10.0 Introduction

Frequency response is the quantitative measure of the output spectrum of a system or device in response to a stimulus, and is used to characterize the dynamics of the system. It is a measure of magnitude and phase of the output as a function of frequency, in comparison to the input.

A first order IIR system

Consider a causal system whose input and output satisfy the difference equation

$$y[n] - ay[n-1] = x[n]$$

Taking z-transform on both the sides we get

$$Y(z) - az^{-1}Y(z) = X(z)$$

$$\therefore H(z) = \frac{1}{1-az^{-1}}$$

Since the Roc $|z|>|a|$

$$\therefore \quad h[n] = a^n u[n]$$

But for those classes of systems whose input and output satisfy a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad ....(i)$$

Taking z- transform on both the sides we get,

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

or, equivalently

$$\frac{y(z)}{x(z)} = H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad ....(ii)$$

$H(z)$ in above equation is a ratio of polynomials in z^{-1} because equation (i) consists of a linear combination of delay terms.

We can express equation (ii) in factored form as

$$H(z) = \left(\frac{b_o}{a_0}\right)\frac{\prod_{K=1}^{M}\left(1-c_k z^{-1}\right)}{\prod_{K=1}^{N}\left(1-d_k z^{-1}\right)} \text{ ----- (iii)}$$

Each of the factors $\left[\left(1-c_k z^{-1}\right)\right]$ in the numerator contributes a zero at $z = c_k$ and a pole at $z = 0$.

Similarly, each of the factors $\left(1-d_k z^{-1}\right)$ in the denominator contributes a zero at $z = 0$ and a pole at $z = d_k$.

If we consider the numerator only as system function; $H(z)$ has no poles except at $z = 0$.

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

$$\therefore h[n] = \sum_{k=0}^{M} b_k \delta[n-k]$$

In this case, the impulse response is finite in length i.e. zero outside a finite interval; consequently, these systems are called finite impulse response (FIR) systems.

Note that for FIR systems, the difference equation as equation (i) is identical to the convolution sum, i.e. $y[n] = \sum_{k=0}^{M} b_k x[n-k]$

10.1 Frequency response for rational system functions

If a stable LTI system has a rational system function i.e. if its input and output satisfy a difference equation of the form of equation (i), then its frequency response has the form

$$H\left(e^{jw}\right) = \frac{\sum_{k=0}^{M} b_k e^{-jwk}}{\sum_{k=0}^{N} a_k e^{-jwk}}$$

That is, $H(e^{jw})$ is a ratio of polynomials in the variable e^{-jw}. To determine the magnitude, phase and group delay associated with the frequency response of such systems, it is useful to express $H(e^{jw})$ in terms of the poles and zeros of $H(z)$. Such expression results from substituting $z=e^{jw}$ on equation (iii)

$$H(e^{jw}) = \left(\frac{b_o}{a_0}\right)\frac{\prod_{K=1}^{M}(1-c_k e^{-jw})}{\prod_{K=1}^{N}(1-d_k e^{-jw})}$$

Taking absolute on both the sides,

$$\left|H(e^{jw})\right| = \left(\frac{b_o}{a_0}\right)\frac{\prod_{K=1}^{M}\left|1-c_k e^{-jw}\right|}{\prod_{K=1}^{N}\left|1-d_k e^{-jw}\right|} \quad \text{..............(iv)}$$

Sometimes it is convenient to consider square of magnitude

i.e.

$$\left|H(e^{jw})\right|^2 = H(e^{jw}).H^*(e^{jw})$$

$$\left|H(e^{jw})\right|^2 = \left(\frac{b_o}{a_0}\right)^2\frac{\prod_{K=1}^{M}(1-c_k e^{-jw})(1-c_k^* e^{jw})}{\prod_{K=1}^{N}(1-d_k e^{-jw})(1-d_k^* e^{jw})}$$

Taking 10 log on both of (iv)

$$20 \log\left|H(e^{jw})\right| = 20 \log\left(\frac{b_o}{a_0}\right) \prod_{K=1}^{M}(1-c_k e^{-jw})(1-c_k^* e^{jw})) - \prod_{K=1}^{N}(1-d_k e^{-jw})(1-d_k^* e^{jw})$$

The function $20 \log \left|H(e^{jw})\right|$ is referred to as the log magnitude of $H(e^{jw})$

Gain in db $= 20 \log \left|H(e^{jw})\right|$

Attenuation = -Gain in db

Similarly the phase response for a rational system function has the form:

$$\mathrm{Arg}\left|H(e^{jw})\right| = \mathrm{Arg}\left(\frac{b_o}{a_0}\right) + \prod_{K=1}^{M} Ang(1-c_k e^{-jw}) - \prod_{K=1}^{N} Ang(1-d_k e^{-jw})$$

Here, Arg represents the angle

The corresponding group delay for a rational system function is,

$$grd\left[H(e^{jw})\right] = \sum_{K=1}^{N} \frac{d}{dw}\left(arg\left[1 - d_k e^{-jw}\right]\right) - \sum_{K=1}^{M} \frac{d}{dw}\left(arg\left[1 - c_k e^{-jw}\right]\right)$$

Where arg[ ] represents the continuous phase. An equivalent expression is,

$$grd\left[H(e^{jw})\right] = \sum_{K=1}^{N} \frac{|d_k|^2 - Re\{d_k e^{-jw}\}}{1 + |d_k|^2 - 2Re\{d_k e^{-jw}\}} - \sum_{K=1}^{M} \frac{|c_k|^2 - Re\{c_k e^{-jw}\}}{1 + |c_k|^2 - 2Re\{c_k e^{-jw}\}}$$

Simply the differentiation of the continuous phase is called group delay.

10.2 Frequency Response of A Single Zero Or Pole

To get the further insight into the properties of frequency response let us first examine the properties of a single factor of the form $(1\text{-}re^{j\Theta}e^{-jw})$ where r is the radius and Θ is the angle of the pole or zero in the z plane. This factor is typical of either a pole or zero at a radius r and angle Θ in the z-plane.

$$\left|(1 - re^{j\theta}e^{-jw})\right|^2 = (1 - re^{j\theta}e^{-jw})(1 - re^{j\theta}e^{-jw})^*$$

$$\left|(1 - re^{j\theta}e^{-jw})\right|^2 = (1 - re^{j\theta}e^{-jw})(1 - re^{-j\theta}e^{jw})$$

$$= 1 - re^{-j\Theta}e^{jw} - re^{j\Theta}e^{-jw} + r^2$$

$$\left|(1 - re^{j\theta}e^{-jw})\right|^2 = 1 + r^2 - r(e^{-j(w-\Theta)} + e^{j(w-\Theta)})$$

$$\left|(1 - re^{j\theta}e^{-jw})\right|^2 = 1 + r^2 - 2r\cos(w - \Theta)$$

To compute the magnitude we take 10 log on both the sides,

$$20\log(1 - re^{j\theta}e^{jw}) = 10\log[1 + r^2 - 2r\cos(w - \Theta)] \text{..................(a)}$$

The phase for such a factor is

$$ARG\left[1 - re^{j\theta}e^{-jw}\right] = \tan^{-1}\left[\frac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right] \text{........................(b)}$$

Differentiating the RHS gives the group delay of the factor as,

$$\mathrm{grd}\left[1 - re^{j\theta}e^{-jw}\right] = \frac{r^2 - r\cos(w-\theta)}{1 + r^2 - 2r\cos(w-\theta)} = \frac{r^2 - r\cos(w-\theta)}{\left|1 - re^{j\theta}e^{-jw}\right|^2}$$

..............(c)

The functions in equations (a), (b) and (c) are periodic in w with period 2π;

If we plot all these functions for fixed r and variable w with different values of Θ, we can obtain the magnitude, phase and group delay of the system.

Example 11.1 Plot the magnitude and phase response of the system which has zeroes at $r = 0.9$ and Θ =0

Solution:

For r=0.9 and Θ=0

$$\text{Magnitude} = 10\log[1 + r^2 - 2r\cos(w - \Theta)]$$

$$= 10\log[1 + 0.9^2 - 2*0.9\cos(w)]$$

Similarly,

$$\text{Phase} = \tan^{-1}\left[\frac{r\sin(w-\theta)}{1 - r\cos(w-\theta)}\right]$$

$$= \tan^{-1}\left[\frac{0.9\sin(w)}{1 - 0.9\cos(w)}\right]$$

Calculation of Magnitude and Phase

W	Magnitude(db)	Phase(rad)
0	-20	0
$\pi/4$	-2.6969	1.051
$\pi/2$	2.576	0.732
Π	5.575	0.20035
$5\pi/4$	4.889	-0.3709
$3\pi/2$	2.576	-0.7328
$7\pi/4$	-2.6969	-1.051
2π	-20	0

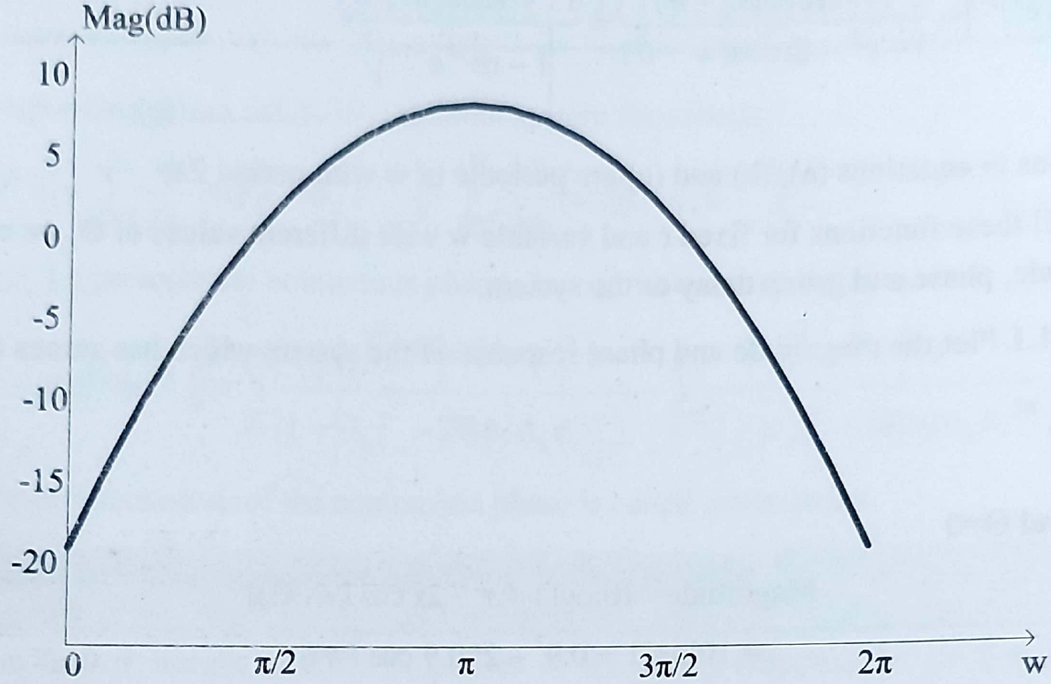

Figure 11.1: Magnitude Response

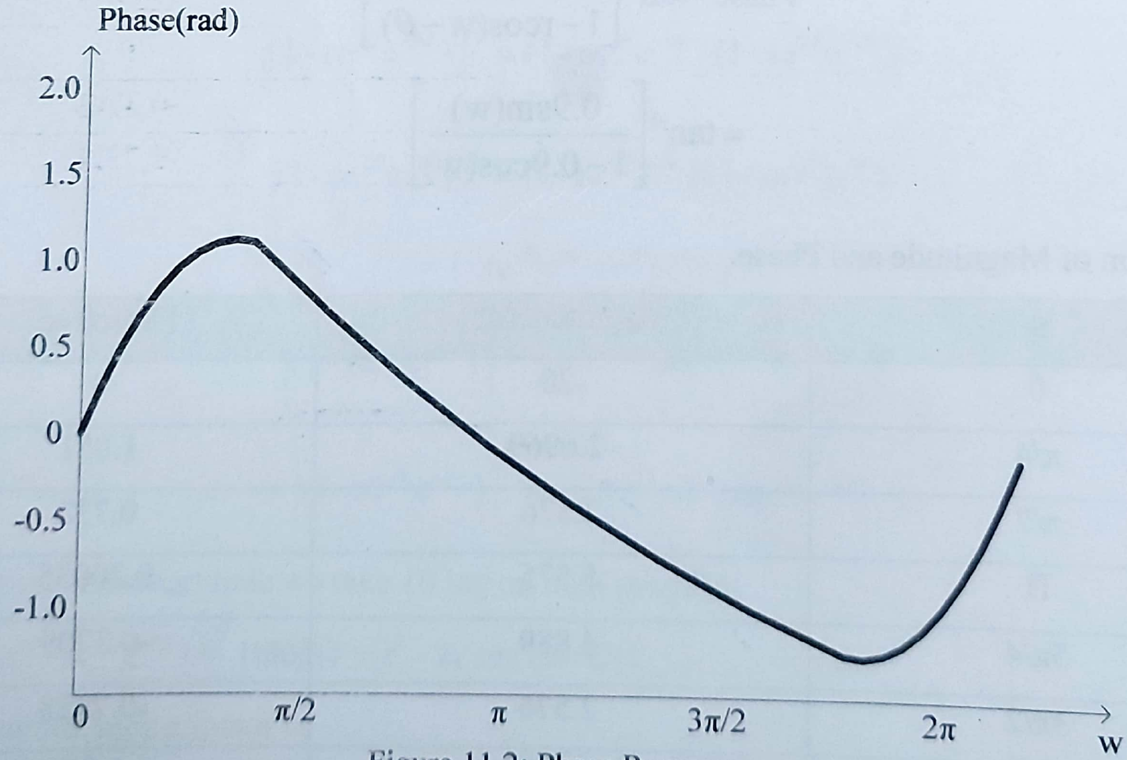

Figure 11.2: Phase Response

Example 11.5 Plot the pole zero in Z-plane and draw Magnitude response (Not to the scale) of the system described by the following difference equation.

y[n]-0.4y[n-1]+0.1y[n-2]=x[n]+0.6x[n-2]

Solution:

Given difference is **y[n]-0.4y[n-1]+0.1y[n-2]=x[n]+0.6x[n-2]**

Taking z-transform on both the sides

$$Y(z) - 0.4z^{-1}Y(z) + 0.1z^{-2}Y(z) = X(z) + 0.6z^{-1}X(z)$$

$$Y(z)[1 - 0.4z^{-1} + 0.1z^{-2}] = X(z)[1 + 0.6z^{-1}]$$

$$\frac{Y(z)}{X(Z)} = H(z) = \frac{1 + 0.6z^{-1}}{1 - 0.4z^{-1} + 0.1z^{-2}}$$

$$H(z) = \frac{Z(Z + 0.6)}{Z^2 - 0.4Z + 0.1}$$

This System has Zeros at: Z=0 and Z=-0.6

This System has poles at: Z=0.2+0.24i and Z= 0.2-0.24i. In polar system we can represent these location of poles as Z=0.31angle 0.87 rad and Z=0.31angle -0.87 rad

Plotting the location of poles and zeros in the Z-plane

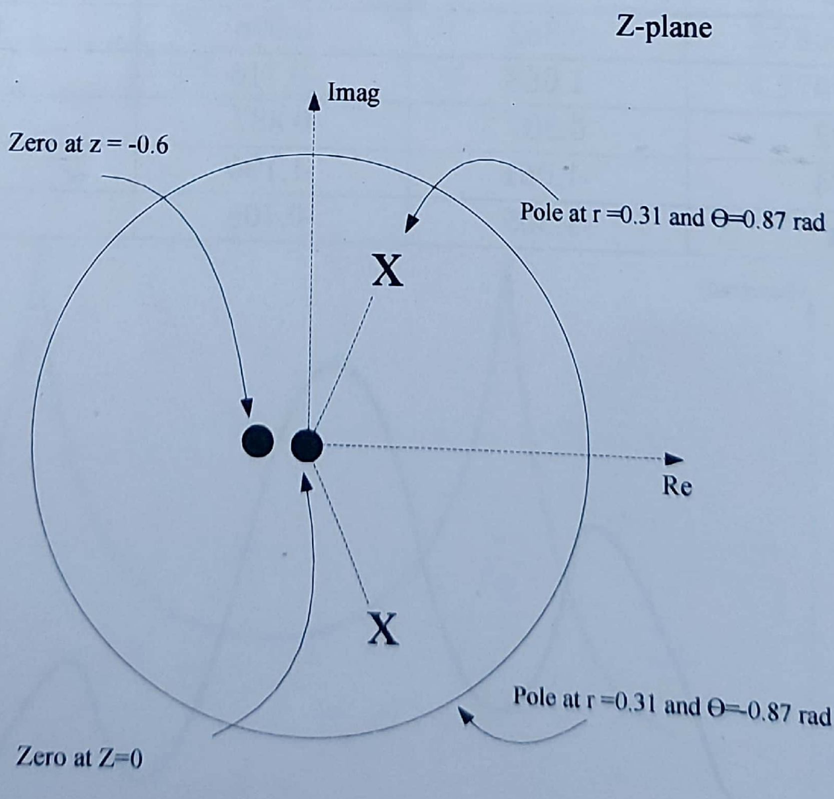

Figure 11.12: Pole-zero plot on the Z-plane

To compute the magnitude,we use the following relation:

$$\text{Magnitude} = 10\log\left[1 + r^2 - 2r\cos(w - \theta)\right]$$

$\text{Magnitude} = 10\log\left[1 + (-0.6)^2 - 2(-0.6)\cos(w - 0)\right]$-10 $\log[1 + 0.31^2 - 2 * 0.31\cos(w - 0.87)] - 10\log\left[1 + 0.31^2 - 2 * 0.31\cos(w + 0.87)\right]$

Let , A=$10\log\left[1 + (-0.6)^2 - 2(-0.6)\cos(w - 0)\right.$

B= $-10\log[1 + 0.31^2 - 2 * 0.31\cos(w - 0.87)]$ and

C= $-10\log\left[1 + 0.31^2 - 2 * 0.31\cos(w + 0.87)\right]$

Hence, Magnitude = A+B+C

Calculation of Magnitude

W	A	B	C	Mag= A+B+C
0	4.08	1.57	1.78	7.43
$\pi/4$	3.44	3.20	-0.47	6.17
$\pi/2$	1.33	2.06	-1.86	1.53
π	-7.95	-1.74	-1.65	-9.81
$5\pi/4$	-2.91	-2.33	-0.04	-5.28
$3\pi/2$	1.33	-1.95	2.29	1.67
$7\pi/4$	3.44	-0.60	3.51	6.35
2π	4.08	1.57	1.78	7.43

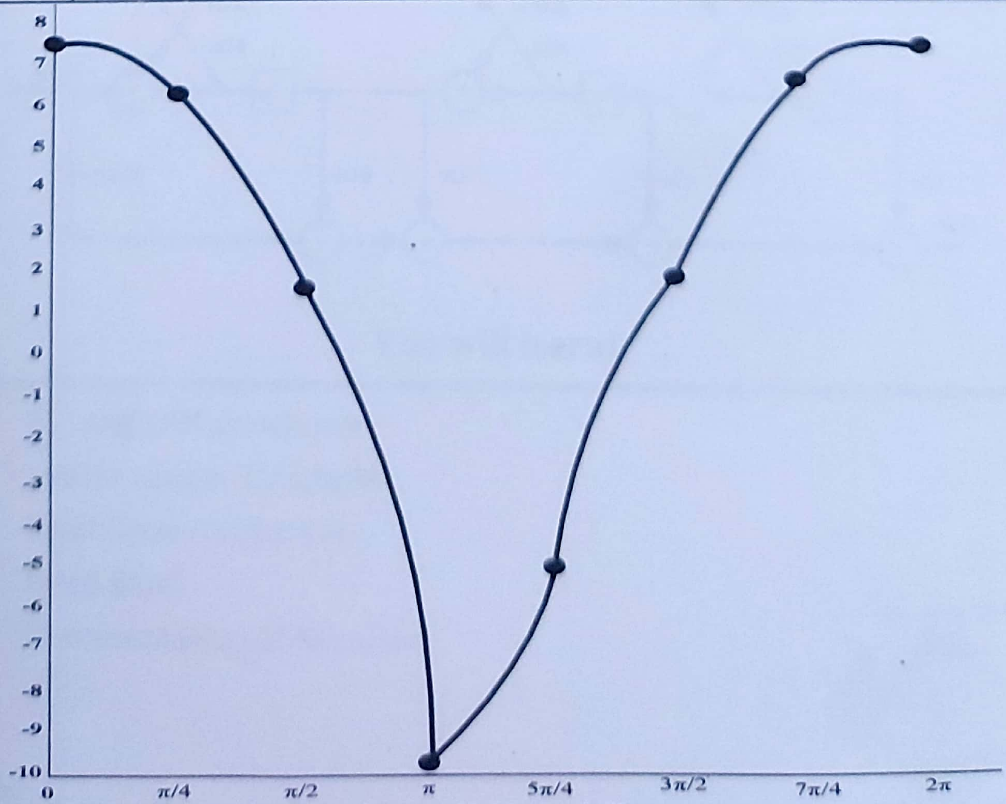

Figure 11.13: Magnitude Plot

Difficult situation:

Let us examine one case where either a pole or a zero is located at r =1, θ= 0 and w=0 in this case magnitude may be obtained as infinite because

$10\log[1+r^2 - 2r\cos(w-\theta)] = 10\log[0] = \text{infinity}$

Which introduces difficulty in plotting the magnitude so, in such a case we assume that $\log[0] \approx \log[0.0000001] = -7$

Exercise

1. Compute the frequency response of first order IIR system.
2. Write algorithm to compute frequency response of the system having multiple poles and zeroes.
3. For a system with poles at $0.3 \pm j0.4$ *and* $0.5 - j0.7$ and zeroes at $0.1 \pm j0.2$and $0.45 \pm j0.25$. Draw the magnitude response and phase response of the system.
4. For the system described by the following difference equation: $y(n) - 0.3y(n-1) = 2x(n-2) + 0.7x(n-1) + 4x(n)$. Plot the magnitude and phase response of the system.