EXAMPLE 6.1. Determine whether the following systems are causal or non-causal:

(i) $y(t) = x(-t)$

(ii) $\dfrac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

Solution : (i) Given $y(t) = x(-t)$

In above equation, if we substitute $t = -2$, then

$$y(-2) = x(-(-2)) = x(2)$$

This means that the output depends upon the future inputs. Therefore, the given system is a **non-causal system.**

(ii) Given $\dfrac{dy(t)}{dt} + 10y(t) + 5 = x(t)$

From above equation, it may be observed that the output $y(t)$ depends upon present inputs. Therefore, this is a **causal system.**

EXAMPLE 6.2. Determine whether the following systems are causal or non-causal:

(i) $y(t) = x(t) \cos(t + 1)$

(ii) $y(t) = x(2t)$

Solution : (i) Given $y(t) = x(t) \cos(t + 1)$

* By recording the entire history of a physiological signal such as EEG or EKG, for example, we can process it to extract key features using a noncausal system. Recorded images typically are processed using non-causal systems for enhancement and feature extraction.

From above equation, it may be observed that the output $y(t)$ depends upon the present input $x(t)$. Also, a cosine function can be calculated at $(t + 1)$.

Thus, the given system is a **causal system.**

(*ii*) Given $y(t) = x(2t)$

From above equation, it may be observed that

If we substitute $t = 2$,

then, $$y(2) = x(2 \times 2) = x(4)$$

This means that the output at $t = 2$ depends upon future input $x(4)$. Therefore, this is a **non-causal system.***

EXAMPLE 6.3. Determine if the systems described by the following input-output equations are causal or non-causal.

(*i*) $y(n) = x(n) - x(n-1)$

(*ii*) $y(n) = a \cdot x(n)$

Solution : The given systems are causal systems because the output $y(n)$ depends only on the present and past inputs. This means that these systems are physically realizable.

EXAMPLE 6.4. Determine if the systems described by the following input-output equations are causal or non-causal

(*i*) $y(n) = x(n) + 3x(n + 4)$,

(*ii*) $y(n) = x(n^2)$

Solution : The given systems are non-causal systems because the output depends on future values of the input.

SOLVED EXAMPLES

EXAMPLE 6.9. A discrete time system is described by the following expression:

$$y(n) = y^2(n-1) + x(n)$$

Now, a bounded input of $x(n) = 2\delta(n)$ is applied to this system. Assuming that the system is initially relaxed, check whether this system is stable or unstable.

Solution : Given that the input $x(n) = 2\delta(n)$ is applied.

We know that, $\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$

Therefore, $x(n) = \begin{cases} 2 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$

Also, we have $y(0) = y^2(0-1) + x(0)$

Since system is initially relaxed, $y^2(0-1) = y^2(-1) = 0$

and $x(0) = 2$ as illustrated above.

Hence, $y(0) = 2$

Now, $y(1) = y^2(1-1) + x(1) = y^2(0) + x(1)$

Since $x(n) = 0$ when $n \neq 0$, above equation becomes,

$$y(1) = y^2(0) = 2^2$$

Similarly, we have $y(2) = y^2(2-1) + x(2) = y^2(1) = (2^2)^2 = 2^4 = 2^{2^2}$

$$y(3) = y^2(3-1) + x(3) = y^2(2) = \left(2^4\right)^2 = 2^8 = 2^{2^3}$$

$$y(4) = y^2(4-1) + x(4) = y^2(3) = \left(2^8\right)^2 = 2^{16} = 2^{2^4}$$

Similarly, $y(n) = 2^{2^n}$

Important Conclusion:

Here as $n \to \infty$, $y(n) \to \infty$. Hence, the input $x(n) = 2\delta(n)$ is bounded but output $y(n)$ is not bounded for all 'n'. Therefore, we conclude that the given system is unstable.

EXAMPLE 6.10. Determine if the following system described by,

$$y(t) = \sin[x(t+2)];$$

is memoryless, causal, linear, time invariant, stable.

(Anna University, Chennai, Sem. Exam; 2005-06)

Solution: (*i*) Given that

$$y(t) = \sin[x(t+2)]$$

(*A*) Memoryless

As present output depends on future inputs, the system is not memoryless (Static).

(*B*) Casual

As output depends on future input *i.e.* $x(t+2)$, the system is non-causal.

(*C*) Time invariant

Given that $y(t) = \sin[x(t+2)]$

Progress is impossible without change; and who cannot change their minds cannot change anything.

— G.B. Shaw

Step I : Delay the input by k samples and denote corresponding output by $y(t, k)$.

$\therefore$ $$y(t, k) = \sin [x(t - k + 2)] \quad ...(i)$$

Step II : Replace t by $t - k$ throughout the given equation.

$\therefore$ $$y(t - k) = \sin [x(t - k + 2)] \quad ...(ii)$$

Step III : Comparing equation (i) and (ii), we get

Here, $$y(t, k) = y(t - k)$$

Thus, the system is time invariant.

***(D)* Linear**

$$y(t) = \sin [x(t + 2)]$$

Step I : When input is zero, output $y(t)$ is also zero so we will check remaining conditions.

Step II : Let us consider two separate inputs $x_1(t)$ and $x_2(t)$.

Let us apply it to the system *i.e.*,

$$x_1(t) \xrightarrow{T} y_1(t) = \sin [x_1(t + 2)]$$

and $$x_2(t) \xrightarrow{T} y_2(t) = \sin [x_2(t + 2)]$$

Add $y_1(t)$ and $y_2(t)$ to get $y'(t)$ *i.e.*,

$\therefore$ $$y'(t) = y_1(t) + y_2(t)$$

$\therefore$ $$y'(t) = \sin [x_1(t + 2)] + \sin [x_2(t + 2)] \quad ...(i)$$

Step III : Let us combine two inputs and apply it to the system, *i.e.*,

$$[x_1(t) + x_2(t)] \xrightarrow{T} \sin [x_1(t + 2) + x_2(t + 2)]$$

Therefore, $$y''(t) = \sin [x_1(t + 2) + x_2(t + 2)] \quad ...(ii)$$

Step IV : Let us compare equations (iii) and (iv) as under:

Now, since $y'(t) \neq y''(t)$, the system is **non-linear.**

***(E)* Stable**

For every bounded value of $x(t + 2)$, its sinc value is always bounded (finite). So given system is **stable.**

EXAMPLE 6.11. Consider the system

$$y(t) = x^2(t - t_0) + 2$$

Determine whether the system is

(*i*) linear; (*ii*) stable; (*iii*) causal; Justify your answer.

(BPTU,. Orissa, Sem. Exam., 2002-2003) (05 marks)

Solution : Given the input-output relationship of a continuous-time system as under:

$$y(t) = x^2(t - t_0) + 2 \quad ...(i)$$

(i) The corresponding outputs for two continuous-time inputs $x_1(t)$ and $x_2(t)$ are

$$y_1(t) = x_1^2(t - t_0) + 2$$

$$y_2(t) = x_2^2(t - t_0) + 2$$

Further, a linear combination of two inputs results in the following output:

$$y_3(t) = T[x_3^2(t - t_0) + 2]$$

or $$y_3(t) = T[[Ax_1(t - t_0) + Bx_2(t - t_0)]^2 + 2]$$

or $$y_3(t) = T[A^2x_1^2(t - t_0) + B^2x_2^2(t - t_0) + 2ABx_1(t - t_0)\, x_2(t - t_0) + 2]$$

$$y_3(t) = A^2x_1^2(t - t_0) + B^2x_2^2(t - t_0) + 2ABx_1(t - t_0)\, x_2(t - t_0) + 2 \quad ...(ii)$$

where $T = 1$

Also, a linear combination of the two outputs results in the following output

$$Ay_1(t) + By_2(t) \quad ...(iii)$$

Substituting equation (*i*) in equation (*iii*), we get

$$Ay_1(t) + By_2(t) = A[x_1^2(t - t_0) + 2] + B[x_2^2(t - t_0) + 2]$$

or $$Ay_1(t) + By_2(t) = Ax_1^2(t - t_0) + 2A + Bx_2^2(t - t_0) + 2B$$

or $$Ay_1(t) + By_2(t) = Ax_1^2(t - t_0) + Bx_1^2(t - t_0) + 2(A + B) \quad ...(iv)$$

Because, both outputs found in equations (*ii*) and (*iv*) are not equal, **therefore, the system is non-linear.**

(*ii*) Given system is

$$y(t) = x^2(t - t_0) + 2$$

The above given system is a causal system because its output does not depend upon future value of input signal, but depends on past value of input.

(*iii*) Given system is

$$y(t) = x^2(t - t_0) + 2 \quad ...(v)$$

Now, let us check the stability of the given system. For this, we shall use the simple bounded inputs such as an unit step function or a constant.

Substituting $x(t) = 1$ in equation (*v*), we get

$$y(t) = 1 + 2 = 3$$

Here, $y(t)$ is bounded.

Important Conclusion:

Hence, we conclude that this system is Bounded Input Bounded Output (BIBO) stable since bounded input $x(t) = 1$, produces bounded output $y(t) = 3$ **Ans.**

EXAMPLE 6.12. Check the following systems for stability:

(*i*) $y(t) = t\,x(t)$

(*ii*) $y(t) = e^{x(t)}$

Solution : (*i*) We have a system which is characterised by

$$y(t) = t\,x(t) \quad ...(i)$$

Now let us check the stability of the above system by using simple bounded inputs such as a unit step function or constant.

Substituting input $x(t) = 1$ = constant in equation (*i*) yields

$$y(t) = t\,x(t) = t \quad ...(ii)$$

Here, $y(t)$ is unbounded.

Therefore, we can say that this system is BIBO unstable because Bounded input $x(t) = 1$, produces unbounded output $y(t) = t$.

(*ii*) We have a system which is characterised by

$$y(t) = e^{x(t)} \quad ...(iii)$$

Now, let us assume that input $x(t)$ is a bounded input, *i.e.*

$$|x(t)| < B < \infty \qquad \text{for all time } t \quad ...(iv)$$

Then output of above system will be given by

$$|y(t)| < e^B < \infty \qquad \text{for all time } t \quad ...(v)$$

where B is the arbitrary positive number.

Important Conclusion:
We conclude that above system is a BIBO stable because it produces bounded output from bounded input.

EXAMPLE 6.13. Input-output relationship for some continuous-time systems is given as under:

(i) $y(t) = x(t-2) + x(2-t)$

(ii) $y(t) = \cos 3(t)\, x(t)$

Check for the following properties:

(a) Memoryless or with memory

(b) Causality

Solution : (i) Given a continuous-time system

$$y(t) = x(t-2) + x(2-t)$$

(a) It is not a memoryless system because its output y(t) at some time depends on both future and past values of inputs. Hence, it is a system with memory.

(b) It is a non-causal system because its output $y(t)$ at some time depends on future values of input $x(n)$.

(ii) Given a continuous-time system

$$y(t) = [\cos(3t)]\, x(t)$$

(a) It is a memoryless system because its output $y(t)$ at some time depends only on that time. Output of this system does not depend on past and future values of input.

(b) It is also a causal system because its output $y(t)$ at some time does not depend on future values of input $x(t)$.

EXAMPLE 6.14. Consider the system whose input-output relation is given by the linear equation

$$y = ax + b$$

where x and y are the input and output of this system, respectively and a and b are constants. Is this system linears.

Solution : If $b \neq 0$, then the system is not linear because $x = 0$ implies $y = b \neq 0$.

If $b = 0$, **then the system is linear. Ans.**

EXAMPLE 6.15. Determine if the systems described by the followed input-output equations are linear or non linear.

(i) $y(n) = nx(n)$

(ii) $y(n) = x^2(n)$ ***(Cochin University, Kerala, Sem. Exam. 2004-05) (05 marks)***

Solution: (i) $y(n) = nx(n)$

$$y_1(n) = nx_1(n)$$

$$y_2(n) = nx_2(n)$$

Therefore, we have

$$y'(n) = a_1 y_1(n) + a_2 y_2(n) = a_1 nx_1(n) + a_2 nx_2(n) \quad ...(i)$$

Now, let us add the input first and then pass through the system, *i.e.*,

$$y''(n) = n[a_1 x_1(n) + a_2 x_2(n)] = a_1 nx_1(n) + a_2 nx_2(n) \quad ...(ii)$$

Since $y'(n) = y''(n)$, **the system is linear.**

(ii) $y(n) = x^2(n)$

$$y_1(n) = x_1^2(n)$$

$$y_2(n) = x_2^2(n)$$

Therefore, we have $y'(n) = a_1\, x_1^2\,(n) + a_2\, x_2^2\,(n)$ ...(iii)

Now, let us add the inputs and pass the addition through the system *i.e.*,

$$y''(n) = [a_1 x_1(n) + a_2 x_2(n)]^2 = a_1^2 x_1^2(n) + a_2^2 x_2^2(n) + 2a_1\, a_2\, x_1(n)x_2(n) \quad ...(iv)$$

Since $y'(n) \neq y''(n)$, **the system is non linear.**

EXAMPLE 6.16. Determine if the systems described by the following input-output equations are causal or non causal.

(*i*) $y(n) = x(n) - x(n-1)$

(*ii*) $y(n) = x(n)^2$

(*iii*) $y(n) = x(-n)$

(WBTU, Kolkata, Sem. Exam. 2004-05) (05 marks)

Solution: (*i*) $y(n) = x(n) - x(n-1)$

This system is *causal* because the output is dependent on the present and past values of $x(n)$.

(*ii*) $y(n) = x(n)^2$

This is a non causal system because the output $y(n)$ is dependent on the future values.

For example, at $n = 2$, we have

$$y(2) = x(2)^2 = x(4)$$

(*iii*) $y(n) = x(-n)$

This is a causal system because the output depends only on the past values.

EXAMPLE 6.17. Consider a discrete-time system with input $x(n)$ and output $y(n)$ related by

$$y(n) = x(n)\, x(n-2)$$

(*i*) Is this system memoryless ?

(*ii*) Is this system linear ?

Solution: (*i*) This system is not memoryless because it needs the values of the past samples to calculate output at any value of n due to the term $x(n-2)$.

(*ii*) Check for linearity :

$$y(n) = x(n)\, x(n-2)$$

i.e. $\quad y_1(n) = x_1(n)\,.\,x_1(n-2)$

and $\quad y_2(n) = x_2(n)\,.\,x_2(n-2)$

Therefore, $\quad y'(n) = a_1\, x_1(n)\, x_1(n-2) + a_2\, x_2(n)\, x_2\,(n-2)$

Now, let us add $x_1(n)$ and $x_2(n)$ and pass the addition through the system *i.e.*,

$$y''(n) = [a_1\, x_1(n) + a_2\, x_2(n)]\, [a_1\, x_1(n-2) + a_2\, x_2(n-2)]$$

Since $\quad y'(n) \neq y''(n)$, **the given system is nonlinear.**