Unit 6. Fuzzy Logic

Fuzzy Logic is a computational paradigm that handles reasoning in situations where information is uncertain, imprecise, or vague. It is based on the concept of degrees of truth, where values range between 0 and 1, instead of binary true/false logic.

6.1. Classical vs Fuzzy Logic

Classical Logic:

- Uses binary truth values: True (1) or False (0).
- Operates with crisp sets where elements either belong to a set or do not.
- Rules are rigid and deterministic.
- Example: "If temperature = 30°C, fan speed = medium."

Fuzzy Logic:

- Allows degrees of truth, where values range between 0 and 1.
- Works with fuzzy sets, where elements have varying degrees of membership.
- Rules are flexible and approximate.
- Example: "If temperature is warm, fan speed is medium."

Comparison Table:

Aspect	Classical Logic	Fuzzy Logic
Truth Values	Binary (0 or 1)	Continuous ([0, 1])
Set Membership	Crisp sets	Fuzzy sets
Rule Flexibility	Rigid	Flexible
Application Areas	Deterministic systems	Uncertain, vague systems
Example Output	"Hot or not"	"80% hot"

6.2. Fuzzy Sets and Membership Functions

Fuzzy Sets:

A fuzzy set is a collection of elements with degrees of membership defined by a membership function. Every element is associated with value, which is between 0 to 1 based on the certainty.

Membership Functions:

• **Definition**: A membership function maps each element of a universe of discourse to a value in the interval [0,1] where:

- 0 indicates no membership,
- 1 indicates full membership,
- Values between 0 and 1 indicate partial membership.

• **Types of Membership Functions**:

I. **Triangular**: Defined by a triangular shape with three parameters (a,b,c). It is linear on both sides.

A **triangular membership function** is one of the simplest and most commonly used membership functions in fuzzy logic. It is defined by three parameters: a, b, and c, where a ≤ b ≤c. These parameters represent the "foot" points (start and end of the triangle) and the "peak" point (maximum membership value = 1).

The formula for a triangular membership function is:

$$\mu(x) = \begin{cases} 0, & x \leq a \text{ or } x \geq c \\ \frac{x-a}{b-a}, & a < x \leq b \\ \frac{c-x}{c-b}, & b < x < c \end{cases}$$

Let's define a fuzzy set "**Medium Temperature**" using a triangular membership function with:

- $a = 20$, $b = 30$, $c = 40$.

Interpretation:

- At $x = 20$, membership is 0 (no membership).
- At $x = 30$, membership is 1 (fully belongs to the set).
- At $x = 40$, membership is 0 (no membership).
- For $x = 25$, membership is calculated as:

$$\mu(25) = \frac{25-20}{30-20} = 0.5$$

The function forms a triangle:

- Base: From 20 to 40,
- Peak: At 30, where membership = 1.

II. **Trapezoidal**: Similar to triangular but with a flat top, defined by four parameters (a, b,c,d).

A **trapezoidal membership function** is an extension of the triangular membership function. It is defined by four parameters: a, b, c, and d, where a ≤b ≤c ≤d. The function has a flat top, making it suitable for representing concepts with a range of full membership values.

The formula for a trapezoidal membership function is:

$$\mu(x) = \begin{cases} 0, & x \leq a \text{ or } x \geq d \\ \frac{x-a}{b-a}, & a < x \leq b \\ 1, & b < x \leq c \\ \frac{d-x}{d-c}, & c < x < d \end{cases}$$

Example:

Let’s define a fuzzy set **"Comfortable Temperature"** using a trapezoidal membership function with:

- a=20, b=25, c=30, d=35.

Interpretation:

- At x=20 or below, membership is 0 (not comfortable).
- Between 25 and 30, membership is 1(fully comfortable).
- Between 20 and 25, membership gradually increases from 0 to 1.
- Between 30 and 35, membership gradually decreases from 1 to 0.
- At x=35 or above, membership is 0 (not comfortable).

Calculation Example:

- At $x = 23$:

$$\mu(23) = \frac{23-20}{25-20} = 0.6$$

- At $x = 32$:

$$\mu(32) = \frac{35-32}{35-30} = 0.6$$

The function forms a trapezoid:

- Base: From 20 to 35,
- Flat top: From 25 to 30, where membership = 1.

III. **Sigmoidal**: S-shaped function, useful for smooth transitions.

A **sigmoidal membership function** is an S-shaped curve often used in fuzzy systems to represent a smooth transition between full membership and non-membership. It is defined by two parameters: the **slope** (a) and the **midpoint** (c), which controls the steepness and the center of the transition, respectively.

The sigmoidal membership function is expressed as:

$$\mu(x) = \frac{1}{1 + e^{-a(x-c)}}$$

Where:

- $a > 0$: The curve increases from 0 to 1 (positive slope).
- $a < 0$: The curve decreases from 1 to 0 (negative slope).
- c: The value of x where the membership is 0.5 (midpoint).

Example:

Let's define a fuzzy set **"Warm Temperature"** using a sigmoidal membership function with:

- $a = 1$ (positive slope),
- $c = 25$ (midpoint).

Interpretation:

- At $x = 25$, membership = 0.5 (midpoint of transition).
- As $x > 25$, membership increases towards 1 (warmer).
- As $x < 25$, membership decreases towards 0 (colder).

Calculation Example:

- At $x = 20$:

$$\mu(20) = \frac{1}{1 + e^{-1(20-25)}} = \frac{1}{1 + e^{5}} \approx 0.0067$$

(very low membership, cold).

- At $x = 30$:

$$\mu(30) = \frac{1}{1 + e^{-1(30-25)}} = \frac{1}{1 + e^{-5}} \approx 0.9933$$

(very high membership, warm).

- The sigmoidal curve starts at nearly 0 for very low x, transitions smoothly through 0.5 at $c = 25$, and approaches 1 for high x.

6.3. Fuzzy Operations

Basic Operations on Fuzzy Sets:

Intersection (AND):

Consider 2 Fuzzy Sets denoted by A and B, then let’s consider Y be the Intersection of them, then for every member of A and B, Y will be:

degree_of_membership(Y)= min(degree_of_membership(A), degree_of_membership(B))

The First Fuzzy Set is A: {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}

The Second Fuzzy Set is B: {'a': 0.9, 'b': 0.9, 'c': 0.4, 'd': 0.5}

Fuzzy Set Intersection is A B: {'a': 0.2, 'b': 0.3, 'c': 0.4, 'd': 0.5}

Union (OR):

Consider 2 Fuzzy Sets denoted by A and B, then let’s consider Y be the Union of them, then for every member of A and B, Y will be:

degree_of_membership(Y)= max(degree_of_membership(A), degree_of_membership(B))

The First Fuzzy Set is A: {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}

The Second Fuzzy Set is B: {'a': 0.9, 'b': 0.9, 'c': 0.4, 'd': 0.5}

Fuzzy Set Union is(AUB): {'a': 0.9, 'b': 0.9, 'c': 0.6, 'd': 0.6}

Complement (NOT):

Consider a Fuzzy Sets denoted by A, then let's consider Y be the Complement of it, then for every member of A, Y will be:

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degree_of_membership(Y)= 1 - degree_of_membership(A)
```

The Fuzzy Set is : {'a': 0.2, 'b': 0.3, 'c': 0.6, 'd': 0.6}

Fuzzy Set Complement is : {'a': 0.8, 'b': 0.7, 'c': 0.4, 'd': 0.4}

Difference :
Consider 2 Fuzzy Sets denoted by A and B, then let's consider Y be the difference of them, then for every member of A and B, Y will be:

degree_of_membership(Y)= min(degree_of_membership(A), 1- degree_of_membership(B))

The First Fuzzy Set is : {"a": 0.2, "b": 0.3, "c": 0.6, "d": 0.6}

The Second Fuzzy Set is : {"a": 0.9, "b": 0.9, "c": 0.4, "d": 0.5}

Fuzzy Set Difference is : {"a": 0.1, "b": 0.1, "c": 0.6, "d": 0.5}

6.4. Fuzzy Rule-Based Systems

Definition:

A Fuzzy Rule-Based System (FRBS) uses fuzzy logic to model relationships between inputs and outputs using fuzzy if-then rules.

1. **Fuzzification**:
   - Converts crisp (precise) input values into fuzzy sets using membership functions.
   - For example, a temperature value of 30°C might belong to the fuzzy sets "Warm" and "Hot" with different degrees of membership (e.g., 0.7 in "Warm" and 0.3 in "Hot").
2. **Rule Base**:

- A collection of **IF-THEN** rules that relate fuzzy input variables to fuzzy output variables.
    - Example Rule:
      IF Temperature is Warm **AND** Humidity is High **THEN** Fan Speed is Medium.
    - These rules are often derived from expert knowledge or observational data.
3. **Inference Engine**:
    - Evaluates the rules in the rule base and determines the fuzzy output based on the fuzzy inputs.
    - Common inference methods include **Mamdani Inference** and **Sugeno Inference**.
4. **Defuzzification**:
    - Converts the fuzzy output into a crisp value to make it interpretable or actionable.
    - Common defuzzification methods include **Centroid of Area (COA)**, **Mean of Maximum (MoM)**, and **Weighted Average**.
5. **Knowledge Base**:
    - Combines the rule base and the membership functions of the fuzzy sets.

Applications of Fuzzy Rule-Based Systems:

- **Control Systems**: HVAC systems, washing machines, and automated cars.
- **Decision Support**: Medical diagnosis, financial modeling, and customer behavior analysis.
- **Pattern Recognition**: Image processing and speech recognition.
- **Optimization Problems**: Resource allocation and scheduling.

Advantages:

- Handles uncertainty and imprecision effectively.
- Mimics human reasoning by working with linguistic terms (e.g., low, medium, high).
- Easy to understand and modify when rules are interpretable.

Disadvantages:

- Rule explosion can occur with many input variables.
- Performance depends on the quality of the rules and membership functions.
- Computationally intensive for large-scale systems.

6.5. Fuzzification and Defuzzification

Fuzzification:

Fuzzification is the process of converting a crisp (precise) input value into a fuzzy value by determining its degree of membership in one or more fuzzy sets. This process allows fuzzy logic systems to handle uncertainty and vagueness by transforming precise numerical data into linguistic terms (e.g., "low," "medium," "high").

Steps in Fuzzification:

1. **Define the Universe of Discourse**:
    - The range of possible values for the input variable.
    - Example: For temperature, the range could be 0°C to 50°C.
2. **Define Fuzzy Sets and Membership Functions**:
    - A fuzzy set is a collection of values within the universe of discourse that have varying degrees of membership.
    - Membership functions assign a degree of membership (between 0 and 1) to each value in the fuzzy set.
3. **Determine the Degree of Membership**:
    - For a given crisp input, calculate its membership value in each fuzzy set using the membership functions.

Example

Problem: Fuzzify the temperature input x = 30^0C

1. **Universe of Discourse**:

   Temperature range: $0^\circ C$ to $50^\circ C$.

2. **Define Fuzzy Sets**:

   - "Cold": Triangular membership function with points (0, 0, 25).
   - "Warm": Triangular membership function with points (20, 30, 40).
   - "Hot": Triangular membership function with points (35, 50, 50).

3. **Crisp Input**:

   X=30^0C

4. **Membership Calculation**:

Cold:

$$\mu_{\text{Cold}}(30) = 0 \quad (\text{as } 30 > 25)$$

Warm:

$$\mu_{\text{Warm}}(30) = \frac{30 - 20}{30 - 20} = 1$$

Hot:

$$\mu_{\text{Hot}}(30) = \frac{50 - 30}{50 - 35} = \frac{20}{15} = 0.67$$

Output of Fuzzification:

The input x = 30^0C has:

- **Cold**: 0 (not cold at all)
- **Warm**: 1 (fully warm)
- **Hot**: 0.67 (partially hot)

Defuzzification:

- The fuzzy output set is converted into a crisp value.
- There are multiple methods for defuzzification:
  - **Centroid Method (Center of Gravity)**: Finds the "center of mass" of the fuzzy set.
  - **Mean of Maximum (MOM)**: Calculates the average of all values at which the membership function reaches its maximum.
  - **Bisector Method**: Splits the fuzzy set into two equal areas.
  - **Smallest of Maximum (SOM)**: Takes the smallest value with the maximum membership.
  - **Largest of Maximum (LOM)**: Takes the largest value with the maximum membership.

Example: Centroid Method

Problem:

A fuzzy logic system evaluates the performance of employees based on "effort" and "skill". Based on fuzzy rules, the system outputs a fuzzy set representing the performance score. The fuzzy set has the following membership function:

- Performance score (crisp values): **10, 20, 30, 40, 50, 60, 70**
- Membership values (fuzzy values): **0.1, 0.4, 0.8, 1.0, 0.7, 0.3, 0.1**

Solution:

Define the Membership Function: The output fuzzy set is:

```
Score:      10   20   30   40   50   60   70
Membership: 0.1  0.4  0.8  1.0  0.7  0.3  0.1
```

Calculate the Centroid: The formula for the centroid method is:

$$C = \frac{\sum(\mu(x) \cdot x)}{\sum \mu(x)}$$

Where $\mu(x)$ is the membership value, and x is the crisp value.

- Numerator: $\sum(\mu(x) \cdot x) = (0.1 \cdot 10) + (0.4 \cdot 20) + (0.8 \cdot 30) + (1.0 \cdot 40) + (0.7 \cdot 50) + (0.3 \cdot 60) + (0.1 \cdot 70)$

$$= 1 + 8 + 24 + 40 + 35 + 18 + 7 = 133$$

- Denominator: $\sum \mu(x) = 0.1 + 0.4 + 0.8 + 1.0 + 0.7 + 0.3 + 0.1 = 3.4$

Calculate the Crisp Output:

$$C = \frac{133}{3.4} \approx 39.12$$

- $C = 39.12$ indicates that the employee's **performance score** (a crisp value) is approximately **39.12**.

Fuzzy inference System: Mamdani

he **Mamdani inference engine** is a widely used method in **fuzzy logic systems** for reasoning and decision-making. It was introduced by Ebrahim Mamdani in 1975 and is particularly well-suited for control systems and decision-making processes that involve a human-like reasoning style.

Key Features of the Mamdani Inference Engine

1. **Fuzzy Rule Base**:
    - Rules are written in the format:
        - `If <condition1> AND/OR <condition2> THEN <output>`
    - Example:
        - `If temperature is high AND humidity is high, THEN fan speed is fast.`
2. **Fuzzification**:
    - Converts **crisp inputs** (precise numerical values) into **fuzzy sets**.
    - Uses **membership functions** (e.g., triangular, trapezoidal, or Gaussian) to map the input values into degrees of membership.
3. **Inference Mechanism**:
    - Uses **logical operations** (AND, OR, NOT) to evaluate the rules in the rule base.
    - Combines the fuzzy inputs to determine the fuzzy output for each rule.
    - Typically uses the **min operator** (for AND) and **max operator** (for OR).
4. **Aggregation**:
    - Combines the fuzzy outputs of all rules into a single fuzzy set.
    - Uses a **fuzzy union** (often the max operation) to aggregate the contributions of all rules.
5. **Defuzzification**:
    - Converts the aggregated fuzzy output into a **crisp value**.
    - Common methods for defuzzification:
        - **Centroid of Area (CoA)**: Finds the center of gravity of the fuzzy set.
        - **Mean of Maximum (MoM)**: Averages the values with the maximum membership degree.

Example of Mamdani Inference

Scenario: Controlling a fan's speed based on temperature and humidity.

Inputs:

- Temperature: 35°C
- Humidity: 70%

Rule Base:

1. If temperature is low AND humidity is low, THEN fan_speed is low.
2. If temperature is medium OR humidity is medium, THEN fan_speed is medium.
3. If temperature is high AND humidity is high, THEN fan_speed is high.

Steps:

1. **Fuzzify the inputs**:
   - Temperature (35°C) → "Medium" (0.6), "High" (0.4)
   - Humidity (70%) → "Medium" (0.3), "High" (0.7)
2. **Evaluate rules**:
   - Rule 1: Low AND Low → Not triggered (0 degree of membership).
   - Rule 2: Medium OR Medium → Degree of membership = max(0.6, 0.3) = 0.6.
   - Rule 3: High AND High → Degree of membership = min(0.4, 0.7) = 0.4.
3. **Aggregate outputs**:
   - Combine fuzzy outputs using max operation.
4. **Defuzzify**:
   - Calculate the crisp fan speed using CoA or another method.