirements is by definition nonlinear.

Obviously, for a relaxed system, $y_{zi}(n)$=0, thus requirement 2, which is the defination of linearity is sufficient.

Example : Determine if the recursive system defined by the difference equation $y(n) = ay(n-1) + x(n)$ is linear.

Solution: we can expressed this difference equation as $y(n) = y_{zi}(n) + y_{zs}(n)$ so the first requirement for linearity is satisfied.

To check for the second requirement. Let us assume that $x(n) = c_1x_1(n) + c_2x_2(n)$. Thus

$$y_{zs}(n) = \sum_{k=0}^{n} a^k[\,c_1x_1(n-k) + c_2x_2(n-k)]$$

$$= c_1\sum_{k=0}^{n} a^k x_1(n-k) + c_2\sum_{k=0}^{n} a^k x_2(n-k)$$

$$= c_1 y_{zs}^{(1)}(n) + c_2 y_{zs}^{(2)}(n)$$

Hence $y_{zs}(n)$ satisfies the principle of superposition, and thus the system is zero-state linear.

Now let us assume that $y(-1) = c_1 y_1(-1) + c_2 y_2(-1)$ we obtain

$$\begin{aligned} y_{zi}(n) &= a^{n+1}[c_1 y_1(-1) + c_2 y_2(-1)] \\ &= c_1 a^{n+1} y_1(-1) + c_2 a^{n+1} y_2(-1) \\ &= c_1 y_{zi}^{(1)}(n) + c_2 y_{zi}^{(2)}(n) \end{aligned}$$

Hence the system is zero-input linear.

Since the system satisfies all three conditions for linearity, it is linear.

In general, a recursive syste, described by the linear difference equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

satisfies all three conditons in the definitation of linearity, and therefore it is linear.

Clearly the system descibed by the equation

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m)$$

is time invariant because the coefficients a_k and b_k are constants. On the other hand, if one or more of these coefficient depends on time, the system is time variant, since its properties changes as a funtion of time. Thus we conclude that the recursive system described by a linear constant-coefficient equation is linear and time invariant.

A linear time-invariant recursive system described by the linear constant coefficient difference equation, it suffies to state that such a system is BIBO stable if and only if for every bounded input and every bounded initial conditon, the total system response is bounded.

Example :Determine if the recursive system defined by the difference equation $y(n) = ay(n-1) + x(n)$ is stable.

Solution: $y(n) = ay(n-1) + x(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k)\,,\ n \geq 0$

let us assume that the input signal $x(n)$ is bounded in amplitude, that is, $|x(n)| \leq M_x < \infty$ for all $n \geq 0$. Thus

$$|y(n)| \leq |a^{n+1}y(-1)| + \left|\sum_{k=0}^{n} a^k x(n-k)\right|, \qquad n \geq 0$$

$$\leq |a|^{n+1}|y(n-1)| + M_x \sum_{k=0}^{n} |a|^k\,, \qquad n \geq 0$$

$$\leq |a|^{n+1}|y(n-1)| + M_x \frac{1-|a|^{n+1}}{1-|a|} = M_y, \qquad n \geq 0$$

If n is finite, the bound M_y is finite and the output is bounded independently of the value of a. however, as $n \to \infty$, the bound M_y remains finite ony if $|a| < 1$ because $|a|^n \to 0$ as $n \to \infty$. Then $M_y = M_x/(1-|a|)$

Thus the system is stable only if $|a| < 1$.

Questions: Find the system stable or not

1. $y(n) = x(n) + 1$
2. $y(n) - y(n-1) = x(n) + x(n-1)$

The task of finding the system stable or not becomes more difficult for higher order systems. Other simple and more efficient techniques exist for investigating the stability of recursive systems.

Solution of Linear Constant-Coefficient Difference Equations

Given a linear constant-coefficient difference equation as the input-output relationship describing a linear time-invariant system, our objective is to determine an explicit expression for the output $y(n)$. This method is direct method. An alternative method based on the z-transform is called indirect method.

Basically, the goal is to determine the output $y(n), n \geq 0$, of the system given a specific input $x(n), n \geq 0$ and a set of initial conditions. The direct solution method assumes that the total solution is the sum of two parts:

$$y(n) = y_h(n) + y_p(n)$$

The part $y_h(n)$ is known as the homogeneous or complementary solution, whereas $y_p(n)$ is called the particular solution.

The homogeneous solution of a difference equation

We first obtain the solution of homogenous difference equation by assuming that the input $x(n) = 0$.

$$\sum_{k=0}^{N} a_k y(n-k) = 0$$

The procedure for solving a linear constant coefficient difference equation direclty is very similar to the procedure for solving a linear constant-coefficient differential equation. Basically, we assume that the solution is in the form of an exponential, that is

$$y_h(n) = \lambda^{\,n}$$

Thus

$$\sum_{k=0}^{N} a_k \lambda^{\,n-k} = 0$$

$$\lambda^{n-N}(\lambda^N + a_1\lambda^{N-1} + a_2\lambda^{N-2} + \cdots + a_{N-1}\lambda + a_N) = 0$$

The polynomial in parantheses is called the characteristic polynomial of the system. In general, it has N roots, which we denote $\lambda_1,\ \lambda_2, \dots \lambda_N$. The roots can be real or complex valued. In practice the coefficients $a_1, a_2, \dots, a_N$ are usually real.

Complex-valued roots occur as complex-conjugate pairs. Some of the N roots may be identical, in which case we have multiple-order roots.

For a moment, let us assume that the roots are distinct, that is, there are no multiple-order roots. Then the most general solution to the homogeneous difference is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + C_3\lambda_3^n + \cdots + C_N\lambda_N^n$$

Where C_1, C_2, $C_3, \ldots, C_N$ are weighting coefficients.

Thess coefficients are determined form the initial conditons specified for the system. Since the input $x(n) = 0$, can be used to obtain the zero-input response of the sysem.

Example1: Determine the homogeneous solution of the system described by the first-order difference equation.

$$y(n) + a_1 y(n-1) = x(n)$$

Solution: The assumed solution obtained by setting $x(n) = 0$ is $y_h(n) = \lambda^n$

When we substitute this solution, we obtain [with $x(n) = 0$]

$$\lambda^n + a_1 \lambda^{n-1} = 0 = \lambda^{n-1}(\lambda + a_1)$$

Thus $\lambda = -a_1$

Therefore, the solution to the homogeneous difference equation is

$$y_h(n) = C\lambda^n = \mathrm{C}(-a_1)^n$$

The zero-input response of the system can be determined with $x(n) = 0$

$y(0) + a_1 y(-1) = 0$ and $y(0) = -a_1 y(-1)$

On the other hand $y_h(0) = \mathrm{C}(-a_1)^0 = C, thus\ C = -a_1 y(-1)$

And hence the zero-input response of the system is

$$y_{zi}(n) = (-a_1)^{n+1} y(-1), \qquad n \geq 0$$

Example2: Determine the zero-input response of the system described by the homogeneous second-order difference equation.

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

Solution: The solution to the homogeneous equation is $y_h(n) = \lambda^n$

Thus the characteristic equation becomes

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - 3\lambda - 4) = 0$$

Therefore, the roots are $\lambda = -1, 4$ and the general form of the solution to the homogeneous equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n = C_1(-1)^n + C_2(4)^n$$

The zero-input response of the system can be obtained from the homogeneous solution by evaluationg the constants in above equation, giving the initial conditions $y(-1)$ and $y(-2)$. From difference equation we have

$$y(0) = 3y(-1) + 4y(-2)$$

$$y(1) = 3y(0) + 4y(-1) = 13y(-1) + 12y(-2)$$

On the other hand from homogeneous equation

$$y_h(n) = C_1(-1)^n + C_2(4)^n$$

$$y_h(0) = C_1 + C_2$$

$$y_h(1) = -C_1 + 4C_2$$

Equating we get, $C_1 + C_2 = 3y(-1) + 4y(-2)$

$$-C_1 + 4C_2 = 13y(-1) + 12y(-2)$$

Solving we get $C_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2)$ and $C_1 = \frac{16}{5}y(-1) + \frac{16}{5}y(-2)$

Putting these values we get

$$y_h(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + [\frac{16}{5}y(-1) + \frac{16}{5}y(-2)](4)^n, n \geq 0$$

If we assume initial conditons zero that means $y(-1) = 5, and\ y(-2) = 0$ then,

$$y_h(n) = (-1)^{n+1} + (4)^{n+1}, \qquad n \geq 0$$

The Particular solution of the Difference equation.

The particular solution $y_p(n)$ is required to satisfy the difference equation for the specific input signal $x(n), n \geq 0$. In other words $y_p(n)$ is any solution satisfying

$$\sum_{k=0}^{N} a_k y_p(n-k) = \sum_{m=0}^{M} b_m x(n-m), \qquad a_0 = 1$$

Table provides the general form of the particular solution for several types of excitation.

Input Signal $x[n]$	Particular Solution $y_p[n]$
A (constant)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \ldots + K_M$
$A^n n^M$	$A^n (K_0 n^M + K_1 n^{M-1} + \ldots + K_M)$
$\begin{Bmatrix} A\cos(\omega_o n) \\ A\sin(\omega_o n) \end{Bmatrix}$	$K_1 \cos(\omega_o n) + K_2 \sin(\omega_o n)$

A particular solution is basically used to calculate the zero state response y_{zs} of the system. The following steps are used to calculate the forced response of the system.

i. We find the nature of zero input response from the roots of characteristics polynomials. This is $y_h(n)$ or y_{zi}
ii. Then, we assume the particular solution $y_p(n)$.
iii. Next, we determine the coefficients in the zero input reponse. Then, the forced response wil be $y_{zs} = y_h(n) + y_p(n)$

Example1 : Find out the particular solution for the following differential equation $y(n) + 3y(n-1) = x(n)$ assume $x(n) = u(n)$.

Solution: Here, input $x(n) = u(n)$ is unit step. Now, corresponding to this input, we will assume the particular solution as under:

$$y_p(n) = ku(n)$$

Here, k is some constant.

Now, the given equation is, $y(n) + 3y(n-1) = x(n)$

Thus $ku(n) + 3ku(n-1) = u(n)$

Now, we calculate the value of k. We know that input is unit step, $u(n)$. Its value is 1 for $n > 0$. Putting $n = 1$ (remember that while putting value of "n" any terms should not vanish).

$$ku(1) + 3ku(0) = u(1)$$

Or $k + 3k = 1$ therefore $k = \frac{1}{4}$

Putting this value we get,

$$y_p(n) = \frac{1}{4}u(n)$$

Example 2 : Determine the particular solution of the difference equation

$$y(n) = \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) + x(n)$$

While the forcing function $x(n) = 2^n, n \geq 0$ and zero elsewhere.
Solution: The form of the particular solution is

$$y_p(n) = k2^n, \qquad n \geq 0$$

Upon substitution of $y_p(n)$ into the differene equation, we obtain

$$k2^n u(n) = \frac{5}{6}k2^{n-1}u(n-1) - \frac{1}{6}k2^{n-2}u(n-2) + 2^n u(n)$$

To determine the vlaue of k, we can evaluate this equation for any $n \geq 2$, where none of the terms vanish, thus we obtain

$$4k = \frac{5}{6}(2\mathrm{k}) - \frac{1}{6}k + 4$$

Hence $\mathrm{k} = \frac{8}{5}$, therefore, the particular solution is

$$y_p(n) = \frac{8}{5}2^n, \qquad n \geq 0$$

The total solution of the difference equation:

The linearity property of the linear constant-coefficient difference equation allows us to add the homogeneous solution and the particular solution in obtain the total solution. Thus

$$y(n) = y_h(n) + y_p(n)$$

Example : Determine the total solution $y(n), n \geq 0$, to the difference equation

$$y(n) + a_1 y(n-1) = x(n)$$

Where $x(n)$ is a unit step sequence [i.e., $x(n) = \mathrm{u(n)}$] and $y(-1)$ is the initial condition.

Solution: The homogeneous solution is

$$y_h(n) = C(-a_1)^{\,n}$$

The particular solution of the difference equation is

$$y_p(n) = ku(n)$$

Where k is a scale factor, upon substutution of this assumed solution we obtain

$$ku(n) + a_1 ku(n-1) = u(n)$$

To determine k, we must evaluate this equation for any $n \geq 0$, where none of the term vanish. Thus $k + a_1 k = 1$. Therefore, $k = \frac{1}{1+a_1}$

And the particular solution is

$$y_h(n) = \frac{1}{1 + a_1}$$

Consequently, the total solution is

$$y(n) = C(-a_1)^{\,n} + \frac{1}{1 + a_1}, \qquad n \geq 0$$

Where the constant C is determined to satisfy the initial conditon $y(-1)$.

In particular, suppose that we wish to obtain the zero-state response of the system described by the first order difference equation. Then we set $y(-1) = 0$. To evaluate C we evaluate the difference equation at $n = 0$ obtaining $y(0) + a_1 y(-1) = 1$ thus $y(0) = 1$

On the other hand at $n = 0$, $y(0) = C(-a_1)^{0} + \frac{1}{1+a_1}$

Comparing we get $C = \frac{a_1}{1+a_1}$

Substituting for C yields the zero-state response of the system

$$y_{zs}(n) = \frac{a_1}{1+a_1}(-a_1)^{n} + \frac{1}{1+a_1}, \qquad n \geq 0$$

$$= \frac{(-a_1)^{n+1}}{1+a_1} + \frac{1}{1+a_1}, n \geq 0$$

$$= \frac{1-(-a_1)^{n+1}}{1+a_1}, n \geq 0$$

If we evaluate the parameter C under the condition that $y(-1) \neq 0$, then total solution will include the zero-input response as well as the zero-state response of the system. In this case $y(0) + a_1 y(-1) = 1$, $y(0) = -a_1 y(-1) + 1$

On the other hand at $n = 0$, $y(0) = C(-a_1)^{\,0} + \frac{1}{1+a_1}$

Comparing these two relations, we obtain

$$C + \frac{1}{1 + a_1} = -a_1 y(-1) + 1$$

$$C = -a_1 y(-1) + \frac{a_1}{1 + a_1}$$

Finally, if we substitute this value of C, we obtain

$$y(n) = (-a_1)^{n+1} y(-1) + \frac{1 - (-a_1)^{n+1}}{1 + a_1}, n \geq 0$$

$$= y_{zi}(n) + y_{zs}(n)$$

Example: Detetmine the response $y(n), n \geq 0$, of the system described by the second order difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

When the input sequence is

$$x(n) = 4^n u(n)$$

Solution: we have already determine the solution to the homogeneous difference equation for this system. (see slide 130). Thus we have

$$y_h(n) = C_1(-1)^n + C_2(4)^n$$

The particular solution is

$$y_p(n) = k(4)^4 u(n)$$

Since this is same as homogeneous solution, so we assume that

$$y_p(n) = kn(4)^4 u(n)$$

Solving we ger $y_p(n) = \frac{6}{5}(4)^4 nu(n)$

Thus total solution is

$$y(n) = C_1(-1)^n + C_2(4)^n + \frac{6}{5}(4)^4 u(n)$$

Calculating $C_1 = -\frac{1}{25}$ and $C_2 = \frac{26}{25}$

$$y_{ZS}(n) = -\frac{1}{25}(-1)^n + \frac{26}{25}(4)^n + \frac{6}{5}(4)^4 u(n)$$

the total response is the response to arbitrary initial condition is sum of above eqation with below eqution.

$$y_{zi}(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + [\frac{16}{5}y(-1) + \frac{16}{5}y(-2)](4)^n, n \geq 0$$