Digital Signal Analysis And Processing

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2. Review of Z-Transform 6 Hrs

- **2.1 Defination z-transform,Convergence of Z-transform, and Region of convergence**
- **2.2 Properties of Z-Transform (Linearity, Time shift, Multiplication by exponential sequence, Differentiation, Time reversal, Convolution, Multiplication)**
- **2.3 Inverse Z-transform-by long division, by partial fraction expansion**

2. Review of Z-Transform

- Transform techniques are an important tool in the analysis of signals and linear time invariant (LTI) systems.
- The z-transform plays the same role in the analysis of discrete time signals and LTI systems as the Laplace transform does in the analysis of continuous time signal and LTI systems.
- In the z-domain (complex z-plane) the convolution of two time domain signals is equivalent to multiplication of their corresponding z-transforms. This property greatly simplifies the analysis of the response of an LTI system to various signals.
- In addition, the z-transform provides us with a means of characterizing an LTI system, and its response to various signals, by its pole-zero locations.
- Because of convergence condition, in many cases, the discrete time Fourier transform of a sequence may not exist, and it is not possible to make use of such frequency domain characterization in these cases.
- A generalization of the discrete time Fourier transform leads to the z-transform, which may exist for many sequences for which the discrete time Fourier transform does not exist.
- Moreover, the use of z-transform techniques permits simple algebraic manipulations.
- Consequently, the z-transform has become an important tool in the analysis and design of digital filters.

2.1 Defination z-transform

The Fourier transform of a sequence $x(n)$ was defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

The z-transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Where z is the complex variable. The above relation is sometimes called the direct z-transform because it transforms the time domain signal $x(n)$ into its complex plane representation $X(z)$. For convience, the z-transform of a signal $x(n)$ is denoted by

$$X(z) \equiv Z\{x(n)\}$$

Whereas the relationship between x(n) and X(z) is indicated by

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

Since the z-transform is an infinite power series, it exists only for those values of z for which this series converges. The region of convergence (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value. Thus any time we cite a z-transform we should also indicate its ROC.

Examples: Determine the z-transform of the following finite duration signals.

a. $x_1(n)$= (1, 2, 5, 7, 0, 1)
b. $x_2(n)$= (1, 2, $\underline{5}$, 7, 0, 1)
c. $x_3(n) = \delta(n)$
d. $x_4(n) = \delta(n-k), k > 0$
e. $x_5(n) = \delta(n+k), k > 0$

Solution:

a. $X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$, ROC: entire z-plane except $z = 0$.
b. $X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$, ROC: entire z-plane except $z = 0$ and $z = \infty$.
c. $X_3(z) = 1$ [$\delta(n) \xleftrightarrow{z} 1$] ROC: entire z-plane
d. $X_4(z) = z^{-k}$ [$\delta(n-k) \xleftrightarrow{z} z^{-k}$], $k > 0$. ROC: entire z-plane except $z = 0$
e. $X_5(z) = z^{k}$ [$\delta(n+k) \xleftrightarrow{z} z^{k}$], $k > 0$. ROC: entire z-plane except $z = \infty$

It is seen that the ROC of a finite duration signal is the entire z-plane , except possibly the point $z = 0$ and/or $z = \infty$. These points are excluded, because $z^k (k > 0)$ becomes unbounded for $z = \infty$ and $z^{-k} (k > 0)$ becomes unbounded for $z = 0$.

The z-transform, defined by the equation

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

is often referred to as the two sided or bilateral z-transform, in contrast to the one-sided or unilateral z-transform, which is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

Clearly, the bilateral and unilateral transforms are equivalent only $if\ x(n) = 0, for\ n < 0$.

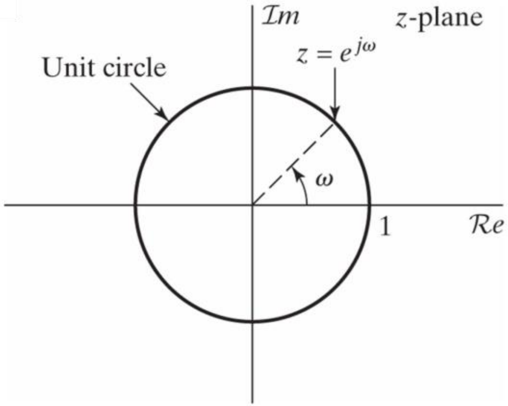
We can express the complex variable z in polar form as $z = re^{j\omega}$

Thus

$$X\left(re^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n} = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

Which can be interpreted as Fourier transform of the product of the original signal sequence $x(n)$ and the exponential sequence r^{-n}. Obviously, for $r = 1$, this equation reduces to the Fourier transform of $x(n)$.

Since the z-transform is a function of a complex variable, it is convenient to describe and interpret it using the complex z-plane. In the z-plane, the contour corresponding to $|z| = 1$ is a circle of unit radius as in figure. This contour is referred to as the unit circle. The z-transform evaluated on the unit circle corresponds to the Fourier transform. Note ω is the angle between the vector to a point z on the plane unit circle and the real axis of the complex z-plane.



If we evaluate $X(z)$ at points on the unit circle in the z-plane beginning at $z = 1$, (i.e., $\omega = 0$), through $z = j$ (i.e., $\omega = \frac{\pi}{2}$) to $z = -1$ (i.e., $\omega = \pi$), we obtian the Fourier transform for $0 \leq \omega \leq \pi$. Continuing around the unit circle would correspond to examining the Fourier transform from $\omega = \pi$ to $\omega = 2\pi$ or equivalently, from $\omega = -\pi$ to $\omega = 0$.

The Fourier transform was displayed in a linear frequency axis. Interpreting the Fourier transform as the z-transform on the unit circle in the z-plane corresponds conceptually to wrapping the linear frequency axis around the unit circle with $\omega = 0$ at $z = 1$ and $\omega = \pi$ at $z = -1$.

With this interpretation, the inherent periodicity in frequency of the Fourier transform is captured naturally, since a change of angle of 2π radians in the z-plane corresponds to traversing the unit circle once and returning to exactly the same point.

Convergence of Z-transform, and Region of convergence

- The power series representing the Fourier transform does not converge for all sequences; i.e., the the infinite sum may not always be finite.
- Similarly, the z-transform does not converge for all sequences or for all values of z.
- For any given sequence, the set of values of z for which the z-transform converges is called the region of convergence, which we abbreviate ROC.
- If the sequence is absolutely summable, the Fourier transform converges to a continuous function of ω. Thus

$$\sum_{n=-\infty}^{\infty} |x(n)r^{-n}| < \infty$$

for convergence of the z-transform.

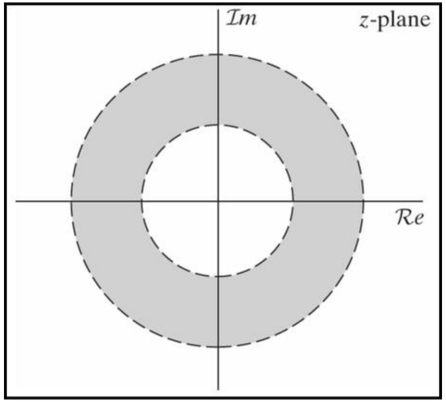
- Because of the multiplication of the sequence by the real exponential r^{-n}, it is possible for the z-transform to converge even if the Fourier transform doesnot.

- For example, the sequence $x(n) = u(n)$ is not absolutely summable, and therefore, the Fourier transform does not converges.
- However, $r^{-n}u(n)$ is absolutely summable if $r > 1$.
- This means that z-transform for the unit step exists with a region of convergence $|z| > 1$.
- Convergence of the power series of z-transform depends only on $|z|$, since $|X(z)| < \infty$ if

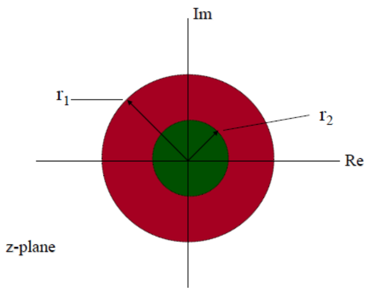
$$\sum_{n=-\infty}^{\infty} |x(n)||z^{-n}| < \infty\,,$$

- i.e., the region of convergence of the power series of z-transform consists of all values of z such that the inequality holds.
- Thus, if some values of z, say $z = z_1$, is in the ROC, then all values of z on the circle defined by $|z| = |z_1|$ will also be in the ROC.

- As, one consequence of this, the region of convergence will consist of a ring in the z-plane centered about the origin.
- Its outer boundary will be a circle (or the ROC may extend outward to infinity), and its inner boundary will be a circle (or it may extend inward to include the origin).
- Figure shows the region of convergence (ROC) as a ring in the z-plane. For specific cases, the inner boundary can extend inward to the origin, and the ROC becomes a disk. For other cases, the outer boundary can extend outward to infinity.
- If the ROC includes the unit circle this of course implies convergence of the z-transform for $|z| < 1$, or equivalently, the Fourier transform of the sequence converges.
- Conversely, if the ROC does not include the unit circle, the Fourier transform does not converge absolutely.



The ROC as a Ring in the z-plane



- The z-transform and all its derivaties must be continuous function of z within the ROC (follow a Laurent series).
- $X(z)$ can be represented in a Laurent (one type of power) series and a rational function of polynomial in z.
- The ROC of a causal signal is the exterior of a circle of some radius r_2 while the ROC of an anticausal is the interior of a circle of some radius r_1.
- Among the most important and useful z-transforms are those for which $X(z)$ is a rational function inside the region of convergence, i.e.,

$$X(z) = \frac{P(z)}{Q(z)}$$

where $P(z)$ and $\mathrm{Q}(z)$ are polynomials in z.

The values of z for which $X(z) \ = 0$ are called the **zeros** of $X(z)$

The values of z for which $X(z) = \infty$ are referred to as the **poles** of $X(z)$.

- Poles may occurs at $z = 0 \; or \; z = \infty$.
- For rational z-transforms, a number of important relationship exist between the locations of poles of $X(z)$ and the region of convergence of the z-transform.

Example 1: Right-sided Exponential Sequence. Consider the signal $x(n) = a^n u(n)$. Because it is nonzero only for $n \geq 0$, this is an example of a right-sided sequence.

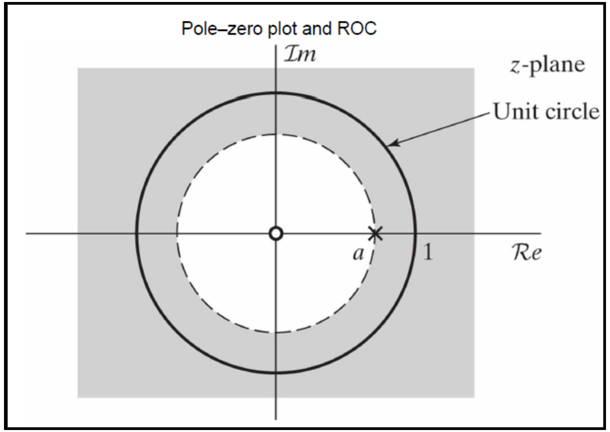
$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of convergence is the range of values of z for which $|az^{-1}| < 1$ or equivalently, $|z| > |a|$. Inside the region of convergence, the infinite series converges to

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \ |z| > |a|$$

The z-transform has a region of convergence for any finite value of $|a|$. The Fourier transform of $x(n)$, on the other hand, converges only if $|a| < 1$. For $a = 1, x(n)$ is the unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, \qquad |z| > 1$$



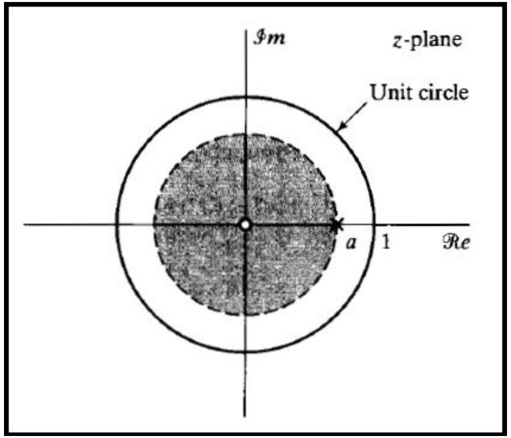
The pole-zero plot and the region of convergence are shown in figure where a “o” denotes the zero and “x” the pole.

Example 2: Left-sides Exponential Sequence: Consider the signal $x(n) = -a^n u(-n-1)$. Since the sequence is nonzero for $n \leq -1$, this is a left-sided sequence. Then

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$



If $|a^{-1}z| < 1$ or, equivalently, $|z| < |a|$, the sum converges, and

$$X(z) = 1 - \frac{1}{1 - a^{-1}z} = \frac{z}{z-a}, \qquad |z| < |a|$$

The pole and zero plot and region of convergence is shown in figure. Note that for $|a| < 1$, the sequence $-a^n u(-n-1)$ grows exponentially as $n \to \infty$, and thus, the Fourier transform does not exist.

- Comparing above two examples, we see that the sequences and, the infinite sums are different;
- However, the algebraic expressions for $X(z)$ and the corresponding pole-zero plots are identical.
- The z-transforms differ only in the region of convergence.
- This emphasizes the need for specifying both the algebraic expression and the region of convergence for the z-transform of a sequence.

Example 3: Sum of Two Exponential Sequences

Consider a signal that is sum of two real exponentials:

$$x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)$$

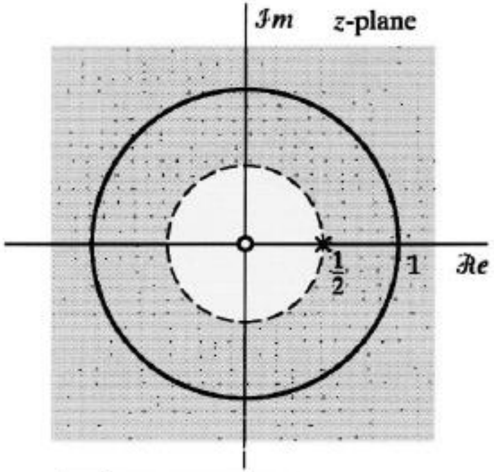
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} \{\left(\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{3}\right)^n u(n)\} z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n} + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u(n) z^{-n}$$

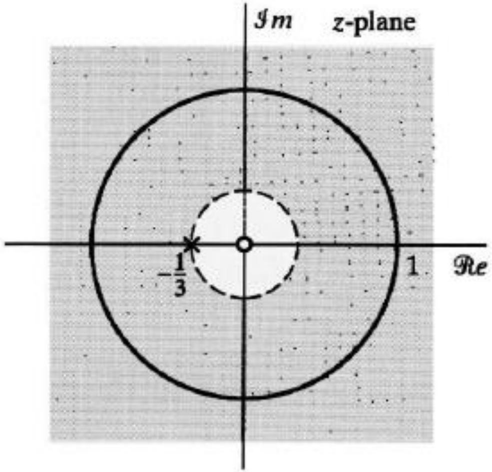
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{3} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}}$$

$$= \frac{2(1 - \frac{1}{12} z^{-1})}{(1 - \frac{1}{2} z^{-1})(1 + \frac{1}{3} z^{-1})} = \frac{2(z - \frac{1}{12})}{(z - \frac{1}{2})(z + \frac{1}{3})}$$

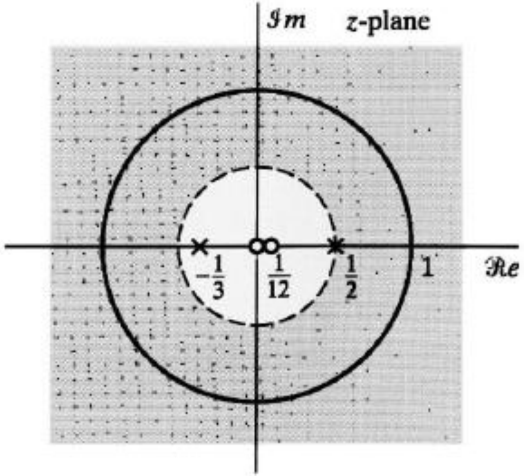
For convergence of $X(z)$, both sums must converge, which requires that both $\left|\frac{1}{2}z^{-1}\right| < 1$ and $\left|(-\frac{1}{3})z^{-1}\right| < 1$ or, equivalently, $|z| > \frac{1}{2}$ and $|z| > \frac{1}{3}$.
Thus, the region of convergence is the region of overlap, $|z| > \frac{1}{2}$. The pole-zero plot and ROC for the z-transform of each of the individual terms and for the combined signal are shown in figure.



$1/(1 - \frac{1}{2}z^{-1})$, $|z| > \frac{1}{2}$.



$1/(1 + \frac{1}{3}z^{-1})$, $|z| > \frac{1}{3}$



$1/(1 - \frac{1}{2}z^{-1}) + 1/(1 + \frac{1}{3}z^{-1})$, $|z| > \frac{1}{2}$.

SOME COMMON z-TRANSFORM PAIRS

Sequence	Transform	ROC
1. $\delta[n]$	1	All z
2. $u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3. $-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
4. $\delta[n-m]$	z^{-m}	All z except 0 (if $m>0$) or ∞ (if $m<0$)
5. $a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
6. $-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
7. $na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
8. $-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$

Sequence	Transform	ROC
9. $[\cos\omega_0 n]u[n]$	$\frac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
10. $[\sin\omega_0 n]u[n]$	$\frac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
11. $[r^n\cos\omega_0 n]u[n]$	$\frac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
12. $[r^n\sin\omega_0 n]u[n]$	$\frac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
13. $\begin{cases} a^n, & 0\le n\le N-1, \\ 0, & \text{otherwise} \end{cases}$	$\frac{1-a^Nz^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$

Example 4: Two-Sided Exponential Sequence

Consider the sequence

$$x(n) = \left(-\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(-n-1)$$

Note that this sequence grows exponentially as $n \to \infty$. Using general result of table.

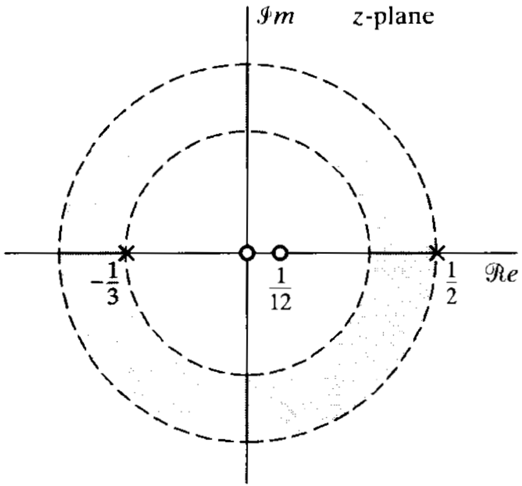
$$\left(-\tfrac{1}{3}\right)^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1+\frac{1}{3}z^{-1}}, \qquad |z| > \tfrac{1}{3},$$

$$-\left(\tfrac{1}{2}\right)^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1-\frac{1}{2}z^{-1}}, \qquad |z| < \tfrac{1}{2}.$$

Thus

$$X(z) = \frac{1}{1+\frac{1}{3}z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}, \quad \tfrac{1}{3} < |z|, \quad |z| < \tfrac{1}{2},$$

$$= \frac{2\left(1-\frac{1}{12}z^{-1}\right)}{\left(1+\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} = \frac{2z\left(z-\frac{1}{12}\right)}{\left(z+\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}.$$



Pole–zero plot and region of convergence

Example: Finite-Length Sequence: Consider the signal

$$x(n) = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

Then

$$X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n = \frac{1-(az^{-1})^n}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

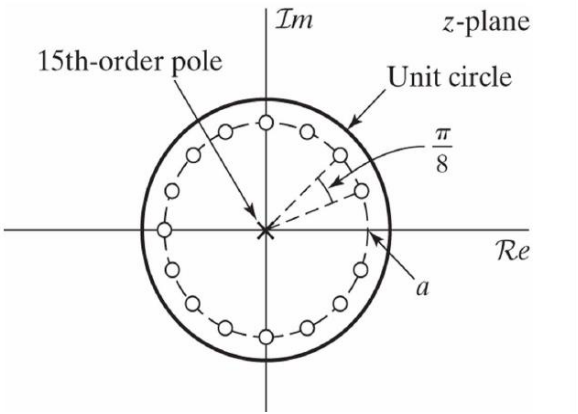
The ROC is determined by the set of values of z for which

$$\sum_{n=0}^{N-1} |az^{-1}|^n < \infty$$

Since there are only a finite number of nonzero terms, the sum will be finite as long as az^{-1} is finite, which in turns requires only that $|a| < \infty$ and $z \neq 0$.

Thus, assuming that $|a|$ is finite, the ROC includes the entire z-plane, with the exception of the origin ($z = 0$).

Pole zero plot for above example with $N = 16$ and a real such that $0 < a < 1$. The region of convergence for this example consists of all values of z except $z = 0$.



Example: The ROC of convergence of the z-transform of

$$x(n) = \left(\frac{1}{2}\right)^{n-1} u(n) + (2 + 3j)^{n-2} u(-n-1)$$

$x(n)$ is two-sided, with two poles. Its ROC is the ring between the two poles , $\frac{1}{2} < |z| < \left|\frac{1}{2+3j}\right|$ or $\frac{1}{2} < |z| < \frac{1}{\sqrt{13}}$

Properties of the Region of Convergence for the z-Transform

- The properties of the region of convergence depend on the nature of the signal.
- We assume that specifically that the algebraic expression for the z-transform is a rational function and that $x(n)$ has finite amplitude, except possibly at $n = \infty$ or $n = -\infty$

1. **Property 1**: The ROC is a ring or disk in the z-plane centered at the origin; i.e., $0 \leq r_R < |z| < r_L \leq \infty$. This property results from the fact that convergence of z-transform equation for a given $x(n)$ is dependent only on $|z|$.

2. **Property 2:** The Fourier transform of x(n) converges absolutely if and only if the ROC of the z-transform of x(n) includes the unit circle. This property is a consequence of the fact that the z-transform equation reduces to the Fourier transform when $|z| = 1$.

3. **Property 3:** The ROC cannot contain any poles. This property follows from the recognition that $X(z)$ is infinite at a pole and therefore, by definition, doesnot converge.

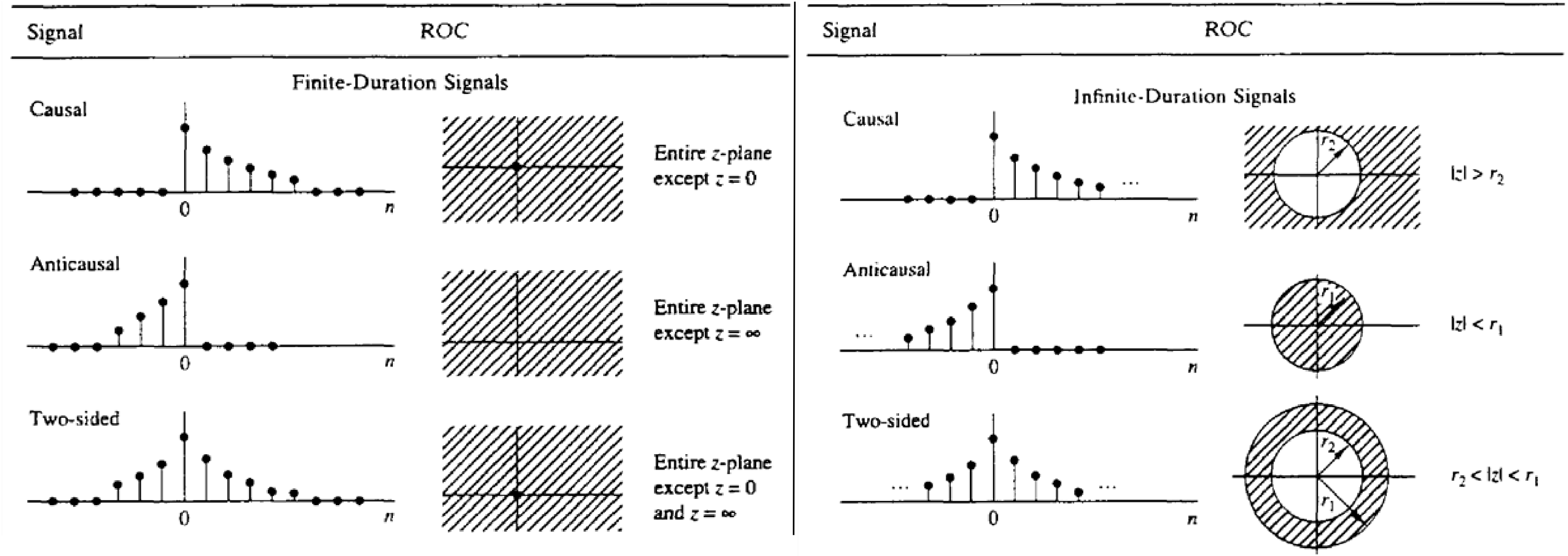
4. **Property 4:** If $x(n)$ is a finite duration sequence, i.e., a sequence that is zero except in a finite interval $-\infty < N_1 \leq n \leq N_2 < \infty$, then the ROC is the entire z-plane, except possibly $z = 0$ or $z = \infty$.

5. **Property 5:** If $x(n)$ is a right-sided sequence, i.e., a sequence that is zero for $n < N_1 < \infty$, then the ROC extends outward form the outermost (i.e., largest magnitude) finite pole in $X(z)$ to (and possibly including) $z = \infty$.

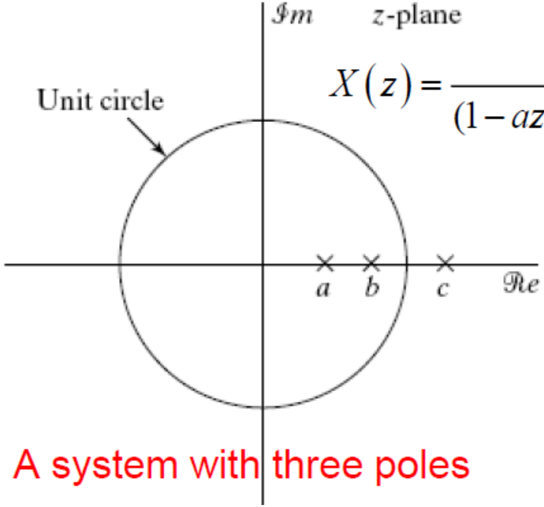
6. **Property 6:** If $x(n)$ is a left-sided sequence, i.e., a sequence that is zero for $n > N_2 > -\infty$, then the ROC extends inward form the innermost (smallest magnitude) nonzero pole in $X(z)$ to (and possibly including) $z = 0$.

7. **Property 7:** A two-sided sequence is an infinite duration sequence that is neither right sided nor left sided. If $x(n)$ is a two sided sequence, the ROC will consist of a ring in the z-plane, bounded on the interior and exterior by a pole and, conistent with property 3, not containing any poles.

8. **Property 8:** The ROC must be a connected region.



Signal
ROC
Finite-Duration Signals
Causal
Entire z-plane except z = 0
Anticausal
Entire z-plane except z = ∞
Two-sided
Entire z-plane except z = 0 and z = ∞
Signal
ROC
Infinite-Duration Signals
Causal
|z| > r2
Anticausal
|z| < r1
Two-sided
r2 < |z| < r1
0
n

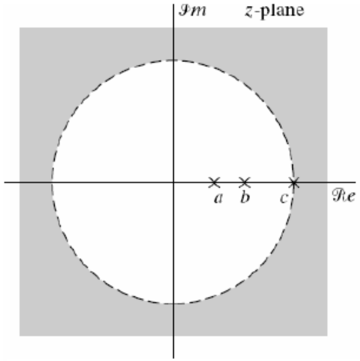


$$X(z)=\frac{kz^{-3}}{(1-az^{-1})(1-bz^{-1})(1-cz^{-1})}$$

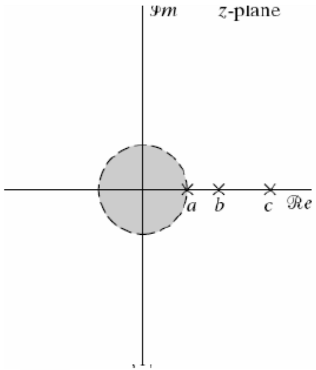
ROC ?

$x[n]=?$

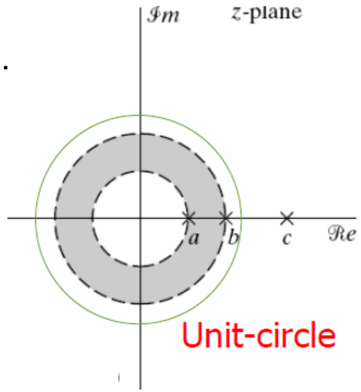
A system with three poles



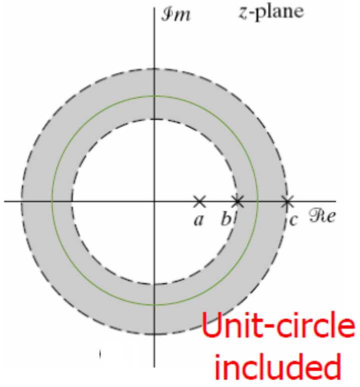
ROC to a
right-sided sequence



ROC to a
left-handed sequence



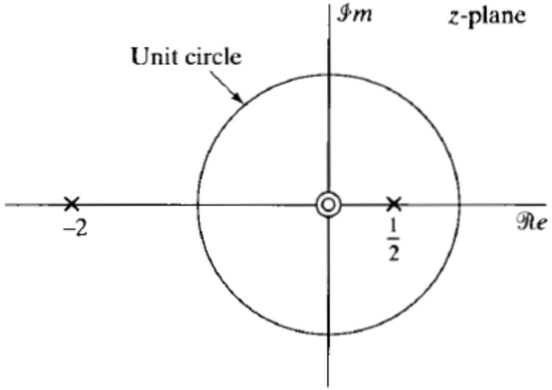
ROC to a
two-sided sequence
not stable



ROC to another
two-sided sequence
stable

Stability, Causality, and the ROC

Consider a system with impulse response $h(n)$ for which the z-transform $H(z)$ has pole-zero plot as in figure.

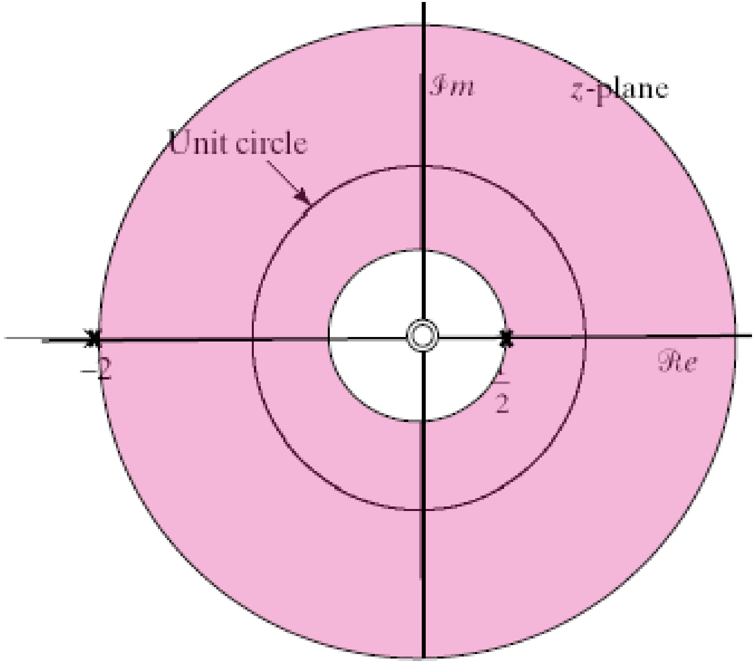


Three possible ROC's consistent with properties 1-8 that can be associated with this pole-zero plot.

1. If we state that the system is stable (or equivalently, that h(n) is absolutely summable and therefore has a Fourier transform), then the ROC must include the unit circle. Thus the stability of the system and properties 1-8 imply that the ROC is the region $\frac{1}{2} < |z| < 2$. Note as a consequence, h(n) is two sided, and therefore, the system is not causal.
2. If we state that the system is causal, and therefore that $h(n)$ is right sided, then property 5 would require that the ROC be the region $|z| > 2$. Under this condition, the system would not be stable, i.e., for this specific pole-zero plot, there is no ROC that would imply that the system is both stable and causal.

1. Stable system: (ROC include unit circle)

ROC: $\frac{1}{2} < |z| < 2$, the impulse response is two-sided, system is non-causal. stable.

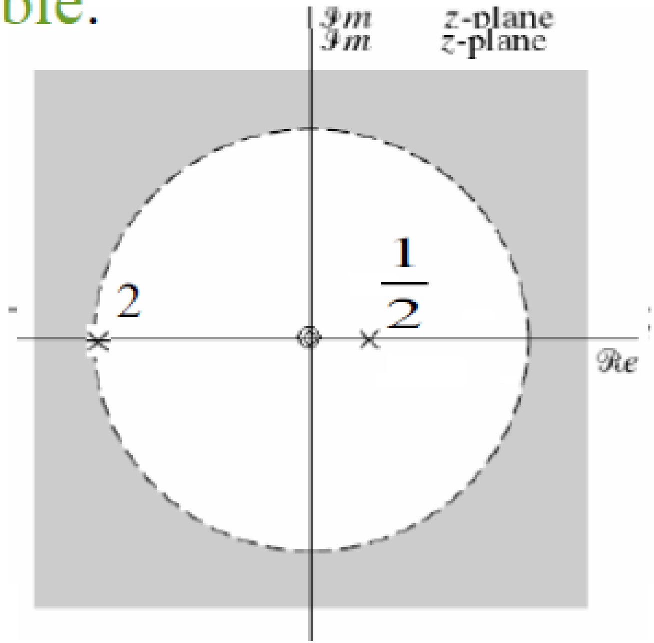


2. Causal System (Right sided sequence)

- ROC: $|z| > 2$,the impulse response is right-sided.

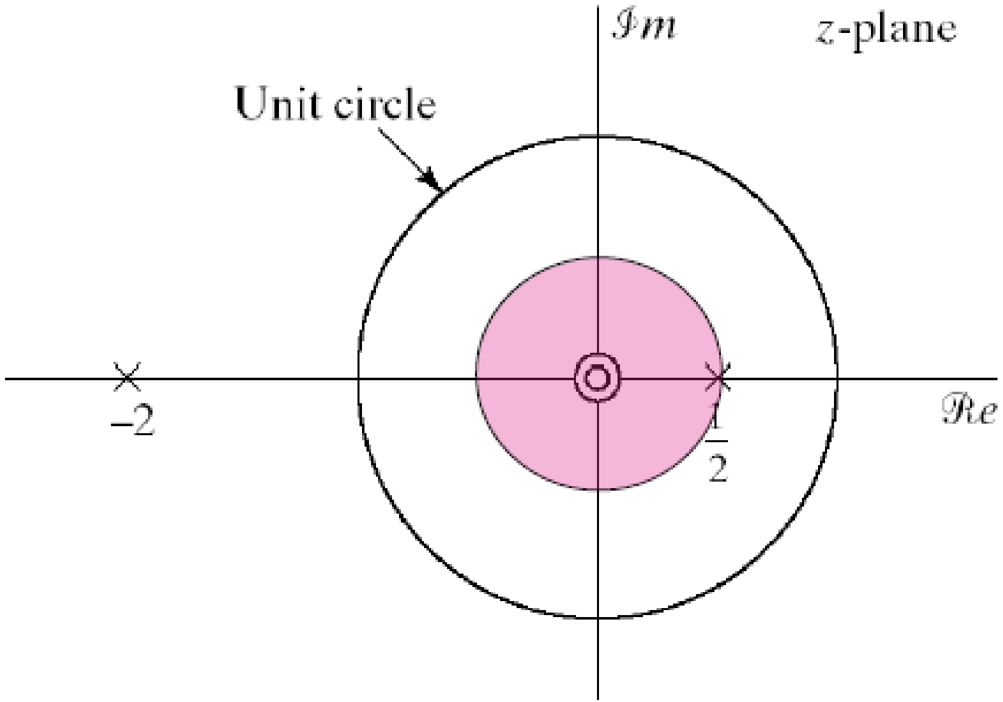
  system is causal but unstable.

- A system is causal and stable if all the poles are inside the unit circle, and the ROC extends outward from the outermost pole to z=∞.



3. Non-causal and unstable

$|z| < \frac{1}{2}$ the impulse response is left sided, system is non-causal, unstable, since ROC does not include unit circle.



2.2 Properties of Z-Transform (Linearity, Time shift, Multiplication by exponential sequence, Differentiation, Time reversal, Convolution, Multiplication)

- The z-transform is a very powerful tool for the study of discrete time signals and systems.
- The power of this transform is a consequence of some very important properties that the transform possesses.
- In these properties $X(z)$ denotes the z-transform of $x(n)$, and the ROC of $X(z)$ is indicated by R_x i.e.,

$$x[n] \overset{Z}{\longleftrightarrow} X(z), \qquad \text{ROC} = R_x.$$

- For properites that involve two sequences and associated z-transform, the transfom pairs will be denoted as

$$x_1[n] \overset{Z}{\longleftrightarrow} X_1(z), \qquad \text{ROC} = R_{x_1},$$

$$x_2[n] \overset{Z}{\longleftrightarrow} X_2(z), \qquad \text{ROC} = R_{x_2}.$$

1. Linearity

The linearity property states that

$$a x_1[n] + b x_2[n] \xleftrightarrow{z} a X_1(z) + b X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2},$$

The ROC is at least the intersection of the individual regions of convergence.

Proof:

$$Z[ax_1(n) + bx_2(x)] = \sum_{n=-\infty}^{\infty} [\,ax_1(n) + bx_2(x)]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} ax_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} bx_2(n)\,z^{-n} = a\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

Example 1: Determine the z-transform and the ROC of the signal

$$x(n) = [3(2)^n - 4(3)^n]u(n)$$

Solution: if we define the signals as $x_1(n) = (2)^n u(n)$ and $x_2(n) = (3)^n u(n)$

Thus $x(n) = 3x_1(n) + 4x_2(n)$. Using linearity properties $\mathrm{X}(Z) = 3X_1(Z) + 4X_2(z)$

Therefore

$$X_1(Z) = \frac{1}{1 - 2z^{-1}} \; ROC{:}\ |z| > 2$$

And

$$X_2(Z) = \frac{1}{1 - 3z^{-1}} \; ROC{:}\ |z| > 3$$

$$Z\{\delta(n)\} = 1$$

$$Z\{u(n)\} = \frac{z}{z - 1}$$

$$Z\{a^n n(n)\} = \frac{z}{z - a}$$

Since the intersection of the ROC of $X_1(Z)$ and $X_2(Z)$ is $|z| > 3$. Thus

$$X(Z) = \frac{3}{1 - 2z^{-1}} + \frac{1}{1 - 3z^{-1}} ROC{:}\ |z| > 3$$

Example 2 : Determine the z-transform of the signals

a. $x(n) = \cos(\omega_0 n)u(n)$

b. $x(n) = \sin(\omega_0 n)\, u(n)$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Solution : we have

$$\cos(\omega_0 n)u(n) = \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

$$\sin(\omega_0 n)u(n) = \frac{1}{2j} e^{j\omega_0 n} u(n) - \frac{1}{2j} e^{-j\omega_0 n} u(n)$$

and

$$Z\left[e^{j\omega_0 n} u(n)\right] = \frac{1}{1 - e^{j\omega_0} z^{-1}}, \qquad ROC\text{: } |z| > 1$$

$$Z\left[e^{-j\omega_0 n} u(n)\right] = \frac{1}{1 - e^{-j\omega_0} z^{-1}}, \qquad ROC\text{: } |z| > 1$$

$$Z[\cos(\omega_0 n)u(n)] = \frac{1}{2}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{1-e^{-j\omega_0}z^{-1}}\right], \qquad ROC\colon |z| > 1$$

$$= \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, \qquad ROC\colon |z| > 1$$

$$Z[\sin(\omega_0 n)u(n)] = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega_0}z^{-1}} - \frac{1}{1-e^{-j\omega_0}z^{-1}}\right], \qquad ROC\colon |z| > 1$$

$$= \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\sin\omega_{0_0} + z^{-2}}, \qquad ROC\colon |z| > 1$$

2. Time Shifting

According to the time shifting property,

$$x[n-n_0] \xleftrightarrow{z} z^{-n_0}X(z), \qquad \text{ROC} = R_x \text{(except for the possible addition or deletion of } z=0 \text{ or } z=\infty\text{)}.$$

The quantity n_0 is an integer. If n_0 is positive, the original sequence $x(n)$ is shifted right, and if n_0 is negative, $x(n)$ is shifted left. The ROC can be changed, since the factor z^{-n_0} can alter the number of poles at $z=0$ or $z=\infty$.

Proof: let $y(n)=x(n-n_0)$, the corresponding z-transform is

$$Y(z)=\sum_{n=-\infty}^{\infty} x(n-n_0)z^{-n}$$

Suppose $m=n-n_0$

Then,

$$Y(z)=\sum_{m=-\infty}^{\infty} x(m)z^{-(m+n_0)} = z^{-n_0}\sum_{m=-\infty}^{\infty} x(m)z^{-m} = z^{-n_0}\mathrm{X}(\mathrm{z})$$

Example 1: Find the inverse z-transform of

$$X(z) = \frac{1}{z - \frac{1}{4}}, |z| > \frac{1}{4}$$

Solution: From the ROC, the sequence is right-sided. Rewriting $X(z)$ as

$$X(z) = \frac{z^{-1}}{1 - \frac{1}{4}z^{-1}} = z^{-1}\left(\frac{1}{1 - \frac{1}{4}z^{-1}}\right), \qquad |z| > \frac{1}{4}$$

Using time shifting property, the factor z^{-1} is a time shift of one sample to the right of the sequence, thus

$$x(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1)$$

Example 2 : Determine the transform of the signal

$$x(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & elsewhere \end{cases}$$

Solution: This signal can be expressed in terms of two unit step signals

$$x(n) = u(n) - u(n-N)$$

Thus

$$X(z) = Z[u(n) - u(n-N)] = Z[u(n)] + Z[u(n-N)]$$

$$= \frac{1}{1-z^{-1}} + z^{-N}\frac{1}{1-z^{-1}}, \qquad ROC\colon\ |z| > 1$$

$$= \frac{1-z^{-N}}{1-z^{-1}}, ROC\colon\ |z| > 1$$

Alternatively:

$$X(z) = \sum_{n=0}^{N-1} 1.z^{-n} = 1 + z^{-1} + z^{-2} + \cdots + z^{-(N-1)} = \begin{cases} N, & if\ z = 1 \\ \dfrac{1-z^{-N}}{1-z^{-1}}, & if\ z \ne 1 \end{cases}$$

Since $x(n)$ has finite duration, its ROC is the entire z-plane, except $z = 0$.

3. Multiplication by an Exponential Sequence

The exponential multiplication property is expressed mathemetically as

$$z_0^n x[n] \xleftrightarrow{z} X(z/z_0), \qquad \text{ROC} = |z_0| R_x.$$

The notation ROC = $|z_0|R_x$ denotes that the ROC is R_x scaled by $|z_0|$; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then $|z_0|R_x$ is the set of values of z such that $|z_0|r_R < |z| < |z_0|r_L$.

Proof:

$$Z[z_0^n x(n)] = \sum_{n=-\infty}^{\infty} z_0^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(z_0^{-1}z)^{-n} = X(z_0^{-1}z) = X(\frac{z}{z_0})$$

Since the ROC of $X(z)$ is $r_R < |z| < r_L$, the ROC of $X(\frac{z}{z_0})$ is $r_R < \left|\frac{z}{z_0}\right| < r_L$ or $|z_0|r_R < |z| < |z_0|r_L$. As a consequence of the exponential multiplication property, all the pole-zero locations are scaled by a factor z_0, since, if $X(z)$ has a pole at $z = z_1$, then $X(z_0^{-1}z)$ will have a pole at $z = z_0 z_1$.

If z_0 is a positive real number, the scaling can be interpreted as a shrinking or expanding of the z-plane, i.e., the pole and zero locations change along radial lines in the z-plane.

If z_0 is complex with unity magnitude, so that $z_0 = e^{j\omega_0}$, the scaling corresponds to a rotation in the z-plane by an angle ω_0, i.e., the pole and zero locations change in position along circles centered at the origin.

This in turn can be interpreted as a frequency shift or translation, associated with the modulation in the time domain by the complex exponential sequence $e^{j\omega_0 n}$. That is, if the Fourier transform exists, this property has the form

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X(e^{j(\omega-\omega_0)}).$$

Example : Determine the z-transform of the signals

a. $x(n) = a^n \cos(\omega_0 n)u(n)$

b. $x(n) = a^n sin(\omega_0 n)u(n)$

Solution: We have

$$Z[\cos(\omega_0 n)u(n)] = \frac{1 - z^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, \qquad ROC: |z| > 1$$

$$Z[\sin(\omega_0 n)u(n)] = \frac{z^{-1}\sin\omega_0}{1 - 2z^{-1}\sin\omega_0 + z^{-2}}, \qquad ROC: |z| > 1$$

Thus using multiplication by exponential sequence, we get,

$$Z[a^n\cos(\omega_0 n)u(n)] = \frac{1 - az^{-1}\cos\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2z^{-2}}, \qquad ROC: |z| > |a|$$

$$Z[a^n\sin(\omega_0 n)u(n)] = \frac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\sin\omega_0 + a^2z^{-2}}, \qquad ROC: |z| > |a|$$

4. Differentiation of $X(z)$

The differentiation property states that $nx[n] \xleftrightarrow{z} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$

Proof: This property is verified by differentiating the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty} x(n)\frac{d(z^{-n})}{dz}$$

$$= -z\sum_{n=-\infty}^{\infty} (-n)x(n)z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} nx(n)z^{-n} = Z[nx(n)]$$

Example 1 : Determine the inverse z-transform of

$$X(z) = \log(1 + az^{-1}), \qquad |z| > |a|$$

Solution: we first differentiate the non-rational to obtain a rational expression:

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1 + az^{-1}}$$

Using the differentiation property,

$$Z[nx(n)] = -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1 + az^{-1}}, \qquad |z| > |a|$$

$$= az^{-1}\left[\frac{1}{1 - (-a)z^{-1}}\right], \qquad |z| > |a|$$

Thus the inverse is obtained by combined use of linearity and time shifting property as

$$nx(n) = a(-a)^{n-1}u(n-1)$$

Thus

$$x(n) = (-1)^{n+1}\frac{a^n}{n}u(n-1)$$

Example 2 : Determine the z-transform of the sequence

$$x(n) = na^n u(n) = n[a^n u(n)]$$

Solution: we have

$$x_1(n) = a^n u(n)]$$

And

$$X_1(z) = \frac{1}{1 - az^{-1}}, \qquad ROC\colon |z| > |a|$$

Thus using differentiation property

$$Z[nx_1(n)] = -z\frac{dX(z)}{dz} = -z\frac{d\left[\frac{1}{1 - az^{-1}}\right]}{dz} = \frac{az^{-1}}{(1 - az^{-1})^2}, \qquad ROC\colon |z| > |a|$$

If we set $a = 1$, we find the z-transform of the unit ramp signal $nu(n)$

$$Z[nu(n)] = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad ROC\colon |z| > 1$$

5. Conjugation of Complex Sequence

The conjugation property is expresses as $x^*[n] \xleftrightarrow{z} X^*(z^*), \qquad \text{ROC} = R_x.$

6. Time Reversal

By time reversal property,

$$x^*[-n] \xleftrightarrow{z} X^*(1/z^*), \qquad \text{ROC} = \frac{1}{R_x}.$$

The notation ROC $= \frac{1}{R_x}$ implies that R_x is inverted; i.e., if R_x is the set of values of z such that $r_R < |z| < r_L$, then the ROC is the set of values of z such that $\frac{1}{r_L} < |z| < \frac{1}{r_R}$.
If the sequence x(n) is real or we do not conjugate a complex sequence, the result becomes

$$x[-n] \xleftrightarrow{z} X(1/z), \qquad \text{ROC} = \frac{1}{R_x}.$$

Example : Determine the z-transfrom of the signal

$$x(n) = u(-n)$$

Solution:

We know

$$X(z) = Z[u(n)] = \frac{1}{1 - z^{-1}}, ROC\colon\ |z| > 1$$

Thus

$$X(z) = Z[u(-n)] = \frac{1}{1 - z}, \qquad ROC\colon\ |z| < 1$$

Similarly

$$x(n) = a^{-n} u(-n)$$

Which is the time reversal of $a^n u(n)$. Thus the time reversal property

$$X(z) = \frac{1}{1 - az} = -\frac{a^{-1} z^{-1}}{1 - a^{-1} z^{-1}}, \qquad ROC\colon |z| < |a^{-1}|$$

7. Convolution of Sequences

According to the convolution property

$$x_1[n] * x_2[n] \overset{z}{\longleftrightarrow} X_1(z)X_2(z), \qquad \text{ROC contains } R_{x_1} \cap R_{x_2}.$$

Proof: The $y(n)$ be the convolution $x_1(n)$ and $x_2(n)$, i.e.,

$$x_1(n) * x_2(n) = y(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

Thus,

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n)z^{-n} = \sum_{n=-\infty}^{\infty} \{\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\}z^{-n}$$

If we interchange the order of summation,

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}$$

Changing the index of summation in the second sum for n to $m = n - k$, we obtain

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1(k)\{\sum_{m=-\infty}^{\infty} x_2(m)\, z^{-m}\}z^{-k}$$

Thus, for the values of z inside the regions of convergence of both $X_1(n)$ and $X_2(n)$, we can write

$$Y(z) = X_1(n)X_2(n)$$

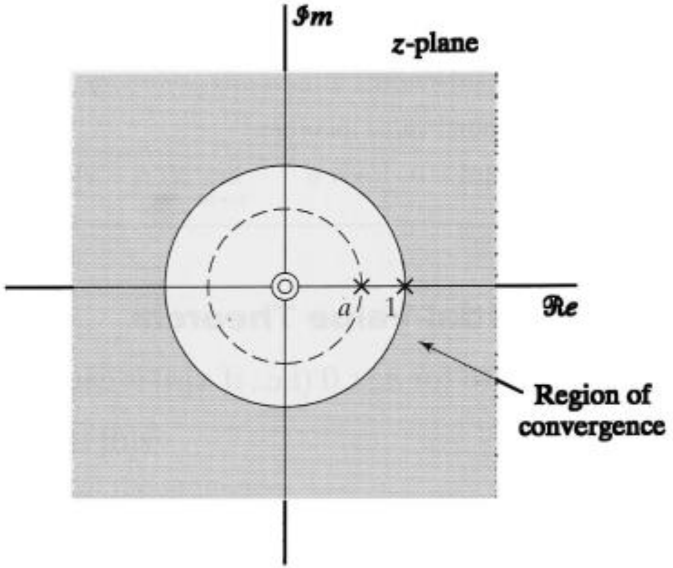
Where the ROC includes the intersection of the regions of convergence of $X_1(n)$ and $X_2(n)$.

The convolution property plays a particularly important role in the analysis of LTI systems. Specifically, as a convolution of this property, the z-transform of the input and the z-transform of the system impulse response. The z-transform of the impulse response of an LTI system is typically referred to as the system function.

Example 1 : Evaluate the convolution using the z-transformof $x_1(n) = a^n u(n)$ and $x_2 = u(n)$.

The corresponding z-transforms are

$X_1(z) = \frac{1}{1-az^{-1}},\ ROC\colon |z| > |a|$



And

$X_2(z) = \frac{1}{1-z^{-1}},\ ROC\colon |z| > 1$

If $|a| < 1$, the z-transform of the convolution of $x_1(n)$ and $x_2(n)$ is then

$$Y(z) = \frac{1}{(1 - az^{-1})(1 - z^{-1})} = \frac{z^2}{(z - a)(z - 1)}, \qquad ROC\colon |z| > 1$$

The poles and zeros of $Y(z)$ are plotted in figure. The ROC is seen to be the overlap region.

The sequence $y(n)$ can be obtained by determining the inverse z-transform. Expanding $Y(z)$ in a partial fraction we get

$$Y(z) = \frac{1}{1-a}\left(\frac{1}{1-z^{-1}} - \frac{a}{1-az^{-1}}\right), \qquad ROC: |z| > 1$$

Therefore,

$$y(n) = \frac{1}{1-a}\left[u(n) - a^{n+1}u(n)\right]$$

Example 2: Compute the convolution $x(n)$ of the signals

$$x_1(n) = (\underset{\uparrow}{1}, -2, 1)$$

$$x_2(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & elsewhere \end{cases}$$

Solution: Using z-transform we get

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$

$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

Using convolution property

$$X(z) = X_1(z)X_2(z) = (1 - 2z^{-1} + z^{-2})(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$= (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5})$$

$$-(2z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-4} + 2z^{-5} + 2z^{-6})$$

$$+(z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7})$$

$$= 1 - z^{-1} - z^{-6} + z^{-7}$$

Hence $x(n) = (\underset{\uparrow}{1}, -1,\ 0,\ 0,\ 0,\ 0,\ -1,\ 1)$

The convolution property is one of the most powerful properties of the z-transform because it converts the convolution of two signals (time domain) to multiplication of their transforms. Computation of the convolution of two signals using the z-transform, requires the following steps:

1. Compute the z-transform of the signals to be convolved.

(time domain to z-domain)

$$X_1(z) = Z[x_1(n)]$$
$$X_2(z) = Z[x_2(n)]$$

2. Multiply the two z-transforms.

(z-domain)

$$X(z) = X_1(z)X_2(z)$$

3. Find the inverse z-transform of $X(z)$.

(z-domain to time domain)

$$x(n) = Z^{-1}[X(z)]$$

This procedure is, in many cases ,computationally easier than the direct evaluation of the convolution summation.

8. Multiplication in time domain or Complex convolution theorem

Let $Z[x_1(n)] = X_1(z)$ and $Z[x_2(n)] = X_2(z)$.

Now, the complex convolution theorem states that,

$$Z[x_1(n)x_2(n)] = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

Where C is a closed contour that encloses the origin and lies within the region of convergence common to both $X_1(v)$ and $X_2(\frac{1}{v})$.

Although this property will not be used immediately, it will useful in filter design based on the window technique, where we multiply the impulse response of an IIR system by a finite duration "window" which serves to truncate the impulse response of IIR system.

Summary of Properties of Ƶ-Transform

Note : $X(z) = \mathcal{Z}\{x(n)\}$; $X_1(z) = \mathcal{Z}\{x_1(n)\}$; $X_2(z) = \mathcal{Z}\{x_2(n)\}$; $Y(z) = \mathcal{Z}\{y(n)\}$

Property		**Discrete time signal**	**Ƶ-transform**
Linearity		$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(z) + a_2 X_2(z)$
Shifting ($m \geq 0$)	$x(n)$ for $n \geq 0$	$x(n-m)$	$z^{-m} X(z) + \sum_{i=1}^{m} x(-i)\, z^{-(m-i)}$
		$x(n+m)$	$z^{m} X(z) - \sum_{i=0}^{m-1} x(i)\, z^{m-i}$
	$x(n)$ for all n	$x(n-m)$	$z^{-m} X(z)$
		$x(n+m)$	$z^{m} X(z)$
Multiplication by n^m (or differentiation in z-domain)		$n^m x(n)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
Scaling in z-domain (or multiplication by a^n)		$a^n x(n)$	$X(a^{-1} z)$
Time reversal		$x(-n)$	$X(z^{-1})$
Conjugation		$x^*(n)$	$X^*(z^*)$
Convolution		$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)$	$X_1(z)\, X_2(z)$

Summary of Properties of Ƶ-Transform

Note : $X(z) = Ƶ\{x(n)\}$; $X_1(z) = Ƶ\{x_1(n)\}$; $X_2(z) = Ƶ\{x_2(n)\}$; $Y(z) = Ƶ\{y(n)\}$

Corrrelation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\,y(n-m)$	$X(z)\,Y(z^{-1})$
Initial value	$x(0) = \underset{z \to \infty}{\mathrm{Lt}}\, X(z)$	
Final value	$x(\infty) = \underset{z \to 1}{\mathrm{Lt}}\,(1 - z^{-1})\,X(z)$ $= \underset{z \to 1}{\mathrm{Lt}}\,\frac{(z-1)}{z}\,X(z)$ if $X(z)$ is analytic for $\lvert z \rvert > 1$	
Complex convolution theorem	$x_1(n)\,x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)\,X_2\left(\frac{z}{v}\right) v^{-1}\,dv$
Parseval's relation	$\sum_{n=-\infty}^{+\infty} x_1(n)\,x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(z)\,X_2^*\left(\frac{1}{z^*}\right) z^{-1}\,dz$	

The Inverse z-Transform

- One of the important roles of the z-transform is in the analysis of discrete time linear systems.
- Often, this analysis involves finding the z-transform of sequences and, after some manipulation of the algebraic expressions, finding the inverse z-transform.
- There are number of formal and informal ways of determining the inverse z-transform from a given algebraic expression and associated regions of convergence.
- There is a formal inverse z-transform expression that is based on the Chauchy integral theorem.
- But we consider some less formal procedure, specifically the inspection method, partial fraction expression, and power series expansion.
- The inspection method is simply inspection using the table.

2.3 Inverse Z-transform-by long division, by partial fraction expansion

1. Long Division Method (Power Series Expansion)

The defining expression for the z-transform is a Laurent series where the sequence values $x(n)$ are the coefficient of z^{-n}. Thus, if the z-transform is given as a power series in the form

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)z^{-2} + x(-1)z^{-1} + x(0) + x(1)z^{1} + x(2)z^{2} + \cdots$$

We can determine any particular values of the sequence by finding the coefficient of the appropriate power of z^{-1}. This approach is very useful for finite length sequences where $X(z)$ may have no simpler form than a polynomial in z^{-1}

When $X(z)$ is rational, the expansion can be performed by long division.

- ❖ If ROC is the exterior of a circle, the signal is to be a causal, thus we seek a power series expansion in negative powers of z.
- ❖ If ROC is the interior of a circle, the signal is to be an anti-causal, we obtain a power series expansion in positive powers of z.

Example 1: Determine the inverse z-transform of

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

When

a. ROC: $|z| > 1$
b. ROC: $|z| < 0.5$

Solution:

a. Since the ROC is the exterior of a circle, we expect $x(n)$ to be a causal signal. Thus we seek a power series expansion in negative powers of z. By dividing the numerator of $X(Z)$ by its denominator, we obtain the power series as thus $X(x) = 1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \frac{81}{16}z^{-4} + \cdots$

Therefore

$x(n) = (\underline{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{81}{16}, \ldots)$

$$
\begin{array}{r|l}
 & 1 + 1.5z^{-1} + 1.75z^{-2} + 1.875z^{-3} + 1.9375z^{-4} + \ldots\ldots\ldots \\
\hline
1 - 1.5z^{-1} + 0.5z^{-2} & 1 \\
 & \underset{(-)}{1} \underset{(+)}{-} 1.5z^{-1} \underset{(-)}{+} 0.5z^{-2} \\
\cline{2-2}
 & \quad 1.5z^{-1} - 0.5z^{-2} \\
 & \quad \underset{(-)}{1.5z^{-1}} \underset{(+)}{-} 2.25z^{-2} \underset{(-)}{+} 0.75z^{-3} \\
\cline{2-2}
 & \qquad 1.75z^{-2} - 0.75z^{-3} \\
 & \qquad \underset{(-)}{1.75z^{-2}} \underset{(+)}{-} 2.625z^{-3} \underset{(-)}{+} 0.875z^{-4} \\
\cline{2-2}
 & \qquad\quad 1.875z^{-3} - 0.875z^{-4} \\
 & \qquad\quad \underset{(-)}{1.875z^{-3}} \underset{(+)}{-} 2.8125z^{-4} \underset{(-)}{+} 0.9375z^{-5} \\
\cline{2-2}
 & \qquad\qquad 1.9375z^{-4} - 0.9375z^{-5} \\
 & \qquad\qquad \vdots
\end{array}
$$

Note: In each step of the long division process. We eliminate the lowest-power term of z^{-1}.

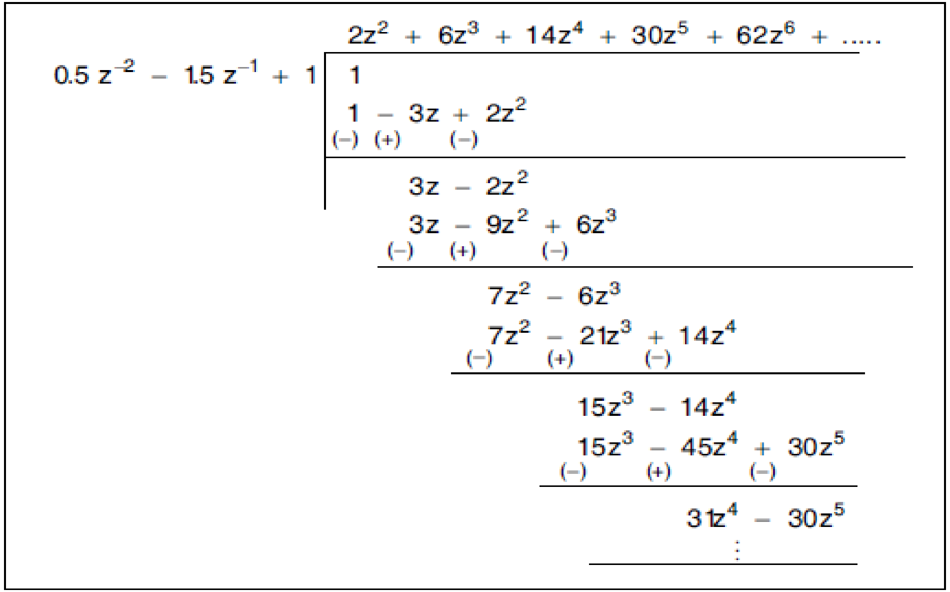
b. In this case the ROC is the interior of a circle. Consequently, the signal x(n) is anticausal. To obtain a power series expansion in positive powers of z we perform the long division in the following way:

Thus $X(z) = 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \cdots$

In this case $x(n) = 0$ for $\mathrm{n} \geq 0$

Thus

$x(n) = (\ldots, 62, 30, 14, 6, 2, 0, \underline{0})$



We observe that in each step of the long division process, the lowest power term of z is eliminated. In case of anticausal signal we Simply carry out the long division by writing down the two polynomials in "reverse" order (i.e., starting with the most negative term on the left).

Example 2: Determine the z-transform

$$X(z) = \frac{1}{1 - az^{-1}}$$

a. For $|z| > |a|$

b. For $|z| < |a|$

Solution:

a. Since the region of convergence is the exterior of a circle, the sequence is a right-sided one. Furthermore, $X(z)$ approaches a finite constant as z approaches infinity, the sequence is causal. Thus, we divide, so as to obtain a series in power of z^{-1}. Carrying out the long division, we obtain

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + \cdots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2z^{-2}} \\ & \qquad a^2z^{-2} \quad \cdots \end{array}$$

$$X(z) = \frac{1}{1 - az^{-1}} = 1 + \mathrm{a}z^{-1} + az^{-2} + \cdots$$

Hence,

$$x(n) = a^n \mathrm{u(n)}$$

b. Because of the region of convergence, the sequence is a left-sided one, since $X(z)$ at $z = 0$ is finite, the sequence is zero for $n > 0$. Thus, we divide, so as to obtain a series in power of z as follows:

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^{2} \\
\hline
-az^{-1}+1 & 1 \\
 & 1 - a^{-1}z \\
\cline{2-2}
 & a^{-1}z \\
 & a^{-1}z - a^{-2}z^{2} \\
\cline{2-2}
 & a^{-2}z^{2}
\end{array}
$$

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|.$$

$$= \frac{z}{-a + z}$$

$$
\begin{array}{r|l}
 & -a^{-1}z - a^{-2}z^{2} - \cdots \\
\hline
-a + z & z \\
 & z - a^{-1}z^{2} \\
\cline{2-2}
 & a^{-1}z^{2}
\end{array}
$$

$$X(z) = \frac{1}{1 - az^{-1}} = -a^{-1}z + a^{-2}z^{2} + \cdots = \sum_{n=-1}^{-\infty} -a^{n}z^{-n}$$

Hence, $x(n) = -a^{n}\mathrm{u}(-\mathrm{n} - 1)$

Example 3: consider the z-transform

$$X(z) = \log(1 + az^{-1})$$

Solution:

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}, \qquad |x| < 1$$

Thus

$$X(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(az^{-1})^n}{n}, |az^{-1}| < 1$$

Therefore

$$x(n) = \begin{cases} \dfrac{(-1)^{n+1}(a)^n}{n}, & n \geq 0 \\ 0, & n < 0 \end{cases} = \frac{(-1)^{n+1}(a)^n}{n} u(n-1)$$

2. Partial fraction expansion

Let $X(z)$ be z transform of $x(n)$, and $X(z)$ be a rational function of z. Now the function $X(z)$ can be expressed as a ratio of two polynomials in z as

$$X(z) = \frac{N(z)}{D(z)}$$

Where $N(z) =$ Numerator polynomial of $X(z)$

$\mathrm{D}(z) =$ Denominator polynomial of $X(z)$

Let us divide both sides of equation by z, then

$$\frac{X(z)}{z} = \frac{N(z)}{zD(z)} = \frac{Q(z)}{D(z)}$$

Where, $Q(z) = \frac{N(z)}{z}$

On factorizing the denominator polynomial, we get

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z - p_1)(z - p_2) \ldots (z - p_N)}$$

Where $p_1, p_2, \ldots p_z$ are roots of denominator polynomial (as well as poles of $X(z)$).

The above equation can be expressed as a series of sum terms by partial fraction expansion as

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

Where, $A_1, A_2, \dots A_N$ are residues, thus

$$X(z) = A_1 \frac{z}{z - p_1} + A_2 \frac{z}{z - p_2} + \cdots + A_N \frac{z}{z - p_N}$$

Now, the inverse z-transform of this equation is obtained by comparing each term with standard z-transform pair. Two popular z-transform pairs are useful for inverse z-transform as

- ❖ If a^n is causal (or right sided) signal then, $Z\{a^n u(n)\} = \frac{z}{z-a}$; with ROC $|z| > |a|$
- ❖ If a^n is anticausal (or left sided) signal then, $Z\{-a^n u(-n-1)\} = \frac{z}{z-a}$; with ROC $|z| < |a|$

Let r_1 be the magnitude of the largest pole and the ROC be $|z| > r_1$(where r_1 is the radius of a circle in z-plane), then the inverse z transform will causal sequence as

$$x(n) = (A_1 p_1^n + A_2 p_2^n + \cdots + A_N p_N^n)u(n)$$

Let r_2 be the magnitude of the smallest pole and the ROC be $|z| < r_2$(where r_2 is the radius of a circle in z-plane), then the inverse z transform will anticausal sequence as

$$x(n) = (-A_1 p_1^n - A_2 p_2^n - \cdots - A_N p_N^n)u(-n-1)$$

Sometimes the specified ROC will be in between two circles of radius r_x and r_y, where $r_x < r_y$ (i.e., ROC is $r_x < |z| < r_y$). Now in this case, the terms with magnitude of poles less than r_x will give rise to causal signal and the terms with magnitude of pole greater than r_y will give rise to anticausal signal so that the inverse z-transform of $X(z)$ will give a two sided signal.

On factorizing the denominator polynomial we get three cases.

Case i: <u>When roots (or poles) are real and distinct</u>

In this case

$$\frac{X(z)}{z} = \frac{A_1}{z - p_1} + \frac{A_2}{z - p_2} + \cdots + \frac{A_N}{z - p_N}$$

The residues are evaluated as

$$A_1 = (z - p_1)\frac{X(z)}{z}\bigg|_{z=p_1}$$

$$A_2 = (z - p_2)\frac{X(z)}{z}\bigg|_{z=p_2}$$

$$\vdots$$

$$A_N = (z - p_N)\frac{X(z)}{z}\bigg|_{z=p_N}$$

Case ii: When roots (or poles) have multiplicity

Let one of the pole has a multiplicity of q. (i.e., repeats q times). In this case

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)\ldots(z-p_x)^q \ldots (z-p_N)}$$

$$= \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \cdots + \frac{A_{x0}}{(z-p_x)^q} + \frac{A_{x1}}{(z-p_x)^{q-1}} + \cdots + \frac{A_{x(q-1)}}{(z-p_x)} + \cdots + \frac{A_N}{(z-p_N)}$$

Where A_{x0}, A_{x1},…, $A_{x(q-1)}$ are residues of repeated root (or pole), $z = p_x$

The residues of distinct roots are evaluated as in case i.

The residue A_{xr} of repeated root is obtained as

$$A_{xr} = \frac{1}{r!}\frac{d^r}{dz^r}\left[(z-p_x)^q \frac{X(z)}{z}\right]\Bigg|_{z=p_x}, where\ r = 0, 1, 2, \ldots (q-1)$$

Case iii: When roots (or poles) are complex conjugate

Let $\frac{X(z)}{z}$ has one pair of compelx conjugate. In this case

$$\frac{X(z)}{z} = \frac{Q(z)}{D(z)} = \frac{Q(z)}{(z-p_1)(z-p_2)\ldots(z^2+az+b)\ldots(z-p_N)}$$

$$= \frac{A_1}{z-p_1} + \frac{A_1}{z-p_1} + \cdots + \frac{A_x}{z-(x+jy)} + \frac{A_x^*}{z-(x-jy)} + \cdots + \frac{A_N}{z-p_N}$$

The residues of real and non-repeated roots are evaluated as in case I

The residue A_x is evaluated as that of case I and the residue A_x^* is the complex conjugate of A_x.

Example : Determine the inverse z-transform of the function,

$$X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$$

Given that, $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} = \dfrac{z^{-2}(3z^2 + 2z + 1)}{z^{-2}(z^2 - 3z + 2)} = \dfrac{3z^2 + 2z + 1}{(z - 1)(z - 2)}$.

$$\therefore \frac{X(z)}{z} = \frac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}$$

Let, $\dfrac{X(z)}{z} = \dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)} = \dfrac{A_1}{z} + \dfrac{A_2}{z - 1} + \dfrac{A_3}{z - 2}$

Now, $A_1 = z\dfrac{X(z)}{z}\Big|_{z=0} = z\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=0} = \dfrac{1}{(-1)\times(-2)} = 0.5$

$A_2 = (z - 1)\dfrac{X(z)}{z}\Big|_{z=1} = (z - 1)\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=1} = \dfrac{3 + 2 + 1}{1 \times (1 - 2)} = -6$

$A_3 = (z - 2)\dfrac{X(z)}{z}\Big|_{z=2} = (z - 2)\,\dfrac{3z^2 + 2z + 1}{z(z - 1)(z - 2)}\Big|_{z=2} = \dfrac{3 \times 2^2 + 2 \times 2 + 1}{2 \times (2 - 1)} = 8.5$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z - 1} + \frac{8.5}{z - 2}$$

$$\therefore X(z) = 0.5 - 6\,\frac{z}{z - 1} + 8.5\,\frac{z}{z - 2}$$

On taking inverse Z-transform of X(z) we get,

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5\,(2)^n\,u(n) = 0.5\,\delta(n) + [-6 + 8.5(2)^n]\,u(n)$$

$\mathcal{Z}\{\delta(n)\} = 1$

$\mathcal{Z}\{u(n)\} = \dfrac{z}{z - 1}$

$\mathcal{Z}\{a^n n(n)\} = \dfrac{z}{z - a}$

Questions:

1. Determine the inverse z-transform of the following z-domain functions.

a) $X(z) = \dfrac{3z^2 + 2z + 1}{z^2 + 3z + 2}$ b) $X(z) = \dfrac{z - 0.4}{z^2 + z + 2}$ c) $X(z) = \dfrac{z - 4}{(z - 1)(z - 2)^2}$

2. Determine the inverse z-transform of the following function.

a) $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$ b) $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

c) $X(z) = \dfrac{1 + z^{-1}}{1 - z^{-1} + 0.5z^{-2}}$ d) $X(z) = \dfrac{1}{(1 + z^{-1})(1 - z^{-1})^2}$

1.

c) Given that, $X(z) = \dfrac{z-4}{(z-1)(z-2)^2}$

By partial fraction expansion we get,

$$X(z) = \frac{z-4}{(z-1)(z-2)^2} = \frac{A_1}{z-1} + \frac{A_2}{(z-2)^2} + \frac{A_3}{(z-2)}$$

$$A_1 = (z-1)\left.\frac{z-4}{(z-1)(z-2)^2}\right|_{z=1} = \left.\frac{z-4}{(z-2)^2}\right|_{z=1} = \frac{1-4}{(1-2)^2} = -3$$

$$A_2 = (z-2)^2\left.\frac{z-4}{(z-1)(z-2)^2}\right|_{z=2} = \left.\frac{z-4}{z-1}\right|_{z=2} = \frac{2-4}{2-1} = -2$$

$$A_3 = \left.\frac{d}{dz}\left[(z-2)^2\frac{z-4}{(z-1)(z-2)^2}\right]\right|_{z=2} = \left.\frac{d}{dz}\left[\frac{z-4}{z-1}\right]\right|_{z=2}$$

$$= \left.\frac{(z-1)-(z-4)}{(z-1)^2}\right|_{z=2} = \left.\frac{3}{(z-1)^2}\right|_{z=2} = \frac{3}{(2-1)^2} = 3$$

$$\therefore X(z) = \frac{-3}{z-1} - \frac{2}{(z-2)^2} + \frac{3}{z-2} = -3\frac{1}{z}\frac{z}{z-1} - \frac{1}{z}\frac{2z}{(z-2)^2} + 3\frac{1}{z}\frac{z}{z-2}$$

$$= -3z^{-1}\frac{z}{z-1} - z^{-1}\frac{2z}{(z-2)^2} + 3z^{-1}\frac{z}{z-2}$$

Multiply and dvide by z

$\mathcal{Z}\{u(n)\} = \dfrac{z}{z-1}$; $\mathcal{Z}\{a^n u(n)\} = \dfrac{z}{z-a}$; $\mathcal{Z}\{na^n u(n)\} = \dfrac{az}{(z-a)^2}$

If $\mathcal{Z}\{x(n)\} = X(z)$ then by time shifting property $\mathcal{Z}\{x(n-1)\} = z^{-1} X(z)$

$\therefore \mathcal{Z}\{u(n-1)\} = z^{-1}\dfrac{z}{z-1}$; $\mathcal{Z}\{a^{(n-1)} u(n-1)\} = z^{-1}\dfrac{z}{z-a}$

and $\mathcal{Z}\{(n-1)\, a^{(n-1)} u(n-1)\} = z^{-1}\dfrac{az}{(z-a)^2}$

On taking inverse $\mathcal{Z}$-transform of X(z) using standard transform and shifting property we get,

$$x(n) = -3\,u(n-1) - (n-1)\,2^{n-1}\,u(n-1) + 3 \times 2^{n-1}\,u(n-1)$$

$$= \left[-3 - (n-1)\,2^{n-1} + 3(2)^{n-1}\right]u(n-1)$$

2. **a) Given that,** $X(z) = \dfrac{1}{1 - 1.5\,z^{-1} + 0.5z^{-2}}$

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{1 - \frac{1.5}{z} + \frac{0.5}{z^2}} = \frac{z^2}{z^2 - 1.5z + 0.5} = \frac{z^2}{(z-1)(z-0.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}$$

By partial fraction expansion, X(z)/z can be expressed as,

$$\frac{X(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\left.\frac{z}{(z-1)(z-0.5)}\right|_{z=1} = \frac{1}{1-0.5} = 2$$

$$A_2 = (z-0.5)\left.\frac{X(z)}{z}\right|_{z=0.5} = (z-0.5)\left.\frac{z}{(z-1)(z-0.5)}\right|_{z=0.5} = \frac{0.5}{0.5-1} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$\therefore X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}$$

$$\boxed{\begin{array}{l} \mathcal{Z}\{a^n\,u(n)\} = \dfrac{z}{z-a}\ ;\ \text{ROC } |z| > |a| \\[2ex] \mathcal{Z}\{u(n)\} = \dfrac{z}{z-1}\ ;\ \text{ROC } |z| > 1 \end{array}}$$

On taking inverse $\mathcal{Z}$-transform of X(z), we get,

$$x(n) = 2\,u(n) - 0.5^n u(n) = [2 - 0.5^n]\,u(n)$$

2. **b) Given that,** $X(z) = \dfrac{z^2}{z^2 - z + 0.5}$

$$X(z) = \frac{z^2}{z^2 - z + 0.5} = \frac{z^2}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A}{z - 0.5 - j0.5} + \frac{A^*}{z - 0.5 + j0.5}$$

$$A = (z - 0.5 - j0.5)\left.\frac{X(z)}{z}\right|_{z = 0.5 + j0.5}$$

$$= (z - 0.5 - j0.5)\left.\frac{z}{(z - 0.5 - j0.5)\,(z - 0.5 + j0.5)}\right|_{z = 0.5 + j0.5}$$

$$\therefore A = \frac{0.5 + j0.5}{0.5 + j0.5 - 0.5 + j0.5} = \frac{0.5 + j0.5}{j1.0} = 0.5 - j0.5$$

$$\therefore A^* = (0.5 - j0.5)^* = 0.5 + j0.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5 - j0.5}{z - 0.5 - j0.5} + \frac{0.5 + j0.5}{z - 0.5 + j0.5}$$

$$X(z) = \frac{(0.5 - j0.5)\,z}{z - (0.5 + j0.5)} + \frac{(0.5 + j0.5)\,z}{z - (0.5 - j0.5)}$$

On taking inverse Z-transform of X(z) we get,

$$x(n) = (0.5 - j0.5)\,(0.5 + j0.5)^n\,u(n) + (0.5 + j0.5)\,(0.5 - j0.5)^n\,u(n)$$

The roots of quadratic $z^2 - z + 0.5 = 0$ are,

$$z = \frac{1 \pm \sqrt{1 - 4 \times 0.5}}{2}$$

$$= 0.5 \pm j\,0.5$$

$$z\{a^n\,u(n)\} = \frac{z}{z - a}\ ;$$

ROC $|z| > |a|$

2. **d) Given that,** $X(z) = \dfrac{1}{(1 + z^{-1})(1 - z^{-1})^2}$

$$X(z) = \frac{1}{(1 + z^{-1})(1 - z^{-1})^2} = \frac{1}{z^{-1}(z + 1)\, z^{-2}(z - 1)^2} = \frac{z^3}{(z + 1)(z - 1)^2}$$

$$\therefore \frac{X(z)}{z} = \frac{z^2}{(z + 1)(z - 1)^2}$$

By partial fraction expansion, we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z + 1} + \frac{A_2}{(z - 1)^2} + \frac{A_3}{z - 1}$$

$$A_1 = (z + 1)\left.\frac{X(z)}{z}\right|_{z=-1} = (z + 1)\left.\frac{z^2}{(z + 1)(z - 1)^2}\right|_{z=-1} = \left.\frac{z^2}{(z - 1)^2}\right|_{z=-1} = \frac{(-1)^2}{(-1 - 1)^2} = 0.25$$

$$A_2 = (z - 1)^2\left.\frac{X(z)}{z}\right|_{z=1} = (z - 1)^2\left.\frac{z^2}{(z + 1)(z - 1)^2}\right|_{z=1} = \left.\frac{z^2}{z + 1}\right|_{z=1} = \frac{1}{1 + 1} = 0.5$$

$$A_3 = \left.\frac{d}{dz}\left[(z - 1)^2\frac{X(z)}{z}\right]\right|_{z=1} = \left.\frac{d}{dz}\left[(z - 1)^2\frac{z^2}{(z + 1)(z - 1)^2}\right]\right|_{z=1}$$

$$= \left.\frac{d}{dz}\left[\frac{z^2}{z + 1}\right]\right|_{z=1} = \left.\frac{(z + 1)\,2z - z^2}{(z + 1)^2}\right|_{z=1} = \frac{(1 + 1) \times 2 - 1}{(1 + 1)^2} = \frac{3}{4} = 0.75$$

$$\therefore \frac{X(z)}{z} = \frac{0.25}{z + 1} + \frac{0.5}{(z - 1)^2} + \frac{0.75}{z - 1}$$

$$X(z) = 0.25\frac{z}{z + 1} + 0.5\frac{z}{(z - 1)^2} + 0.75\frac{z}{z - 1}$$

$$= 0.25\frac{z}{z - (-1)} + 0.5\frac{z}{(z - 1)^2} + 0.75\frac{z}{z - 1}$$

On taking inverse $\mathcal{Z}$-transform of X(z) we get,

$$x(n) = 0.25(-1)^n + 0.5n\,u(n) + 0.75\,u(n)$$

$$= [0.25(-1)^n + 0.5n + 0.75]\,u(n)$$

$$\mathcal{Z}\{a^n u(n)\} = \frac{z}{z - a}$$

$$\mathcal{Z}\{n\,u(n)\} = \frac{z}{(z - 1)^2}$$

$$\mathcal{Z}\{u(n)\} = \frac{z}{z - 1}$$

3. Determine the inverse Z-transform of $X(z) = \dfrac{1}{1 - 0.8 z^{-1} + 0.12 z^{-2}}$

a) if ROC is, $|z| > 0.6$ b) if ROC is, $|z| < 0.2$ c) if ROC is, $0.2 < |z| < 0.6$

Solution

Given that, $X(z) = \dfrac{1}{1 - 0.8 z^{-1} + 0.12 z^{-2}} = \dfrac{1}{z^{-2}(z^2 - 0.8z + 0.12)} = \dfrac{z^2}{(z - 0.6)(z - 0.2)}$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z - 0.6)(z - 0.2)}$$

The roots of quadratic $z^2 - 0.8z + 0.12 = 0$ are,

$$z = \frac{0.8 \pm \sqrt{0.8^2 - 4 \times 0.12}}{2} = \frac{0.8 \pm 0.4}{2} = 0.6,\ 0.2$$

By partial fraction expansion technique we get,

$$\frac{X(z)}{z} = \frac{z}{(z - 0.6)(z - 0.2)} = \frac{A_1}{z - 0.6} + \frac{A_2}{z - 0.2}$$

$$A_1 = (z - 0.6)\left.\frac{X(z)}{z}\right|_{z = 0.6} = (z - 0.6)\left.\frac{z}{(z - 0.6)(z - 0.2)}\right|_{z = 0.6} = \frac{0.6}{0.6 - 0.2} = 1.5$$

$$A_2 = (z - 0.2)\left.\frac{X(z)}{z}\right|_{z = 0.2} = (z - 0.2)\left.\frac{z}{(z - 0.6)(z - 0.2)}\right|_{z = 0.2} = \frac{0.2}{0.2 - 0.6} = -0.5$$

$$\therefore \frac{X(z)}{z} = \frac{1.5}{z - 0.6} - \frac{0.5}{z - 0.2}$$

$$\therefore X(z) = 1.5\frac{z}{z - 0.6} - 0.5\frac{z}{z - 0.2}$$

Now, the poles of $X(z)$ are $p_1 = 0.6$, $p_2 = 0.2$

a) ROC is |z| > 0.6

The specified ROC is exterior of the circle whose radius corresponds to the largest pole, hence x(n) will be a causal (or right sided) signal.

$$\boxed{\mathcal{Z}\{a^n\,u(n)\} = \frac{z}{z-a}\ ;\ \text{ROC}\ |z| > |a|}$$

$$\therefore x(n) = 1.5(0.6)^n\,u(n) - 0.5\,(0.2)^n\,u(n)$$

b) ROC is |z| < 0.2

The specified ROC is interior of the circle whose radius corresponds to the smallest pole, hence x(n) will be an anticausal (or left sided) signal.

$$\therefore x(n) = 1.5(-(0.6)^n\,u(-n-1)) - 0.5\,(-(0.2)^n\,u(-n-1))$$

$$= -1.5\,(0.6)^n\,u(-n-1) + 0.5\,(0.2)^n\,u(-n-1)$$

$$\boxed{\mathcal{Z}\{-a^n\,u(-n-1)\} = \frac{z}{z-a}\ ;\ \text{ROC}\ |z| < |a|}$$

c) ROC is 0.2 < |z| < 0.6

The specified ROC is the region in between two circles of radius 0.2 and 0.7. Hence the term corresponds to the pole, $p_1 = 0.6$ will be anticausal signal (because $|z| < 0.6$) and the term corresponds to the pole, $p_2 = 0.2$, will be a causal signal (because $|z| > 0.2$).

$$\therefore x(n) = 1.5(-(0.6)^n\,u(-n-1)) - 0.5\,(0.2)^n\,u(n)$$

$$= -1.5(0.6)^n\,u(-n-1)) - 0.5\,(0.2)^n\,u(n)$$