Digital Signal Analysis And Processing

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4. Discrete Filter Structure 6 Hours

- 4.1 FIR Filter, Structure for FIR Filter (Direct form, Cascade, Lattice)
- 4.2 IIR Filter, Structure of IIR Filter (Direct form I,Direct form II, Cascade, Lattice, Lattice Ladder)
- 4.3 Quantization effect (truncation, rounding), limit cycles and scaling

4. Discrete Filter Structure

- A discrete time system is a system that accepts a discrete time signal as input and processes it, and delivers the processed discrete time signal as output.
- Mathematically, a discrete time system is represented by a difference equation.
- Physically, a discrete time system is realized or implemented either as a digital hardware (like special purpose Microprocessor/Microcontorller) or as a software running on a digital hardware (like PC-Personal Computer)
- The processing of the discrete time signal by the digital hardware involves mathematical operations like addition, multiplication, and delay.
- Also the calculations are performed either by using fixed point arithmetic or floating point arithmetic.
- The time taken to process the discrete time signal and the computational complexity, depends on number of calculations involved and the type of arithemetic used for compuation.
- These issues are addressed in structure realization of discrete time systems.
- From the implementation point of view, the discrete time systems are basically classified as FIR and IIR systems.
- The various structures for FIR and IIR systems, attempt to reduce the computational complexity, errors in computation and the memory requirement of the system.

Discrete Time FIR and IIR Systems

- The impulse response or the frequency response classify digital filters.
- An impulse response is the response of a filter to an input impulse: $x(0) = 1$ and $x(i) = 0$ for $i \neq 0$.
- The Fourier Transform of the impulse response is the frequency response of the filter.
- If the impulse response of the filter falls to zero after a finite period of time, it is an FIR(Finite Impulse Response) filter.
- However, if the impulse response exists indefinitely, its is an IIR (Infinite Impulse Response) filter.
- For FIR filters the output values depend on the current and the previous input values, whereas for IIR filters the output values also depends on the previous output values.
- When a discret time system is designed by considering all the infinite samples of the impulse response, then the system is called IIR (Infinite Impusle Response) sytem.
- When a discrete time system is desiged by choosing only finite samples (usually N-samples) of the impulse response, then the system is called FIR (Finite Impulse Response) system.

FIR System	IIR System
Finite duration impulse response	Infinite duration impulse response
For causal FIR system, $h(n) = 0 \; for \; n < 0 \; \& \; n \geq M-1$, then convolution output becomes $$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$	For IIR system, $h(n) = 0 \; for \; n < 0$ then convolution output becomes $$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$
FIR system has finite memory length	IIR system has infinite memory length
FIR system can be realized with finite number of addition, multiplication and memory location	IIT system can't realized with convolution sum, because it requires infinite number of addition, multiplication so, described by difference equation
Example: Moving average system $$y(n) = \frac{1}{M}\sum_{k=0}^{M} x(n-k)$$	Example: Accumulator average system $$y(n) = \frac{1}{n+1}\sum_{k=0}^{\infty} x(k)$$

Structure for the Realization of Discrete Time Systems

Le us consider the Linear time-invariant discrete-time systems characterized by the general linear constant-coefficient difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

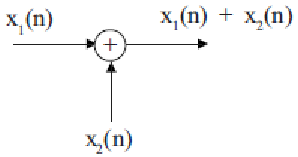
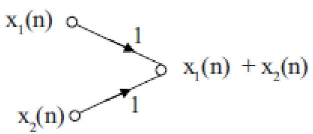
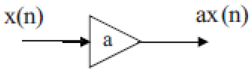
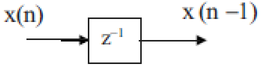
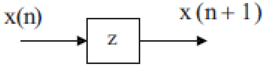
And by means of the z-transform, such linear time-invariant discrete-time systems are also characterized by the rational system function

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

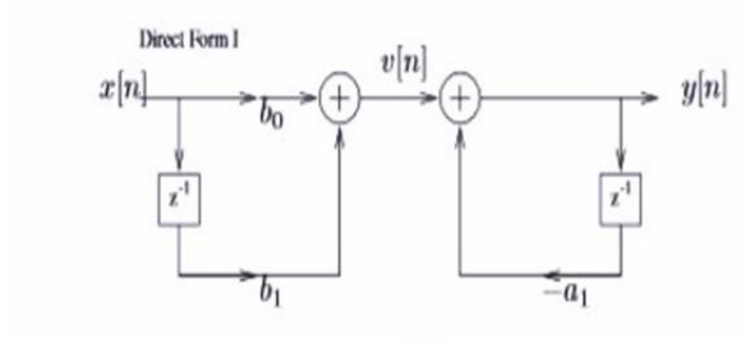
The implementation of a linear time invariant discrete time system by iteratively evaluating a recurrence formula obtained from a difference equation requires that delayed values of the output, input, and intermediate sequences be available.

The delay of sequence values implies the need for storage of past sequence values. Also, we must provide means for multiplication of the delayed sequence value by the coefficients, as well as means for adding the resulting products.

➢ The basic elements required for the impelementation of a linear time invariant discrete time system are adders, multipliers, and memory for storing delayed sequence values. Figure shows symbolic representation of the basic elements of the blocks diagram and signal flow graph.

Element	Block diagram representation	Signal flow graph representation
Adder		
Constant multiplier		
Unit delay element		
Unit advance element		

Implementation of LTI system



Let us consider the first-order system

$y(n) = -a_1 y(n-1) + b_0 x(n) + b_1 x(n-1)$

Which is realized in figure. This realization uses separate delays (memory represented by z^{-1}) for both the input and output signal samples and is called a $direct\ form\ I\ structure$.

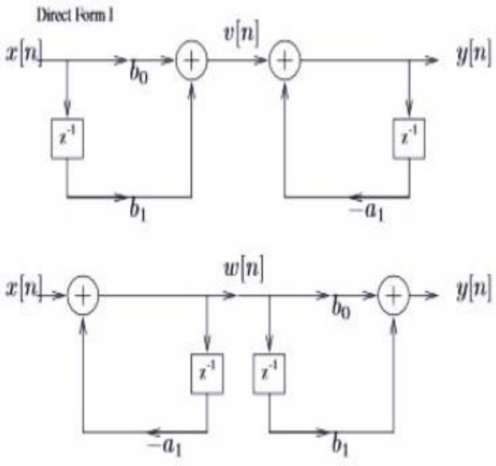
From figure we have

$v(n) = b_0 x(n) + b_1 x(n-1)$

And $y(n) = -a_1 y(n-1) + v(n)$

The first equation is a nonrecursive system and the second equation is a recursive system.

If we interchange the order of the cascaded linear time-invariant systems, the overall system response remains the same. Thus if we interchange the order of the recursive and nonrecursive systems.

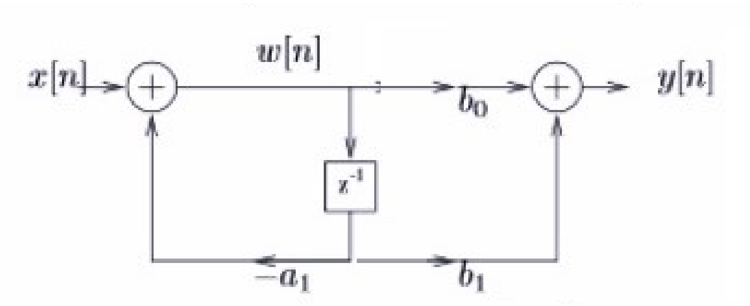


We obtain an alternative structure as in figure (below)
From this figure we obtain the two difference equations.

$$w(n) = -a_1 w(n-1) + x(n)$$
$$y(n) = b_0 w(n) + b_1 w(n-1)$$

which provide an alternative algorithm for computing the output of the system described by the single difference equation.
The two delay elements contain the same input w(n) and hence the same output w(n-1). Consequently these two elements can be merged into one delay.

This new realization requires only one delay for the auxiliary quantity $w(n)$ and hence it is more efficient in terms of memory requirements. It is called the $Direct\ form\ II\ structure$ and it is used extensively in practical applications.



These structures can readily be generalized for the general linear time-invariant recursive system described by the difference equation

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{m=0}^{M} b_m x(n-m)$$

This structure requires $M+N$ delays and $N+M+1$ multiplications.

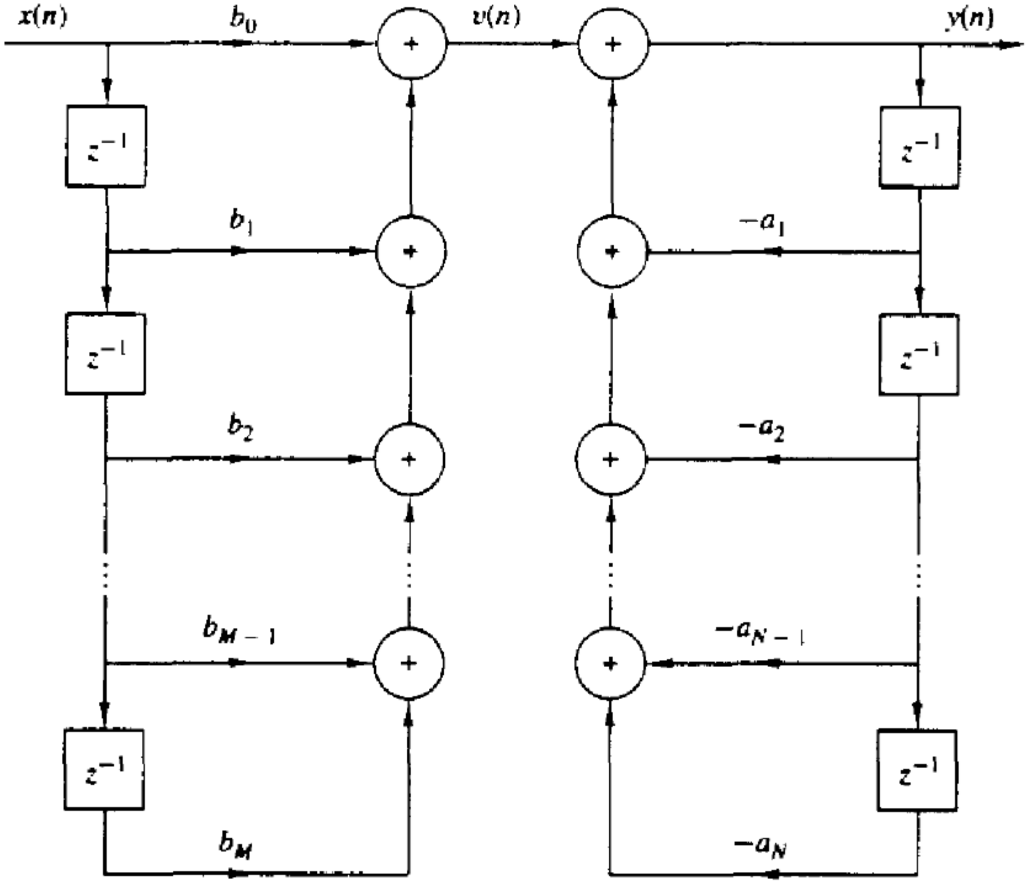
Figure show the direct form I structure for this system.

It can be viewed as the cascade of a non recursive system

$$v(n) = \sum_{k=0}^{M} b_k x(n-k)$$

And a recursive system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + v(n)$$



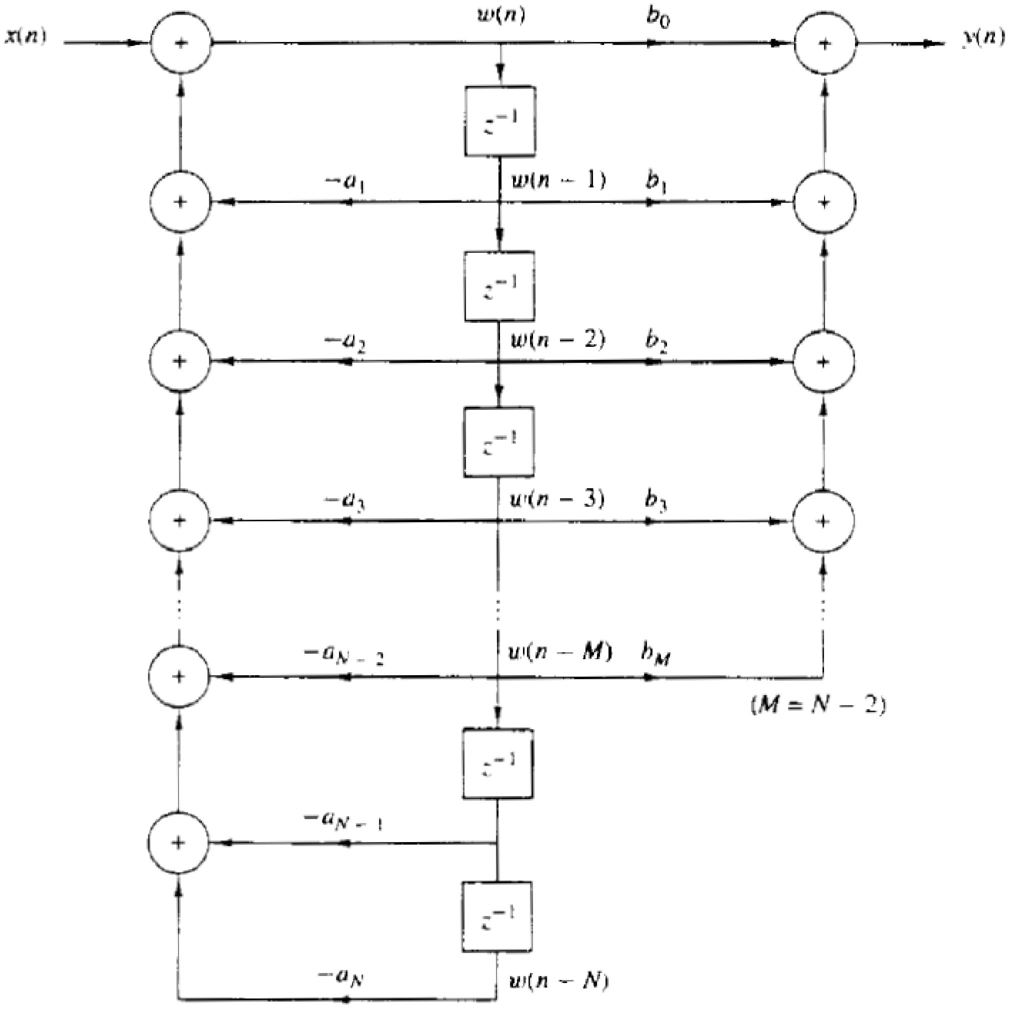
By revesing the order of these two systems, we obtain the direct form II structure as shown in figure for $N > M$. This structure is the cascade of a recutsive system

$$w(n) = -\sum_{k=1}^{N} a_k w(n-k) + x(n)$$

Followed by a non-recursive system

$$y(n) = \sum_{k=0}^{M} b_k w(n-k)$$

The direct form II structure requires M+N+1 multiplication and max{M, N} delays.



4.1 Structure for FIR Filter (Direct, Cascade, Lattice)

In general, an FIR system is described by the difference equation

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$

Or, equivalently, by the system function

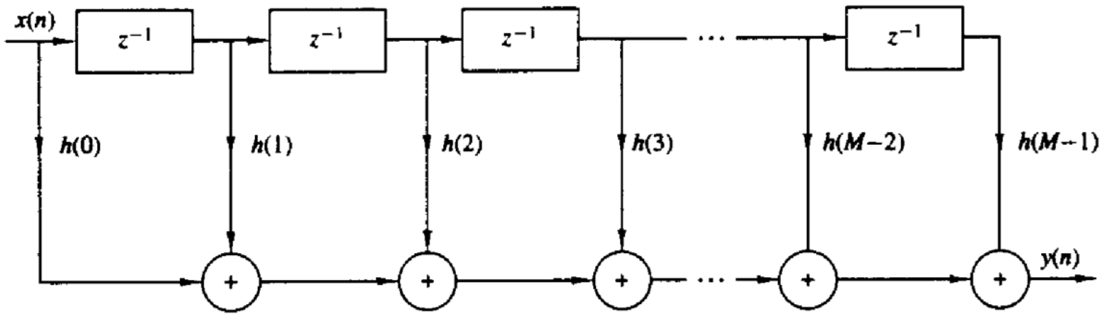
$$H(z) = \sum_{k=0}^{M-1} b_k z^{-k}$$

Furthermore, the unit sample response of the FIR system is identical to the coefficietnts $\{b_k\}$, that is

$$h(n) = \begin{cases} b_n, & 0 \le n \le M-1 \\ 0, & otherwise \end{cases}$$

Direct-Form Structure

The direct-form realization follows immediately from the nonrecursive difference equation or the convolution summation



Direct-form realization of FIR system.

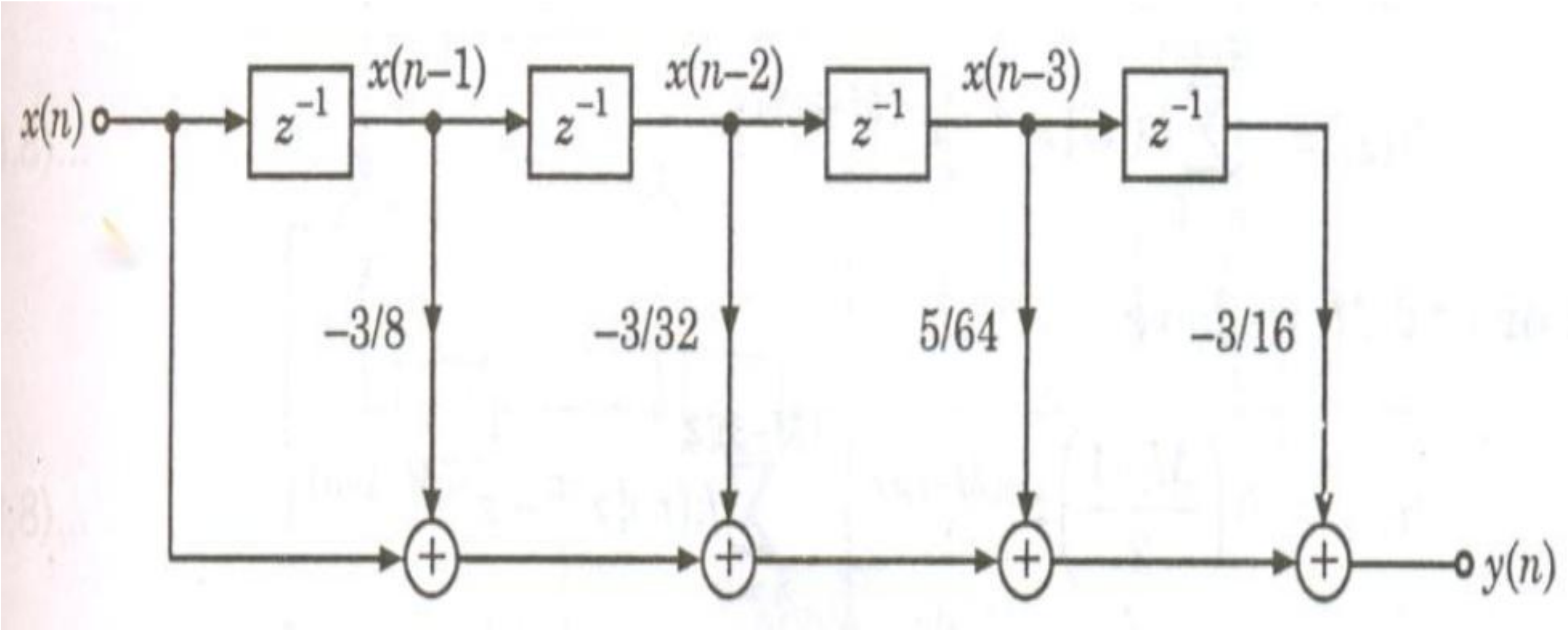
$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

The structure is illustrated in figure above.

We observe that this structure requires $M-1$ memory locations for storing $M-1$ previous inputs, and has a complexity of M multiplication and $M-1$ additions per output points. The structure resembles a tapped delay line or a transversel system. Thus the direct form realization is often called a transversal or tapped-delay line filter.

Example : Determine the Direct Form Realization of an FIR digital filter given by

$$H(z) = 1 - \frac{3}{8}z^{-1} - \frac{3}{32}z^{-2} + \frac{5}{64}z^{-3} - \frac{3}{16}z^{-4}$$

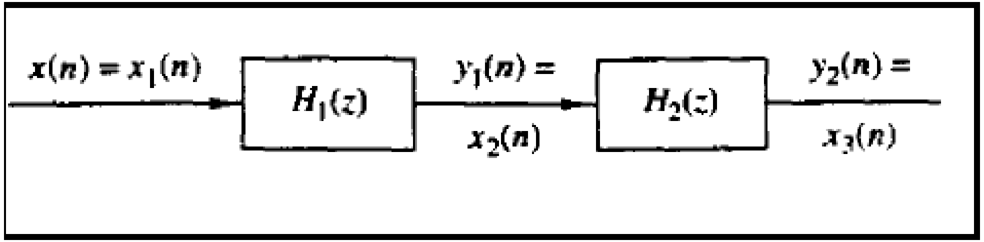


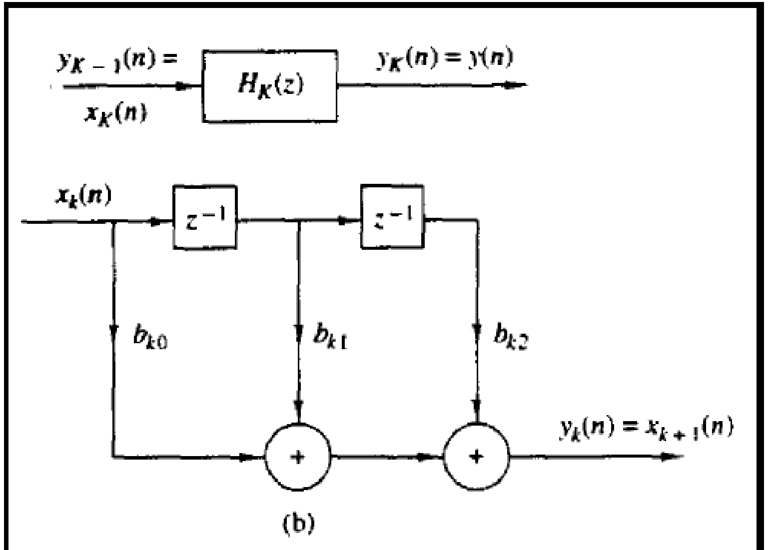
Cascade-Form Structures

The cascade form for FIR systems is obtained by factoring the polynomial system function into second-order FIR system. That is, we represent $H(z)$ as

$$H(z) = \sum_{n=0}^{M} h(n) z^{-n} = \prod_{k=1}^{M_s} (b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2})$$

Where $M_s = [\frac{M+1}{2}]$ is the largest integer contained in $\frac{M+1}{2}$. The figure is shown below.



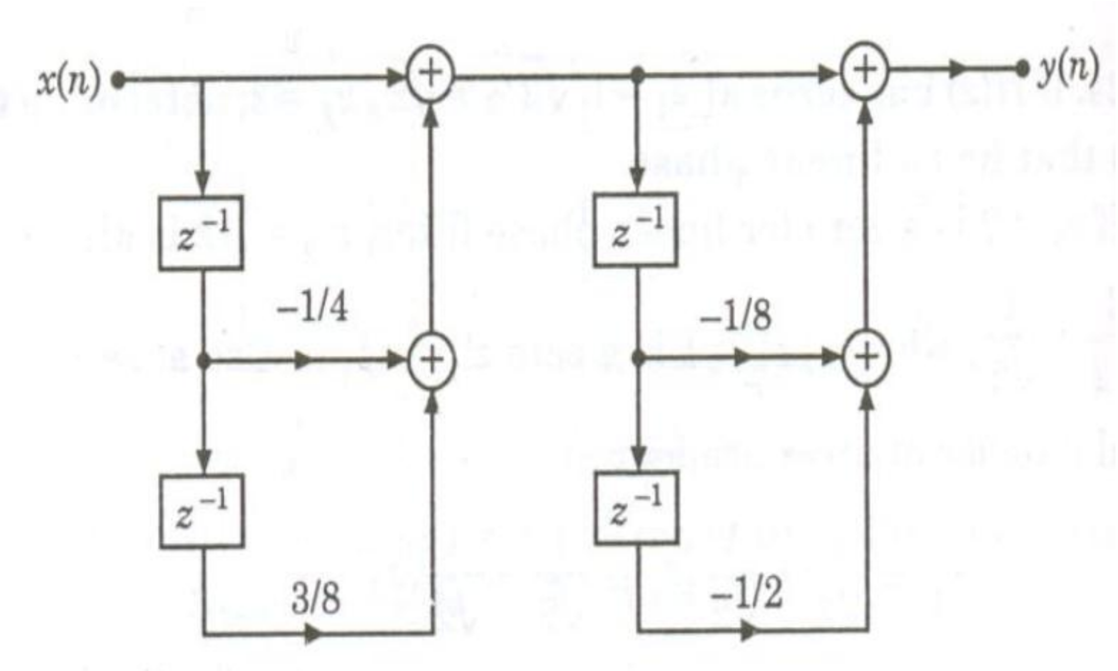


(b)

Cascade realization of an FIR system.

Example : Determine the cascade form realization for the transfer function for an FIR digital filter

$$H(z) = (1 - \frac{1}{4}z^{-1} + \frac{3}{8}z^{-2})(1 - \frac{1}{8}z^{-1} - \frac{1}{2}z^{-2})$$



Lattice

- Lattice filters are used extensively in digital speech processing and in the implementation of adaptive filters.
- Assume that we have a filter with transfer function $H(z)$. We can write

$$H_m(z) = A_m(z), \qquad m = 0,1,2,\ldots,M-1$$

Where $A_m(z)$ is a polynomial with $A_0(z) = 1$ and

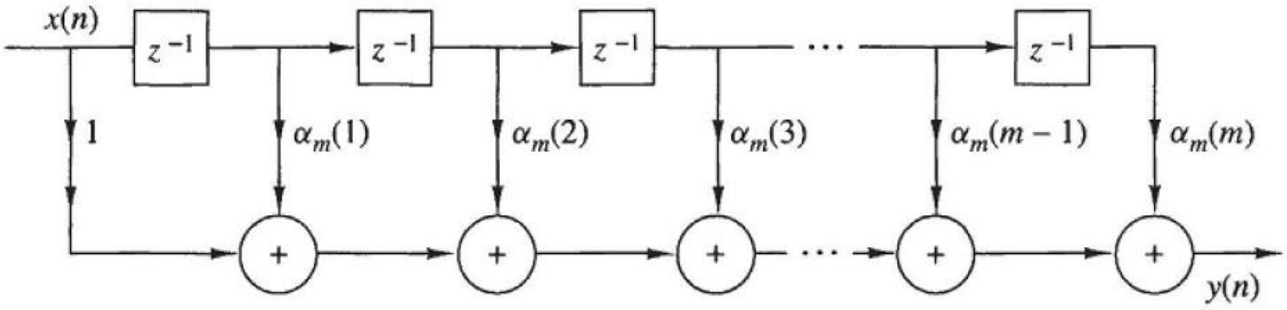
$$A_m(z) = 1 + \sum_{k=1}^{m} \alpha_m(k) z^{-k}, \qquad m \geq 1$$

For mathematical convenience, we define $\alpha_0(k) = 1$.

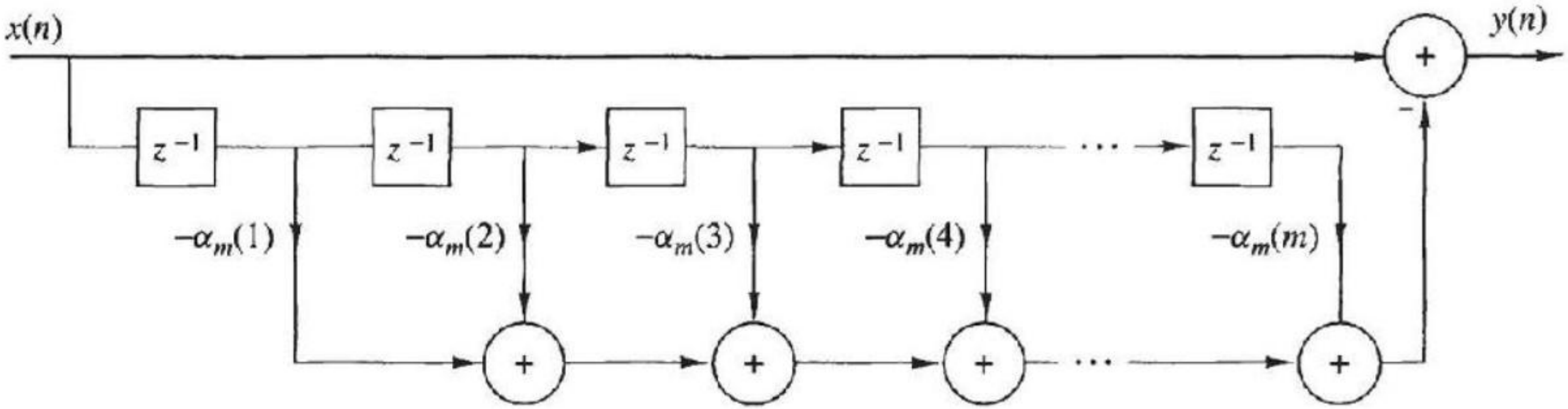
If $x(n)$ is the input sequence to the filter $A_m(z)$ and $y(n)$ is the output sequence, we have

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k) x(n-k)$$

The direct form implementation can be expressed as



Or



This structure is called a prediction error filter.

$$y(n) = x(n) + \sum_{k=1}^{m} \alpha_m(k)x(n-k)$$

Now suppose that we have a filter of order $m = 1$. The output of such filter is

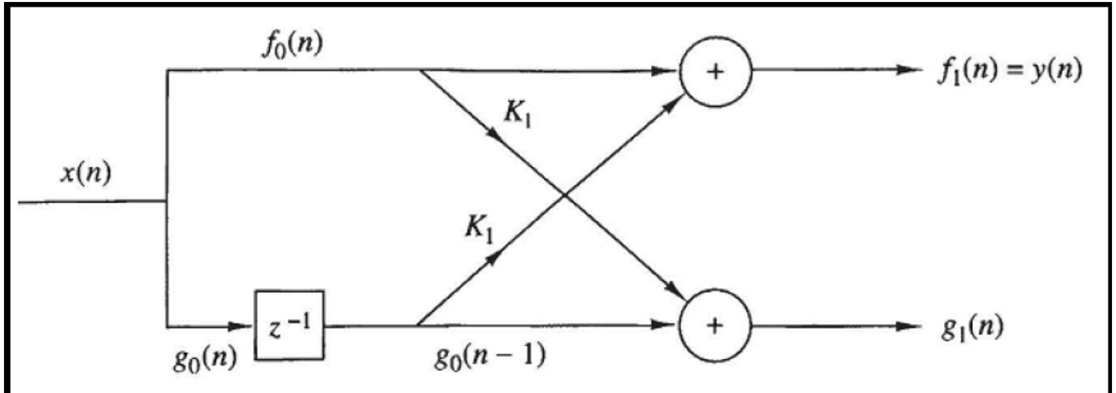
$$y(n) = x(n) + \alpha_1(1)x(n-1)$$

This output can also can be obtained from a first-order or single lattice filter as in figure by exciting both of the inputs by $x(n)$ and selecting the output from the top branch.

The output is exactly, if

$K_1 = \alpha_1(1)$

The parameter K_1 is called a reflection coefficient.

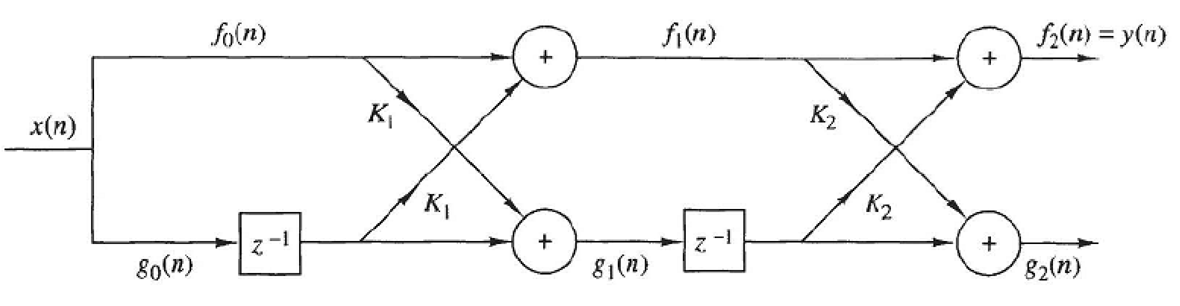


$$f_0(n) = g_0(n) = x(n)$$
$$f_1(n) = f_0(n) + K_1 g_0(n-1) = x(n) + K_1 x(n-1)$$
$$g_1(n) = K_1 f_0(n) + g_0(n-1) = K_1 x(n) + x(n-1)$$

Next, let us consider an FIR filter for which $m = 2$. In this case the output from a direct-form structure is

$$y(n) = x(n) + \alpha_2(1)x(n-1) + \alpha_2(2)x(n-2)$$

By cascading two lattice stages as shown in figure it is possible to obtain the same output.



Two-stage lattice filter.

Here,

$$f_1(n) = x(n) + K_1 x(n-1)$$

$$g_1(n) = K_1 x(n) + x(n-1)$$

And

$$f_2(n) = f_1(n) + K_2 g_1(n-1)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

Or

$$f_2(n) = x(n) + K_1 x(n-1) + K_2[K_1 x(n-1) + x(n-2)]$$

$$= x(n) + K_1(1 + K_2)x(n-1) + K_2 x(n-2)$$

Comparing we get,

$$\alpha_2(2) = K_2$$
$$\alpha_2(1) = K_1(1 + K_2)$$

Or,

$$K_2 = \alpha_2(2)$$
$$K_1 = \frac{\alpha_2(1)}{1 + \alpha_2(2)}$$

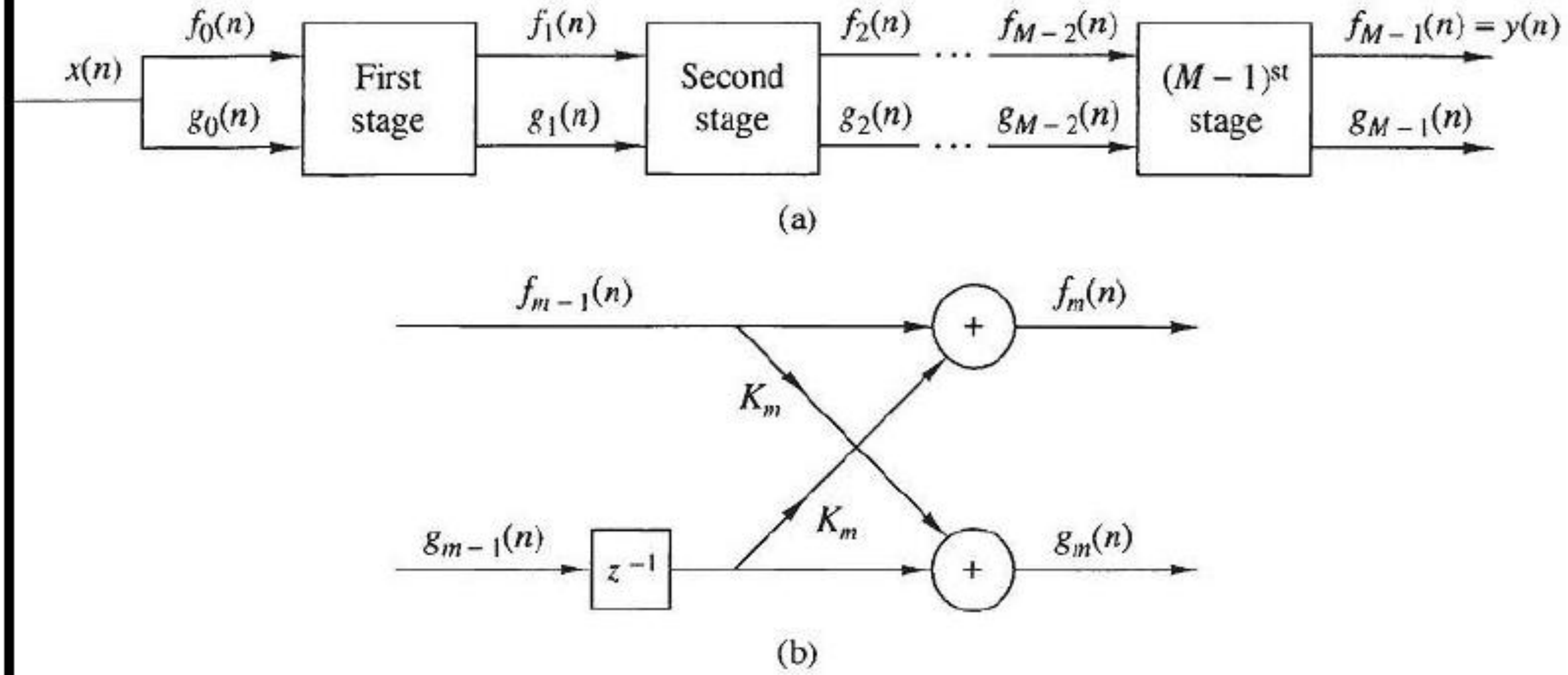
Thus the reflection coefficients K_1 and K_2 of the lattice can be obtained from the coefficient $\alpha_m(k)$ of the direct-form realization.

In general the mth order direct-form FIR filter and m-order or m-stage lattice filter are described by

$$f_0(n) = g_0(n) = x(n)$$
$$f_m(n) = f_{m-1}(n) + K_m g_{m-1}(n-1), \qquad m = 1,2,\ldots,M-1$$
$$g_m(n) = K_m f_{m-1}(n) + g_{m-1}(n-1), \qquad m = 1,2,\ldots M-1$$

Then the output of the $(M-1)$ stage filter corresponds to the output of an $(M-1)$ order FIR filter,that is,

$$y(n) = f_{M-1}(n)$$



$(M-1)$-stage lattice filter.

Conversion of Lattice coefficients to direct-form filter coefficients

The direct-form FIR filter coefficient $\alpha_m(k)$ can be obtained from the lattice coefficients K_i by using the following relations:

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1,2,\ldots,M-1$$

$$B_m(z) = K_m A_{m-1}(z) + z^{-1} B_{m-1}(z), \qquad m = 1,2,\ldots,M-1$$

$$B_m(z) = z^{-m} A_m(z^{-1}), \qquad m = 1,2,\ldots,M-1$$

The solution is obtained recursively, beginning with $m = 1$. Thus we obtain a sequence of $(M-1)$ FIR filters, one for each value of m.

Example: Given a three-stage lattice filter with coefficients $K_1 = \frac{1}{4}, K_2 = \frac{1}{2}, K_3 = \frac{1}{3}$, determine the FIR filter coefficients for the direct –form structure.

Solution:

For $m = 1$, we have

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$A_1(z) = A_0(z) + K_1 z^{-1} B_0(z) = 1 + K_1 z^{-1} = 1 + \frac{1}{4} z^{-1}$$

Hence the coefficient of an FIR filter corresponding to the single-stage lattice are $\alpha_1(0) = 1, \alpha_1(1) = \frac{1}{4}$.

Since $B_m(z)$ is the reverse polynomial of $A_m(z)$ that is,

$$B_1(z) = \frac{1}{4} + z^{-1}$$

For $m = 1$, we have $A_2(z) = A_1(z) + K_2 z^{-1} B_1(z) = 1 + \frac{1}{4} z^{-1} + \frac{1}{2} z^{-1} \left(\frac{1}{4} + z^{-1} \right)$

$$= 1 + \frac{1}{4} z^{-1} + \frac{1}{8} z^{-1} + \frac{1}{2} z^{-2}$$

$$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2}$$

Hence the FIR filter parameters corresponding to the two-stage lattice are $\alpha_2(0) = 1, \alpha_2(1) = \frac{3}{8}$, $\alpha_2(1) = \frac{1}{2}$ and $B_2(z) = \frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}$

Finally for $m = 2$, we have $A_3(z) = A_2(z) + K_3 z^{-1} B_2(z)$

$$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{3} z^{-1} \left(\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2} \right)$$

$$= 1 + \frac{3}{8} z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{6} z^{-1} + \frac{1}{8} z^{-2} + \frac{1}{3} z^{-3}$$

$$= 1 + \frac{13}{24} z^{-1} + \frac{5}{8} z^{-2} + \frac{1}{3} z^{-3}$$

Consequently, the desired direct-form FIR filter is characterized by the coefficients

$\alpha_3(0) = 1, \alpha_3(1) = \frac{13}{24}$, $\alpha_3(1) = \frac{5}{8}$, $\alpha_3(2) = \frac{1}{3}$

Conversion of direct-form FIR filter coefficients to Lattice Coefficients

To obtain K_{m-1} we need the polynomials $A_{m-1}(z)$. Since in general, K_m is obtained from the polynomial $A_m(z)$ for $m = M-1, M-2, \ldots, 1$. consequently, we need to compute the polynomials $A_m(z)$ starting from $m = M-1$ and "stepping down" successively to $m = 1$.

We have,

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$
$$A_m(z) = A_{m-1}(z) + K_m[B_m(z) - K_m A_{m-1}(z)]$$

Thus

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}, \qquad m = M-1, M-2, \ldots 1$$

Thus we compute all lower degree polynomials $A_m(z)$ beginning with $A_{m-1}(z)$ and obtain the desired lattice coefficients from the relation $K_m = \alpha_m(m)$.

The procedure works as long as $|K_m| \neq 1$ for $m = 1,2,3, \ldots, M-1$.

Example 1: Draw the lattice structure for the following FIR filter function

$$H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$

Solution: Here $A_3(z) = H(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$

So, $\alpha_3(3) = \frac{1}{3}$, $\alpha_3(2) = \frac{5}{8}$, $\alpha_3(1) = \frac{13}{24}$, $\alpha_3(0) = 1$

we know $K_m = \alpha_m(m)$, thus $K_3 = \alpha_3(3) = \frac{1}{3}$

Furthermore $B_3(z) = \frac{1}{3} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}$

The step-down relationship with $m = 3$ is

$$A_2(z) = \frac{A_3(z) - K_3B_3(z)}{1 - K_3^2} = \frac{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3} - \frac{1}{3}[\frac{1}{3} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}]}{1 - \frac{1}{9}}$$

$$= \frac{9}{8}\left[\frac{8}{9} + \frac{8}{24}z^{-1} + \frac{4}{9}z^{-2}\right] = 1 + \frac{3}{8}z^{-1} + \frac{1}{2}z^{-2}$$

$$A_2(z) = 1 + \frac{3}{8}z^{-1} + \frac{1}{2}z^{-2}$$

Hence $K_2 = \alpha_2(2) = \frac{1}{2}$, $\alpha_2(1) = \frac{3}{8}$

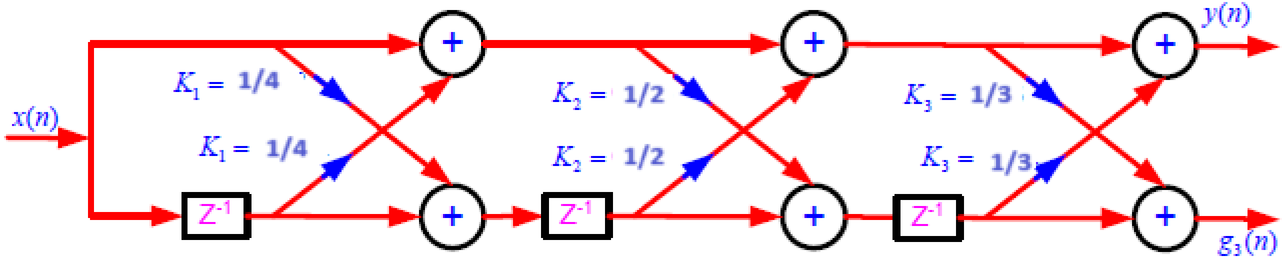
and $B_2(z) = \frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}$,

By repeating the step-down recursion, we obtain

$$A_1(z) = \frac{A_2(z) - K_2B_2(z)}{1-K_2^2} = \frac{1+\frac{3}{8}z^{-1}+\frac{1}{2}z^{-2}-\frac{1}{2}[\frac{1}{2}+\frac{3}{8}z^{-1}+z^{-2}]}{1-\frac{1}{4}} = \frac{4}{3}\left[\frac{3}{4} + \frac{3}{16}z^{-1}\right]$$

$$= 1 + \frac{1}{4}z^{-1}$$

Hence $K_1 = \alpha_1(1) = \frac{1}{4}$



$K_1 = \frac{1}{4}, K_2 = \frac{1}{2}, K_3 = \frac{1}{3}$

To check for stability, $|K_m| < 1$, thus the system is stable.

Form above example we obtain a formula for recursively computing K_m, beginning with $m = M - 1$ and stepping down to $m = 1$.

For $m = M - 1, M - 2, \ldots, 1$

We have

$$K_m = \alpha_m(m) \qquad\qquad \alpha_{m-1}(0) = 1$$

$$\alpha_{m-1}(k) = \frac{\alpha_m(k) - K_m \beta_m(k)}{1 - K_m^2}$$

$$= \frac{\alpha_m(k) - \alpha_m(m)\alpha_m(m-k)}{1 - \alpha_m^2}, 1 \leq k \leq m - 1$$

Which is again the recursion.

Example 2: Draw the lattice structure for the following FIR filter function

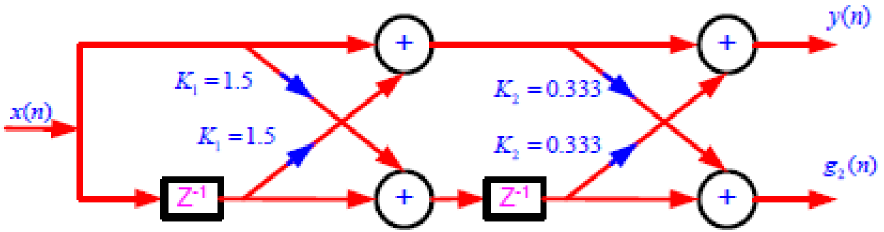
$$H(z) = 1 + 2z^{-1} + \frac{1}{3}z^{-2}$$

Solution:

$$\alpha_2(0) = 1, \alpha_2(1) = 2, \alpha_2(2) = \frac{1}{3}$$

We have $K_m = \alpha_m(m)$ so $K_2 = \alpha_2(2) = \frac{1}{3}$

And $K_1 = \frac{\alpha_2(1)}{1+\alpha_2(2)} = \frac{2}{1+\frac{1}{3}} = \frac{2}{\frac{4}{3}} = \frac{3\times 2}{4} = \frac{3}{2}$



Lattice filter

4.2 Structure of IIR Filter (Direct form I and II, Cascade, Lattice, Lattice Ladder)

In general, the time domain representation of an N^{th} order IIR system is,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

And the z-domain representation of an N^{th} order IIR system is,

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

The above two representation of IIR system can be viewed as a computational procedure (or algorithm) to determine the output sequence $y(n)$ from the input sequence $x(n)$. Also, in the above representation the value of M gives the number of zeros and the value of N gives the number of poles of the IIR system.

The computations in the above equation can be arranged into various equivalent sets of difference equations, with each set of equations defining a computational procedure or algorithm for implementing the system.

The main advantages of rearranging the sets of difference equations is to reduce the computational complexity, memory requirements and finite word length effects in computations.

For each set of equations, we can construct a block diagram consisting of delays, adders and multipliers. Such block diagrams are referred as realizations of system or equivalently as a structure for realizing system.

Some of the block diagram representation of the system gives a direct relation between the time domain equation and the z-domain equation.

The difference types of structure for realizing the IIR sysems are

i. Direct Form I & II
ii. Cascade Form
iii. Parallel
iv. Lattice & Lattice ladder

Direct-Form I & II Structures

The rational system function that characterizes an IIR system can be viewed as two systems in cascade, that is,

$$H(z) = H_1(z)H_2(\mathrm{z})$$

Where $H_1(z)$ consists of the zeros of $H(z)$, and $H_1(z)$ consists of the poles of $H(z)$,

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$

And

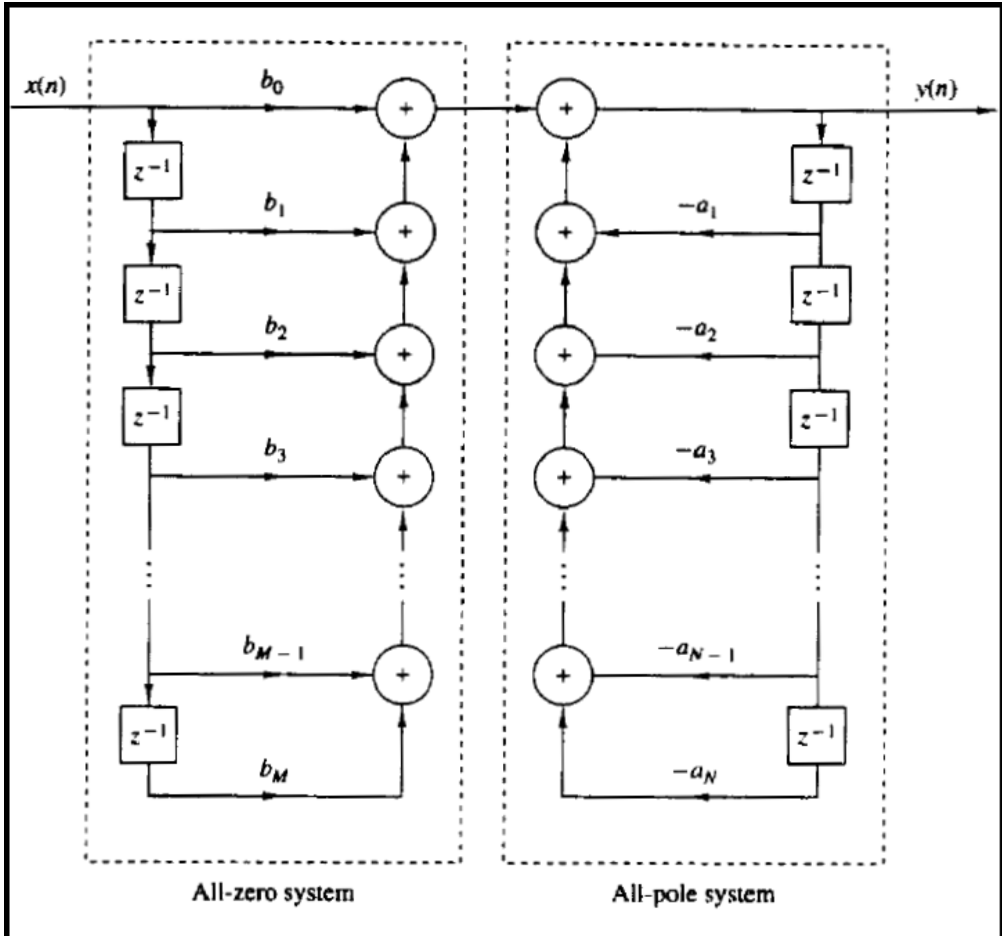
$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$H_1(z)$ is an FIR system its direct-form can be realized. By attaching the all-pole system in cascade with $H_1(z)$, we obtain the direct form I realization as:

This realization requires
$M + N + 1$ multiplication,
$M + N$ additions and
$M + N + 1$ memory locations.

Note: b_k are positive and
a_k are negative

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$



Direct form I realization.

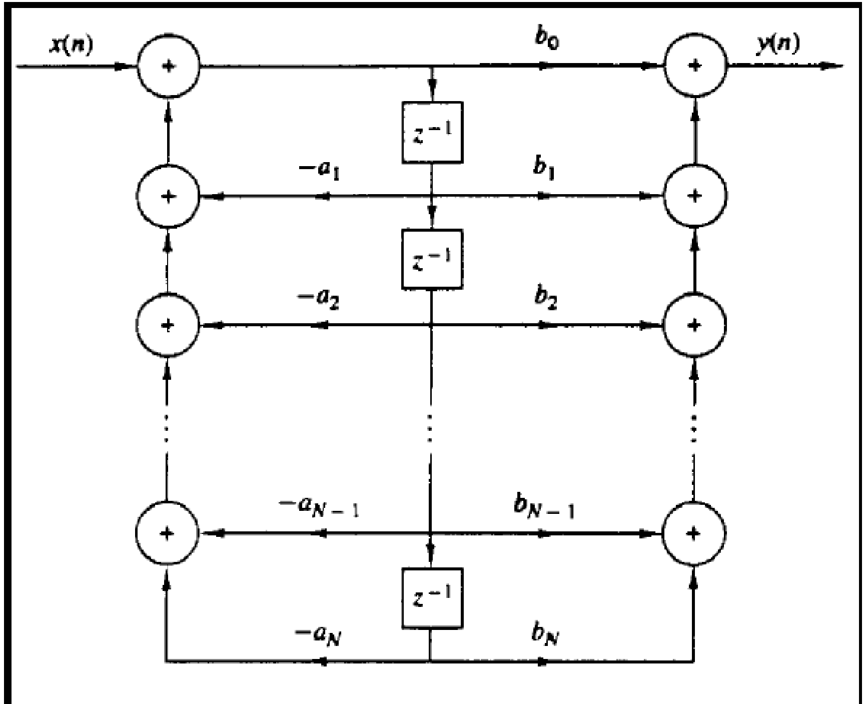
If the all pole filter $H_2(\mathrm{z})$ is placed before the all zero filter $H_1(\mathrm{z})$, a more compact structure is obtained which involves only a single delay line or a single set of memory locations. The resulting structure is called direct form II realization and is depicted as in figure.

This structure requires $M + N + 1$ multiplications, $M + N$ additions, and the maximum of $\{M, N\}$ memory locations.

This direct form II realization minimizes the number of memory locations, it is said to be canonic.

This direct form structures are extremely sensitive to parameter quantization, in general and are not recommended in practical applications.

A small change in filter coefficient due to parameter quantization, results in a large change in the locations of the poles and zeros of the system.

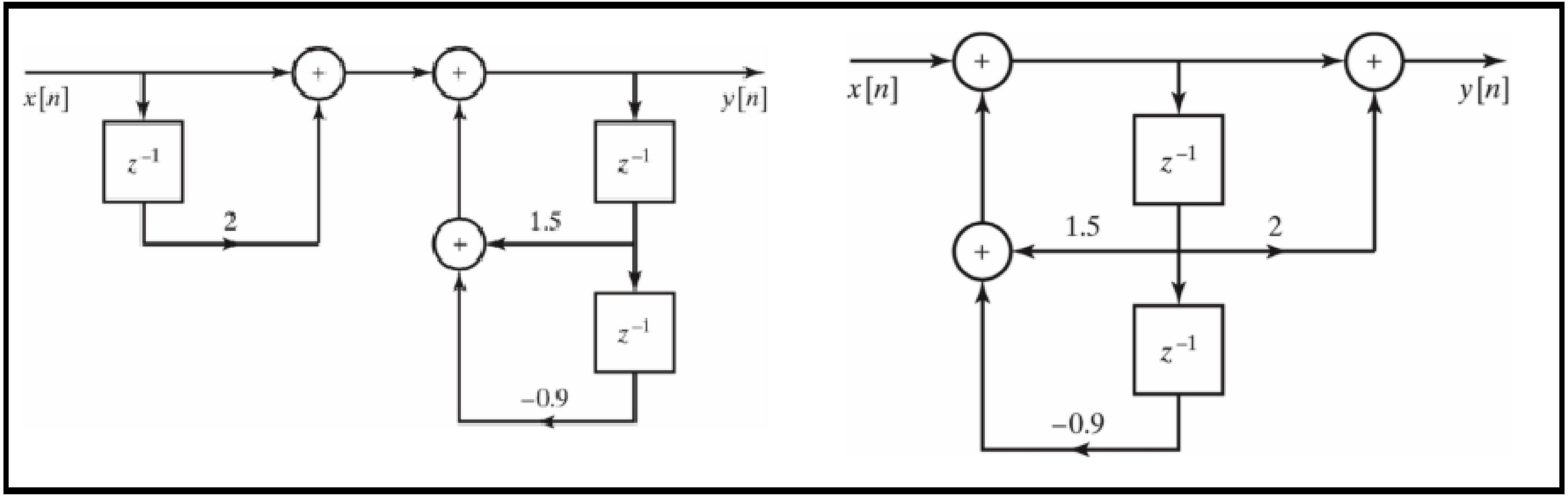


Direct form II realization ($N = M$).

Example

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

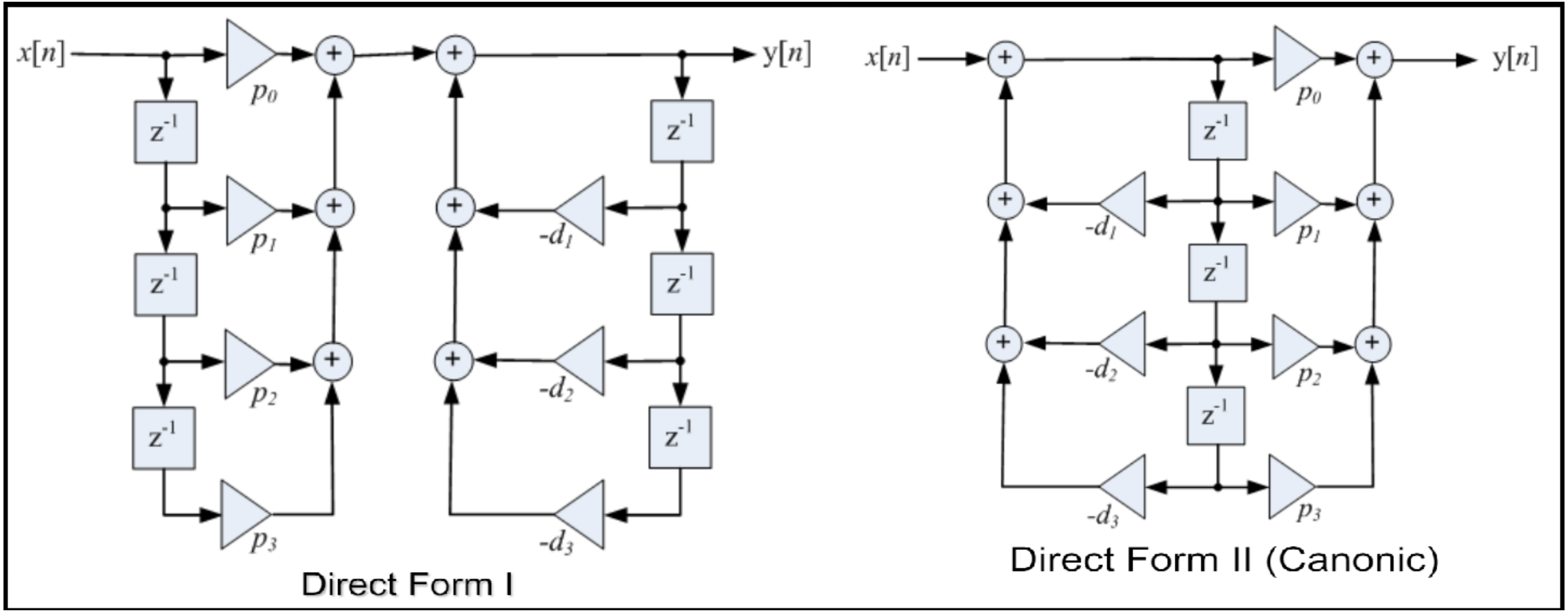
Solution: Figure Left direct form I and right direct form II.



Questions: Consider a third ordre IIR described by transfer function

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3}}{1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3}}$$

Implement as direct form I and right direct form II



Cascade-Form

The higher order IIR system given by

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

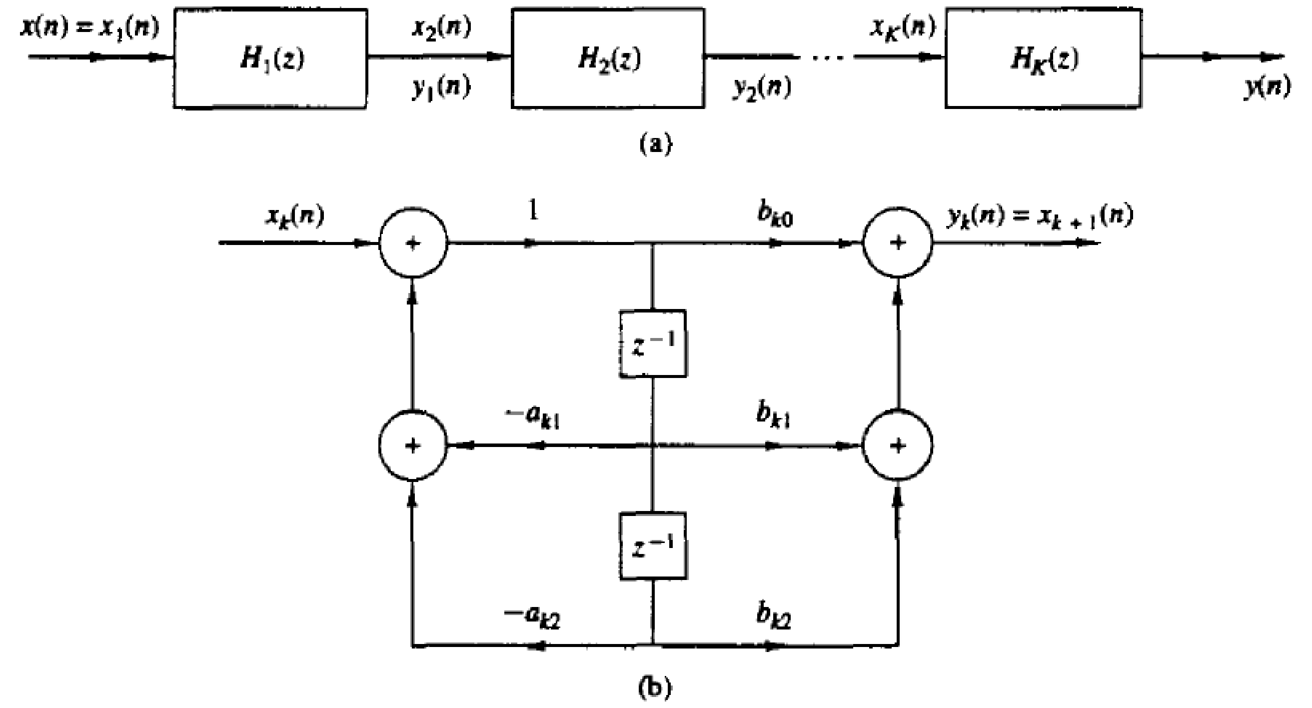
with $N > M$, can be factored into a cascade of second order subsystems, such that $H(z)$ can be expressed as

$$H(z) = \prod_{k=1}^{K} H_k(z) = \prod_{k=1}^{K} \frac{b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}}{1 + a_{k1} z^{-1} + a_{k2} z^{-2}}$$

Where $K = {}^{(N+1)}/_{2}$.

Each of the second order subsystems with system can be realized in either form I, or direct form II. Since there are many ways to pair the poles and zeros of $H(z)$ into a cascade of second-order sections, and several ways to order the resulting subsystems, it is possible to obtain a variety of cascade realizations.

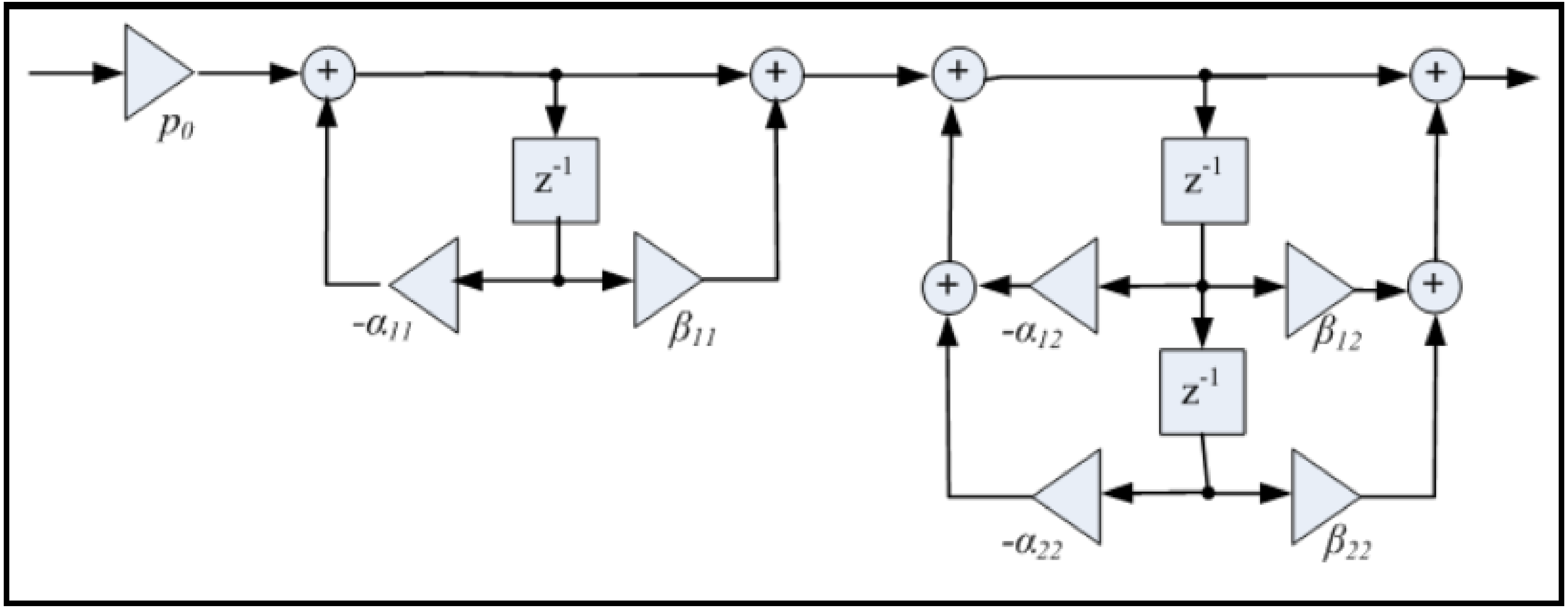
The general form of the cascade structure is illustrated in figure where we use the direct form II structure for each subsystems. This is signal flow diagram.



Cascade structure of second-order systems and a realization of each second-order section.

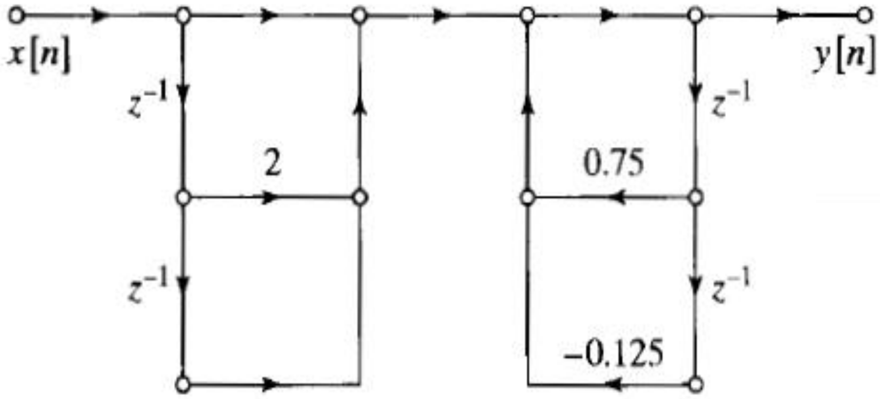
Cascade Form for third order transfer function

$$H(z) = p_0\left(\frac{1 + \beta_{11}z^{-1}}{1 + \alpha_{11}z^{-1}}\right)\left(\frac{1 + \beta_{12}z^{-1} + \beta_{22}z^{-2}}{1 + \alpha_{12}z^{-1} + \alpha_{22}z^{-2}}\right)$$

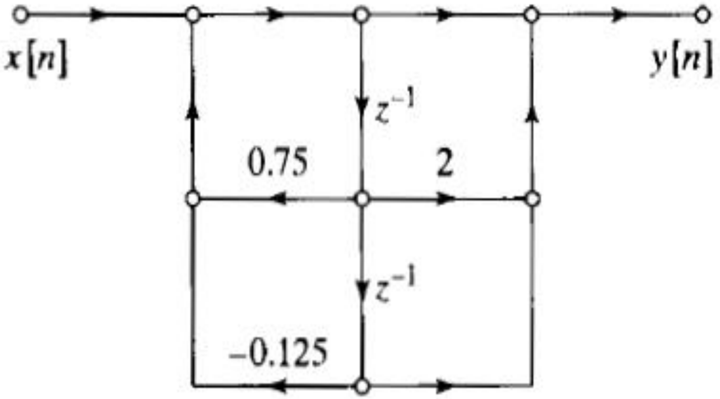


Example : Realize the transfer function using direct form I, direct form II, and Cascade form (in signal flow diagram)

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$



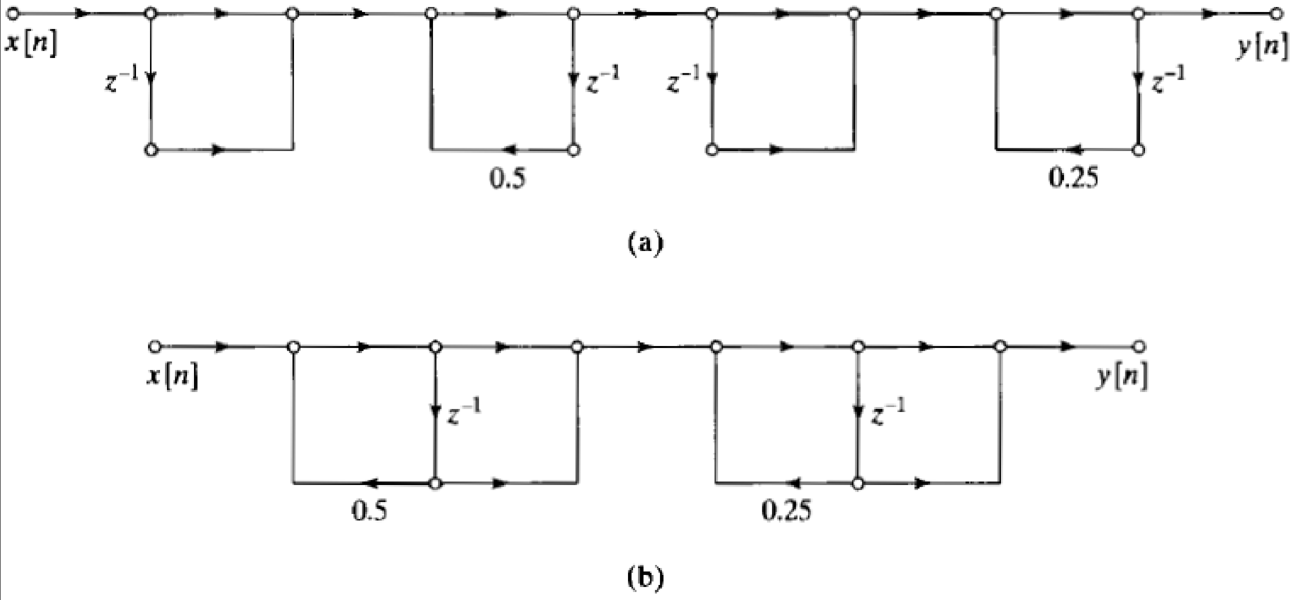
Direct form I structure



Direct form II structure

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - .025z^{-1})}$$

Here we use the first order system



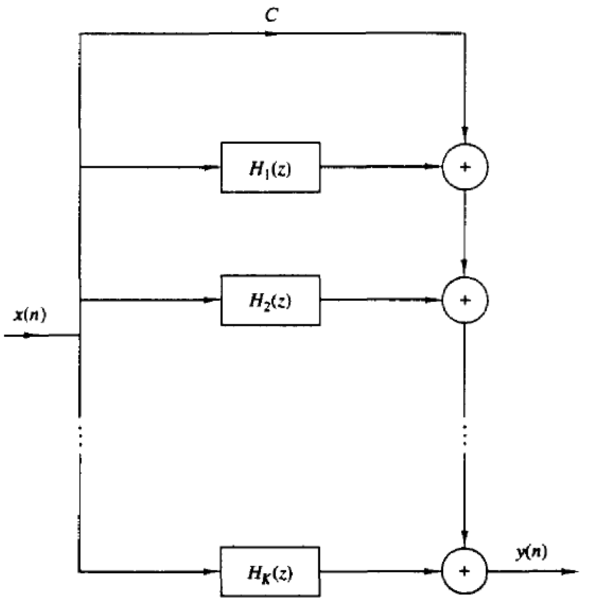
Cascade structures (a) Direct form I subsections. (b) Direct form II subsections.

Parallel Form

A parallel-form realization of an IIR system can be obtained by performing a partial-fraction expansion of $H(z)$. We assume that $N \geq M$ and that the poles are distinct. Then,

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$

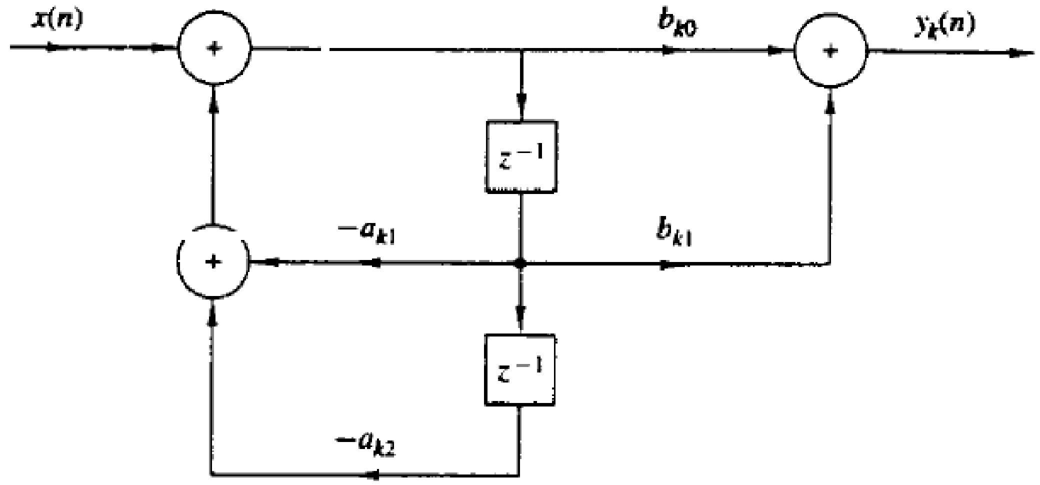
Where p_k are the poles, and A_k are the coefficients (residues) in the partial fraction expansion and C is the constant. The structure is shown in figure. It consists of parallel bank of single-pole filters



Parallel structure of IIR system.

If the poles of $H(z)$ is complex valued, then the subsystem is realized in second order having the form

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}$$



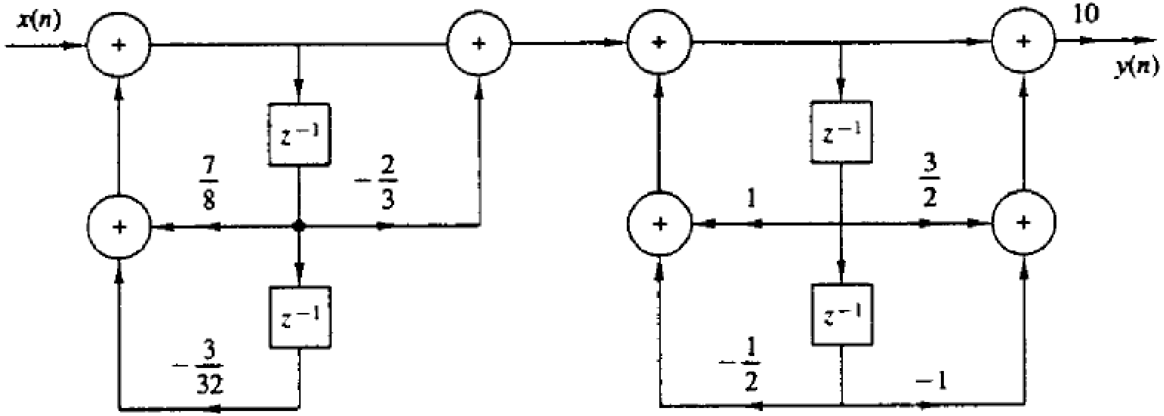
Structure of second-order section in a parallel IIR system realization.

Example 1: Determine the cascade and parallel realization for the system described by the system.

$$H(z) = \frac{10(1-\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})(1+2z^{-1})}{\left(1-\frac{3}{4}z^{-1}\right)\left(1-\frac{1}{8}z^{-1}\right)\left[1-\left(\frac{1}{2}+j\frac{1}{2}\right)z^{-1}\right]\left[1-\left(\frac{1}{2}-j\frac{1}{2}\right)z^{-1}\right]}$$

$$H(z) = 10\frac{(1-\frac{2}{3}z^{-1})}{(1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2})}\frac{(1+\frac{3}{2}z^{-1}-z^{-2})}{(1-z^{-1}+\frac{1}{2}z^{-2})}$$

Cascade realization is as in figure.



For Parallel-form realization $H(z)$ must be expanded in partial fractions.

$$H(z) = \frac{A_1}{1-\frac{3}{4}z^{-1}} + \frac{A_2}{1-\frac{1}{8}z^{-1}} + \frac{A_3}{1-(\frac{1}{2}+j\frac{1}{2})z^{-1}} + \frac{A_3^*}{1-(\frac{1}{2}-j\frac{1}{2})z^{-1}}$$

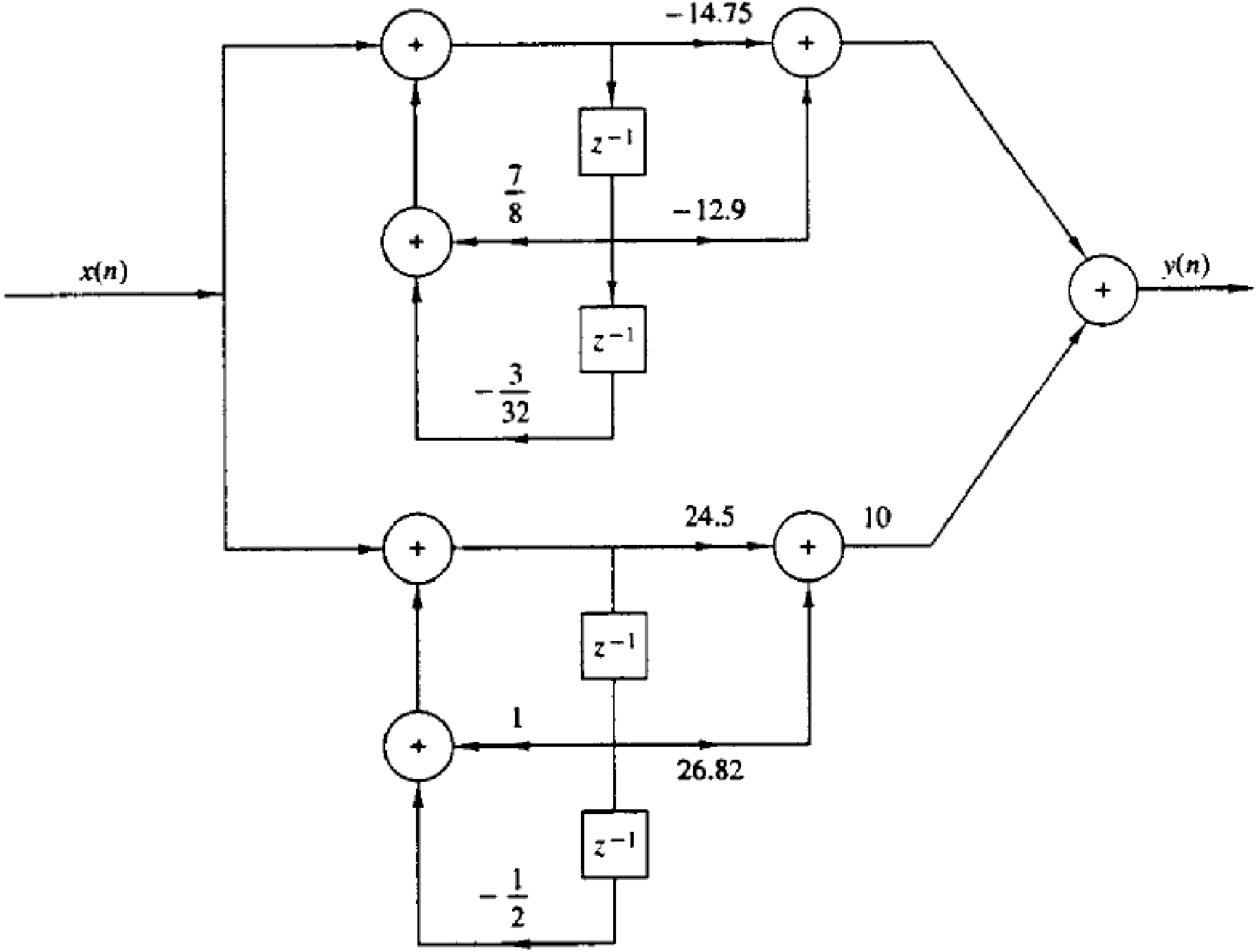
Calculating we get,

$$A_1 = 2.93, A_2 = -17.68, A_3 = 12.25 - j14.57, A_3^* = 12.25 + \mathrm{j}14.57$$

Upon recombining pairs of poles, we obtain

$$H(z) = \frac{-14.75 - 12.90z^{-1}}{1-\frac{7}{8}z^{-1}+\frac{3}{32}z^{-2}} + \frac{24.50 + 26.82z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}$$

The parallel form realization

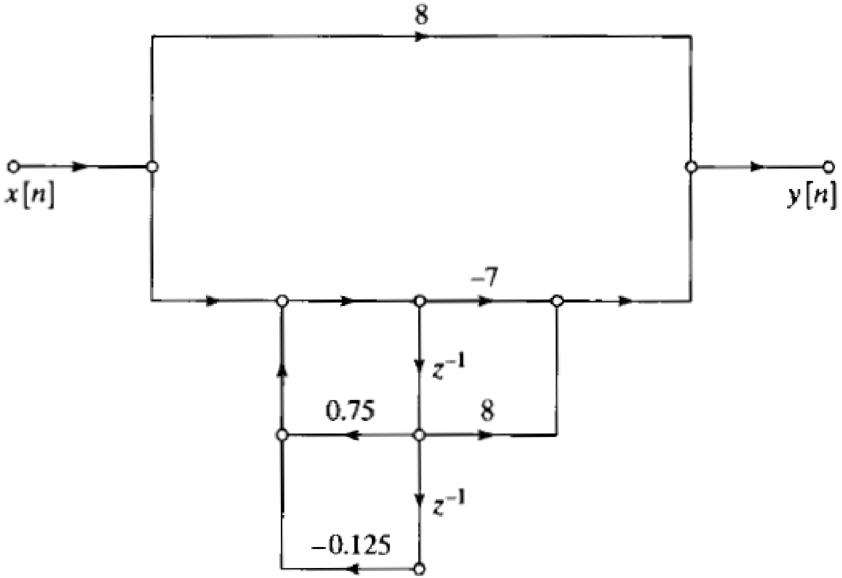


Example 2 : Realize the parallel form of

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Solution:

$$H(z) = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

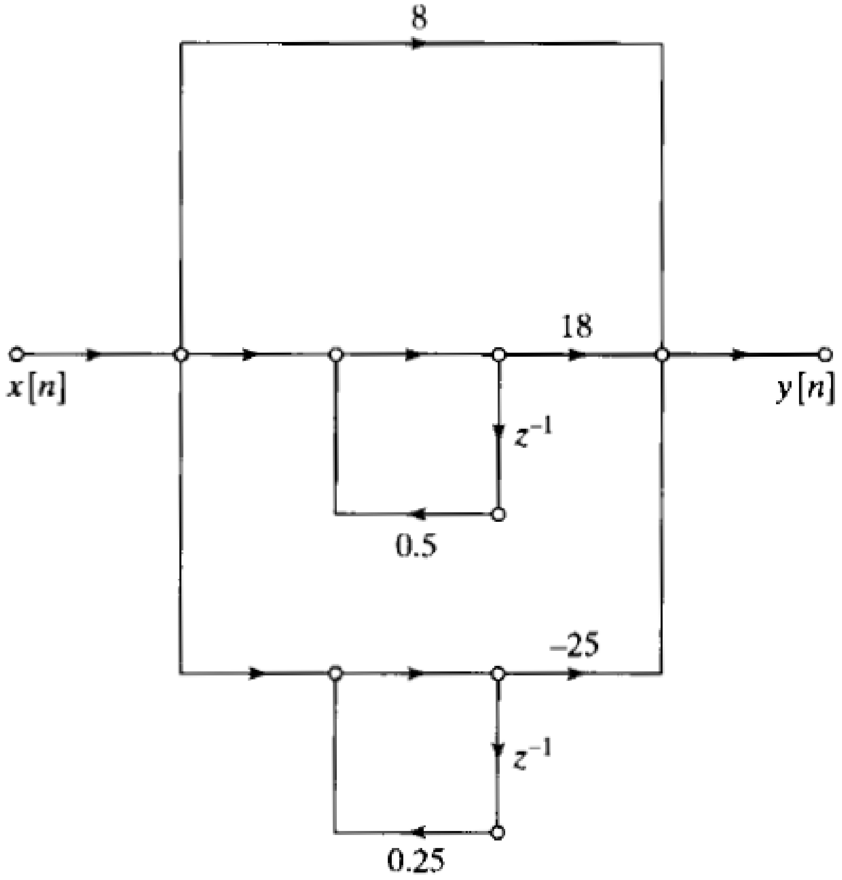


The parallel form realization with second order in signal flow is shown in figure.

Since all poles are real, we can obtain an alternative parallel form by expanding $H(z)$.

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}}$$

The resulting parallel form in signal flow with first order is as in figure.



Lattice and Lattice-Ladder Structure

An IIR system can be expressed as $H(z) = H_1(z)H_2(z)$

With an all-zero part

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$$

And all-pole part

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{1}{A_N(z)}$$

All pole IIR system → Lattice structure

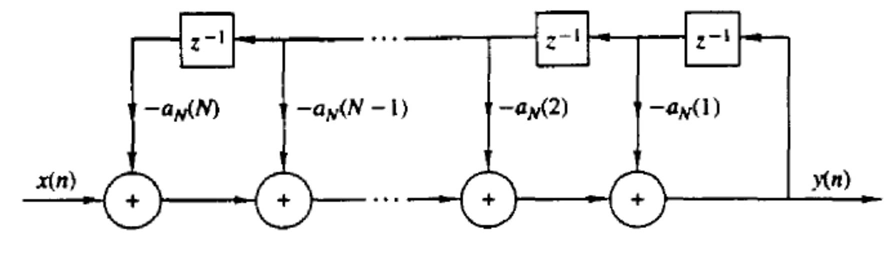
Pole-zero IIR system →Lattice-Ladder structure

Lattice Structure

Let us consider with an all-pole system with $H_2(z)$ system function.

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{1}{A_N(z)}$$

The direct form realization of this system is show in figure.



Direct-form realization of an all-pole system.

The difference equation for this IIR system is

$$y(n) = -\sum_{k=1}^{N} a_N(k) y(n-k) + x(n)$$

$$x(n) = y(n) + \sum_{k=1}^{N} a_N(k) y(n-k)$$

Thus to obtain a lattice structure for an all-pole IIR system we interchange the roles of input and output from all-zero FIR lattice.

Let us consider when N=1, for the figure as shown

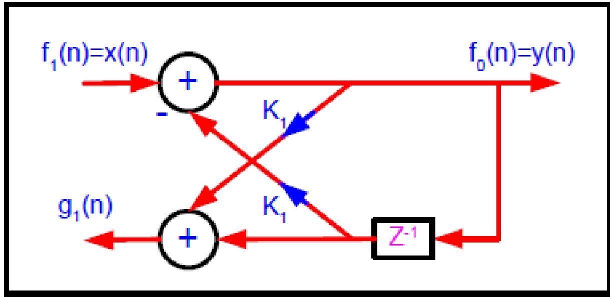
$$x(n) = f_1(n)$$

$$f_0(n) = f_1(n) - K_1 g_0(n-1)$$

$$g_1(n) = K_1 f_0(n) + g_0(n-1)$$

$$y(n) = f_0(n) = x(n) - K_1 y(n-1)$$

And $\quad g_1(n) = K_1 y(n) + y(n-1)$



Thus this equation represents a first order all-pole IIR system.

Now let consider the case N=2, which corresponds to the structure as figure below.

$$f_2(n) = x(n)$$

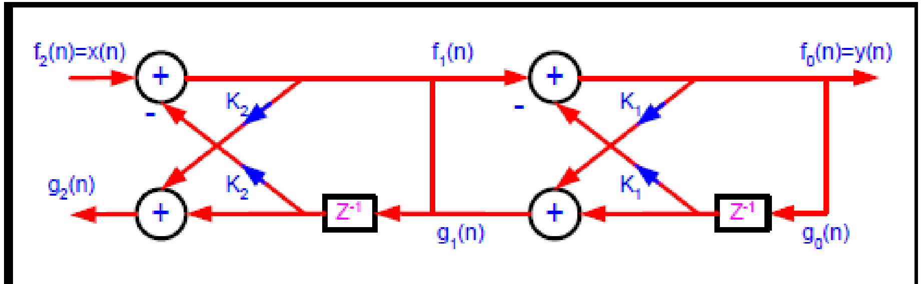
$$f_1(n) = f_2(n) - K_2 g_1(n-1)$$

$$g_2(n) = K_2 f_1(n) + g_1(n-1)$$

$$f_0(n) = f_1(n) - K_1 g_0(n-1)$$

$$g_1(n) = K_1 f_0(n) - g_0(n-1)$$

$$y(n) = f_0(n) = g_0(n)$$



Single stage Lattice filter

Substituiting we get,

$$y(n) = f_0(n) = f_1(n) - K_1 g_0(n-1) = f_2(n) - K_2 g_1(n-1) - K_1 g_0(n-1)$$

$$= f_2(n) - K_2 K_1 [f_0(n-1) - g_0(n-2)] - K_1 g_0(n-1)$$

$$= x(n) - K_2 K_1 y(n-1) + K_2 K_1 y(n-2) - K_1 y(n-1)$$

$$= -K_1(1+K_2) y(n-1) - K_2 y(n-2) + x(n)$$

Similarly,

$$g_2(n) = K_2 y(n) + K_1(1+K_2)\mathrm{y(n-1)} + \mathrm{y(n-2)}$$

Clearly, the diffference equation represents a two-pole IIR system, and the relation in this equation is the input-output equation for a two-zero FIR system.

Note: The coefficeints for the FIR system are identical to those in the IIR system except that they occur in reverse order.

In general, these conclusions hold for any N. The system function for the all-pole IIR system is

$$H_a(z) = \frac{Y(z)}{X(z)} = \frac{F_0(z)}{F_m(z)} = \frac{1}{A_m(z)}$$

Similarly, the system function of the all zero (FIR) system is

$$H_b(z) = \frac{G_m(z)}{Y(z)} = \frac{G_m(z)}{G_0(z)} = B_m(z) = z^{-m}A_m(Z^{-1})$$

And the previous relations for FIR lattices still hold

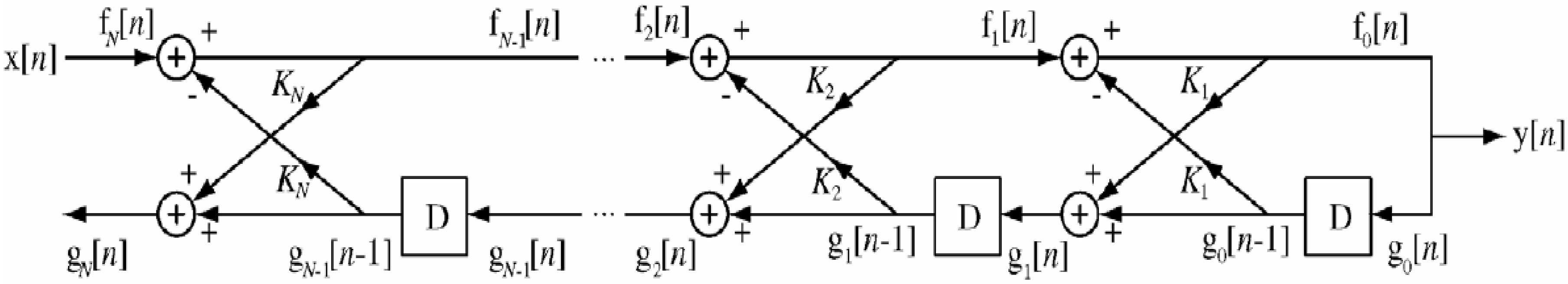
$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z), \qquad m = 1,2,\dots,M-1$$

$$B_m(z) = z^{-m}A_m(z^{-1}), \qquad m = 1,2,\dots,M-1$$

$$A_{m-1}(z) = \frac{A_m(z) - K_m B_m(z)}{1 - K_m^2}, \qquad m = M-1, M-2, \dots 1$$

The N-order lattice is shown in figure.



This is a lattice filter with similar to the FIR lattice with the role of input and output changed.

The all-zero and all pole lattice structures are characterized by the same set of lattice parameters, namely, $K_1, K_2, \dots, K_N$, but in reverse order as before.

The two lattice structure differ only in the interconnections of their signal flow graphs, the reverse (-) signs of the signal arriving from the buttom.

The all-pole lattice structure is a stable system if and only if its parameters $|K_m| < 1$ for all m.

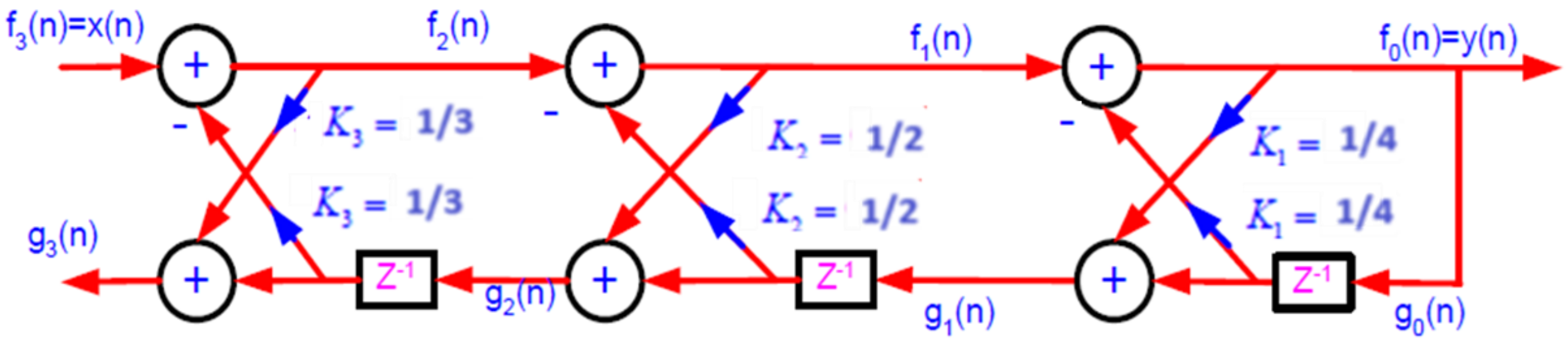
Example : Draw the lattice structure for the following IIR filter function

$$H(z) = \frac{1}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

Solution : Let $H(z) = \frac{1}{A_m(z)}$ Thus $A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$

We get $K_1 = \frac{1}{4}, K_2 = \frac{1}{2}, K_3 = \frac{1}{3}$ (see slide 28, 29)

The lattice structure is as shown in figure.



Lattice Ladder

The all-pole lattice provides the basic building block for lattice-type structures that implement IIR systems that contain both poles and zeros.

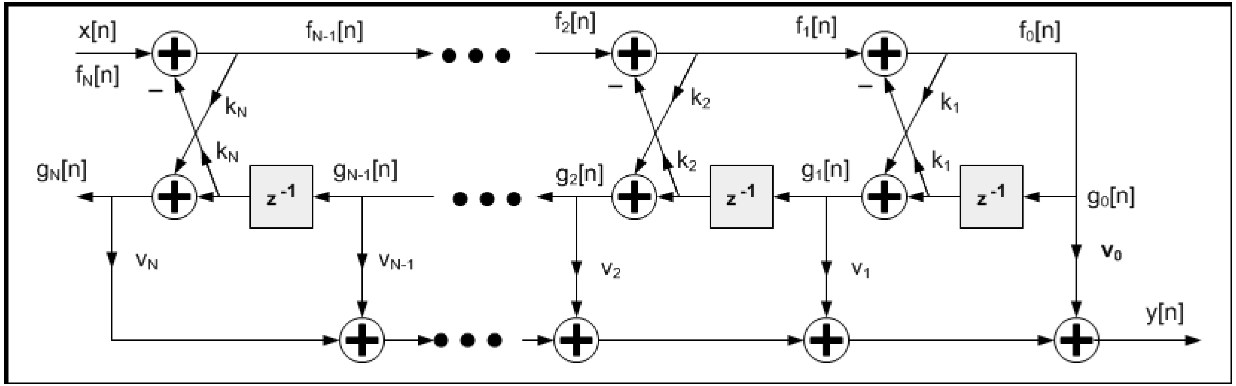
Let us consider an IIR system with system function

$$H(z) = \frac{\sum_{k=0}^{M} c_M(k) z^{-k}}{1 + \sum_{k=1}^{N} a_N(k) z^{-k}} = \frac{C_M(z)}{A_N(z)}$$

The lattice-ladder structure is shown in figure for $M = N$. Its output is for

$$y(n) = \sum_{m=0}^{M} v_m g_m(n)$$

Where v_m are the parameters that determines the zeros of the system.



The system function corresponding is

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{m=0}^{M} v_m \frac{G_m(z)}{X(z)}$$

Since $X(z) = F_N(z)$ and $F_0(z) = G_0(z)$, thus

$$H(z) = \sum_{m=0}^{M} v_m \frac{G_m(z)}{G_0(z)} \frac{F_0(z)}{F_N(z)} = \sum_{m=0}^{M} v_m \frac{B_m(z)}{A_N(z)} = \frac{\sum_{m=0}^{M} v_m B_m(z)}{A_N(z)}$$

Thus comparing we get

$$C_M(z) = \sum_{m=0}^{M} v_m B_m(z)$$

This relationship can be used to determine the weighting coefficients v_m.
Thus the coefficient of the numerator polynomial $C_M(z)$ determine the ladder parameters v_m where as the coefficients in the denominator polynomial $A_N(z)$ determine the lattice parameters K_m.

Given the polynomial $C_M(z)$ and $A_N(z)$, where N>M, the parameter of the all-pole lattice are determined first.

The ladder parameters are determined as

$$C_m(z) = \sum_{m=0}^{M-1} v_m B_m(z) + v_m B_m(z)$$

Or

$$C_m(z) = C_{m-1}(z) + v_m B_m(z)$$

Thus $C_M(z)$ can be computed recursively from the reverse polynomials $B_m(z)$, $m = 1,2,\ldots,M$. Since $B_m(m) = 1$ for all m, so

$$v_m = c_m(m), \qquad \mathrm{m} = 0,1,\ldots,\mathrm{M}$$

And

$$C_{m-1}(z) = C_m(z) - v_m B_m(z)$$

And running this recursive relation backward in m (i.e., $m = M, M-1, \ldots .2$), we obtain $C_m(m)$ and therefore the ladder parameter.

- The lattice-ladder filter structures requires the minimum amount of memory but not the minimum number of multiplications.
- The modularity, the build-in stability characteristics embodied in the coefficients K_m, and its robustness to finite-word-length effects make the lattice structure very attractive in many practical applications including speech processing systems, adaptive filtering, and geophysical signal processing.

Example Convert the following IIR filter into lattice ladder structure

$$H(z) = \frac{1 + 2z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

Solution: Let $H(z) = \frac{C_m(z)}{A_m(z)}$ Thus $A_3(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$

We get $K_1 = \frac{1}{4}, K_2 = \frac{1}{2}, K_3 = \frac{1}{3}$ (as previous)

$$B_3(z) = \frac{1}{3} + \frac{5}{8}z^{-1} + \frac{13}{24}z^{-2} + z^{-3}$$

$$B_2(z) = \frac{1}{2} + \frac{3}{8}z^{-1} + z^{-2}$$

$$B_1(z) = \frac{1}{4} + z^{-1}$$

$$C_3(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

since $v_m = c_m(m)$, thus $v_3 = c_3(3) = 1$

$$C_3(z) = \sum_{m=0}^{3} v_m B_m(z) = v_0 B_0(z) + v_1 B_1(z) + v_2 B_2(z) + v_3 B_3(z)$$

$$= v_0 \times 1 + v_1 \left[\frac{1}{4} + z^{-1}\right] + v_2 \left[\frac{1}{2} + \frac{3}{8} z^{-1} + z^{-2}\right] + v_3 \left[\frac{1}{3} + \frac{5}{8} z^{-1} + \frac{13}{24} z^{-2} + z^{-3}\right]$$

$$= \left(v_0 + \frac{1}{4} v_1 + \frac{1}{2} v_2 + \frac{1}{3} v_3\right) + \left(v_1 + \frac{3}{8} v_2 + \frac{5}{8} v_3\right) z^{-1} + \left(v_2 + \frac{13}{24} v_3\right) z^{-2} + v_3 z^{-3}$$

Comparing we get,

$$v_3 = 1$$

$$v_2 + \frac{13}{24} v_3 = 2, thus\ \ v_2 = 2 - \frac{13}{24} \times 1 = 1.4583$$

$$v_1 + \frac{3}{8} v_2 + \frac{5}{8} v_3 = 2, \text{thus } v_1 = 0.8281$$

$$v_0 + \frac{1}{4} v_1 + \frac{1}{2} v_2 + \frac{1}{3} v_3 = 1, \text{thus} v_0 = -0.2698$$

Alternative way to find the ladder coefficient

$$B_3(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

Thus $\beta_0 = 1, \beta_1 = 2, \beta_2 = 2, \beta_3 = 1$

Also

$\alpha_3(3) = \frac{1}{3}$, $\alpha_3(2) = \frac{5}{8}$, $\alpha_3(1) = \frac{13}{24}$, $\alpha_3(0) = 1$

$\alpha_2(2) = \frac{1}{2}$, $\alpha_2(1) = \frac{3}{8}$

$\alpha_1(1) = \frac{1}{4}$

The ladder coefficient is obtained by,

$$v_m = \beta_m - \sum_{i=m+1}^{M} v_i \alpha_i(i-m), \qquad m = M-1, \dots .0$$

For $\mathrm{m} = 3, v_3 = \beta_3 = 1$

For $\mathrm{m} = 2$,

$$v_2 = \beta_2 - \sum_{i=2+1}^{3} v_i \alpha_i (i-2) = 2 - \sum_{i=3}^{3} v_3 \alpha_3 (3-2) = 2 - v_3 \alpha_3 (1) = 2 - 1 \times \frac{13}{24} = 1.4583$$

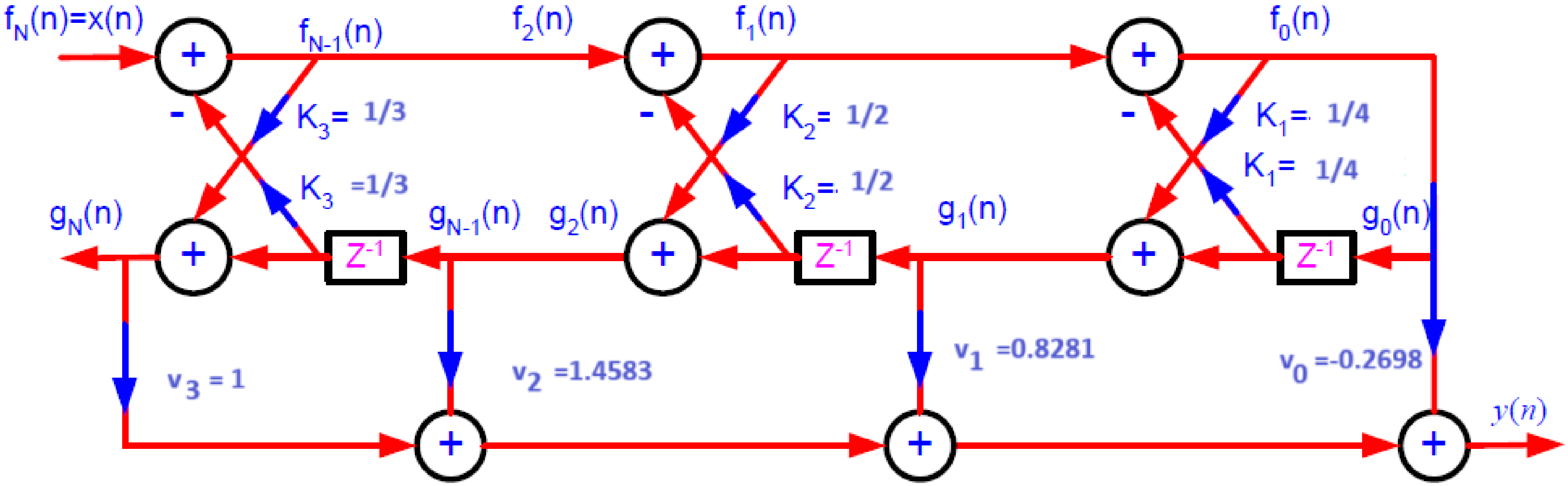
For $\mathrm{m} = 1$,

$$v_1 = \beta_1 - \sum_{i=2}^{3} v_i \alpha_i (i-1) = \beta_1 - [v_2 \alpha_2 (1) + v_3 \alpha_3 (2)] = 2 - \left[1.458 \times \frac{3}{8} + 1 \times \frac{5}{8}\right] = 0.8281$$

For $\mathrm{m} = 0$,

$$v_0 = \beta_0 - \sum_{i=1}^{3} v_i \alpha_i (i) = \beta_0 - [v_1 \alpha_1 (1) + v_2 \alpha_2 (2) + v_3 \alpha_3 (3)]$$

$$= 1 - \left[0.8281 \times \frac{1}{4} + 1.4583 \times \frac{1}{2} + 1 \times \frac{1}{3}\right] = -0.2698$$



Lattice filter

Assignment :5 marks(7th jan 2025)

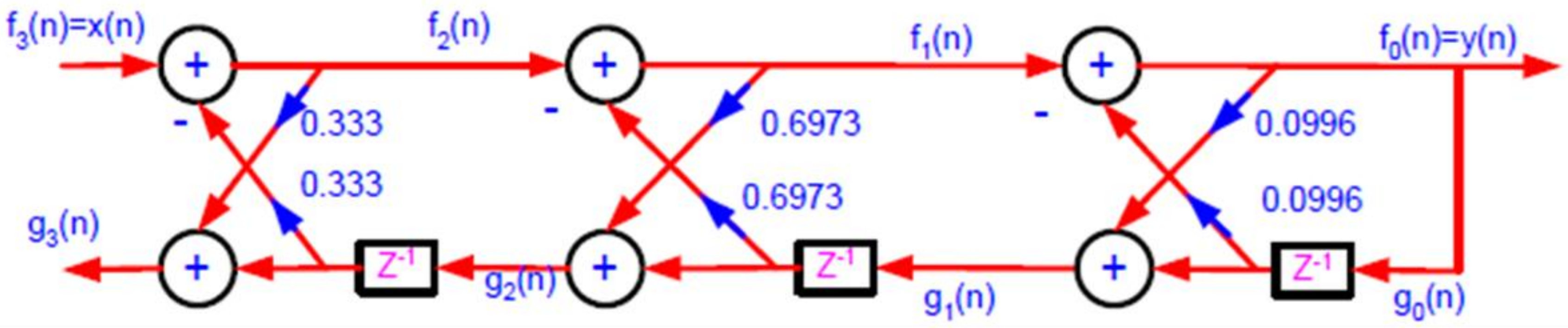
1: Draw the lattice structure for the following IIR filter function

$$H(z) = \frac{1}{1 + \frac{2}{5}z^{-1} + \frac{3}{4}z^{-2} + \frac{1}{3}z^{-3}}$$

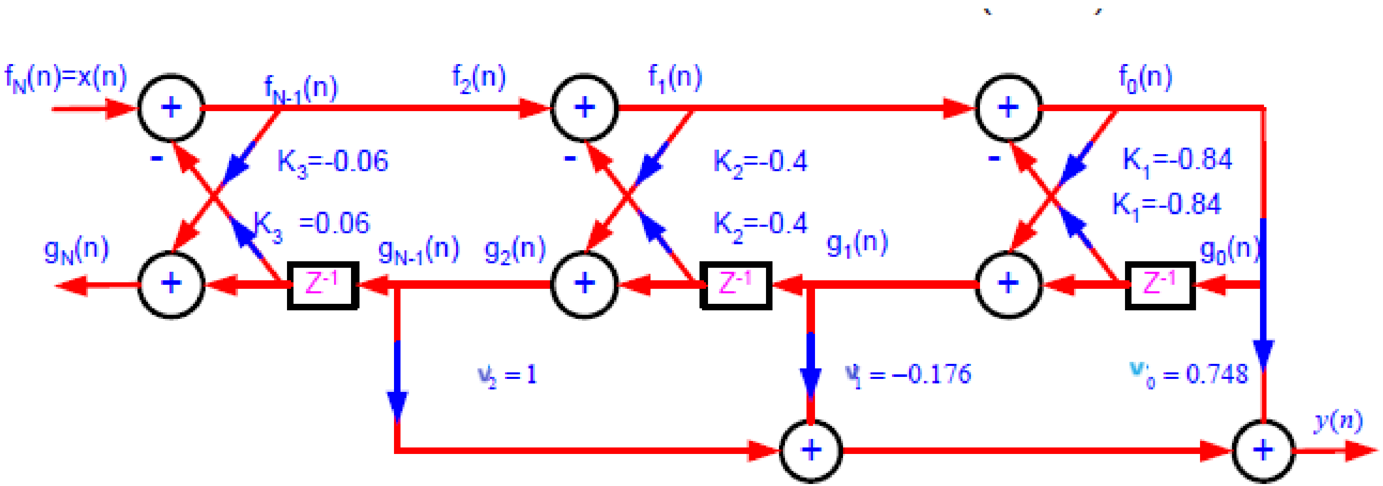
2: Realize the IIR transfer function using lattice ladder structure

$$H(z) = \frac{1 + z^{-1} + z^{-2}}{(1 + 0.5z^{-1})(1 + 0.3z^{-1})(1 + 0.4z^{-1})}$$

1 answer $K_1 = 0.0996, K_2 = 0.6937, K_3 = \frac{1}{3}$



2 answer



4.3 Quantization effect (truncation, rounding), limit cycles and scaling

- In the realization of FIR and IIR filters in hardware or in software on a general purpose computer, the accuracy with which filter coefficients can be specified is limited by the word length of the computer or the register provided to store the coefficient.
- Since, the coefficient used in implementing a given filter are not exact, the poles and zeros of system function will, in general, be different from the desired poles and zeros.
- Consequently, we obtain a filter having a frequency response of the filter with un-quantized coefficients.
- The sensitivity of the filter frequency response characteristics to quantization of the filter coefficients is minimized by realizing a filter having a large number of poles and zeros as an interconnection of second order filter sections.
- This leads to the parallel form and cascade form realizations in which the basic blocks are second order filter sections.
- Limit-cycle oscillations occurs in IIR filters as a result of quantization effects in fixed point multiplication and rounding.
- The lattice and lattice-ladder filter structure are known to be robust in fixed-point implementations.

Questions:

1. Obtain the parallel form realization of following IIR filter

$$H(z) = \frac{1 + \frac{1}{4}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2})}$$

2. Convert the following IIR filter into lattice ladder structure

$$H(z) = \frac{1 + z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

3. Convert the following in Direct form- II realization described by

$$H(z) = \frac{1 + z^{-1} + 2z^{-2} + 3z^{-3}}{1 + 15z^{-1} + 15z^{-2} + 12z^{-3}}$$

4. Realize the transfer function using cascade realization

$$X(z) = \frac{2(z + 2)}{z(z - 0.1)(z + 0.5)(z - 0.4)}$$

5. Draw the lattice structure of the given transfer function

$$H(z) = 1 + \frac{3}{8}z^{-1} + \frac{5}{4}z^{-2} + \frac{3}{2}z^{-3}$$

6. Obtain the parallel form realization of following filter

$$H(z) = \frac{3(2z^2 + 5z + 4)}{(2z + 1)(z + 2)}$$

7. Compute Lattice coefficients and draw lattice structure for given IIR system. Also check stability of the system

$$H(z) = \frac{1}{(1 - 0.525z^{-1} + 0.6125z^{-2} + 0.3z^{-3})}$$

8. Define Digital Filters. Differentiate between IIR & FIR Digital Filters.
9. Realize the following system by its direct form I and form II structure.

$$y[n] \; = \; x[n] - \; 2x[n - 1] \; + \; 0.5x\,[n - 2] \; + \; 0.8y\,[n - 2]$$

10. What are the advantages of FIR filters compared to the IIR filters?
11. Determine the direct form realizations of the following difference equation

$$2y(n) + 3y(n - 1) + 5y(n - 2) = x(n) + 2x(n - 3)$$

12. Obtain the lattice ladder structure and parallel realization of the discrete time system described by the difference equation and compare with cascade and parallel realization.

$$y(n) = -\frac{3}{4}y(n - 1) + \frac{1}{4}y(n - 2) + x(n) + \frac{1}{2}x(n - 1)$$