# An Introduction to Automata Theory & Formal Languages

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# 1

CHAPTER

# Introduction to Concept of Automata

**INSIDE THIS CHAPTER**

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## 1.1. HISTORICAL ASPECT OF AUTOMATA

Automata theory is the study of abstract computing devices, or "machines". Before there were computers, in 1930's, Alen turing introduced an abstract machine that has all the capabilities of today's computers, at least as far as in what they could compute. In the 1940's and 1950's simpler kinds of machines, which we today call "finite-automata", were studied by a number of researchers. These automata, originally proposed to model brain function, turned out to be extremely useful for a variety of others purposes. Also in the late 1950's the linguist N. Chomsky began the study of formal "grammars". While not strictly machines, these grammars have close relationships to abstract automata and serve today on the basis of some important software components, including parts of compilers.

In 1969, S. Cook extended turing's study of what could and what could not be computed. Cook separated those problems that can be solved, efficiently by computer, from those problems that *m* problems can be solved, but in practice take so much time, that computers are useless for all but very small instances of the problem. The later class of problem's is called "intractable", or NP-hard. All of this theoritical development bears directly on what Computer Scientists do today.

In computer science we find many examples of finite state system and the theory of finite state systems, and the theory of finite automata as a useful design tool for these systems. A primary example is a switching circuit. A switching circuit is composed of a finite number of gates, each of which can be in one or two conditions, usually denoted by 0 and 1.

We can assume another situation, in case of electric switch only two situation are possible that is "on" and "off". Let us say 1 for on and 0 for off. We can interchange the situation by pushing the switch, it can be seen in Fig. 1.1.

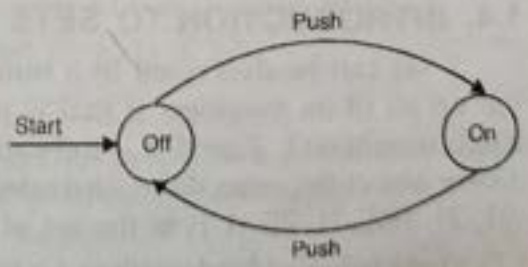


**Fig. 1.1.**

Automation is also tempting to view the human brain as a finite state system. The number of brain cell or neurons are limited, probably $2^{35}$ at most. Although there is evidence to the contrary, that the state of each neuron can be described by a small number of bits, if so then finite state theory applies to the brain also.

## 1.2. THE STUDY OF AUTOMATA THEORY IS FRUITFUL AND FUTURISTIC

There are several reasons why the study of automata and complexity is an important and core part of computer science. In this section we will introduce several reasons which will definitely motivate the computer science students to study automata theory. Let us list some important aspects of the automata theory.

(1) Automata theory plays an important role when we are making software for designing and checking the behaviour of digital circuit.

(2) The "Lexical analyser" of the typical compiler, that is the compiler component that breaks the input text into logical units, such as identifiers, keywords and punctuation.

(3) Software for scanning large bodies of text, such as collections of web pages, to find occurance of words, phrases or other patterns.

(4) Automata theory is key to software for verifying systems of all types that have a finite number of distinct states, such as communication protocols or protocol for secure exchange of information.

(5) Automata theory is most useful concept of software for natural language processing.

## 1.3. DEVELOP YOUR FEELINGS WITH THE AUTOMATA

Truly speaking it is very necessary to understand the subject like automata theory, that you should develop feelings about this subject.

And moreover you must try to implement it in your day to day life. In the whole book you will read about states so let us develop feelings for this term.

Sometime when you are sitting alone thinking about your previous life. Some events you certainly remember, which plays important role in your life. Some of them are not relevent and it is difficult to you to remember those events. Same theory applies in the case of finite state machines, when we design a system or machine, two kinds of points come into picture, some of them certainly affect the output of the system and others never affect the output of the system. So we can skip these irrelevent points. Those points which affect the output of any system are called states (with the assumption that system can design in finite number of steps). Clearly number of states will be finite. The advantage of having only a finite number of states is that we can implement the system with a fixed number of resources.

## 1.4. INTRODUCTION TO SETS

A set can be described in a number of different ways. The simplest is to list up all of its members if that is possible. For example {1, 2, 3} is the set of three numbers 1, 2, and 3. { indicates the beginning of the set, and } its end. Every object between them separated by commas is a member of the set. Thus {{1, 2}, {{3}, 2}, 2}, {1 }} is the set of the elements {1, 2}, {{3}, 2} and {1}.

A set can also be described by listing the properties that its members must satisfy. For example, $\{ x \mid 1 \le x \le 2$ and x is a real number. } represents the set

of real numbers between 1 and 2, and $\{x \mid x$ is the square of an integer and $x < 100\}$ represents the set $\{0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100\}$.

A third way to describe a set is to give a procedure to generate the members of the set. In this representation, first, basic elements of the set are presented. Then a method is given to generate elements of the set from known elements of the set. Thirdly a statement is given that excludes undesirable elements (which may be included in the set otherwise) from the set. for example the set of natural numbers $N$ can be defined recursively as the set that satisfies the following (1), (2), and (3) :

(1) $0 \in N$.

(2) For any number $x$ if $x \in N$, then $x + 1 \in N$.

(3) Nothing is in $N$ unless it is obtained from (1) and (2).

Following this definition, the set of natural numbers $N$ can be obtained as follows :

First by (1), 0 is put into $N$.

Then by (2), since 0 is in $N$, $0 + 1$ (=1) is in $N$.

Then by (2) again, $1 + 1$ (= 2) is in $N$.

Proceeding in this manner all the natural numbers are put into $N$.

Note that if we don't have (3), 0.5, 1.5, 2.5, ... can be included in $N$, which is not what we want as the set of natural numbers.

### 1.4.1. Equality of Sets

Two sets are equal if and only if they have the same elements. More formally, for any sets $A$ and $B$, $A = B$ if and only if $\forall x\ [x \in A \leftrightarrow x \in B]$.

Thus for example $\{1, 2, 3\} = \{3, 2, 1\}$, that is the order of elements does not matter, and $\{1, 2, 3\} = \{3, 2, 1, 1\}$, that is duplications do not make any difference for sets.

### 1.4.2. Subset

A set $A$ is a subset of a set $B$ if and only if everything in $A$ is also in $B$.

More formally, for any sets $A$ and $B$, $A$ is a subset of $B$, and denoted by $A \subseteq B$, if and only if $\forall x[x \in A \rightarrow x \in B]$.

If $A \subseteq B$, and $A \neq B$, then $A$ is said to be a proper subset of $B$ and it is denoted by $A \subset B$.

For example $\{1, 2\} \subseteq \{3, 2, 1\}$

Also $\{1, 2\} \subset \{3, 2, 1\}$.

### 1.4.3. Cardinality

If a set $S$ has $n$ distinct elements for some natural number $n$, $n$ is the cardinality (size) of $S$ and $S$ is a finite set. The cardinality of $S$ is denoted by $|S|$.

For example the cardinality of the set $\{3, 1, 2\}$ is 3.

### 1.4.4. Empty Set

A set which has no elements is called an empty set. More formally, an empty set, denoted by $\emptyset$, is a set that satisfies the following : $\forall x\ x \notin \emptyset$, where $\notin$ means "is not in" or "is not a member of".

Note that $\emptyset$ and $\{\emptyset\}$ are different sets. $\{\emptyset\}$ has one element namely $\emptyset$ in it. So $\{\emptyset\}$ is not empty. But $\emptyset$ has nothing in it.

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### 1.4.5. Universal Set

A set which has all the elements in the universe of discourse is called a universal set. More formally, a universal set, denoted by $U$, is a set that satisfies the following : $\forall x\ x \in U$.

Three subset relationships involving empty set and universal set are listed below as theorems without proof. Note that the set $A$ in the next four theorems are arbitrary. So $A$ can be an empty set or universal set.

**Theorem 1 :** For an arbitrary set $A$ $\quad A \subseteq U$.

**Theorem 2 :** For an arbitrary set $A$ $\quad \emptyset \subseteq A$.

**Theorem 3 :** For an arbitrary set $A$ $\quad A \subseteq A$.

### 1.4.6. Power Set

The set of all subsets of a set $A$ is called the power set of $A$ and denoted by $2^A$ or $\rho(A)$.

For example for $A = \{1, 2\}$, $\rho(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

For $B = \{\{1, 2\}, \{\{1\}, 2\}, \emptyset\}$, $\rho(B) = \{\emptyset, \{\{1, 2\}\}, \{\{\{1\}, 2\}\}, \{\emptyset\}, \{\{1, 2\}, \{\{1\}, 2\}\}, \{\{1, 2\}, \emptyset\}, \{\{\{1\}, 2\}, \emptyset\}, \{\{1, 2\}, \{\{1\}, 2\}, \emptyset\}\}$.

Also, $\rho(\emptyset) = \{\emptyset\}$ and $\rho(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

**Theorem 4 :** For an arbitrary set $A$, the number of subsets of $A$ is $2^{|A|}$.

**Example 1.1. List the members of the following sets :**

**(a)** $\{x \mid x$ **is a real number such that** $x^2 = 4\}$

**(b)** $\{x \mid x$ **is an integer such that** $x^2 = 2\}$.

**Solution.** (a) $\{-2, 2\}$, (b) $\emptyset$

**Example 1.2. Determine whether each of the following pairs of sets is equal.**

**(a)** $\{1, 2, 1, 3, 1, 2\}, \{2, 3, 1\}$ **(b)** $\{\{1\}\}, \{1, \{1\}\}$ **(c)** $\emptyset, \{\emptyset\}$

**Solution.** (a) Yes (b) No (c) No

**Example 1.3. For each of the following sets, determine whether 1 is an element of that set.**

**(a)** $\{x \in R \mid x$ **is an integer greater than 1**$\}$

**(b)** $\{x \in R \mid x$ **is the square of an integer**$\}$

**(c)** $\{1, 2, \{1\}\}$

**(d)** $\{\{1\}, \{\{1\}\}\}$

**(e)** $\{\{1, 2\}, \{1, \{1\}\}\}$

**(f)** $\{\{\{1\}\}\}$.

**Solution.** (a) No (b) Yes (c) Yes (d) No (e) No (f) No

**Example 1.4. Suppose that $A$, $B$ and $C$ are sets such that $A \subset B$ and $B \subseteq C$. Show that $A \subseteq C$.**

**Solution.** Let $x$ be an arbitrary element of $A$. We are going to show that $x \in C$. Since $A \emptyset B$, this implies that $x \in B$. Since $B \subseteq C$, we see that $x \in C$. Since $x \in A$ implies that $x \in C$, it follows that $A \subseteq C$.

**Example 1.5. Find the power set of each of the following sets :**

(a) $\{a\}$ (b) $\{a, \{a\}\}$ (c) $\{\emptyset, \{\emptyset\}\}$

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**Solution.**

(*a*) $\{\emptyset, \{a\}\}$ (*b*) $\{\emptyset, \{a\}, \{\{a\}\}, \{a, \{a\}\}\}$

(*c*) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

### 1.4.7. Set Operations

Sets can be combined in a number of different ways to produce another set. Here some basic operations are introduced and their properties are discussed.

(*i*) Union of sets
(*ii*) Intersection of sets
(*iii*) Difference of sets
(*iv*) Complement of set
(*v*) Ordered pair, ordered *n*-tuple
(*vi*) Equality of ordered *n*-tuples
(*vii*) Cartesian product of sets

#### 1.4.7.1. Union of Sets

The union of sets $A$ and $B$, denoted by $A \cup B$, is the set defined as

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

For example if $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

For example if $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$.

Note that elements are not repeated in a set.

#### 1.4.7.2. Intersection of Sets

The intersection of sets $A$ and $B$, denoted by $A \cap B$, is the set defined as

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

For example if $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A \cap B = \{1, 2\}$.

For example if $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A \cap B = \emptyset$.

#### 1.4.7.3. Difference of Sets

The difference of sets $A$ and $B$, denoted by $A - B$, is the set defined as

$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

For example if $A = \{1, 2, 3\}$ and $B = \{1, 2, 4, 5\}$, then $A - B = \{3\}$ and if $A = \{1, 2, 3\}$ and $B = \{4, 5\}$, then $A - B = \{1, 2, 3\}$.

Note that in general $A - B \neq B - A$.

#### 1.4.7.4. Complement of Sets

For a set $A$, the difference $U - A$, where $U$ is the universe, is called the complement of $A$ and it is denoted by $\overline{A}$. Thus $\overline{A}$ is the set of everything that is not in $A$.

The fourth set operation is the Cartesian produce. We first define an ordered pair and Cartesian produce of two sets using it. Then the Cartesian produce of multiple sets is defined using the concept of *n*-tuple.

#### 1.4.7.5. Ordered Pair

An ordered pair is a pair of objects with an order associated with them. If objects are represented by $x$ and $y$, then we write the ordered pair as $<x, y>$.

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Two ordered pairs $<a, b>$ and $<c, d>$ are equal if and only if $a = c$ and $b = d$. For example, the ordered pair $<1, 2>$ is not equal to the ordered pair $<2, 1>$.

### 1.4.7.6. Cartesian Product

The set of all ordered pairs $<a, b>$, where $a$ is an element of $A$ and $b$ is an element of $B$, is called the Cartesian produce of $A$ and $B$ and is denoted by $A \times B$.

**Example 1.6.** Let $A = \{1, 2, 3\}$ and $B = \{a, b\}$. Then

$$A \times B = \{<1, a>, <1, b>, <2, a>, <2, b>, <3, a>, <3, b>\}.$$

**Example 1.7.** For the same $A$ and $B$ as in Example 1.6,

$$B \times A = \{<a, 1>, <a, 2>, <a, 3>, <b, 1>, <b, 2>, <b, 3>\}.$$

As you can see in these examples, in general, $A \times B \neq B \times A$ unless $A = \emptyset$, $B = \emptyset$ or $A = B$. Note that $A \times \emptyset = \emptyset \times A = \emptyset$ because there is no element in $\emptyset$ to form ordered pairs with elements of $A$.

The concept of Cartesian produce can be extended to that of more than two sets. First we are going to define the concept of *ordered n-tuple*.

### 1.4.7.7. Ordered n-tuple

An ordered $n$-tuple is a set of $n$ objects with an order associated with them (rigorous definition to be filled in). If $n$ objects are represented by $x_1, x_2, \ldots, x_n$, then we write the ordered $n$-tuple as $<x_1, x_2, \ldots, x_n>$.

### 1.4.7.8. Cartesian Product

Let $A_1, \ldots, A_n$ be $n$ sets. Then the set of all ordered $n$-tuples $<x_1, \ldots, x_n>$, where $x_i \in a_i$ for all $i$, $1 \leq i \leq n$, is called the Cartesian produce of $A_1, \ldots, A_n$, and is denoted by $A_1 \times \ldots \times A_n$.

**Example 1.8.** Let $A = \{1, 2\}$, $B = \{a, b\}$ and $C = \{5, 6\}$. Then

$$A \times B \times C = \{<1, a, 5>, <1, a, 6>, <1, b, 5>, <1, b, 6>, <2, a, 5>, <2, a, 6>, <2, b, 5>, <2, b, 6>\}.$$

**Equality of $n$-tuples :** Two ordered $n$-tuples $<x_1, \ldots, x_n>$ and $<y_1, \ldots, y_n>$ are equal if and only if $x_i = y_i$ for all $i$, $1 \leq i \leq n$.

For example the ordered 3-tuple $<1, 2, 3>$ is not equal to the ordered $n$-tuple $<2, 3, 1>$.

**Example 1.9. Let $A$ be a set. Show that $\emptyset \times A = A \times \emptyset = \emptyset$.**

**Solution.** $\emptyset \times A = \{(x, y) \mid x \in \emptyset \text{ and } y \in A\} = \emptyset = \{(x, y) \mid x \in A \text{ and } y \in \emptyset\} = A \times \emptyset$

**Example 1.10. How many different elements does $A \times B$ have if $A$ has m elements and $B$ has $n$ elements ?**

**Solution.** $mn$.

**Example 1.11. Let $A = \{1, 2, 3, 4\}$ and $b = \{0, 3, 5\}$. Find**

**(a)** $A \cup B$ **(b)** $A \cap B$

**(c)** $A - B$ **(d)** $B - A$

**Solution.**

(a) $\{0, 1, 2, 3, 4, 5\}$ (b) $\{3\}$

(c) $\{1, 2, 4\}$ (d) $\{0, 5\}$

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**Example 1.12. What can you say about the sets $A$ and $B$ if the following are true ?**

**(a)** $A \cup B = A$ **(b)** $A - B = A$

**Solution.** (a) $B \subseteq A$. For since $A \cup B = A$ every element of $B$ is also in $A$. But $A$ may have more elements than $B$.

(b) $A \cap B = \emptyset$. For $A - B = A$ means that subtracting elements of $B$ does not change $A$. Hence no elements of $B$ are in $A$. Hence $A$ and $B$ have no elements in common.

**Example 1.13.** Let $A$ and $B$ be subsets of a universal set $U$. Show that $A \subseteq B$ if and only if $\overline{B} \subseteq \overline{A}$.

**Solution.**

$$A \subseteq B \Leftrightarrow \forall x(x \in A \rightarrow x \in B) \Leftrightarrow \forall x(x \notin B \rightarrow x \notin A) \Leftrightarrow \forall x(x \in \overline{B} \rightarrow x \in \overline{A}) \Leftrightarrow \overline{B} \subseteq \overline{A}$$

### 1.4.7.9. Properties of Set Operation

1. $A \cup \emptyset = A$
   $A \cap U = A$ ...Identity Laws
2. $A \cup U = U$
   $A \cap \emptyset = \emptyset$ ...Domination Laws
3. $A \cup A = A$
   $A \cap A = A$ ...Idempotent Laws
4. $A \cup B = B \cup A$
   $A \cap B = B \cap A$ ...Commutative Laws
5. $(A \cup B) \cup C = A \cup (B \cup C)$
   $(A \cap B) \cap C = A \cap (B \cap C)$ ...Associative Laws
6. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ ...Distributive Laws
7. If $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$, and $A \cap C \subseteq B \cap D$.
8. If $A \in B$, then $A \cup B = B$ and $A \cap B = A$.
9. $A \cup (B - A) = A \cup B$.
10. $A \cap (B - A) = \emptyset$.
11. $A - (B \cup C) = (A - B) \cap (A - C)$ (cf. $\overline{A \cup B} = \overline{A} \cap \overline{B}$)
    $A - (B \cap C) = (A - B) \cup (A - C)$ (cf. $\overline{A \cap B} = \overline{A} \cup \overline{B}$) ...DeMorgan's Laws
12. $B = A$ if and only if $A \cup B = U$ and $A \cap B = \emptyset$
13. $\overline{\overline{A}} = A$.

**Additional Properties :**

14. $A \subseteq A \cup B$
15. $A \cap B \subseteq A$

## 1.5. LOGIC

In this section, we will introduce some of the basic concepts of mathematical logic. These logical arguments will usually be applied to mathematics, they employ the same technique that are used by a doctor in clinic or a by lawyer in a courtroom. Another reason for the importance of this section is that the circuits that make up digital computers are designed using the same algebra of propositions that we will be discussing.

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### 1.5.1. Propositions

A proposition is a sentence to which one and only one of the terms true or false can be meaningfully applied.

For example :

"Five is odd" → True

"$5 \in \{1, 3, 2\}$ → False

and "$45 \geq 22$" → True

are propositions.

### 1.5.2. Logical Operators

There are several ways in which we commonly combine simple statements into compound ones. To avoid any confusion, we will precisely define each one's meaning and introduce its standard symbol. Let us discuss these logical operators as follows :

1. If $p$ and $q$ are propostions, **their conjunction (AND)**, $p$ and $q$ (denoted $p \wedge q$) is defined in following truth table :

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

2. If $p$ and $q$ are propositions, their **disjunction (OR)** is $p$ or $q$, denoted $p \vee q$, and is defined in following truth table.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

3. If $p$ is a proposition, its **negation (NOT)**, not $p$, is denoted $\sim p$ is defined in following truth table.

| $p$ | $\sim p$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

4. The **conditional operator if $p$ then $q$**, denoted $p \rightarrow q$, is defined by following truth table :

| $p$ | $q$ | $p \rightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 1 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

**The only case in which a conditional proposition is false is when the condition is true and the conclusion is false.**

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5. If $p$ and $q$ are propositions, the **Biconditional operator (If and only if)** $p$ **if and only if** $q$, denoted $p \leftrightarrow q$, is defined by following :

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| 0 | 0 | 1 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

$p \leftrightarrow q$ is true when $p$ and $q$ have the same truth values.

### 1.5.3. Tautology

An expression involving logical variables that is true in all cases is called a tautology. For example following are tautologies.

(*i*) $\sim(p \wedge q) \leftrightarrow (\sim p) \vee (\sim q)$

(*ii*) $p \vee \sim p$

(*iii*) $(p \wedge q) \rightarrow p$

(*iv*) $q \rightarrow (p \vee q)$

(*v*) $(p \vee q) \leftrightarrow (q \vee p)$

### 1.5.4. Contradiction

An expression involving logical variables that is false for all cases of its truth table is called a contradiction. For example $p \wedge \sim p$ and $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ are contradictions.

## 1.6. RELATIONS

Complete understanding of a set is possible only if the structure of the set is made clear by the inter-relationships between its elements. For example, the individuals in crowd can be compared by height, by nationality, or through several other criteria. **In mathematics, such comparisons are called relations.**

"Let $A$ and $B$ be set. A relation from $A$ into $B$ is any subset of $A \times B$"

For example, let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ then

$\{(1, 4), (2, 4), (3, 5)\}$ is a relation from $A$ into $B$.

Let $A = \{2, 3, 5, 6\}$ and define a relation $R$ from $A$ into $A$ by $(x, y) \in R$ if and only if $x$ divides evenly into $y$. So $R = \{(2, 2), (3, 3), (5, 5), (6, 6), (2, 6), (3, 6)\}$.

### 1.6.1. Properties of the Relation

There are following three basic properties of Relation :

(*a*) Reflexive (*b*) Antisymmetric (*c*) Transitive.

Let $A$ be a set and $R$ is a relation on $A$ then

(*i*) $R$ is called **reflexive** if and only if ${}_aR_a$ for all $a \in A$.

(*ii*) $R$ is called **antisymmetric** if and only if whenever ${}_aR_b$ and $a \neq b$ then ${}_bR_a$ is false; or equivalently whenever ${}_aR_b$ and ${}_bR_a$ then $a = b$.

(*iii*) $R$ is called **transitive** if and only if whenever ${}_aR_b$ and ${}_bR_c$ then ${}_aR_c$.

### 1.6.2. Partial Ordering

A relation on a set $A$ **which is reflexive, antisymmetric and transitive** is called a partial ordering on $A$. A set on which there is a partial ordering defined is called a **Partial order set or poset.**

For example, for a set $A$, $\rho(A)$ together with the relation $\subseteq$ is a poset, can be prove by showing following three properties :

(*i*) Let $B \in \rho(A)$. We must show that $\rho \subseteq B$. This is true by the definition of subset. hence the relation is reflexive.

(*ii*) Let $B_1, B_2 \in \rho(A)$ and assume that $B_1 \subseteq B_2$ and $B_1 \neq B_2$ clearly $B_2 \not\subseteq B_1$ so relation is antisymmetric.

(*iii*) Let $B_1, B_2, B_3 \in \rho(A)$ and assume that $B_1 \subseteq B_2$ and $B_2 \subseteq B_3$. So by the defition $B_1 \subseteq B_3$ so relation is transitive.

Above three properties are enough to show that for a set $A$, power set of $A$ ($\rho(A)$) with relation $\subseteq$ is a poset.

### 1.6.3. Symmetric Relation

Let $R$ be a relation on a set $A$. $R$ is called symmetric if and only if whenever ${}_aR_b$, it follows ${}_bR_a$ for $a, b \in A$.

For example consider the relation = defined on $R$. Certainly $a = b \Rightarrow b = a$, so = is a symmetric relation on $R$.

### 1.6.4. Equivalence Relation

A relation $R$ on a set $A$ is called an equivalence relation if and only if it is reflexive, symmetric and transitive.

The classic example of an equivalence relation is the relation = on $R$.

Let $Z^*$ be the set of non-zero integers. One of the most basic equivalence relation in mathematics is the relation $q$ on $Z \times Z^*$ defined by $(a, b)q\ (c, d)$ if and only if $ad = bc$.

## 1.7. COUNTING PRINCIPLES

It is frequently necessary to count how many ways certain choices can be made. In computer science, for instance, we may want to estimate the number of times a certain procedure is called during the execution of a program, in order to assess the program's efficiency, or in theory of games, one might want to access the probability of a certain outcome given the current position, and this usually involves counting the number of ways in which certain combinations can arise.

In this section we collect some of the basic methods for counting the number of ways in which events can happen. We begin with rules of sum and product.

### 1.7.1. Sum Rule

We say **events are mutually exclusive or incompatible** if they cannot occur simultaneously. the rule of sum then says that the number of ways in which either of two mutually exclusive events can occur is equal to the sum of the number of ways in which each can occur separately. In set theoretical terms this says that if $A$ and $B$ are disjoint sets then $|A \cup B| = |A| + |B|$.

### 1.7.2. Product Rule

The rule of product says that number of ways in which two independent events $E_1$ and $E_2$ can occur is the product of the number of ways in which $E_1$ and $E_2$ can occur separately.

## 1.8. ALPHABETS

Everything in mathematics is based on symbols. This is also true for automata theory of computation. The symbols are generally letters and digits.

"Alphabets are defined as a finite set of symbols"

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An example of an alphabet is a set of decimal numbers.

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

Here $\Sigma$ is representation for the alphabet.

Another example is a binary alphabet that is :

$$\Sigma = \{0, 1\}$$

Similarly $\Sigma = \{A, B, \ldots Z\}$ is a alphabet.

$\Sigma = \{a, b, \ldots z\}$ is also a alphabet}.

## 1.9. STRINGS

"A string (or sometimes word) is a finite sequence of symbols selected from some alphabet". For example, if $\Sigma$ $\{a, b\}$ is an alphabet then *abab* is string over alphabet $\Sigma$. A string is generally denoted by $w$.

The **"length"** of the string is denoted by $|w|$ and it is, the number of positions for the symbol in the string. For instance string

$$w = 01101 \text{ has length}$$

$$|w| = 5.$$

**It is common to say that the length of the string is "the number of symbols" in the string; this statement is generally accepted but not strictly correct. Thus, there are only two symbols, 0 and 1, in the string 01101, but there are five positions for symbols, and its length is 5.**

The **"empty"** string is the string with zero occurance of symbols. This string is represented by $\in$, or $e$ or $\lambda$, or $\gamma$ **but we will follow $\in$ in this particular book.**

The set of strings, including the empty, over an alphabet $\Sigma$ is denoted by $\Sigma^*$, that is suppose $\Sigma = \{0, 1\}$ then $\Sigma^*$ is a set containing empty and all combinations of 0's and 1's. Thus

$$\Sigma^* = \{\in, 0, 1, 01, 10, 000, 010, 0000, 1111000 \ldots\}$$

Similarly $\{0\}^* = \{\in, 0, 00, 000 \ldots\}$,

$$\{1\}^* = \{\in, 1, 11, 111 \ldots\}$$

Note that $\in$ is in $\Sigma^*$, regardless of what alphabet $\Sigma$ is.

**That is $\in$ is only string whose length is 0.**

if $\Sigma = \{0, 1\}$, then $\Sigma^1 = \{0, 1\}$, $\Sigma^2 = \{00, 01, 10, 11\}$,

$$\Sigma^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$$

and so on. Note that there is slight confusion between $\Sigma$ and $\Sigma^1$. The formal is an alphabet; its members 0 and 1 are symbols. The latter is a set of strings; its members are the strings 0 and 1, each of which is of length 1. We shall not try to use seprate notations for the two sets, relying on context to make it clear whether $\{0, 1\}$ or similar sets are alphabets or set of strings.

Some times, we wish to exclude the empty string from the set of strings. The set of non-empty strings from alphabet $\Sigma$ is denoted by $\Sigma^+$. Thus, two appropriate equivalence are :

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \ldots$$

$$\Sigma^* = \Sigma^+ \cup \{\in\}.$$

**Concatenation of strings.** Let $w_1$ and $w_2$ be two strings. Then $w_1 w_2$ denotes the concatenation of $w_1$ and $w_2$, that is, the string formed by making a copy of $w_1$ and followed it by a copy of $w_2$.

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if
$$w_1 = abc$$
$$w_2 = xyz$$
then
$$w_1 w_2 = abc\ xyz$$

In the **set form string** $w_2$ is defined as :

$$w_2 = \begin{cases} \epsilon & \text{for } i = 0 \\ w^{i-1} w & \text{for } i \geq 1 \end{cases}$$

that is if $w = aba$

then
$$w^3 = (aba)^3$$
$$w^3 = aba\ aba\ aba = (aba)^2\ aba.$$

**Reverse of the string.** Reverse of the string can be achieved by flipping over last symbols :

if
$$w = abc$$
then $w^R = cba$, where $w^R$ is reverse of $w$, $|w| = |w^R|$

## 1.10. LANGUAGES

"A set of strings all of which are chosen from some $\Sigma^*$, where $\Sigma$ is a particular alphabet, is called a language". If $\Sigma$ is an alphabet, and $L \subseteq \Sigma^*$, then $L$ is said to be language over alphabet $\Sigma$.

Notice that a language over $\Sigma$ need not include strings with all the symbols of $\Sigma$, so once we have established that $L$ is a language over $\Sigma$, we also know it is a language over any alphabet which is a superset of $\Sigma$.

An example is english, where the collection of legal english words is a set of strings over the alphabet that consists of all the letters. Another example is C or any other programming language, where the legal programs are a subset of the possible strings that can be formed from the alphabet of the language. This alphabet is a subset of the ASCII characters. The exact alphabet may differ slightly among different programming languages, but generally includes the upper- and lower-case letters; the digits, punctuation, and mathmatical symbols.

However, there may be some other languages that appear when we study automata. Some examples are as follows :

(*a*) The language of all strings consisting of $n$ 0's followed by $n$ 1's, for some $n \geq 0$ : $\{\epsilon, 01, 0011, 000111, ...\}$

(*b*) The set of strings over 0's and 1's with equal number of each : $\{\epsilon, 01, 10, 0011, 0101, 1001 ...\}$.

(*c*) The set of binary numbers whose value is a prime : $\{10, 11, 101, 111, 1011, ...\}$

(*d*) $\Sigma^*$ is a language for any alphabet $\Sigma$.

(*e*) $\phi$, the empty language, is a language over any alphabet.

(*f*) $\{\epsilon\}$, the language consisting of only the empty string, is also a language over any alphabet. Notice that $\phi \neq \{\epsilon\}$; the former has no strings and latter has one string.

"The only important constraint on what can be a language is that all alphabet are finite. Thus languages, although they can have an infinite number of strings, are restricted to consist of strings drawn from one fixed, finite alphabet.

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### 1.10.1. Concatenation of Languages

Concatenation of languages can be defined as :

if $L_1$ and $L_2$ are two languages, their concatenation

$L = L_1 . L_2$ where $L = \{w/w = xy, x \in L_1, y \in L_2\}$.

That is, the string in $L_1 L_2$ are formed by choosing a string in $L_1$ and followed by a string in $L_2$, in all possible combinations.

### 1.10.2. Languages in Set Forms

It is very common to describe a language using a "set-form" :

**{w/some logical view about w}.**

This expression is read "the set of words $w$ such that (whatever is said about $w$ to the right of the vertical bar)." For example :

(a) $\{w/w$ consists an equal number of 0's and 1's$\}$

(b) $L = \{w/w$ is a binary integer that is prime$\}$

It is common to replace $w$ by some expression with parameters and describe the strings in the language by stating conditions on the parameters. Here are some examples :

$\{a^n b^n/n \geq 1\}$. Read "the set of a to the $n$, $b$ to the $n$ such that $n$ is greater than or equal to 1," this language consists of the strings {ab, aabb, aaabbb, ...}. Notice that as with alphabet, we can raise a single symbol to a power $n$ in order to represent $n$ copies of that symbol.

## 1.11. KLEENE CLOSURE

Given an alphabet, we wish to define a language in which any string of letters form $\Sigma$ is a word, even the null string. This language we shall call the closure of the alphabet. It is denoted by writing a star, after the name of the alphabet as a superscript :

$$\Sigma^*$$

This notation is some times known as the "kleene star" after the name of logician who was one of the founders of this subject.

**Definition.** "If $S$ is a set of words then by $S^*$ we mean the set of all finite strings formed by concatenating words from $S$, where any word may be used as often we like, and where the null string is also included".

For example if $\Sigma = \{a\}$

then $\Sigma^* = \{\epsilon, a, aa, aaa, aaaa, ...\}$

if $\Sigma = \{0, 1\}$

then $\Sigma^* = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001 ...\}$

We can think of the kleene star as an operation that makes an infinite language of strings of letters out of an alphabet. When we say "infinite language", we mean infinitely many words, each of which of finite length.

## 1.12. FUNCTION

A function is an association of each object of one kind with a unique object other kinds.

"A function from a set $S_1$ to a set $S_2$ is a binary relation R on $S_1$ and $S_2$ with the following property : for each $x \in S_1$ there is **exactly one** ordered pair in $R$ with first component of $S_2$.

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Let us see a example :

Let $S_1$ be set of cities and $S_2$ is the set of states and we are defining two relations as follows :

$$R_1 = \{(x, y) : x \in S_1, y \in S_2, \text{ and } x \text{ is a city in state } y\},$$
$$R_2 = \{(x, y) : x \in S_2, y \in S_1, \text{ and } y \text{ is a city in state } x\}.$$

Then relation $R_1$ is a function, since each city is in one and only one state, but $R_2$ is not a function, since some states have more than one city.

Generally function is represented as

$$f : S_1 \to S_2$$

that is it indicate $f$ is a function from $S_1$ to $S_2$. We call $S_1$ the domain and $S_2$ as range. If $x \in S_1$. We write $f(x)$ for the element $y$ of $S_2$ such that $(x, y) \in f$, since $f$ is function, there is exactly one $y \in S_2$ with this property so $f(x)$ denotes a unique object. The object $f(x)$ is called the image of $x$ under $f$.

Now let us discuss various types of functions :

### 1.12.1. On-to-function

A function from set $S_1$ to $S_2$ is said to be an on to function if every element of $S_2$ is the image of one or more element of $S_1$.

Let $S_1 = \{a, b, c, d, e\}$

$S_2 = \{\alpha, \beta, \gamma\}$

Here $f$ is defined as follows :

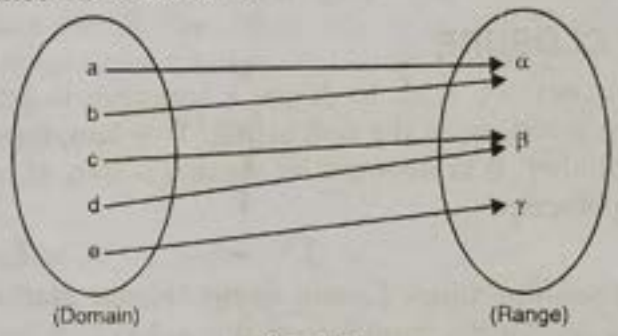


Fig. 1.2.

then $f$ is onto function.

### 1.12.2. One to One Function

A function from $S_1$ to $S_2$ is said to be a one to one function if no two elements of $S_1$ have the same image in $S_2$.

Let $S_1 = \{a, b, c, d, e\}$

and $S_2 = \{\alpha, \beta, \gamma, \delta, \psi\}$

and function $f$ is defined as follows :

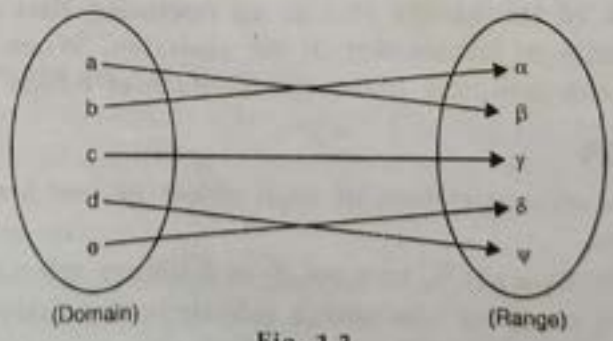


Fig. 1.3.

Here $f$ is said to be one to one function.

### 1.12.3. One to One On-to Function

A function from set $S_1$ to set $S_2$ is said to be one to one onto function if it is both onto and one to one functions.

In the literature an on-to function is also called a surjection. A one to one function is also called an injection. In the similar fashion A one to one onto function is also called bijection.

## 1.13. THREE FUNDAMENTAL PROOF TECHNIQUES

There are three basic fundamental technique to proof. These three fundamental principles, are important to proof any problem.

### 1.13.1. The Principle of Mathmatical Induction

The principle of mathmatical induction states that any set of natural numbers containing zero, and with the property that it contains $n + 1$ whenever if contains all numbers up to end including $n$, must in fact be the set of all natural numbers.

We can use this induction method as follows :

Let us assume that, we want to show that property p holds for all natural numbers. Then to prove it follow the following three steps :

**Basic Step.** First show that property $P$ is true for 0 (zero).

**Induction Hypothesis.** Assume property $P$ holds for 0, 1, 2, ... $n$.

**Induction Step.** Using induction hypothesis show that $P$ is true for $n + 1$.

Then by the induction principle, $P$ is true for all natural numbers.

**Example 1.14. Prove by mathmatical induction**
**$n^4 - 4n^2$ is divisible by 3 for $n \geq 0$.**

**Solution. Basic Step.** For $n = 0$

$$n^4 - 4n^2 = 0 \text{ which is divisible by 3.}$$

**Induction hypothesis.** Let $n^4 - 4n^2$ is divisible by 3

**Induction Step.** $(n + 1)^4 - 4(n + 1)^2$

$$\begin{aligned}
&= ((n + 1)^2)^2 - 4(n + 1)^2 \\
&= (n^2 + 2n + 1)^2 - (2n + 2)^2 \\
&= (n^2 + 2n + 1 + 2n + 2)\ (n^2 + 2n + 1 - 2n - 2) \\
&= (n^2 + 4n + 3)\ (n^2 - 1) \\
&= n^4 + 4n^3 + 3n^2 - 3 - 4n - n^2 \\
&= (n^4 - 4n^2) + 6n^2 - 3 + 4(n^3 - x)
\end{aligned}$$

Now we have to prove that $(n^3 - n)$ is divisible by 3, Let us again use Induction :

**Basic Step.** For $n = 0$

$n^3 - n = 0$ which is divisible by 3.

**Induction hypothesis.** Let $n^3 - n$ is divisible by 3.

**Induction Step.** $(n + 1)^3 - (n + 1)$

$$\begin{aligned}
&= (n^3 + 3n^2 + 3n + 1) - (n + 1) \\
&= (n^3 + 3n^2 + 3n + 1 - n - 1 \\
&= n^3 + 3n^2 + 2n \\
&= (n^3 - n) + 3n^2 + 3n \text{ which is divisible by 3.}
\end{aligned}$$

So we can say that $(n^4 - 4n^2) + 6n^2 - 3 + 4\ (n^3 - n)$ is also divisible by 3. Hence proved.

### 1.13.2. The Pigeonhole Principle

If $S_1$ and $S_2$ are non-empty finite sets and $|S_1| > |S_2|$, then there is no one-to-one function from $S_1$ to $S_2$ (that is there is no bijection).

In other words, if we attempt to pair off the element of $S_1$ (the "pigeons") with element of $S_2$ (the "pigeonholes"), sooner or later we will have to put more than one pigeon in a pigeonhole.

For example

if $S_1 = \{a_1, a_2, a_3\}$ then $|S_1| = 3$

if $S_2 = \{(a_1, a_2), (a_2, a_3)\}$ then $|S_2| = 2$

This would only holds for an undirected graph that is no cycles.

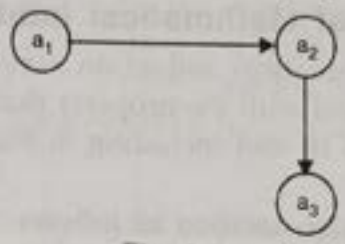


Fig. 1.4.

### 1.13.3. The Diagonalization Principle

Consider set $S_i$ with members $k \in N$ as indicated ( ➤ ) below :

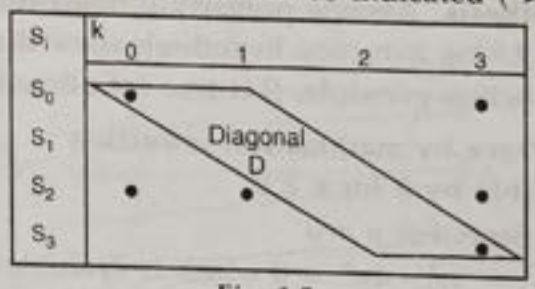


Fig. 1.5.

Diagonal set $D = \{0, 3, ...\}$ and

$\overline{D}$ (Complement of diagonal) $= \{1, 2, ...\}$

Now we can define the diagonal principle as follows :

" Let $R$ be a binary relation on a set $S$, and let $D$, the diagonal set of $R$, be $\{x : x \in S \text{ and } (x, x) \notin R\}$. For each $x \in S$; let $R_x = \{b : b \in S \text{ and } (x, y) \in R\}$. Then $D$ is distinct from each $R_x$.

## 1.14. RECURSIVE DEFINITION

Sets which have too many elements to list them up, and for which there are no convenient or obvious predicates to specify their elements can often be defined using a recursive definition (also called inductive definition). It essentially gives a procedure to generate the members of the set one by one starting with some subset of its elements. In this type of definition, first a collection of elements to be included initially in the set is specified. These elements can be viewed as the seeds of the set being defined. Next, the rules to be used to generate elements of the set from elements already known to be in the set (initially the seeds) are given. These rules provide a method to construct the set element by element starting with the seeds. These rules can also be used to test elements for the membership in the set.

A recursive definition of a set always consists of three distinct clauses :

1. The basis clause (or simply basis) of the definition establishes that certain objects are in the set. This part of the definition specifies the "seeds" of the set from which the elements of the set are generated using the methods given in the inductive clause. The set of elements specified here is called basis of the set being defined.
2. The inductive clause (or simply induction) of the definition establishes the ways in which elements of the set can be combined to produce new elements of the set. The inductive clause always asserts that if objects are elements of the set, then they can be combined in certain specified ways to create other objects. Let us call the objects used to create a new object the parents of the new object, and the new object is their child .
3. The extremal clause asserts that unless an object can be shown to be a member of the set by applying the basis and inductive clauses a finite number of times, the object is not a member of the set.

Let us see some of the examples of Recursive Definition of Set

**Example 1.15.** Definition of the Set of Natural Numbers $N$.

**Solution.** The set $N$ is the set that satisfies the following three clauses:

**Basis Clause.** $0 \in N$.

**Inductive Clause.** For any element $x$ in $N$, $x + 1$ is in $N$.

**Extremal Clause.** Nothing is in $N$ unless it is obtained from the Basis and Inductive Clauses.

The basis for this set $N$ is $\{0\}$. The $x + 1$ in the Inductive Clause is the parent of $x$, and $x$ is the child of $x + 1$.

Following this definition, the set of natural numbers $N$ can be obtained as follows :

First by the Basis Clause, 0 is put into $N$.

Then by the Inductive Clause, since 0 is in $N$, $0 + 1$ (= 1) is in $N$. 0 is the parent of 1, and 1 is the child of 0.

Then by the Inductive Clause again, $1 + 1$ (= 2) is in $N$. 1 is the parent of 2, and 2 is the child of 1.

Proceeding in this manner all the "natural numbers" are put into $N$.

Note that if we don't have the Extremal Clause, 0.5, 1.5, 2.5, ... can be included in $N$, which is not what we want as the set of natural numbers.

**Example 1.16. Definition of the Set of Nonnegative Even Numbers *NE*.**

**Solution.** The set $NE$ is the set that satisfies the following three clauses:

**Basis Clause.** $0 \in NE$

**Inductive Clause.** For any element $x$ in $NE$, $x + 2$ is in $NE$.

**Extremal Clause.** Nothing is in $NE$ unless it is obtained from the Basis and Inductive Clauses.

**Example 1.17. Definition of the Set of Even Integers *EI*.**

**Solution.** The set $EI$ is the set that satisfies the following three clauses :

**Basis Clause.** $0 \in EI$

**Inductive Clause.** For any element $x$ in $EI$, $x + 2$, and $x - 2$ are in $EI$.

**Extremal Clause.** Nothing is in $EI$ unless it is obtained from the Basis and Inductive Clauses.

**Example 1.18. Definition of the Set of Strings $S$ over the alphabet $\{a, b\}$ excepting empty string. This is the set of strings consisting of $a$'s and $b$'s such as *abbab*, *bbabaa*, etc.**

**Solution.** The set $S$ is the set that satisfies the following three clauses :

**Basis Clause.** $a \in S$, and $b \in S$.

**Inductive Clause.** For any element $x$ in $S$, $ax \in S$, and $bx \in S$.

Here $ax$ means the concatenation of a with $x$.

**Extremal Clause.** Nothing is in $S$ unless it is obtained from the Basis and Inductive Clauses.

### 1.14.1. Recursive Definition of Function

Some functions can also be defined recursively.

**Condition :** The domain of the function you wish to define recursively must be a set defined recursively.

**How to define function recursively :** First the values of the function for the basis elements of the domain are specified. Then the value of the function at an element, say $x$, of the domain is defined using its value at the parent(s) of the element $x$.

A few examples are given below.

They are all on functions from integer to integer except the last one.

**Example 1.19. The function $f(n) = n!$ for natural numbers $n$ can be defined recursively as follows.**

**Solution.** The function $f$ is the function that satisfies the following two clauses:

**Basis Clause.** $f(0) = 0! = 1$

**Inductive Clause.** For all natural number $n$, $f(n + 1) = (n + 1) f(n)$.

Note that here Extremal Clause is not necessary, because the set of natural numbers can be defined recursively and that has the extremal clause in it. So there is no chance of other elements to come into the function being defined.

Using this definition, 3 ! can be found as follows:

Since $0! = 1$, $1! = 1 * 0! = 1 * 1 = 1$,

Hence $2! = 2 * 1! = 2 * 1 = 2$

Hence $3! = 3 * 2! = 3 * 2 * 1 = 6$.

**Example 1.20. The function $f(n) = 2n + 1$ for natural numbers $n$ can be defined recursively as follows.**

**Solution.** The function $f$ is the function that satisfies the following two clauses :

**Basis Clause.** $f(0) = 1$

**Inductive Clause.** For all natural number $n$, $f(n + 1) = f(n) + 2$.

See above for the extremal clause.

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## EXERCISE

1. Let $L = \{ab, aa, baa\}$. Which of the following strings are in $L^*$ : *abaab aaabaa, aaaabaaaa, baaaaabaaaab, baaaaabaa.*
2. Give some examples of strings in, and not in these sets, where $\Sigma = \{a, b\}$.
   (a) $\{w : \text{for sum } x \in \Sigma, w = x\, x^R x\}$
   (b) $\{w : ww = www\}$
   (c) $\{w : \text{for some } \mu, v \in \Sigma^*, \mu vw = wv\mu\}$
   (d) $\{w : \text{for some } \mu \in \Sigma^*, www = \mu\mu\}$.
3. Which of the following are true? Explain.
   (a) $baa \in a^* b^* a^* b^*$ (b) $b^* a^* \cap a^* b^* = a^* \cup b^*$
   (c) $a^* b^* \cap b^* c^* = \phi$ (d) $abcd \in (a(cd)^* b)^*$.
4. Write the following explicitly :
   (i) $\{1\} \times \{1, 2\} \times \{1, 2, 3\}$
   (ii) $\phi \times \{1, 2\}$.
5. Let $f : A \to B$ and $g : B \to C$. Let $h : A \to C$ be their composition. In each of the following cases state necessary and sufficient conditions on $f$ and $g$ for $h$ to be as specified.
   (i) On to.
   (ii) One to one.
   (iii) A bijection.
6. Prove the following by principle of induction :
   (i) $\sum_{k=1}^{n} K^2 = \frac{n(n+1)(2n+1)}{6}$
   (ii) $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{(n+1)}$.
7. Prove the following by the principle of induction :
   (i) $2i > i$ for all $i > 1$
   (ii) $1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$
8. If $X = \{a, b\}$ and $Y = \{b, c\}$, find
   (i) $(X \cup Y)^*$ (ii) $(X \cap Y)^c$
   (iii) $(X^* \cup Y^*)$ (iv) $X^* \cap Y^*$
   (v) $X - Y$ (vi) $(X - Y)^*$
   (vii) $(Y - X)^*$.
9. Let $f : \{a, b\}^* \to \{a, b\}^*$ be given by $f(x) = ax$ for every $x \in \{a, b\}^*$. Show that $f$ is one to one not on to.
10. Prove the following by induction $2^x \geq x^2$ if $x \geq 4$.

*****

2

CHAPTER

# Finite Automata

INSIDE THIS CHAPTER

**2.1.** Introduction **2.2.** Definition of Deterministic Finite Automata **2.3.** Processing of strings by DFA **2.4.** Transition Graph **2.5.** Generalised Transition Graphs **2.6.** Simpler Notations for DFA's **2.7.** The Language of DFA **2.8.** Extended Transition Function to Strings **2.9.** Nondeterminism **2.10.** The Language of NFA **2.11.** Transformation of NFA to DFA **2.12.** NFA with $\epsilon$-transitions **2.13.** Acceptance of a string by NFA with $\epsilon$-transitions **2.14.** Conversion of NFA with $\epsilon$-transitions to NFA without $\epsilon$-transitions **2.15.** Minimization of DFA's **2.16.** Myhill-Nerode Theorem **2.17.** Implementation of Myhill-Nerode theorem

## 2.1. INTRODUCTION

In this book, we are discussing the mathematical model of the computers and algorithms. In next two chapter we are going to define powerful models of computation, more and more sophisticated device for accepting the generating languages, which are restricted model of the actual computers, called finite automata or finite state machine. These machines are very similar to the Central Processing Unit of a computer. Absence of the memory makes these machines more restricted model.

Computer is also deterministic, by which we mean that, on reading one particular input instruction, the machine converts itself from the state it was, into some particular other state, where the resultant state is completely determined by the prior state and the input instruction. Some sequence of instruction may lead to success and some may not. Success is entirely determined by the sequence of inputs. Either program will work or not.

**Finite automation is called "finite" because number of possible states and number of letter in the alphabet are both finite, and "automation" because the change of the state is totally governed by the input. It is deterministic, since, what state is next is automatic not will-full, just as the motion of the hands of the clock is automatic, while the motion of hands of a human is presumably the result of desire and thought.**

Before discussing the mathematical model let us discuss the pictorial representation of finite machine. Strings are fed into device by means of an input tape, which is divided into square, with one symbol in each square. The main part of the machine is a "black box", which is responsible for all proceedings. Let us say "black box" is the finite control, finite control can serve, that what symbol is written at any position on the input tape by means of a movable reading head. Initially the head is placed at the left most square of the tape and the finite control is set is designated initial state.

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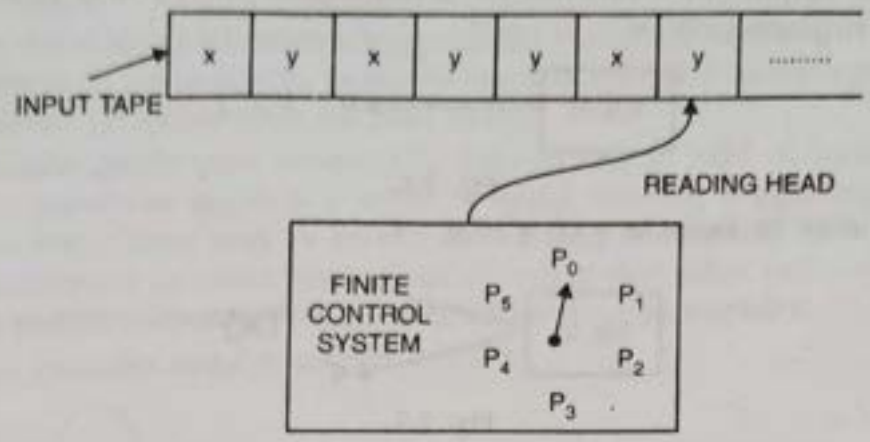


Fig. 2.1.

$P_0, P_1, P_2, P_3, P_4, P_5$ are states in finite control system $x$ and $y$ are input symbols.

At regular interval the automation reads one symbol from the input tape and then enters in a new state that depends only on the current state and the symbol just read.

After reading an input symbol, reading head moves one square to the right on the input tape, so that on the next move, it will read the symbol in next tape square. This process is repeated again and again.

The automation then indicates approval or disapproval.

If it winds up *in one of a set of final states* the input strings is considered to be accepted. The language accepted by the machine is the set of strings, it accepts.

## 2.2. DEFINITION OF DETERMINISTIC FINITE AUTOMATA

When we analyse above discussion mathematically the following formal version obtained.

A deterministic finite automata is a quintuple

$$M = (Q, \Sigma, \delta, q_0, F)$$

where

$Q$ : Is a non-empty finite set of states presents in the finite control. $(q_0, q_1, q_2, \ldots)$

$\Sigma$ : Is a non-empty finite set of input symbols which can be passed to the finite state machine. $(a, b, c, d, e, \ldots)$

$q_0$ : Is a starting state, one of the state in $Q$.

$F$ : Is a non-empty set of final states or accepting states, set of final states belongs to $Q$.

$\delta$ : Is a function called transition function that takes two arguments a state and a input symbol, it returns a *single* state.

Let '$q$' is the state and '$a$' be input symbol passed to the transition function as :

$$\delta(q, a) = q'$$

$q'$ is the output of the function *i.e.*, a single state only, $q'$ may be $q$. We can also view it graphically as :

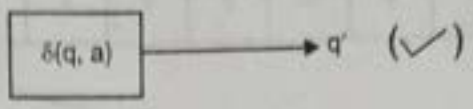


**Fig. 2.2.**

but $q'$ may be same as $q$ *i.e.* $q' = q$

**Fig. 2.3.**

case discussed in Fig. 2.3 is not valid case in deterministic finite automata.

**Some people prefer to call this just a finite acceptor because its sole job is to accept certain input strings and rejects others. It does not do anything like printing machine. Some time it is also referred as language recognizer because FA merely recognizes whether the input strings is in the language as we can also recognize that some body is speaking *Russian* or *English* without necessarily understanding what it means.**

## 2.3. PROCESSING OF STRINGS BY DFA

Suppose $a_1, a_2, a_3, \ldots, a_n$ is a sequence of input symbols. We start out with deterministic finite automata having $q_0, q_1, q_2, q_3, \ldots, q_n$ states where $q_0$ is the initial state and $q_n$ is the final state and transition function and processed as

$$\begin{aligned} \delta(q_0, a_1) &= q_1 \\ \delta(q_1, a_2) &= q_2 \\ \delta(q_2, a_3) &= q_3 \\ &\vdots \\ \delta(q_{n-1}, q_n) &= q_n \end{aligned}$$

Input $a_1, a_2, \ldots, a_n$ is said to be "**accepted**" since $q_n$ is the member of the final states, and if not then it is "**rejected**". By adopting this approach we can define the language of a particular DFA, and can also check that particular string is accepted or rejected by given FA.

**"So a string is said to be accepted by the given FA only when it is processed by transition function δ in such a manner that it ends at the final state".**

Before discussing the formal notations for the FA, let us know about the transition graph :

## 2.4. TRANSITION GRAPH

A transition graph is a collection of three things :

1. A finite set of states, at least one of which is designated as the start

state and some (may be none) of which are designated as

final states

2. An alphabet $\Sigma$ of possible input letters from which input strings are formed.
3. A finite set of transitions (edge labels) that show how to go from some states to some others, based on reading specified substrings of input letters (possibly even the null string).

Transition graph were invented by John Myhill in 1957. A *successful path* through a transition graph is a series of edges forming a path beginning at some start state (there may be several start states) and ending at a final state. If we concatenate in order the strings of letters that label each edge in the path, we produce a word that is accepted by the this machine.

Let us consider some examples of TG's :

(*a*)

This picture above represents a TG that accepts nothing, not even the null string, to be accept any thing, it must be final state.

(*b*) The machine

Fig. 2.4.

accept null string.

(*c*)

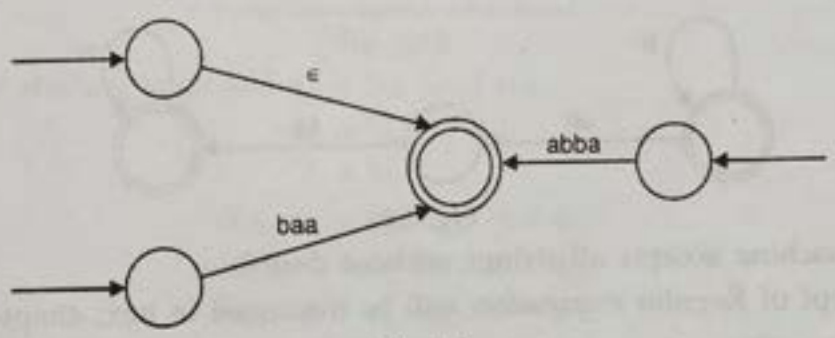


Fig. 2.5.

above TG accepts ∈, *baa* and *abba*.

(*d*)

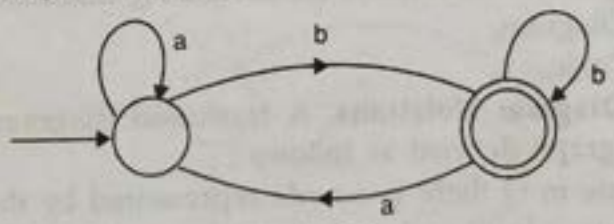


Fig. 2.6.

The language accepted by the TG is all words ending in *b* over alphabet $\Sigma = \{a, b\}$.

## 2.5. GENERALISED TRANSITION GRAPHS

The ultimate step liberating state-to-state transition is to allow the input to progress from one place to another by contributing a substring restricted to being a word in a predetermined language. For example :

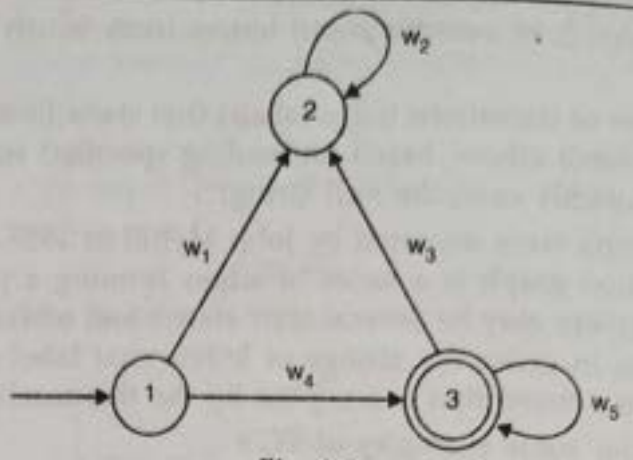


Fig. 2.7.

We can travel from start to state 2 by reading any (of course finite) word from the (possibly infinite) set of choices $w_1$ and, similarly, between all other states.

### 2.5.1. Definition of Generalised Transition Graph

A generalised transition graph (GTG) is a collection of three things :

1. A finite set of states, of which at least one is a start state and some (may be none) are final states.
2. An alphabet $\Sigma$ of input letters.
3. Directed edges connecting some pair of states, each labelled with a regular expression.

**Example :**

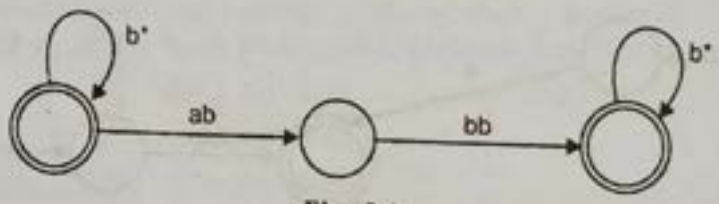


Fig. 2.8.

This machine accepts all strings without double $a$.

(Concept of Regular expression will be discussed in next chapter)

## 2.6. SIMPLER NOTATIONS FOR DFA's

There are two preferred notations for describing automata :

1. Transition Diagram,
2. Transition Table.

**1. Transition Diagram Notations.** A transition diagram for DFA, $M = (Q, \Sigma, \delta, q_0, F)$ is a graph defined as follows :

(*a*) For each state in $Q$ there is a node represented by the circle.

(*b*) For each state $q$ in $Q$ and each input symbol $a$ in $\Sigma$, let $\delta(q, a) = P$. Then transition diagram has an arc from node $q$ to node $P$, labelled $a$. If there are several input symbols that cause transition from $q$ to $P$, then the transition diagram can have one arc, labelled by the list of these symbols as :

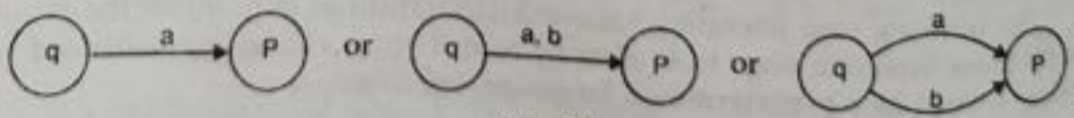


Fig. 2.9.

(c) If any state $q$ in $Q$ is the starting state then it represented by the circle with arrow as

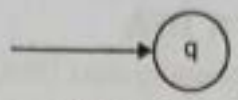


Fig. 2.10.

(d) Nodes corresponding to accepting states (those are in $F$) are marked by a double circle. States not in $F$ have a single circle.

Fig. 2.11.

**2. Transition Tables.** A transition table is a Conventional, tabular representation of a function like $\delta$ that takes two arguments and returns a state. The rows of the table correspond to the states, and the columns correspond to the input. The entry for one row corresponding to state $q$ and the column corresponding to input $a$ is the state $\delta(q, a)$. For example :

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $q_1$ | $q_2$ |
| $q_1$ | $q_2$ | $q_0$ |
| $*q_2$ | $q_2$ | $q_2$ |

Fig. 2.12.

$q_0$ is the starting state and $q_2$ is the final state,

$$Q = \{q_0, q_1, q_2\}$$
$$F = \{q_2\}$$
$$\delta(q_0, a) = q_1, \delta(q_0, b) = q_2$$
$$\delta(q_1, a) = q_2, \delta(q_1, b) = q_0$$
$$\delta(q_2, a) = q_2, \delta(q_2, b) = q_2$$

Let us draw the transition diagram for the transition table shown in Fig. 2.12.

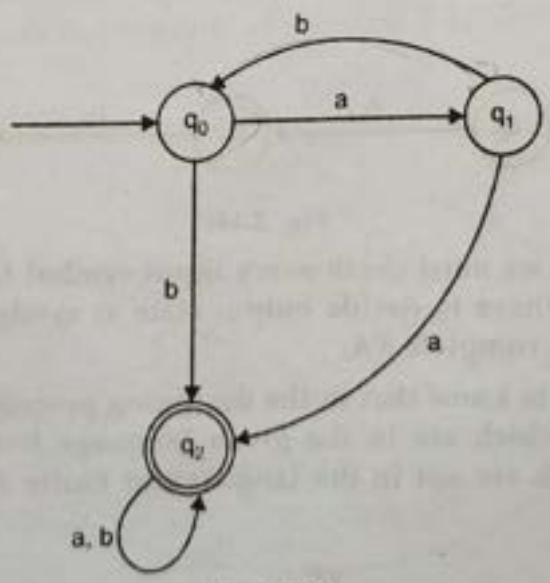


Fig. 2.13.

So clearly transition diagram can be find from transition table and viceversa is also true.

## 2.7. THE LANGUAGE OF A DFA

It have been explained informally that DFA defines a language : The set of all strings that result in a sequence of state transition from start state to an accepting state.

Now we can define the language of a DFA $M = (Q, \Sigma, \delta, q_0, F)$. This language is denoted by $L(M)$, and defined by

$$L(M) = \left\{ w / \hat{\delta}(q_0, w) \text{ is in } F \right\}$$

* $\hat{\delta}$ (extended transition function) will be explained afterwords.

That is the language of $M$ is the set of strings ($w$) that take the start $q_0$ to one of the accepting states.

**If $L$ is $L(M)$ for some deterministic finite automata, then we say $L$ is a regular language.**

Now for practical understanding of FA, let us discuss the some examples.

**Example 2.1. Design a FA that accepts set of strings such that every string ends in 00, over alphabet**

$$\{0, 1\} \textit{ i.e., } \Sigma = \{0, 1\}$$

**Solution.** Let $M = (Q, \Sigma, \delta, q_0, F)$ be the DFA

$q_0$ : initial state.

$\Sigma$ : $\{0, 1\}$ is given.

So according to the problem we have to design a FA which accepts string $w = w'\,00$.

Where $w'$ can be any combination of 0's and 1's for example 011100, 111100, 00, 100, 01010100 etc.

Now first we will decide the approach to design FA. It is not an easy task to think FA as a whole. So first we have to fulfill minimum condition i.e. every string end in 00.

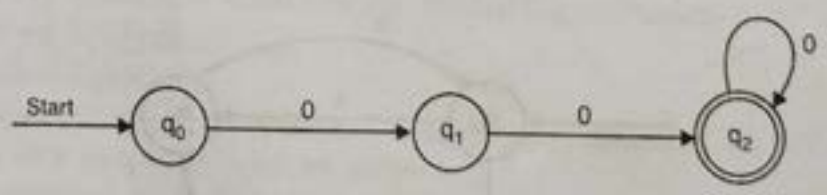


**Fig. 2.14.**

**Being the DFA we must check every input symbol for output state from every state. So we have to decide output state at symbol 1 from $q_0$, $q_1$ and $q_2$. Then it will be complete FA.**

**It is important to know that in the designing procedure, FA must accept all those strings which are in the given language but should not accept those strings which are not in the language of Finite Automata.**

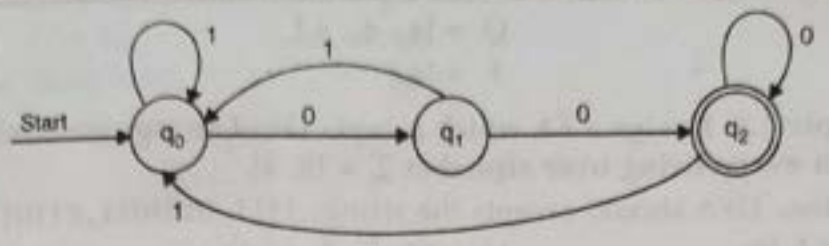


Fig. 2.15.

So Fig. 2.14 satisfied the minimum condition and Fig. 2.15 is the transition diagram of required FA.

$$Q = \{q_0, q_1, q_2\}$$
$$F = \{q_2\}$$

## Facts in Designing Procedure of FA

On the basis of example 2.1, we can decide some *facts or observation* for FA.

(*a*) In the designing of the FA, first of all we have to analyse the set of strings *i.e.* language which is accepted by the FA.

(*b*) Make sure that every state is check for the output state for every input symbol of $\Sigma$.

(*c*) Make sure that no state must have two different output state for a single input symbol.

(*d*) Make sure, that there is one initial state and at least one final state in transition diagram of FA.

Consideration of above discussed points in the designing, will minimize the error.

**Example 2.2. Construct a FA that accepts set of strings where the number of 0's in every string is multiple of three over alphabet $\Sigma = \{0, 1\}$.**

**Solution.** Multiple of three means numbers of 0's in the strings may be 0, 3, 6, 9, 12, ... in the number at any position.

Let FA be $M = (Q, \Sigma, \delta, q_0, F)$

Here it is necessary to consider acceptance of Null ($\epsilon$) string since it contains number of 0's zero times which is divisible by three so it must be very clear that here initial state *i.e.* $q_0$ is also final state. Here positions of 0's and 1's are irrelevent but number of 0's should be divisible by 3.

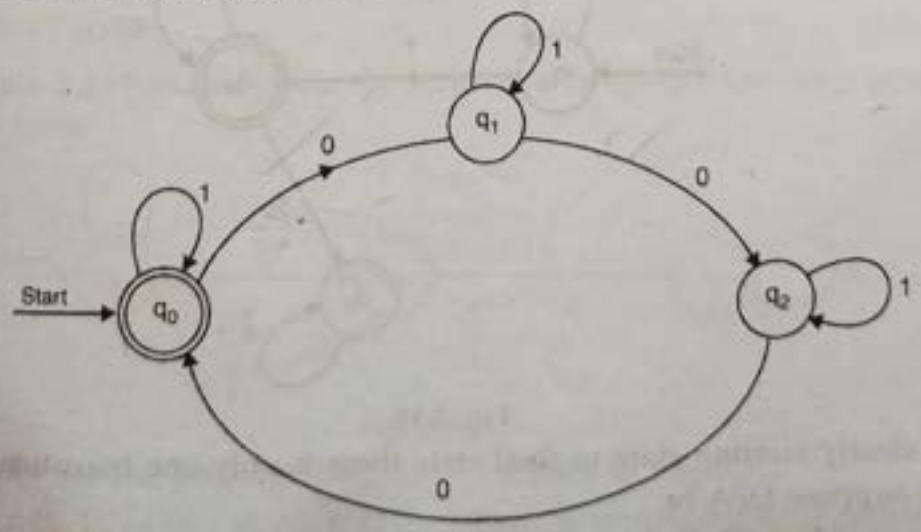


Fig. 2.16.

$$Q = \{q_0, q_1, q_2\}$$
$$F = \{q_0\}$$

**Example 2.3. Design a FA which accepts set of strings containing exactly four 1's in every string over alphabet $\Sigma = \{0, 1\}$.**

**Solution.** DFA should accepts the strings 1111, 0101011, 0110110000 .....

Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

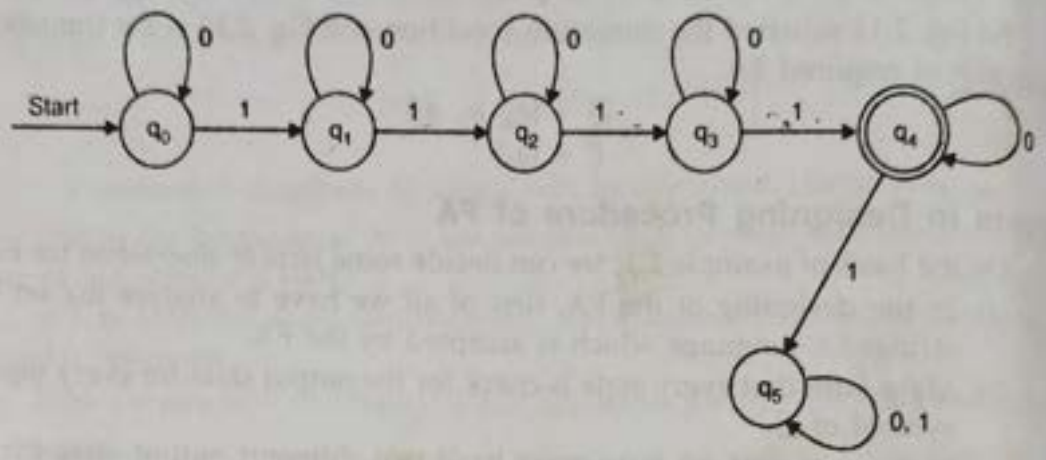


**Fig. 2.17.**

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$
$$F = \{q_4\}$$

From $q_0$ to $q_4$ transitions four 1's are already there and $q_4$ is final state so 1 from $q_4$ should go on a trap path, that is why $q_5$ is there so $q_5$ is called trap or dead state.

*"Dead states are those non final states which transits in itself for all the input symbol. So $q_5$ is the example of dead state."*

**Example 2.4. Design a FA that accepts strings containing exactly 1 over alphabet {0, 1}**

**Solution.** String set according to the problem, which is accepted by the FA is like

1, 0001, 01000, 0001000, ..... .

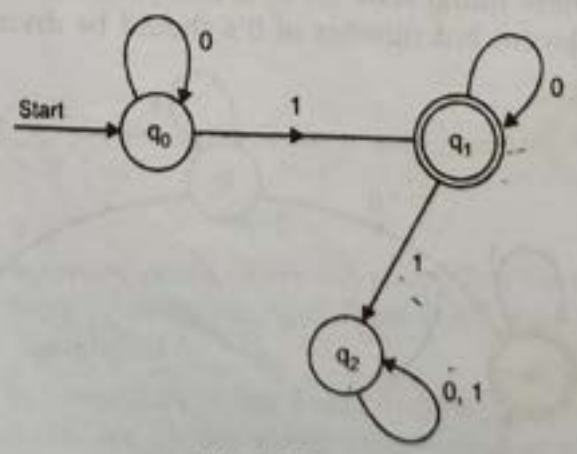


**Fig. 2.18.**

So clearly starting state to final state there is only one transition of 1.

Let suppose DFA be

$$M = (Q, \Sigma, \delta, q_0, F)$$

$Q = \{q_0, q_1, q_2\}$
$F = \{q_1\}$

$q_2$ is the dead state.

Transition table for above FA

| $\delta/\Sigma$ | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $q_0$ | $q_1$ |
| $* q_1$ | $q_1$ | $q_2$ |
| $q_2$ | $q_2$ | $q_2$ |

$\rightarrow$ non final state which transit in self loop for every input symbol in $\Sigma$, so it is dead state.

**Example 2.5. Design a FA which accepts the language**

**$L = \{w \in (0, 1)^*$/second symbol of $w$ is '0' and fourth input is 1$\}$.**

**Solution.** So after analysing the problem we find that every string of the language is like

0001, 1011, 10110000, .....

Let FA is $M = (Q, \Sigma, \delta, q_0, F)$

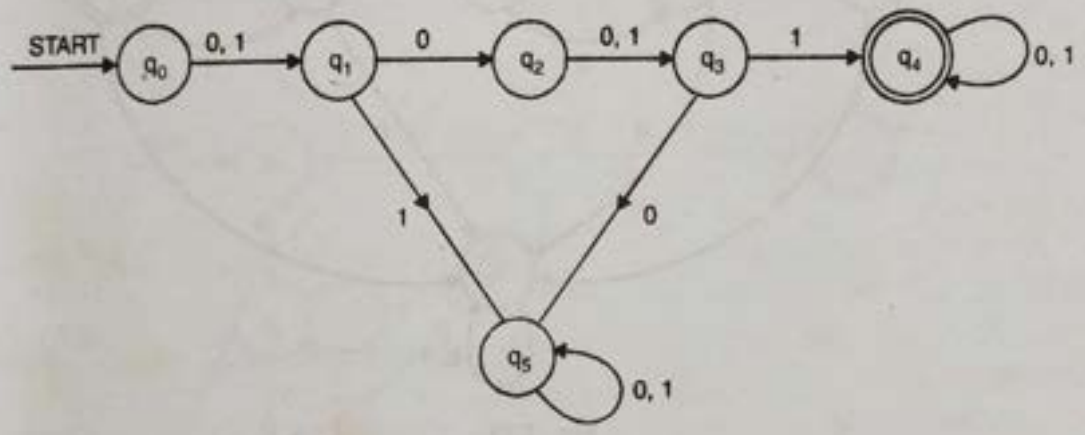


**Fig. 2.19.**

$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$
$F = \{q_4\}$

$q_5$ is dead state.

**Example 2.6. For the given FA write the language and also give the transition table**

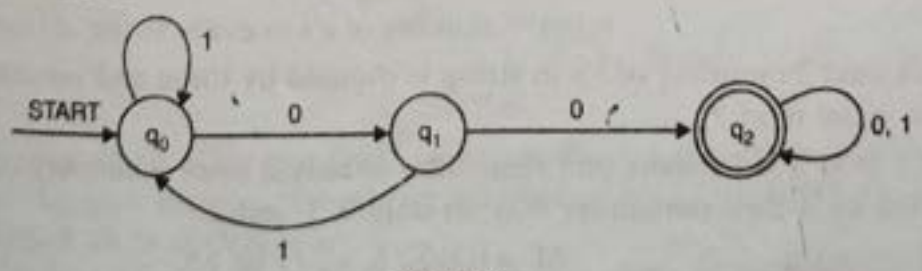


**Fig. 2.20.**

**Solution.** Let the language be $L$ for FA shown in Fig. 2.20.

$L = \{w \in (0, 1)^*$/every string $w$ of the language containing 00 as substring$\}$.

Transition table :

| $\delta/\Sigma$ | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $q_1$ | $q_0$ |
| $q_1$ | $q_2$ | $q_0$ |
| $* q_2$ | $q_2$ | $q_2$ |

**Example 2.7. Design FA for the language**

$$L = \{(01)^i 1^{2j} \mid i \geq 1, j \geq 1\}.$$

**Solution.** By analysing language $L$, it is clear that FA will accepts string start with any number of 01 (not empty) and end with even number of 1's.

Let FA be

$$M = (Q, \Sigma, \delta, q_0, F)$$

$q$ : initial state.

$\delta$ : is transition function.

$\Sigma = \{0, 1\}$ given.

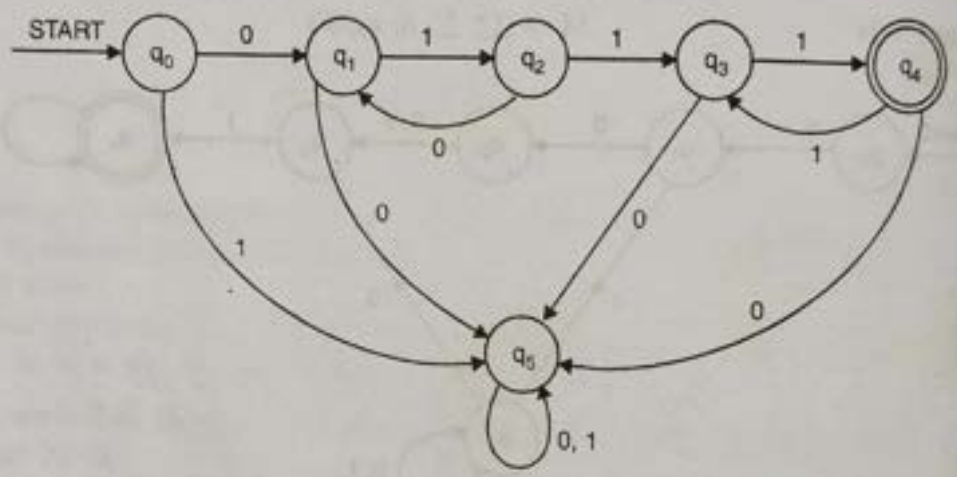


Fig. 2.21.

$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$

$F = \{q_4\}$

$q_5$ is dead state.

**Example 2.8. Design DFA for the language**

$$L = \{w \in (a, b)^* / n_b(w) \bmod 3 > 1\}.$$

**Solution.** Let us first analyse the language $L$.

$n_b(w)$ : number of $b$'s in every string of language

$L_b(w)$ mod 3 : number of $b$'s in string is divided by three and remainder is result of $n_b(w)$ mod 3.

$n_b(w)$ mod 3 > 1 means that remainder is only 2 since when any number is divided by 3 then remainder may be only 0, 1 and 2.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be FA

$\Sigma = \{a, b\}$

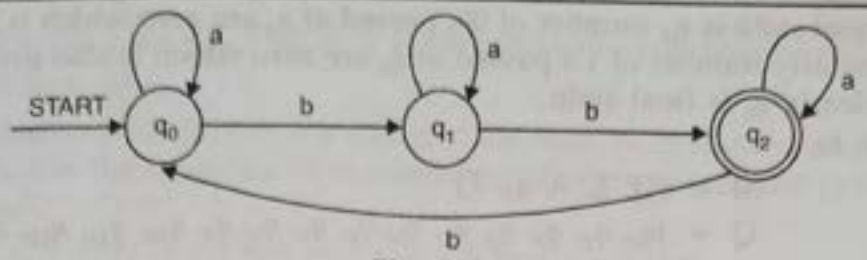


Fig. 2.22.

It will accept string, having $2b$'s, $5b$'s and $8b$'s so remainder is always two.

$$Q = \{q_0, q_1, q_2\}$$
$$F = \{q_2\}$$

**Example 2.9. Design a FA over alphabet $\Sigma = \{0, 1\}$, which accepts the set of strings either start with 01 or end with 01.**

**Solution.** By the analysis of problem, it is clear that FA will accepts the strings such as 01, 011111, 01000, 000101, 00011101, 111101..... .

Let FA $\quad M = (Q, \Sigma, \delta, q_0, F)$

$\Sigma = \{0, 1\}$ given

$q_0$ is initial state.

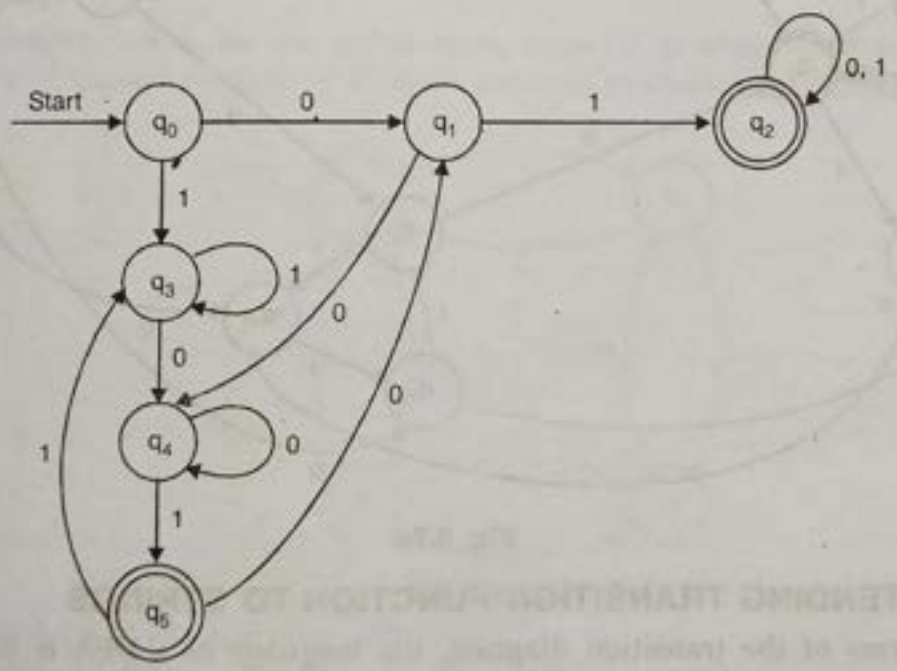


Fig. 2.23.

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$
$$F = \{q_2, q_5\}$$

**Example 2.10. Give the DFA accepting the set of strings over alphabet $\Sigma = \{0, 1\}$, such that in each string number of 0's is divisible by five *and* number of 1's is divisible by 3.**

**Solution.** Here two conditions *i.e.* number of 0's is divisible by five *and* number of 1's is divisible by 3 must satisfied simultaneously. So clearly only those state of FA will be final which satisfied both the condition simultaneously.

Let initial state is $q_0$, number of 0's passed at $q_0$ are zero which is divisible by five similarly number of 1's passed at $q_0$ are zero which is also divisible by three so clearly $q_0$ is final state.

Let FA be

$$M = (Q, \Sigma, \delta, q_0, F)$$
$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12}, q_{13}, q_{14}\}$$
$$F = \{q_0\}$$

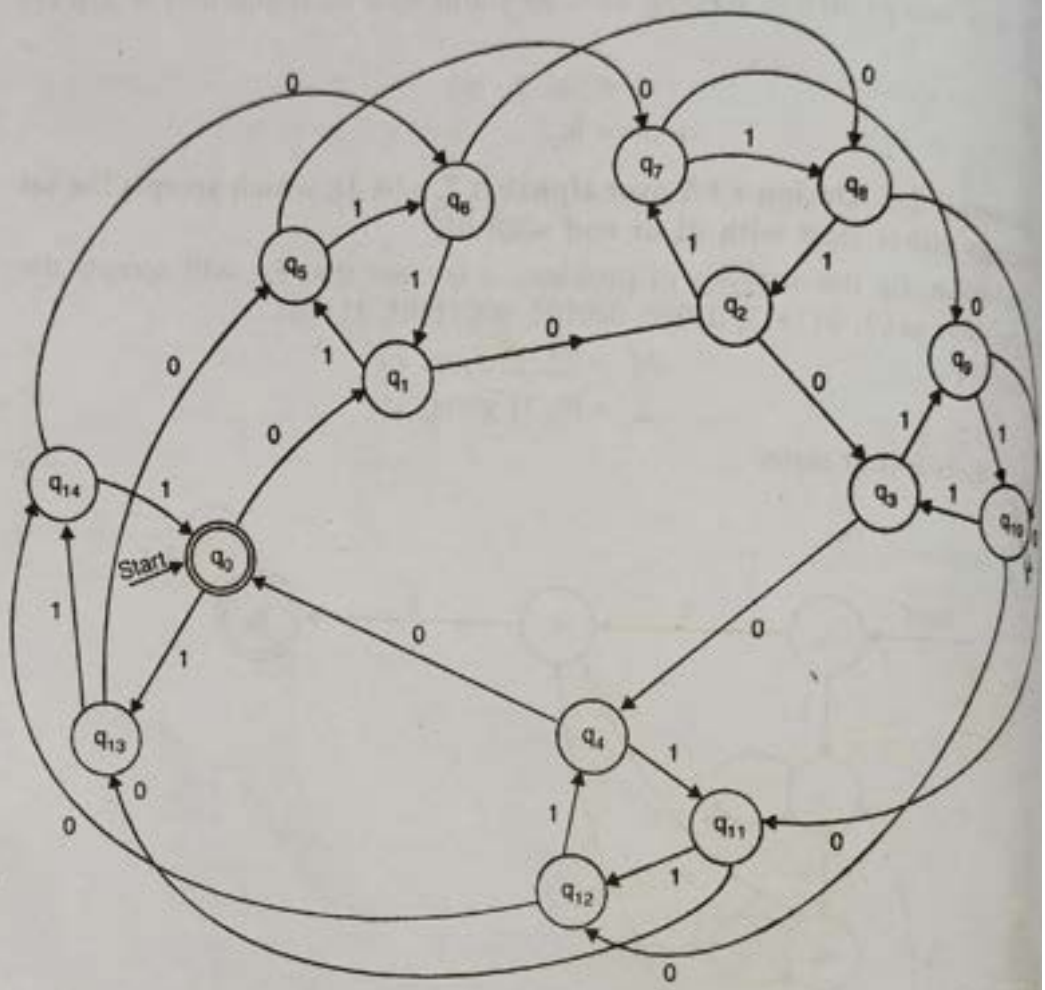


Fig. 2.24.

## 2.8. EXTENDING TRANSITION FUNCTION TO STRINGS

In terms of the transition diagram, the language of a DFA is the set of labels along the paths that lead from the start state to any accepting state.

Now, we need to make the notation of the language of FA precise. To do so we define extended transition function that describes what happens when we start in any state and follow any sequence of inputs. If $\delta$ is our transition function, the extended function constructed from $\delta$ will be called $\hat{\delta}$.

The extended transition function is the function that takes a state $q$ and a string $w$ and returns a state $P$, the state that automation reaches when starting in state $q$ and processing the sequence of inputs $w$. We define $\hat{\delta}$ by induction on the length of the input string as follows :

**Basis :** $\hat{\delta}(q, \in) = q$. That is, if we are in state $q$ and read no input, then we are still in state $q$.

**Induction.** Suppose $w$ is a string of the form $xa$, that is $a$ is the last symbol of $w$ and $x$ is the substring of $w$, consisting of all except the last symbol '$a$'. For example

$w = 1101$ is broken into $x = 110$ and $a = 1$ then

$$\hat{\delta}(q, w) = \delta(\hat{\delta}(q, x), a)$$

*i.e.*, to compute $\hat{\delta}(q, w)$, first compute $\hat{\delta}(q, x)$, the state that the automation is in after processing all but the last symbol of $w$. Suppose this state is $P'$; that is $\hat{\delta}(q, x) = P'$. Then $\hat{\delta}(q, w)$ is what we get by making a transition from state $P'$ on input $a$, the last symbol of $w$.

$$\hat{\delta}(q, w) = \delta(P', a) = P$$

Let see this concept through an example.

**Example 2.11. Design a FA which accepts the language $L = \{w/w$ has both an even number of 0's and an even number of 1's over alphabet $\Sigma = \{0, 1\}\}$.**

**Solution.** Let $q_0$ be the initial state, since '0' is even so at initial state number of 0's and number of 1's both are even so clearly $q_0$ will be final state

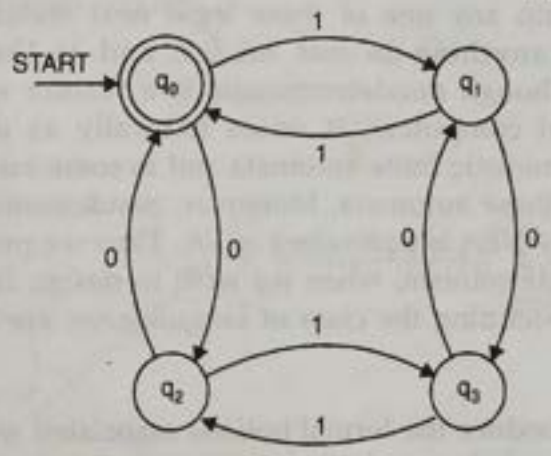


**Fig. 2.25.**

Let FA be $M = (Q, \Sigma, \delta, q_0, F)$

$\Sigma = \{0, 1\}$

$q_0$ : both 1 and 0's are even

$q_1$ : 0's are even and 1's are odd

$q_2$ : 1's are even and 0's are odd

$q_3$ : both are odd.

Check 110101 string on above FA *i.e.* it is accepted by FA or rejected.

Suppose extended transition function is $\hat{\delta}$,

$$\hat{\delta}(q_0, \in) = q_0$$

$$\hat{\delta}(q_0, 1) = \delta(\hat{\delta}(q_0, \in), 1) = \delta(q_0, 1) = q_1$$

$$\hat{\delta}(q_0, 11) = \delta(\hat{\delta}(q_0, 1), 1) = \delta(q_1, 1) = q_0$$

$$\hat{\delta}(q_0, 110) = \delta(\hat{\delta}(q_0, 11), 0) = \delta(q_0, 0) = q_2$$

$$\hat{\delta}(q_0, 1101) = \delta(\hat{\delta}(q_0, 110), 1) = \delta(q_2, 1) = q_3$$

$$\hat{\delta}(q_0, 11010) = \delta(\hat{\delta}(q_0, 1101), 0) = \delta(q_3, 0) = q_1$$

$$\hat{\delta}(q_0, 110101) = \delta(\hat{\delta}(q_0, 11010), 1) = \delta(q_1, 0) = q_0$$

and $q_0$ is final state so string 110101 is accepted by FA in Fig. 2.25.

## 2.9. NONDETERMINISM

Nondeterminism means a choice of moves for an automaton. Rather than prescribing a unique move in each situation, we allow a set of possible moves. Formally, we achieve this by defining the transition function so that its range is a set of possible states.

### 2.9.1. Why Nondeterminism

A "nondeterministic" finite automation (NFA) has the power to be in several states at once. This ability is often expressed as an ability to "guess" something about its output. Each NFA accepts a language that is also accepted by some DFA. That is the NFA's accept exactly the regular language, just as DFA's do. We can always convert a NFA to DFA. It is clear now, we can permit several possible "next states" for a given combination of current state and input symbol. The automation, as it reads the input string, may choose at each step to go into any one of these legal next states, the choice is not deterministic by anything in our model, and is therefore said to be nondeterminstic. Though nondeterministic is a feature we do not normally associate with real computers, it arises naturally as an extension of the behaviour of deterministic finite automata and in some cases greatly simplifies the description of these automata. Moreover, nondeterminism is an essential feature of FA, every NFA is equivalent to FA. Thus we profit from the greater flexibility of nondeterminism, when we wish to design language recognizers without actually extending the class of language we are able to describe.

### Definition

Now let me introduce the formal notions associated with nondeterministic finite automata. The difference between DFA's and NFA's will be pointed out as we do.

Let $M$ be NFA then $M$ is defined as

$$M = (Q, \Sigma, \delta, q_0, F)$$

where
- $Q$ : is a non empty finite set of states in NFA.
- $\Sigma$ : is a non-empty finite set of input symbols.
- $q_0$ : initial state of NFA and it is the member of $Q$.
- $F$ : a non-empty finite set of final states in $Q$.
- $\delta$ : is the transition function that takes a states and $Q$ and an input symbol in $\Sigma$ as arguments and returns a subset of $Q$.

that is $Q \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^Q$

So in nondeterministic accepter, the range of $\delta$ is in the power set $2^Q$, so its value is not a single element of $Q$, but a subset of it. This subset defines the set of possible states that can be reached by transition.

Let $q$ be an state of $Q$ and $a \in \Sigma$ then

We can transit from a state $q$ on same input $a$ to different states $q_0, q_1, \ldots, q_n$ in $Q$. This not possible in the case of FA.

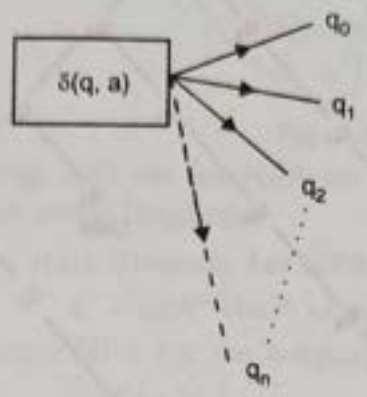


**Fig. 2.26.**

**Example 2.12. Design a NFA for the language $L$ = all strings over {0, 1} that have at least two consecutive 0's or 1's.**

**Solution.** By the analysis it is clear that NFA will accepts the strings of the pattern 00, 11, 101000, 101100, ..... .

Let NFA be $M = (Q, \Sigma, \delta, q_0, F)$

$\Sigma = \{0, 1\}$

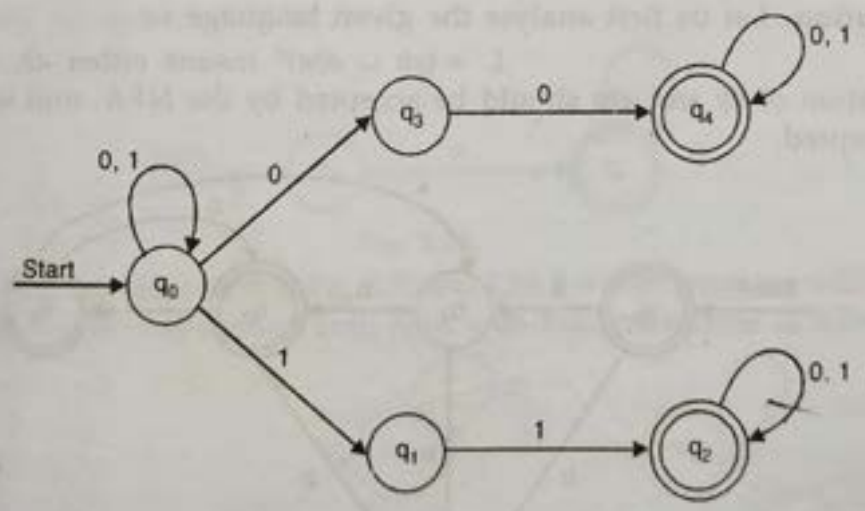


**Fig. 2.27.**

Note that state $q_3$ does not go anywhere for an input of 1. **We use the terminology that the path "dies" if in $q_3$ getting an input 1.**

The tree of states this NFA is in for the input 0100011.

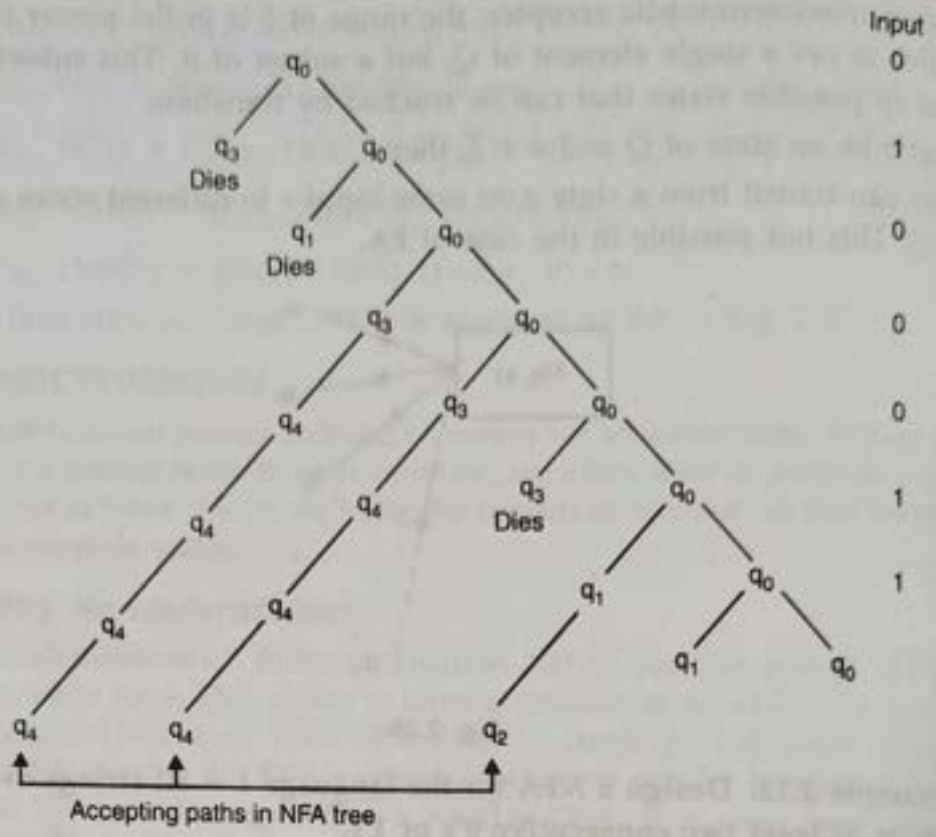


Fig. 2.28.

So it is clear from figure 2.28 that for a particular string there may b various paths, some may lead it to accepted states, and some may lead it to no accepted states and some may lead it to the blocked paths as $q_0 \rightarrow q_3$ as $q_3$ die

**Example 2.13. Design a NFA for the language $L = (ab \cup aba)^*$.**

**Solution.** Let us first analyse the given language *i.e.*

$L = (ab \cup aba)^*$ means either $ab$, $aba$ or a combination of $ab$ and $aba$ should be accepted by the NFA. null string ($\epsilon$) also accepted.

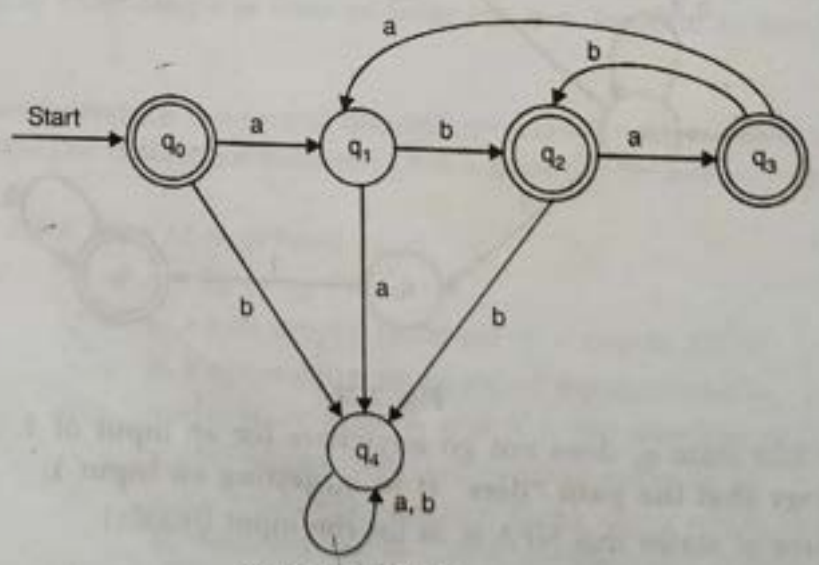


Fig. 2.29. DFA for the $L$.

If DFA for language $L$.

Let us now design NFA for $L$.

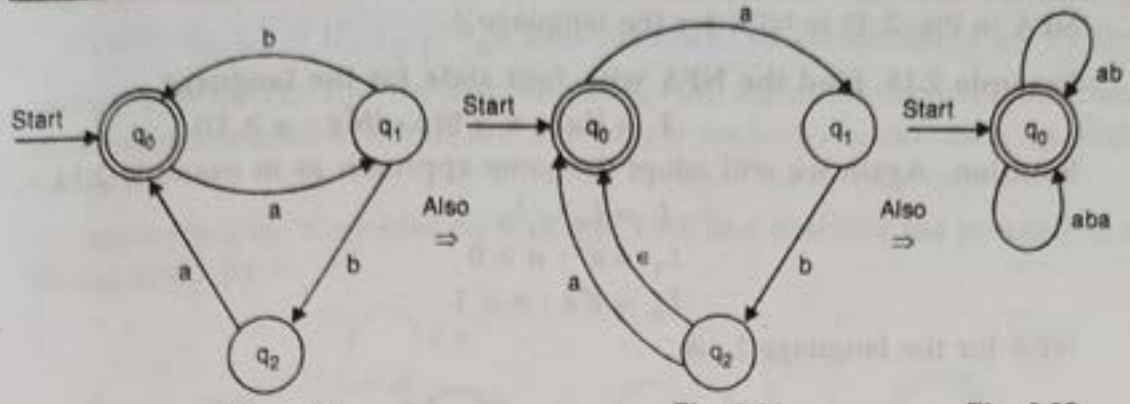


Fig. 2.30. Fig. 2.31. Fig. 2.32.

Fig. 2.30, Fig. 2.31 and Fig. 2.32 are the NFA for the language $L$. Thus we can have different NDFA for same language.

**Example 2.14. Draw the state diagram for NFA accepting language**

$$L = (ab)^* (ba)^* \cup aa^*.$$

**Solution.** We can construct NFA for the language $L$ in two paths *i.e.*

$$L = L_1 \cup L_2$$
$$L_1 = (ab)^* (ba)^*$$
$$L_2 = aa^*$$

First we can construct NFA for $L_1$

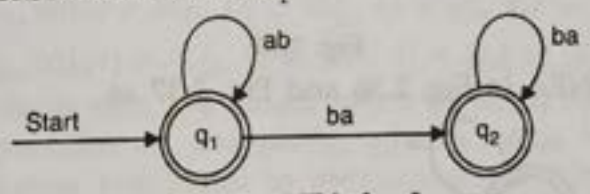


**Fig. 2.33. NFA for $L_1$.**

Secondly we construct NFA for $L_2$.

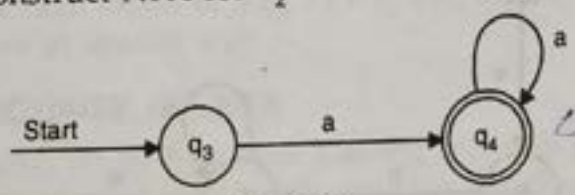


**Fig. 2.34.**

Now we can combine NFA in Fig. 2.33 and 2.34. For combining we will introduce another state $q_0$ and will connect both NFA with Null transitions as follows :

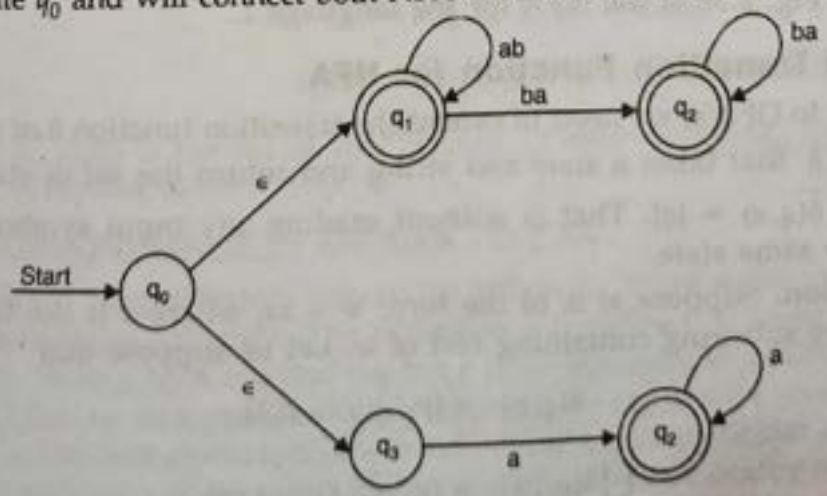


**Fig. 2.35.**

NFA in Fig. 2.35 is NFA for the language $L$.

**Example 2.15. Find the NFA with four state for the language**

$$L = \{(a^n : n \geq 0) \cup (b^n a : n \geq 1)\}.$$

**Solution.** Again we will adopt the same approach as in example 2.14.

$$L = L_1 \cup L_2$$
$$L_1 = a^n : n \geq 0$$
$$L_2 = b^n a : n \geq 1$$

NFA for the language $L_1$ is

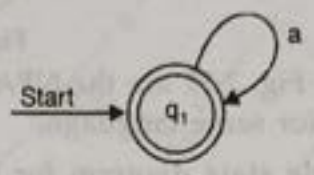


**Fig. 2.36.**

NFA for the language $L_2$ is

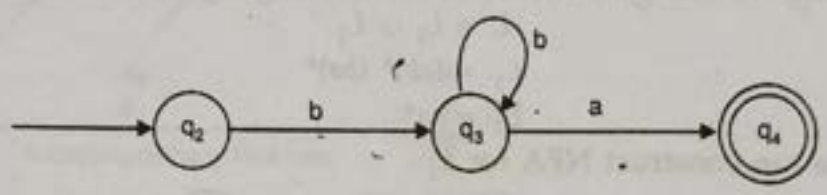


**Fig. 2.37.**

Now combine NFA in Fig. 2.36 and Fig. 2.37 as

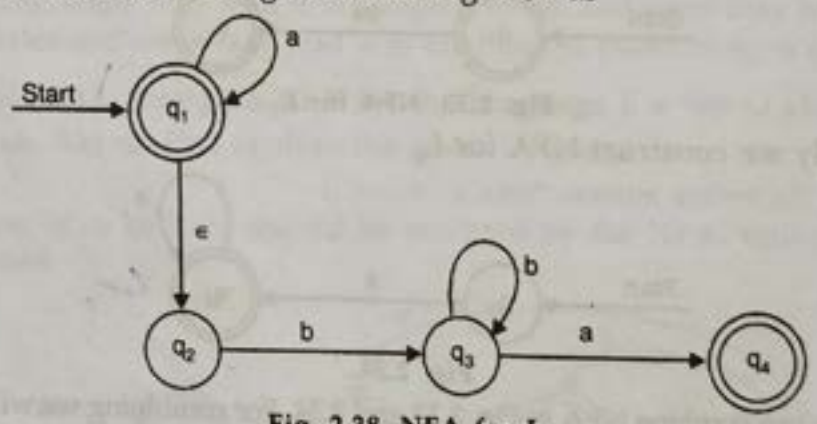


**Fig. 2.38. NFA for $L$.**

Finally Fig. 2.38 is the NFA for the language $L$.

## Extended Transition Function for NFA

Similar to DFA's, we need to extend the transition function $\delta$ of an NFA t a function $\hat{\delta}$ that takes a state and string and return the set of states.

**Basis.** $\hat{\delta}(q, \epsilon) = \{q\}$. That is without reading any input symbols, we a only in the same state.

**Induction.** Suppose $w$ is of the form $w = xa$, where $a$ is the last symb and $x$ is the substring containing rest of $w$. Let us suppose that

$$\hat{\delta}(q, x) = \{p_1, p_2, \ldots p_k\},$$

Let $$\bigcup_{i=1}^{k} \delta(p_i, a) = \{r_1, r_2, r_3, \ldots, r_m\}$$

Then $\hat{\delta}(q, w) = \{r_1, r_2, \ldots r_m\}$. Less formally, we compute $\hat{\delta}(q, w)$ by first computing $\hat{\delta}(q, x)$, and by then following any transition from any of these states that is labeled $a$. Let us see an example for better under standing of the concept.

**Example 2.16. Consider NFA in Fig. 2.39, and describe the processing of string 00101 by it.**

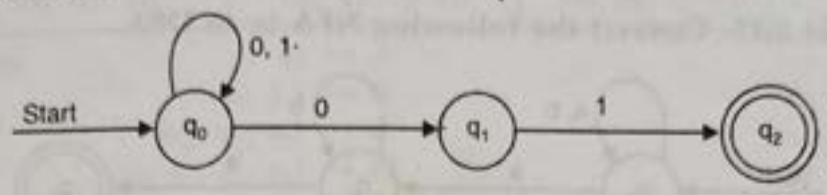


**Fig. 2.39.**

**Solution.** It is NFA accepting all strings end in 01. Let us use $\hat{\delta}$ to describe the processing of input 00101 by NFA. A summary of the steps is.

**Step 1.** $\hat{\delta}(q_0, \in) = \{q_0\}$

**Step 2.** $\hat{\delta}(q_0, 0) = \delta(q_0, 0) = \{q_0, q_1\}$

**Step 3.** $\hat{\delta}(q_0, 00) = \delta(q_0, 0) \cup \delta(q_1, 0) = \{q_0, q_1\} \cup \phi = \{q_0, q_1\}$

**Step 4.** $\hat{\delta}(q_0, 001) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$

**Step 5.** $\hat{\delta}(q_0, 0010) = \delta(q_0, 0) \cup \delta(q_2, 0) = \{q_0, q_1\} \cup \phi = \{q_0, q_1\}$

**Step 6.** $\hat{\delta}(q_0, 00101) = \delta(q_0, 1) \cup \delta(q_1, 1) = \{q_0\} \cup \{q_2\} = \{q_0, q_2\}$.

Step 1 is the basis rule. We obtain step 2 by applying $\delta$ to the done state $q_0$, that is in the previous set, and get $\{q_0, q_1\}$ as a result. Step 3 is obtained by taking the union over two states in previous set of what we get when we apply $\delta$ to the input 0. That is, $\delta(q_0, 0) = \{q_0, q_1\}$, while $\delta\{q_1, 0\} = \phi$.

For step 4, we take union of $\delta(q_0, 1) = q_0$ and $\delta(q_1, 1) = \{q_2\}$. So step 5 and step 6 are obtained in similar way.

## 2.10. THE LANGUAGE OF NFA

As we discussed, an NFA accepts a string $w$ if it is possible to make any sequence of choice of next state while reading the characters of $w$, and go from the start state to any accepting state.

If $M$ is a NFA

$$M = (Q, \Sigma, \delta, q_0, F) \text{ then}$$

$$L(M) = \{w / \hat{\delta}(q_0, w) \cap F \neq \phi\} \text{ is the Language of}$$

$M$ *i.e.* NFA. $L(M)$ is the set of strings $w$ in $\Sigma^*$ such that $\hat{\delta}(q_0, w)$ contains at least one accepting state.

## 2.11. TRANSFORMATION OF NFA TO DFA

For every non deterministic finite automata there exist an equivalent deterministic finite automata. The equivalence is defined in terms of language acceptance. Since a NFA is a nothing but a finite automata in which zero, one or more transition on an input symbol is permitted, we can always construct a finite automata which will simulate all the moves of NFA on a particular input symbol in parallel, then get a finite automata in which there will be exactly one transition on every input symbol, hence it will be DFA equivalent to NFA.

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Since the DFA equivalent of NFA is to simulate the moves in parallel st of a DFA will be combination of one or more states of NFA, hence every st of a DFA will be represented by some subset of set of states of NFA, a **therefor the transition of NFA to DFA is normally called a subset constructi**

So transformation of NFA to DFA in value, finding subset of set of states NFA. Let us implement this concept in the form of set of steps as algorith through an example.

**Example 2.17. Convert the following NFA in to DFA.**

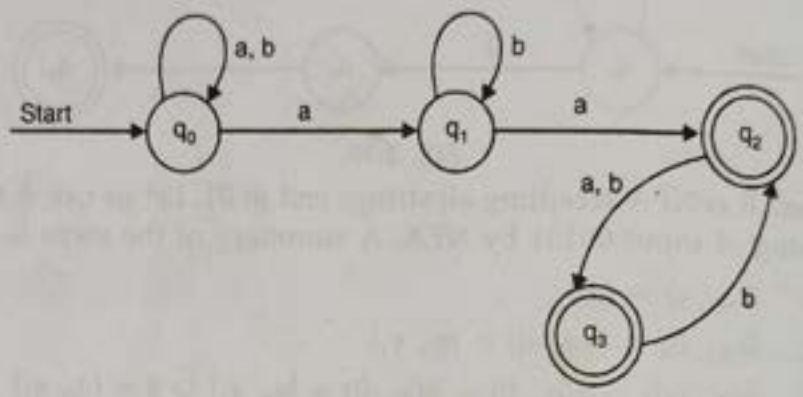


**Fig. 2.40.**

**Solution.**

**Step 1.** Seek all the transition from starting state $q_0$ for every symbol in *i.e.* $(a, b)$. If you get a set of states for same input then consider that set as ne single state as :

$$\delta(q_0, a) = \{q_0, q_1\}$$

in Fig. 2.40 $q_0$ transit to both $q_0$ and $q_1$ for input $a$.

$$\delta(q_0, b) = \{q_0\}$$

**Step 2.** In step 1 we are getting a new state $\{q_0, q_1\}$. Now repeat step 1 fo this new state only *i.e.* check all transitions of $a$ and $b$ (that is $\Sigma$) from $\{q_0, q_1\}$ as

$$\delta(\{q_0, q_1\}, a) = \delta(q_0, a) \cup \delta(q_1, a) = \{q_0, q_1\} \cup \{q_2\}$$
$$= \{q_0, q_1, q_2\}$$ new state

$$\delta(\{q_0, q_1\}, b) = \delta(q_0, b) \cup \delta(q_1, b) = \{q_0, q_1\}$$

**Step 3.** Repeat step 2 till you are getting any new state. All those state which consists at least one accepting state of given NFA as member state wi be considered as final states.

$\{q_0, q_1, q_2\}$ is new state.

$$\delta(\{q_0, q_1, q_2\}, a) = \delta(q_0, a) \cup \delta(q_1, a) \cup \delta(q_2, a)$$
$$= \{q_0, q_1\} \cup \{q_2\} \cup \{q_3\}$$
$$= \{q_0, q_1, q_2, q_3\} \quad \text{(New state}$$
$$\delta(\{q_0, q_1, q_2\}, b) = \delta(q_0, b) \cup (q_1, b)\ \delta(q_2, b)$$
$$= q_0 \cup q_1 \cup q_3 = \{q_0, q_1, q_3\} \quad \text{(New state}$$
$$\delta(\{q_0, q_1, q_2, q_3\}, a) = \{q_0, q_1, q_2, q_3\} \quad \text{(Old state}$$
$$\delta(\{q_0, q_1, q_2, q_3\}, b) = \{q_0, q_1, q_3, q_2\} \quad \text{(Old state}$$
$$\delta(\{q_0, q_1, q_3\}, a) = \{q_0, q_1, q_2, q_3\} \quad \text{(Old state}$$
$$\delta(\{q_0, q_1, q_3\}, b) = \{q_0, q_1, q_3, q_2\} \quad \text{(Old state}$$

Let draw transition table

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_0\}$ |
| $\{q_0, q_1\}$ | $\{q_0, q_1, q_2\}$ | $\{q_0, q_1\}$ |
| $*\{q_0, q_1, q_2\}$ | $\{q_0, q_1, q_2, q_3\}$ | $\{q_0, q_1, q_3\}$ |
| $*\{q_0, q_1, q_2, q_3\}$ | $\{q_0, q_1, q_2, q_3\}$ | $\{q_0, q_1, q_2, q_3\}$ |
| $*\{q_0, q_1, q_3\}$ | $\{q_0, q_1, q_2, q_3\}$ | $\{q_0, q_1, q_2, q_3\}$ |

Let us say

$$q_0 \rightarrow A$$
$$\{q_0, q_1\} \rightarrow B$$
$$\{q_0, q_1, q_2\} \rightarrow C$$
$$\{q_0, q_1, q_2, q_3\} \rightarrow D$$
$$\{q_0, q_1, q_3\} \rightarrow E$$

$A$ is initial state and $C$, $D$ and $E$ are final states since they contain $q_2$ and $q_3$ as member which are final states of NFA.

So

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow A$ | $B$ | $A$ |
| $B$ | $C$ | $B$ |
| $*C$ | $D$ | $E$ |
| $*D$ | $D$ | $D$ |
| $*E$ | $D$ | $D$ |

Let us draw transition diagram

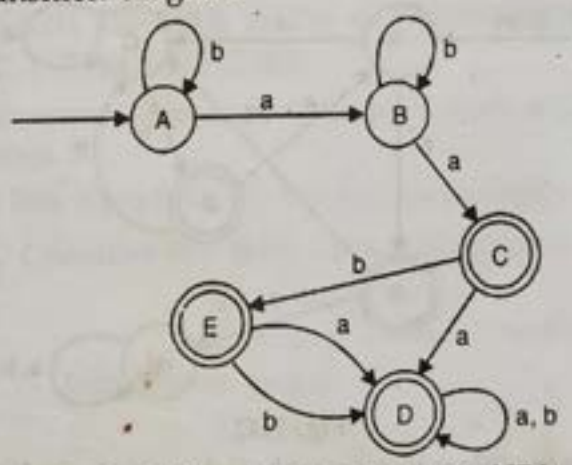


**Fig. 2.41. DFA equivalent of given NFA.**

It is a DFA.

**Example 2.17. Given NFA is**

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $\{q_0, q_1\}$ | $\{q_2\}$ |
| $q_1$ | $\{q_0\}$ | $\{q_1\}$ |
| $*q_2$ | $\phi$ | $\{q_0, q_1\}$ |

**Convert it in to DFA.**

**Solution.** Do the same process as in example 2.16.

$$\delta(q_0, a) = \{q_0, q_1\} \quad \text{(New)}$$
$$\delta(q_0, b) = \{q_2\} \quad \text{(New)}$$

$$\delta(\{q_0, q_1\}, a) = \delta(q_0, a) \cup \delta(q_1, a) = \{q_0, q_1\} \cup \{q_0\}$$
$$= \{q_0, q_1\} \quad \text{(Old)}$$
$$\delta(\{q_0, q_1\}, b) = \delta(q_0, b) \cup \delta(q_1, b) = \{q_1\} \cup \{q_1\}$$
$$= \{q_1, q_2\} \quad \text{(New)}$$
$$\delta(q_2, a) = \{\phi\} \quad \text{(Empty set)}$$
$$\delta(q_2, b) = \{q_0, q_1\} \quad \text{(Old)}$$
$$\delta(\{q_1, q_2\}, a) = \delta(q_1, a) \cup \delta(q_2, a) = \{q_0\} \cup \{\phi\}$$
$$= \{q_0\} \quad \text{(Old)}$$
$$\delta(\{q_1, q_2\}, b) = \delta(q_1, b) \cup \delta(q_2, b) = \{q_1\} \cup \{q_0, q_1\}$$
$$= \{q_0, q_1\} \quad \text{(Old)}$$

Let say

$q_0 \rightarrow A$ (Initial state)

$q_2 \rightarrow B$ (Final state)

$\{q_0, q_1\} \rightarrow C$

$\{q_1, q_2\} \rightarrow D$ (Final state)

Transform table for DFA

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow A$ | $C$ | $B$ |
| $*B$ | $\{\phi\}$ | $C$ |
| $C$ | $C$ | $D$ |
| $*D$ | $A$ | $C$ |

Transition diagram for DFA

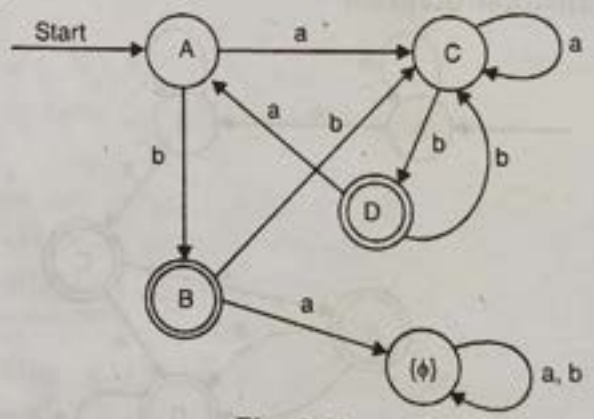


**Fig. 2.42.**

Here $\{\phi\}$ state is nothing but interpreted as if there is a input '$a$' on state $B$ then there is no path for it and machine will be hang.

## 2.12. NFA WITH ∈–TRANSITIONS

If a FA is modified to permit transition without input symbols, along with zero, one or more transition on input symbols, then we get a NFA with ∈ transitions, because the transition made without symbols are called as ∈-transitions.

Consider the NFA given by following diagram.

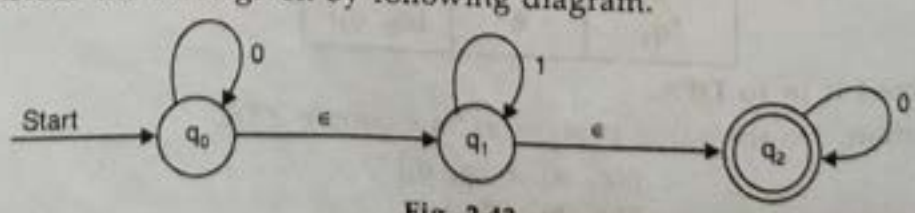


**Fig. 2.43.**

This is a NFA with $\epsilon$-transitions because it is possible to make transition from state $q_0$ to $q_1$ without consuming any of the input symbols. Being a finite Automata, NFA with $\epsilon$-transitions will also be denoted as five tuple.

$$M = (Q, \Sigma, \delta, q_0, F)$$

where $Q$, $\Sigma$, $q_0$ and $F$ are having usual meaning, and transition function $\delta$ defines a mapping from

$Q \times (\Sigma \cup \epsilon)$ to $2^Q$.

## 2.13. ACCEPTANCE OF A STRING BY NFA WITH $\epsilon$-TRANSITIONS

A string $w$ in $\Sigma^*$ will be accepted by NFA with $\epsilon$-transition, if there exist at least one path corresponding $w$, which starts in an initial state, and ends in one of the final states. Since this path may formed by $\epsilon$-transition as well non $\epsilon$-transitions to find out wheather $w$ is accepted or not by the NFA with $\epsilon$ moves, we require to defined a function $\epsilon$-closure $(q)$, where $q$ is a state of the automata.

The function $\epsilon$-closure $(q)$ is defined as follows :

$\epsilon$-closure $(q)$ = set of all those states of the automata (NFA with $\epsilon$-transitions) which can be reached from $q$ on a path labeled by $\epsilon$ *i.e.* without consuming any input symbol.

Consider the NFA discussed in Fig. 2.43 then

$$\epsilon\text{-closure } (q_0) = \{q_0, q_1, q_2\}$$
$$\epsilon\text{-closure } (q_1) = \{q_1, q_2\}$$
$$\epsilon\text{-closure } (q_2) = \{q_2\}$$

## 2.14. CONVERSION OF NFA WITH $\epsilon$-TRANSITIONS TO NFA WITHOUT $\epsilon$-TRANSITIONS

There is a conversion algorithm from NFA with $\epsilon$-transition $s$ to $a$ NFA without $\epsilon$-transitions !

We will learn this algorithm by solving an example.

**Example 2.18. Consider the NFA with $\epsilon$-transition $M = (Q, \Sigma, \delta, q_0, F)$**

$$Q = \{q_0, q_1, q_2\}$$
$$\Sigma = \{a, b, c\} \text{ and } \epsilon \text{ moves.}$$
$$\text{Initial state} = \{q_0\}$$
$$F = \{q_2\}$$

Transition Table

| $\delta/\Sigma$ | $a$ | $b$ | $c$ | $\epsilon$ |
|---|---|---|---|---|
| $\rightarrow q_0$ | $\{q_0\}$ | $\{\phi\}$ | $\{\phi\}$ | $\{q_1\}$ |
| $q_1$ | $\{\phi\}$ | $\{q_2\}$ | $\{\phi\}$ | $\{q_2\}$ |
| $* q_2$ | $\{\phi\}$ | $\{\phi\}$ | $\{q_2\}$ | $\{\phi\}$ |

and transition diagram is following

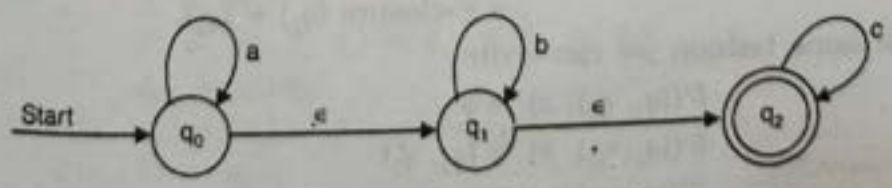


**Fig. 2.44. NFA with $\epsilon$-transition.**

**Solution.**

**Step 1.** Find the states of NFA without $\epsilon$- transition including initial state and final states as follows :

(*i*) The language accepted by the above NFA with $\epsilon$-transform is the set of strings over $\{a, b, c\}$ including the null string and all strings with any number of $a$'s followed by any number of $b$'s and follows by any number of $c$'s.

(*ii*) There will be the same number of states but the names can be constructed by writing the state name as the set of states in the $\epsilon$-closure.

(*iii*) Initial state will be $\epsilon$-closure of initial state of NFA with $\epsilon$-transition in Fig. 2.44 as

$$\epsilon\text{-closure } (q_0) = \{q_0, q_1, q_2\} \quad \text{(new initial state for NFA, without } \epsilon \text{ transition)}$$

(*iv*) Rest of the states are

$$\epsilon\text{-closure } (q_1) = \{q_1, q_2\} \quad \text{(New state)}$$
$$\epsilon\text{-closure } (q_2) = \{q_2\} \quad \text{(New state)}$$

(*v*) The final states of NFA without $\epsilon$-transition are all those new states which contains final state of NFA with $\epsilon$-transition as member.

So $\{q_0, q_1, q_2\}$, $\{q_1, q_2\}$ and $\{q_2\}$ all are final states.

So if NFA without $\epsilon$-transition is

$$M' = (Q', \Sigma, \delta', q_0', F')$$
$$Q' = (\{q_0, q_1, q_2\}, \{q_1, q_2\}, \{q_2\})$$
$$q_0' = \{q_0, q_1, q_2\}$$
$$F' = (\{q_0, q_1, q_2\}, \{q_1, q_2\}, \{q_2\})$$

**Step 2.** Now we have to decide $\delta'$ to find out the transitions as follows :

Very carefully consider each old machine transition in each row, you can ignore $\phi$ enteries and $\epsilon$-transitions column. In the old machine $\delta(q_0, a) = q_0$ thus in new machine *i.e.* in $M'$, $\delta'(\{q_0, q_1, q_2\}, a) = \{q_0, q_1, q_2\}$ this is just because the new machine accepts the same language as the old machine and must at least have the same transitions for the new state names. We can allow interpret it as follows :

$$\begin{aligned}
\delta'(\{q_0, q_1, q_2\}, a) &= \epsilon\text{-closure } (\delta(q_0, q_1, q_2), a) \\
&= \epsilon\text{-closure } (\delta(q_0, a) \cup \delta(q_1, a) \cup \delta(q_1, a)) \\
&= \epsilon\text{-closure } (q_0 \cup \phi \cup \phi) \\
&= \epsilon\text{-closure } (q_0) = \{q_0, q_1, q_2\}
\end{aligned}$$

Similarly

$$\begin{aligned}
\delta'(\{q_0, q_1, q_2\}, b) &= \epsilon\text{-closure } (\delta(q_0, q_1, q_2), b) \\
&= \epsilon\text{-closure } (\delta(q_0, b) \cup \delta(q_1, b) \cup \delta(q_2, b)) \\
&= \epsilon\text{-closure } (\phi \cup q_1 \cup \phi) \\
&= \epsilon\text{-closure } (q_1) = \{q_1, q_2\}.
\end{aligned}$$

$$\begin{aligned}
\delta'(\{q_0, q_1, q_2\}, c) &= \epsilon\text{-closure } (\delta(q_0, q_1, q_2), c) \\
&= \epsilon\text{-closure } (\delta(q_0, c) \cup \delta(q_1, c) \cup \delta(q_1, c) \\
&= \epsilon\text{-closure } (\phi \cup \phi \cup q_2) \\
&= \epsilon\text{-closure } (q_2) = \{q_2\}.
\end{aligned}$$

In the same fashion we can write

$$\delta'(\{q_1, q_2\}, a) = \phi$$
$$\delta'(\{q_1, q_2\}, b) = \{q_1, q_2\}$$
$$\delta'(\{q_1, q_2\}, c) = \{q_2\}$$

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$$\delta'(q_2, a) = \phi$$
$$\delta'(q_2, b) = \phi$$
$$\delta'(q_2, c) = \{q_2\}$$

So transition table for NFA without $\epsilon$-transition will be as follows :

| $\delta'/\Sigma$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $\rightarrow$ * $\{q_0, q_1, q_2\}$ | $\{q_0, q_1, q_2\}$ | $\{q_1, q_2\}$ | $\{q_2\}$ |
| * $\{q_1, q_2\}$ | $\phi$ | $\{q_1, q_2\}$ | $\{q_2\}$ |
| * $\{q_2\}$ | $\phi$ | $\phi$ | $\{q_2\}$ |

Let us say $\{q_0, q_1, q_2\}$ as $q_x$

$\{q_1, q_2\}$ as $q_y$

and $\{q_2\}$ as $q_z$

So transition diagram is as follows :

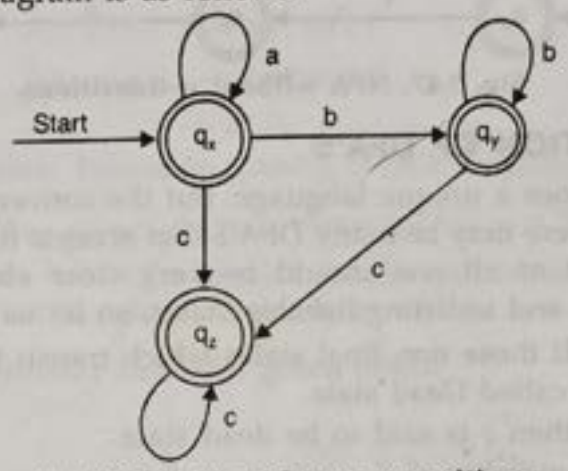



**Fig. 2.45. NFA without $\epsilon$-transitions.**

Let us see one more example.

**Example 2.19. Consider the NFA given by following diagram**

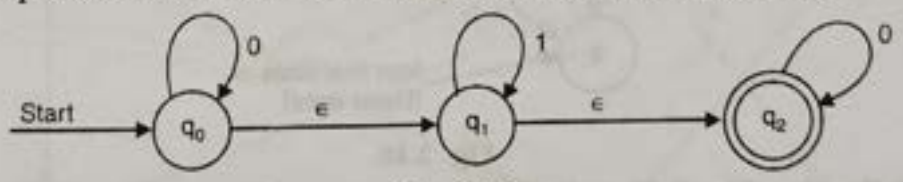



**Fig. 2.46.**

**Find the equivalent NFA without $\epsilon$-transitions.**

**Solution.** Let NFA without $\epsilon$-transition be

$$M' = (Q', \Sigma, q_0', \delta', F')$$

So as we see in example 2.19.

$$Q' = (\epsilon\text{-closure } (q_0), \epsilon\text{-closure } (q_1), \epsilon\text{-closure } (q_2))$$
$$= (\{q_0, q_1, q_2\}, \{q_1, q_2\}, \{q_2\}) \equiv (q_a, q_b, q_c)$$
$$\Sigma = \{0,1\}$$
$$F' = (q_a, q_b, q_c) \text{ as } q_a \rightarrow \{q_0, q_1, q_2\}$$
$$q_b \rightarrow \{q_1, q_2\}$$
$$q_c \rightarrow \{q_2\}$$

and $\delta'(q_a, 0) = q_a$

$\delta'(q_a, 1) = q_b$

$\delta'(q_b, 0) = q_c$
$\delta'(q_b, 1) = q_b$
$\delta'(q_c, 0) = q_c$
$\delta'(q_c, 1) = \phi$

So transition table is as follows :

| $\delta'/\Sigma$ | 0 | 1 |
|---|---|---|
| $\rightarrow^* q_a$ | $q_a$ | $q_b$ |
| $^* q_b$ | $q_c$ | $q_b$ |
| $^* q_c$ | $q_c$ | $\phi$ |

and transition diagram will be as follows :

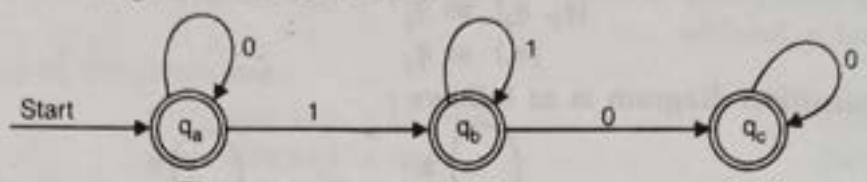



**Fig. 2.47. NFA without ∈-transitions.**

## 2.15. MINIMIZATION OF DFA'S

Any DFA defines a unique language, but the converse is not true. For a given language, there may be many DFA's that accepts it. To under stand the minimization first of all one should be very clear about the dead state, inaccessible states and indistinguishable states, so let us discuss them first.

**Dead state.** All those non final states which transit to itself for all input symbols in $\Sigma$, are called Dead state.

Let $\Sigma = \{a, b\}$ then $q$ is said to be dead state.

If $q$ is not the member of $F$

and $\delta(q, a) = q$ *i.e.,* $\delta(q, b) = q$

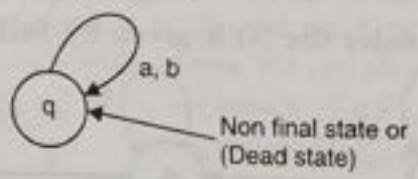



**Fig. 2.48.**

in the following Fig. 2.49 $q_4$ is dead state.

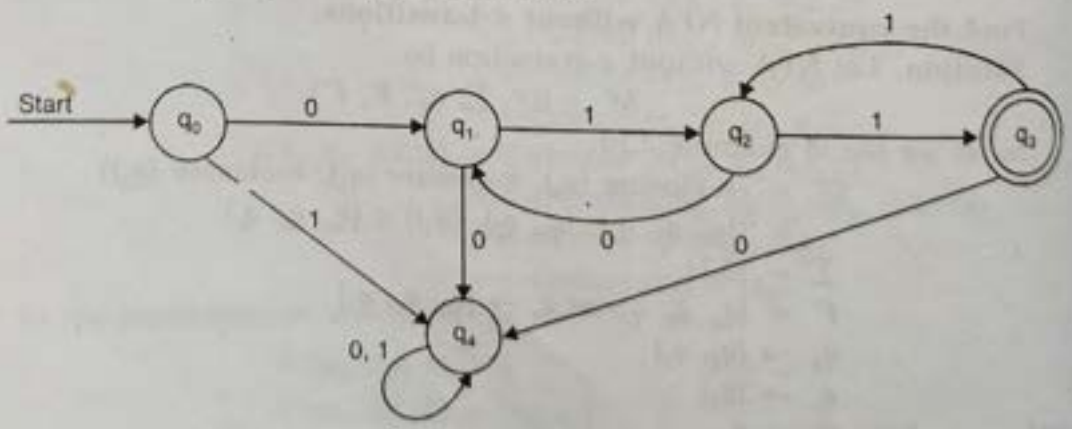



**Fig. 2.49.**

**Inaccessible states.** All those states which can never be reached from initial state are called inaccessible states. For example :

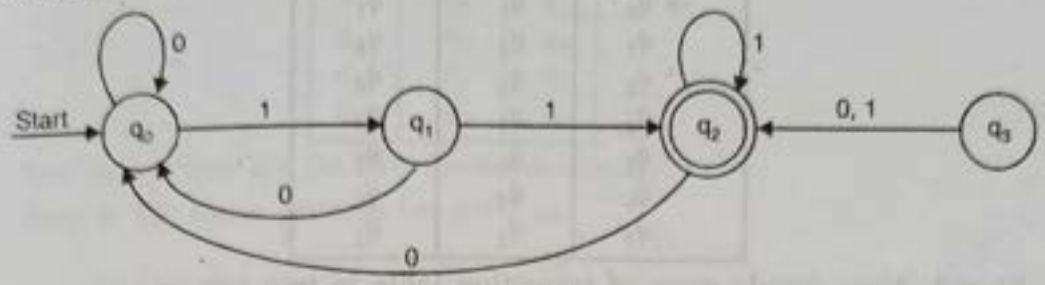



**Fig. 2.50.**

In Fig. 2.50 $q_3$ is the state which is not reachable from $q_0$ (initial state). So $q_3$ is the unreachable state.

**Indistinguishable states.** Two states $q$ and $r$ of DFA are called indistinguishable if

$$\hat{\delta}(q, w) \in F \Rightarrow \hat{\delta}(r, w) \in F$$

and

$$\hat{\delta}(q, w) \notin F \Rightarrow \hat{\delta}(r, w) \notin F$$

for all $w \in \Sigma^*$.

**Distinguishable states.** Two states $q_1$ and $q_2$ of DFA are called distinguishable if $\hat{\delta}(q_1, w) \in F$ and $\hat{\delta}(q_2, w) \notin F$ or vice versa; for all $w \in \Sigma^*$.

Now let us discuss the steps involve in the minimization process of a given DFA. Here we will consider an example.

**Example 2.20. Minimize the DFA given below.**

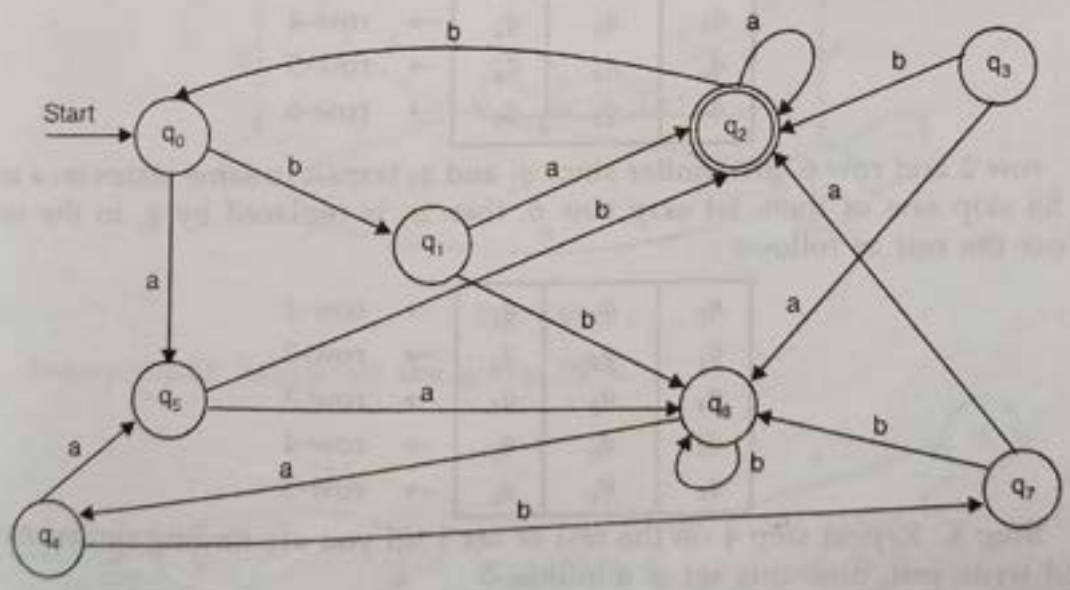



**Fig. 2.51.**

**Solution.** Let us discuss the solution step by step.

**Step 1.** Remove the unreachable states from the given DFA. The DFA given in Fig. 2.51, $q_3$ is unreachable state so remove it.

**Step 2.** Draw the transition table for rest states, after removing $q_3$. (unreachable states).

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $q_5$ | $q_1$ |
| $q_1$ | $q_2$ | $q_6$ |
| $* q_2$ | $q_2$ | $q_0$ |
| $q_4$ | $q_5$ | $q_7$ |
| $q_5$ | $q_6$ | $q_2$ |
| $q_6$ | $q_4$ | $q_6$ |
| $q_7$ | $q_2$ | $q_6$ |

**Step 3.** Now divide rows of transition table in two sets as :

(*i*) One set contains only those rows, which starts from non-final states i.e.

| | | |
|---|---|---|
| $q_0$ | $q_5$ | $q_1$ |
| $q_1$ | $q_2$ | $q_6$ |
| $q_4$ | $q_5$ | $q_7$ |
| $q_5$ | $q_6$ | $q_2$ |
| $q_6$ | $q_4$ | $q_6$ |
| $q_7$ | $q_2$ | $q_6$ |

**Set-1 (say)**

(*ii*) Other set contains only those rows, which starts from final states.

$q_2$ $q_2$ $q_0$ **Set-2 (say)**

**Step 4.** Apply step 4 on both the sets individually. Now first consider set-1

| | | | | |
|---|---|---|---|---|
| $q_0$ | $q_5$ | $q_1$ | $\rightarrow$ | row-1 |
| $q_1$ | $q_2$ | $q_6$ | $\rightarrow$ | row-2 |
| $q_4$ | $q_5$ | $\cancel{q_7}$ $q_1$ | $\rightarrow$ | row-3 |
| $q_5$ | $q_6$ | $q_2$ | $\rightarrow$ | row-4 |
| $q_6$ | $q_4$ | $q_6$ | $\rightarrow$ | row-5 |
| ~~$q_7$~~ | ~~$q_2$~~ | ~~$q_6$~~ | $\rightarrow$ | row-6 |

row 2 and row 6, are similar since $q_1$ and $q_7$ transit to same states on $a$ and $b$. So skip one of them let skip row 6, then $q_7$ is replaced by $q_1$ in the rest. Write the rest as follows :

| | | | | |
|---|---|---|---|---|
| $q_0$ | $q_5$ | $q_1$ | $\rightarrow$ | row-1 |
| $q_1$ | $q_2$ | $q_6$ | $\rightarrow$ | row-2 |
| $q_4$ | $q_5$ | $q_1$ | $\rightarrow$ | row-3 |
| $q_5$ | $q_6$ | $q_2$ | $\rightarrow$ | row-4 |
| $q_6$ | $q_4$ | $q_6$ | $\rightarrow$ | row-5 |

**Step 5.** Repeat step 4 on the rest of set 1 till you are finding similar row and write rest, now this set is minimised.

| | | |
|---|---|---|
| $q_0$ | $q_5$ | $q_1$ |
| $q_1$ | $q_2$ | $q_6$ |
| ~~$q_4$~~ | ~~$q_5$~~ | ~~$q_1$~~ |
| $q_5$ | $q_6$ | $q_2$ |
| $q_6$ | ~~$q_4$~~ $q_0$ | $q_6$ |

as row-1 and row-3 are similar

Now minimised set is

| | | |
|---|---|---|
| $q_0$ | $q_5$ | $q_1$ |
| $q_1$ | $q_2$ | $q_6$ |
| $q_5$ | $q_6$ | $q_2$ |
| $q_6$ | $q_0$ | $q_6$ |

See, now there are no two rows are similar.

**Step 6.** Do same process for set 2 as :

$q_2 \quad q_2 \quad q_0$

So only single row, already minimised.

**Step 7.** Now combine set 1 and set 2 as :

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $q_5$ | $q_1$ |
| $q_1$ | $q_2$ | $q_6$ |
| $q_5$ | $q_6$ | $q_2$ |
| $q_6$ | $q_0$ | $q_6$ |
| $^*q_2$ | $q_2$ | $q_6$ |

Now it is the transition table of minimised DFA.

Transition diagram of minimised DFA :

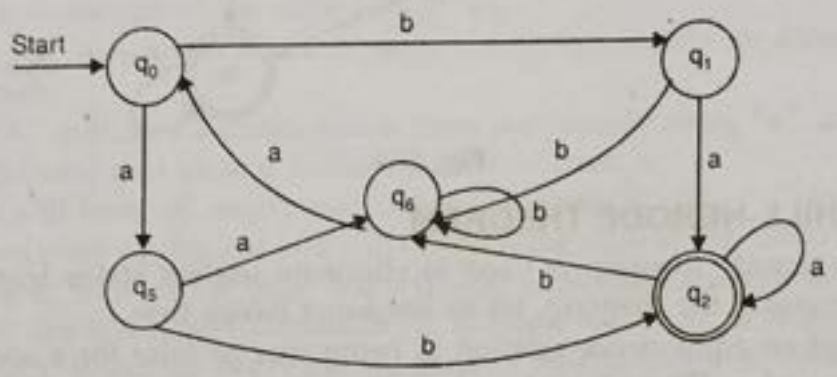



**Fig. 2.52.**

**Example 2.21. Minimise the given DFA.**

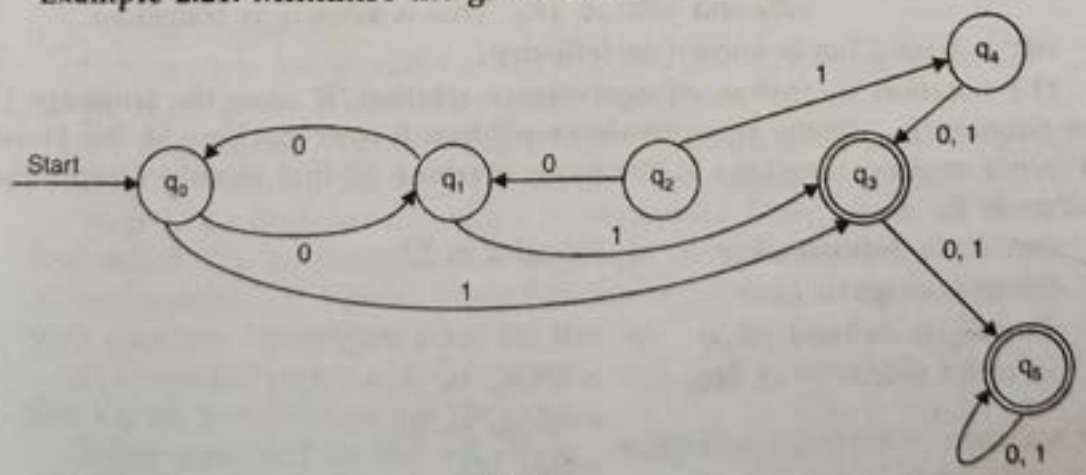



**Fig. 2.53.**

**Solution.** DFA given in Fig. 2.53 is having $q_2$ and $q_4$ as unreachable states, so remove them and draw the transition table for rest of the states.

| | $\delta/\Sigma$ | 0 | 1 |
|---|---|---|---|
| set 1 | $\rightarrow q_0$ | $q_1$ | $q_3$ |
| | $q_1$ | $q_0$ | $q_3$ |
| set 2 | $*q_3$ | $\cancel{q_5}$ $q_3$ | $\cancel{q_5}$ $q_3$ |
| | $\cancel{*q_5}$ | $\cancel{q_5}$ | $\cancel{q_5}$ |

Set 1 has no, two similar rows so we have to minimise only set 2. Now transition table for minimised DFA is as follows :

| $\delta/\Sigma$ | 0 | 1 |
|---|---|---|
| $\rightarrow q_0$ | $q_1$ | $q_3$ |
| $q_1$ | $q_0$ | $q_3$ |
| $*q_3$ | $q_3$ | $q_3$ |

Transition diagram is as follows :

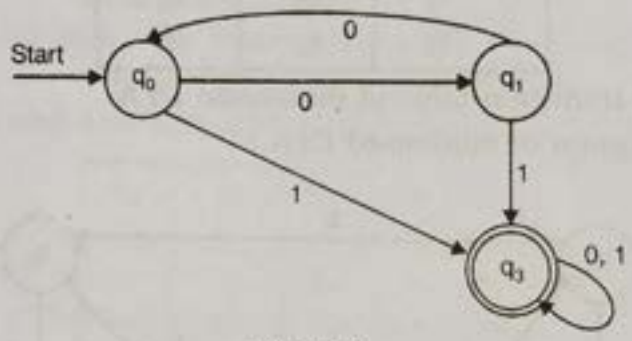



Fig. 2.54.

## 2.16. MYHILL-NERODE THEOREM

Myhill-Nerode theorem is used to eliminate useless states from a DFA. Before discussing the theorem, let us see some basics first.

Think of an equivalence relation as being true or false for a specific pair of strings $x$ and $y$. Thus $xRy$ is true for some set of pairs $x$ and $y$. We will use a relation $R$ such that $xRy \Leftrightarrow yRx$, $x$ has a relation to $y$ if and only if $y$ has the same relation to $x$. This is known as symmetric.

$xRy$ and $yRz \Rightarrow xRz$. This is known as transitive.

$xRx$ is true. This is known as reflexive.

The notation $R_L$ means an equivalence relation '$R$' over the language $L$. The notation $R_M$ means an equivalence relation $R$ over machine $M$. We know for every regular language $L$ there is a machine $M$ that exactly accepts the strings in $L$.

Our $R_L$ is defined $xR_Ly \Leftrightarrow$ for all $z$ in $\Sigma^*$
($xz$ in $L \Leftrightarrow yz$ in $L$)

Our $R_M$ is defined $xR_My \Leftrightarrow xzR_Myz$ for $z$ in $\Sigma^*$.

In other words
$$\begin{aligned}\delta(q_0, xz) &= \delta(\delta(q_0, x), z)\\ &= \delta(\delta(q_0, y), z)\\ &= \delta(q_0, yz)\end{aligned}$$

for $x$, $y$ and $z$ strings in $\Sigma^*$.

**$R_M$ divides the set $\Sigma^*$ into equivalence classes, one class for each state reachable in that particular state of $M$ from the starting state $q_0$. To get $R_L$ from this we have to consider only the final reachable states of $M$.**

**Statement of the Myhill-Nerode Theorem.**

The Myhill-Nerode theorem says the following three statements are equivalent :

1. The set $L$, a subset of $\Sigma^*$, is accepted by a DFA. (We know this means $L$ is regular language).
2. $L$ is the union of some of the equivalence classes of a right invariant (with respect to concatenation) equivalence relation of finite index.
3. Let equivalence relation $R_L$ be defined by : $xR_Ly$ if and only if for all $z$ in $\Sigma^*$, $xz$ is in $L$ exactly when $yz$ is in $L$. Then $R_L$ is of finite index.

## 2.17. IMPLEMENTATION OF MYHILL-NERODE THEOREM

**Step 1.** Start with a FA = $(Q, \Sigma, \delta, q_0, F)$ as usual remove from $Q$, $F$ and $\delta$ all states that cannot be reached from $q_0$.

**Step 2.** Build a two dimensional matrix labelling the right side $q_0, q_1, q_2, \ldots$ running down and denote this as the "$P$" first subscript. Label the top as $q_0, q_1, q_2, \ldots$ and denote this as "$q$" second subscript.

Put dashes in the major diagonal and the lower triangular part of the matrix (every thing below the diagonal) we will always use the upper triangular part because $xR_My = yR_Mx$ is symmetric. We will also use $(p, q)$ to index into the matrix with the subscript of the state called "$p$" always less than the subscript of the state called "$q$".

**Step 3.** We can have one of three things in a matrix location where there is no dash.

A "X" indicates a distinct state from our initialization, "$x$", at the location $(p, q)$ indicate that state $p$ is distinguishable from $q$.

We will level all empty matrix location with "0". The "0" location means the $p$ and $q$ are equivalent and will be the same state in the minimum machine.

**Step 4.** Now put at $(p, q)$ in $F \times (Q - F)$ such that subscript of $p$ < subscript of $q$. Do not write over dashes. If $(p, q)$ has a dash, put X in $(q, p)$.

**Step 5.** Begin for every pair of distinct states $(p, q)$ in $F \times F$ and for every pair of distinct state $(p, q)$ in $(Q - F) \times (Q - F)$ do

{

if for any input symbol, $(r, s)$ has X or $x$ then put $x$ at $(p, q)$, check $(s, r)$ if $(r, s)$ has a dash.

(where : $r = \delta(p, a)$ and $s = \delta(q, a)$ : $\Sigma$ is the input symbol in $\Sigma$.)

- If $(p, q)$ has a dash then mark $x$ at $(q, p)$
- You do not have to write another '$x$' if one is there already.

**Step 6.** Mark all the empty position '0' in upper triangle.

**Step 7.** To find out the states in minimised DFA, check all the rows of final matrix. The minimised machine will contain the states equal to the number of rows contains "0" entries (including the initial state if row starts with initial state does not contain any '0' entry).

If a row start with $q_x$ is having "0" at $q_y$ and $q_z$ positions then $\{q_x, q_y, q_z\}$ will a state in minimum machine.

Initial state will be that set which contains initial state as member state. (If there is no "0" entry at the row, starts with the initial state, then initial

state of new minimised machine will be same as in old machine). Final states of minimised machine are all those sets which contains the any accepting state of old machine as member state.

**Step 8.** For finding the transition between the states of new machine, we will adopt the same strategy as we adopt in the minimization process discussed earlier, start with the initial state of $M$ and check transition of it for all input symbols in $\Sigma$. Repeat this process till you are getting any new state.

**Example 2.22. Minimize the given DFA by using Myhill-nerode theorem.**

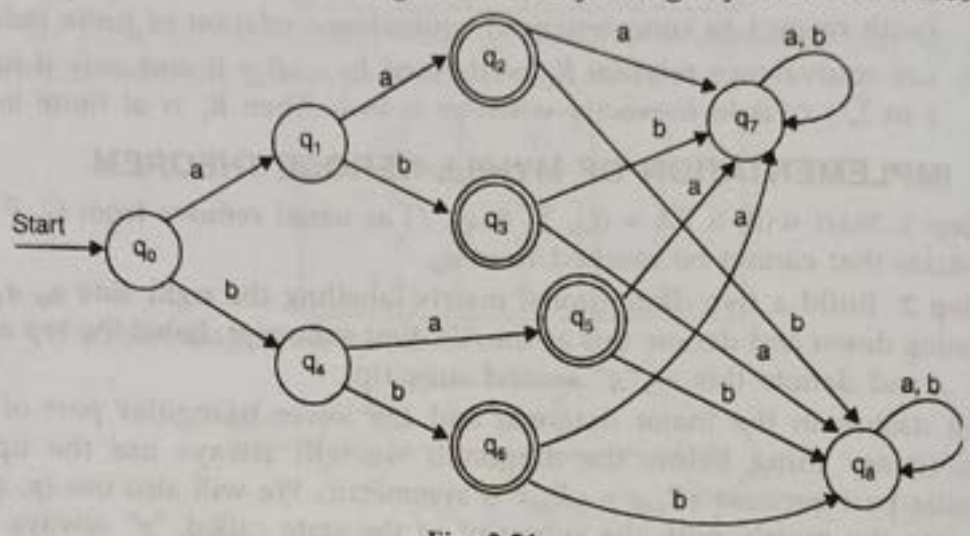



**Fig. 2.54.**

**Solution.** There is no unreachable state in Fig. 2.54 so we can directly apply Myhill-nerode theorem as follows :

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8\}$$
$$\Sigma = \{a, b\}$$
$$\text{Initial state} = q_0$$
$$F = \{q_2, q_3, q_5, q_6\}$$
$$\{Q - F\} = \{q_0, q_1, q_4, q_7, q_8\}$$

We use an ordered $F \times F = \{(q_2, q_3), (q_2, q_5), (q_2, q_6), (q_3, q_5), (q_3, q_6), (q_5, q_6)\}$

and $(Q - F) \times (Q - F) = \{(q_0, q_1), (q_0, q_4), (q_0, q_7), (q_0, q_8), (q_1, q_4), (q_1, q_7), (q_1, q_8), (q_4, q_7), (q_4, q_8), (q_7, q_8)\}$

Transition table will be following :

| $\delta/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow q_0$ | $q_1$ | $q_4$ |
| $q_1$ | $q_2$ | $q_3$ |
| $*\ q_2$ | $q_7$ | $q_8$ |
| $*\ q_3$ | $q_8$ | $q_7$ |
| $q_4$ | $q_5$ | $q_6$ |
| $*\ q_5$ | $q_7$ | $q_8$ |
| $*\ q_6$ | $q_7$ | $q_8$ |
| $q_7$ | $q_7$ | $q_7$ |
| $q_8$ | $q_8$ | $q_8$ |

**Step 1.** Now build the matrix labeling the "$p$" rows $q_0, q_1, q_2$ ... and labeling the "$q$" columns $q_0, q_1$, ... and put the dashes on the diagonal and below the diagonal

| | $q_0$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $q_0$ | — | | | | | | | | |
| $q_1$ | — | — | | | | | | | |
| $q_2$ | — | — | — | | | | | | |
| $q_3$ | — | — | — | — | | | | | |
| $q_4$ | — | — | — | — | — | | | | |
| $q_5$ | — | — | — | — | — | — | | | |
| $q_6$ | — | — | — | — | — | — | — | | |
| $q_7$ | — | — | — | — | — | — | — | — | |
| $q_8$ | — | — | — | — | — | — | — | — | — |

**Fig. 2.55.**

**Step 2.** Now mark $X$ at $(p, q)$ in upper triangle such that $p$ in $F$ and $q$ in $(Q - F)$ as subscript of $p$ is lower then the subscript of $q$ or $p$ in $(Q - F)$ and $q$ in $F$ and $p$ is lower then $q$.

$$F = \{q_2, q_3, q_5, q_6\}$$
$$Q - F = \{q_0, q_1, q_4, q_7, q_8\}$$

$F \times (Q - F)$ or $(Q - F) \times F$ such that $p$ is lower then $q$.

$= \{(q_2, q_4), (q_2, q_7), (q_2, q_8), (q_3, q_4), (q_3, q_7), (q_3, q_8), (q_5, q_7), (q_5, q_8), (q_6, q_8), (q_6, q_7), (q_0, q_2), (q_0, q_3), (q_0, q_5), (q_0, q_6), (q_1, q_2), (q_1, q_3), (q_1, q_5), (q_1, q_6), (q_4, q_5), (q_4, q_6)$

Now mark × in the matrix given in Fig. 2.56.

| | $q_0$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $q_0$ | — | | × | × | | × | × | | |
| $q_1$ | — | — | × | × | | × | × | | |
| $q_2$ | — | — | — | | × | | | × | × |
| $q_3$ | — | — | — | — | × | | | × | × |
| $q_4$ | — | — | — | — | — | × | × | | |
| $q_5$ | — | — | — | — | — | — | | × | × |
| $q_6$ | — | — | — | — | — | — | — | × | × |
| $q_7$ | — | — | — | — | — | — | — | — | |
| $q_8$ | — | — | — | — | — | — | — | — | — |

**Fig. 2.56.**

**Step 3.** Mark '$x$' and '$0$' in the matrix :

First we are considering $Q - F$

$$Q - F = \{q_0, q_1, q_4, q_7, q_8\}, \text{ so ordered pair of}$$
$$(Q - F) \times (Q - F) = \{(q_0, q_1), (q_0, q_4), (q_0, q_7), (q_0, q_8), (q_1, q_4), (q_1, q_7), (q_1, q_8), (q_4, q_7), (q_4, q_8), (q_7, q_8)\}$$

• Now we will select $(p, q)$ from $(Q - F) \times (Q - F)$ and find $(r, s)$ as $r = \delta(p, a)$ $s = \delta(q, a)$, if $(r, s)$ is $X$ or $x$ then $(p, q)$ will be $x$. If $(r, s)$ is neither $X$ nor $x$ then $(p, q)$ will be '0'.

→ for $(q_0, q_1)$, $r = \delta(q_0, a) = q_1$, $s = \delta(q_1, a) = q_2$.

It can be check from matrix in Fig. 2.56 that $(q_1, q_2)$ is × so $(q_0, q_1)$ will be $x$.

→ for $(q_0, q_4)$, $r = \delta(q_0, a) = q_1$, $s = \delta(q_4, a) = q_5$
$(q_1, q_5)$ is also × so $(q_0, q_4)$ will be $x$.

→ for $(q_0, q_7)$, $r = \delta(q_0, q_7) = q_1$, $s = \delta(q_7, a) = q_7$
$(q_1, q_7)$ is not × but it is $x$ so $(q_0, q_7)$ will be $x$.

→ for $(q_0, q_8)$, $r = \delta(q_0, a) = q_1$, $s = \delta(q_8, a) = q_8$
$(q_1, q_8)$ is not × but it is $x$ so $(q_0, q_8)$ will be $x$.

→ for $(q_1, q_4)$, $r = \delta(q_1, a) = q_2$, $s = \delta(q_4, a) = q_5$
$(q_2, q_5)$ is neither × nor $x$ so $(q_1, q_4)$ will be "0".

→ for $(q_1, q_7)$, $r = \delta(q_1, a) = q_2$, $s = \delta(q_7, a) = q_7$
$(q_2, q_7)$ is × so $(q_1, q_7)$ will be $x$.

→ for $(q_1, q_8)$, $r = \delta(q_1, a) = q_2$, $s = \delta(q_8, a) = q_8$
$(q_2, q_8)$ is × so $(q_1, q_8)$ will be $x$.

→ for $(q_4, q_7)$, $r = \delta(q_4, a) = q_5$, $s = \delta(q_7, a) = q_7$
$(q_5, q_7)$ is × so $(q_4, q_7)$ will be $x$.

→ for $(q_4, q_8)$, $r = \delta(q_4, a) = q_5$, $s = \delta(q_8, a) = q_8$
$(q_5, q_8)$ is × so $(q_4, q_8)$ will be $x$.

→ for $(q_7, q_8)$, $r = \delta(q_7, a) = q_7$, $s = \delta(q_8, a) = q_8$
$(q_7, q_8)$ is neither × nor $x$ so $(q_7, q_8)$ will be "0".

Now consider $F = \{q_2, q_3, q_5, q_6\}$

Ordered pair of $F \times F = \{(q_2, q_3), (q_2, q_5), (q_2, q_6), (q_3, q_5), (q_3, q_6)\ (q_5, q_6)\}$

→ for $(q_2, q_3)$, $r = \delta(q_2, a) = q_7$, $s = \delta(q_3, a) = q_8$
$(q_7, q_8)$ is neither × nor $x$ so $(q_2, q_3)$ will be "0".

→ for $(q_2, q_5)$, $r = \delta(q_2, a) = q_7$, $s = \delta(q_5, a) = q_7$
$(q_7, q_7)$ is dash so $(q_2, q_5)$ will be "0".

→ for $(q_2, q_6)$, $r = \delta(q_2, a) = q_7$, $s = \delta(q_6, a) = q_7$
$(q_7, q_7)$ is dash so $(q_2, q_6)$ is "0".

→ for $(q_3, q_5)$, $r = \delta(q_3, a) = q_8$, $s = \delta(q_5, a) = q_7$
$(q_8, q_7)$ is dash so check $(q_7, q_8)$, it is also neither × nor $x$ so $(q_3, q_5)$ is "0" entry.

→ for $(q_3, q_6)$, $r = \delta(q_3, a) = q_8$, $s = \delta(q_6, a) = q_7$
$(q_3, q_6)$ is also "0" entry.

→ for $(q_5, q_6)$, $r = \delta(q_5, a) = q_7$, $s = \delta(q_6, a) = q_7$
$(q_7, q_7)$ is dash so $(q_5, q_6)$ will be "0" entry

Rest of the enteries in upper triangle will be "0". Now fill the "$x$" and "0" in matrix given in Fig. 2.56.

| | $q_0$ | $q_1$ | $q_2$ | $q_3$ | $q_4$ | $q_5$ | $q_6$ | $q_7$ | $q_8$ |
|---|---|---|---|---|---|---|---|---|---|
| $q_0$ | — | $x$ | × | × | $x$ | × | × | $x$ | $x$ |
| $q_1$ | — | — | × | × | 0 | × | × | $x$ | $x$ |
| $q_2$ | — | — | — | 0 | × | 0 | 0 | × | × |
| $q_3$ | — | — | — | — | × | 0 | 0 | × | × |
| $q_4$ | — | — | — | — | — | × | × | $x$ | $x$ |
| $q_5$ | — | — | — | — | — | — | 0 | × | × |
| $q_6$ | — | — | — | — | — | — | — | × | × |
| $q_7$ | — | — | — | — | — | — | — | — | 0 |
| $q_8$ | — | — | — | — | — | — | — | — | — |

Fig. 2.57.

The "0" at $(q_1, q_4)$ means $(q_1, q_4)$ is a state in minimum machine.

So if resulting minimum machine is $M' = (Q', \Sigma, \delta', q_0', F')$

$$Q' = (\{q_0\}, \{q_1, q_4\}, \{q_2, q_3, q_5, q_6\}, \{q_7, q_8\})$$
$$F' = (\{q_2, q_3, q_5, q_6\})$$
$$q_0' = q_0$$

Now consider Fig. 2.54 (given DFA) and draw the transition table for minimum machine as

$$\delta'(\{q_0\}, a) = \{q_1, q_4\}$$

Since $\delta(q_0, a) = q_1$ in given DFA, $q_1$ is in $\{q_1, q_4\}$ state of minimised machinery on the same way, the transition table for $M'$ is

| $\delta'/\Sigma$ | $a$ | $b$ |
|---|---|---|
| $\rightarrow \{q_0\}$ | $\{q_1, q_4\}$ | $\{q_1, q_4\}$ |
| $\{q_1, q_4\}$ | $\{q_2, q_3, q_5, q_6\}$ | $\{q_2, q_3, q_5, q_6\}$ |
| $*\{q_2, q_3, q_5, q_6\}$ | $\{q_7, q_8\}$ | $\{q_7, q_8\}$ |
| $\{q_7, q_8\}$ | $\{q_7, q_8\}$ | $\{q_7, q_8\}$ |

So minimized machine is

$\{q_0\}$ — $A$
$\{q_1, q_4)$ — $B$
$\{q_2, q_3, q_5, q_6\}$ — $C$
$\{q_7, q_8\}$ — $D$

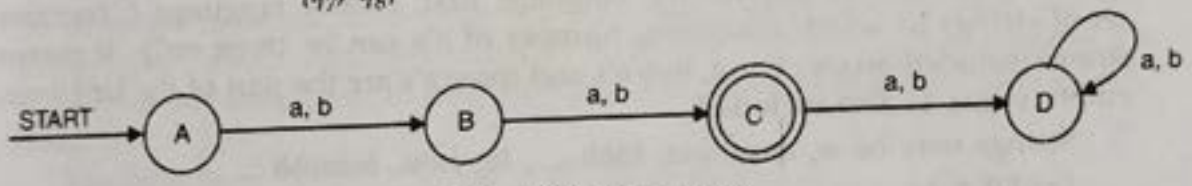



**Fig. 2.58. Minimized DFA.**

At the end of these discussion we can discuss some more examples to consolidate our concepts.

**Example 2.23. Design a DFA for the language**

$$L = \{w : n_a = 1, w \in (a, b)^*\}$$

**Solution.** Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

When we analyse the language, then it contains a set of string, every string contains exactly one '$a$' only, that is $a$, $bbb$ ... $a$, $bbb$ ... $abbb$ ..., and $abbb$... . Then transition diagram will be as follows :

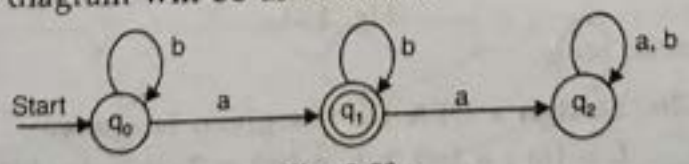



**Fig. 2.59.**

Here $q_2$ is dead state

$$Q = \{q_0, q_1, q_2\}$$
$$\text{intial state} = \{q_0\}$$
$$\Sigma = \{a, b\}$$
$$F = \{q_1\}.$$

**Example 2.24. Design a DFA for the language**

$$L = \{w : n_a(w) \geq 1, w \in (a, b)^*\}.$$

**Solution.** Given language is

$$L = \{w : n_a(w) \geq 1, w \in (a, b)^*\}$$

Clearly language is the set of strings, in which every string containing at least one '*a*', any where in the string such as *a*, *bbbb* ... *a*, *aaaa*, *bababa*, *aaabbb* ... .

Let DFA be

$$M = (Q, \Sigma, \delta, q_0, F)$$
$$\Sigma = (a, b)$$
$$Q = (q_0, q_1)$$
$$F = \{q_1\}$$

Transition Diagram is

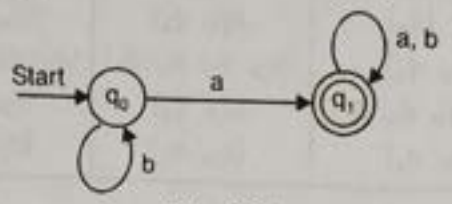



**Fig. 2.60.**

**Example 2.25. Design the determinstic finite automata for the given language**

$$L = \{w : n_a(w) \leq 3, w \in \{a, b\}^*\}.$$

**Solution.** Let us analyse the language first, clearly language *L* contains set of strings in which maximum number of *a*'s can be three only. It means strings included no *a*'s , one *a*, two *a*'s and three *a*'s are the part of the language, empty string is also included.

Strings may be $\in$, *a*, *aa*, *aaa*, *bbbb* ... , *ba*, *baba*, *baaabbb* ... .

Let DFA be

$$M = (Q, \Sigma, \delta, q_0, F)$$
$$\Sigma = (a, b) \text{ given}$$
$$Q = (q_0, q_1, q_2, q_3, q_4)$$
$$F = (q_0, q_1, q_2, q_3)$$

Transition diagram is shown in Fig. 2.61.

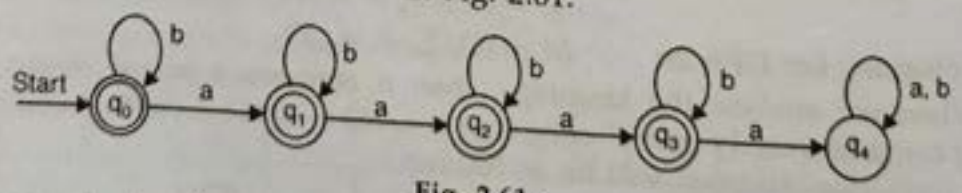



**Fig. 2.61.**

Here $q_4$ is dead state.

**Example 2.26. Design a DFA for the given language**

$$L = \{w : n_a(w) \geq 1, n_b(w) = 2, w \in \{a, b\}^*\}.$$

**Solution.** Let us analyse the language first, since here conditions are defined as $n_a(w) \geq 1$ and $n_b(w) = 2$ means every string of the language must contain at least one '*a*' exactly two *b*'s such as *abb*, *bab*, *baaaaa* ... *b*, *aaaaa* ... *baaab*, *bba* and many more which are satisfying both the conditions.

Let us assume that DFA be $m = (Q, \Sigma, \delta, q_0 F)$

where
$$Q = (q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8)$$
$$F = (q_3, q_8)$$
$$\Sigma = (a, b)$$

Transition diagram is shown in Fig. 2.62.

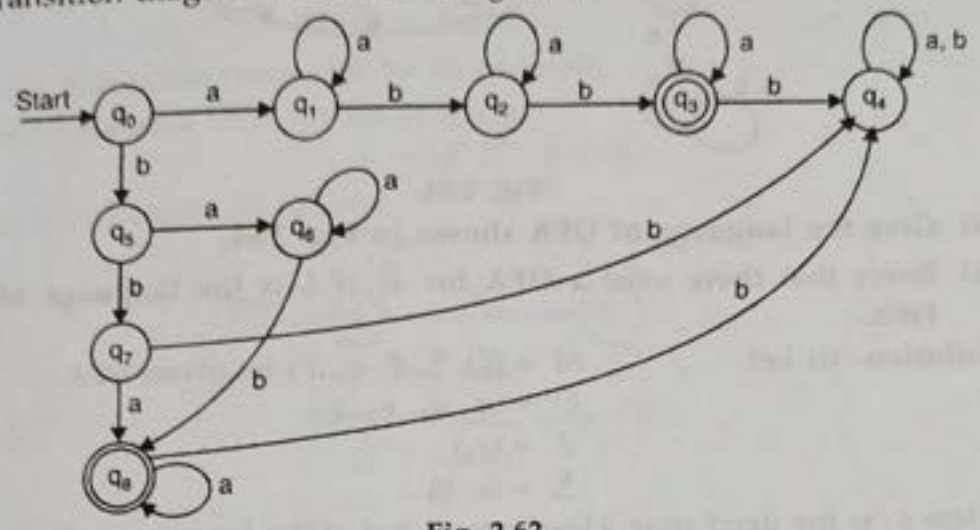



Fig. 2.62.

**Example 2.27. Design a DFA for the language**
$$L = \{w : n_a(w) = 2, n_b(w) > 2, w \in \{a, b\}^*\}.$$

**Solution.** Given language is $L = \{w : n_a(w) = 2, n_b(w) > 2, w \in (a, b)^*\}$

Now it is clear from $n_a(w) = 2$ and $n_b(w) > 2$ that strings, which are belongs to language must satisfied both the condition simultaneously that is every string must contain exactly two $a$'s in it (in any position) and number of $b$'s must be more then two (in any position), such as *abbbbabbb, aabbbbb, aabbbb, bbbaa*.

Let DFA be
$$M = (Q, \Sigma, \delta, q_0, F)$$
$$Q = (q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12})$$
$$\Sigma = (a, b)$$
$$F = (q_5)$$

Transition diagram is shown in Fig. 2.63.

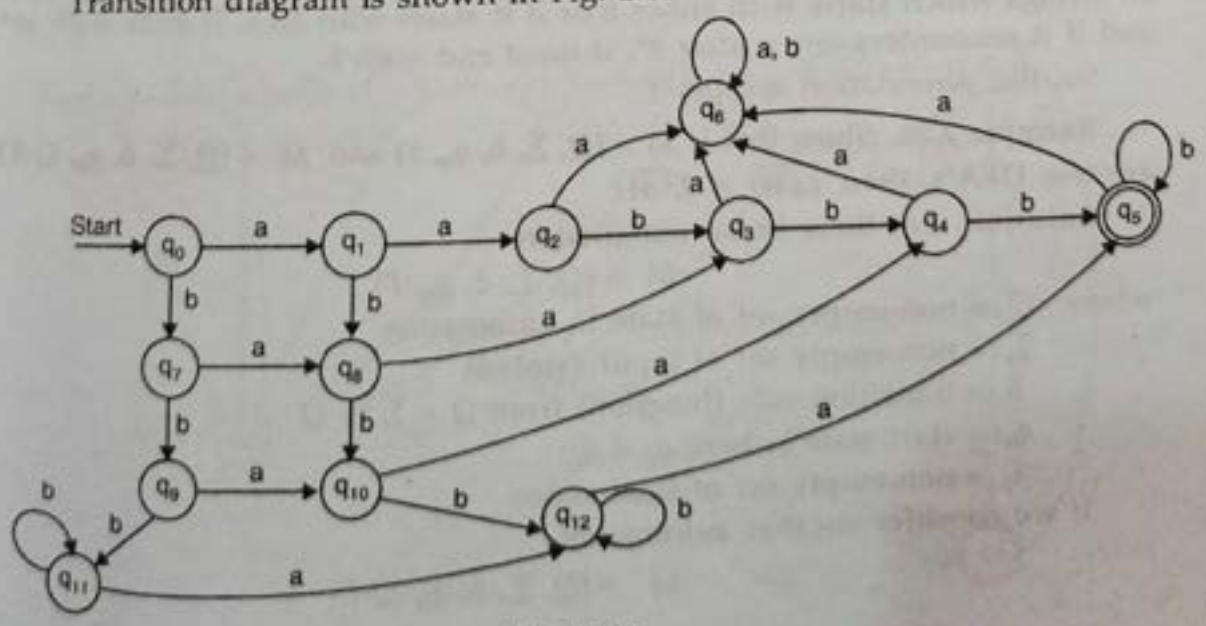



Fig. 2.63.

**Example 2.28. Given DFA is**

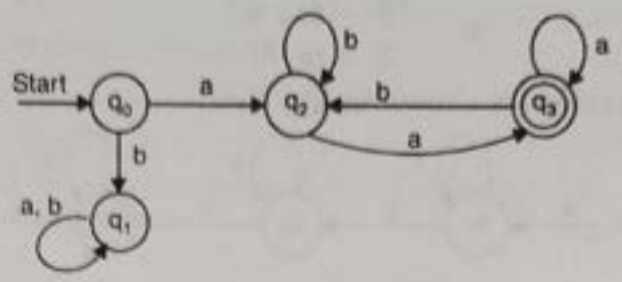



**Fig. 2.64.**

**(i) Give the language of DFA shown in Fig. 2.64.**

**(ii) Prove that there exist a DFA for $\overline{L}$ if $L$ is the language of given DFA.**

**Solution.** (i) Let

$$M = (Q, \Sigma, \delta, q_0, F) \text{ be given DFA}$$
$$Q = (q_0, q_1, q_2, q_3)$$
$$F = (q_3)$$
$$\Sigma = (a, b)$$

Here $q_1$ is the dead state, clearly no string of the language will start from $b$ character. So every string starts with '$a$' and end with '$a$'.

(ii) Now make $q_3$ as non-final state and $q_0, q_1, q_2$ as final states and prove the new automation obtained accepts $\overline{L}$ *i.e.* it does not accepts *awa*.

Now it can be show through the following transition diagram.

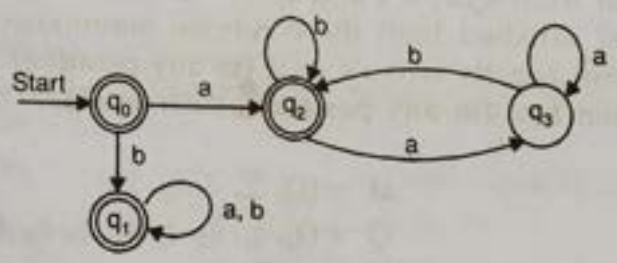



**Fig. 2.65.**

This automation accepts $b(a, b)^*$, $\in$, and $ab^*\ aa^*\ bb^*$ or $ab^*$. Clearly it accepts all strings which starts with either $b$ or if it starts with an $a$, it ends with $ab^*$ and if it encounters any $a$ after $b^*$, it must end with $b$.

So, this automation accepts $\overline{L}$.

**Example 2.29. Show that if $M = (Q, \Sigma, \delta, q_0, F)$ and $\overline{M} = (Q, \Sigma, \delta, q_0, Q\text{-}F)$ are two DFA's, then $\overline{L(M)} = L(\overline{M})$.**

**Solution.** Let there be an automation

$$M = (Q, \Sigma, \delta, q_0, F)$$

where $Q$ = non-empty set of state in automation

$\Sigma$ = non-empty set of input symbols

$\delta$ = transition rule (function) from $Q \times \Sigma$ to $Q$

$q_0$ = start state (where $q_0 \in Q$)

$F$ = non-empty set of final states.

If we consider another automation

$$\overline{M} = (Q, \Sigma, \delta, q_0, Q\text{-}F)$$

where all $Q$, $\Sigma$, $\delta$ and $q_0$ are same as previous, but the final states are the other state than $F$, then the new automation can not reach finally at the state ($F$) in this automation as it used to be in the automation $M$. So, this automation will accept other language, but it will not accept the language which is accepted by $M$.

So $$\overline{L(M)} = L(\overline{M})$$

We can also understand this by an example.

Let us consider a DFA for the language

$$L = \{a^n b^0 : n \geq 0\}.$$

Transition diagram as follows :

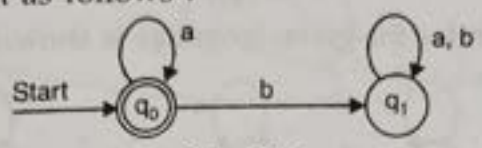



**Fig. 2.66.**

Let say
$$M = \{Q, \Sigma, \delta, q_0, F)$$
$$Q = (q_0, q_1)$$
$$q_0 = q_0$$
$$F = \{q_0, q_1\} - \{q_0\} = \{q_1\}$$

Now making

So we will get

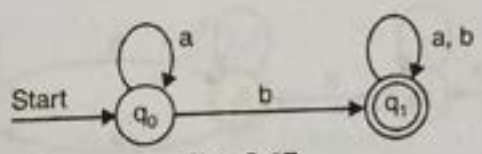



**Fig. 2.67.**

This automation (shown in Fig. 2.67) accepts all strings, except $a^n b^0$, $n \geq 0$.

**Example 2.30. Design a DFA for the language**

$$L = \{ab^5 w b^4 : w \in \{a, b\}^*\}.$$

**Solution.** Let DFA be
$$M = (Q, \Sigma, \delta, q_0, F)$$

where
$$Q = (q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11})$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_{10}\}$$

Transition diagram is following :

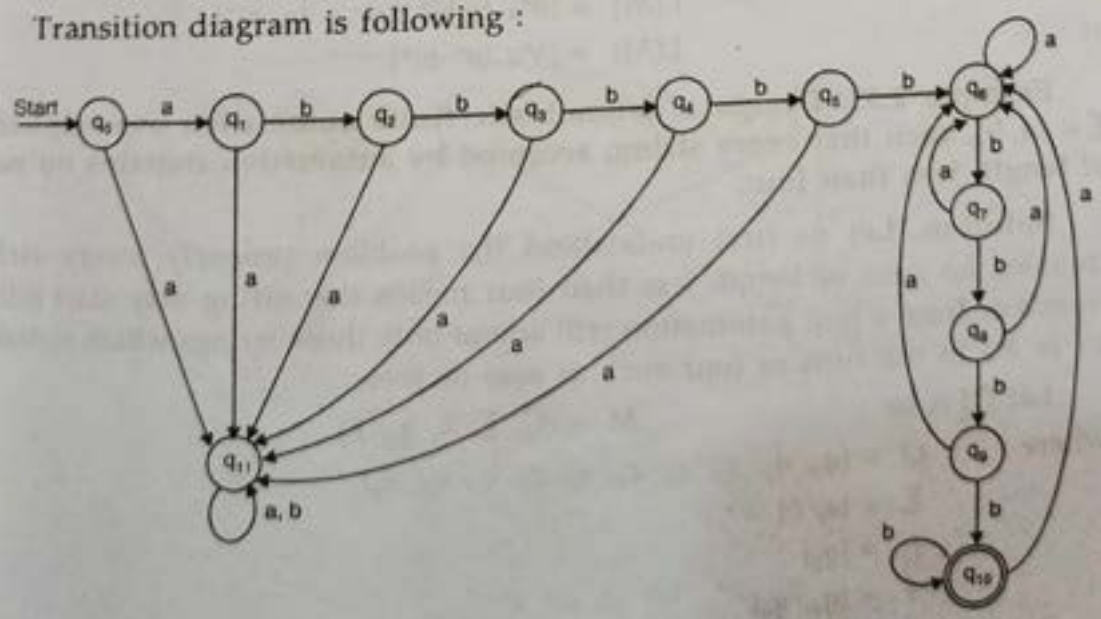



**Fig. 2.68.**

**Example 2.31. Design a DFA for the language**

$$L = \{w_1\ ab\ w_2 : w_1, w_2 \in (a, b)^*\}.$$

**Solution.** It is clearly known after analysis of language $L$ that every string at least one $ab$ in string at any position.

Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

where

$$Q = \{q_0, q_1, q_2\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2\}$$

Transition diagram for the given language is shown in Fig. 2.69, as follows :

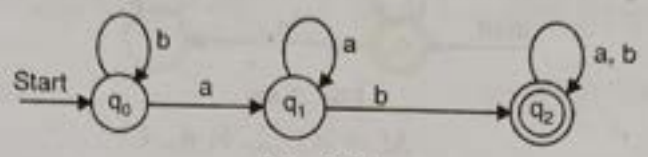



Fig. 2.69.

**Example 2.32. Give the description of the language for the DFA, shown in Fig. 2.70.**

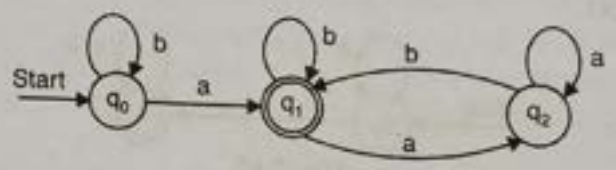



Fig. 2.70.

**Solution.** Given DFA $M = (Q, \Sigma, \delta, q_0, F)$

$$Q = \{q_0, q_1, q_2\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_1\}$$

Language of DFA can be defined as follows :

$$L(M) = \{b^*a\ \{wb\}^* : w \in \{a, b\}^*\}$$

or

$$L(M) = \{b^*a\ \{a^*\ b\}^*\}.$$

**Example 2.33. Design a determinstic finite automation over alphabet $\Sigma = \{a, b\}$, such that every string, accepted by automation contains no runs of length less than four.**

**Solution.** Let us first undetstand the problem properly, every string contains no runs of length less than four means that string may start either from $a$ or from $b$ but, automation will accept only those strings which contains $a$'s or $b$'s in the runs of four such as $aaaa$ or $bbbb$.

Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

where

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8, q_9\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_4, q_8\}$$

Transition diagram is shown in Fig. 2.71, as follows :

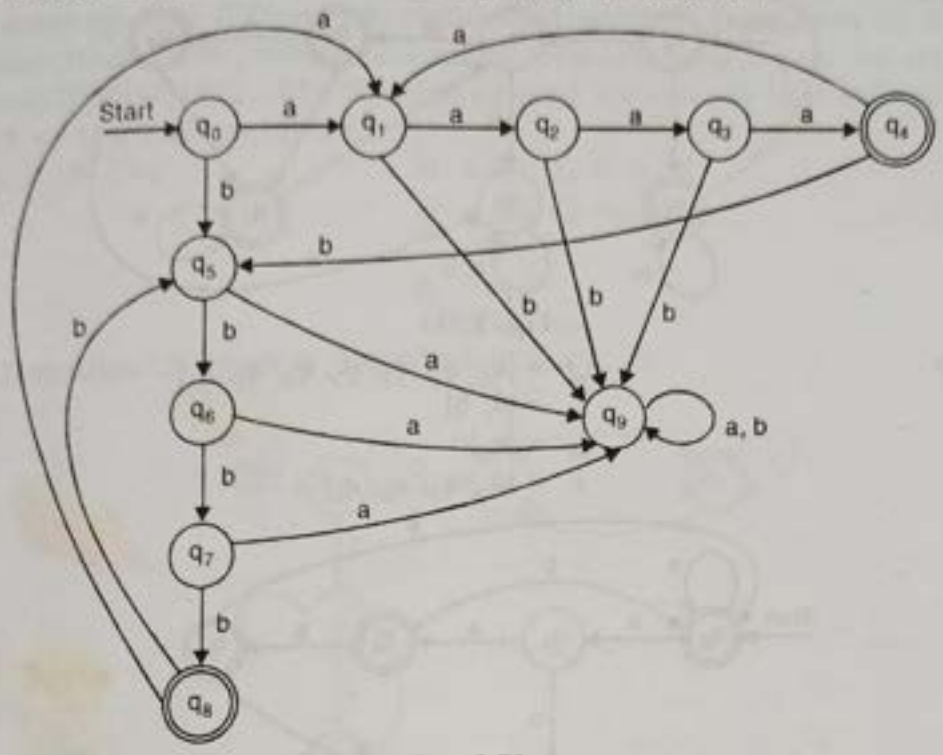



**Fig. 2.71.**

Here $q_0$ is dead state.

**Example 2.34. Design a DFA for the language**

**$L = \{w$ : every run of a's has length either two or three$\}$.**

**Solution.** We can solve this problem with same logic as we have solve Example 2.34.

Automation will accepts only those strings, if an '$a$' encountered then it must follow an '$a$' or two $a$'s, that is the meaning of "every run of $a$'s has length either two or three.

Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

Here this automation can have three different transition diagrams, as follows :

(*a*)

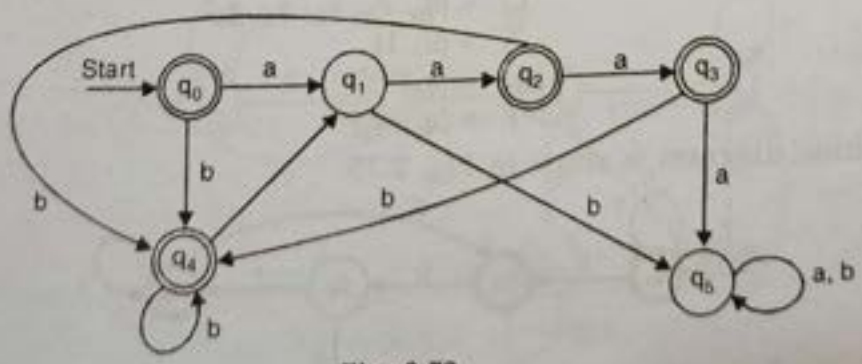



**Fig. 2.72.**

$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$

$\Sigma = \{a, b\}$

$q_0 = \{q_0\}$

$F = \{q_2, q_3, q_4, q_0\}$

(b)

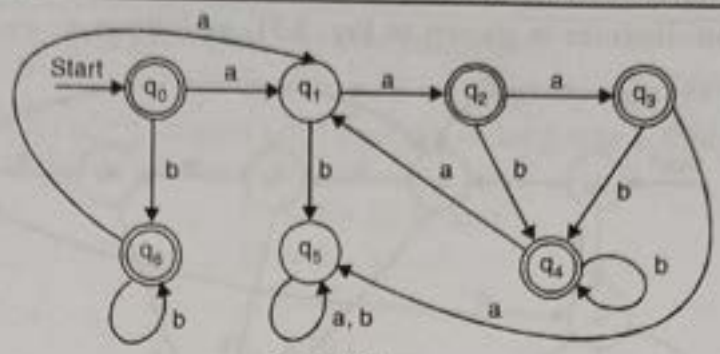



Fig. 2.73.

Here
$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5, q_6\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2, q_3, q_6, q_0\}.$$

(c)

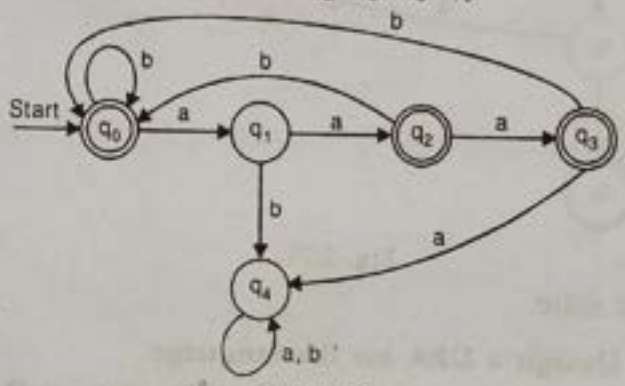



Fig. 2.74.

Here
$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2, q_3, q_0\}.$$

**Example 2.35. Design a DFA, which accepts strings, in which every 00 is followed immediately by a 1. For example, the strings 001, 0010, 00100111001 are in the language, but 0001 and 00100 are not.**

**Solution.** Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

where
$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\Sigma = \{0, 1\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_0, q_1\}$$

Transition diagram is show in Fig. 2.75.

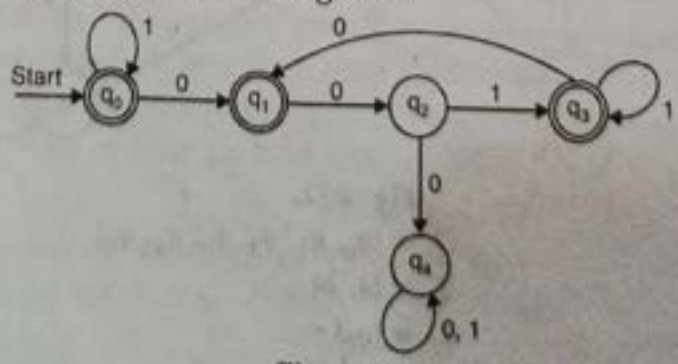



Fig. 2.75.

Here $q_4$ is dead state.

**Example 2.36. Design a DFA for the language, contains strings in which left most symbol differ from right most symbol. $\Sigma$ is given {0, 1}.**

**Solution.** Here problem is straight forward. Language is the set of strings such as 01110001101 , 01, 10, or in general we can say that strings are either $0\,w\,1$ or $1\,w\,0$ where $w \in (0, 1)^*$.

Let DFA be

$$M = (Q, \Sigma, \delta, q_0, F)$$
$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\Sigma = \{0, 1\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2, q_4\}.$$

Transition diagram is described in Fig. 2.76.

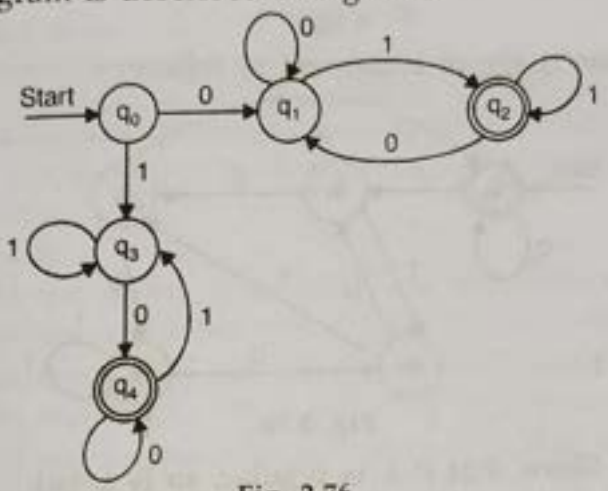



Fig. 2.76.

**Example 2.37. Design a determinstic finite automation which accepts set of strings such that every string containing 00 as a substring but not 000 as sub-string.**

**Solution.** Let DFA be

$$M = (Q, \Sigma, \delta, q_0, F)$$

where

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$
$$\Sigma = \{0, 1\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2, q_3, q_5\}$$

Transition diagram is defined in Fig. 2.77 as follows :

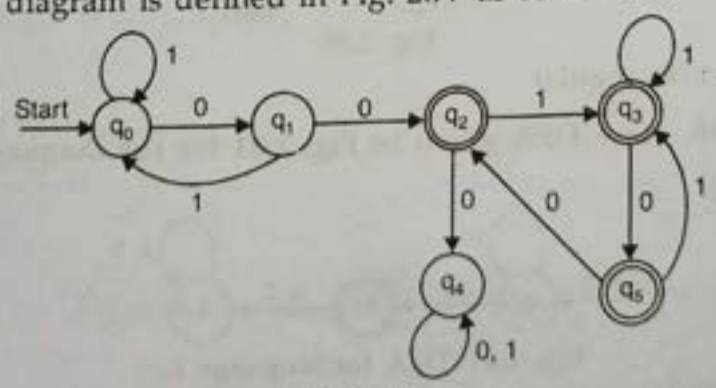



Fig. 2.77.

Here $q_4$ is dead state.

**Example 2.38. Construct a DFA that accepts strings on {0, 1}, if and only if the value of the string, interpreted as a binary representation of an integer, is zero module five. For example, 0101 and 1111, representing the integers 5 and 15, respectively, are to be accepted.**

**Solution.** Let us undetstand problem first. We have to design a automation which accepts those binary strings, when we interpreted them in integer then those integers are divisible by 5 that means after dividing 5 reminder is 0, it is the only meaning of the "is zero module five". So clearly integer interpretation should be 0, 5, 15, 20, 25, ... So binary strings which are accepted by automation are 0, 101, 1111 ...

Let DFA be $M = (Q, \Sigma, \delta, q_0, F)$

where

$$Q = \{q_0, q_1, q_2, q_3, q_5\}$$
$$\Sigma = \{0, 1\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_0\}$$

and finally transition diagram is defined as follows :

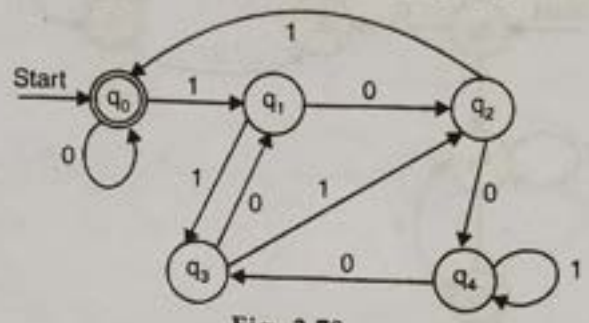



**Fig. 2.78.**

**Example 2.39. Show that if $L$ is regular, so is $L-\{\epsilon\}$.**

**Solution.** Let us assume that $L$ is regular and $w$ is the string such that $w \in L$. So automation will be

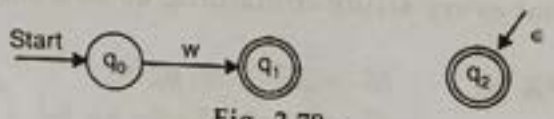



**Fig. 2.79.**

If we substracted $\epsilon$ from $L$ then we have $L - \epsilon = L^+$

Transition diagram can be defined as :

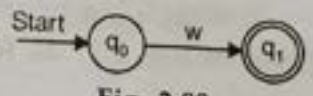



**Fig. 2.80.**

So, $L - \epsilon$ is also regular.

**Example 2.40. Here DFA given in Fig. 2.81 for the language $L$ find the DFA for $L^2$.**

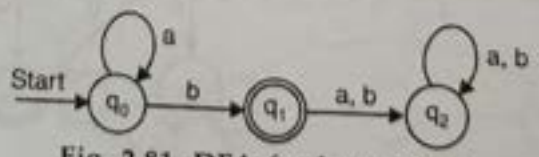



**Fig. 2.81. DFA for language $L$.**

**Solution.** We have given DFA for language $L$ in Fig. 2.81($a$).

The language for DFA given is

$$L = \{a^n b : n \geq 0\}$$

Let as assume that DFA for $L^2$ is

$$M = (Q, \Sigma, \delta, q_0, F)$$

$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{a, b\}$$
$$q_0 = \{q_0\}$$
$$F = \{q_2\}$$

Then transition diagram for $L^2$ is defined as follows :

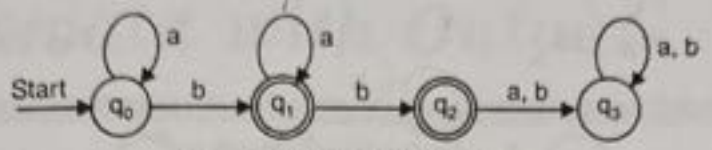



**Fig. 2.81(*a*). DFA for $L^2$.**

Here $q_3$ is dead state.

## EXERCISE

1. Build an FA that accepts only the works *baa*, *ab* and *abb* and no other strings longer or shorter.
2. For the $\Sigma = \{a, b\}$
   (*i*) Build an FA that accepts only those words that have more than four letters.
   (*ii*) Build an FA that accepts only those words that have fewer than four letters.
   (*iii*) Build an FA that accepts only those words with exactly four letters.
   (*iv*) Build an FA that accepts only those words that do not end with *ba*.
   (*v*) Build an FA that accepts only those words that begin or end with a double letter.
3. Describe the language accepted by the following FA's :

(*i*)

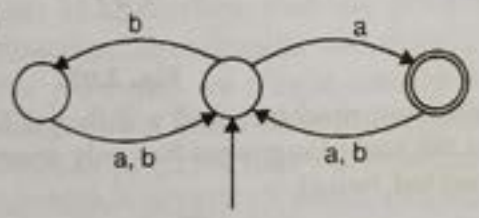



**Fig. 2.82.**

(*ii*)

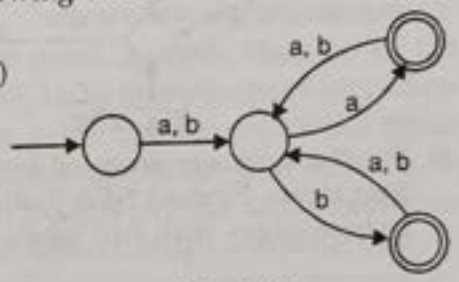



**Fig. 2.83.**

4. Convert each of the following NFA's in to FA's

(*i*)

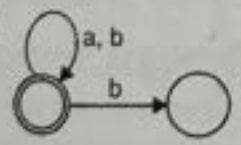



**Fig. 2.84.**

(*ii*)

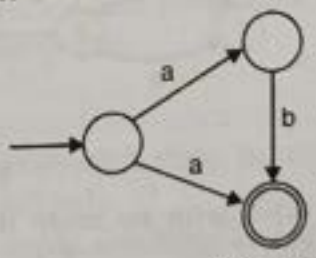



**Fig. 2.85.**

(*iii*)

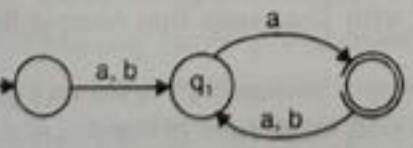



**Fig. 2.86.**

(*iv*)

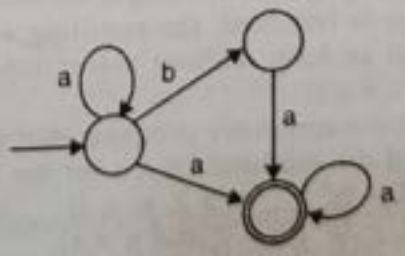



**Fig. 2.87.**

(v)

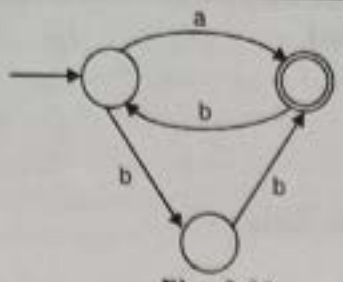



Fig. 2.88.

(vi)

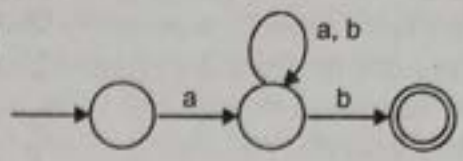



Fig. 2.89.

(vii)

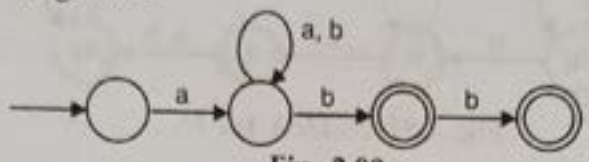



Fig. 2.90.

(viii)

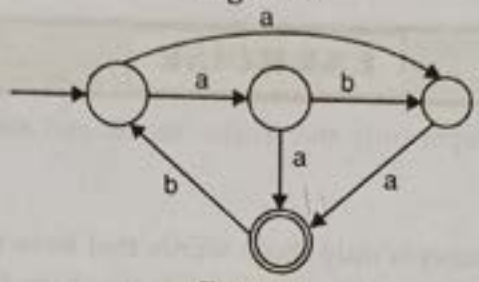



Fig. 2.91.

5. Convert the following NFA–ϵ's into NFA without ϵ-transitions.

(i)

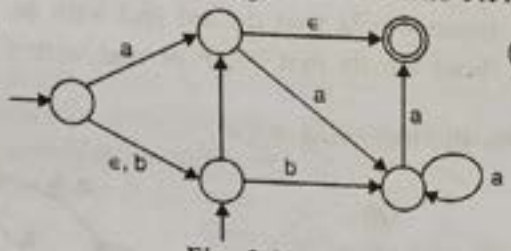



Fig. 2.92.

(ii)

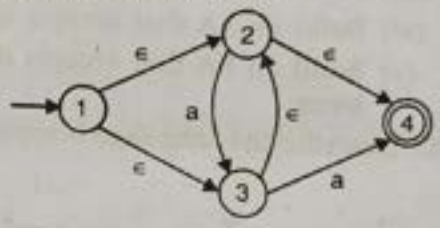



Fig. 2.93.

6. For the language accepted by the following machine, find a different FA with four states. Find an NFA that accepts the same languages has only seven edges (where edges with two labels are counted twice).

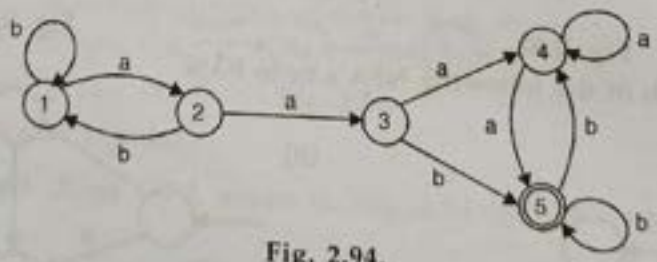



Fig. 2.94.

7. Design a NFA with no more than five states for the set $\{abab^n : n \geq 0\} \cup \{aba^n : n \geq 0\}$.
8. Find an NFA that accepts $\{a\}^*$ and is such that if in its transition graph a single edge is removed, the resulting automation accepts $\{a\}$.
9. Find an NFA without ϵ-trantion and with final state that accepts the set $\{a\} \cup \{b^n : n \geq 1\}$.
10. Write a computer program that produces a minimal DFA for any given DFA.
11. Find the minimal DFA's for the following languages below :

(i) $L = \{a^n b^m : n \geq 2, m \geq 1\}$ (ii) $L = \{a^n : n \geq 0, n \neq 3\}$

(iii) $L = \{a^n : n \neq 2 \text{ and } n \neq 4\}$

*****

# 3

CHAPTER

# Automata with Output

INSIDE THIS CHAPTER

3.1. Introduction 3.2. Moore Machine 3.3. Mealy Machine 3.4. Equivalence of Mealy and Moore Machine

## 3.1. INTRODUCTION

One limitation of the finite automation as we have defined it in previous chapter is that output is limited to a binary signal : accept (1)/not accepted (0). The only explicit task a machine has done so far is to recognize a language. Computers as we know, often have the more useful function of performing calculations and conveying results. In this chapter, we expand the notion of machine task.

If we assume that all the printing of output is to be done at the end of the program run, at which time we have an instruction that dumps a buffer that has been assembled, then we have a maximum on the number of charaters that dumps a buffer that has been assembled, then we have a maximum on the number of characters that the program can print, namely, the size of the buffer. However, theoritically we should be able to have outputs of any finite length. For example, we might simply want to print out a copy of the input string, which could itself be arbitrarily long. These questions that have to be faced if we are to claim that our mathematical nodels of FAs and Transition graphs represent actual physical machines. In this chapter we will discuss two different model of FAs with output capabilities. These were created by G.H. mealy in 1955 and, independently, by E.F. Moore in 1956. We will present these two models, prove that they are equivalent and give some examples of how they arise in the "logic" section of a computer.

## 3.2. MOORE MACHINE

It is the FA in which output is associated with each state. In moore machine every state of this finite machine has a fix output.

Mathematically moore machine is a six-tuple machine and defined as

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$

$M_0$ is the six-tuple moore machine in which

$Q$ : A nonempty finite set of state in $M_0$.

$\Sigma$ : A nonempty finite set of input symbols.

$\Delta$ : A nonempty finite set of outputs.

$\delta$ : It is transition function which takes two arguments as in finite automata, one is input state and another input symbol. The output

of this function is a single state, so clearly $\delta$ is the function which is responsible for the transitions in $M_0$.

$q_0$ : is the initial state of $M_0$ and $q_0 \in Q$.

$\lambda'$ : is a mapping function which maps $Q$ to $\Delta$, giving the output associated with each state.

Let $M_0$ be a moor machine and $\Sigma$ is $a_1, a_2, a_3, \ldots, a_n$, $n \geq 0$ then output of $M_0$ for $\Sigma$ is $\lambda'(q_0), \lambda'(q_1), \lambda'(q_2) \ldots \lambda'(q_n)$, where $q_0, q_1, q_2, \ldots q_n$ is the sequence of states such that

$$\delta(q_{i-1}, a_i) = q_i \text{ for } 1 \leq i \leq n$$

It is to be noticed that moore machine gives output $\lambda'(q_0)$ for input $\epsilon$, *i.e.* in case of moore machine length of output sequence is $n + 1$.

### 3.2.1. Representation of Moore Machine

Moore machine can be represented by transition table as well as transition diagram same as finite automata.

Let $M_0$ be moore machine having following transition table $\Sigma = \{0, 1\}$

| Present state | Next state at input | | Output |
|---|---|---|---|
| | $a = 0$ | $a = 1$ | |
| $\rightarrow q_0$ | $q_3$ | $q_1$ | 0 |
| $q_1$ | $q_1$ | $q_2$ | 1 |
| $q_2$ | $q_2$ | $q_3$ | 0 |
| $q_3$ | $q_3$ | $q_0$ | 0 |

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$
$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{0, 1\}$$
$$\Delta = \{0, 1\}$$
$$\lambda'(q_0) = 0, \lambda'(q_1) = 1, \lambda'(q_2) = 0 \text{ and } \lambda'(q_3) = 0$$

$q_0$ is the initial state. There is no concept of final state in moore machine since we are considering output for the each corresponding state.

We can also viewed it by transition diagram as

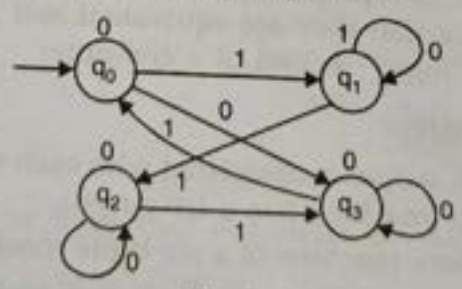



**Fig. 3.1.**

By the analysing moore machine transition diagram of Fig. 3.1. We can easily noticed that it is FA without any final state and output is there for every corresponding state. Output for every state is written just above the corresponding state.

### 3.2.2. Processing of the String through Moore Machine

Let us process the string (input) $w = 0111$ by the moore machine discussed in Fig. 3.1.

$$\xrightarrow{\text{Start}} \overset{0}{q_0} \xrightarrow[\text{Input}]{0} \overset{0}{q_3} \xrightarrow[\text{Input}]{1} \overset{0}{q_0} \xrightarrow[\text{Input}]{1} \overset{1}{q_1} \xrightarrow[\text{Input}]{1} \overset{0}{q_2}$$

The output string will be $w' = \lambda'(q_0)\ \lambda'(q_3)\ \lambda'(q_0)\ \lambda'(q_1)\ \lambda'(q_2)$

$$= 00010$$

$$|\ w\ | = 4$$

$$|\ w'\ | = 5$$

So clearly if the length of input string is $n$ then the length of output state will be $(n + 1)$. Since for the input $\epsilon$ (null string), the output is $\lambda'(q_0) = 0$.

Let us discuss some example for better understand.

**Example 3.1. Design a moore machine which determine the residue mod-3 for each binary string treated as binary integer.**

**Solution.** We have to analyse the problem. First of all you must understand the meaning of residue mod-3 for each binary string treated as binary integer. Binary string means $\Sigma = \{0, 1\}$ is given, so binary string is combination of 0's and 1's. Binary integer means, you must find decimal equivalent of input string as

| *Binary* | *Decimal equivalent* |
|---|---|
| 0000 | 0 |
| 0001 | 1 |
| 0010 | 2 |
| 0011 | 3 |
| 0100 | 4 |
| 0101 | 5 |
| 0110 | 6 |
| 0111 | 7 |
| 1000 | 8 |

residue mod-3 means reminder, when decimal number divided by the 3.

So now things are clear, we have to design a moore machine which prints the reminder, when decimal equivalent of input string is divided by the 3.

So clearly $\Delta = \{0, 1, 2\}$

Since 0, 1 and 2 are the possible reminder when a decimal number is divided by the three. So we need three state in moore machine.

Let moore machine is $M_0$

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$

$$Q = \{q_0, q_1, q_2\}$$

$$\lambda'(q_0) = 0$$

$$\lambda'(q_1) = 1$$

$$\lambda'(q_2) = 2$$

$$\Sigma = \{0, 1\}$$

and $q_0$ is initial state.

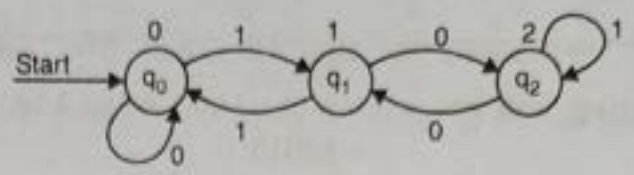



**Fig. 3.2. Transition diagram of $M_0$.**

Let us assume input string is 111, decimal equivalent is 7, 7/3 = 1(reminder). When we pass 111 by $M_0$ it ends at $q_1$ and $\lambda'(q_1) = 1$ which reminder of 7/3.

**Example 3.2. Design a Moore machine which counts the occurrence of substring *aab* in input string.**

**Solution.** Let Moore machine be

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$
$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{a, b\}$$

We will design this machine in such a way that, machine prints out the character 0 except for state $q_3$, which prints a 1. To get $q_3$, we must come from state $q_2$ and have just read '$b$'. To get state $q_2$, we most read at least two $a$'s in a row, having started in any state.

$$\Delta = \{0, 1\}$$
$$\lambda'(q_0) = 0, \lambda'(q_1) = 0, \lambda'(q_2) = 0, \lambda'(q_3) = 1.$$

The following machine will counts *aab* occurrence for us.

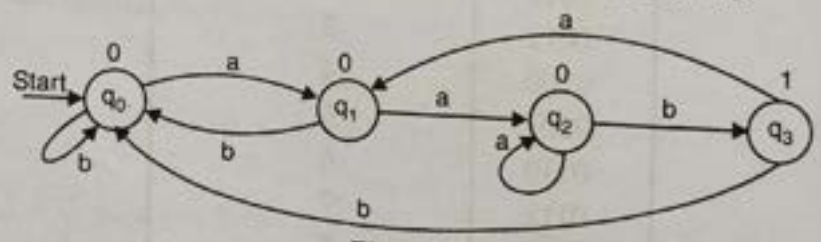



**Fig. 3.3.**

Let us pass string $w = aaab\ baabaa$

$$q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{b} q_0 \xrightarrow{b} q_0 \xrightarrow{a} q_1 \xrightarrow{a} q_2 \xrightarrow{b} q_3 \xrightarrow{a} q_1 \xrightarrow{a} q_2$$

Output string $w' = \lambda'(q_0)\ \lambda'(q_1)\ \lambda'(q_2)\ \lambda'(q_2)\ \lambda'(q_3)\ \lambda'(q_0)\ \lambda'(q_0)\ \lambda'(q_1)\ \lambda'(q_2)\ \lambda'(q_3)\ \lambda'(q_1)\ \lambda'(q_2)$

$= 000010000100$

There are number of 1's are two so occurrence of *aab* will be 2.

**Example 3.3. Construct a moore machine which calculate residue mod-4 for each binary string treated as binary integer.**

**Solution.** This problem can be solved as we solved example 3.1. When we divide any number by 4 then reminder can be 0, 1, 2 and 3, so clearly moore machine will have four state.

Let moore machine is $M_0$

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$
$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{0, 1\}$$
$$\Delta = \{0, 1, 2, 3\}$$

$\lambda'(q_0) = 0$, $\lambda'(q_1) = 1$, $\lambda'(q_2) = 2$ and $\lambda'(q_3) = 3$

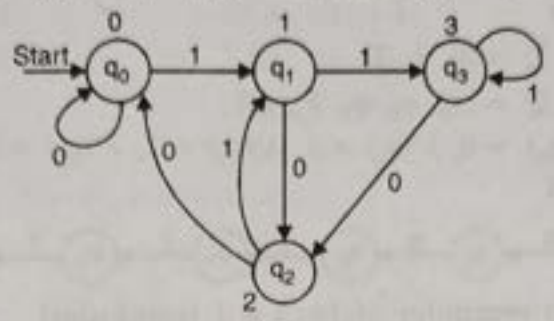



**Fig. 3.4.**

Let us pass the string 11, decimal equivalent is 3 and reminder of 3/4 is 3.

$$q_0 \xrightarrow{1} q_1 \xrightarrow{1} q_3$$

and $\lambda'(q_3) = 3$, which is reminder of 3/4.

**Example 3.4. Give the moore machine for the input from (0 + 1 + 2)* which prints the residue module 5 of the input treated as a ternary (base 3, with digits 0, 1, 2) number.**

**Solution.** First of all, let us analyse the problem, as input is from (0 + 1 + 2)* that input string will be combination of 0, 1 and 2. Like 012, 210, 11102, ... residue module 5 means reminder when a decimal divided by 5. Clearly reminder may be 0, 1, 2, 3, 4. Another important pointed is that conversion of string in two decimal number as

For example input string is 0 1 2 1, then decimal equivalent of it is $w$(say)

$$\begin{array}{c} 0\ 1\ 2\ 1 \\ \downarrow\downarrow\downarrow\downarrow \\ 3\ 2\ 1\ 0 \end{array}$$

$$\begin{aligned} w &= 0 \times 3^3 + 1 \times 3^2 + 2 \times 3^1 + 1 \times 3^0 \\ &= 0 + 9 + 6 + 1 \\ &= 16 \end{aligned}$$

According problem 16/5 = 1 (reminder). So far input 0121 moore machine should print as output, when last character that is 1 is passed through machine.

It should be clear that moore machine will have 5 states since there may be five possible reminder when any number is divided by the 5 possibilities are 0, 1, 2, 3 and 4 transition diagram of this moore machine will be following :

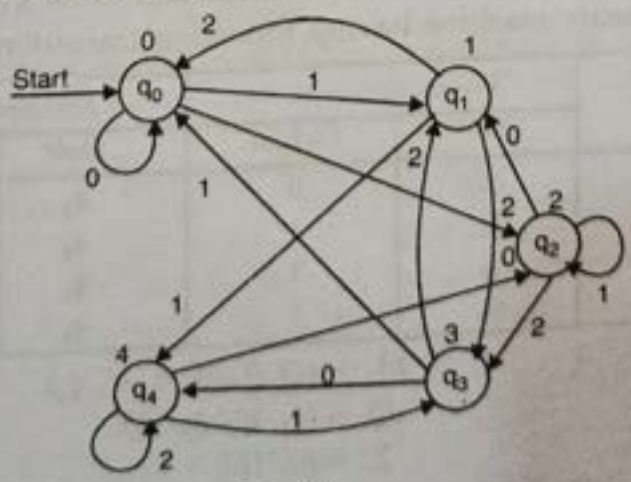


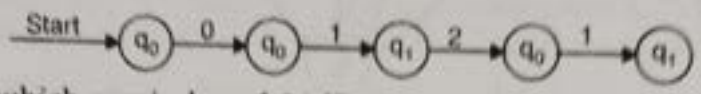

**Fig. 3.5.**

$$M_0 = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$

$q_0$ is initial state

$$\Sigma = \{0, 1, 2\}$$
$$Q = \{q_0, q_1, q_2, q_3, q_4\}$$
$$\lambda'(q_0) = 0, \lambda'(q_1) = 1, \lambda'(q_2) = 2, \lambda'(q_3) = 3, \lambda'(q_4) = 4.$$

Let us pass 0121




and $\lambda'(q_1) = 1$ which reminder of $16/5 = 1$ (reminder)

## 3.3. MEALY MACHINE

It is finite automata in which output is associated with each transition. In mealy machine every transition for a particular input symbol has a fix output.

Mathematically moore machine is a six-tuple machine and defined as

$$M_e = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$

$Q$ : A nonempty finite set of state in $M_e$.

$\Sigma$ : A nonempty finite set of input symbols.

$\Delta$ : A nonempty finite set of outputs.

$q_0$ : initial state of $M_e$.

$\delta$ : It is a transition function which takes two arguments as in moore machine, one is input state and another is input symbol. The output of this function is a single state, so clearly $\delta$ is the function which is responsible for the transition in $M_e$.

$\lambda'$ : is a mapping function which maps $Q \times \Sigma$ to $\Delta$, giving the output associated with each transition.

So $\lambda'$ maps $Q \times \Sigma \rightarrow \Delta$, that is $\lambda'(q, a)$ gives the output associated with the transition from state $q_0$ on input $a$. The output of $M_e$ in response to input $a_1, a_2, a_3 \ldots a_n$ is $\lambda'(q_0, a_1), \lambda'(q_1, a_2) \ldots \lambda'(q_{n-1}, a_n)$ where $q_0, q_1, q_2, \ldots q_n$ is the sequence of states such that $\delta(q_i - 1, a_i) = q_i$ for $1 \le i \le n$. Clearly in this case length of output sequence is $n$.

### 3.3.1. Representation of Mealy Machine

Mealy machine can also be represented by transition table, as well as transition diagram, same as moore machine and finite automata.

Let $M_e$ be mealy machine having following transition table.

| *Present state* | *For input a = 0* | | *For input a = 1* | |
|---|---|---|---|---|
| | *State* | *Output* | *State* | *Output* |
| → $q_0$ | $q_3$ | 0 | $q_2$ | 0 |
| $q_2$ | $q_1$ | 1 | $q_4$ | 0 |
| $q_3$ | $q_2$ | 1 | $q_1$ | 1 |
| $q_4$ | $q_4$ | 1 | $q_3$ | 0 |

$$M_e = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$
$$Q = (q_1, q_2, q_3, q_4)$$
$$\Sigma = \{0, 1\}$$
$$\Delta = \{0, 1\}$$

$q_0$ is initial state and $\delta$ is transition function.

$\lambda'(q_1, 0) = 0$ $\quad$ $\lambda'(q_1, 1) = 0$

$\lambda'(q_2, 0) = 1$ $\quad$ $\lambda'(q_2, 1) = 0$

$\lambda'(q_3, 0) = 1$ $\quad$ $\lambda'(q_3, 1) = 1$

$\lambda'(q_4, 0) = 1$ $\quad$ $\lambda'(q_4, 1) = 0$

Transition diagram for $M_e$ is following :

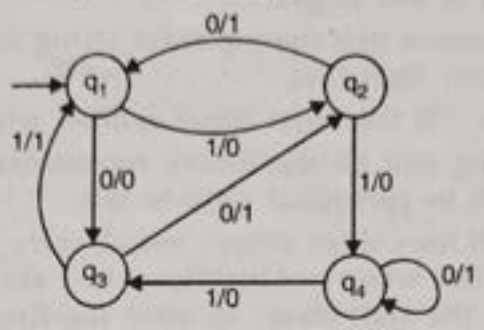



**Fig. 3.6.**

By the analysing mealy machine transition diagram of Fig. 3.6. We can easily noticed that, it is also finite automata without any final state and output is there for input-symbol on corresponding transition. Output for every input symbol is represented as level on each transition in the form input symbol/ corresponding output. For example in Fig. 3.6 arc between $q_1$ and $q_2$ is labelled by 0/1, 0 is input symbol and 1 is output symbol.

### 3.3.2. Processing of the String through Mealy Machine

Let us process the string $w$ = 0111 by the mealy machine, describe in Fig. 3.6.

$$\xrightarrow{\text{Start}} q_1 \xrightarrow{0/1} q_2 \xrightarrow{1/0} q_4 \xrightarrow{1/0} q_3 \xrightarrow{1/1} q_1$$

So output string will be 1001

$$w_1 = \underline{1001}$$
$$|w| = 4$$
$$|w_1| = 4$$

So the length of input string is same as length of output string. It is difference with moore machine.

**Example 3.5. Design a melay machine, which prints 1's complement of input bit string over alphabet $\Sigma = \{0, 1\}$.**

**Solution.** If input string is 101110 then 1's complements of 101110 will be 010001, so we have to design a mealy machine which will point output 010001 for input string 101110.

Let $\quad M_e = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$

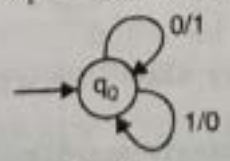



**Fig. 3.7.**

$Q = \{q_0\}$

$\Sigma = \{0, 1\}$ given

$q_0$ is also initial state.

Let us pass input $w = 011$ to $M$,

$$q_0 \xrightarrow{0/1} q_0 \xrightarrow{1/0} q_0 \xrightarrow{1/0} q_0$$

output string is $w' = 100$ which 1's complement of $w = 011$.

**Example 3.6. Design mealy machine, which marks as a increment machine that assumes that its input is a binary number and prints out the binary number that is one larger.**

**Solution.** We assume that the input bit string is a binary number fed is backward, that is, unit digit first.

If input string is 100 then first input symbol will be 0 not 1.

The output string will be the binary representation of the number one greater and also will be generated right to left.

The machine will have three states : initial, carry, no carry. The carry state represents the overflow when two bits equal to 1 are added — we print 0 and we carry a 1. From the start state, we read the first bit. If we read a 0, we print a 1 and we do not owe a carry bit. If we read a 1, we print 0 and we do owe a carry bit. If at any point in the process we are in no-carry (which means that do not owe a carry), we print the next bit just as we read it and remain in no-carry. However if at some points in the process we are in owe-carry, the situation is different. If we read a 0, we print a 1 and go to the no-carry state. If we are in owe-carry and we read a 1, we print 0 and we loop back to owe-carry. The complete picture for this machine is following

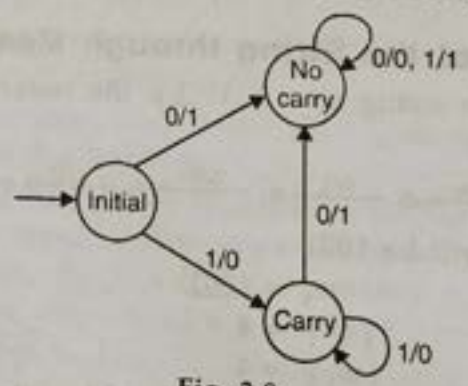
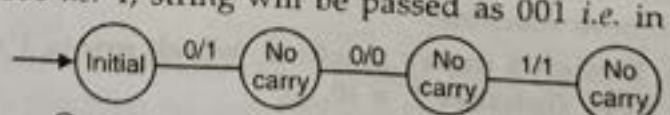


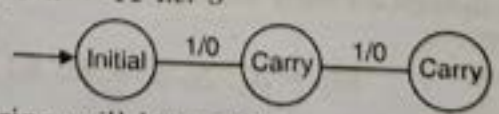

**Fig. 3.8.**

Let us pass 100 *i.e.* 4, string will be passed as 001 *i.e.* in reverse order.




Output string = 101 *i.e.* 5 which 4 + 1

Now input string is $w = 11$ *i.e.* 3




Output string will be = 100

Since process end in "**carry state**" so we put 1 infront 00 and number becomes 100.

**Example 3.7. Construct a mealy machine for regular expression $(0 + 1)^*$ $(00 + 11)$.**

**Solution.** Let us first analyse the problem, check the string set that is language of $(0 + 1)^*$ $(00 + 11)$. It is set of strings either end with 00 or end with 11, like 00, 11, 1011, 010100, .. .

Here we define a three state mealy machine that use it's state to remember the last symbol read, emits output $y$ whenever the current input matches the previous one, and emits '$n$' otherwise. Here $y$ is for the string belongs to regular expression and '$n$' otherwise.

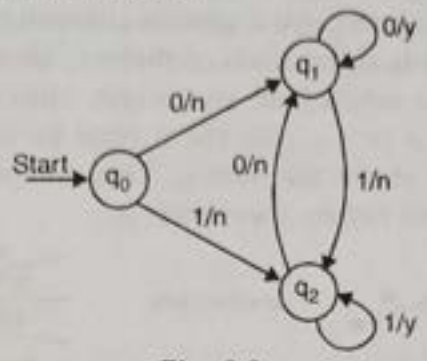



**Fig. 3.9.**

Let us pass input string $w$ through mealy machine construct in Fig. 3.9.

$$w = 01100$$

$$q_0 \xrightarrow{0/n} q_1 \xrightarrow{1/n} q_2 \xrightarrow{1/y} q_2 \xrightarrow{0/n} q_1 \xrightarrow{0/y} q_1$$

When we pass last input symbol '0' from $q_1$ we get y as output so string belongs to set of strings of regular expressions.

**Example 3.8. Construct a mealy machine which calculate residue mod-4 for each binary string treated as binary integer.**

**Solution.** This problem is same as example 3.3 but there we have to design moore machine now for the same logic we want to design mealy machine.

Let mealy machine is

$$M_e = (Q, \Sigma, \Delta, \delta, \lambda', q_0)$$
$$Q = \{q_0, q_1, q_2, q_3\}$$
$$\Sigma = \{0, 1\}$$
$$\Delta = \{0, 1, 2, 3\}$$

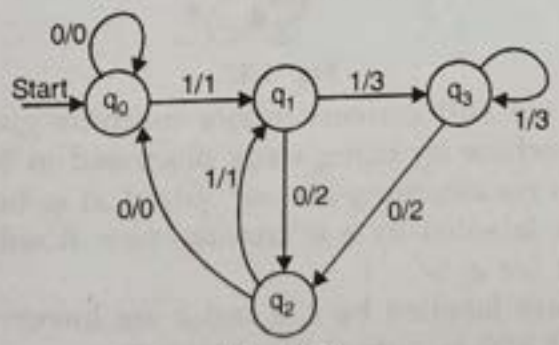



**Fig. 3.10.**

Let us pass the string 11, decimal equivalent of 11 is 3 and reminder of 3/4 is 3.

$$q_0 \xrightarrow{1/1} q_1 \xrightarrow{1/3} q_3$$

So when we pass last then output is 3 which is reminder of 3/4.

## 3.4. EQUIVALENCE OF MEALY AND MOORE MACHINE

Given the mealy machine $M_e$ and the moore machine $M_0$, which prints the automatic start state character '$a$', we will say that these two machines are equivalent if for every input string the output string from $M_0$ is exactly '$a$' concatenated with the output from $M_e$.

### 3.4.1. Mealy Equivalent of given Moore Machine

For every given moore machine $M_0$, there exist a equivalent mealy machine $M_e$. We will learn to convert $M_0$ into $M_e$ in various steps.

**Step 1.** Let $M_0$ is given moore machine, select any particular state of $M_0$, call it $q_6$. It gives instruction to print a certain character, call it $P$. Let us consider all the edges that enter this state. Each of them is labelled with an input letter. Let us change this, now relabel all the edges coming into $q_6$. If they were previously labelled $a$ or $b$ or $c$ ..., let them now be labelled $a/p$ or $b/p$ or $c/p$ ... and let us erase $P$ from above the state $q_6$. This means that we shall be printing a $P$ on the incoming edges before they enter $q_6$.

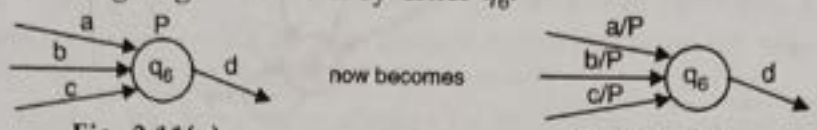



Fig. 3.11(*a*). Fig. 3.11(*b*).

**Step 2.** We leave the outgoing edge from $q_6$ alone. They will be relabelled to print the character as associated with state to which they lead. The outgoing edged in Fig. 3.11(*a*), which is labelled $\delta$ is not change in Fig. 3.11(*b*).

**Step 3.** Repeat step 2 and step 3 for every state $q_0, q_1, q_2$ ... of $M_0$, we turn $M_0$ into melay machine $M_e$.

The symbol that used to printed automatically when the machine started in state $q_0$ is no longer the first output character, but this does not stop the rest of the output string from being the same.

**Example 3.9. Convert given moore machine into equivalent mealy machine.**

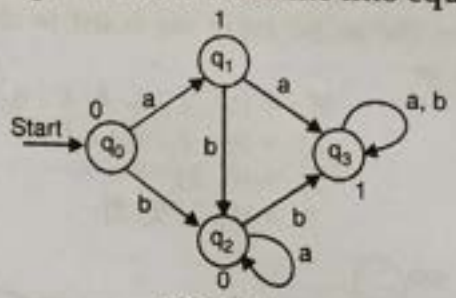



Fig. 3.12.

**Solution.** Now we will convert moore machine given in Fig. 3.12 into equivalent mealy machine by using steps discussed in 3.4.1, as follows :

- at $q_0$ there is no incoming arc, so output at $q_0$ becomes useless
- at $q_1$ one arc labelled by a is coming, now it will be labelled by $a/1$, since output for $q_1$ was 1.
- at $q_2$ three arcs labelled by $b$, $a$ and $b$ are incoming, no they becomes $b/0$, $a/0$ and $b/0$ as output for $q_2$ was 0.

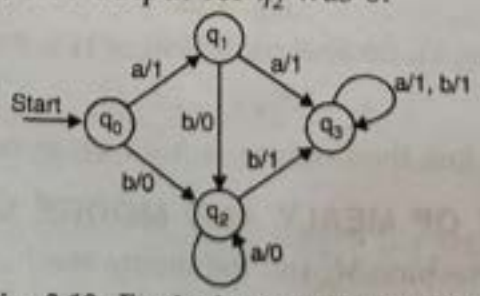



**Fig. 3.13. Equivalent mealy machine.**

- at $q_3$ arcs was labelled by $a$, $b$, $a$ and $b$, now the becomes $a/1$, $b/1$, $a/1$ and $b/1$ since output for $q_3$ was 1.

### 3.4.2. Moore Equivalent of given Mealy Machine

For every given mealy machine $M_e$ there exist a equivalent moore machine $M_0$.

We cannot just do the reverse of previous procedure. If we were try to push the printing instruction from the edge as it is in $M_e$ to the inside of the state as it should be for a moore machine, we might end up with a conflict. We will handle these conflicts in steps, as follows :

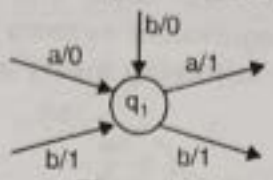



Fig. 3.14.

**Step 1.** Let us consider the situation in which two edges might come into the same state but have different printing instructions, as follows

We need then are two copies of the same state. The edge $a/0$ will go into $q_1^1$ ($q_1$ copy 1, a state that prints a '0') and edge $b/1$ will go into $q_1^2$ ($q_1$ copy 2, a state that print a '1'). The edge labelled $b/0$ will also go into $q_1^1$. Above these states, we include the printing instructions 0 and 1. The arrow coming out of each of these copies of what used to be $q_1$ must be the same as the edges coming out of $q_1$ originally. We get two sets of the output edges each equal to the original outgoing edges, but one set of the original incoming edges is divided between the two copies.

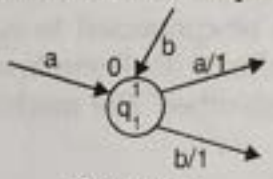



Fig. 3.15.

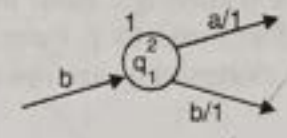



Fig. 3.16.

It can be noticed from Fig. 3.15 and Fig. 3.16 that instruction 0 is printed above $q_1^1$ and instruction 1 is printed above the $q_1^2$.

Repeat the this step for all the states in given mealy machine.

**Step 2.** Another conflict may come on the states, those are having loop. For example

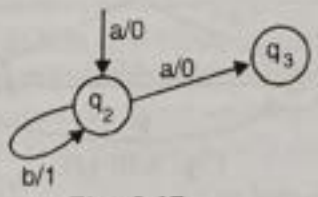



Fig. 3.17.

What happens in the preceding example is that the edge labelled $a/0$ has to enter a version of $q_2$ that prints 0. Let us call $q_2^1$. The loop labelled $b/1$ at $q_3$ has to enter a version of $q_2$ that prints a 1. We call this $q_2^2$. When we enter $q_2$ from edge $a/0$, we enter $q_2^1$, but we must also be able to loop with $b$'s while staying in a $q_2$-like state. Therefore, an edge labelled $b$ must correct $q_2^1$ to $q_2^2$. Because we must be allowed to repeat as many $b$'s as we want, there must be a $b$-loop at state $q_2^2$. Each $b$-loop we go around prints another 1 when it reenters $q_3^2$. As with all such clones of an original state, they must both be corrected to $q_3$ by $a/0$ as shown in Fig. 3.18.

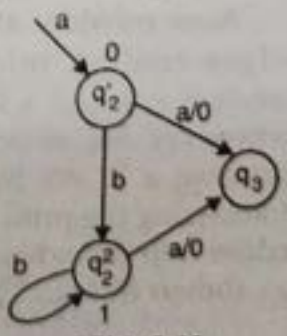



Fig. 3.18.

**Step 3.** If there is ever a state that has no edges entering it, we can assign if any printing instruction we want, even if this state is the start state.

**Step 4.** Let us repeat this process *i.e.* from step 1 to step 3 for each state of $M_e$, $q_0, q_1, q_2, q_3, \ldots$. This will produced $M_0$. If we have to make copies of the start state in $M_e$, we can let any one of them be the start state in $M_0$ because they all give the identical directions for proceeding to other states. Having a choice of start states means that the conversion of $M_e$ into $M_0$ is not unique. We should except this because may $M_e$ is equivalent to more then one $M_0$.

**Example 3.10. Convert the following mealy machine into moore machine.**

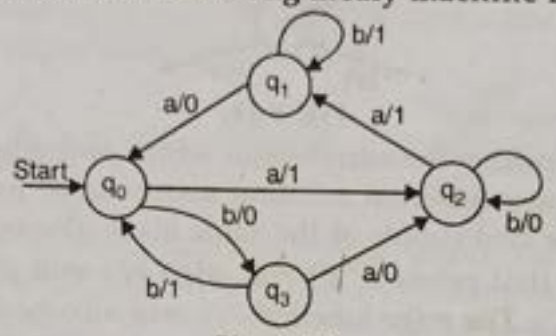



**Fig. 3.19.**

**Solution.** We can begin the conversion process anywhere because the algorithm does not specify the order of replacing states, so let us start with $q_0$.

Two edges comes into this state, one labelled $a/0$ and one labelled $b/1$. Therefore, we need two copies of this state, one that prints a '0' called $q_0'$ and one that print a 1 (called $q_0^2$). Both these states must be connected to $q_2$ through an edge labelled $a/1$ and to $q_3$ through an edge labelled $b/0$. There is no loop at $q_0$, so these two clones of $q_0$ are not connected to each other. The machine becomes,

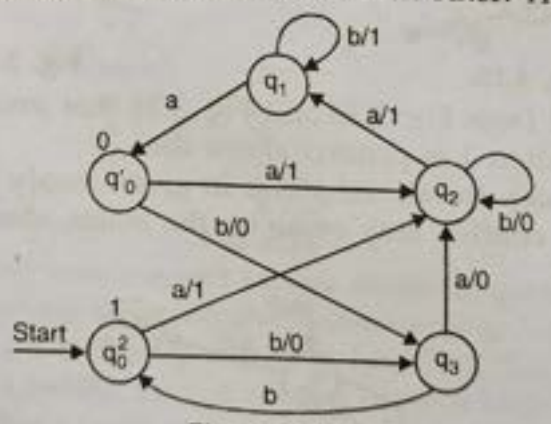



**Fig. 3.20 (*a*).**

Let us select arbitraily $q_0^2$ as start state.

Now consider state $q_1$, has only two edges coming into it : one from $q_2$ labelled $a/1$ and a loop labelled $b/1$. So whenever we enter $q_1$, we are always printing a 1. We have no trouble hour transfering the print instructions from the edges into state. Now machine looks like as shown in Fig. 3.20(*b*).

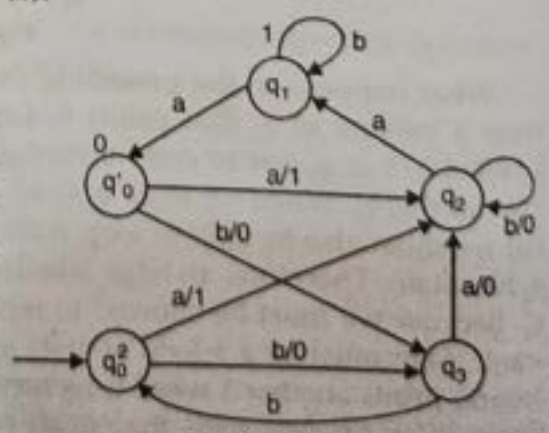



**Fig. 3.20 (*b*).**

Now handle the state $q_3$. Two edges come into it, both labelled $b/0$, so we change the state to $q_3$ (output is 0) and simply edges labels to $b$ alone :

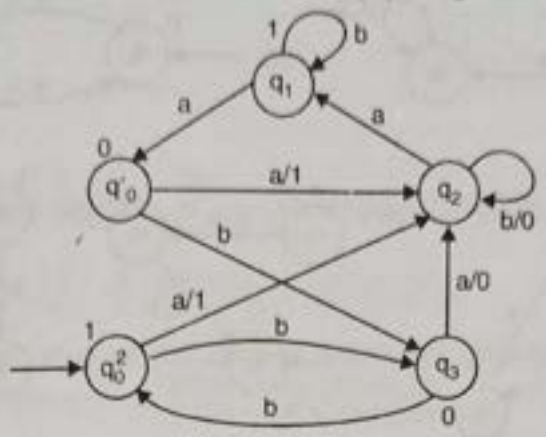



**Fig. 3.20 (c).**

Now consider last state $q_2$. It has some 0-printing edges entering it and some 1-printing edges. Therefore, we must split it into two copies, $q_2^1$ and $q_2^2$. Let us first print a 0 and the second print $a_1$. The two copies will be connected by a $b$-edge going from $q_2^2$ to $q_2^1$ (to print a 0). There will also be a loop at $q_2^1$. The final $M_0$ will be following :

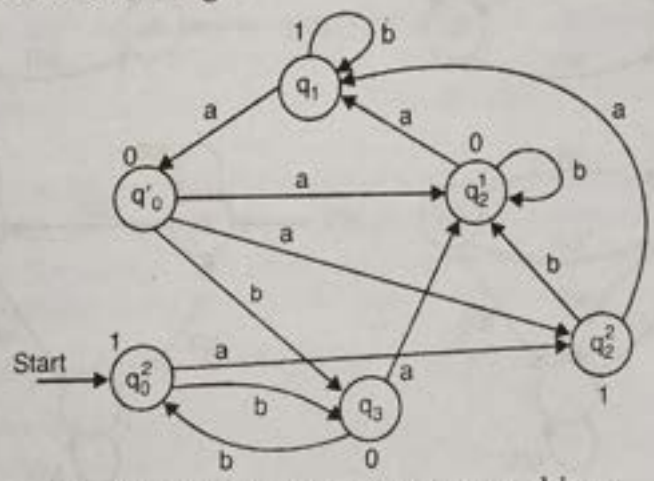



**Fig. 3.20 (d). Equivalent moore machine.**

## EXERCISE

1. Each of the following is a moore machine with alphabet $\Sigma = \{a, b\}$ and output alphabet $\Delta = \{0, 1\}$. Given the transition and output tables. Draw the machines.

(i)

| | $a$ | $b$ | *output* |
|---|---|---|---|
| $q_0$ | $q_1$ | $q_2$ | 1 |
| $q_1$ | $q_1$ | $q_1$ | 0 |
| $q_2$ | $q_1$ | $q_0$ | 1 |

(ii)

| | $a$ | $b$ | *output* |
|---|---|---|---|
| $q_0$ | $q_0$ | $q_2$ | 0 |
| $q_1$ | $q_1$ | $q_0$ | 1 |
| $q_2$ | $q_2$ | $q_1$ | 1 |

(iii)

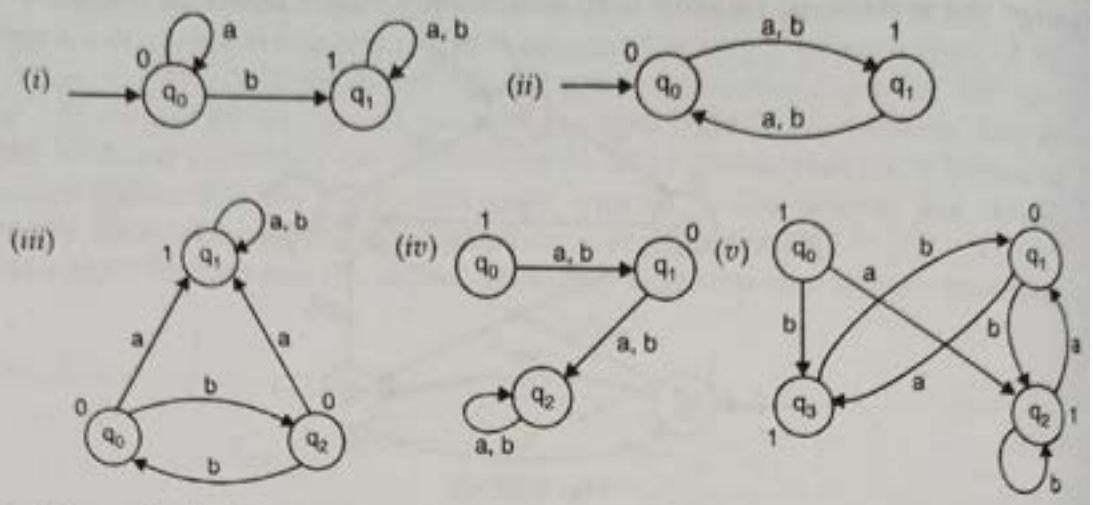
| | $a$ | $b$ | *output* |
|---|---|---|---|
| $q_0$ | $q_3$ | $q_2$ | 0 |
| $q_1$ | $q_1$ | $q_0$ | 0 |
| $q_2$ | $q_2$ | $q_3$ | 1 |
| $q_3$ | $q_0$ | $q_1$ | 0 |

(iv)

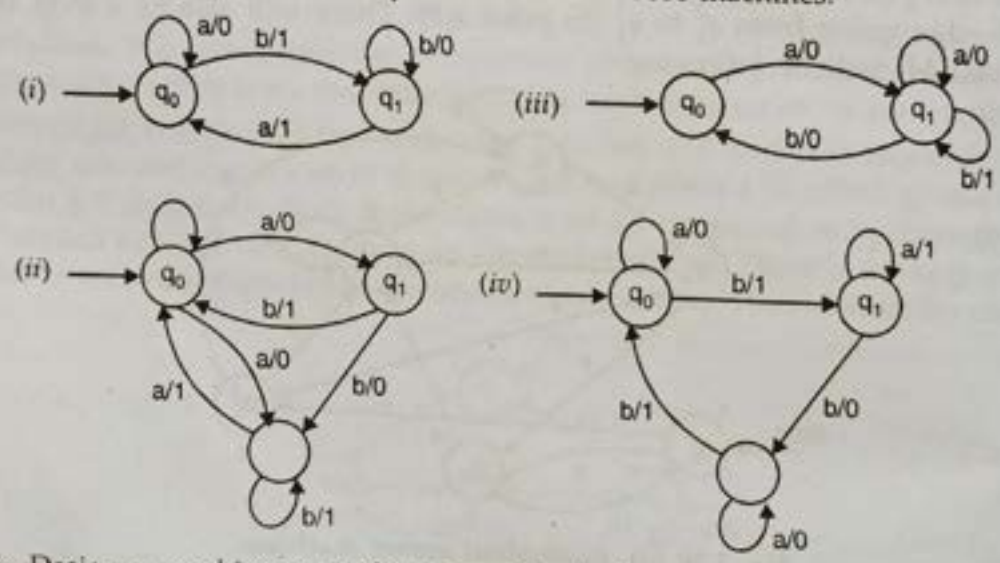
| | $a$ | $b$ | *output* |
|---|---|---|---|
| $q_0$ | $q_1$ | $q_2$ | 0 |
| $q_1$ | $q_2$ | $q_3$ | 0 |
| $q_2$ | $q_3$ | $q_4$ | 1 |
| $q_3$ | $q_4$ | $q_4$ | 0 |
| $q_4$ | $q_0$ | $q_0$ | 0 |

CamScanner

2. Convert the following moore machine into mealy machines :




3. One each of the moore machine in problem 1 and 2, run the input sequence *aabab*. What are their respective outputs ?
4. Convert the following mealy machine into moore machines.




5. Design a machine to perform a parity check on the input string; that is, the output string ends in 1 if the total number of 1-bits in the input string is odd so if the total number of 1-bits in the string is even (the front part of the output string is ignored).
6. Construct a moore machine which converts binary number to octal number.
7. Give the mealys moore machine for the following processes. For input from $(0 + 1)^*$, if the input ends in 101, output *A*; if the input ends in 110 output *B*, otherwise output *C*.
8. Construct a melay machine which can output EVEN(E) ODD(0) according as the total number of 1's encountered is even or odd. The input symbols are 0 and 1.
9. Given a bit string of length '*n*', the shift-left-cyclic operation places the first bit at the end, leaving the rest of the bits unchanged. For example, SLC (100110) = 001101, build a mealy machine with input and output alphabet {0, 1, $} such that for any string *x* when we input the *n* + 1 bits as *x*$, we get output the *n* + 1 bit string $SLC(*x*).

*****

# 4 CHAPTER

# Regular Expression and Languages

**INSIDE THIS CHAPTER**

**4.1.** Regular Expression **4.2.** Comparative Study of Regular Expression, Regular Sets and Finite Automata **4.3.** Construction of FA for Regular Expression **4.4.** Construction of Regular Expression from DFA **4.5.** Algebraic Laws for Regular Expressions

## 4.1. REGULAR EXPRESSION

We have already studied Finite Automata. Now we switch our attention from machine like description of language to an algebraic description : "the regular expression." We shall find that, regular expressions can define exactly the same languages that the various forms of automata describe : the regular languages.

However, regular expression offer something that automata do not : a declarative way to express the strings, we want to accept. Thus regular expression as the input language for many system. For example, lexical-analyser generators, such as Lex or Flex.

### 4.1.1. The Operators of Regular Expression

Before describing the regular notation, we need to learn the three operations on languages that operators of regular expression represent. These operations are :

1. The union of two languages $L_1$ and $L_2$, denoted $L_1 \cup L_2$, is the set of string that are in either $L_1$ or $L_2$, or both.
   For example, if $L_1 = \{001, 10, 111\}$ and $L_2 = \{\epsilon, 001\}$ then $L_1 \cup L_2 = \{\epsilon, 10, 001, 111\}$.
2. The concatenation of languages $L_1$ and $L_2$ is a set of strings that can be formed by taking any string in $L_1$ and concatenating it with any strings in $L_2$. We denote concatenation of languages with a dot.
   For example, if $L_1 = \{001, 10, 111\}$ and $L_2 = \{\epsilon, 001\}$, then $L_1 . L_2 = \{001, 10, 111, 001001, 10001, 111001\}$.
3. The closure (or star, or kleene closure) of a language $L$ is denoted by $L^*$ and represents the set of those strings that can be formed by taking any number of strings from $L$, possibly with repetitions (that is same string can be repeated more then once) and concatenating all of them. For example, $L = \{0, 1\}$, then $L^*$ is all strings of 0's and 1's including null string.
   More formally, $L^*$ is the infinite union $U_i \geq L^i$, where $i \geq 0$.

4. The positive closure of a language $L$ is denoted by $L^+$ and represent the set of those strings that can be formed by taking any number of strings from $L$, possibly with repetitions and concatenating all of them, excluding null string *i.e.*

$$L^+ = L^* - \epsilon$$

More formally, $L^+$ is the infinite union $U_i \geq 1\ L^i$, where $i > 0$.

### 4.1.2. Definition of Regular Expression

The set of regular expression is defined by the following rules :

1. Every letter of $\Sigma$ can be made into a regular expression, null string $\epsilon$ itself is a regular expression.
2. If $r_1$ and $r_2$ are regular expression, then
   (*i*) $(r_1)$ (*ii*) $r_1 r_2$
   (*iii*) $r_1 + r_2$ (*iv*) $r_1^*$
   (*v*) $r_1^+$ are also regular expression.
3. Nothing else is regular expression.

### 4.1.3. Building Regular Expression

The algebra of regular expressions follows this path using constants and variables that denote languages, and operators for three operations of union, dot and star. We can describe the regular expression recursively, as follows :

In this definition, we not only describe what are the legal regular expression, but for each regular expression $r$, we describe the language, it represents, which we denote as $L(r)$. We can consider some facts :

1. The constants $\epsilon$ (null string) and $\phi$ (empty set) are regular expression, denoting the languages $\{\epsilon\}$ and $\phi$, respectively.

That is, $L(\epsilon) = \{\epsilon\}$, and $L(\phi) = \phi$.

2. If $a$ is any symbol, then $a$ is regular expression. This expression denotes the language $\{a\}$. That is $L(a) = \{a\}$.

3. A variable, usually capitalized and italic such as $L$ is a variable, representing any language.

### 4.1.4. Inductive Step

There are four parts to the inductive step, one for each of three operators and one for the introduction of parantheses.

1. If $r_1$ and $r_2$ are regular expressions, then $r_1 + r_2$ is a regular expression denoting the union of $L(r_1)$ and $L(r_2)$.

That is, $L(r_1 + r_2) = L(r_1) \cup L(r_2)$

2. If $r_1$ and $r_2$ are regular expression, then $r_1 r_2$ is a regular expression denoting the concatenating of $L(r_1)$ and $L(r_2)$

That is, $L(r_1 r_2) = L(r_1) \,.\, L(r_2)$

3. If $r$ is a regular expression then $r^*$ is a regular expression denoting the closure of $L(r)$. That is,

$$L(r^*) = (L(r))^*$$

4. If $r$ is a regular expression, then $(r)$, a parenthesized $r$, is also a regular expression, denoting the same language as $r$.

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Formally, $L((r)) = L(r)$.

Now let us discuss some examples of building regular expression for better understanding.

**Example 4.1. Write the regular expression over alphabet $(a, b, c)$ containing at least one $a$ and at least one $b$.**

**Solution.** Let us first analyse the language of regular expression.

Problem is very simple and the language will have set of strings, like *ab*, *abbbb*, *abab*, *aaaab*, *baaaa* .... . So strings are $\underline{1}\ a\ \underline{2}\ b\ \underline{3}$, any combination of $a$'s, $b$'s and $c$'s is possible at the places 1, 2, and 3 any combination of $a$, $b$, $c$ means $(a + b + c)^*$.

So regular expression is

$$r = (a + b + c)^* a(a + b + c)^* b(a + b + c)^* + (a + b + c)^* b(a + b + c)^* a(a + b + c)^*$$

If $r = r_1 + r_2$ (say)

then minimum string of $r_1$ is $ab$ and minimum string of $r_2$ is $ba$.

**Example 4.2. Write the regular expression for the set of strings of 0's and 1's whose tenth symbol from the right end is 1.**

**Solution.** $\Sigma = \{0, 1\}$ given the set of strings, according to the required regular expression are 111000000000, 100I1010101010 .... . So clearly we concern about only tenth position and it should be 1, rest 9 places from the right are either 0 or 1. Similarly positions 11th onward from right end may be either 0 or 1.

So regular expression is

$$r = (0 + 1)^* 1(0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)(0 + 1)$$

$$r = (0 + 1)^* 1(0 + 1)^9.$$

**Example 4.3. Write the regular expression for the set of strings of an equal number of 0's and 1's such that in every prefix, the number of 0's differs from the number of 1's by at most 1.**

**Solution.** Clearly the set of strings is having strings like 1010101010, 10101010, 0110, 1001 ...

So regular expresion will be

$$r = (01 + 10)^*.$$

**Example 4.4. Write a regular expression for the set of strings of 0's and 1's not containing 101 as a substring.**

**Solution.** When we analyse the set of strings then it becomes clear that string may start with '0' and 0's may be repeated, wherever one is encountered then no single 0 must follow it so strings are like 0000010, 00000111111, 1001001 ....

Regular expression is $r = (0^*1^*00)^* 0^* 1^*$.

**Example 4.5. Write the regular expression over alphabet $\{a, b\}$ for the set of strings with even number of $a$'s followed by odd number of $b$'s that is for the language**

$$L = \{a^{2n} b^{2m+1} : n \geq 0, m \geq 0\}.$$

**Solution.** Clearly every string of language $L$ will start with even number of $a$'s. String should end with odd number of $b$'s.

Regular expression is $r = (aa)^* (bb)^* b$.

**Example 4.6. Write the regular expression for the language**

$$L = \{w \in \{0, 1\}^* : w \text{ has no pair of consecutive zeros}\}.$$

**Solution.** Clearly whenever a 0 occurs, it must be followed by a 1. Such a substring may be proceeded and followed by any number of 1's, that is $(1^*011^*)^*$. But strings ending in 0 or consisting of all 1's are unaccounted here, that is $1^* (0 + \epsilon)$.

So regular expression is $r_1 = (1^*011^*)^* (0 + \epsilon) + 1^*(0 + \epsilon)$

Another possible solution is $r_2 = (1 + 01)^* (0 + \epsilon)$

**"Although the two expression look different, both answers are correct, as they denote the same language. Generally, there are an unlimited number of regular expressions for any given language."**

**Example 4.7. Write the regular expression for the language**

$$L = \{a^n b^m / n \geq 4, m \leq 3\}$$

**Solution.** Language contain the set of strings such that every string start with atleast 4 $a$'s and end at the most 3 $b$'s.

So regular expression will be $r = a^4 a^* (\epsilon + b + b^2 + b^3)$.

**Example 4.8. Write the regular expression for the language**

$$L = \{a^n b^m : (n + m) \text{ is even}\}$$

**Solution.** We know $(n + m)$ can be even in two cases

**Case 1.** $n$ and $m$ both are even.

**Case 2.** $n$ and $m$ both are odd.

Let regular expression for case 1 is $r_1$ then

$$r_1 = (aa)^* (bb)^*$$

Let regular expression for case 2 is $r_2$ then

$$r_2 = (aa)^* a (bb)^* b$$

Let regular expression for language $L$ is $r$ then

$$r = r_1 + r_2$$

$$r = (aa)^* (bb)^* + (aa)^* a (bb)^* b.$$

**Example 4.9. Write the regular expression for the language**

$$L = \{ab^n w : n \geq 3, w \in (a, b)^+\}$$

**Solution.** Clearly every string of language $L$ starts with a followed by atleast three $b$'s, followed by atleast one $a$ or one $b$, that is strings are like $ab^3a$, *abbbbbba, abbbbbbbb, abbbb....a*

So regular expression is : $r = ab^3b^* (a + b)^+$

Here + is a positive closure *i.e.* $(a + b)^+ = (a + b)^* - \epsilon$.

**Example 4.10. Write the regular expression for the language**

$$L = \{w : | w | \bmod 3 = 0\}, w \in (a, b)^*\}$$

**Solution.** Let us first understand the meaning of $| w | \bmod 3 = 0$, when length of string belongs to the language $L$, is divided by 3 then reminder should be 0, that is length of $w$ must be 0, 3, 6, 9 .... Clearly multiple of three.

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Regular expression is : $r = ((a + b)^3)^*$

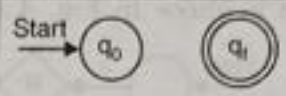
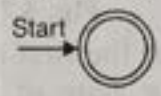
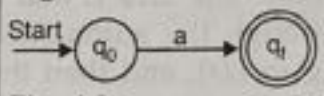
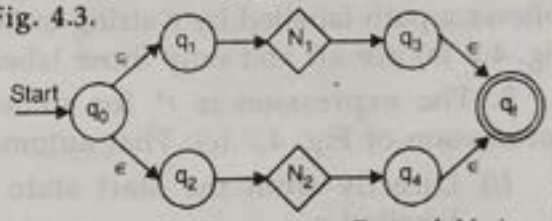
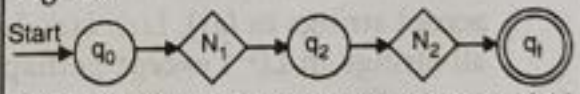
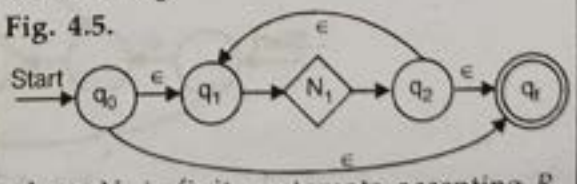
**Example 4.11. Write the regular expression for the language**

$$L = \{w \in (a, b)^* : n_a(w) \bmod 3 = 0\}$$

**Solution.** $n_a(w) \bmod 3 = 0$ means, number of $a$'s in string should be 0, 3, 6, 9 ... .

So regular expression is : $r = (b^*ab^*ab^*ab^*)^*$

## 4.2. COMPARATIVE STUDY OF REGULAR EXPRESSION, REGULAR SETS AND FINITE AUTOMATA

| *Regular expression* | *Regular set* | *Finite automata* |
|---|---|---|
| $\phi$ | $\{\ \}$ |   **Fig 4.1.** $q_f$ **: is the final state.** |
| $\epsilon$ | $\{\epsilon\}$ |   **Fig. 4.2.** |
| Every $a$ in $\Sigma$ is a regular expression. | $\{a\}$ |   **Fig. 4.3.** |
| If $r_1$ and $r_2$ are regular expression then $(r_1 + r_2)$ or $r_1/r_2$ is regular expression. | $R_1 \cup R_2$ (where $R_1$ and $R_2$ are regular sets corresponding to $r_1$ and $r_2$ respectively). |   Where $N_1$ is FA accepting $R_1$ and $N_2$ is a finite automata accepting $R_2$. **Fig. 4.4.** |
| $r_1r_2$ is a regular expression. | $R_1R_2$ (where $R_1$ and $R_2$ are regular sets corresponding to $r_1$ and $r_2$ respectively). |   Where $N_1$ is finite automata for $R_1$ and $N_2$ is FA for $R_2$. **Fig. 4.5.** |
| $r^*$ is a regular expression if $r$ is regular expression. | $R^*$ (where $R$ is a regular set corresponding to $r$). |   where $N_1$ is finite automata accepting $R$. **Fig. 4.6.** |

## 4.3. CONSTRUCTION OF FA FOR REGULAR EXPRESSION

We can construct NFA with $\epsilon$ transmission by considering following four facts.

1. The expression is $r + s$ for some smaller expression $r$ and $s$. The following automation serves, where $R$ is automation for $r$ and $S$ is automation for $s$. That is, starting at the new start state, we can go to the start state of either the automation for $r$ or the automation for $S$. We can reach the accepting state of one of these automata, following a path labelled by some string in $L(r)$ and

$L(s)$, respectively. Once we reach the accepting state of the automation for $r$ or $s$, we can follow one of the $\epsilon$-arcs to the accepting state of the new automation. The language for the automation in Fig. 4.7 ($a$) is $L(r) \cup L(s)$.

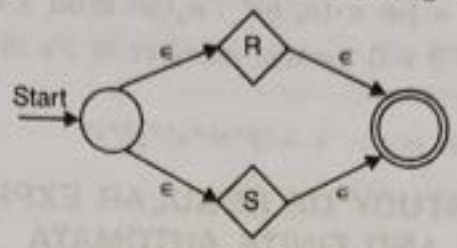



**Fig. 4.7 ($a$).**

2. The expression $rs$ for some smaller expression $r$ and $s$. The automation for the concatenation is show in Fig. 4.7 ($b$).

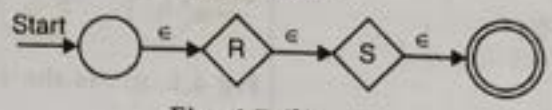



**Fig. 4.7 ($b$).**

Note that the start state of the first automation becomes the start state of the whole, and the accepting state of the second automation becomes the accepting of the whole. The idea is that the only paths from start to accepting state to go first through the automation of $r$, where it must follow a path labelled by a string in $L(r)$, and then through the automation for $s$, where it follows a path labelled by a string in $L(s)$. Thus, the path in the automation of Fig. 4.7 ($b$) are all and only those labelled by strings in $L(R)$ $L(s)$.

3. The expression is $r^*$ for some smaller expression $r$. Then we use automation of Fig. 4.7 ($c$). That automation allow us to go either :

($i$) Directly from the start state to the accepting state along a path labelled $\epsilon$.

($ii$) To the start state of automation for $r$, through that automation one or more times, and then to accepting state. This set of paths allow us to accept strings in $L(r)$, $L(r)$, $L(r)$, $L(r)$, $L(r)$, $L(r)$, and so on, thus covering all strings in $L(r^*)$ except perhaps $\epsilon$ which was covered by the direct arc to the accepting state.

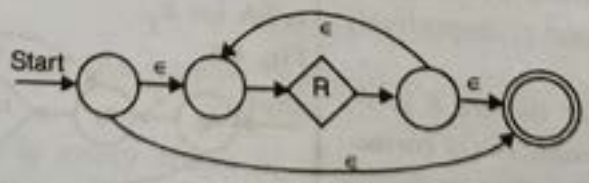



**Fig. 4.7 ($c$).**

where $R$ is automation for $r$.

4. The expression is ($r$) for the smaller expression $r$. The automation for $r$ also serves as the automation for ($r$), since the parentheses do not change the language defined by the expression.

**Example 4.12. One the basis of above discussion find the automation for regular expression $a \,.\, (a + b)^* \,.\, b \,.\, b$.**

**Solution.** Let us solve this problem step by step :

**Step 1.** The basic regular expression involved are $a$ and $b$, we start with automation for $a$ and automation for $b$. Since brackets are evaluated first. We

first constructed automation for $(a + b)$, using automation for $a$ and automation $b$ shown in Fig. 4.8($a$)

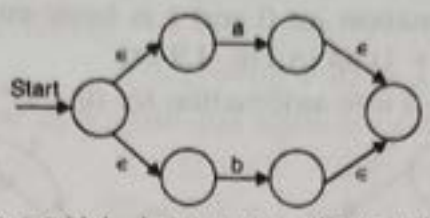



**Fig. 4.8($a$). Automation for $(a + b)$.**

**Step 2.** Since closure is required to take next, we construct automation for $(a + b)^*$ using automation for $(a + b)$ as shown in Fig. 4.8($b$).

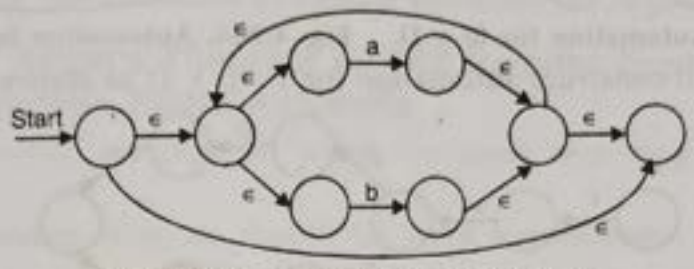



**Fig. 4.8($b$). Automation for $(a + b)^*$**

**Step 3.** Next we construct the automation for $a \,.\, (a + b)^*$ as shown in Fig. 4.8 ($c$).

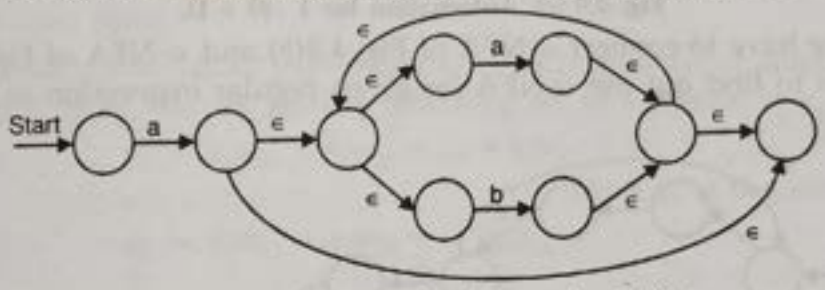



**Fig. 4.8($c$). Automation for $a \,.\, (a + b)^*$.**

**Step 4.** Next we construct the automation for $a \,.\, (a + b)^* \,.\, b$ by using automation for $a \,.\, (a + b)^*$.

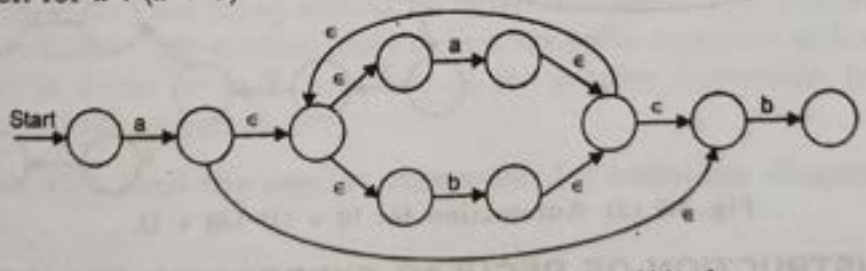



**Fig. 4.8($d$). Automation for $a \,.\, (a + b)^* \,.\, b$.**

**Step 5.** Now finally we can construct automation for $a \,.\, (a + b)^* \,.\, b \,.\, b$ by using automation for $a \,.\, (a + b)^* \,.\, b$.

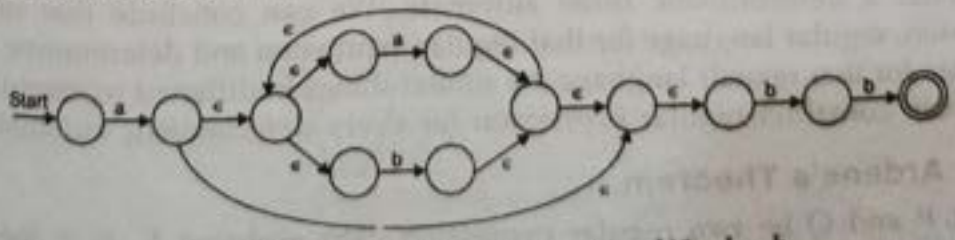



**Fig. 4.8($e$). Automation for $a \,.\, (a + b)^* \,.\, b \,.\, b$.**

It is required automation for given regular expression.

**Example 4.13. Construct the ∈-NFA for the regular expression (0 + 1)* 1 (0 + 1).**

**Solution.** Here automation for 0 and 1 is basic automation. First we will make automation for (0 + 1) as in Fig. 4.9 (*a*).

Now we will convert it into automation for (0 + 1)* as shown in Fig. 4.9(*b*).

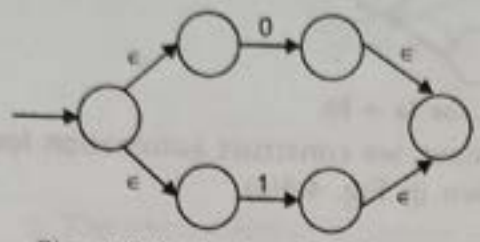



**Fig. 4.9(*a*). Automation for (0 + 1).**

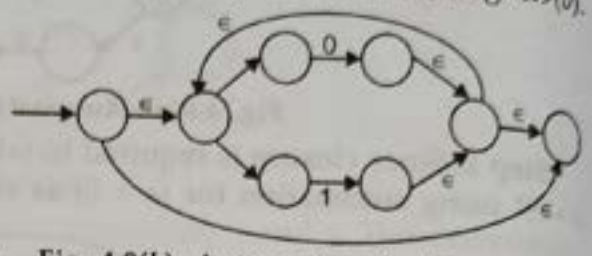



**Fig. 4.9(*b*). Automation for (0 + 1)*.**

Now we will construct automation for 1 . (0 + 1) as shown in Fig. 4.9(*c*).

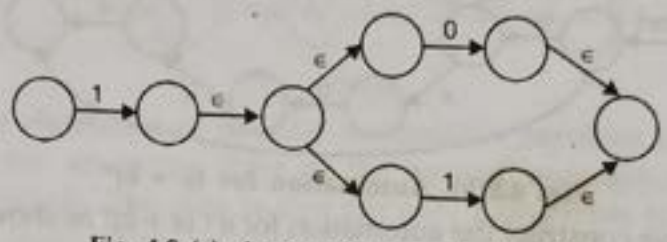



**Fig. 4.9 (*c*). Automation for 1 . (0 + 1).**

Now we have to connect ∈-NFA of Fig. 4.9(*b*) and ∈-NFA of Fig. 4.9 by a ∈-transition to find out the ∈-NFA for given regular expression as shown in Fig. 4.9(*d*).

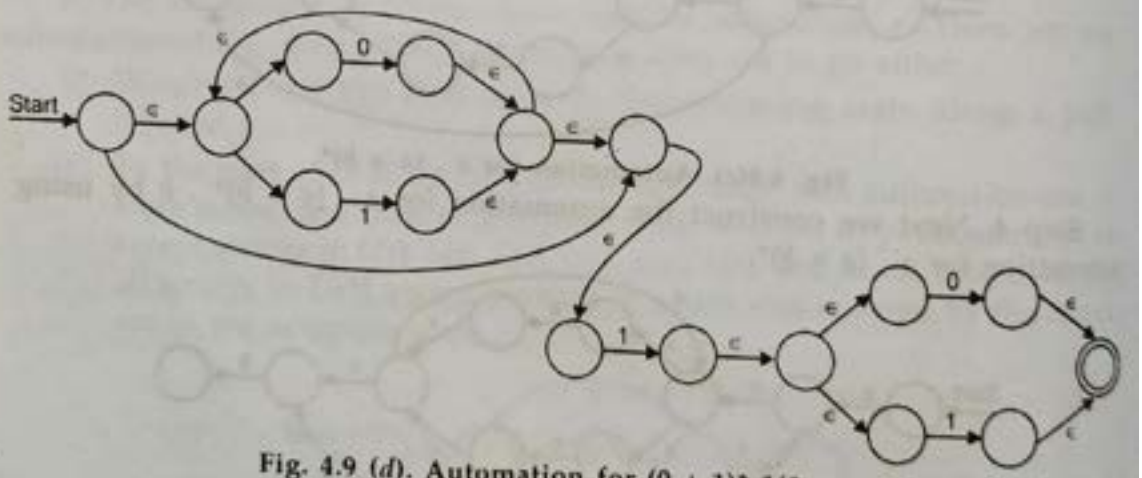



**Fig. 4.9 (*d*). Automation for (0 + 1)* 1(0 + 1).**

## 4.4. CONSTRUCTION OF REGULAR EXPRESSION FROM DFA

Now it is clear that regular expression is mathematical expression for a given regular language. We have already know that for every regular language there exist a deterministic finite automate. We can conclude that regular expression, regular language for that regular expression and deterministic finite automata for that regualr language are similar things in different representation. So we can construct regular expression for every deterministic automata.

### 4.4.1. Ardens's Theorem

Let $P$ and $Q$ be two regular expression over alphabet $\Sigma$. If $P$ does not contain null string $\in$, then

$$R = Q + RP$$

has a unique solution that is $R = QP^*$

It can be understand as : $R = Q + RP$

Put the value of $R$ in R.H.S.

$$R = Q + (Q + RP)P = Q + QP + RP^2$$

When we put the value of $R$ again and again we got the following equation.

$$R = Q + QP + QP^2 + QP^3 \ldots$$
$$R = Q(1 + P + P^2 + P^3 \ldots$$
$$R = Q(\epsilon + P + P^2 + P^3 + \ldots$$
$$R = QP^*$$

(By the definition of closure operation for regular expression.)

### 4.4.2. Use of Arden's Theorem to find Regular Expression of a Deterministic Finite Automata

There are certain assumptions which are made regarding the transition system :

(*i*) The transition diagram should not have $\epsilon$-transitions.

(*ii*) It must have only a single initial state.

(*iii*) It vertices are $q_1 \ldots q_n$.

(*iv*) $q_i$ is final state.

(*v*) $w_{ij}$ denotes the regular expression representing the set of labels of edges from $q_i$ to $q_j$. We can get the following set of equation in $q_1 \ldots q_n$.

$$q_1 = q_1 w_{11} + q_2 w_{21} + \ldots + q_n w_{n1} + \epsilon \quad \ldots(1)$$

(since $q_1$ is the initial state)

$$q_2 = q_1 w_{12} + q_2 w_{22} + \ldots q_n w_{n2} \quad \ldots(2)$$

$$\vdots \qquad\qquad \vdots$$

$$q_n = q_1 w_{1n} + q_2 w_{2n} + \ldots + q_n w_{nn} \quad \ldots(n)$$

We solve these equation for $q_i$ in terms of $w_{ij}$'s and it will be required regular expression. One thing should be noted that we add $\epsilon$ (null string) in the equation starts with starting state $q_1$ and we solve equation to find out $q_i$ (final state) in terms of $w_{ij}$'s, it is one of the regular expression for given deterministic finite automata.

**Example 4.14. Find the regular expression for transition diagram given in Fig. 4.11.**

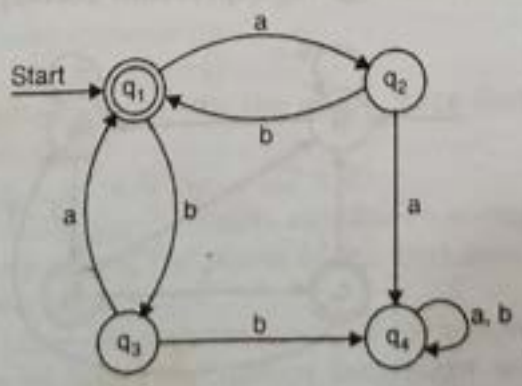



**Fig. 4.11.**

**Solution.** Now let us from the equations :

$$q_1 = q_2 b + q_3 a + \epsilon$$
$$q_2 = q_1 a$$
$$q_3 = q_1 b$$
$$q_4 = q_2 a + q_3 b + q_4 a + q_4 b$$

Put $q_2$ and $q_3$ in $q_1$ as

$$q_1 = q_1 ab + q_1 ba + \epsilon$$
$$q_1 = \epsilon + q_1(ab + ba)$$
$$q_1 = \epsilon(ab + ba)^*$$

So required regular expression is $(ab + ba)^*$.

**Example 4.15. Construct a regular expression corresponding to the state diagram given in Fig. 4.12.**

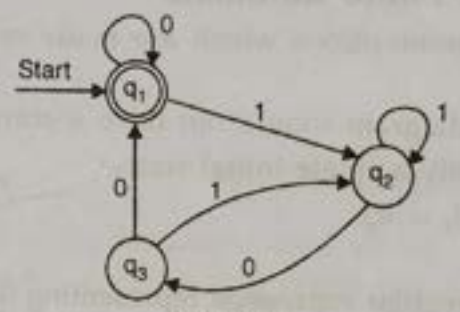



**Fig. 4.12.**

**Solution.** Let us form the equations :

$$q_1 = q_1 0 + q_3 0 + \epsilon$$
$$q_2 = q_1 1 + q_2 1 + q_3 1$$
$$q_3 = q_2 0$$
$$q_2 = q_1 1 + q_2 1 + (q_2 0)\, 1 = q_1 1 + q_2(1 + 01)$$
$$q_2 = q_1\, 1\, (1 + 01)^*$$

So $q_1 = q_1 0 + q_3 0 + \epsilon = q_1 0 + q_2 00 + \epsilon = q_1 0 + (q_1 1\ (1101)^*)00 + \epsilon$

$$q_1 = q_1(0 + 1(1 + 01)^*00) + \epsilon$$
$$q_1 = \epsilon(0 + 1(1 + 01)^*00)^*$$
$$q_1 = (0 + 1(1 + 01)^*\, 00)^*$$

So regular expression is $(0 + 1(1 + 01)^*\, 00)^*$.

**Example 4.16. Find the regular expression corresponding to Fig. 4.13.**

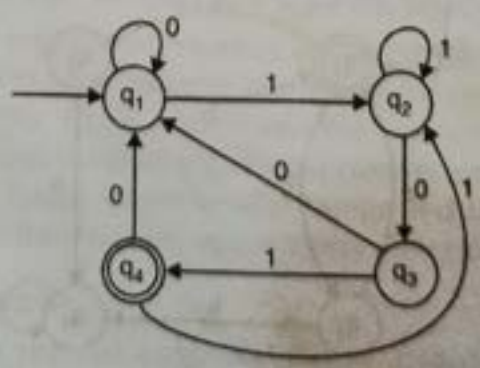



**Fig. 4.13.**

**Solution.** Now let us from the equations

$$q_1 = q_1 0 + q_3 0 + q_4 0 + \epsilon$$
$$q_2 = q_1 1 + q_2 1 + q_4 1$$
$$q_3 = q_2 0$$
$$q_4 = q_3 1$$

Now $q_4 = q_3 1 = q_2 01$

Using $q_2$ equation, we get $q_2 = q_1 1 + q_2 1 + q_2 011 = q_1 1 + q_2(1 + 011)$

$$q_2 = q_1 1(1 + 011)^* = q_1(1(1 + 011)^*)$$

Put $q_3$ and $q_4$ in $q_1$ equation $q_1 = q_1 0 + q_2 00 + q_2 010 + \epsilon$

$$= q_1 0 + q_2(00 + 010) + \epsilon$$
$$= q_1 0 + q_1 1(1 + 011)^* (00 + 010) + \epsilon$$
$$q_1 = \epsilon(0 + 1(1 + 011)^* (00 + 010))^*$$

Put $q_2$ in $q_4$ equation $q_4 = q_1 1(1 + 011)^* 01 = q_1(1(1 + 01)^* 01)$

$$q_4 = (0 + 1(1 + 011)^* (00 + 010))^* (1(1 + 011)^* 01)$$

So required regular expression is

$$(0 + 1(1 + 011)^* (00 + 010))^* (1(1 + 011)^* 01)$$

## 4.5. ALGEBRAIC LAWS FOR REGULAR EXPRESSIONS

In this section, we will see a collection of algebraic laws that bring the issue of when two regular laws are equivalent. Instead of examining specific regular expressions, we shall consider pairs of regular expressions with variables as arguments. Two expressions with variables are equivalent if whatever languages we substitute for the variables, the result of the two expressions are the same language.

Like arithmatic expressions, the regular expressions have a number of laws that work for them. Many of these are similar to the laws for arithmatic, if we think of union as addition and concatenation as multiplication. However, there are a few places where the analogy breaks down, and there are also some laws that apply to regular expressions but have no analog for arithmatic, especially when the closure operator is involved. The next sections from a catalog of the major laws.

### 4.5.1. Associativity and Commutativity

Commutativity is the property of an operator that says that we can switch the order of its operands and get the same result. For example, the associative law of multiplication is $(x \times y) \times z = x \times (y \times z)$. Here are three laws of these types that hold for regular expression :

$$P + Q = Q + P.$$

This law, the commutative law for the union, says that we take union of two languages in either order.

$$(P + Q) + R = P + (Q + R)$$

This law, the associative law for union says that we may take the union of three languages either by taking the union of the first two initially, or taking the union of the last two initially.

Note that, together with the commutative law for union, we conclude that we can take the union of any collection of languages with any order and grouping, and the result will be the same. Intuitively, a string is in $L_1 \cup L_2 \cup \dots + L_K$ if and only if it is in one or more of the $L_i$'s.

$$(PQ)R = P(QR)$$

This law, the associative law for concatenation, says that we can concatenate three languages by concatenating either for first two or the last two initially.

### 4.5.2. Identities

An identity for an operator is a value such that when the operator is applied to identity and some other value, the result is the same value. For instance, 0 is the identity for addition, since $0 + m = m + 0 = m$, and 1 is the identity for the multiplication, since $1 \times x = x \times 1 = x$. There are three laws for regular expression involving these concepts; we list them below.

- $\phi + L = L + \phi = L$. This law assets that $\phi$ is the identity for union.
- $\epsilon L = L\epsilon = L$. This law assets that $\epsilon$ is the identity for concatenation.
- $\phi L = L\phi = \phi$. This law assets that $\phi$ is the annihilater for concatenation.

### 4.5.3. Distributed Law

A distributed law involves two operators, and assets that one operator can be pushed down to be applied to each argument of the operator individually. The most common example is the distributed law of multiplication over addition, that is

$$x \times (y + z) = x \times y + x \times z$$

However, there is an analogous law for regular expression, that we most state in two forms. Since concatenation is not commutative. These laws are:

- $P(Q + R) = PQ + PR$. This law, is the left distributed law of concatenation over union.
- $(Q + R)P = QP + RP$. This law, is the right distributed law of concatenation over union.

### 4.5.4. The Idempotent Law

An operator is said to be idempotent if the result of applying it to two of the same values as arguments is that value. The common arithmatic operators are not idempotent, $x + x \neq x$ in general and $x \times x \neq x$ in general (although there are some values of $x$ for which the equality holds, such as $0 + 0 = 0$). However, union and intersection are common examples of idempotent operators. Thus for the regular expression, we may assert the following law:

- $L + L = L$. This law, the idempotent law for union, states if we take the union of two identical expression, we can replace them by one copy of the expression.

### 4.5.5. Laws for Closure

There are number of laws involving the closure operators. We shall list them here, and give some explanation for why they are true.

1. $(L^*)^* = L^*$. This law says that closing an expression that is already closed does not change the language. The language of $(L^*)^*$ is all strings created for concatenating strings in the language $L^*$. But those strings are themselves composed of strings from $L$. Thus, the string in $(L^*)^*$ is also a concatenation of strings from $L$ and therefore in the language of $L^*$.
2. $\phi^* = \epsilon$. The closure of $\phi$ contains only the string $\epsilon$.
3. $\epsilon^* = \epsilon$. It is easy to check that only string that can be formed by concatenating any number of copies of the empty string is the empty string itself.

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4. $L^+ = LL^* = L^*L$. Recall that positive closure of $L$, $L^+$ is defined to be $L + LL + LLL + \ldots$. Also, $L^* = \epsilon + L + LL + LLL + \ldots$

   Thus $LL^* = L + LL + LLL + LLLL + \ldots$

   That proves $L^+ = LL^*$.

5. $L^* = L^+ + \epsilon$. The proof is easy, since the expansion of $L^+$ includes every term in the expansion of $L^*$ except $\epsilon$. Note that if the language $L$ contains the string $\epsilon$ then the additional $+\epsilon$ is not needed; that is $L^+ = L^*$ in this special case.

6. $L? = \epsilon + L$. This rule is really the definition of the operator ?

### 4.5.6. Identities for Regular Expression

On the basis of above discussion we will see some identitites for the regular expression. Let $p$, $q$, $r$ are three regular expression. Two regular expression $p$ and $q$ are equivalent (we say $p = q$) if $p$ and $q$ represent the same set of strings.

Some identities are following :

$I^1$ $\phi + r = r$

$I^2$ $\phi r = r\phi = \phi$

$I^3$ $\epsilon r = r\epsilon = r$

$I^4$ $\epsilon^* = \epsilon$ and $\phi^* = \epsilon$

$I^5$ $r + r = r$

$I^6$ $r^*r^* = r^*$

$I^7$ $rr^* = r^*r = r^+$

$I^8$ $(r^*)^* = r^*$

$I^9$ $\epsilon + rr^* = r^* = \epsilon + r^*r$

$I^{10}$ $(pq)^*p = p(qp)^*$

$I^{11}$ $(p + q)^* = (p^*q^*)^* = (p^* + q^*)^*$

$I^{12}$ $(p + q)r = pr + qr$ and $r(p + q) = rp + rq$

**Example 4.17. Prove that $(1 + 00^*1) + (1 + 00^*1)\ (0 + 10^*1)^*\ (0 + 10^*1) = 0^*1(0 + 10^*1)^*$**

**Solution.** L.H.S. $= (1 + 00^*1)\ (\epsilon + (0 + 10^*1^*(0 + 10^*1)$ using $I^{12}$

$= (1 + 00^*1)\ (0 + 10^*1)^*$ using $I^9$

$= (\epsilon + 00^*)1\ (0 + 10^*1)^*$ using $I^{12}$ for $(1 + 00^*1)$

$= 0^*1(0 + 10^*1)^*$ using $I^9$

$=$ R.H.S.

Hence it is proved.

## EXERCISE

1. Give the English description for the following regular expression :
   (a) $r = (aa)^*\ (bb)^*\ b$
   (b) $r = (0 + 1)^*00\ (0 + 1)^*$
   (c) $r = ((0 + 1)\ (0 + 1)^*)^*\ 00(0 + 1)^*$
2. Find the regular expression for the following :
   (a) $L = \{a^n b^m,\ n \geq 1,\ m \geq 1,\ nm \geq 3\}$
   (b) $L = \{ab^n w : n \geq 3,\ w \in \{ab\}^+\}$
   (c) $L = \{a^n : n \geq 0,\ n \neq 3\}$

3. Write regular expression for the following language over alphabet {0, 1} :
   (a) All string ending in 01
   (b) All strings not ending in 01
   (c) All strings containing an even number of 0's
   (d) All strings with at most two occurrences of the substring 00.
   (e) All strings not containing the substring 10.
4. Prove that
   (a) $(r_1^*)^* = r_1^*$ (b) $r_1^*(r_1 + r_2)^* \equiv (r_1 + r_2)^*$
   (c) $(r_1 + r_2)^* \equiv (r_1^* r_2^*)^*$
   For all regular expression $r_1$ and $r_2$. Here $\equiv$ stands for equivalence in the sense of the language generated.
5. Give the English description for the following regular expression :
   (a) $r = a(a + b)^* ab$ (b) $r = a^* b + b^* a$ (c) $r = (aa + b)^* (bb + a)^*$
6. Prove the following identity :
   $(a^* ab + ba)^* a^* = (a + ab + ba)^*$
7. Construct the transition system for the following regular expressions :
   (a) $r = 1 + 00 + 010^*1$ (b) $r = (ab + c^*)^* b$
8. Construct a finite automata for the regular expression
   $$r = (a + b)^* abb.$$
9. Find the regular expression for the following transition diagram.

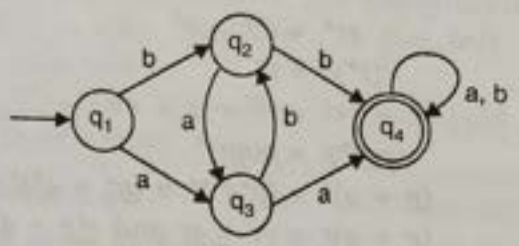


Fig. 4.13.

10. Find the regular expression corresponding to the automation given in Fig. 4.14.
11. Find the regular expression representing the following sets :
    (i) The set of all strings over {0, 1}, having at most one pair of 0's or at most 1 pair of 1's.
    (ii) The set of all strings over {0, 1} in which there are atleast two occurrence of 1 between any two occurance of a.

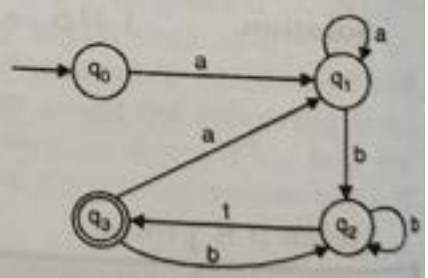


Fig. 4.14.

    (iii) The set of all strings over {a, b} ending with bb and begining with a.
    (iv) The set of all strings over {0, 1} not containing 101 as substring.
12. Construct a regular expression defining each of the following language over the alphabet $\Sigma = \{a, b\}$
    (a) All strings that do not end in a double letter.
    (b) All strings in which letter b never triples.
    (c) All strings in which a is tripled or b is tripled.
    (d) All strings that have an even number of a's and an odd number of b's.
    (e) All strings that have an odd number of a's and an odd number of b's.

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# 5 CHAPTER

# Properties of Regular Languages

**INSIDE THIS CHAPTER**

5.1. Introduction 5.2. Proving Languages Not to be Regular 5.3. Closure Properties of Regular Languages 5.4. Decision Properties of Regular Languages

## 5.1. INTRODUCTION

This chapter explores the properties of regular languages. Our first tool for this exploration is a way to prove that certain languages are not regular. One important kind of fact about the regular languages is called a "closure property." These properties let us build recognize for languages that are constructed from other languages by certain operations. As an example, the intersection of two regular languages is also regular. Thus, given automata that recognize two different regular languages, we can construct mechanically or automation that recognizes exactly the intersection of these two languages.

Some other important facts about regular languages are called "decision properties." Our study of these properties gives us algorithms for answering important questions about automata.

## 5.2. PROVING LANGUAGES NOT TO BE REGULAR

We have established that the class of Languages known as the regular languages for at least four different descriptions. They are the languages accepted by DFA's, NFA's, and by ∈-NFA's; they are also the languages defined by regular expression.

Not every language is regular. In this section, we shall introduce a powerful technique, known as the "Pumping Lemma", for showing certain language not to be regular.

### 5.2.1. The Pumping Lemma for Regular Languages

Pumping Lemma is a powerful tool for proving certain language non-regular. It's also useful in the development of algorithm to answer certain questions concerning, finite automata, such as weather the language accepted by a given FA is finite or infinite.

**Statement : Let $L$ be a regular language. Then there exist a constant $n$ (which depends on $L$) such that for every string $w$ in $L$, such that $|w| \geq n$, we can break $w$ into three substrings, $w = xyz$, such that :**

**(1) $y \neq \epsilon$**

**(2) $|xy| \leq n$**

**(3) For all $i \geq 0$, the string $xy^iz$ is also in $L$.**

That is we always find a nonempty string $y$ not too far from the beginning of $w$ that can be "pumped"; that is, repeating $y$ any number of times, keeps the resulting string in the language.

**Proof.** Suppose $L$ is regular. Then $L = L(M)$ for some DFA, $M$. Suppose $M$ has $n$ states. Now, consider any string $w$ of length $n$ or more, say $w = a_1 a_2 \ldots a_m$, where $m \geq n$ and each $a_i$ is an input symbol, for $i = 0, 1, 2, \ldots, m$ define state $p_i$ to be $\hat{\delta}(q_0, a_1 a_2 \ldots a_i)$, where $\delta$ is the transition function of $M$, and $q_0$ is the start state of $M$. That is $p_i$ is the state, $M$ is in after reading the first $i$ symbols of $w$.

By the pigeonhole principle, it is not possible for the $n + 1$ different $p_i$'s for $i = 0, 1, \ldots, n$ to be distinct, since there are only $n$ different states. Thus we can find two different integers $i$ and $j$, with $0 \leq i < j \leq n$, such that $p_i = p_j$. Now we can break $w = xyz$ as follows :

1. $x = a_1 a_2 \ldots a_i$.
2. $y = a_{i+1} a_{i+2} \ldots a_j$.
3. $z = a_{j+1} a_{j+2} \ldots a_m$.

That is $x$ takes us to $p_i$ once; $y$ takes us from $p_i$ back to $p_i$ (since $p_i$ is also $p_j$), and $z$ is balance of $w$. The relationship among the strings and states are discussed in Fig. 5.1.

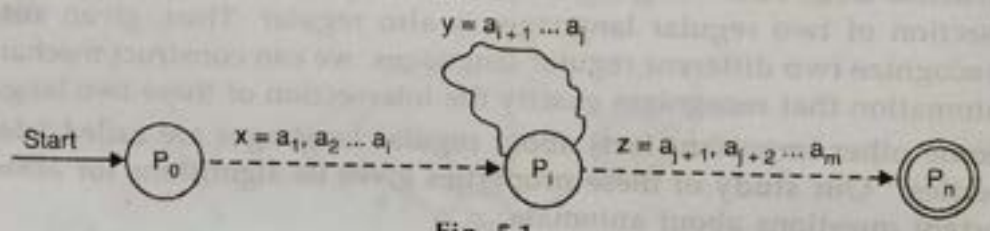


**Fig. 5.1.**

Note that $x$ may be empty, in the case that $i = 0$. Also, $z$ may be empty if $j = n = m$. However $y$ cannot be empty, since $i$ is strictly less than $j$.

Now consider what happens if the automation $M$ receives $xy^iz$ for any $i \geq 0$. If $i = 0$, then automation $M$ goes from the start state $q_0$ to $P_i$ on input $x$. Since $p_i$ is also $p_j$, it must be that $M$ goes from $p_i$ to the accepting state in Fig. 5.1 on input $z$. Thus, in accepts $xz$.

If $i > 0$, then $M$ goes from $p_0$ to $p_i$ on input string $x$, circles from $p_i$ to $p_i$, $i$ times on input $y_i$, and then goes to the accepting state on input $z$. Thus, for any $i \geq 0$, $xy^iz$ is also accepted by $M$; that is, $xy^iz$ is in $L$.

We can also understand these transactions as follows :

For $i = 0$

$$\hat{\delta}(p_0, x) = \hat{\delta}(p_i, x) = p_i$$

$$\hat{\delta}(p_i, z) = P_n$$

For $i > 0$

$$\hat{\delta}(q_0, x) = \hat{\delta}(p_0, x) = p_i$$

$$\hat{\delta}(p_i, y) = p_i$$

In general

$$\hat{\delta}(p_i, y^j_{i \geq 0}) = p_i$$

$$\hat{\delta}(p_i, z) = p_n$$

### 5.2.2. Application of the Pumping Lemma

The pumping lemma is extremely useful in proving that certain sets are non-regular. The general methodology in its application is an "adversary argument" of the following :

1. Select the language $L$, you wish to prove non-regular.
2. The "adversary" picks $n$, the constant mentioned in the pumping lemma. Once the adversary has picked $n$, he may not change it.
3. Select a string $z$ in $L$. Your choice may depend implicitly on the value of $n$ chosen.
4. The adversary breaks $z$ into $u$, $v$ and $w$, subject to the constants that $|uv| \leq n$ and $|v| \geq 1$.
5. You achieve a contradiction to the pumping lemma by showing, for any $u$, $v$ and $w$ determined by the adversary that there exit $i$ for which $uv^iw$ is not in $L$. It may then concluded and $L$ is not regular. Your selection of $i$ may depend on, $n$, $u$, $v$ and $w$.

Now let us discuss some example to see the application of pumping lemma.

**Example 5.1. Prove that Language $L = \{a^nb^n$ for $n = 0, 1, 2, 3, \ldots\}$ is not regular.**

**Solution.** Let us see how we could apply the pumping lemma directly to this case.

The pumping lemma says that there must be strings $x$, $y$ and $z$ such that all words of the form $xy^nz$ are in $L$.

How do we break this into three pieces $x$, $y$ and $z$ ?

**Case 1.** If middle part $y$ is made off entirely of $a$'s, as

$$x\ aaaa \ldots z$$

How if we pump it as $xyyz$, $xyyyz$ then number of $a$'s increases, but in language $L = \{a^nb^n$ for $n = 0, 1, 2, 3, \ldots\}$ $a$'s and $b$'s are equal so it is not allowed.

**Case 2.** If middle part $y$ is made off entirely of $b$'s as

$$x\ bbbbb \ldots z$$

For the same reason, it is also not allowed.

**Case 3.** $y$ part is made of some positive number of $a$'s and some positive number of $b$'s. This would mean that $y$ contain the substring $ab$.

$$x \ldots aaaaabbbbb \ldots z$$

Then $xyyz$ would have two copies of the substring $ab$. But every word in $L$ contains substring $ab$ exactly once. Therefore $xyyz$ cannot be a word in $L$.

This proves that the pumping lemma cannot apply to $L$ and therefore $L$ is not regular.

**Example 5.2. Prove that $L = \{a^nb\ a^n$ for $n = 0, 1, 2, \ldots\}$ is not regular.**

**Solution.** If this language were regular then there would exist three strings $x$, $y$ and $z$ such that $xyz$ and $xyyz$ were both words in this language. We can show that this is impossible.

**Observation 1.** If the $y$ string contained the $b$, then $xyyz$ would contain two $b$'s which is not allowed, according to the language $L$.

**Observation 2.** If the $y$ string is all $a$'s than the $b$ in the middle of the word $xyz$ is in the $x$-side or $z$-side. In either case, $xyyz$ has increased the number of $a$'s either infront of the $b$ or after $b$, but not both.

**Conclusion.** Therefore, $xyyz$ does not have its $b$ in the middle and is not in the form $a^n ba^n$. This language cannot be pumped and is therefore not regular.

**Example 5.3. Prove that $L = \{a^n b^n\ ab^{n+1}$ for $n = 1, 2, 3, ...\}$ is not regular.**

**Solution.** We are going to show that this language too is not regular by showing that if $xyz$ is in this language for any three strings $x$, $y$ and $z$, then $xyyz$ is not in this language :

**Observation 1.** For every word in this language, if we know the total number of $a$'s, we can calculate the exact number of $b$'s (twice the total number of $a$'s − 1).

And conversely, if we know the total number of $b$'s, we can uniquely calculate the number of $a$'s (add 1 and divide by 2). So no two different words have the same number of $a$'s and $b$'s.

**Observation 2.** All words in this language have exactly two substrings two substrings equal to $ab$ and one equal to $ba$.

**Observation 3.** If $xyz$ and $xyyz$ are both in this language, then $y$ cannot contain either substring $ab$ or the substring $ba$ because then $xyyz$ would have too many.

**Conclusion 1.** It must be a solid dump of $a$'s or solid dump of $b$'s.

**Conclusion 2.** If $y$ is solid $a$'s, then $xyz$ and $xyyz$ are different words with same total $b$'s, violating observation 1. If $y$ is solid $b$'s, then $xyz$ and $xyyz$ are different words with the same number of $a$'s violating observation 1.

**Conclusion 3.** It is impossible for both $xyz$ and $xyyz$ to be in this language for any string $x$, $y$ and $z$. Therefore, the language is unpumpable and not regular.

**Example 5.4. Prove that language which contains set of strings of balanced parentheses is not regular.**

**Solution.** Let us suppose language is regular, and $n$ be a constant according to pumping lemma. We can divide each string in three parts $xyz$ as $|\ xy\ | \le n$ and $y \ne \epsilon$. So $xy^iz$ must be in $L$ for $i \ge 0$.

**Observation.** It is clear that language will contains the strings like ( ( ) ( ) ( ) ), ( ) ( ) ( ), ( ( ( ( ) ) ) ) ....

Let us select a string $w$ = ( ( ( ... ( ) .... ) ) ), where there are $n$ left parenthesis by $n$ right parenthesis. This string is in $L$, and is of length at least $n$. So we can divide $w$ in three parts $w = xyz$ as $|\ xy\ | \le n$ and $y \ne \epsilon$.

**Conclusion.** Not let us choose $i = 0$. The resulting string $xz$ is not in $L$, reason is very clear.

Since $w = xyz\ |\ xy\ | < n$.

It means $y$ will be made of only left parenthesis. So $xz$ is not in $L$ since it has fewer left parenthesis then right parenthesis. This contradict the pumping lemma, so our original assumption, that $L$ was regular, must have been incorrect.

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**Example 5.5. Prove that $L = \{0^n1^m2^n, n, m \geq 0\}$ is not regular.**

**Solution.** We assume that $L$ is regular and try to obtain a contradiction.

**Observation.** Suppose $L$ is regular, let $n$ be constant provided by the pumping lemma. Let $w$ be the string $0^n2^n$. (Here $m = 0$ which is allowed). This string is in $L$, and is of length at least $2n$.

So we can write $w = xyz$, $|xy| \leq n$
$y \neq \epsilon$

Clearly $xy$ will contain only 0's since in $w = 0^n2^n$ first $n$ placed are occupied by the 0's only. So $y$ will also made of 0's only.

**Conclusion.** According to pumping lemma $xy^iz$ is also in $L$ for $i \geq 0$. Let us choose $i = 0$. The resulting string $xz$, is not in $L$ since it has fewer 0's then 1's. This contradicts the pumping lemma, so our original assumption that $L$ was regular, must have been incorrect.

**Example 5.6. Prove that language $L = \{0^n1^m \mid n \leq m\}$ is not regular.**

**Solution.** Let us analyse the language first, $L = \{0^n1^m \mid n \leq m\}$ means that it contains set of strings, like 01, 011, 0111, 0011, 00111, ....

We will adopt same approach as we assume $L$ is regular then try to obtain a contradiction to pumping lemma.

Let $L$ be regular, and $p$ be the constant provided by the pumping lemma. Let $w$ be the string of $L$

$$w = 0^p1^p.$$

The string $w$ is in $L$ and is of length at least $p$.

Let us divide $w$ in three parts $w = xyz$, $|xy| \leq p$ and $y$ never empty.

**Observation.** Since $|xy| < p$, $y$ consists entirely of 0's and since $y$ is not $\epsilon$ there is at least one 0 in $y$. If $xyz \in L$ then according to pumping lemma $xy^kz$ must be in $L$ for all $k \geq 0$.

**Conclusion.** So let us choose $k$ to be 2. The resulting string $xyyz$, is not in $L$, since it has more 0's than 1's. This contradicts the pumping lemma, so our original assumption, that $L$ was regular, must have been incorrect.

**Example 5.7. Prove that $L = \{0^n1^{2n} \mid n \geq 1\}$ is not regular.**

**Solution.** Let us assume that $L$ is regular and $p$ be the constant provided by the pumping lemma. Let $w$ be the string of $L$

$$w = 0^p1^{2p}$$

**Observation.** The string $w$ is in $L$, and its length at least $p$. So $w$ can be written as $xyz$ with $|xy| \leq p$ and $y$ not $\in$ according to pumping lemma if $xyz$ is in $L$ then $xy^kz$ is in $L$ for $k \geq 0$.

**Conclusion.** Since $|xy| \leq p$, $y$ consists entirely of 0's, and since $y$ is not $\epsilon$ there is at least one 0 in $y$. So now let us choose $k$ to be $2p + 1$. (Note that we are allowed to have our $k$ depend on the $p$. The value $p$ was given to us by pumping lemma.)

$$xyz \in L$$

$$\Rightarrow \quad xy^kz \in L$$

Now $\quad xy^{2p+1}z \notin L$

Since the number of 0's is at least $2p + 1$ but the number of 1's is exactly $2p$.

This contradicts the pumping lemma, so our original assumption, that $L$ was regular, must have been incorrect.

**Example 5.8. Prove that language $L = \{0^n \mid n$ is a perfect square$\}$ is not regular.**

**Solution.** Let us suppose $L$ be regular, let $p$ be a constant provided by the pumping lemma. Let $w$ be the string $O^{p^2}$. This string is in $L$ and is of length at least $p$. So $w$ can be written as $xyz$ with $|xy| \le p$ and $y$ not $\epsilon$, and each $xy^kz$ also in $L$. Since $|xy| \le p$, $y$ consists of no more than $p$ 0's, and since $y$ is not $\epsilon$ so there is atleast one '0' in $y$.

So now let us choose $k$ to be 2. We claim that the resulting string $xyyz$ is not in $L$.

Now let us see the proof of this claim :

We want to show that the length of $xyyz$ is not a perfect square. In fact what we do is show that it must lie strictly in between 2 consecutive perfect squares, namely, $p^2$ and $(p-1)^2$.

So let $q$ stand for the length of $y$. Then the length of $xyyz$ is $p^2 + q$. So it suffices to show that $p^2 < p^2 + q < (p+1)^2$. The first in equality holds since $q > 0$ (since $y$ was not $\epsilon$). For second inequality, we compute : $(p+1)^2 = p^2 + 2p + 1$, and this greater than $p^2 + q$ since $q < p$ (since $|xy| \le p$). The claim is established, so $xyyz$ is not in $L$.

This contradicts the pumping lemma, so our original assumption, that $L$ was regular, must have been incorrect.

**Example 5.9. Prove that $L = \{0^n \mid n$ is perfect cube$\}$ is not regular.**

**Solution.** We will apply same approach that is we assume that $L$ is regular and try to obtain contradiction by applying pumping lemma.

**Observation.** Let us assume $L$ is regular, let $p$ be a constant provided by the pumping lemma. Let $w$ be the string $O^{p^3}$. This string is in $L$, and is of length at least $p$. So $w$ can be written as $xyz$ with $|xy| \le p$ and $y$ not $\epsilon$.

**Conclusion.** Pumping lemma say that if $xyz \in L$ then $xy^kz$ also in $L$. $y$ contains at least one 0 in $y$ since $y$ is not $\epsilon$.

Let us choose $K = 2$. We claim that resulting string $xyyz$ is not a perfect cube.

Now let us see the proof for this claim :

We want to show that the length of $xyyz$ is not a perfect cube. Let $q$ be the length of $y$, then the length of $xyyz$ is $p^3 + q$. So it suffices to show that $p^3 < p^3 + q < (p+1)^3$. The first inequality holds since $q > 0$ (since $y$ was not $\epsilon$). For second inequality, we compute $(p+1)^3 = p^3 + 3p^2 + 3p + 1$, and this is greater than $p^3 + q$ since $q < p$ (since $|xy| \le p$). The claim is established, so $xyyz$ is not in $L$.

This contradicts the pumping lemma, so our original assumption, that $L$ was regular, must have been incorrect.

**Example 5.10. Let** $\Sigma = \{a, b\}$. **Show that**

$$L = \{w \in \Sigma^* : n_a(w) < n_b(w)\}$$

**is not regular.**

**Solution.** Let us assume that $L$ is regular and $p$ is a constant provided by the pumping lemma. Let $w = a^p b^{p+1}$ be a string in $L$.

**Observation.** Let us write $w$ as $xyz$ where $|xy| \le p$ and $y$ not $\epsilon$. $w$ is in $L$ and $y$ consists of no more than $p$ $a$'s since $|xy| \le p$. According to pumping lemma $xy^kz$ will be in $L$ for $k \ge 0$.

**Conclusion.** Let us choose $k = 2$ then $xyyz$ is not in $L$, since as we pump $y$ by two then number of $a$'s increases which violates the condition $n_a(w) < n_b(w)$.

So we got a contradiction for the pumping lemma so our assumption that $L$ is regular must have been incorrect.

**Example 5.11. Prove that set of all strings begining with a nonempty string of the form $ww$, is not regular. It is given that $\Sigma = \{0, 1\}$.**

**Solution.** Let us assume that given language $L$ is regular. Because the regular language over any alphabet are closed under intersection (we will discuss this theorem in Section 5.3).

So clearly language $L' = L \cap L$ is regular.

Now let us assume that $n$ be constant given by the pumping lemma for language $L'$.

Let $$w = 10^n\, 10^n \in L'$$

By the pumping lemma, we can write $w = xyz$ such that $xy \le n$, $y \ne \epsilon$, and for all $k \in N$, $xy^kz \in L'$. Now let us consider $xy^2z$ (that is choose $k = 2$).

**Conclusion.** If $y$ contains a 1, then $xy^2z$ certain more than two 1's and hence is not in $L'$.

If $y$ contains no 1's then $xy^2z = 10^{n+1}10^n$ for $i > 1$, and hence is not in $L'$ (since $y \ne \epsilon$).

In either case, we obtain a contradiction. We conclude that $L$ is not regular.

**Example 5.12. Prove that language $L$, which is the set of all strings beginning with a string $w$ of length at least 3 such that $w = w^R$, is not regular. It is given that alphabet $\Sigma = \{0, 1\}$.**

**Solution.** Let us assume $L$ is not regular. Because the set of regular languages over any alphabet is closed under intersection, the language $L' = L \cap L$ is regular, (let say 001* 01*00).

Let $n$ be the constant guaranteed by the pumping lemma for $L'$. Let $w = 001^{n+1}01^{n+1}00 \in L'$. By the pumping lemma, we can write $w = xyz$ such that $xy \le n$, $y \ne \epsilon$ and for all $k \in N$, $xy^kz \in L'$.

**Conclusion.** Because every string in $L'$ has exactly five 0s, $y$ can contain no 0's, for otherwise, $xy^0z$ would not be in $L'$. Therefore $xy^iz = 001^{n+i+1} + 01^{n+1}00$ for some $i > 0$. Clearly $xy^2z$ contains no prefix of length at least three that is palindrome so $xy^2z \notin L'$—a contraction. Therefore, $L$ is not regular.

**Example 5.13. Prove that language $L = \{0^{2^K} \mid K \in N\}$ is not regular.**

**Solution.** By contradiction.

Let us assume that $L$ is regular. Let $n$ be constant provided by the pumping lemma.

Let $w = 0^{2^i}$ for some $i \in N$ such that $2^i > n$. Clearly $w \in L$, by the pumping lemma we can write $w = xyz$ such that $|xy| \le n$, $y \ne \epsilon$, and for all $k$, $k \in N$, $xy^kz \in L$.

Clearly, $0 < |y| < 2^i$, by adding $2^i$ to each term, we obtained

$$2^i < |y| + 2^i < 2^{i+1}$$

Thus $$xy^2z = 0^{2^{|+|y|}}$$
$$\notin L$$
a contradiction. Therefore $L$ is not regular.

**Example 5.14. Prove that language $L = \{0^K\}$ $K$ is prime number $\}$ is not regular.**

**Solution.** By contradiction. Let us assume that $L$ is regular, let $n$ be constant guaranteed by pumping lemma. Let $w = 0^p$ where $p$ is some prime number so smaller then $n$. Clearly $w$ is in $L$.

By pumping lemma we can write $w = xyz$ such that $|xy| < n$, $y \neq \epsilon$ and for all $k \in N$ $xy^kz \in L$. Then

$$xy^{p+1}z = 0^{p+p|y|}$$
$$= 0^{p(1+|y|)}$$
$$\notin L \text{ because } |y| > 0$$
a contradiction. Therefore, $L$ is not regular.

**Example 5.15. Prove that $L = \{x \in (0, 1)^* \mid x = x^R\}$ is not regular.**

**Solution.** We will apply same approach of proving by contradiction.

Let us assume that $L$ is regular and $n$ be a constant guaranteed by pumping lemma.

Let $w = 0^n 10^n \in L$.

By the pumping lemma we can write

$w = xyz$ such that $|xy| \leq n$, $y \notin \epsilon$ and for all $k \in N$ $xy^kz \in L$.

Because $|xy| \leq n$ we have

$$xy^2z = 0^{n+|y|}10^n \quad \text{For } k = 2$$
$$\notin L \text{ because } |y| > 0$$
So it is contradiction, hence $L$ is not regular.

## 5.3. CLOSURE PROPERTIES OF REGULAR LANGUAGES

In this section, we shall prove several theorems of the form "if certain languages are regular, and a language $L$ is formed from them by certain operations, then $L$ is also regular. These theorems are often called closure properties of the regular languages, since they show that the class of regular languages is closed under the operation mentioned. Here is a summary of the principle closure properties of regular languages.

1. The union of two regular languages is regular.
2. The intersection of two regular languages is regular.
3. The complement of a regular language is regular.
4. The difference of two regular languages is regular.
5. The reversal of a regular language is regular.
6. The closure of a regular language is regular.
7. The concatenation of regular language is regular.
8. A homomorphism (substitution of strings for symbols) of a regular language is regular.
9. The inverse homomorphism of a regular language is regular.

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**Theorem 5.1.** If $L_1$ and $L_2$ are regular languages, then so are $L_1 \cup L_2$, $L_1L_2$ and $L_1^*$.

We can say that family of regular language is closed under union, concatenation, and star-closure.

**Proof.** If $L_1$ and $L_2$ are regular, then there exist regular expression $r_1$ and $r_2$ such that $L_1 = L(r_1)$ and $L_2 = L(r_2)$.

By the definition, $r_1 + r_2$, $r_1r_2$ and $r_1^*$ are regular expression denoting the languages $L_1 \cup L_2$, $L_1L_2$, and $L_1^*$, respectively. Thus, closure under union, concatenation and star-closure is immediate.

**Theorem 5.2.** If $L$ is regular then complement of $L$ that is $\overline{L}$ is also regular. In other words, the set of regular languages is closed under complementation.

**Proof.** To show closure under complementation, let $M = (Q, \Sigma, \delta, q_0, F)$ be a *dfa* that accepts $L$, then *dfa*

$$\overline{M} = (Q, \Sigma, \delta, q_0, Q - F)$$

accepts $\overline{L}$.

We can justify it as if $M$ accepts the language $L$ then some of the states of this FA are final states and, must likely, some are not. Let us reverse the final status of each state, that is if it was a final state, make it a non-final state, and if it was a non-final state, make it a final state. If an input string formally ended in a non-final state, it now ends in a final state and vice versa. This new machine we have built accepts all input strings that were not accepted by the original FA (all the words in $\overline{L}$) and rejects all the input strings that the FA used to accept (the words in $L$). Therefore in machine accepts exactly the language $\overline{L}$. So $\overline{L}$ is also regular language.

**Example 5.16. An FA that accepts only the strings *aba* and *abb* is shown adjoining figure, find FA for the complement of *L i.e.* accepted every string other then *aba* and *abb*.**

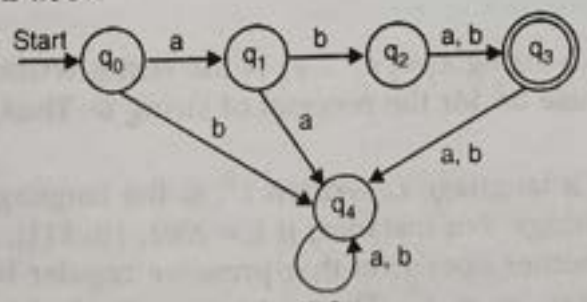


**Fig. 5.2.**

**Solution.** Apply Theorem 5.2. to find out the required FA which accepts strings other then *aba* and *abb* that is every non-final state will be final state and every final state will be non-final.

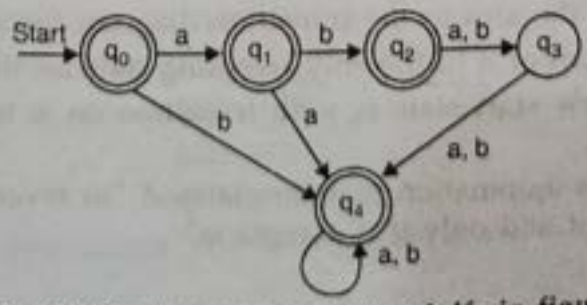


**Fig. 5.3. DFA for the complements of *dfa* in figure 5.2.**

**Theorem 5.3.** If $L_1$ and $L_2$ are regular languages, then $L_1 \cap L_2$ is also regular language. In other words the set of the regular language is closed under intersection.

**Proof.** By Demorgon's law for sets of any kind (regular languages or not):

$$L_1 \cap L_2 = \overline{(\overline{L_1} + \overline{L_2})}$$

This can be justify by the Venn diagrams as follows :

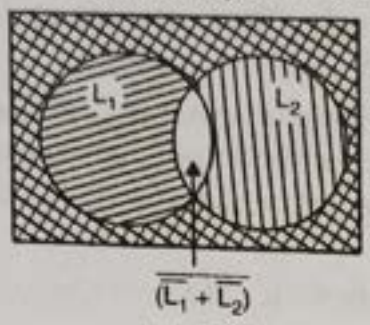


**Fig. 5.4.**

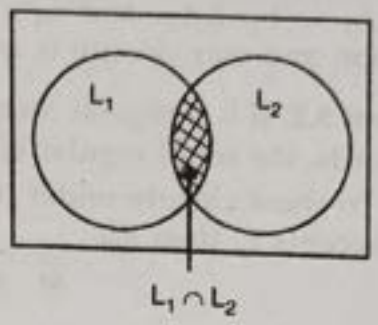


**Fig. 5.5**

So it is clear from Fig. 5.4 and Fig. 5.5 that

$$L_1 \cap L_2 = \overline{(\overline{L_1} + \overline{L_2})}$$

Because $L_1$ and $L_2$ are regular so $\overline{L_1}$ and $\overline{L_2}$ are regular (from Theorem 5.2). So is $\overline{L_1} + \overline{L_2}$ (by Theorem 5.1). And because $\overline{L_1} + \overline{L_2}$ is regular then so is $\overline{(\overline{L_1} + \overline{L_2})}$ which means $L_1 \cap L_2$ is regular.

**Theorem 5.4.** If $L$ and $m$ are regular languages, then so is $L - m$.

**Proof.** Observe that $L - m = L \cap \overline{m}$. Here $\overline{m}$ is also regular since $m$ is regular (according to Theorem 5.2) and $L \cap \overline{m}$ is regular according to the Theorem 5.3. Therefore $L - m$ is regular.

### 5.3.1. Reversal

The reversal of a string $a_1 a_2 a_3 \ldots a_n$ is the string written backwards, that is $a^n a^{n-1} \ldots a_1$. We use $w^R$ for the reversal of string $w$. Thus, $1101^R$ is 1011 and $\epsilon^R = \epsilon$.

The reversal of a language $L$, written $L^R$, is the language consisting of the reversal of all its strings. For instance, if $L = \{001, 10, 111\}$, then $L^R = \{100, 01, 111\}$. Reversal is another operation that preserve regular languages; that is if $L$ is regular language, so is $L^R$. There are two simple proofs, one based on automata and one based on regular expression.

Given a language $L$ that is $L(A)$ for some finite automation, perhaps with non-determinism and $\epsilon$-transitions, we may construct automation for $L^R$ by :

1. Reverse all the arcs in the transition diagram for $A$.
2. Make the start of $A$ be the only accepting state for the new automation.
3. Create a new start state $p_0$ with transition on $\epsilon$ to all the accepting states of $A$.

The result is an automation that simulates $A$ "in reverse", and therefore accepts a string $w$ if and only if $A$ accepts $w^R$.

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**Theorem 5.5.** If $L$ is a regular language, so is $L^R$.

**Proof.** Let us assume that $L$ is defined by the regular expression $r$. The proof is a structural induction on the size of $r$. We show that there is another regular expression $r^R$ such that $L(r^R) = (L(r))^R$; that is the language of $r^R$ is the reversal of the language of $r$.

**Basis.** If $r$ is $\epsilon$, $\phi$, or $a$, for some symbol $a$, then $r^R$ is the same as $r$. That is, we know $\{\epsilon\}^R = \{\epsilon\}$, $\phi^R = \phi$ and $\{a\}^R = \{a\}$.

**Induction.** There are three cases, depending on the form of $r$.

1. $r = r_1 + r_2$. Then $r^R = r_1^R + r_2^R$. The justification is that the reversal of the union of two languages is obtained by computing the reversal of the two languages and taking the union of those languages.
2. $r = r_1 \cdot r_2$. Then $r^R = r_2^R r_1^R$. Note that we reverse the order of the two languages, as reversing the language themselves.
3. $r = r_1^*$. Then $r^R = (r_1^R)^*$. The justification is that any string $w$ in $L(r)$ can be written as $w_1, w_2, \ldots, w_n$, where each $w_i$ is in $L(r)$. But

$$w^R = w_n^R w_{n-1}^R \ldots w_1^R.$$

Each $w_i^R$ is in $L(r)^R$, so $w^R$ is in $(r_1^R)^*$.

**Example 5.17. Let $L$ be defined by the regular expression $(0 + 1)0^*$, then find $L^R$.**

**Solution.**

$$L = (0 + 1)\, 0^*$$

$L^R = (0^*)^R\, (0 + 1)^R$, by the rule for the concatenation.

### 5.3.2. Homomorphism

A string homomorphism is a function on strings that works by substituting a particular string for each symbol.

Suppose $\Sigma$ and $\Sigma'$ are alphabets. Then function

$$h : \Sigma \longrightarrow \Sigma'$$

is called a "homomorphism". In other words, a homomorphism is a substitution in which a single letter is replaced with a string. The domain of the function $h$ is extended to strings in an obvious fashion, if

$$w = x_1 x_2 \ldots x_n,$$

then

$$h(w) = h(x_1)\, h(x_2)\, h(x_3) \ldots h(x_n)$$

If $L$ is a language on $\Sigma$, then its homomorphic image is defined as

$$h(L) = \{h(w) : w \in L\}.$$

**Example 5.18. Let $\Sigma = \{0, 1\}$ and $\Sigma' = \{0, 1, 2\}$ and defined by**

$$h(0) = 01$$
$$h(1) = 112$$

**Find $h(010)$ and homomorphic image of $L = \{00, 010\}$.**

**Solution.** Given $\Sigma = \{0, 1\}$, $\Sigma' = \{0, 1, 2\}$

$h$ is defined as :

$$h(0) = 01$$
$$h(1) = 112$$

So

$$h(010) = 0111201.$$

The homomorphic image of $L = \{00, 010\}$ is the language $h(L) = \{0101, 0111201\}$.

If we have a regular expression $r$ for a language $L$, then a regular expression for $h(L)$ can be obtained by simply applying the homomorphism to each $\Sigma$ symbol of $r$.

**Example 5.19. $\Sigma = \{0, 1\}$, and $\Sigma' = \{1, 2, 3\}$. Define $h$ by**

$$h(0) = 3122$$
$$h(1) \ 132$$

**if $L$ is regular language denoted by**

$$r = (0 + 1^*)\ (00)^*$$

**the find out the regular expression for language $h(L)$.**

**Solution.** Given
$$\Sigma = \{0, 1\}$$
$$\Sigma' = \{1, 2, 3\}$$
and $h$ is defined as
$$h(0) = 3122$$
$$h(1) = 132$$

$r = (0 + 1^*)\ (00)^*$; where $r$ is regular expression for language $L$.

Let regular expression for the language $h(L)$ be $r'$, then

$$r' = (h(0) + h(1)^*)\ (h(0)\ h(0))^*$$

Let us put $h(0)$ and $h(1)$ in $r'$.

$$r' = (3122 + (132)^*)\ (3122\ 3122)^*$$

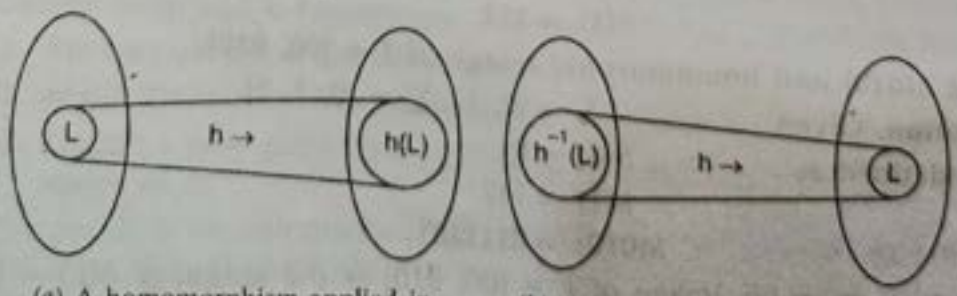
denotes then regular language $h(L)$.

**Theorem 5.6.** Let $h$ be a homomorphism. If $L$ is a regular, then its homomorphic image $h(L)$ is also regular. The family of regular languages is therefore closed under arbitrary homomorphisms.

**Proof.** Let $L$ be regular language denoted by some regular expression $r$. We find $h(r)$ by substituting $h(a)$ for each symbol $a \in \Sigma$ or $r$. It can be shown directly by an appeal to the definition of a regular expression that the result is a regular expression. It is equally easy to see that expression denotes $h(L)$. All we need to do is to show that for every $w \in L(r)$. The corresponding $h(w)$ is in $L(h(r))$ and conversely that for every $x$ in $L(h(r))$ there is a $w$ in $L$, such that $x = h(w)$. Leaving the detail as an exercise, we claim that $h(L)$ is regular.

### 5.3.3. Inverse Homomorphism

Homomorphism may also be applied "backwards," and in this made they also preserve regular language. That is, suppose $h$ is a homomorphism from some alphabet $\Sigma$ to strings in another (possible the same) alphabet $\Sigma'$. Let $L$ be a language over alphabet $\Sigma'$. Then $h^{-1}(L)$, read "$h$ inverse of $L$," is the set of strings $w$ in $\Sigma^*$ such that $h(w)$ is in $L$. Fig. 5.6 suggests the effect of a homomorphism on a language $L$ in part ($a$), and the effect of an inverse homomorphism in part ($b$).




($a$) A homomorphism applied in forward direction.

($b$) A homomorphism applied into inverse direction.

**Fig. 5.6.**

**Theorem 5.7.** If $h$ is a homomorphism from alphabet $\Sigma$ to alphabet $\Sigma'$, and $L$ is a regular over $\Sigma'$, then $h^{-1}(L)$ is also regular language.

**Proof.** The proof starts with a DFA A for $L$. We construct from A and $h$ a DFA for $h^{-1}(L)$ using the plan suggested by Fig. 5.7. This DFA uses the states of A but translates the input symbol according to $h$ before deciding on the next state.

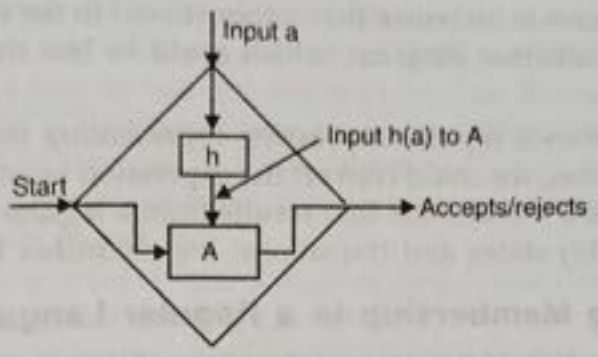



**Fig. 5.7.**

The DFA for $h^{-1}(L)$ applies to its input, and then simulates the DFA for $L$.

Formally, let $L$ be $L(A)$, where DFA $A = (Q, \Sigma', \delta, q_0, F)$. Define a DFA $B = (Q, \Sigma, \delta', q_0, F)$, where transition function $\delta'$ is constructed by the rule $\delta'(q, a) = \hat{\delta}(q, h(a))$. That is, the transition $B$ makes on input $a$ is the result of the sequence of transitions that $a$ makes on the string of symbols $h(a)$. Remember that $h(a)$ could be $\epsilon$, it could be one symbol, or it could be many symbols, but $\delta$ is properly defined to take care of all these cases.

It is an easy induction on $|w|$ to show that $\hat{\delta}'(q_0, w) = \hat{\delta}(q_0, h(w))$. Since the accepting states of $A$ and $B$ are the same, $B$ accepts $w$ if and only if $A$ accepts $h(w)$. Put another way, $B$ accepts exactly those strings what are in $h^{-1}(L)$.

## 5.4. DECISION PROPERTIES OF REGULAR LANGUAGES

In this section we consider how one answers important questions about regular languages. First, we must consider what it means to ask a question about a language. The typical language is infinite, so you cannot present the strings of the language to someone and ask a question that requires them to inspect the infinite set of strings. Rather, we present a language by giving one of the finite representations for it that we have developed : a DFA, an NFA, an $\epsilon$-NFA or a regular expression.

Of course the language so described will be regular and in fact there is no way at all to represent completely arbitrary languages. Let us consider some of the fundamental questions about languages :

1. Is the language described empty ?
2. Is a particular string $w$ is in the described language ?
3. Do two descriptions of a language actually describe the same language ? This question is often called "Equivalence" of languages.

### 5.4.1. Testing Emptiness of Regular Languages

If our representation is any kind of finite automation, the emptiness question is whether there is any path whatsoever from the start state to some accepting state, if so, language is nonempty, while if the accepting states are all separated from the start state, then the language is empty.

The start state is surely reachable from the start state, if there is an arc from $q$ to $p$ with any label (an input symbol, or $\epsilon$ if the automation is an $\epsilon$-NFA) then $p$ is reachable. In that manner we can compute the set of reachable states. If any accepting state is among them, we answer "no" (the language of the automation is not empty), and otherwise we answer "yes". Note that the reachability calculation takes no more time that $0(n^2)$ if the automation has $n$ states, and in fact it is no worse than proportional to the number of arcs in the automation is transition diagram, which could be less than $n^2$ and cannot be more than $0(n^2)$.

If we are given a regular expression representing the language $L$, rather than an automation, we could convert the expression to an $\epsilon$-NFA and proceed as above. Since the automation that results from a regular expression of length $n$ has at most $0(n)$ states and transitions, the algorithm takes $0(n)$ time.

### 5.4.2. Testing Membership in a Regular Language

The next question of importance is, given a string $w$ and a regular language $L$, is $w$ in $L$. While $w$ is represented explicitly, $L$ is represented by an automation or regular expression.

If $L$ is represented by a DFA, the algorithm is simple. Simulate the DFA processing the string of input symbol $w$, beginining in the start state. If the DFA ends in an accepting state, the answer is "yes"; otherwise the answer is "no". If $|w| = n$, and the DFA is represented by a suitable data structure, such as two-dimensional array that is the transition table, then each transition requires constant time, and the entire test takes $0(n)$ time.

**Theorem 5.8.** Whether two regular sets are identical is solvable.

**Proof.** Let us take two finite automate ($M_1$ and $M_2$) and examine a picture of the sets they accept.

If the intersection is the same as both sets then indeed they are identical. Or, on the other hand, if the areas outside the intersection are empty then both sets ($s(M_1)$ and $s(M_2)$) are identical.

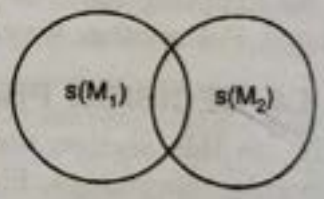



**Fig. 5.8.**

Let us examine these outside areas.

The picture in Fig. 5.9 ($a$) represents the set accepted by $M_1$ and rejected by $M_2$ while that in Fig. 5.9 ($b$) is the set which $M_2$ accepts and $M_1$ rejects.

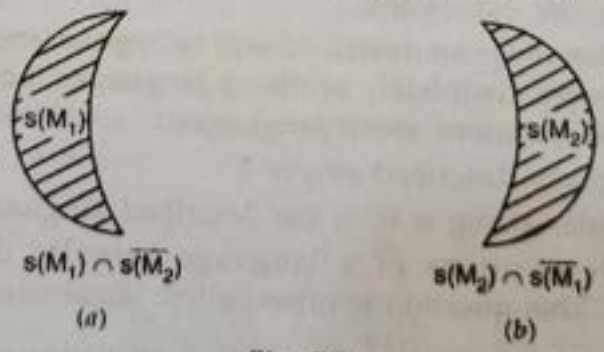



**Fig. 5.9.**

If these two areas (or sets) are empty then sets accepted by $M_1$ and $M_2$ are exactly the same. This means that the equivalence problem for $s(M_1)$ and $s(M_2)$ is exactly same as the emptiness problem for :

$$|s(M_1) \cap s(\bar{M}_2)| \cup |s(M_2) \cap s(\bar{M}_1)|$$

So, if we can solve the emptiness problem for the above set, then we can solve the equivalence problem for $s(M_1)$ and $s(M_2)$. Since the regular sets are closed under union, complement and intersection; the above set is a regular set. And we know that emptiness for the class of regular sets is solvable. Hence it proved.

**Theorem 5.9.** Whether a regular set is finite is solvable.

**Proof.** We know that if a finite automation accepts any strings at all then some will be of length less than the size of the state set. (This does not help directly but it gives a hint as to what we need for this theorem.)

If we were to find a string accepted by finite automation $m$ which was longer than or equal to the size of it's state set, we could use the inflation aspect of the pumping lemma to show that machine $m$ must accept on infinite number of strings. This means that :

"a finite automation accepts only strings of length less than the size of its set of states, if and only if it accepts a finite set."

Thus, to solve the finiteness problem for $M = (Q, \Sigma, \delta, q_0, F)$, we need to determine whether or not :

$L(M) - |$ strings of length $< |Q|\ | = \phi$

A question now arises as to how many inputs we must examine in order to tell it $M$ will accept on input longer than or equal to the size of its state set. The answer is that we only need to consider input strings upto twice the size of the state set.

**Theorem 5.10.** If a finite automation accepts any strings, it will accepts one of length less than the size of its state set.

**Proof.** Let $m$ be a finite automation which accepts the string $x$ and that the length of $x$ is no less than the size of $M$'s state set. Assume further that $M$ accepts no string shorter than $x$. This is the opposite of our theorem.)

Immediately the pumping lemma asserts that there are strings $u$, $v$ and $w$ such that $uvw = x$, $v \neq \in$ and $uw \in L(M)$. Since $v \neq \in$, $uw$ is shorter than $uvw = x$. Thus $M$ accepts shorter strings than $x$ and the theorem follows :

## EXERCISE

1. Use the pumping lemma to show that each of these languages is non-regular :
   (i) $\{a^n b^{n+1}\} = \{abb, aabbb, aaabbbb \ldots\}$
   (ii) $L = \{a^n b^n a^n \mid n > 0\}$
   (iii) $L = \{a^n b^{2n} \mid n > 0\}$
   (iv) $L = \{a^n b a^n \mid n > 0\}$
   (v) $L = \{a^n b^n a^m$ where $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots\}$
2. Define the language square as follows :
   $$\text{Square} = \{a^n \text{ where } n \text{ is a square}, n > 0\}$$
   $$= \{a, aaaa, aaaaaaaaa \ldots\}$$
   Using the pumping lemma to prove that square is non-regular.
3. Prove that $L = \{a^n b^n$ where $n$ is square and $n > 0\}$ is non-regular.
4. Define the language double prime as follows
   Double prime $= \{a^p b^p$ where $p$ is any prime$\}$
   prove that double prime is non-regular.
5. Define the language double factorial as follows :
   $$\text{DF} = \{a^{n!} b^{n!} \mid n > 0\}$$
   prove that DF is non-regular.
6. Let $L_1, L_2, L_3 \ldots$ be an infinite sequence of regular languages.

(*i*) Let $L$ be the infinite union of all these language taken together. Is $L$ necessarily regular ?

(*ii*) Is the infinite intersection of all these languages necessarily regular ?

7. Consider the following language :

$$\bar{P} = \{a^n \text{ where } n \text{ is not prime}\}$$
$$= \{\epsilon, a, aaaa, aaaaaa, aaaaaaaaa, \ldots\}$$

prove that $\bar{P}$ is non-regular.

8. Consider the language $L$ defined as

$L = \{a^n$ where $n$ is any integer with an even number of digits in base $10\}$
$= \{\epsilon, a^{10}, a^{11}, a^{12}, \ldots\}$

prove that $L$ is non-regular.

9. (*i*) Show that if we add a finite set of words to a regular language, the result is a regular language.

(*ii*) Show that if we subtract a finite set of words from a regular language, the result is a regular language.

10. Prove that following languages are not regular.

(*i*) $L = \{a^n b^l a^k : k \geq n + 1\}$ (*ii*) $L = \{a^i b^j a^k : k \neq i + j\}$

(*iii*) $L = \{a^i b^j a^k : i = j \text{ or } j = k\}$ (*iv*) $L = \{a^i b^j : i \leq 1\}$

(*v*) $L = \{w : n_a(w) \neq n_b(w)\}$

11. Prove that following language regular ?

(*i*) $L = \{xyy^R z : x, z, y \in \{0, 1\}^*\}$

(*ii*) $L = \{xyy^R z : x, z, y \in \{0, 1\}^+, |x| \geq |z|\}$

12. Suppose that we know that $L_1 \cup L_2$ and $L_1$ are regular. Can we conclude from this that $L_2$ is regular.

13. Prove that following languages over alphabet $\{0, 1\}$ are not regular.

(*i*) The set of strings of 0's and 1's, beginning with a 1, such that when interpreted as an integer, that integer is prime.

(*ii*) The set of strings of the form $w\,1n$, where $w$ is a string of 0's and 1's of length $n$.

(*iii*) The set of strings of 0's and 1's of the form $w\bar{w}$, where $\bar{w}$ is formed from $w$ by replacing all 0's by 1's, and vice-versa that is, $\overline{011} = 100$, and 011 100 is an example of string in the language.

14. Given an algorithm to tell whether a regular language $L$ is infinite.

15. Let $A$ be any FA with $N$ states. Then :

(*i*) If $A$ accepts an input string $w$ such that $N \leq$ length $(w) < 2N$, then $A$ accepts an infinte language.

(*ii*) If $A$ accepts many words, then $A$ concepts some word $w$ such that $N \leq$ length $(w) < 2N$.

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# 6 CHAPTER

# Context-Free Grammars and Languages

**INSIDE THIS CHAPTER**

6.1. Grammars 6.2. Context-Free Grammars 6.3. Parse Tree 6.4. Parsing an Example of Context Free Grammar 6.5. Ambiguity in Grammars and Languages.

## 6.1. GRAMMARS

All of us know that a grammar is nothing but a set of rules to define valid sentences in any languages. In this chapter we introduce the context-free grammars, which generates context-free languages. Context-free languages have great practical significance in defining programming languages and in simplifying the translation for programming languages.

Initially Linguists were trying to define precisely valid sentences and to give structural description for these sentences. They tried to define rules for natural languages like Hindi, English etc. Noam Chomsky gave a mathematical model for the grammars in 1956. Although it was useless to describe natural languages but it becomes very useful for computer languages.

The original motivation for grammars was the description of natural languages. We can write rules for the grammar of natural languages as follows :

<sentence> → <noun phrase> <verb phrase>

<noun phrase> → <article> <noun>

According to above set of rules "The girl eats" is valid sentence.

It can be easily observed that, **some words in the grammar works as terminator for the sentence, these symbols are called terminal symbol. Rest of the symbols of the vocabulary are called non-terminals.** So it is clear that terminals and non-terminals forms vocabulary for any language.

Let us now formalise the idea of a grammar and how it is used. There are four important components in a grammatical description of a language :

1. There is a finite set of symbols that form the strings of the language being defined. We call these alphabets terminal symbols, represented by $V_t$.
2. There is a finite set of variables, also called sometimes non-terminals or syntactic categories. Each variable represents a language; *i.e.*, a set of strings, represented by $V_n$.
3. One of the variable represents the language being defined; it is called the start symbol. Generally it is denoted by $S$.

4. There is a finite set of rules or productions that represents the recursive definition of a langauge. Each production consists of :
   (*a*) A variable that is being (partially) defined by the production. This variable is often called the head of the production.
   (*b*) The production symbol $\rightarrow$
   (*c*) A string of zero or more terminals and variables. This string, called the body of the production, represents one way to form strings in the language of the variable of the head. So, we leave terminals unchanged and substitute for each variable of the body any string that is known to be in the language of that variable.

After this discussion, we are able to define context free grammar.

## 6.2. CONTEXT-FREE GRAMMARS

Now let us formalize the concept of context free grammar, as we discussed intuitive notions for context free grammar in previous section.

Mathmatically context-free grammar is defined as follows :

"A grammar $G = (V_n, V_t, P, S)$ is said to be context-free"

where $V_n$ : A finite set of non-terminals, generally represented by capital letters, $A, B, C, D, \ldots$ .

$V_t$ : A finite set of terminals, generally represented by small letters, like, $a, b, c, d, e, f, \ldots$

$S$ : Starting non-terminal, called start symbol of the grammar. $S$ belongs to $V_n$.

$P$ : Set of rules or productions in CFG.

$G$ is context-free and all productions in $P$ have the form

$$\alpha \rightarrow \beta$$

where

$$\alpha \in V_n \text{ and } \beta \in (V_t \cup V_n)^*$$

Every regular grammar is context-free, so a regular language is also a context-free one. It is already proved by pumping lemma that language $\{a^n b^n / n > 0\}$ is not regular, but it is possible to design a context-free-grammar for these languages (we will design it in next section). **So now it is very clear that "the family of regular language is a proper subset of the family of context-free language".**

Context-free grammars derive their name from the fact that the substitution of the variable on the left of a production can be made any time, such a variable appears in the sentential form. It does not depend on the symbols in the rest of the sentential form (the context). This feature is the consequence of allowing only a single variable on the left side of the production.

### 6.2.1. Derivations

We now define the notations to represent a derivation. First we define two notations $\underset{G}{\Rightarrow}$ and $\overset{*}{\underset{G}{\Rightarrow}}$. If $\alpha \rightarrow \beta$ is a production of $P$ in CFG and $a$ and $b$

are strings in $(V_n \cup V_t)^*$, then

$$a\ \alpha\ b \underset{G}{\Rightarrow} a\ \beta\ b.$$

We say that the production $\alpha \to \beta$ is applied to the string $a\ \alpha\ b$ to obtain $a\ \beta\ b$ or we say that $a\ \alpha\ b$ directly drives $a\ \beta\ b$.

Now suppose $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m$ are string in $(V_n \cup V_t)^*$,

$$m \geq 1 \text{ and } \alpha_1 \underset{G}{\Rightarrow} \alpha_2, \alpha_2 \underset{G}{\Rightarrow} \alpha_3, \alpha_3 \underset{G}{\Rightarrow} \alpha_4, \ldots, \alpha_{m-1} \underset{G}{\Rightarrow} \alpha_m$$

Then we say that $\alpha_1 \overset{*}{\underset{G}{\Rightarrow}} \alpha_m$, *i.e.*, we say $\alpha_1$, drives $\alpha_m$ in grammar $G$. If $\alpha$ drives $\beta$ by exactly $i$ steps, we say $\alpha \overset{i}{\underset{G}{\Rightarrow}} \beta$.

### 6.2.2. Language of Context-Free Grammar

If $G = (V_n, V_t, P, S)$ is a CFG, the language of $G$, denoted by $L(G)$, is the set of terminal strings that have derivations from the start symbol. That is

$$L(G) = \left\{ w \text{ in } V_t \,/\, S \overset{*}{\underset{G}{\Rightarrow}} w \right\}$$

If a language $L$ is the language of some context-free grammar, then $L$ is said to be context-free language, or CFL.

### 6.2.3. Sentential Forms

Derivations from the start symbol produce strings that have a special rule. We call these "Sentential forms". That is, if $G = (V_n, V_t, P, S)$ is a CFG, then any string $\alpha$ in $(V_n \cup V_t)^*$ such that $S \overset{*}{\Rightarrow} \alpha$ is a sentential form.

Note that the language $L(G)$ is those sentential forms that are in $V_t^*$, *i.e.*, they consists solely of terminals.

Now let us discuss some examples.

**Example 6.1. Consider a grammar $G = (V_n, V_t, P, S)$ where $V_n = \{S\}$, $V_t = \{a, b\}$ and set of productions $P$ is given by**

$$P = \{S \to aSb$$
$$S \to ab$$
$$\}.$$

Here $S$ is the only non-terminal which is the starting symbol for the grammar; '$a$' and '$b$' are terminals. There are two productions $S \to aSb$ and $S \to ab$. Now we will show how the strings $a^2b^2$ can be derived.

$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aabb \\ &\Rightarrow a^2b^2 \end{aligned}$$

Here we need to apply the first production then second production.

By applying first production $n - 1$ times, followed by an application of second production, we get

$$\begin{aligned} S &\Rightarrow aSb \\ &\Rightarrow aaSbb \\ &\Rightarrow : \end{aligned}$$

$$\Rightarrow a^{n-1}Sb^{n-1}$$
$$\Rightarrow a^n b^n$$

Hence we can say that language for the above grammar is

$$L(G) = \{a^n b^n / n >= 1\}.$$

**Example 6.2. Following is a CFG for the language**

$$L = \{wcw^R / w \in (a, b)^*\}$$

**Solution.** Let G be CFG for language $L = \{wcw^R / w \in (a, b)^*\}$

$$G = (V_n, V_t, P, S)$$

Here

$$V_n = \{S\}$$
$$V_t = \{a, b, c\}$$

and $P$ is given by

$$S \rightarrow aSa$$
$$S \rightarrow bSb$$
$$S \rightarrow c$$

Let us check that $abbcbba$ can be drived from the given CFG.

$$S \Rightarrow aSa \quad \text{(use the } S \rightarrow aSa)$$
$$\Rightarrow abSba \quad \text{(use the } S \rightarrow bSb)$$
$$\Rightarrow abbSbba \quad \text{(use the } S \rightarrow bSb)$$
$$\Rightarrow abbcbba \quad \text{(use the } S \rightarrow c)$$

So string $abbcbba$ can be derived from given CFG.

### 6.2.4. Bacos Naur Form (BNF)

While linguists were studying CFG's computer scientists began to describe programming languages by notation called Bacus-Naur form or Bacus Normal Form, which is the CFG notation with minor changes in format and some shorthand. For example, the above grammar can be rewritten in BNF as

$$S \rightarrow bA/aB$$
$$A \rightarrow bAA/aS/a$$
$$B \rightarrow aBB/bS/a$$

Hence BNF is a shorthand notation for context free grammar.

**Example 6.3. Write a CFG, which generates string of balanced parenthesis.**

**Solution.** This grammar will accept the balanced right and left parenthesis. For example ( ) ( ) is acceptable, ((( ))) is also accepted.

Let us design the CFG for this

Let CFG be

$$G = (V_n, V_t, P, S) \text{ where}$$
$$V_n = \text{set of non-terminals} = \{S\},$$
$$V_t = \text{set of terminals} = \{(, )\}$$

and set of production $P$ is given by

$$S \rightarrow SS$$
$$S \rightarrow (S)$$
$$S \rightarrow \epsilon$$

CS CamScanner

Now we see what this grammar generates :

$$\begin{aligned} S &\Rightarrow SS \\ &\Rightarrow (S)\ S \\ &\Rightarrow (S)\ SS \\ &\Rightarrow (S)\ (S)\ S \\ &\Rightarrow (S)\ (S)\ (S) \\ &\Rightarrow (\ )\ (S)\ (S) \\ &\Rightarrow (\ )\ (\ )\ (S) \\ &\Rightarrow (\ )\ (\ )\ (\ ) \end{aligned}$$

Thus the above grammar always generates balanced pairs of parenthesis.

**Example 6.4. Write a CFG, which generates palindrome for binary numbers.**

**Solution.** Grammar will generate palindrome for binary numbers, that is 00, 010, 11, 101, 11100111 ...

Let CFG be $G = (V_n, V_t, P, S)$

where $V_n$ = set of non-terminal = $\{S\}$

$V_t$ = set of terminals = $\{0, 1\}$

and production rule $P$ is defined as

$$S \rightarrow 0S0/1S1$$
$$S \rightarrow 0/1/\epsilon$$

Obviously this grammar generates palindrome for binary numbers, it can be seen by the following derivation

$$\begin{aligned} S &\Rightarrow 0S0 \\ &\Rightarrow 01S10 \\ &\Rightarrow 010S010 \\ &\Rightarrow 0101010 \end{aligned}$$

which is a palindrome.

**Example 6.5. Write a CFG for the regular expression**

$$r = 0^* 1(0 + 1)^*$$

**Solution.** Let us analyse regular expression

$$r = 0^* 1 (0 + 1)^*$$

Clearly regular expression is the set of strings which starts with any number of 0's, followed by a one and end with any combination of 0's and 1's.

Let CFG be $G = (V_n, V_t, P, S)$

where $V_n = \{S, A, B\}$

$V_t = \{0, 1\}$

and productions $P$ are defined as

$$S \rightarrow A1B$$
$$A \rightarrow 0A/\epsilon$$
$$B \rightarrow 0B/1B/\epsilon$$

Let us see the derivation of the string 00101

$$\begin{aligned} S &\Rightarrow A1B \\ &\Rightarrow 0A10B \\ &\Rightarrow 00A101B \\ &\Rightarrow 00101 \end{aligned}$$

So clearly G is CFG for regular expression $r$.

**Example 6.6. Write a CFG which generates strings having equal number of $a$'s an $b$'s.**

**Solution.** Let CFG be

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S\}$$
$$V_t = \{a, b\}$$

where $P$ is defined as

$$S \rightarrow aSbS / bSaS / \in$$

Let us derive a string

$$w = \text{bbabaabbaa}$$

$$\begin{aligned} S &\Rightarrow b\underline{S}aS \\ &\Rightarrow bb\underline{S}aSaS \\ &\Rightarrow bbaSb\underline{S}aSaS \\ &\Rightarrow bbaSba\underline{S}bSaSaS \\ &\Rightarrow bba\underline{S}baaSbSbSaSaS \\ &\Rightarrow bbabaa\underline{S}bSbSaSaS \\ &\Rightarrow bbabaaba\underline{S}bSaSaS \\ &\Rightarrow bbabaabb\underline{S}aSaS \\ &\Rightarrow bbabaabba\underline{S}aS \\ &\Rightarrow bbabaabbaa\underline{S} \\ &\Rightarrow bbabaabbaa \end{aligned}$$

**Example 6.7. Design a CFG, which can generate string, having any combination of $a$'s and $b$'s, except null string.**

**Solution.** Let CFG be $G = (V_n, V_t, P, S)$

$$V_n = \{S\}$$
$$V_t = \{a, b\}$$

Productions $P$ are defined as

$(P_1$, say) $S \rightarrow aS$

$(P_2$, say) $S \rightarrow bS$

$(P_3$, say) $S \rightarrow a$

$(P_4$, say) $S \rightarrow b$

We can produce the word $baab$ as follows :

$$\begin{aligned} S &\Rightarrow bS \text{ (by } P_2) \\ &\Rightarrow baS \text{ (by } P_1) \\ &\Rightarrow baaS \text{ (by } P_1) \\ &\Rightarrow baab \text{ (by } P_4) \end{aligned}$$

The language generated by this CFG is the set of all possible strings of the letters $a$ and $b$ except for the null string, which we cannot generate.

We can generate any word by the following algorthim :

At the beginning, the working string is the start symbol $S$ select a word to be generated. Read the letters of the desired word from left to right, one at a time. If an $a$ is read that is not the last letter of the word, apply $P_1$ to the working string. If '$b$' is read that is not the last letter of the word, apply $P_2$ to the working string.

If the last letter is read and it is an $a$, apply $P_3$ to the working string. If the last letter is read and it is $b$, apply $P_4$ to the working string.

Productions 3 and 4 can be used only once and one of them can be used. For example, to generated $babb$, we apply in order productions $P_2$, $P_1$, $P_2$ and $P_4$ as below

$$S \Rightarrow bS \Rightarrow baS \Rightarrow babS \Rightarrow babb.$$

**Example 6.8. Design CFG for regular expression**

$$r = (a + b)^* \, aa \, (a + b)^*.$$

**Solution.** Let CFG be

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, T\}$$
$$V_t = \{a, b\}$$

Productions $P$ are defined as

$$S \rightarrow TaaT \quad \text{(say } P_1\text{)}$$
$$T \rightarrow aT \quad \text{(say } P_2\text{)}$$
$$T \rightarrow bT \quad \text{(say } P_3\text{)}$$
$$T \rightarrow \in \quad \text{(say } P_4\text{)}$$

Last three productions, *i.e.*, $P_2$, $P_3$ and $P_4$ allows us to generate any word we want from terminal $T$. If the non-terminal $T$ appears in any working string, we can apply productions to turn in to any string we want. Therefore, the words generated from $S$ have the form any substring $aa$ any substring

or $\quad (a + b)^* \, aa \, (a + b)^*$

which is the language of the all words with a double '$a$' in them somewhere.

For example to generate baabaab, we can proceed as follows :

$$\begin{aligned} S &\Rightarrow TaaT \\ &\Rightarrow bTaaT \\ &\Rightarrow baTaaT \\ &\Rightarrow baaTaaT \\ &\Rightarrow baabTaaT \\ &\Rightarrow baab\in aaT \\ &\Rightarrow baabaaT \\ &\Rightarrow baabaabT \\ &\Rightarrow baabaab\in \\ &\Rightarrow baabaab \end{aligned}$$

There are other sequences that can also derive the word $baabaab$.

**Example 6.9. Design a CFG for the regular expression $r = (a + b)^*$.**

**Solution.** Let G be CFG

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S\}$$
$$V_t = \{a, b\}$$

$P$ are desined as

$$S \to aS$$
$$S \to bS$$
$$S \to a$$
$$S \to b$$
$$S \to \epsilon$$

The word $ab$ can be generated by the derivation

$$\begin{aligned} S &\Rightarrow aS \\ &\Rightarrow abS \\ &\Rightarrow ab\epsilon \\ &\Rightarrow ab \end{aligned}$$

or by the derivation

$$\begin{aligned} S &\Rightarrow aS \\ &\Rightarrow ab. \end{aligned}$$

**Example 6.10. The difference between even 'palindrome' and 'odd palindrome' (whose definitions is obvious) is that when we are finally ready to get rid of $S$ in the even palindrome working string, we must replace it with $\epsilon$. If we were force to replace it with an $a$ or $b$ instead, we would create a central letter and the result would be a grammar for odd palindrome as follows :**

$$S \to aSa$$
$$S \to bSb$$
$$S \to a$$
$$S \to b$$

**Solution.** If we allow the option of turning the central $S$ into either $\epsilon$ or a letter, we would have a grammar for the entire language palindrome :

$$S \to aSa$$
$$S \to bSb$$
$$S \to a$$
$$S \to b$$
$$S \to \epsilon$$

The language $\{a^n b^n\}$ and palindrome are amazingly similar in grammartical structure while the first is nearly a regular expression and the other is for from it.

**Example 6.11. Write a CFG for the language $L(G) = \{ww^r : w \in \{0, 1\}^*\}$.**

**Solution.** Let grammar be G.

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S\}$$
$$V_t = \{0, 1\}$$

Productions are defined as

$$S \to 0S0 \qquad (P_1)$$
$$S \to 1S1 \qquad (P_2)$$
$$S \to \epsilon \qquad (P_3)$$

Let us derive a string $\underset{w}{\underline{001}}\,\underset{w^r}{\underline{100}}$ by above CFG.

$$\begin{aligned} S &\Rightarrow 0S0 && \text{(by } P_1\text{)} \\ &\Rightarrow 00S00 && \text{(by } P_1\text{)} \\ &\Rightarrow 001S100 && \text{(by } P_2\text{)} \\ &\Rightarrow 001100 && \text{(by } P_3\text{)} \end{aligned}$$

**Example 6.12. Design a CFG for the language**

$$L(G) = \{ab\,(bbaa)^n\,bba\,(ba)^n : n \geq 0\}.$$

**Solution.** Let CFG be G

$$G = (V_n, V_t, P, S)$$
$$V_n = (S, X, Y)$$
$$V_t = \{a, b\}$$

Productions are defined as $S \to abX$

$$X \to bbYa$$
$$Y \to aaXb$$
$$Y \to \epsilon$$

or we can also defined production as :

$$S \to abX$$
$$X \to bbaaXba$$
$$X \to bba$$

Let us derive a string $ab(bbaa)^2\,bba(ba)^2$

$$\begin{aligned} S &\Rightarrow ab\underline{X} \\ &\Rightarrow abbb\underline{Y}a \\ &\Rightarrow abbbaa\underline{X}ba \\ &\Rightarrow abbbaabb\underline{Y}aba \\ &\Rightarrow abbbaabbaa\underline{X}baba \\ &\Rightarrow ab(bbaa)^2bbYa(ba)^2 \\ &\Rightarrow ab(bbaa)^2bb\epsilon a(ba)^2 \\ &\Rightarrow ab(bbaa)^2bba(ba)^2 \end{aligned}$$

or

$$\begin{aligned} S &\Rightarrow abX \\ &\Rightarrow abbbaaXba \\ &\Rightarrow abbbaabbaaXbaba \\ &\Rightarrow ab(bbaa)^2bba(ba)^2. \end{aligned}$$

**Example 6.13. Design a CFG for the language**

$$L = \{a^n\,b^m : n \neq m\}.$$

**Solution.** If $n \neq m$ then there are only two cases are possible.

**Case 1.** $n > m$

Let us say langauge $L$ on condition $n > m$ is

$$L_1 \text{ and } L_1 = \{a^n b^m : n > m\}.$$

Let us say $G_1$ be the CFG for the language $L_1$,

$$G_1 = (V_n^1, V_t^1, P^1, S^1)$$
$$V_n^1 = \{S_1, A, S^1\}$$
$$V_t^1 = \{a, b\}$$

then productions $P^1$ are defined as

$$S^1 \to AS_1$$
$$S_1 \to aS_1b/\in$$
$$A \to aA/a$$

**Case 2.** $n < m$

Let us say $L$ on condition $n < m$ is $L_2$

$$L_2 = \{0^n 1^m : n < m\}$$

and

Let CFG for $G_2$ be $G_2$

$$G_2 = (V_n^2, V_t^2, P^2, S^2)$$
$$V_n^2 = \{S^2, S_2, B\}$$
$$V_t^2 = \{a, b\}$$

$P^2$ are defined as follows :

$$S^2 \to S_2B$$
$$S_2 \to aS_2b/\in$$
$$B \to bB/b$$

by the help (by the combining $G_1$ and $G_2$) of $G_1$ and $G_2$ we can write the CFG for $L$ as

$$S \to S^1/S^2$$

where $S$ is start symbol of CFG for $L$.

**Example 6.14. Design a CFG for the language**

$$L = \{0^n 1^n/n \geq 0) \cup (1^n 0^n/n \geq 0)\}.$$

**Solution.** We can assure $L$ as

$$L = L_1 \cup L_2$$
$$L_1 = \{0^n 1^n/n \geq 0\}$$

Let say $G_1$ be CFG for the language $L_1$

$$G_1 = (V_n, V_t, P, S)$$
$$V_n = \{S_1\}$$
$$S = \{S_1\}$$
$$V_t = \{0, 1\}$$

and $P$ is defined as

$$S_1 \to 0S_11/\in$$

and

$$L_2 = \{1^n 0^1/n \geq 0\}$$

Let say $G_2$ be CFG for language $L_2$

$$G_2 = (V_n, V_t, P, S)$$
$$V_n = \{S_2\}$$

$$S = \{S_2\}$$
$$V_t = \{0, 1\}$$

$P$ is defined as

$$S_2 \rightarrow 1S_20/\epsilon$$

Since $L = L_1 \cup L_2$, suppose G be CFG for language $L$, with starting non-terminal $S$.

Then productions in $G$ will be

$$S \rightarrow S_1/S_2$$
$$S_1 \rightarrow 0S_11/\epsilon$$
$$S_2 \rightarrow 1S_20/\epsilon.$$

**Example 6.15. Design CFG for $\Sigma = \{a, b\}$ that generates the set of**

**(a) all strings with exactly one $a$.**

**(b) all strings with at least one $a$.**

**(c) all strings with at least 3 $a$'s.**

**Solution.** (a) Let CFG be X $G_1$

$$G_1 = (V_n, V_t, P, S)$$
$$V_n = \{S, A\}$$
$$V_t = \{a, b\}$$

$P$ is defined as

$$S \rightarrow AaA$$
$$A \rightarrow bA/\epsilon$$

Let us derive a string $bbaab$

$$S \Rightarrow AaA$$
$$\Rightarrow bAaA$$
$$\Rightarrow bbAaA$$
$$\Rightarrow bbAabA$$
$$\Rightarrow bbAab\epsilon$$
$$\Rightarrow bb\underline{A}ab$$

How we don't have any choice for derving second $G$ so string $bbaab$ cannot be derive from $G_1$.

(b) Let CFG be $G_2$

$$G_2 = (V_n, V_t, P, S)$$
$$V_n = \{S, A\}$$
$$V_t = \{a, b\}$$

$P$ is defined as

$$S \rightarrow AaA$$
$$A \rightarrow aA/bA/\epsilon$$

Let us derive string $baab$

$$S \Rightarrow AaA$$
$$\Rightarrow bAaA$$
$$\Rightarrow baAaA$$
$$\Rightarrow ba\epsilon AaA$$
$$\Rightarrow baaA$$

CamScanner

$$\Rightarrow baabA$$
$$\Rightarrow baab\epsilon$$
$$\Rightarrow baab$$

(c) Let CFG be $G_3$

and

$$G_3 = (V_n, V_t, P, S)$$
$$V_n = \{S, A\}$$
$$V_t = \{a, b\}$$

$P$ is defined as

$$S \rightarrow AaAaAaA$$
$$A \rightarrow aA/bA/\epsilon.$$

**Example 6.16. Give the simple description of the language generated by the grammar with production**

$$S \rightarrow aA$$
$$A \rightarrow bS$$
$$S \rightarrow \epsilon.$$

**Solution.** Given CFG is

$$S \rightarrow aA \quad \text{(say } P_1\text{)}$$
$$A \rightarrow bS \quad \text{(say } P_2\text{)}$$
$$S \rightarrow \epsilon \quad \text{(say } P_3\text{)}$$

From the production $P_3$, CFG can generate null string. Now let us see production $P_1$ (*i.e.* $S \rightarrow aA$), once we use it then we have to $A \rightarrow bS$. Now again we have to use either production $P_1$ ($S \rightarrow aA$) or production $P_3$ ($S \rightarrow \epsilon$), so clearly the langauge of above CFG will be as follows :

$$L = \{(ab)^n / n \geq 0\}$$

Clearly language $L$ is the set of strings. Starts with $ab$ followed by any number of $ab$'s. Null string ($\epsilon$) also included in language $L$.

**Example 6.17. Give the simple description of the language generated by the grammar with productions.**

$$S \rightarrow SS$$
$$S \rightarrow (S)$$
$$S \rightarrow \epsilon.$$

**Solution.** Given CFG is

$$S \rightarrow SS$$
$$S \rightarrow (S)$$
$$S \rightarrow (\epsilon)$$

The language generated by the CFG is set of strings, contains balanced set of paranthesis that is for every left paranthesis there is a right paranthesis and strings like ( ) ( ) ( ) ... or ((( ))) ( ) ( ) ((( ))) ... are the part of the language of CFG.

**Example 6.18. Write the CFG for the language**

$$L = \{a^n b^n c^m d^m / n >= 1, m >= 1\}.$$

**Solution.** Let CFG for the language $L$ be $G$

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, X, Y\}$$
$$V_t = \{a, b\}$$

Productions $P$ is defined as follows :

$$S \rightarrow XY \qquad \text{(Production } P_1\text{)}$$
$$X \rightarrow aXb/ab \qquad \text{(Production } P_2\text{)}$$
$$Y \rightarrow cYd/cd \qquad \text{(Production } P_3\text{).}$$

When we use $P_1$ (*i.e.* $S \rightarrow XY$) then $X$ can generate equal number of $a$'s and $b$'s with no $a$'s follows when any $b$ is encountered. Similarly $Y$ can generate strings of equal number of $c$'s and $d$'s with no $c$'s follows when any $d$ is encountered.

Let us derive a string aaabbbcd that $n = 3$, $m = 1$

$$\begin{aligned} S &\Rightarrow XY \\ &\Rightarrow aXbY \\ &\Rightarrow aaXbbY \\ &\Rightarrow aaabbbY \\ &\Rightarrow aaabbbcd \end{aligned}$$

which required string.

**Example 6.19. Design a CFG for the language**

$$L = \{a^n\, b^m\, c^m\, d^n / n \geq 1,\ m >= 1\}.$$

**Solution.** Let us analyse the language $L$ first, language $L$ is the set of strings which contain equal number of $a$'s and $d$'s and equal number of $b$'s and $c$'s in between $a$'s and $d$'s. There is no '$a$' followed by $b$, $c$ and, there is no '$b$' followed by $c$ and, there is no $c$ followed by $d$. Similarly there is no $b$, $c$, and $d$ before any '$a$', there is no '$c$' and $d$ before any '$b$', finally there is no $d$ before any $c$.

On the base of above description let us design a CFG for the language.

Let us assume CFG be $G$

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, A\}$$
$$V_t = \{a, b, c, d\}$$

and $P$ is defined as

$$S \rightarrow aSd \qquad \text{(say } P_1\text{)}$$
$$S \rightarrow aAd \qquad \text{(say } P_2\text{)}$$
$$A \rightarrow bAc \qquad \text{(say } P_3\text{)}$$
$$A \rightarrow bc \qquad \text{(say } P_4\text{)}$$

Let us derive a string *aabbbcccdd* from grammar $G$.

$$\begin{aligned} S &\Rightarrow aSd && \text{(by } P_1\text{)} \\ &\Rightarrow aaAdd && \text{(by } P_2\text{)} \\ &\Rightarrow aabAcdd && \text{(by } P_3\text{)} \\ &\Rightarrow aabbAccdd && \text{(by } P_3\text{)} \\ &\Rightarrow aabbbcccdd && \text{(by } P_4\text{)} \end{aligned}$$

which is required string.

**Example 6.20. Design a CFG for the language $L = \{a^n\, b^{2n} / n >= 0\}$.**

**Solution.** Let CFG be $G$ for the language $L$

$$G = (V_n, V_t, P, S)$$

$$V_n = \{S\}$$
$$V_t = \{a, b\}$$

and $P$ is defined as

$$S \rightarrow aSbb / \in$$

Let us derive a string *aaabbbbbb*

$$\begin{aligned} S &\Rightarrow aSbb \\ &\Rightarrow aaSbbbb \\ &\Rightarrow aaaSbbbbbb \\ &\Rightarrow aaa\in bbbbbb \\ &\Rightarrow aaabbbbbb \end{aligned}$$

which is required string.

Now let us derive another string *abbb*.

$$S \Rightarrow aSbb$$

From this derivation only it is clear that third $b$ is not possible against a single '$a$'.

**Example 6.21. Write the CFG for the language $L = \{a^{2n} b^m / n > 0, m > = 0\}$.**

**Solution.** Let us analyse the language first language $L$ is the set of strings, if string starts with a then number of $a$'s are even, followed by any number of times $b$'s. There is no '$a$' in any string after first '$b$' is encountred similarly there is no '$b$' before any '$a$' in the string.

Let suppose CFG for the $L$ is $G$

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, A, B\}$$
$$V_t = \{a, b\}$$

Productions $P$ is defined as

$$S \rightarrow aaAB \quad \text{(say } P_1\text{)}$$
$$A \rightarrow aaA / \in \quad \text{(say } P_2\text{)}$$
$$B \rightarrow bB / \in \quad \text{(say } P_3\text{)}$$

Let us derive the string *aaaabbb*

$$\begin{aligned} S &\Rightarrow aaAB && \text{(by } P_1\text{)} \\ &\Rightarrow aaaaAB && \text{(by } P_2\text{)} \\ &\Rightarrow aaaa\in B && \text{(by } P_2\text{)} \\ &\Rightarrow aaaabB && \text{(by } P_3\text{)} \\ &\Rightarrow aaaabbB && \text{(by } P_3\text{)} \\ &\Rightarrow aaaabb\in && \text{(by } P_3\text{)} \\ &\Rightarrow aaaabb \end{aligned}$$

which is required string.

**Example 6.21(*a*). Write CFG for**

$$L = \{(a^n b^n c^m d^m / n > = 1, m > = 1) \cup (a^n b^m c^m d^n / n > = 1, n > = 1)\}$$

**Solution.** Let suppose $L = L_1 \cup L_2$

where

$$L_1 = \{a^n b^n c^m d^m / n > = 1, m > = 1\}$$

Let CFG for $L_1$ is $G_1$

$$G_1 = (V_n, V_t, P_1, S_1)$$
$$V_n = \{S_1, A, B\}$$
$$V_t = \{a, b, c, d\}$$

and $P_1$ is defined as follows

$$S_1 \rightarrow AB$$
$$A \rightarrow aAb/ab$$
$$B \rightarrow cBd/cd$$
$$L_2 = \{a^n b^m c^m d^n / n >= 1, m >= 1\}$$

Let us assume that CFG for $L_2$ is $G_2$

$$G_2 = (V_n, V_t, P_2, S_2)$$

and

$$V_n = \{S_2, D, E\}$$
$$V_t = \{a, b, c, d\}$$

$P_2$ is defined as follows

$$S_2 \rightarrow aDd$$
$$D \rightarrow aDd/E$$
$$E \rightarrow bEc/bc$$

By the help of the $G_1$ and $G_2$, we can define CFG for the language $L$, since $L = L_1 \cup L_2$

$$\text{so } G = G_1 \cup G_2$$

Let suppose

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, S_1, S_2, A, B, D, E\}$$
$$V_t = \{a, b, c, d\}$$

and $P$ is defined as

$$S \rightarrow S_1/S_2 \qquad (\text{for } G = G_1 \cup G_2)$$
$$S_1 \rightarrow AB$$
$$A \rightarrow aAb/ab$$
$$B \rightarrow cBd/cd$$
$$S_2 \rightarrow DE$$
$$D \rightarrow aDd/E$$
$$E \rightarrow bEc/bc$$

**Example 6.22. Write a CFG for language $L = \{0^i 1^j 2^k / k \le i \text{ or } k \le j\}$.**

**Solution.** In the same fashion as we solve Example 6.21, we approach this problem.

Let us assume $L = L_1 \cup L_2$

where

$$L_1 = \{0^i 1^j 2^k \ / k \le i\}$$

and

$$L_2 = \{0^i 1^j 2^k / k \le j\}$$

Let as assume that CFG for $L_1$ is $G_1$

$$G_1 = (V_n, V_t, P_1, S_1)$$
$$V_n = \{S_1, X, Y\}$$
$$V_t = \{0, 1\}$$

and $P_1$ is defined as

$$S_1 \rightarrow 0S_12/0Y2$$
$$Y \rightarrow C/0Y/\epsilon$$
$$C \rightarrow 1C/\epsilon$$

Let us assume that CFG for $L_2$ is $G_2$

$$G_2 = (V_n, V_t, P_2, S_2)$$
$$V_n = \{S_2, A, B\}$$
$$V_t = \{0, 1\}$$

$P_2$ is defined as follows

$$S_2 \rightarrow AB$$
$$A \rightarrow 0A/\epsilon$$
$$B \rightarrow 1B2/1B/\epsilon$$

We can define CFG for $L$ (since $L = L_1 \cup L_2$)

Let us assume that CFG for $L$ is $G$

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S_1, S_2, X, Y, Z, B\}$$
$$V_t = \{0, 1\}$$

$P$ is defined as

$$S \rightarrow S_1/S_2$$
$$S_1 \rightarrow 0S_12/0Y2$$
$$Y \rightarrow C/0Y/\epsilon$$
$$C \rightarrow 1C/\epsilon$$
$$S_2 \rightarrow AB$$
$$A \rightarrow 0A/\epsilon$$
$$B \rightarrow 1B2/1B/\epsilon.$$

**Example 6.23. Write the CFG for the langauge**

$$L = \{0^i 1^j 2^k / i = j \text{ or } j = k\}.$$

**Solution.** Let us assume $L = L_1 \cup L_2$

where

$$L_1 = \{0^i 1^j 2^k / i = j\}$$

and

$$L_2 = \{0^i 1^j 2^k / j = k\}$$

Let us consider $L_1$ first, let CFG for $L_1$ be $G_1$

$$G_1 = (V_n, V_t, P_1, S_1)$$
$$V_n = \{S_1, A, B\}$$
$$V_t = \{0, 1, 2\}$$

$P_1$ is defined as follows

$$S_1 \rightarrow AB$$
$$A \rightarrow 0A1/\epsilon$$
$$B \rightarrow 2B/\epsilon$$

Now let us assume that CFG for language $L_2$ is $G_2$

$$G_2 = (V_n, V_t, P_2, S_2)$$
$$V_n = (S_2, C, D)$$
$$V_t = (0, 1, 2)$$

Productions are defined as follows :

$$S_2 \rightarrow CD$$
$$C \rightarrow 0C/\epsilon$$
$$D \rightarrow 1D2/\epsilon$$

By the help $G_1$ and $G_2$ we can define CFG for the language $L$, Let it is $G$.

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, S_1, S_2, A, B, C, D\}$$
$$V_t = \{0, 1, 2\}$$

Productionsa are defined as follows

$$S \rightarrow S_1/S_2$$
$$S_1 \rightarrow AB$$
$$A \rightarrow 0A1/\epsilon$$
$$B \rightarrow 2B/\epsilon$$
$$S_2 \rightarrow CD$$
$$C \rightarrow 0C/\epsilon$$
$$D \rightarrow 1D2/\epsilon$$

which is required CFG.

**Example 6.24. Find the CFG for the following language**

$$L = \{a^n b^{2n} c^m / n, m >= 0\}.$$

**Solution.** Let us analyse the language first, $L$ is language, which contain a set of strings such that every string may start from $a$ or $c$ but not by $b$.

If string starts with '$a$' then number of $a$'s must follow $b$'s and the, number of $b$'s are twice then the number of $a$'s. If string does start with a then it starts with $c$, followed by any number of $c$'s. There should be no a after any $b$ or any $c$, and no $b$ after any $c$. Similarly there should be no $b$ or $c$ before '$a$' and no $c$ before any $b$.

On the basic of above discussion, let us assume that CFG for the languageis $G$.

$$G = (V_n, V_t, P, S)$$
$$V_n = \{S, A, B\}$$
$$V_t = \{a, b, c\}$$

$P$ is defined as follows

$$S \rightarrow AB$$
$$A \rightarrow aAbb/\epsilon$$
$$B \rightarrow cB/\epsilon$$

which is required CFG.

**Example 6.25. Write the CFG for the language**

$$L = \{a^n b^m c^{2m} / n, m >= 0\}.$$

**Solution.** This problem is very similar to Example 6.25.

Let us assume CFG be $G$

$$G = (V_n, V_t, P, S)$$
$$V_n = (S, A, B)$$
$$V_t = (a, b, c)$$

$P$ is defined as follows

$$S \to AB$$
$$A \to aA / \epsilon$$
$$B \to bBcc / \epsilon$$

which is required grammar.

### 6.2.5. Left most and Right most Derivations

In order to restrict the number of choices of replacement of variables, if at each step we replace the "left most" variable by one of its production bodies such a derivation is called a "left most derivation", and we indicate that a derivation is left most by using relation $\underset{lm}{\Rightarrow}$ and $\overset{*}{\underset{lm}{\Rightarrow}}$ for one or many steps, respectively.

Similarly, it is possible that at each step the "right most" variable is replaced by one of its bodies. If so, we call the derivation "right most" and use symbols $\underset{rm}{\Rightarrow}$ and $\overset{*}{\underset{rm}{\Rightarrow}}$ to indicate one or many right most derivation steps, respectively.

**Example 6.26. (*i*) Write a CFG for solving simple expression, such + and *.**
**(*ii*) Also write CFG for regular expression**

$$r = (a + b)\ (a + b + 0 + 1)^*.$$

**(*iii*) Derive the string (which is defined in part (*ii*)) $a^*\ (a + b00)$ by applying left most derivation and right most derivation.**

**Solution.** (*i*) We need two variables in this grammar. One, which we call $E$, represents the expression, it is the start symbol and represents the language of expressions we are defining. The other variable, $I$, reprsents the identifiers.

Let CFG be

$$G = (V_n, V_t, P, S)$$
$$V_n = \{E, I\}$$
$$V_t = \{+, *, (, )\}$$

$P$ is defined as

$$E \to I$$
$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$

Here $I$ can derive any terminal symbol as

$$I \to \text{terminal.}$$

(*ii*) Given regular epression is

$$(a + b)\ (a + b + 0 + 1)^*$$

Let CFG be

$$G = (V_n, V_t, P, S)$$
$$V_n = \{E, I\}$$
$$V_t = \{+, *, (, ), a, b, 0, 1\}$$

$P$ is defined as follows :

$$E \to I$$
$$E \to E * E$$
$$E \to E + E$$

$$E \to (E)$$
$$I \to a$$
$$I \to b$$
$$I \to Ia$$
$$I \to Ib$$
$$I \to I0$$
$$I \to I1$$
$$I \to 0$$
$$I \to 1.$$

(*iii*) String is $a * (a + b00)$, and we want to derive it by using left most and right most derivations, from CFG defined in part (*ii*)

By using leftmost derivation :

$$E \underset{lm}{\Rightarrow} \underline{E} * E \underset{lm}{\Rightarrow} \underline{I} * E \underset{lm}{\Rightarrow} a * \underline{E} \underset{lm}{\Rightarrow} a * (\underline{E}) \underset{lm}{\Rightarrow} a * (\underline{E} * E) \underset{lm}{\Rightarrow} a * (\underline{I} + E) \underset{lm}{\Rightarrow}$$

$$a * (a + \underline{E}) \underset{lm}{\Rightarrow} a * (a + \underline{I}0) \underset{lm}{\Rightarrow} a * (a + \underline{I}00) \underset{lm}{\Rightarrow}$$

$$\underset{lm}{\Rightarrow} a * (a + b00)$$

which is required string.

By using right most derivation :

$$E \underset{rm}{\Rightarrow} E * \underline{E} \underset{rm}{\Rightarrow} E * (\underline{E}) \underset{rm}{\Rightarrow} E * (E + \underline{E}) \underset{rm}{\Rightarrow} E * (E + \underline{I}) \underset{rm}{\Rightarrow} E * (E + \underline{I}0) \underset{rm}{\Rightarrow} E * (E + \underline{I}00) \underset{rm}{\Rightarrow}$$

$$E * (\underline{E} + b00) \underset{rm}{\Rightarrow} E * (\underline{I} + b00) \underset{rm}{\Rightarrow} \underline{E} * (a + b00) \underset{rm}{\Rightarrow}$$

$$\underset{rm}{\Rightarrow} \underline{I} * (a + b00) \underset{rm}{\Rightarrow} b * (a + b00)$$

which is required string.

## 6.3. PARSE TREE

There is a tree representation for derivation that has proved extremely useful. It is the second way of showing derivations, independent of the order in which productions are used, is also called "derivation tree". "A parse tree is an odered tree in which nodes are labeled with the left sides of productions and in which the children of a node represent its corresponding right sides".

### 6.3.1. Definition

Let $G = (V_n, V_t, P, S)$ be a context-free grammar. An ordered tree for this CFG, G is a derivation tree if and only if it has the following properties.

(*a*) The root is labeled by the starting non-terminal of the CFG that is $S$.

(*b*) Every leaf of the orderd ordered tree has a label from $V_t \cup \{\epsilon\}$.

(*c*) Every Interior node of ordered tree has a label from $V_n$.

(*d*) Let us assume that a vertex has label $X \in V_n$, and its children are labeled (from left to right) $y_1, y_1, y_3, \ldots, y_n$, then production must contain a production of the form

$$X \to y_1, y_2, \ldots, y_n.$$

(*e*) A leaf labeled $\epsilon$ has no siblings, that is, vertex with a child labeled $\epsilon$ can have no other children. Clearly if the leaf is labeled $\epsilon$, then it must be the only child of its parent.

### 6.3.2. The Yield of Parse Tree

If we look at the leaves of any parse tree and can concate them from the left, we get a string, called the yield of the tree, which is always a string that is derived from the root will be proved shortly of special importance are those parse trees such that :

(*a*) The yield is a terminal string. That is all leaves are labeled either with a terminal or with $\epsilon$.

(*b*) The root is labeled by the start symbol.

Now let us see some examples based on the parse tree.

**Example 6.27. Consider the CFG**

$$S \to XX$$
$$X \to XXX/bX/Xb/a$$

**Find the parse tree for the string *bbaaaab*.**

**Solution.** Given CFG is

$$S \to XX$$
$$S \to XXX/bX/Xb/a$$

and string is $w = bbaaaab$

We begin with $S$ and apply the production $S \to XX$.

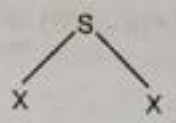



**Fig. 6.1.**

To the left hand $X$, let us apply the production $X \to bX$. To the right hand $X$, let us apply $X \to XXX$.

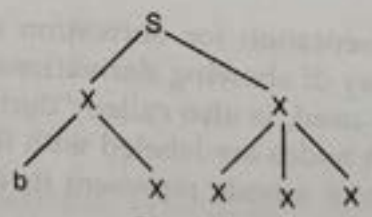



**Fig. 6.2.**

The $b$ that we have on the bottom line is a terminal, so it does not descend further. In the terminology of trees, it is called a "terminal node". Let the four $A$'s left to right, undergo the production $A \to bA$, $A \to a$, $A \to a$, and $A \to A$ respectively. We now have

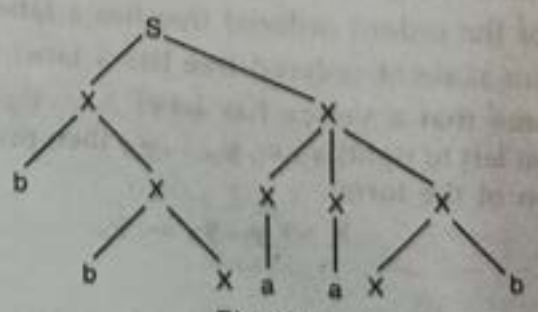



**Fig. 6.3**

Let us finish off the genration of a word with the production $X \to a$ and $X \to a$:

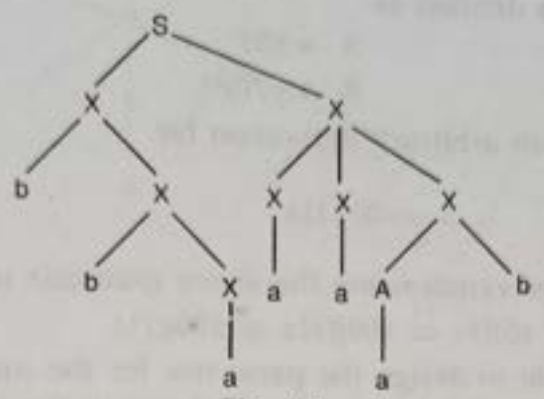



**Fig. 6.4.**

Reading from left to right, we see the word we have produced is *bbaaaab*.

These tree diagrams are called "syntax trees", "parse trees", "generation" trees, "production trees" or derivation trees.

**Example 6.28. Consider the grammar *G*, with production**

$$S \to aXY$$
$$X \to bYb$$
$$Y \to X/\epsilon.$$

**Solution.** Let us see a partial derivation tree for *G*, this tree gives *abbbb* as yield.

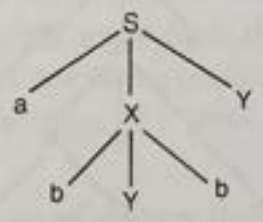



**Fig. 6.5.**

Now extend this partial derivative tree for the string *abbbb*.

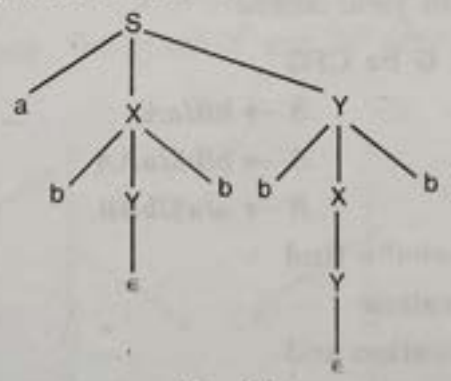



**Fig. 6.6.**

the yield is $ab\epsilon bb\epsilon b$ that is *abbbb*.

**Example 6.29. Write the CFG for the language $L = \{x\ 0^n y 1^n z / n > = 0\}$, and give the parse tree for the string x000 y11z.**

**Solution.** Let the CFG for the language *L* is

$$G = (V_n, V_t, P, S)$$

where

$$V_n = \{S, B\}$$

$$V_t = \{x, y, z, 0, 1\}$$

and production $P$ is defined as

$$S \to xBz$$
$$B \to y/0B1.$$

Now construct an arbitrary derivation for

$$S \overset{*}{\underset{G}{\Rightarrow}} x00\,y11z.$$

One possible derivation using the above grammar is

$$S \Rightarrow x\underline{B}z \Rightarrow x0B1z \Rightarrow x00\underline{B}11z \overset{*}{\Rightarrow} x00y11z$$

Now we are able to design the parse tree for the string $x00y11z$ from the above CFG. Parse tree is shown in Fig. 6.7.

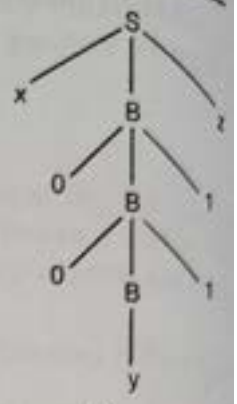



Fig. 6.7.

**Example 6.30. Consider the CFG 'G' whose productions are**

$$S \to aAS/a$$
$$A \to SbA/SS/ba.$$

**show that $S \overset{*}{\Rightarrow} aabbaa$ and construct a derivation tree whose yield is $aabbaa$.**

**Solution.** $S \Rightarrow a\underline{A}S \Rightarrow a\underline{S}bAS \Rightarrow aab\underline{A}S \Rightarrow aabbaS \Rightarrow a^2b^2a\underline{S} \Rightarrow a^2b^2a^2$

The derivation tree is given in Fig. 6.8.

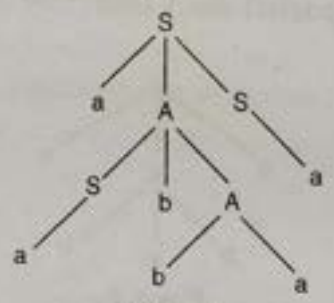



Fig. 6.8.

Derivation tree with yield $aabbaa$.

**Example 6.31. Let $G$ be CFG**

$$S \to bB/aA,$$
$$A \to b/bS/aAA$$
$$B \to a/aS/bBB.$$

**For the string $bbaababa$ find**

**(i) left most derivation**

**(ii) rightmost derivation and**

**(iii) parse tree.**

**Solution.** (i) Left most derivation for string $w = bbaababa$ is

$S \Rightarrow b\underline{B} \Rightarrow bb\underline{B}B \Rightarrow bba\underline{B} \Rightarrow bbaa\underline{S} \Rightarrow b^2a^2b\underline{B} \Rightarrow b^2a^2ba\underline{S} \Rightarrow b^2a^2bab\underline{B} \Rightarrow b^2a^2baba$

(ii) The right most derivation is

$S \Rightarrow b\underline{B} \Rightarrow bb\underline{B}B \Rightarrow bbBa\underline{S} \Rightarrow bbBab\underline{B} \Rightarrow b^2Baba\underline{S} \Rightarrow b^2Babab\underline{B} \Rightarrow b^2\underline{B}ababa \Rightarrow b^2a^2baba$

The derivation tree is following in Fig. 6.9.

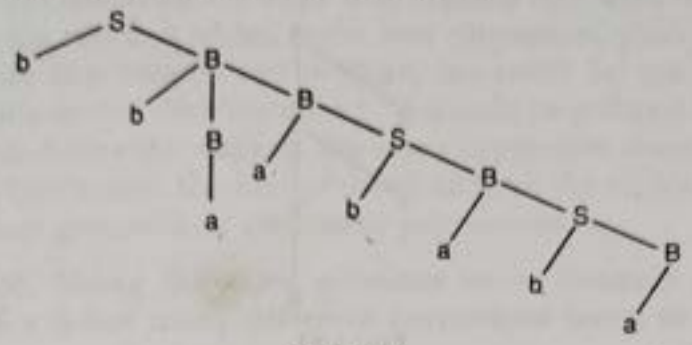



**Fig. 6.9.**

Yield is *bbaababa*.

## 6.4. AMBIGUITY IN GRAMMARS AND LANGUAGES

Grammars can be used to put structure on programs and documents. The assumption was that a grammar uniquely determines a structure for each string in its language. However not every grammar does provide unique structure.

When a grammar fails to provide unique structures, it is sometimes possible to redesign the grammar to make the structure unique for each string in the language. Unfortunately some times we cannot do so. That is, there are some CFL's that are "inherently ambigious"; every grammar for the language puts more than one structure for some strings in the language.

**Definition.** "A CFG is called ambigious if for at least one word in the lanauge that it generates, there are two possible derivations of the word that correspond to different syntax trees. If a CFG is not ambiguous, it is called unambiguous.

**Example 6.32. The language of all non null strings of *a*'s can be defined by a CFG as follows :**

$$S \rightarrow aS/Sa/a$$

**Solution.** In this case, the word $a^3$ can be generated by four different tree's :

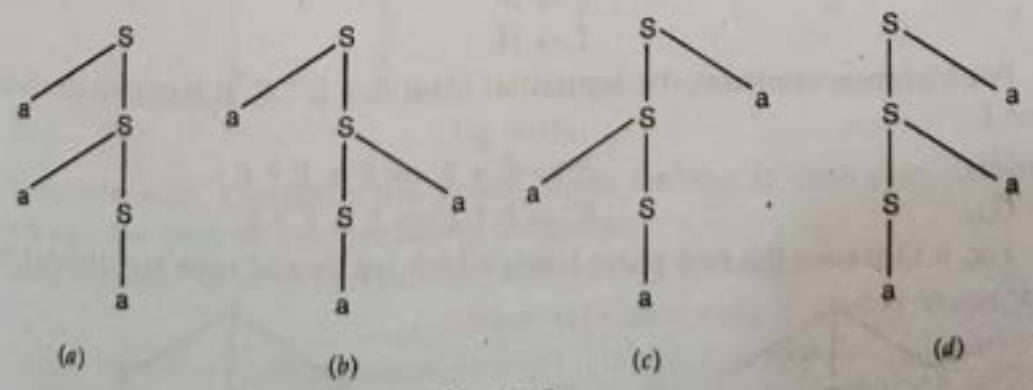



**Fig. 6.10.**

The CFG is therefor ambigious.

However, the same language can also be defined by the CFG

$$S \rightarrow aS/a$$

for which $a^3$ has only one production :

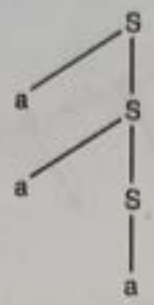



Fig. 6.11.

**Example 6.33. The CFG $S \to aSb/SS/\epsilon$ is ambigous. The sentence $aabb$ has the two derivation trees shown in Fig. 6.12 (*a*) and 6.12 (*b*).**

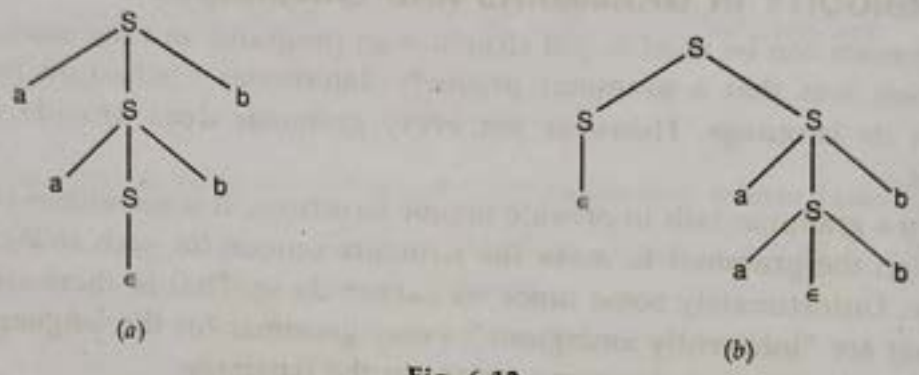



Fig. 6.12.

**Example 6.34. Consider the grammar**

$$E \to I$$
$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$I \to a$$
$$I \to b$$
$$I \to Ia$$
$$I \to Ib$$
$$I \to I0$$
$$I \to I1$$

For instance, consider the sentential form $E + E * E$. It has two derivation from $E$ :

(1) $$E \Rightarrow E + E \Rightarrow E + E * E$$

(2) $$E \Rightarrow E * E \Rightarrow E + E * E$$

Fig. 6.13 shows the two parse trees, which we should note are distinct trees

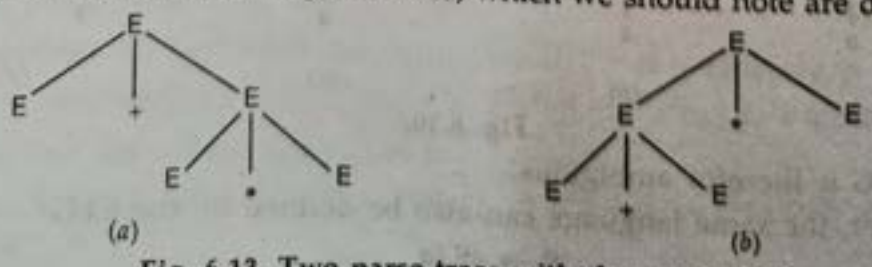



**Fig. 6.13. Two parse trees with the same yield.**

The difference between two derivations is significant. As the structure of the expression is concerned (1) says that second and third expression are multiplied, and the result is added to the first expression, while derivation (2) adds the first two expressions and multiply the result by the third one. For example, according to first derivation 2 + 5 * 3 should be grouped 2 + (5 * 3) = 17, while the second derivation suggest the same expression should be grouped (2 + 5) * 3 = 21. Obviously, the first of these, and not the second, matches our notation of correct grooping of arithmatic expressions.

**Example 6.35. Using the same grammar as in Example 6.34, we find that strng $a + b \times b$ has many different derivations trees, as follows.**

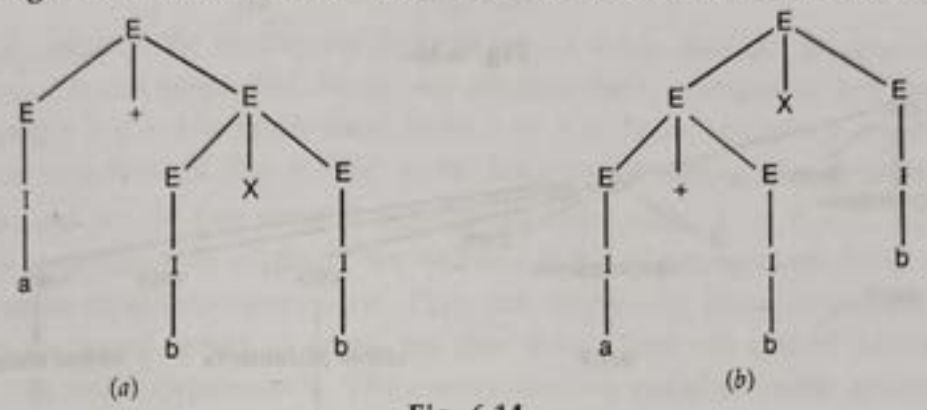
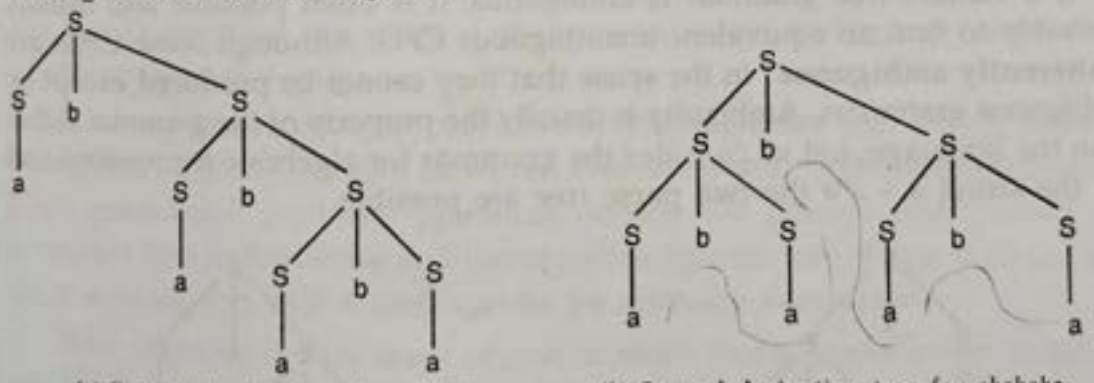



**Fig. 6.14.**

**Example 6.36. If CFG a is $S \rightarrow SbS/a$, show that $G$ is ambigous.**

**Solution.** To prove that $G$ is ambigous, we have to find a string, $w \in L(G)$ which is ambigous.

Consider the $w = abababa \in L(G)$. Then we get two derivation tree for $w$. Thus grammar $G$ is ambiguous.




(a) First derivation tree for *abababa*. (b) Second derivation tree for *abababa*.

**Fig. 6.15.**

**Example 6.37. Consider the productions for the *if then else* statement which are the part of programming language**

**<statement> → if <expression> then <st> | if <expression> then <st> else <st> | <other statement>**

**Expression → exp1 | exp2 | ... exp.**

**<other statement> → st1 | st2 | ... | stn.**

**Prove that grammar is ambigious.**

**Solution.** Consider the statement

if exp1 then if exp2 then st1 else st2

There are two different parse trees as shown in Fig. 6.16 and Fig. 6.17.

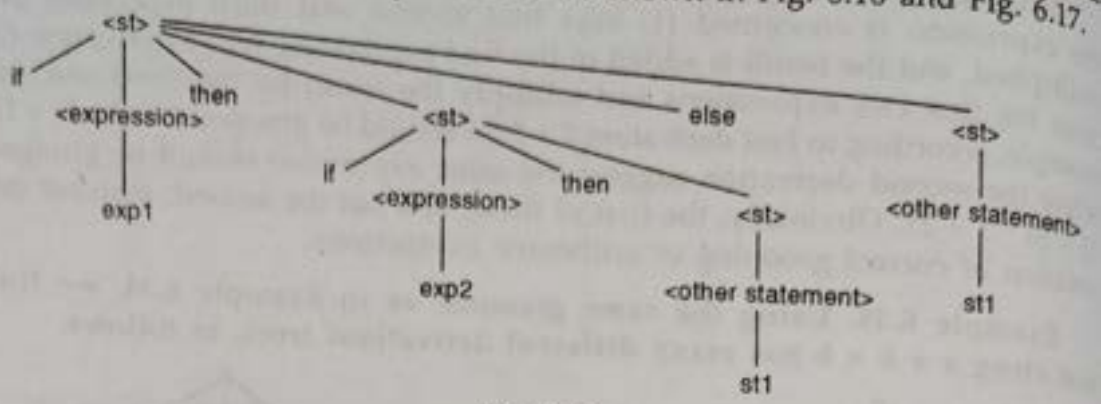



Fig. 6.16.

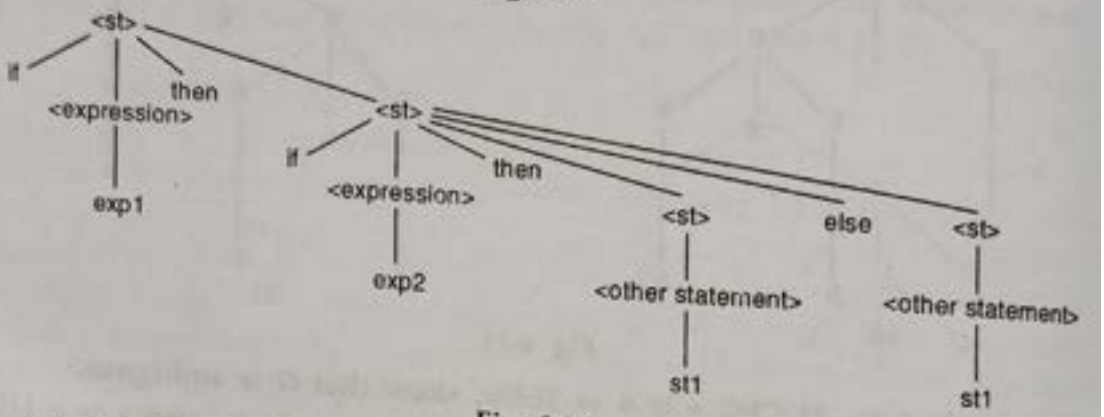



Fig. 6.17.

Hence grammar is ambiguous.

### 6.4.1. Removing Ambiguity from Grammars

If a context free grammar is ambiguous, it is often possible and usually desirable to find an equivalent unambiguous CFG. Although some CFGs are "**inherently ambiguous**" in the sense that they cannot be produced except by ambiguous grammars. Ambiguity is usually the property of the grammar rather than the language. Let us consider the grammar for algebraic expressions and for the string $a + a*a$ the two parse tree are possible.

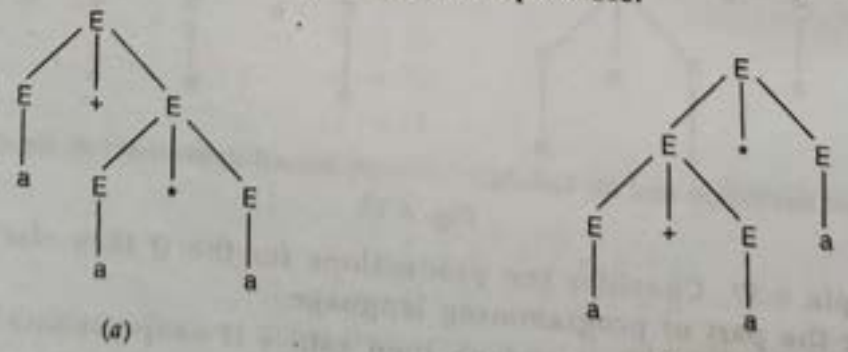



Fig. 6.18.

The first cause of ambiguity in the grammar is that the precedence of the operator + and * is not respected. Both parsing trees in Fig. 6.18 are valid. We need to force only the structure of Fig. 6.18(*a*) to be legal in an ambiguous grammar in which the * operator has higher precedence then + because $a + a*a$ is equavalent to $a + (a*a)$ and $a*a + a$ is equivalent to $(a*a) + a$.

Again, the parse trees for $a + a + a$ are

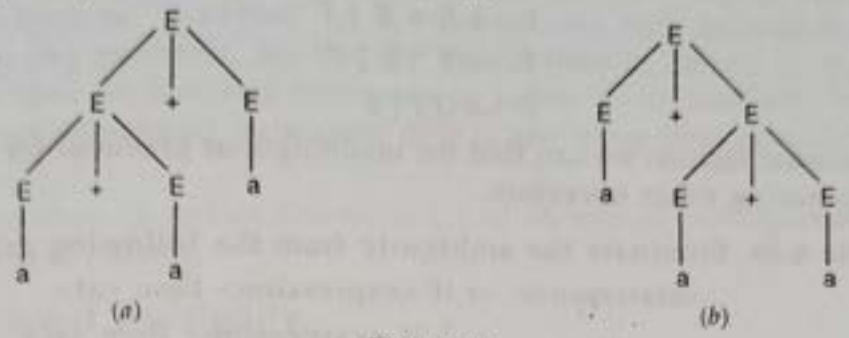



**Fig. 6.19.**

The cause of the ambiguity in string $a + a + a$ is that the associativity it of + operator is not respected. Since, we assume the + operator is left associative the string $a + a + a$ is equivalent to $(a + a) + a$. Thus, again we need to force only the structure of Fig. 6.19(*a*) to be legal in an ambiguous grammar.

Because we do not want $E + E$ (as we have seen, $E \rightarrow E + E$, by itself is enough to produce ambiguity) we will not think of expressions involving + as being sums of other expressions. They are obviously sums of something: Let we use the word **terms** to stand for this things that are added *i.e.* combined using + to create expressions. The corresponding variable in the grammar will be $T$. Expressions can also be products; because the two expressions $a + b * c$ and $a * b + c$ are both sums, it is more appropriate to say that terms can be products. Let us say that "factors" are the things that are multiplied (combined using *) to produce terms. The corresponding variable in the grammar will be $F$.

Since the multiplication have higher precedence over addition. We can say that Expressions are sums of one or more terms, and terms are products of one or more factors.

Now, we must deal with parentheses. We might say that $(E)$ could be an expression, a term or a factor. However, evaluation of a parenthetical expression takes precedence over any operators outside the parentheses. Factors are evaluated first in hierarchy and therefore the appropriate choice is to say that $(E)$ is a factor and that $E$ itself can be an arbitrary expression.

Now, expressions are sums of one or more terms, terms are products of one or more factors, and factors are either parenthetical expressions or single identifiers. Sum of terms might suggest something such as $E \rightarrow T + T \mid T$. (keep in mind that an expression can consist of single term)

However, to obtain the sum of three terms with this approach, we would be forced to try $T \rightarrow T + T$ or something comparable, and again we would have ambiguity. What we say instead is that an expression is either a single term or the sum of a term and another experssion. The only question is whether we want $E \rightarrow E + T$ or $E \rightarrow T + E$. Since + operator is left associative, we would probably choose $E \rightarrow E + T$ as more appropriate. Similarly, we choose the production $T \rightarrow T * F$ rather than $T \rightarrow F * T$.

The resulting unambiguous grammar is

$$E \rightarrow E + T \mid T$$
$$T \rightarrow T * F \mid F$$
$$F \rightarrow (E) \mid a$$

In the similar fashion we can find the unambiguous grammar for algebraic expressions having other operators.

**Example 6.38. Eliminate the ambiguity from the following grammar**

**<statement> → if <expression> then <st>**
**| if <expression> then <st>**
**else <st> | <other statement>**
**<other statement> → $S_1 \mid S_2 \mid S_3 \mid \ldots S_n$**
**<expression> → $E_1 \mid E_2 \mid E_3 \ldots \mid E_n$**

**Solution.** The Grammar is ambiguous since the string "if $E_1$ then if $E_2$ then $S_1$ else $S_2$" has two parse trees.

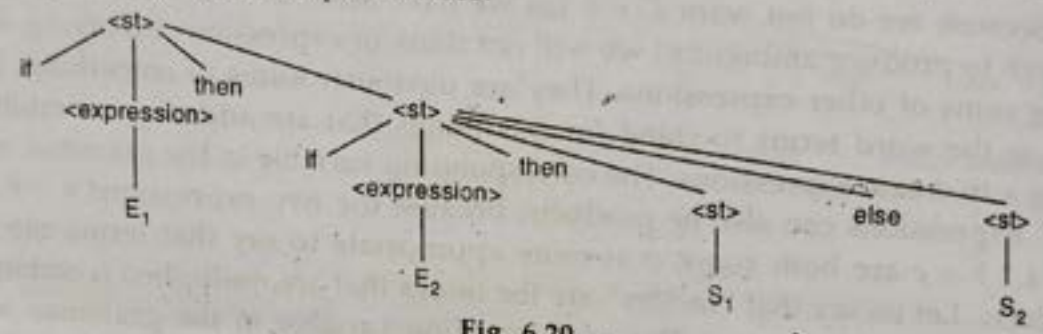



**Fig. 6.20.**

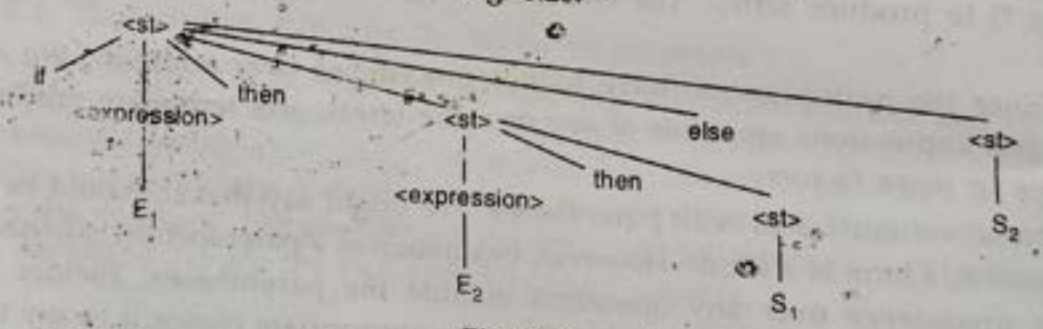



**Fig. 6.21.**

In all programming languages with conditional statements of this form, the first parse tree is preferred. The general rule is "Match each else with the closest previous unmatched then". Then unambiguous grammar is as follows:

<st> → <matched-statement> | <unmatched-statement>
<matched-statement> → if <expression> then <matched-statement>
else <matched statement> | <other statement>
<unmatched-statement> → if <expression> then <st>
| if <expressions> then <matched-statement>
else <unmatched-statement>
<expression> → $E_1 \mid E_2 \mid \ldots E_n$
<unmatched statement> → $S_1 \mid S_2 \ldots S_n$
<matched statement> → $S_1 \mid S_2 \ldots \mid S_n$

The idea in this grammar is that a statement appearing between a *then* and an *else* must be "matched" *i.e.* it must not end with an unmatched *then* followed by *any statement*, for the *else* would then be forced to match this unmatched *then*. A matched statement is either an if then else statement containing no unmatched statements or it is any other kind of unconditional statement.

This grammar generates the same set of strings as previous one, but it allows only one parsing.

### 6.4.2. Inherent Ambiguity

"If $L$ is a context-free language for which three exists an unambiguous grammar, then $L$ is said to be unambiguous. If every grammar that generates $L$ is ambiguous, then the language is said to be "inherently ambiguous".

Let us consider an example for it :

**Example 6.39. Show that**

$$L = \{a^n b^n c^m\} \cup \{a^n b^m c^m\}$$

**with $n$ and $m$ non-negative, is an inherently ambigous context-free language.**

**Solution.** Let us say $L = L_1 \cup L_2$

where $L_1 = \{a^n b^n c^m\}$

and $L_2 = \{a^n b^m c^m\}$

Let us write CFG for $L_1$, let it is $G_1$, as follows :

$$S_1 \rightarrow S_1 c / A$$
$$A \rightarrow aAb / \in$$

Similarly we can write CFG for $L_2$, Let it is $G_2$, as follows :

$$S_2 \rightarrow aS_2 / B$$
$$B \rightarrow bBc / \in$$

Now with the help of $G_1$ and $G_2$ we can write CFG for the language $L$, as follows

$$S \rightarrow S_1 / S_2$$

where $S$ is starting non-terminal for CFG of language $L$.

The grammar is ambiguous since the string $a^n b^n c^m$ has two distinct derivation, one starting with $S \Rightarrow S_1$, and another with $S \Rightarrow S_2$. It does of course not follows that $L$ is in herently ambiguous as there might exist some other non-ambiguous grammar for it. But in some way $L_1$ and $L_2$ have some conflicting requirements, the first putting a restriction on the number of $a$'s and $b$'s, while the second does the same for $b$'s and $c$'s. A few tries will quickly convince us of the impossiblity of combining these requirements in a single set of rules that cover the case $n = m$ uniquely.

**Now let us discuss some more example about these concepts.**

**Example 6.40. Design a CFG for the language $L = \{a^n b^m : n \leq m + 3\}$.**

**Solution.** Let us solve this problem for $n = m + 3$. Then add more $b$'s.

Let us assume CFG for above concept is $G = (V_n, V_t, P, S)$

where $V_n = \{S, A, B\}$

$V_t = \{a, b, \in\}$

and productions are defined as :

$$S \to aaaA$$
$$A \to aAb/B$$
$$B \to Bb/\in$$

But when we analyse this solution, then it is found incomplete since it creates at least three $a$'s. To take care of these cases $n = 1, 2$ we add

$$S \to \in/aA/aaA$$

So finally solution is

$$S \to \in/aA/aaA$$
$$S \to aaaA$$
$$A \to aAb/B$$
$$B \to Bb/\in$$

**Example 6.41. Find the context-free grammar for the set of all regular expression on the alphabet $\{a, b\}$.**

**Solution.** Let us analyse the regular expression on the alphabet $\{a, b\}$ regular expression by any combination of $a$'s and $b$'s is $r = (a + b)^*$.

Let CFG for regular expression is $G = (V_n, V_t, P, S)$

$V_n = \{S, A, B\}$

$V_t = \{a, b, \in\}$

and $P$ is defined as

$$S \to AB/BA$$
$$A \to aA/\in \quad (P_1 \text{ say})$$
$$B \to bB/\in \quad (P_2 \text{ say})$$
$$(P_3 \text{ say})$$

By the help of $P_1$, $P_2$ and $P_3$ we can derive any combination of $a$'s and $b$'s.

**Example 6.42. Let** $L = \{a^n b^n : n \geq 0\}$

**Show that $L^2$ is context-free.**

**Solution.**

$L = \{a^n b^n : n \geq 0\}$

Let CFG for $L$ is $G = (V_n, V_t, P, S)$

$V_n = \{s\}$

$V_t = \{a, b, \in\}$

$P$ is defined as $S \to aSb$

$S \to \in$

Clearly $L$ is context-free language, since there exist a CFG for given language.

Now consider a grammar, which is CFG and production of this CFG are defined as follows.

$$S_1 \to SS$$

$$S \to aSb$$
$$S \to \epsilon$$

Now, when carefully analyse these production then we find that, it is the CFG for the language $L^2$. So $L^2$ is context free language, provided $L$ is context free.

**Example 6.43. Show that the following grammar is ambiguous.**

$$S \to AB/aaB$$
$$A \to a/Aa$$
$$B \to b.$$

**Solution.** Given CFG is

$$S \to AB/aaB$$
$$A \to a/Aa$$
$$B \to b$$

The string $w = aab$ has two distinct leftmost derivations, as follows :

(1) $S \Rightarrow aa\underline{B} \Rightarrow aab$

(2) $S \Rightarrow \underline{A}B \Rightarrow \underline{A}aB \Rightarrow aa\underline{B} \Rightarrow aab$

Now let us see parse trees for the same

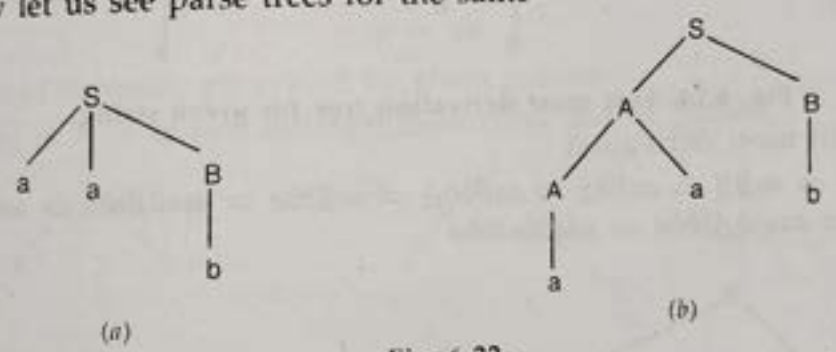



**Fig. 6.22.**

**Example 6.44. Find a derivation free of $a * b + a * b$ given that $a * b + a * b$ is in $L(G)$, where $G$ is given by**

$$S \to S + S/S * S$$
$$S \to a/b.$$

**Solution.** Given grammar is

$$S \to S + S/S * S$$
$$S \to a/b$$

and string is $w = a * b + a * b$

$S \Rightarrow \underline{S} + S \Rightarrow \underline{S} * S + S \Rightarrow a * \underline{S} + S \Rightarrow a + b + \underline{S} \Rightarrow a * b + \underline{S} * S$
$\Rightarrow a * b + a * \underline{S} \Rightarrow a * b + a * b$

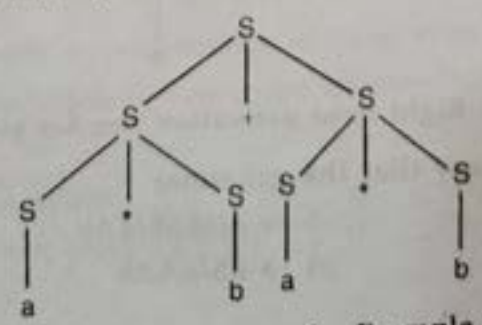



**Fig. 6.23. Derivation tree for Example 6.44.**

**Example 6.45. Consider the grammar**

$$S \rightarrow aB/bA$$
$$A \rightarrow aS/bAA/a$$
$$B \rightarrow bS/aBB/b$$

**For the string *aaabbabbba* find**

**(*i*) The left most derivation and left most derivation tree.**

**(*ii*) The right most derivation and right most derivation tree.**

**Solution.** Given string is *aaabbabbba*.

(*i*) Left most derivation

$S \Rightarrow a\underline{B} \Rightarrow aa\underline{B}b \Rightarrow aaa\underline{B}BB \Rightarrow aaab\underline{B}B \Rightarrow aaabb\underline{B} \Rightarrow aaabba\underline{B}B \Rightarrow aaabbab\underline{B} \Rightarrow$ $aaabbabbS \Rightarrow aaabbabbb\underline{A} \Rightarrow aaabbabbba.$

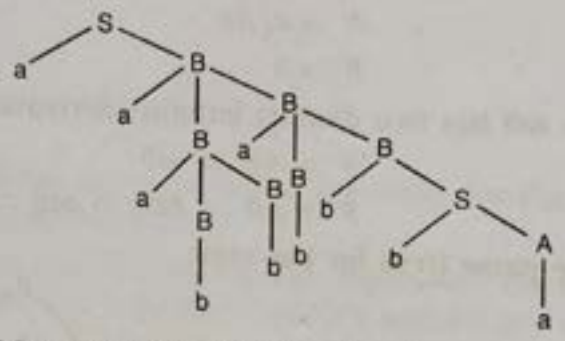



**Fig. 6.24. Left most derivation tree for given string.**

(*ii*) Right most derivation :

$S \Rightarrow a\underline{B} \Rightarrow aaB\underline{B} \Rightarrow aaBb\underline{S} \Rightarrow aaBbb\underline{A} \Rightarrow aa\underline{B}bba \Rightarrow aaaB\underline{B}bba \Rightarrow aaa\underline{B}bbba \Rightarrow$ $aaab\underline{S}bbba \Rightarrow aaabb\underline{A}bbba \Rightarrow aaabbabbba$

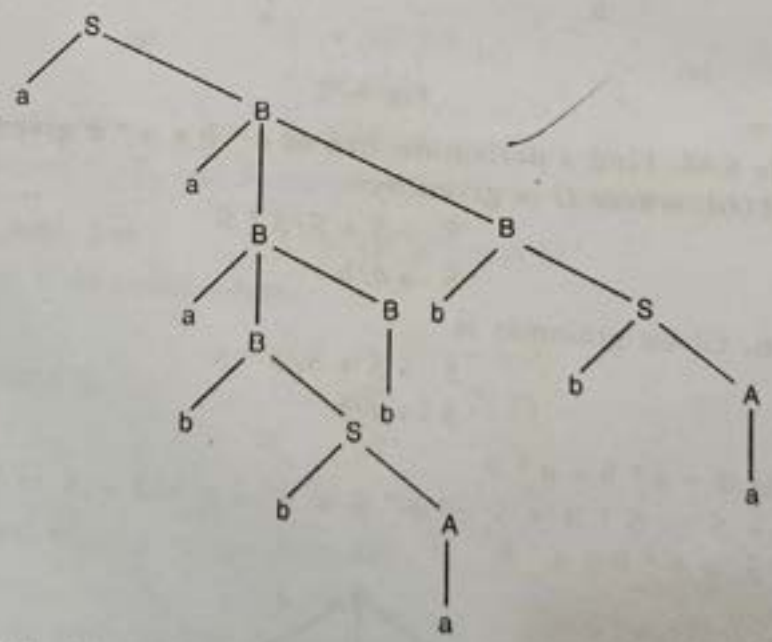



**Fig. 6.25. Right most derivation tree for given string.**

**Example 6.46. Show that the grammar**

$$S \rightarrow a/abSb/aAb$$
$$A \rightarrow bS/aAAb$$

**is ambiguous.**

**Solution.** Given grammar

$$S \to a \qquad (P_1 \text{ say})$$
$$S \to abSb \qquad (P_2 \text{ say})$$
$$S \to aAb \qquad (P_3 \text{ say})$$
$$A \to bS \qquad (P_4 \text{ say})$$
$$A \to aAAb \qquad (P_5 \text{ say})$$

Let us consider a string $w = abab$, it has two different derivations, as follows :

(i) $S \Rightarrow abSb \Rightarrow abab$ (by using $P_2$ and $P_1$)

(ii) $S \Rightarrow aAb \Rightarrow abSb \Rightarrow abab$ (by using $P_3$, $P_4$ and $P_1$)

**Example 6.47. Show that grammar**

$$S \to aB/ab$$
$$A \to aAB/a$$
$$B \to ABb/b$$

**is ambiguous.**

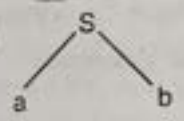
**Solution.** Let us consider a string

$$w = ab$$

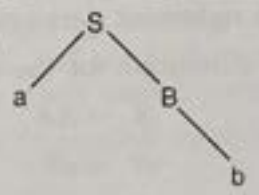
which can be easily generated by given grammar.

Now $w = ab$ as two different derivation tree as follows :

(a)




(b)




So clearly given grammar is ambiguous.

## EXERCISE

1. Find CFG's that generate these regular language over the alphabet $\Sigma = \{a, b\}$.
   (i) The language defined by $(aaa + b)^*$.
   (ii) The language defined by $(a + b)^* (bbb + aaa) (a + b)^*$.
   (iii) All the strings without the substring $aaa$.
   (iv) All strings that end in $b$ and have an even number of $b$'s in total.
   (v) The set of all strings of odd length.
   (vi) All strings with exactly one a or exactly one $b$.
   (vii) All strings with an odd number of $a$'s or even number of $b$'s.
2. Write CFG for the following languages
   (i) $\{a^m b^n / m >= n\}$
   (ii) $\{a^m b^n c^p d^q / m + n = p + q\}$
   (iii) $\{u\ a\ w\ b / \mu, w \in \{a, b\}^*, |\mu| = |w|\}$

3. Consider the grammar

$$G = (V_n, V_t, P, S)$$

where

$$V_n = \{E, I\}$$
$$V_t = \{a, b, c, +, *, (, )\},$$
$$S = E, \text{ and } P \text{ is given by}$$
$$E \to I$$
$$E \to E + E$$
$$E \to E * E$$
$$E \to (E)$$
$$I \to a/b/c$$

show that this grammar is ambiguous by deriving two parse tree for string $a + b * c$.

4. Consider the grammar with production

$$S \to aaB$$
$$A \to bBb/\epsilon$$
$$B \to Aa$$

Show that the string *aabbabba* is not accepted by this grammar.

5. The following grammar generates prefix expression with operands x and y and binary operators +, –, and *

$$E \to +EE/*EE/-EE/x/y.$$

Find the leftmost and rightmost derivations for the string $+ * - xyxy$.

6. (*i*) Find the left most derivation for the word *abba* in the grammar

$$S \to AA$$
$$A \to aB$$
$$B \to bB/\epsilon$$

(*ii*) Find the left most derivation for the word *abbabaabbbabbab* in the CFG

$$S \to SSS/aXb$$
$$X \to ba/bba/abb$$

7. Find the regular expression and defined the language of following CFG's.

(*i*)
$$S \to aX/bS/a/b$$
$$X \to aX/a$$

(*ii*)
$$S \to bS/aX/b$$
$$X \to bX/aS/a$$

(*iii*)
$$S \to aaS/abS/baS/bbS/\epsilon$$

(*iv*)
$$S \to aB/bA/\epsilon$$
$$A \to aS$$
$$B \to bS$$

(*v*)
$$S \to aB/bA$$
$$A \to aB/a$$
$$B \to bA/b$$

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8. Starting with the alphabet

$$\Sigma = \{a, b, (, ), +, *\}$$

Find a CFG that generates all regular expressions.

9. Give the derivation tree for $(((a + b)(c)) + a + b$, using the following grammar

$$E \rightarrow T$$
$$T \rightarrow F$$
$$F \rightarrow I$$
$$E \rightarrow E + T$$
$$T \rightarrow T * F$$
$$F \rightarrow (E)$$
$$I \rightarrow a/b/c$$

10. Show that the language $L = \{wwR : w \in \{a, b\}^*\}$ is not inherently ambiguous.

11. Write the CFG for the following language (with $n \geq 0$, $m \geq 0$)

(i) $L = \{a^n b^m : n \neq m - 1\}$

(ii) $L = \{w \in \{a, b\}^* : n_a(w) = 2n_b(w) + 1\}$.

12. Find the CFG for the following language (with $n \geq 0$, $m \geq 0$).

(i) $L = \{a^n b^m c^k, k \neq n + m\}$

(ii) $L = \{a^n b^n c^k : k \geq 3\}$

(iii) $L = \{w \in \{a, b, c\}^* : n_a(w) + n_b(w) \neq n_c(w)\}$

13. Define what one right mean by properly rested parenthesis structures involving two kind of paranthesis, say ( ) and [ ]. Intuitively, properly rested strings in this situation are ([ ]), ([[ ]]) [( )], but not ([ )] or (( ]]. Using your definition, give a CFG for generating all properly nested parentheses.

14. Find a CFG that generate all the production rules for CFG with $T = \{a, b\}$ and $V = \{A, B, C\}$.

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# Simplified Context-Free Grammar and It's Normal Form

7
CHAPTER

**INSIDE THIS CHAPTER**

7.1. Introduction 7.2. Reduction of Context Free Grammar 7.3. Chomsky Normal Form 7.4. Greibach Normal Form 7.5. Regular Grammars.

## 7.1. INTRODUCTION

We have already seen, that definition of a context grammar imposes no restriction whatsoever on the right side of a production. Eliminating rules of the form $X \to \epsilon$ and $X \to Y$ made the arguments simpler. In many instances, it is desirable to place even more restriction on the grammar. Due to this, we need to look at method for transforming an arbitrary context free grammar into an equivalent one that satisfies certain restriction on its form. In this chapter, we are going to study several transformation and substitutions that will be useful in subsequent discussion.

We will also see normal forms for context free grammar. These are the form of context free gramars, which are broad enough so that any grammar has an equivalent normal form version. We will study two of the most useful of these, the Chomsky Normal Form (CNF) and the Greibach Normal Form (GNF).

## 7.2. REDUCTION OF CONTEXT FREE GRAMMAR

There are several ways in which one can restrict the format of context free grammar without reducing the language generation power of context free grammar. Let $L$ be a non-empty context free language, then it can be generated by a context-free grammar $G$ with the following properties :

(*a*) We must eliminate useless symbols, those variables or terminals that do not appears in any derivation of a terminal string from the start symbol.

(*b*) We must eliminate $\epsilon$-productions, those of the form $X \to \epsilon$ for some variable $X$.

(*c*) We must eliminate unit production, those of the form $X \to Y$ for variables $X$ and $Y$.

### 7.2.1. Eliminating Use Less Symbols

Here we are going to identified those symbols, which do not play any role in the derivation of any string $w$ in $L(G)$ (these symbols are called use less symbols) and then eliminate the identified production, which contains useless symbols, from the context free grammar.

A symbols $Y$ in a context-free grammar is useful it and only if :



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(*a*) $Y \overset{*}{\rightarrow} w$, where $w \in L(G)$ and $w$ in $V_t^*$, that is $Y$ leads to a string of terminals. Here $Y$ is said to be "generating".

(*b*) If there is a derivation $S \overset{*}{\Rightarrow} \alpha Y \beta \overset{*}{\Rightarrow} w$, $w \in L(G)$, for same $\alpha$ and $\beta$, then $Y$ is said to be reachable.

Surely a symbol that is useful will be both generating and reachable. If we eliminate the symbols that are not generating first, and then eliminate from the remaining grammar those symbols that are not reachable. Then, after this process, context free grammar will have only useful symbols.

There for reduction of a given grammar G, involves following steps :

(*a*) Identified non-generating symbols in given CFG and eliminate those productions which contains non-generating symbols.

(*b*) Identified non-reachable symbols in grammar and eliminate those productions which contains non-reachable symbols.

Let us consider examples :

**Example 7.1. Consider a CFG**

$$S \rightarrow AB/a$$
$$A \rightarrow b$$

**Identified and eliminate useless symbols.**

**Solution.** Given CFG is

$$S \rightarrow AB/a$$
$$A \rightarrow b$$

Here by observing the given CFG, it becomes very clear that $B$ is non-generating symbol. Since $A$ derives $b$, $S$ derives $a$ but $B$ is not deriving any string $w$ in the $V_t^*$.

So we can eliminate $S \rightarrow AB$ from the context free grammar, now CFG becomes

$$S \rightarrow a$$
$$A \rightarrow b$$

Here $A$ is non-reachable symbol, since it can not be reached by starting non- terminal $S$.

$$S \searrow a$$

So we can eliminate $A \rightarrow b$, now grammar is

$$S \rightarrow a$$

which is reduced grammar.

**Example 7.2. Remove the useless symbol from the given context free grammar :**

$$S \rightarrow aB/bX$$
$$A \rightarrow BAd/bSX/a$$
$$B \rightarrow aSB/bBX$$
$$X \rightarrow SBD/aBx/ad.$$

**Solution.** First we choose those non-terminals which are deriving to the strings of terminals. The non-terminals $A$ and $X$ directly deriving to the strings of terminal $a$ and $ad$ respectively. Hence they are useful symbols. Since $S \rightarrow bX$

and $X$ is useful symbol, hence $S$ is also useful symbol. But $B$ does not derive any string $w$ in $V_t^*$ so clearly $B$ is a non-generating symbol, so eliminate those productions which contains $B$. Now grammar becomes :

$$S \to bX$$
$$A \to bSX/a$$
$$X \to ad$$

Again here $A$ is non-reachable, since $A$ can not be reached from $S$ (starting non-terminal), it can be understand by following parse tree in Fig. 7.1.

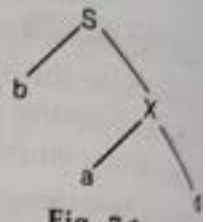


Fig. 7.1.

So eliminate $A \to bSX/a$, then reduced CFG is

$$S \to bX$$
$$X \to ad.$$

**Example 7.3. Consider the following grammar and obtain an equivalent grammar containing no useless grammar symbol.**

$$A \to xyz/Xyzz$$
$$X \to Xz/xYz$$
$$Y \to yYy/Xz$$
$$Z \to Zy/z.$$

**Solution.** Since $A \to xyz$ and $Z \to z$, hence $A$ and $Z$ are directly deriving to the string of terminals. Hence they are useful symbols.

Hence $X$ and $Y$ do not lead to a string of terminals, that is, $X$ and $Y$ are useless symbols. Therefore by eliminating these productions we get :

$$A \to xyz$$
$$Z \to Zy/z$$

Now since $A$ is the starting non-terminal and right side of $A$ does not contain $Z$, it means $Z$ is not reachable. Hence after eliminating $Z \to Zy/z$, grammar becomes

$$A \to xyz.$$

which is a grammar containing no useless symbols.

**Example 7.4. Find the reduced grammar that is equivalent to the CFG given below :**

$$S \to aC/SB$$
$$A \to bSCa$$
$$B \to aSB/bBC$$
$$C \to aBC/ad.$$

**Solution.** Since $C \to ad$, therefore, $C$ is generating symbol. Since $S \to aC$, therefore $S$ is also a useful symbol according to production $A \to bSCa$, $A$ is also generating. Right side of $B \to aSB$ and $B \to bBC$ contains $B$, and $B$ is not terminating so $B$ is not a generating symbol so we can eliminate those productions and grammar becomes

$$S \to aC$$
$$A \to bSCa$$
$$C \to ad$$

Now since $S$ is the start symbol and right side of $S$ does not contain $A$, hence $A$ is not reachable as :

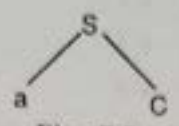


Fig. 7.2.

So by eliminating $A$ we get

$$S \rightarrow aC$$
$$C \rightarrow ad$$

which is reduced grammar equivalent to the given grammar, containing no useless symbol.

### 7.2.2. Removal of Unit Production

Let us first define unit production :

A production of the form

Non-terminal $\rightarrow$ One non-terminal

that is a production of the form $A \rightarrow B$ (where $A$ and $B$, both are non-terminals) is called unit production. Unit production increase the cost of derivation in a grammar. Following algorthim can be used to eliminate the unit production.

**Algorthim :** Removal of unit productions $\Rightarrow$

While (there exist a unit production, $A \rightarrow B$)

{

select a unit production $A \rightarrow B$, such that there exist a production $B \rightarrow \alpha$, where $\alpha$ is a terminal.

For (every non-unit production, $B \rightarrow \alpha$)

Add production $A \rightarrow \alpha$ to the grammar

Eliminate $A \rightarrow B$ from the grammar.

}.

Now let us apply this algorthim on the given CFG's.

**Example 7.5. Consider the context free grammar $G$.**

$$S \rightarrow AB$$
$$A \rightarrow a$$
$$B \rightarrow C/b$$
$$C \rightarrow D$$
$$D \rightarrow E$$
$$E \rightarrow a$$

**remove the unit production.**

**Solution.** Given CFG

$$S \rightarrow AB$$
$$A \rightarrow a$$
$$B \rightarrow C/b$$
$$C \rightarrow D$$
$$D \rightarrow E$$
$$E \rightarrow a$$

Contain three unit productions

$$B \to C$$
$$C \to D$$
$$D \to E$$

Now to remove unit production $B \to C$, we see if there exists a production whose left side has $C$ and right side contains a terminal (*i.e.* $C \to a$), but there is no such productions in G. Similar things holds for production $C \to D$. Now we try to remove unit production $D \to E$, because there is a production $E \to a$. Therefore, eliminate $D \to E$ and introduce $D \to a$, grammar becomes

$$S \to AB$$
$$A \to a$$
$$B \to C/b$$
$$C \to D$$
$$D \to a$$
$$E \to a$$

Now we can remove $C \to D$ by using $D \to a$, we get

$$S \to AB$$
$$A \to a$$
$$B \to C/b$$
$$C \to a$$
$$D \to a$$
$$E \to a$$

Similarly, we can remove $B \to C$ by using $C \to a$, we obtain

$$S \to AB$$
$$A \to a$$
$$B \to a/b$$
$$C \to a$$
$$D \to a$$
$$E \to a$$

Now it can be easily seen that productions $C \to a$, $D \to a$ $E \to a$ are useless because if we start deriving from $S$, these productions will never be used. Hence eliminating them gives,

$$S \to AB$$
$$A \to a$$
$$B \to a/b$$

which is completely reduced grammar.

**Example 7.6. Consider the following unambiguous expression grammar**

$$I \to a/b/Ia/Ib/I0/I1$$
$$F \to I/(E)$$
$$T \to F/T*F$$
$$E \to T/E + T$$

**Identified unit production and remove them.**

**Solution.** In the given grammar

$$F \to I$$
$$T \to F$$
$$E \to T$$

are three unit production.

Unit production $F \to I$ can be removed by the help of

$$I \to a/b/Ia/Ib/I0/I1$$

and grammar becomes

$$I \to a/b/Ia/Ib/I0/I1$$
$$F \to (E)/a/b/Ia/Ib/I0/I1$$
$$T \to F/T*F$$
$$E \to T/E + T$$

Now we can eliminate $T \to F$ by the help of

$$T \to (E)/a/b/Ia/Ib/I0/I1$$

and now grammar becomes

$$I \to a/b/Ia/Ib/I0/I1$$
$$F \to (E)/a/b/Ia/Ib/I0/I1$$
$$T \to T*F/(E)/a/b/Ia/Ib/I0/I1$$
$$E \to T/E + T$$

Now let us remove $E \to T$ by the help

$$T \to T*F/(E)/a/b/Ia/Ib/I0/I1$$

and finally, unit production free grammar is

$$I \to a/b/Ia/Ib/I0/I1$$
$$F \to (E)/a/b/Ia/Ib/I0/I1$$
$$T \to T*F/(E)/a/b/Ia/Ib/I0/I1$$
$$E \to E + T/T * F/(E)/a/b/Ia/Ib/I0/I1$$

**Example 7.7. Identified and remove the unit productions from following grammar**

$$S \to A/bb$$
$$A \to B/b$$
$$B \to S/a$$

**Solution.** Here unit productions are

$$S \to A$$
$$A \to B$$
$$B \to S$$

We list all unit productions and sequences of unit productions.Then we will handle one unit production or sequence at a time and try to prove generating and reachable for each non-terminal involve in the unit productions, as follows :

| | |
|---|---|
| $S \to A$ | generate $S \to b$ |
| $S \to A \to B$ | generate $S \to a$ |
| $A \to B$ | generate $A \to a$ |

$A \to B \to S$ generate $A \to bb$

$B \to S$ generate $B \to bb$

$B \to S \to A$ generate $B \to b$

The new CFG becomes

$$S \to bb/b/a$$
$$A \to b/a/bb$$
$$B \to a/bb/b$$

which hold no unit productions.

### 7.2.3. Removal of ∈-productions and Nullable Non-terminal

Any context-free language in which ∈ is a word, must have some ∈-productions in its grammar since otherwise we could never derive the word ∈ from S. The statement is obvious, but it should be given some justification, mathematically, this is easy; we observe that ∈-productions are only productions that shorten the working string. If we begin with the string S and apply only non-∈-productions, we never develop a word of length zero.

We now form our attention to the eliminate the productions of the form $A \to \in$, which are called ∈-productions. Surely if ∈ is in $L(G)$, we can not eliminate all ∈-productions from G, but if ∈ is not in $L(G)$, we can eliminate all ∈-productions from G.

"In a given CFG, we call a non-terminal N Nullable if

There is a production $N \to \in$ or

There is a derivation that starts at N and leads to ∈ :

$N \Rightarrow \ldots \Rightarrow \in$"

**To eliminate ∈-productions from a grammar G we use the following technique :**

"If $A \to \in$ is a prodoction to be eliminated then we look for all productions, whose right side contains A, and replace each occurrence of A in each of these productions to obtain the non ∈-productions. Now these resultant non ∈-production must be added to the grammar to keep the language generated, the same."

Let us apply this concept :

**Example 7.8. Consider the following grammar**

$$S \to aA$$
$$A \to b/\in$$

**remove the Nullable non-terminal.**

**Solution.** Given CFG is

$$S \to aA$$
$$A \to b/\in$$

Here $A \to \in$ is ∈-production, so let us apply above described concept, put ∈ in place of A, at the right side of productions and add the resulted productions to the grammar.

Here is only one productions $S \to aA$ whose right side contains A. So, replacing A by ∈ in this production, we get $S \to a$; now add this new productions **to keep the language generated by this grammar same.** Therefore, the ∈-free grammar, equivalent to above grammar is

$$S \to aA$$
$$S \to a$$

$$A \to b$$

or

$$S \to aA/a$$
$$A \to b$$

there is no $\epsilon$-production in the above context-free grammar.

**Example 7.9. Consider the following grammar G**

$$S \to ABAC$$
$$A \to aA/\epsilon$$
$$B \to bB/\epsilon$$
$$C \to c$$

**remove the $\epsilon$-production from the above grammar.**

**Solution.** Here we have two $\epsilon$-production to remove $A \to \epsilon$ and $B \to \epsilon$. To eliminate $A \to \epsilon$ from the grammar, the non $\epsilon$-productions to be added are obtained as follows :

List of productions whose right side contain $A$ are :

$$S \to ABAC$$
$$A \to aA$$

So on replacing each occurrence of $A$ by $\epsilon$ (one by one), we get four new productions to be added to the grammar :

$$S \to ABC/BAC/BC$$
$$A \to a$$

add these productions to the grammar and eliminate $A \to \epsilon$, to obtain the following grammar

$$S \to ABAC/ABC/BAC/BC$$
$$A \to aA/a$$
$$B \to bB/\epsilon$$
$$C \to c$$

Now to eliminate $B \to \epsilon$, list all productions whose right side contains $B$, these are :

$$S \to ABAC/BAC/ABC/BC$$
$$B \to bB$$

Replace occurrence of $B$ in each of these productions to obtain non $\epsilon$-productions to be added to the grammar, we get following new productions.

$$S \to AAC/AC/C$$
$$B \to b$$

Add these productions to the grammar and eliminate $B \to \epsilon$ from the grammar, to obtain

$$S \to ABAC/BAC/ABC/BC/AAC/AC/C$$
$$A \to aA/a$$
$$B \to bB/b$$
$$C \to c$$

which is the grammar without any $\epsilon$-production.

**Example 7.10. Consider the following grammar and remove $\epsilon$-productions**

$$S \to aSa$$
$$S \to bSb/\epsilon$$

**Solution.** Here $S \to \epsilon$ is the $\epsilon$-production, so let us replace occurrence of $S$ by $\epsilon$ in the following productions

$$S \to aSa$$
$$S \to bSb$$

We will get following new productions

$$S \to aa/bb$$

add these productions in the grammar and remove $S \to \epsilon$ from the grammar

$$S \to aSa/aa/bb/bSb$$

It is $\epsilon$-production free grammar.

**Example 7.11. Consider the following grammar**

$$S \to a/Xb/aYa$$
$$X \to Y/\epsilon$$
$$Y \to b/X$$

eliminate $\epsilon$-productions.

**Solution.** Here $X \to \epsilon$ is $\epsilon$-productions. So this problem can also solved in similar fashion and grammar without $\epsilon$-productions will be

$$S \to a/Xb/aYa/b/aa$$
$$X \to Y$$
$$Y \to b/X.$$

**Example 7.12. Design a CFG for regular expression $r = (a + b)^* bb (a + b)^*$, which is free from $\epsilon$-productions.**

**Solution.** Let CFG $G$ be required context free grammar with $\epsilon$-productions

$$S \to XY$$
$$X \to Zb$$
$$Y \to bW$$
$$Z \to AB$$
$$W \to Z$$
$$A \to aA/bA/\epsilon$$
$$B \to Ba/Bb/\epsilon$$

Finally here $A \to \epsilon$ and $B \to \epsilon$ are $\epsilon$-productions and $A$, $B$ are nullable non-terminals.

The CFG without $\epsilon$-production can be achieved from $G$ in similar fashion as above discussed. Let it be $G'$, then $G'$ is

$$S \to XY$$
$$X \to Zb/b$$
$$Y \to bW/b$$
$$Z \to AB/A/B$$
$$W \to Z$$
$$A \to aA/bA/a/b$$
$$B \to Ba/Bb/a/b$$

Because $\epsilon$ was not generated by $G$, the new CFG $G'$ generates exactly the same language.

## 7.3. CHOMSKY NORMAL FORM

If a CFG has only production of the form

Non-terminals → string of exactly two non-terminals or of the form

Non-terminals → one terminal

is said to be chomsky normal form or CNF.

### 7.3.1. Conversion to Chomsky Normal Form

"Conversion of the grammar to CNF can be understood by the following example"

Let us conside a grammar

$G = (\{S, A, B\}, \{a, b\}, P, S)$, where $S$ is the start symbol and $P$ is given by

$$S \rightarrow bA/aB$$
$$A \rightarrow bAA/aS/a$$
$$B \rightarrow aBB/bS/a$$

Now let us find an equivalent chomsky normal form of it.

As we know that right side in CNF either contains two non-terminals or one terminal. So it is clear that in the first production above, we have to replace terminal '$b$' by a non-terminal say $C_b$ and '$a$' by $C_a$, hence grammar becomes

$$S \rightarrow C_bA/C_aB$$
$$A \rightarrow C_bAA/C_aS/a$$
$$B \rightarrow C_aBB/C_bS/b$$
$$C_a \rightarrow a$$
$$C_b \rightarrow b$$

Now first rule is in CNF. In second rule $C_bAA$ is not in CNF. So we can replace $AA$ by a non-terminal say $D$ and similarly $BB$ by $E$, we obtain

$$S \rightarrow C_bA/C_aB$$
$$A \rightarrow C_bD/C_aS/a$$
$$B \rightarrow C_aE/C_bS/b$$
$$C_a \rightarrow a$$
$$C_b \rightarrow b$$
$$D \rightarrow AA$$
$$E \rightarrow BB$$

The above grammar is equivalent Chomsky normal form of the given grammar.

**Example 7.13. Change the following grammar in to CNF**

$$S \rightarrow 1A/0B$$
$$A \rightarrow 1AA/0S/0$$
$$B \rightarrow 0BB/1$$

**Solution.** We know that in CNF, either two non-terminal or one terminal is on the right side.

Therefore replace $AA$ by $C_a$ and $BB$ by $C_b$, 1 by $C_1$, 0 by $C_0$, we get

$$S \rightarrow C_1A/C_0B$$
$$A \rightarrow C_1C_a/C_0S/0$$

$$B \to C_0C_b/1$$
$$C_1 \to 1$$
$$C_0 \to 0$$
$$C_a \to AA$$
$$C_b \to BB$$

while the CNF form.

**Example 7.14. Change the following grammar in to CNF**

$$G = (\{S\}, \{a, b, c\}, \{S \to a/b/CSS\}, S).$$

**Solution.** Here productions are

$$S \to a$$
$$S \to b$$
$$S \to CSS$$

Replace $SS$ by $D$, we get

$$S \to a$$
$$S \to b$$
$$S \to CD$$
$$D \to SS$$

which is in CNF form.

**Example 7.15. Change the following grammar in to CNF**

$$S \to abSb/a/aAb$$
$$A \to bS/aAAb.$$

**Solution.** Replace $aA$ by $B_a$ and $Ab$ by $B_b$, we get

$$S \to abSb/a/aB_b$$
$$A \to bS/B_aB_b$$
$$B_a \to aA$$
$$B_b \to Ab$$

Now replace $ab$ by $C$ and $Sb$ by $D$, we get

$$S \to CD/a/aB_b$$
$$A \to bS/B_aB_b$$
$$B_a \to aA$$
$$B_b \to aA$$
$$B_b \to Ab$$
$$C \to ab$$
$$D \to Sb$$

Now replace $a$ by $X_a$ and $b$ by $X_b$, we get

$$S \to CD/a/X_aB_b$$
$$A \to X_bS/B_aB_b$$
$$B_a \to X_aA$$
$$B_b \to X_aB$$
$$C \to AX_b$$
$$D \to SX_b$$

$$X_a \to a$$
$$X_b \to b.$$

It is required CNF form.

**Example 7.16. Convert CFG which is given below in to CNF form.**

$$S \to bA/aB$$
$$A \to bAA/aS/a$$
$$B \to aBB/bS/b.$$

**Solution.** Let us replace $b$ by $C_b$ and $a$ by $C_a$, then CFG becomes

$$S \to C_bA/C_aB$$
$$A \to C_bAA/C_aS/a$$
$$B \to C_aBB/C_bS/b$$
$$C_b \to b$$
$$C_a \to a$$

Now let us replace $C_bA$ by $D$ and $C_aB$ by $E$ then grammar becomes as follows :

$$S \to C_bA/C_aB$$
$$A \to DA/C_aS/a$$
$$B \to EB/C_bS/b$$
$$C_b \to b$$
$$C_a \to a$$
$$D \to C_bA$$
$$E \to C_aB$$

Now every production of the grammar is in the CNF form.

**Example 7.17. Design a CFG for the language $L = \{a^{4n} \mid n \geq 1\}$ and convert that CFG into CNF form.**

**Solution.** Let CFG for the language $L = \{a^{4n}/n \geq 1\}$ is $G$

$$G = (V_n, V_t, P, S)$$

where

$$V_n = \{S\}$$
$$V_t = \{a\}$$

and $P$ is defined as

$$S \to aaaaS/aaaa$$

Now let us convert this CFG in to CNF form, replace each $a$ by $A$ then CFG becomes

$$S \to AAAAS/AAAA$$
$$A \to a$$

or we can also write ,

$$S \to AAAAS$$
$$S \to AAAA$$
$$A \to a$$

then CNF form will be

$$S \to AR_1$$
$$R_1 \to AR_2$$

$$R_2 \rightarrow AR_3$$
$$R_3 \rightarrow AS$$
$$S \rightarrow AR_4$$
$$R_4 \rightarrow AR_5$$
$$R_5 \rightarrow AA$$
$$A \rightarrow a.$$

## 7.4. GREIBACH NORMAL FORM

For every context free language $L$ without $\epsilon$, there exist a grammar in which every production is of the form $A \rightarrow aV$, where '$A$' is a variable, '$a$' exactly one terminal and '$V$' is the string of none or more variables, clearly $V \in V_n^*$.

"In other words if every production of the context free grammar is of the form $A \rightarrow aV/a$, then it is in Greibach Normal Form."

Greibach normal form will be used to construct a push down atuomata that recognize the language generated by a context free grammar.

To convert a grammar to GNF we start with a production in which the left side has a higher numbered variable than first variable in the right side and make replacements in right side.

Generally we see $A \rightarrow A\alpha/\beta$ type production in CFG, we can replace it by following two production :

$$A \rightarrow \beta A'$$
$$A' \rightarrow \alpha A'/\epsilon, \text{ here } A' \text{ is new non-terminal.}$$

By this way we actually remove left-recursion from the grammar.

In general if a grammar contains productions :

$$A \rightarrow A\alpha_1/A\alpha_2/A\alpha_3/.../A\alpha_m/\beta_1/\beta_2/\beta_3/.../\beta$$

Then the left recursion can be eliminated by adding the following productions in place of the above productions.

$$A \rightarrow B_1A'/B_2A'/B_3A'.../B_nA'$$
$$A' \rightarrow \alpha_1A'/\alpha_2A'/.../\alpha_mA'/\epsilon$$

### 7.4.1. Conversion of CFG into GNF

We know that every CFL '$L$' without $\epsilon$ can be generated by a grammar in which every productions are in Greibach Normal form.

We can convert CFG into GNF by following the number of steps discussed below :

**Step 1 :** Every CFG must be in CNF form, if not then convert it into CNF. Now rename all variables ($V_n$) by $A_1, A_2, ..., A_n$ where $A_1$ is start symbol.

**Step 2 :** Now modify the grammar so that every production are in following form :

$$\boxed{\begin{array}{l} A_i \rightarrow a\gamma \\ A_i \rightarrow A_j\gamma \text{ where } j > i \text{ and } \gamma \in V_n^* \end{array}}$$

or

To get CNF in above discussed format, we should follow the method :

Start with $A_1$ and proceed to $A_n$. Suppose upto $A_{m-1}$ the above condition $1 \le i \le m-1$, $A_i \rightarrow A_j\gamma$ then $j > i$ is satisfied and for $A_m$ we have produc

$A_m \to A_j\gamma$ such that $j < m$. To convert $A_m$ production so that it satisfy the above condition we apply the substitute rule.

(that is if $P$ contains productions like $A \to \alpha_1\beta\alpha_2$ and

$$B \to \beta_1 \mid \beta_2 \mid \beta_3 \ldots \beta_n$$

then we can write $\quad A \to \alpha_1\beta_1\alpha_2 \mid \alpha_2\beta_2\alpha_2 \mid \ldots \alpha_1\beta_n\alpha_2)$

Here we can generate new productions by substituting $A_p$ the right hand side of each $A_j$ proproduction *i.e.* if $A_2 \to A_3A_1 \mid a$ and $A_3 \to A_2A_1 \mid b$ then use $A_2$ production in R.H.S. of $A_3$. Now it becomes $A_3 \to A_3A_1 \mid aA_1 \mid b$. By repeating the process at most $m - 1$ times, we obtain productions of the form

$$A_m \to A_p\gamma \text{ where } p \geq m.$$

If $m = p$, we have $A_m \to A_m\gamma$, in left recursive form, which are then replaced by method of elimination of left recursion introducing a new variable $B_m$.

By repeating the step 2 for each variable in $G$, we have only productions of the form

$$A_i \to A_j\gamma \quad \text{where } j > i$$
$$A_i \to a\gamma \quad a \in V_t$$
$$B_i \to \gamma \quad \text{where } \gamma \in (V_n \cup \{B_1B_2 \ldots B_{i-1}\})^*$$

**Step 3.** Since $A_n$ is the highest numbered variable then productions are of the form $A_n \to a\gamma$. Now the leftmost symbol of right side of any production for $A_{m-1}$ must be either terminal or $A_m$ (since all productions have been converted $A_i \to A_j$, $j > i$ or $A_i \to a\gamma$ in step 2) replace $A_m$ on right side of production of $A_{m-1}$ by replacement rule.

Now, the leftmost symbol of the right hand side of productions for $A_{m-1}$, is terminal. Repeat this process for $A_{m-2}$, $A_{m-3}$... $A_1$. Now right hand side of each production for an $A_i$ starts with a terminal symbol.

**Step 4.** In step 2; when we have productions of the form $A_i \to A_i\gamma$, we eliminate left recursion and each time when we eliminate left recursion, we produced new variables $B_i$. Since in first step we have already converted the grammar into CNF and on elimination method of left recursion (if $A \to A\alpha \mid \beta$ we convert it into $A \to \beta B \mid \beta$, $B \to \alpha B \mid \alpha$), we cannot find new variable $B_i$ in right hand side as leftmost symbol of right hand side. Thus, after step 3 every production of $A_i (1 \leq i \leq n)$ is in the form $A_i \to a\gamma$ or $A_i \to A_jA_k$. Again, in production $B_i (B_i \to \alpha,\ B_i \to \alpha B_i)$, all $B_i$ productions have right hand side beginning with terminals or variable $A_i$'s. So, no $B_i$ production can start with another $B_j$. Finally, we apply the substitution rule and get the grammar which is in GNF.

**In the above steps, it is clear to see that we start with grammar in CNF and then we apply substitution rule and elimination of left recession.**

**Example 7.18. Convert the grammar.**

$$S \to AB \mid BC$$
$$A \to aB \mid bA \mid a$$
$$B \to bB \mid cC \mid b$$
$$C \to c$$

**into GNF.**

**Solution.** Here the production $S \to AB \mid BC$ is not in GNF. On applying the substitution rule we immediately get equivalent grammar.

$$S \to aBB \mid bAB \mid aB \mid bBC \mid cCC \mid bC$$

which is in GNF.

**Example 7.19. Convert the grammar.**

$$S \to abaSa \mid aba$$

into GNF.

**Solution.** If we introduce new variables $A$ and $B$ and productions as $A \to a$, $B \to b$ and substitute into given grammar as

$$S \to aBASA \mid aBA$$
$$A \to a,\ B \to b$$

which is in GNF.

**Example 7.20. Convert the grammar**

$$S \to AB$$
$$A \to BS \mid a$$
$$B \to SA \mid b$$

**into GNF.**

**Solution. Step 1 :** The given grammar is in CNF. Let us change the name of variables as $A_1, A_2, A_3$ for $S, A, B$ respectively. Now the given productions are in the form (with $A_1$ as start symbol)

$$A_1 \to A_2A_3$$
$$A_2 \to A_3A_1 \mid a$$
$$A_3 \to A_1A_2 \mid b$$

**Step 2 :** In this step we need productions must be in the form that the R.H.S. of productions must starts with terminal or with higher numbered variable. The productions.

$$A_1 \to A_2A_3$$
$$A_2 \to A_3A_1 \mid a$$
$$A_3 \to b$$

are in required form. But $A_3 \to A_1A_2$ is not. Now we want to convert $A_3 \to A_1A_2$ into required form.

By applying substitution in R.H.S. for $A_1$, we get $A_3 \to A_2A_3A_2$

Again applying substitution rule for $A_2$, we get

$$A_3 \to A_3A_1A_3A_2 \mid aA_3A_2$$

Thus, we get :

$$A_1 \to A_2A_3$$
$$A_2 \to A_3A_1 \mid a$$
$$A_3 \to A_3A_1A_3A_2 \mid aA_3A_2 \mid b.$$

Eliminating the left recursion from $A_3$ production, we get

$$A_3 \to aA_3A_2B_3 \mid bB_3 \mid aA_3A_2 \mid b$$

and

$$B_3 \to A_1A_3A_2B_3 \mid A_1A_3A_2$$

where $B_3$ is a new variable introduced.

Thus, we have set of productions as :

$$A_1 \rightarrow A_2A_3$$
$$A_2 \rightarrow A_3A_1 \mid a$$
$$A_3 \rightarrow aA_3A_2B_3 \mid bB_3 \mid aA_3A_2 \mid b$$
$$B_3 \rightarrow A_1A_3A_2B_3 \mid A_1A_3A_2$$

**Step 3 :** Now all the $A_3$ productions start with terminals. With the help of substitution rule we can replace $A_1$ in R.H.S. of $A_2$ production. We get

$$A_2 \rightarrow aA_3A_2B_3A_1 \mid bB_3A_1 \mid aA_3A_2A_1 \mid bA_1 \mid a.$$

Now, all $A_2$ productions start with terminal. Again we replace $A_2$ from $A_1 \rightarrow A_2A_3$, we have

$$A_1 \rightarrow aA_3A_2B_3A_1A_3 \mid bB_3A_1A_3 \mid aA_3A_2A_1A_3 \mid bA_1A_3 \mid aA_3$$

The resulting set of productions are the following :

$$A_1 \rightarrow aA_3A_2B_3A_1A_3 \mid bB_3A_1A_3 \mid aA_3A_2A_1A_3 \mid bA_1A_3 \mid aA_3$$
$$A_2 \rightarrow aA_3A_2B_3A_1 \mid bB_3A_1 \mid aA_3A_2A_1 \mid bA_1 \mid a$$
$$A_3 \rightarrow aA_3A_2B_3 \mid bB_3 \mid aA_3A_2 \mid b$$
$$B_3 \rightarrow A_1A_3A_2B_3 \mid A_1A_3A_2.$$

**Step 4 :** In step 3 only B productions are not in GNF. Applying substitution rule in production

$$B_3 \rightarrow A_1A_3A_2B_3$$

We get

$$B_3 \rightarrow aA_3A_2B_3A_1A_3A_3A_2B_3 \mid bB_3A_1A_3A_3A_2B_3 \mid aA_3A_2A_1A_3A_3A_2B_3$$
$$\mid bA_1A_3A_3A_2B_3 \mid aA_3A_3A_2B_3$$

And from $B_3 \rightarrow A_1A_2A_3$, we get

$$B_3 \rightarrow aA_3A_2B_3A_1A_3A_3A_2B_3 \mid bB_3A_1A_3A_3A_2B_3 \mid aA_3A_2A_1A_3A_3A_2A_3$$
$$\mid bA_1A_3A_3A_2B_3 \mid aA_3A_3A_2$$

The final sets of productions are

$$A_1 \rightarrow aA_3A_2B_3A_1A_3 \mid bB_3A_1A_3 \mid aA_3A_2A_1A_3 \mid bA_1A_3 \mid aA_3$$
$$A_2 \rightarrow aA_3A_2B_3A_1 \mid aB_3A_1 \mid aA_3A_2A_1 \mid bA_1 \mid a$$
$$A_3 \rightarrow aA_3A_2B_3 \mid bB_3 \mid aA_3A_2 \mid \mathrm{b}$$
$$B_3 \rightarrow aA_3A_2B_3A_1A_3A_2A_3B_3 \mid bB_3A_1A_3A_2A_3B_3 \mid aA_3A_2A_1A_3A_2A_3B_3$$
$$\mid bA_1A_3A_2A_3B_3 \mid aA_3A_2A_3B_3 \mid aA_3A_2B_3A_1A_3A_2A_3$$
$$\mid bB_3A_1A_3A_2A_3 \mid aA_2A_3A_1A_3A_2A_3 \mid bA_1A_3A_2A_3 \mid aA_3A_2A_3$$

In which all productions are in GNF.

**Example 7.21. Consider the following grammar and write its equivalent GNF**

$$S \rightarrow AB$$
$$A \rightarrow aA/bB/b$$
$$B \rightarrow b.$$

**Solution.** We know that in the GNF each production should be of the $A \rightarrow aV/a$ form where $V \in V_n^*$.

We can replace $A$ in $S \to AB$ by $aA$, or $bB$ or $b$.

$$S \to aAB/bBB/bB$$
$$A \to aA/bB/b$$
$$B \to b$$

which is in Greibach normal form.

**Example 7.22. Convert the following grammar**

$$S \to abSb/aa$$

**in to GNF.**

**Solution.** Here we can use a method similar to the one introduced in the construction of Chomsky normal form. We introduce new variables $A$ and $B$ that are essentially synonyms for $a$ and $b$, respectively, substituting for the terminals with their associated variables leads to the equivalent grammar.

$$S \to aBSB/aA$$
$$A \to a$$
$$B \to b$$

which is in GNF.

**Example 7.23. Convert the following grammar into GNF**

$$S \to aAS$$
$$S \to a$$
$$A \to SbA$$
$$A \to SS$$
$$A \to ba.$$

**Solution.** Let us introduced new non-terminals as $B$, $C$ and $D$, then GNF will be

$$A \to aBC$$
$$A \to aBS$$
$$A \to aC$$
$$A \to aS$$
$$A \to bD$$
$$B \to aBCS$$
$$B \to aBSS$$
$$B \to aBSS$$
$$B \to aCS$$
$$B \to aSS$$
$$B \to bDS$$
$$C \to bA$$
$$S \to aB$$
$$S \to a$$
$$D \to a.$$

## 7.5. REGULAR GRAMMARS

A regular grammar may be left linear or right linear.

"If all production of a CFG are of the form $A \to wB$ or $A \to w$, where $A$ and $B$ are variables and $w \in V_T^*$, then we say that grammar is right liner".

"If all the production of a CFG are of the form $A \to Bw$ or $A \to w$, we call it left linear."

A right or left linear grammar is called a regular grammar.

**Example 7.24. Write the left linear and right linear grammar for the regular expression**

$$r = 0(10)^*.$$

**Solution.** Given regular expression $r = 0(10)^*$, it means any string which starts with 0 followed by any number of 10's is in regular expression.

**Left Linear.** Let grammar be $G = (V, V_t, P, S)$

here

$$V_n = \{S\}$$
$$V_t = \{0, 1\}$$

and $P$ is defined as

$$S \to S10/0$$

**Right Linear.** Let grammar be $G' = (V_n, V_t, P, S)$

where

$$V_n = \{S, A\}$$
$$V_t = \{0, 1\}$$

and $P$ is defined as

$$S \to 0A$$
$$A \to 10\ A/\in.$$

**Example 7.25. Write the right linear and left linear regular grammar for the regular expression $r = (ab)^*\ a$.**

**Solution.** Let us analyse the language of this regular expression, clearly it contains set of strings in which every string start with any number of $ab$'s and end with single $a$.

**Right Linear.** Let grammar be $G_1 = (\{S\}, \{a, b\}, S, P)$

where $P_1$ is defined as follows :

$$S \to abS/a$$

**Left Linear.** Let grammar be $G_2 = (\{S, S_1, S_2\}, \{a, b\}, S, P_2)$

with production

$$S \to S_1a$$
$$S_1 \to S_1ab/\in$$

is left-linear. Both $G_1$ and $G_2$ are regular grammars.

### 7.5.1. Equivalence of Regular Grammars and Finite Automata

The regular grammars characterize the regular sets, in the sense that a language is regular if and only if it has a left-linear grammar or if and only if it has a right linear grammar. These results are proved in the next two theorems.

**Theorem 7.1. If $L$ has a regular grammar, then $L$ is a regular set ?**

**Proof.** First, suppose $L = L(G)$ for some right-linear grammar

$$G = (V_n, V_t, P, S).$$

Let us construct an NFA with $\epsilon$-moves, $M = (Q, V_t, \delta, \{S\}, \{\epsilon\})$ that simulates derivates derivation in $G$.

Consists of the symbols $[\alpha]$ such that $\alpha$ is $S$ or a suffix of some right hand side of a production in $P$, we define $\delta$ by :

(*a*) If $A$ is a variable, then $\delta([A], \epsilon) = \{[\alpha]/A \to \alpha$ is a production$\}$.

(*b*) If $a$ is in $V_t$ and $\alpha$ in $V_t^* \cup V_t^* V$, then

$$\delta([a\alpha], a) = \{[\alpha]\}.$$

Then it is easy to show that $\delta([S], w)$ contains $[\alpha]$ if and only if $S \overset{*}{\Rightarrow} xA \Rightarrow xy\alpha$, where $A \to y\alpha$ is a production and $xy = w$, or if $\alpha = S$ and $w = \epsilon$. As $[\epsilon]$ is the unique final state, $M$ accepts $w$ if and only if $S \overset{*}{\Rightarrow} xA \Rightarrow w$ But since every derivation of a terminal string has atleast on step, we see that on accepts $w$ if and only if $G$ generates $w$. Hence every right-linear grammar generates a regular set.

**Theorem 7.2. If $L$ is a regular set, then $L$ is generated by some left-linear grammar and by some right-linear grammar.**

**Proof.** Let $L = L(M)$ for DFA $M = (Q, \Sigma, \delta, q_0, F)$. First suppose that $q_0$ is not a fixed state. Then $L = L(a)$ for right linear grammar $G = (Q, \Sigma, P, q_0)$, where $P$ consists of production $P \to aq$ whenever $\delta(P, a) = q$ and also $P \to a$ whenever $\delta(P, a)$ is a final state. Then clearly $\delta(P, w) = q$ if and only if $P \overset{*}{\Rightarrow} wq$. If $wa$ is accepted by $M$, let $\delta(q_0, w) = P$, implying $q_0 \overset{*}{\Rightarrow} wP$. Also $\delta(P, a)$ is final, so $P \to a$ is a production. Thus $q_0 \overset{*}{\Rightarrow} wa$. Conversely, let $q_0 \overset{*}{\Rightarrow} x$. Then $x = wa$ and $q_0 \overset{*}{\Rightarrow} wP \Rightarrow wa$ for some state (variable) $P$. Then $\delta(q_0, w) = P$, and $\delta(P, a)$ is final. Then $x$ is in $L(M)$. Hence $L(M) = L(G) = L$.

Now let $q_0$ be in $F$, so $\epsilon$ is in $L$. We note that the grammar $G$ defined above generates $L - \{\epsilon\}$. We may modify $G$ by adding a new start symbol $S$ with productions $S \to q_0/\epsilon$. The resulting grammar is still right-linear and generates $L$.

To produce a left-linear grammar for $L$, start with an NFA for $L^R$ and then reverse the right sides of all productions of the resulting right-linear grammar.

### 7.5.2. Conversion of Regular Grammar into Finite Automata

Given a regular grammar $G$, a finite automata accepting $L(G)$ can be obtained as follows :

(*a*) The number of states in the automata will be equal to the number of non-terminals plus one. Each state in automata represents each non-terminal in the regulr grammar. The additional state will be the final state of the automata. The state corresponding to the start symbol of the grammar will be the initial state of automata. If $L(G)$ contains $\epsilon$ that is start symbol in grammar derives to $\epsilon$, then make start state also as final state.

CamScanner

(*b*) The transitions for automata are obtained as follows

(*i*) For every production $A \to aB$ make $\delta(A, a) = B$ that is make an are labelled '*a*' from $A$ to $B$.

(*ii*) For every production $A \to a$ make $\delta(A, a)$ = final state.

(*iii*) For every production $A \to \epsilon$, make $\delta(A, \epsilon) = A$ and $A$ will be final state.

**Example 7.26. Consider the following grammar**

$$S \to 0A/1B/0/1$$
$$A \to 0S/1B/1$$
$$B \to 0A/1S$$

**Find the automation for this grammar.**

**Solution.** The automation for this grammar will have four states : $S$, $A$, $B$ and one additional say $C$, which will be the final state.

**Step 1.** For production $S \to 0A$ and $S \to 1B$

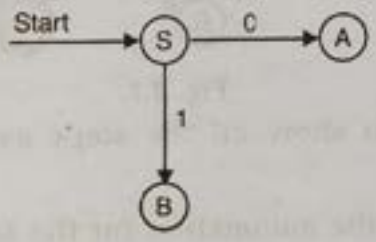


**Fig. 7.3.**

**Step 2.** For production $S \to 0/1$

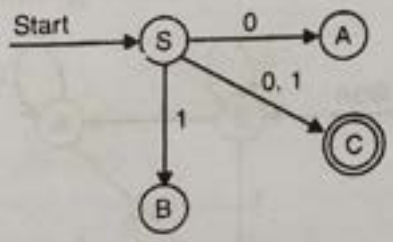


**Fig. 7.4.**

**Step 3.** For production $A \to 0S$

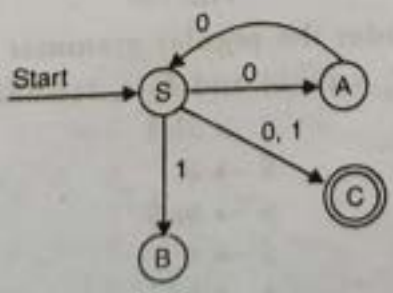


**Fig. 7.5.**

**Step 4.** For the production $A \to 1B$

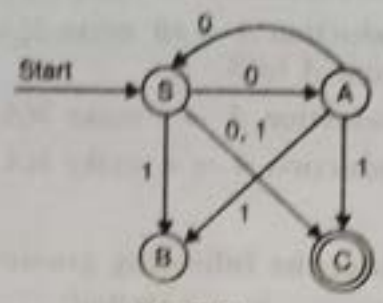


Fig. 7.6.

**Step 5.** For the production $B \to 0A$ and $B \to 1S$

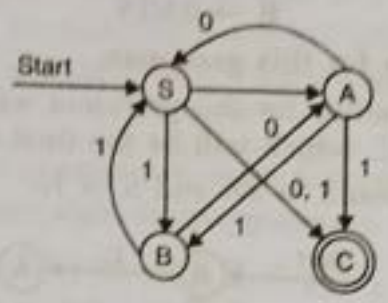


Fig. 7.7.

There is no need to show all the steps every time and FA can be generated directly.

**Example 7.27. Give the automation for the following grammar :**

$$S \to 0S/1A/1$$
$$A \to 0A/1A/0/1.$$

**Solution.** The automation for the above grammar will be

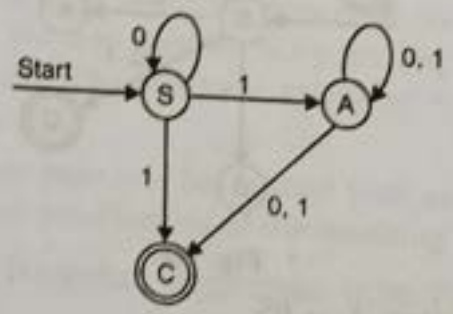


Fig. 7.8.

**Example 7.28. Consider the regular grammar $G = (V_n, V_e, P, S)$, where $V_n = \{A, B, S\}$, $V_t = \{a, b\}$ and $P$ is defined as**

$$S \to abA$$
$$S \to B$$
$$S \to baB$$
$$S \to \epsilon$$
$$A \to bS$$
$$B \to aS$$
$$A \to b$$

**Construct a NFA $M$ such that $L(M) = L(G)$. Trace the transitions of $M$ to accept the string *abba*.**

**Solution.** The NFA $M$ is

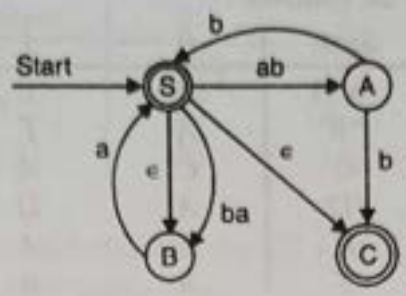


**Fig. 7.9.**

Here $C$ is final state.

Given string $B$ W = *abba*

$$S \xrightarrow{ab} A \xrightarrow{b} S \xrightarrow{\epsilon} B \xrightarrow{a} S \xrightarrow{\epsilon} C$$

**Example 7.29. Find whether the following grammars denote the same language.**

**Grammar $G_1$ is**

$$A \rightarrow 0B/1E$$
$$B \rightarrow 0A/1F/\epsilon$$
$$C \rightarrow 0C/1A$$
$$D \rightarrow 0A/1D/\epsilon$$
$$E \rightarrow 0C/1A$$
$$F \rightarrow 0A/1B/\epsilon$$

**where $A$ is the start symbol for $G_1$.**

**Grammar $G_2$ is :**

$$X \rightarrow 0Y/0/1Z$$
$$Y \rightarrow 0X/1Y/1$$
$$Z \rightarrow 0Z/1X$$

**where $X$ is the start symbol of $G_2$.**

**Solution.** In order to prove that $G_1$ and $G_2$ denote the same language, *i.e.*, $L(G_1) = L(G_2)$, we only need to prove that minimum state automata accepting $L(G_1)$ and the minimal state automata accepting $L(G_2)$ are identical.

The finite state automata for grammar $G_1$ is given below :

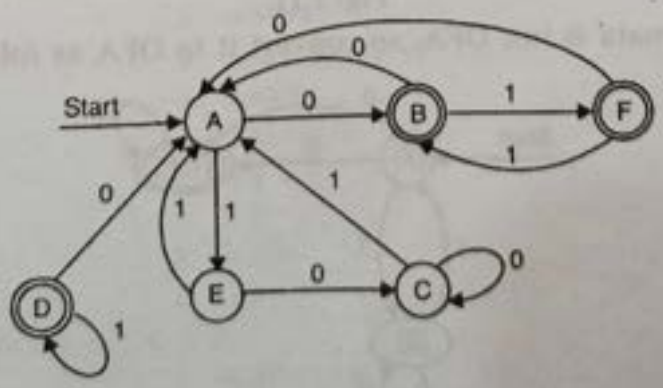


**Fig. 7.10.**

This automation shown in Fig. 7.10 is determinstic. To minimise this, we write it in tabular forms as follows :

| State | 0 | 1 |
|---|---|---|
| → A | B | E |
| *B | A | F |
| C | C | A |
| *D | A | D |
| E | C | A |
| *F | A | B |

Here, two sets of states (final and non-final) are {A, C, E} and {B, D, F}.

Since the output C and E are same, hence replacing every occurrence of E by C, we get

| State | 0 | 1 |
|---|---|---|
| → A | B | C |
| *B | A | F |
| C | C | A |
| *D | A | D |
| *F | A | B |

and now the two sets become {A, C} and {B, D, F}.

Now since in set {B, D, F} each state enters in a final state 'A' at input 0 and enters in a final state at input 1. Hence they can be reduced as follows :

| State | 0 | 1 |
|---|---|---|
| → A | B | C |
| *B | A | B |
| C | C | A |

Similarly automation for grammar $G_2$ is

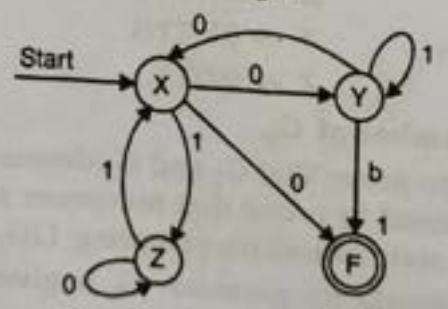


Fig. 7.11.

Above automata is not DFA, so convert it to DFA as follows :

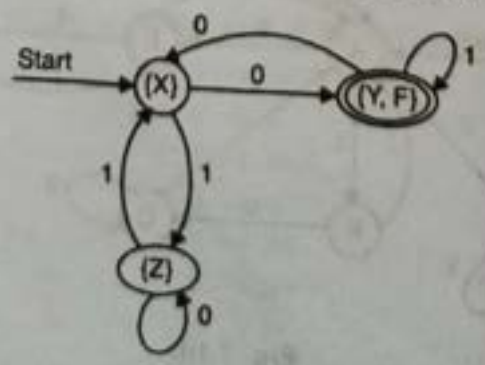


Fig. 7.12.

To minimize it, write it in tabular form

| State | 0 | 1 |
|---|---|---|
| → {X} | {Y, F} | {Z} |
| *{Y, F} | {X} | {Y, F} |
| {Z} | {Z} | {X} |

It is already minimised.

It is clear that these two automation for $G_1$ and $G_2$ are similar, hence $G_1$ and $G_2$ represent the same language.

## EXERCISE

1. Each of the following CFG's has a production using the symbol $\epsilon$ and yet $\epsilon$ is not a word in the language. Using the method, disccussed in this chapter, show that there are other CFGs for these languages that do not use $\epsilon$-productions.

(i) $S \to aX/bX$
$X \to a/b/\epsilon$

(ii) $S \to aX/bS/a/b$
$X \to aX/a/\epsilon$

(iii) $S \to aS/bX$
$X \to aX/\epsilon$

(iv) $S \to XaX/bX$
$X \to XaX/XbX/\epsilon$

2. Each of the following CFG's has unit productions. Using the algorthim presented in this chapter, find CFG's for these same languages that do not have unit productions.

(i) $S \to aX/Yb$
$X \to S$
$Y \to bY/b$

(ii) $S \to AA$
$A \to B/BB$
$B \to abB/b/bb$

(iii) $S \to AB$
$A \to B$
$B \to aB/Bb/\epsilon$

3. Convert the following CFG's to CNF :

(i) $S \to SS/a$

(ii) $S \to aSa/SSa/a$

(iii) $S \to aXX$
$X \to aS/bS/a$

(iv) $E \to E + E$
$E \to E * E$
$E \to (E)$
$E \to 7$

The terminal here are +, *, (, ), 7.

(v) $S \to ABABAB$
$A \to a/\epsilon$
$B \to b/\epsilon$

Note that $\epsilon$ is the word in this language, but when converted in to CNF, the grammar will no longer generate it.

(vi) $S \rightarrow SaS/SaSbS/SbSaS/\epsilon$

(vii) $S \rightarrow AS/SB$

$A \rightarrow BS/SA$

$B \rightarrow SS$

4. Convert the following CFG's with unit production in to CNF :

(i) $S \rightarrow X$

$X \rightarrow Y$

$Y \rightarrow Z$

$Z \rightarrow aa$

(ii) $S \rightarrow SS/A$

$A \rightarrow SS/AS/a$

5. Construct a DFA that accepts the language generated by the grammar

$S \rightarrow abA$

$A \rightarrow baB$

$B \rightarrow aA/bb.$

6. Find the regular grammar that generates the language $L(aa^* (ab + a)^*)$.

7. Construct the left linear grammar and right-linear grammar for the language

$L = \{a^n b^m : n \geq 2, m \geq 3\}$.

8. Find a regular grammar that generates the language on $\Sigma = \{a, b\}$ consisting of all strings with no more than $a$'s.

9. Find a regular grammar for the language

$L = \{a^n b^m : n + m \text{ is even}\}$.

10. Find regular grammars for the following languages on $\{a, b\}$.

(a) $L = \{w : n_a(w) \text{ and } n_b(w) \text{ are both even}\}$

(b) $L = \{w : (n_a(w) - n_b(w)) \bmod 3 = 1\}$.

*****

# 8

CHAPTER

# Pushdown Automata

**INSIDE THIS CHAPTER**

8.1. Introduction 8.2. Definition of Pushdown Automata 8.3. Pushdown Automata and Context-free Grammars

## 8.1. INTRODUCTION

The finite automata that we have studied earlier are not capable to recognise the context free languages such as $\{W C W^R / W \in \Sigma^*\}$. Because by reading a string in this language from left to right, automata must "remember" the string before it encounters symbol 'C' and then compare it to the string after 'C'. But our finite automata is not capable to remember any thing.

From above example it becomes clear that the finite automata can be extended by addition of an auxiliary storage to accept context free languages.

The pushdown automata is essentially a finite automata with control of both an input tape and a stack to store what it has read. A stack is a "first in last out" list, that is, symbols can be entered or removed only at the top of the list. When a new symbol is entered at the top, the symbol previously at the top becomes second and so on.

A familiar example of the stack is the set of plates that we see in cafeteria. We can remove plates only from the top of this set (stack) not from the middle. Similarly we can add plates only at the top of the stack not in the middle.

We shall now formalise the concept of a PDA. The PDA will have 3 things : an input tape, a finite control, and a stack. The device will be non-deterministic, having some finite number of moves in each situation.

We can also see pictorial representation of pushdown automata.

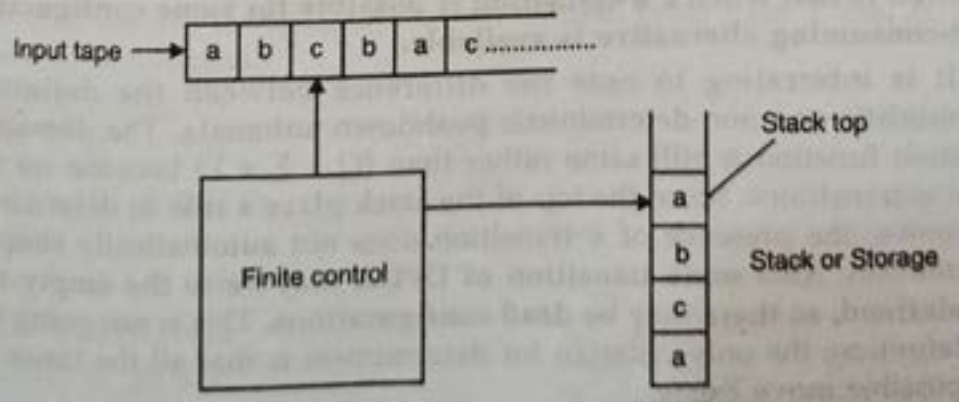


**Fig. 8.1.**

Here is input tape, from which finite control (which works like finite control of finite automata), reads the input, and same time it reads the symbol from

stack top. Now it depends on finite control, that what is the next state and what will happen with stack top.

## 8.2. DEFINITION OF PUSHDOWN AUTOMATA

A pushdown automata is a system, which is mathematically defined as follows :

$$P = (Q, \Sigma, \Gamma, \delta, s, F)$$

where $Q$ : non-empty finite set of states

$\Sigma$ : non-empty finite set of input symbols

$\Gamma$ : is finite set of pushdown symbols

$s$ : is the initial state, $s \in Q$

$F$ : is the set of final states and $F \subseteq Q$

$\delta$ : it is a transition function or transition relation which maps.

$$(Q \times \Sigma^* \times \Gamma^*) \rightarrow (Q \times \Gamma^*)$$

As we have already seen in case of automation, that it may be nondeterministic or deterministic, pushdown automata also behaves in similar fashion.

"If it follows all the properties of non-determinision then it is said to be non-deterministic pushdown automata".

Clearly for the "NPDA" transition relation must behave as follows :

$$(Q \times (\Sigma \cup \{\epsilon\} \times \Gamma^*) \rightarrow \text{finite subset of } (Q \times \Gamma^*).$$

From the above relation it becomes very clear that $\epsilon$-transitions are allowed in case of non-deterministic pushdown automata. Similar to the non-deterministic finite automata, non-deterministic pushdown automata also have the power of guess.

Now let us analyse the case of deterministic pushdown automata.

"A pushdown automata $P = (Q, \Sigma, \Gamma, \delta, S, F)$ is said to be deterministic if it is an automation as defined in definition of PDA, subject to the restrictions that for every $q \in Q$, $a \in (\Sigma \cup \{\epsilon\})$ and $b \in \Gamma$.

(*a*) $\delta(q, a, b)$ contains at most one element,

(*b*) if $\delta(q, \epsilon, b)$ is not empty, then $\delta(q, c, b)$ must be empty for every $c \in \Sigma$.

**The first of these conditions simply requires that for any given input symbol and any stack top, at most one move can be made. The second condition is that when a $\epsilon$-transition is possible for some configuration, no input-consuming alternative is available.**

"It is interesting to note the difference between the definitions of deterministic and non-deterministic pushdown automata. The domain of the transition function is still same rather than $(Q \times \Sigma \times \Gamma)$ because we want to retain $\epsilon$-transitions. Since the top of the stack plays a role in determining the next move, the presence of $\epsilon$-transition does not automatically simply non-determinism. **Also some transition of DPDA may be to the empty set, that is undefined, so there may be dead configurations.** This is not going to affect the definition; the only criterian for determinism is that all the times at most one possible move exists.

### 8.2.1. Explanation for the Moves of Pushdown Automata

Let us see some moves of PDA in the form of mapping, that

$$(Q \times \Sigma^* \times \Gamma^*) \rightarrow (Q \times \Gamma^*).$$

(*a*) The interpretation of $((q, a, z), (p, y)) \in \delta$ (where $p, q, \in Q$, $a$ is an alphabet, $z$ and $y$ in $\Gamma^*$) is that PDA whenever is in state $q$, with $z$ on the top of the stack may read '$a$' from the input tape, replace $z$ by $y$ on the top of the stack and enter state $P$.

(*b*) To push a symbol on the stack, just add it on the top of the stack. This can be achieved by the transition $((q, a, \in), (p, a))$.

(*c*) Similarly to pop a symbol is to remove it from the top of the stack. The transition $((q, a, z), (p, \in))$ pops $z$ from the stack.

(*d*) PDA can also behave as do nothing machine, just read the input from the input and don't make any change in the state and symbol at the stack.

Let PDA in state $P$ and it has any symbol say $x$ on the stack top, now it read a input '$a$' from the input tape then write a move for PDA which behaves as do nothing machine.

$$((P, a, z), (p, z))$$

Clearly PDA just read input '$a$', and without making any change, the state or at stack top, waits for another input from input tape.

Now I think we are able to handle some practical problems :

**Example 8.1. Design a PDA which accepts the language**

**$L = \{w \in \{a, b\}^* / w$ has the equal number of $a$'s and $b$'s$\}$.**

**Solution.** To solve any problem, it becomes very important that we should be able to analyse the language very clearly.

Here Language $L = \{w \in \{a, b\}^* / w$ has equal number of $a$'s and $b$'s$\}$ is having a set of string in which every string is having equal number of $a$'s and $b$'s, there is no concern of the position of $a$'s and $b$'s, the machine can read input '$a$' first or input $b$'s first, $a$ after $b$, $b$ after $a$, $a$ followed by $a$ or $b$ followed by $b$ and of course null string is also accepted.

Now let us assume that PDA be $P$.

$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = \{s, q, f\}$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, c\}$$
$$F = \{f\}$$ $\delta$ is follows.

(Note that we are introducing another symbol other than $\Sigma$ that is '$c$' to make the solution logical.)

1. $((s, \in, \in), (q, c))$
2. $((q, a, c), (q, ac))$
3. $((q, a, a), (q, aa))$
4. $((q, a, b), (q, e))$
5. $((q, b, c), (q, bc))$
6. $((q, b, b), (q, bb))$
7. $((q, b, a), (q, \in))$
8. $((q, \in, c), (f, \in))$

Here we are using symbol $c$ to mark the bottom of the stack. PDA intially pushes '$c$' in to the stack (using transition 1). Now if PDA reads symbol '$a$'

from the input tape and if stack is empty or has '$a$' on the stack top then push '$a$' in to the stack (by using transition 2 and 3). Now if it reads '$a$' from input and '$b$' is on the top of the stack then it simply pops the symbol '$b$' and pushes nothing. (This is achieved by transition 4) same reason holds true for symbol '$b$'. When it has read all the input and finds empty symbol '$c$' on the top of the stack, it enters in state $f$, and pop the symbol $c$. **So final state is defined as, PDA is in state $f$ and stack is empty.**

Take a string *abab* for example. Now if we want to check whether it contains equal number of $a$'s and $b$'s, we simply store $a$'s or $b$'s whatever we read. Now when we read next $a$, we delete a corresponding '$b$' from our store. The same process follows for $b$'s. If at the end we have nothing to read and nothing in store then we can say that the string contains equal number of $a$'s and $b$'s. Following table shows the operation of PDA on the string *abbbaaba*.

| S. No. | State | Unread input | Stock | Transition used |
|---|---|---|---|---|
| 1 | $s$ | *abbbaaba* | $\in$ | . |
| 2 | $q$ | *abbbaaba* | $\in$ | 1 |
| 3 | $q$ | *bbbaaba* | *ac* | 2 |
| 4 | $q$ | *bbaaba* | *c* | 7 |
| 5 | $q$ | *baaba* | *bc* | 5 |
| 6 | $q$ | *aaba* | *bbc* | 6 |
| 7 | $q$ | *aba* | *bc* | 4 |
| 8 | $q$ | *ba* | *c* | 4 |
| 9 | $q$ | *a* | *bc* | 5 |
| 10 | $q$ | $\in$ | *c* | 4 |
| 11 | $f$ | $\in$ | $\in$ | 8 |

This problem can also be solved without introducing the symbol '$c$' at the stack bottom, let us see another solution.

Let PDA be $P_1$

$$P_1 = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = \{s, q, f\}$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b\}$$
$$F = \{f\}$$

and transition relation $\delta$ is defined as follows :

1. $((s, a, \in), (q, a))$
2. $((s, b, \in), (q, b))$
3. $((q, a, a), (q, aa))$
4. $((q, b, b), (q, bb))$
5. $((q, a, b), (q, \in))$
6. $((q, b, a), (q, \in))$
7. $((q, \in, \in), (f, \in))$

It will also work.

**Example 8.2. Design PDA for the language $L = \{wcw^r / w \in \{a, b\}^*\}$.**

**Solution.** By the analysis of the language it becomes clear that $ababcbaba \in L$, but $abc\ ab \notin L$, and $cbc \notin L$.

Let pushdown automata be $P$

$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = \{s, f\}$$
$$\Sigma = \{a, b, c\}$$
$$\Gamma = \{a, b\}$$

$F = \{f\}$ and transition relation $\delta$ is defined as follows :

(1) $((s, a, \in), (s, a))$
(2) $((s, b, \in), (s, b))$
(3) $((s, c, \in), (f, \in))$
(4) $((f, a, a), (f, \in))$
(5) $((f, b, b), (f, \in))$.

This automation operates in the following way. As it reads the first half of its input, if remains its in intial state and uses transition 1 and 2 to transfer symbols from the input string on to the pushdown store. **Note that these transitions are applicable regardless of the current content of the stack.** Since the "string" to be matched on the top of the pushdown store is the empty string, when machine sees *a* '*c*' in the input string, it switches from state *s* to state *f* without operating on the stack. Thereafter only transition 4 and 5 are operative; these permit the removal of the top symbol on the stack, provided that it is the same as the next input symbol. If the input symbol does not match the top symbol on the stack, no further operation is possible. If the automation reaches in this way the configuration $(f, \in, \in)$. Final state, end of input, empty stack then the input was indeed of the form $wc\,w^R$, and the automation accepts on the other hand if automation or if the input is exhausted before the stack is emptied, then it does not accept.

The operation of $P$ can be seen by following table, we have taken the string *abacaba* for example

| S.No. | State | Unread input | Stack | Transition used |
|---|---|---|---|---|
| 1 | $s$ | *abacaba* | $\in$ | – |
| 2 | $s$ | *bababa* | *a* | 1 |
| 3 | $s$ | *acaba* | *ba* | 2 |
| 4 | $s$ | *caba* | *aba* | 1 |
| 5 | $f$ | *aba* | *aba* | 3 |
| 6 | $f$ | *ba* | *ba* | 4 |
| 7 | $f$ | *a* | *a* | 5 |
| 8 | $f$ | $\in$ | $\in$ | 4 |

**Example 8.3. Construct a push down automata to accept the following language**

$$L = \{ww^R : w \in \{a, b\}^*\}.$$

**Solution.** By the analysis of given language it becomes very clear that the strings accepted by the machine are the same as those accepted by the machine of the previous example, except that the symbol *c* that marked the center of the string is missing. **Therefore, the machine must "gess" when it has reached the middle of the input string and change from state 's' to state 'f' in a non-deterministic fashion.**

Let PDA be $P = (Q, \Sigma, \Gamma, \delta, S, F)$

where $Q = \{s, f, f'\}$

$\Sigma = \{a, b\}$

$F = \{f'\}$

and transition relation $\delta$ is defined as follows :

(1) $((s, a, \in), (S, a))$

(2) $((s, b, \in), (S, b))$

(3) $((s, \in, \in), (f, \in)) \longrightarrow$ (middle of the input string)

(4) $((f, a, a), (f, \in))$

(5) $((f, b, b), (f, \in))$

(6) $((f, \in, \in), (f', \in))$

Clearly whenever the machine is in state '$s$' it can non-deterministically choose either to push the next input symbol on the stack or to switch to state $f$ without consuming any input. Move 6 is for accepting the string.

**Example 8.4. Design a PDA for the following language**

$$L = \{a^n b^n : n > 0\}.$$

**Solution.** By the analysis of the language it is clear that PDA for the given language will accept the set of strings in which every string starts with '$a$' followed by any number of $a$'s and followed by same number of $b$'s, clearly '$a$' should not be encountered whenever $b$ is encountered strings like $ab$, $aabb$, $aaabbb$ ... should be accepted and $\in$, $aa$, $bb$, $aabbb$ and $bbaa$ should be rejected.

Let push down automata be $P$

$P = (Q, \Sigma, \Gamma, \delta, S, F)$

where $Q = \{s, f, q\}$

$F = \{f\}$

$\Sigma = \{a, b\}$

$\Gamma = \{a\}$

and transition relation $\delta$ is defined as follows :

(1) $((s, a, \in), (S, a))$

(2) $((s, a, a), (S, aa))$

(3) $(s, b, a), (q, \in))$

(4) $((q, b, a), (q, \in))$

(5) $((q, \in, \in), (f, \in))$

Initially machine is at state $s$, and it can read input a then pushes all $a$'s encountered before any '$b$' to the stack (by using transition relation on 1 and 2). When first '$b$' is encountered, corresponding '$a$' is poped up from the stack and now state becomes $q$, and this process can be repeat with the help of transition relation number '4'. Finally when there is no input an stack is empty PDA charges its state from $q$ to final state '$f$'.

CamScanner

**Example 8.5. In Fig. 8.2 a DFA is given, write the regular expression for it and design a pushdown automata**

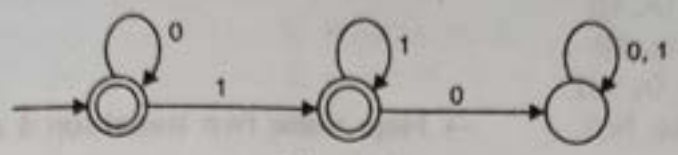


**Fig. 8.2.**

**Solution.** By the observation of DFA defined by the transition diagram in Fig. 8.2, we can write the regular expression for it, let it is $r$ then

$$r = 0^* 1^*$$

Clearly all the strings which start with any number of 0's followed by any number of 1's and there is no '0' after 1 is encountered is the string of regular set.

Let pushdown automata be $P$

$$P = (Q, \Sigma, \Gamma, \delta, s, F)$$
$$Q = \{s, p, r\}$$
$$\Sigma = \{0, 1\}$$
$$\Gamma = \{\in\}$$
$$F = \{s, P\}$$

and the transition relations $\delta$ are defined as follows :

(1) $((s, \in, \in), (s, \in))$
(2) $(s, 0, \in), (s, \in))$
(3) $((s, 1, \in), (p, \in))$
(4) $(p, 1, \in), (p, \in))$
(5) $((p, 0, \in), (r, \in))$

Now let us see the explanation of transitions, initial state of machine is $s$, it is also final state so $\in$ is accepted. Now any number of 0's can be accepted (by the help of transition relation 2). As soon as 1 is encountered state $s$ is changed to $p$ from $s$ (by the help of transition 3). Now any number of 1 can be accepted by the machine with the help of transition number 4. Whenever '0' is encountered after reading 1, state is change to $r$ from $P$ and $r$ state behaves like trap state and here process stopped and string is not accepted, since $r$ is not final state.

**Example 8.6. Design a pushdown automata for the following language, $L = \{a^n b^{2n} : n > 0\}$.**

**Solution.** Language $L = \{a^n b^{2n} : n > 0\}$ is the set of strings, in which every string starts with $a$, ends with $b$, no '$a$' comes after $b$, no $b$ comes before $a$ and number of $b$'s are double than number of $a$'s. For example, strings like *aabbbb*, *abb*, *aaabbbbbbb* will be accepted by PDA and strings like *a*, *b*, *ab*, *ba*, *aab*, *aaabbb* will be rejected.

Let PDA be $P = (Q, \Sigma, \Gamma, \delta, S, F)$

where $Q = (p, q, f, r)$

$$S = \{P\}$$
$$F = \{f\}$$

and transition relations are defined as follows :

(1) $((p, a, \in), (p, a))$
(2) $((p, a, a), (p, a))$
(3) $((p, b, a), (q, b))$
(4) $((q, b, ba), (r, \in))$
(5) $((r, b, a), (q, ba))$ → Here these two transition 4 and 5 represent the looping.
(6) $((r, \in, \in), (f, \in))$

Now let us see explanation for above discussed transition relation. Here first we pushed all the $a$'s by the help of moves 1 and 2. Now as first $b$ is encountered, then it is pushed on the stack by the help of move number 3 and when next '$b$' is encountered then we pop the $ba$ from the stack and change the state from $q$ to $r$. It should be clear that moves 4 and 5 represent loop that is move 4 and 5 works on the value of '$n$'. Finally when input becomes empty and stack also becomes empty then state $r$ changes to final state that is $f$ and string is accepted.

**Example 8.7. Construct a PDA for the regular expression**

$$r = 0^* 1^+.$$

**Solution.** Regular expression is $r = 0^* 1^+$, let us write the language for this regular expression, let us say it is $L$,

$$L = \{0^m 1^n / m \geq 0, n \geq 1\}$$

Clearly regular expression $r$ contains the set of string in which every string starts with any number of 0's but certainly having atleast one 1, and it must be last symbol of the string. As 1 encountered then 0's should not come.

Let PDA be $P$

$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = \{q_0, f\}$$
$$S = \{q_0\}$$
$$F = \{f\}$$
$$\Sigma = \{0, 1\}$$

and $P$ is defined as

(1) $((q_0, 0, \in), (q_0, \in))$
(2) $((q_0, 1, \in), (f, \in))$
(3) $((f, 1, \in), (f, \in))$

It is very important to understand that in above three moves we are not using the stack. Initially machine is at state $q_0$ when it reads 0's then it does nothing, but as soon as 1 is encountered then state changes to final state, with the help of move number '2'. Now after it, machine can accept any number of time 1's in the string by the help of move '3'.

**Example 8.8. Construct the pushdown automata for the following language :**

$$L = \{a^n b^{n+1} / n = 1, 2, 3, \ldots\}.$$

**Solution.** Language $L$ is the set of strings in which number of $b$'s are exactly one greater number of $a$'s and every string starts with '$a$' and ends with '$b$'. No $a$ is encountered after $b$.

Let PDA be $P$

$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = \{p, q, q_1, f\}$$
$$\Sigma = \{a, b\}$$
$$S = \{p\}$$
$$F = \{f\}$$

and transitions relations are defined as follows :

(1) $((p, a, \in), (p, a))$
(2) $((p, a, a), (p, aa))$
(3) $((p, b, a), (q, \in))$
(4) $((q, b, a), (q, \in))$
(5) $((q, b, \in), (q_1, \in))$
(6) $(q_1, \in, \in), (f, \in))$.

Let us analyse the solution, first of all by the help of move number '1' and '2' we are pushing all the $a$'s on the stack then by the help of move '3' and '4' we pop all $a$'s from the stack when machine starts reading $b$'s from the input tape. When machine reads last $b$ (at this time stack is empty), the change the state to $q_1$, and now at state $q_1$, input is null string and stack is empty, then machine goes to final state, so string is accepted.

**Example 8.9. Design a pushdown automata for the following language**

$$L = \{a^n b^m / n > m \geq 0\}.$$

**Solution.** As usual, let us first analyse the language. Here language is the set of strings in which every string starts with '$a$' (there is at least one $a$ symbol in the strings) and if $b$ is there then number of $a$'s are always greater than number of $b$'s and there should be no '$a$' when any $b$ is encountered clearly strings $aaabb$, $a$, $aab$ are in language but $ab$, $abb$, $b$, $aaabbb$ ... are not accepted.

Let PDA for the language is $P$

$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$

where

$$Q = (q_0, q_1, q_2, q_3, f)$$
$$S = \{q_0\}$$
$$F = \{f\}.$$
$$\Sigma = \{a, b\}$$

and here transition relations are defined as follows :

(1) $((q_0, a, \in), (q_1, a))$
(2) $((q_1, a, a), (q_1, a))$
(3) $((q_1, b, a), (q_2, \in))$.
(4) $((q_2, b, a), (q_3, \in))$
(5) $((q_3, b, a), (q_2, \in))$
(6) $((q_2, \in, a), (q_2, \in))$
(7) $((q_2, \in, \in), (f, \in))$
(8) $((q_1, \in, a), (q_1, \in))$
(9) $((q_1, \in, \in), (f, \in))$

Now let us analyse the move of the pushdown machine. Since single '$a$' is in the language so moves '1' and '9' make the acceptance of '$a$' possible. Move 1 and 2 help us to push all the $a$'s on the stack, now as soon as '$b$' is encounttered an '$a$' is poped up and state is changed to $q_2$, move 4 and 5 are showing loop for poping $a$'s when $b$'s are encountered, move '6' helps us to empty stack from the '$a$' since there is no input on the input tape, finally move '7' makes the string acceptable.

## 8.3. PUSHDOWN AUTOMATA AND CONTEXT-FREE GRAMMARS

It should be clear now that PDA can recognise any language for which there exists a CFG. "That is class of language accepted by pushdown automata is exactly the class of context-free languages".

### 8.3.1. Construction of PDA Equivalent of a CFG

Let $G = (V_n, V_t, P, S)$ be a context-free grammar; we must construct a pushdown automata $P$ such that $L(P) = L(G)$. The machine we construct has only two states, $p$ and $q$, and remains permanently in state $q$ after its first move. Also, $p$ uses $V_n$, the set of non-terminals and $V_t$ the set of terminals, as its stack alphabet. We let $P = (Q, \Sigma, \Gamma, \delta, S, F)$

where
$$Q = \{p, q\}$$
$$\Sigma = V_t$$
$$\Gamma = V_n \cup V_t$$
that is the set of terminals and non-terminals.
$$S = p$$
and transition relation $\delta$ is defined as follows :

(1) $\{(p, \in, \in), (q, S))$ (as $S$ is strating non-terminal of CFG)

(2) $((q, \in, A), (q, x))$ for each rule $A \to x$ in CFG.

(3) $((q, a, a), (q, \in))$ for each $a \in V_t$.

The pushdown automata $P$ begins by pushing $S$, the starts symbol of grammar $G$, on its initially empty pushdown store, and entering state $q$ (transition 1). On each subsequent step, it either replaces the top most symbol $A$ on the stack, provided that it is a non-terminal, by the right hand side $x$ of some rule $A \to x$ in grammar (transition of type 2), or POPS the top most symbol from the stack, provide that it is a terminal symbol that matches the next input symbol (transition of type 3).

**Example 8.10. Design a PDA for the following CFG,**
$$G = (V_n, V_t, P, S) \text{ with}$$
$V_n = \{S\}$, $V_t = \{(, )\}$ and $P$ is define $\delta$ as follows :
$$S \to \in$$
$$S \to SS$$
$$S \to (S).$$

**Solution.** Let us assume corresponding PDA will be
$$P = (Q, \Sigma, \Gamma, \delta, S, F)$$
where
$$Q = \{p, q\}$$
$$\Sigma = \{(, )\}$$
$\Gamma = (S, (, )\}$ that is terminals and non-terminals in grammar

$S = p$

$F = \{q\}$, and $\delta$ is as follows.

(1) $((p, \in, \in), (q, S))$

(2) $((q, \in S), (q, \in))$ because $S \to \in$ is a rule of CFG

(3) $((q, \in S), (q, SS))$

(4) $((q, \in, S), (q, (S))$

(5) $((q, (, ( ), (q, \in))$

(6) $((q, ), )), (q, \in))$ for each $a \in V_t$

Let us apply these transition relations on the string

$w = ( ) ( )$

| S. No. | State | Unread input | Stack | Transition used |
|---|---|---|---|---|
| 1 | $p$ | ( ) ( ) | $\in$ | – |
| 2 | $q$ | ( ) ( ) | $S$ | (1) |
| 3 | $q$ | ( ) ( ) | $SS$ | (3) |
| 4 | $q$ | ( ) ( ) | $(S)\ SS$ | (4) |
| 5 | $q$ | ) ( ) | $S)\ S$ | (5) |
| 6 | $q$ | ) ( ) | $)\ S$ | (2) |
| 7 | $q$ | ( ) | $S$ | (6) |
| 8 | $q$ | ( ) | $(S)$ | (4) |
| 9 | $q$ | ) | $S)$ | (5) |
| 10 | $q$ | ) | ) | (2) |
| 11 | $q$ | $\in$ | $\in$ | (6) |

**Example 8.11. Design PDA for the grammar $G = (V_n, V_t, P, S)$ where**

$V_n = \{S\}$

$V_t = \{a, b, c\}$

**and $P$ is defined as**

$S \to aSa$

$S \to bSb$

$S \to c.$

**Solution.** Let PDA be

$P = (Q, \Sigma, \Gamma, \delta, S, F)$

where

$Q = \{p, q\}$

$\Sigma = \{a, b, c\}$

$\Gamma = \{S, a, b, c\}$

$S = \{p\}$

$F = \{q\}$

and $\delta$ is defined as :

(1) $((p, \in, \in), (q, S))$

(2) $((q, \in, S), (q, aSa))$

(3) $((q, \in, S), (q, bSb))$

(4) $((q, \in, S), (q, c))$

(5) $((q, a, a), (q, \in))$

(6) $((q, b, b), (q, \in))$

(7) $((q, c, c), (q, \in))$

Let us apply this on string

$$w = abbcbba$$

| S.No. | State | Unread input | Stack | Transform used |
|---|---|---|---|---|
| 1 | $p$ | $abbcbba$ | $\in$ | – |
| 2 | $q$ | $abbcbba$ | $S$ | (1) |
| 3 | $q$ | $abbcbba$ | $aSa$ | (2) |
| 4 | $q$ | $bbcbba$ | $Sa$ | (5) |
| 5 | $q$ | $bbcbba$ | $bSba$ | (3) |
| 6 | $q$ | $bcbba$ | $Sba$ | (6) |
| 7 | $q$ | $bcbba$ | $bSbba$ | (3) |
| 8 | $q$ | $cbba$ | $Sbba$ | (6) |
| 9 | $q$ | $cbba$ | $cbba$ | (7) |
| 10 | $q$ | $bba$ | $bba$ | (6) |
| 11 | $q$ | $ba$ | $ba$ | (6) |
| 12 | $q$ | $a$ | $a$ | (6) |
| 13 | $q$ | $\in$ | $\in$ | (5) |

### 8.3.2. Construction of CFG Equivalent of a PDA

Let $P = (Q, \Sigma, \Gamma, \delta, q_0)$ be a PDA that accepts a language $L$ by empty stack. Then a CFG $G = (V_n, V_t, P, S)$ that generates $L$ can be constructed using following rules :

**Assumption :** Here we are taking a assumption that stack bottom is indicated by a special symbol \$.

Let $S$ be the start symbol of grammar.

(*i*) For every $q \in Q$, add a production $S \to [q_0 \$ q]$ in $P$. Thus if there are $n$ states in a PDA $P$, then there will be '$n$' new productions added in $P$.

(*ii*) For every $q, r \in Q$, $a \in \{\Sigma \cup \in\}$, $X \in \Gamma$, if $\delta(q, a, X) = (r, \in)$, then add a production

$$[q \times r] \to a \text{ in } P.$$

(*iii*) For every $q, r \in Q$, $a \in \{\Sigma \cup \in\}$, $X \in \Gamma$, and $x \geq 1$, if $\delta(q, a, X) = (r, X_1, X_2, \ldots X_k)$ where $X_1, X_2, \ldots, X_k \in \Gamma$, then for every choice of $q_1, q_2, \ldots, q_k \in Q$, add the production

$$[q\ X\ q_k] \to a[r\ X_1\ q_1]\ [q_1\ X_2\ q_2] \ldots [q_{k-1}\ X_k\ q_k] \text{ in } P.$$

The basic idea behind this construction is to recognize that the current string in the derivation will consist of two parts, the string of input symbols read by the PDA so far and a remaining portion corresponding to the stack contents. Next we introduce variables of the proposed grammar in the form $[P\ X\ q]$ where $P$ and $q$ are states of the PDA. For the variable $[P\ X\ q]$ to be replaced by a symbol '$a$' ($a$ may be $\in$ also), it may be the case there is a move

in PDA that reads a, pops $X$ from the stack, and takes machine from state $P$ to $q$ moves. For example, it may go from state $P$ to state $P_1$, that reads a and replaced $X$ on stack by $Y_1, Y_2, Y_3, \ldots Y_k$. Then after a sequence of moves it may Pop off $Y_1$ and end up in state say $P_2$. Similarly it may pop off $Y_2$ and go into state $P_2$ and through many such steps finally entering into state $P_k$ and popping of symbol $Y_k$. All these intermediate move, in their cumulative effect, are equivalent to derive a portion of strings say w and reaching a state $P_k$. We can represent this fact by the production

$$[P\ X\ q_k] \rightarrow a[r\ X_1\ q_1]\ [q_1\ X_2] \ldots [q_{k-1}\ X_k\ q_k]$$

**Example 8.12. Consider the following PDA and construct equivalent CFG.**

1. $((q_0, a, \$) \rightarrow (q_0, a\$))$
2. $((q_0, b, \$) \rightarrow (q_0, b\$))$
3. $((q_0, a, a) \rightarrow (q_0, aa))$
4. $((q_0, b, b) \rightarrow (q_0, bb))$
5. $((q_0, a, b) \rightarrow (q_0, ab))$
6. $((q_0, b, a) \rightarrow (q_0, ba))$
7. $((q_0, c, \$) \rightarrow (q_1, \$))$
8. $((q_0, c, a) \rightarrow (q_1, a))$
9. $((q_0, c, b) \rightarrow (q_1, b))$
10. $((q_1, a, a) \rightarrow (q_1, \in))$
11. $((q_1, b, b) \rightarrow (q_1, \in))$
12. $((q_1, \in, \$) \rightarrow (q_1, \in))$

**Solution.** CFG using above construction production rules is as follows :

$$S \rightarrow [q_0\ \$\ q_0]/[q_0\ \$\ q_1] \qquad \text{(from rule } (i))$$

$$\left.\begin{array}{l} [q_1\ a\ q_1] \rightarrow a \\ [q_1\ b\ q_1] \rightarrow b \\ [q_1\ \$\ q_1] \rightarrow \in \end{array}\right\} \qquad \text{from rule } (ii)$$

Now we will get remaining productions using rule (*iii*).

$$[q_0\ \$\ q_0] \rightarrow a[q_0\ a\ q_0]\ [q_0\ \$\ q_0]$$
$$[q_0\ \$\ q_1] \rightarrow a[q_0\ a\ q_0]\ [q_0\ \$\ q_1]$$
$$[q_0\ \$\ q_1] \rightarrow a[q_0\ a\ q_1]\ [q_1\ \$\ q_1]$$
$$[q_0\ \$\ q_1] \rightarrow b[q_0\ b\ q_0]\ [q_0\ \$\ q_0]$$
$$[q_0\ \$\ q_1] \rightarrow b[q_0\ b\ q_0]\ [q_0\ \$\ q_1]$$
$$[q_0\ \$\ q_1] \rightarrow b[q_0\ b\ q_1]\ [q_1\ \$\ q_1]$$
$$[q_0\ a\ q_0] \rightarrow a[q_0\ a\ q_0]\ [q_0\ a\ q_0]$$
$$[q_0\ a\ q_1] \rightarrow a[q_0\ a\ q_0]\ [q_0\ a\ q_1]$$
$$[q_0\ a\ q_1] \rightarrow a[q_0\ a\ q_1]\ [q_1\ a\ q_1]$$

$$[q_0\ a\ q_0] \rightarrow b[q_0\ b\ q_0]\ [q_0\ a\ q_0]$$
$$[q_0\ a\ q_1] \rightarrow b[q_0\ b\ q_0]\ [q_0\ a\ q_1]$$
$$[q_0\ a\ q_1] \rightarrow b[q_0\ b\ q_1]\ [q_1\ a\ q_1]$$
$$[q_0\ b\ q_0] \rightarrow a[q_0\ a\ q_0]\ [q_0\ b\ q_0]$$
$$[q_0\ b\ q_1] \rightarrow a[q_0\ a\ q_0]\ [q_0\ b\ q_1]$$
$$[q_0\ b\ q_1] \rightarrow a[q_0\ a\ q_1]\ [q_1\ b\ q_1]$$
$$[q_0\ b\ q_0] \rightarrow b[q_0\ b\ q_0]\ [q_0\ b\ q_0]$$
$$[q_0\ b\ q_1] \rightarrow b[q_0\ b\ q_0]\ [q_0\ b\ q_1]$$
$$[q_0\ b\ q_1] \rightarrow b[q_0\ b\ q_1]\ [q_1\ b\ q_1]$$
$$[q_0\ \$\ q_1] \rightarrow c[q_1\ b\ q_1]$$
$$[q_0\ a\ q_1] \rightarrow c[q_1\ a\ q_1]$$
$$[q_0\ b\ q_1] \rightarrow c[q_1\ b\ q_1]$$

**Example 8.13. Construct PDA for the language**

$$L = \{a^3\ b^n\ c^n : n \geq 0\}.$$

**Solution.** By the observation we can say that language contains only that set of strings, in which every string contains exactly $3a$'s in the starting of string (like *aaa* ...). If $n > 0$, then $a$'s are followed by $b$'s and $b$'s are followed by $c$'s and number of $b$'s and $c$'s are equal.

String like *aaa, aaabc, aaabbbccc* ... are accepted and strings like *a, aa, abc, aabc, aaabbbc, aaaccc, aaacb* are not accepted.

Let PDA be $P = (Q, \Sigma, \Gamma, \delta, S, F)$

where $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\ f\}$

$\Gamma = \{b\}$

$\Sigma = \{a, b, c\}$

$F = \{f\}$

$S = \{q_0\}$

and $\delta$ is defined as follows :

(1) $((q_0, a, \in), (q_1, \in))$
(2) $((q_2, a, \in), (q_2, \in))$
(3) $((q_2, a, \in), (q_3, \in))$
(4) $((q_3, \in, \in), (f, \in))$
(5) $((q_3, b, \in), (q_4, b))$
(6) $(q_4, b, b), (q_4, b))$
(7) $(q_4, c, b), (q_5, \in))$
(8) $((q_5, c, b), (q_5, \in))$
(9) $((q_5, \in, \in), (f, \in))$

Here first three moves read exactly three $a$'s and machine moves to state $q_3$, now if $n = 0$ then $q_3$ change to final state and if $n > 0$ then we push all $b$'s on stack with the help of moves 5 and 6. Now when $c$ encountered then we POP all corresponding $b$'s (by the help of moves 7 and 8). If $b$'s and $c$'s are equal then machine finally transit to final state.

## EXERCISE

1. Let input alphabet be $\Sigma = \{a, b, c\}$ and $L$ be the language of all words in which all the $a$'s come before the $b$'s and there are same number of $a$'s as $b$'s and arbitrarily many $c$'s that can be in front, behind or among the $a$'s and $b$'s. Some words in $L$ are $abc$, $caabcb$, $ccacaabcccbccbc$.
   (*i*) Show the language is not regular.
   (*ii*) Design a PDA for above given language.
2. Let $L$ be some regular language in which all the words happen to have an even length. Let us define the new language second ($L$) to be set of all the words of $L$ second, where second we mean the first and second letters have been interchanged, the third and fourth letters have been interchanged, and so on.
   If $L = \{ba, abba, babb, ...\}$
   Second ($L$) $= \{ab, baab, abbb, ...\}$
   Build a PDA that accepts second ($L$).
3. Construct a PDA accepting the following languages :
   (*i*) $\{0^n1^m0^n / m, n \geq 1\}$
   (*ii*) $\{0^n1^{2n} / n \geq 1\}$
   (*iii*) $\{0^m1^m2^n / m, n \geq 1\}$
   (*iv*) $\{0^m 1^n / m > n \geq 1\}$.
4. Let given language $L = \{a^m b^n / n < m\}$
   (*i*) Construct a PDA for the language
   (*ii*) Write the CFG for the language and
   (*iii*) Convert this CFG into pushdown automata.
5. Starting with the CFG for $\{a^n b^n\}$
   $$S \rightarrow aSb/ab$$
   (*i*) put this CFG into CNF form
   (*ii*) Give the equivalent PDA for this CNF form.
6. For each of the CFGs, construct PDA that accepts the same language they generate.
   (*i*) $S \rightarrow aSbb/abb$
   (*ii*) $S \rightarrow SS/a/b$
   (*iii*) $S \rightarrow XaaX$
   $X \rightarrow aX/bX/\epsilon$
   (*iv*) $S \rightarrow aS/aSbS/a$

(v) $S \rightarrow XY$

$X \rightarrow aX/bX/a$

$Y \rightarrow Ya/Yb/a$

(vi) $S \rightarrow Xa/Yb$

$X \rightarrow Sb/b$

$Y \rightarrow Sa/a$

7. Construct the equivalent PDA for the following CFG's

(i) $S \rightarrow Saa/aSa/aaS$

(ii) $S \rightarrow (S)\ (S)/a$

(iii) $S \rightarrow XaY/YbX$

$X \rightarrow YY/aY/b$

$Y \rightarrow b/bb$

8. Construct npda's that accepts the following languages :

(i) $L_1 = L(aaa^*b)$

(ii) $L_2 = L(aab^*aba^*)$

9. Find the npda for the language :

$$L = \{ab(ab)^n b(ba)^n : n \geq 0\}$$

10. Find the PDA for the following languages :

(i) $L = \{w : n_a(w) < n_b(w)\}$

(ii) $L = \{w : n_a(w) = 2n_b(w)\}$

(iii) $L = \{w : n_a(w) + n_b(w) = n_c(w)\}$.

11. Design a PDA which convert infix to prefix.

12. Design a PDA which convert prefix to postfix.

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CamScanner

# 9

CHAPTER

# Properties of Context-Free Languages

INSIDE THIS CHAPTER

9.1. Introduction 9.2. Periodicity Properties 9.3. The Pumping Lemma for CFL's 9.4. Decidiability 9.5. Finiteness 9.6. Membership—The CYK Algorithm.

## 9.1. INTRODUCTION

We have seen till now that there are two ways to recognise a context-free language. One way is to find a context-free grammar for that CFL and another is to generate a PDA for them. Sometimes one representation is more easier than the other. For example, we can easily generate a PDA for the language in which all strings contains equal number of $a$'s and $b$'s, but it is difficult to find a CFG for this language.

In this section, we study the general properties of CFL. So that we will be more familiar with them. These properties can help us to recognise whether a language is a CFL or not. These properties also help us to generate a CFG for complex CFLS.

**Theorem 9.1. The family of context free languages is closed under union, concatenation and Kleene star-closure.**

**Proof.** Let $L_1$ and $L_2$ be two context-free languages generated by the following context free grammars respectively.

$$G_1 = (V_{n_1}, V_{t_1}, P_1, S_1)$$

and

$$G_2 = (V_{n_2}, V_{t_2}, P_2, S_2)$$

**Union.** Consider the language $L(G)$, generated by the following grammar

$$G = (V_n, V_t, P, S)$$

where

$$V_n = V_{n_1} \cup V_{n_2} \cup \{S\}$$

$$V_t = V_{t_1} \cup V_{t_2}$$

$S$ is the start symbol, and productions $P$ is defined as follows

$$P = \{P_1 \cup P_2 \cup \{S \to S_1/S_2\}\}$$

Now let us choose a string $W \in (V_{t_1} \cup V_{t_2})^*$

It $S_1 \overset{*}{\Rightarrow} W$ or $S_2 \overset{*}{\Rightarrow} W$, and in our grammar

$S \to S_1/S_2$, hence $S$ will lead to $W$.

Hence $G$ is a context-free grammar, so that $L(G)$ is a context free languages that is

$$L(G) = L_1 \cup L_2 \text{ is a CFL.}$$

**Concatenation.** Now again consider a language $L(G)$ generated by the CFG

$$G = (V_n, V_t, P, S)$$

where

$$V_n = V_{n_1} \cup V_{n_2} \cup \{S\}$$

$$V_t = V_{t_1} \cup V_{t_2}$$

$S$ is the starting symbol of this grammar and production $P$ is defined as follows :

$$P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$$

Let $W_1$ is a string, $W_1 \in L_1$ and $W_2$ is a string, $W_2 \in L_2$ we know that $S_1 \overset{*}{\Rightarrow} W_1$ and $S_2 \overset{*}{\Rightarrow} W_2$ but in above grammar $G$, $S \Rightarrow S_1 S_1$ so $S$ will lead to $W_1 W_2$ in broader way the language of grammar $G$ will be $L_1 L_2$, since $L_1$ is CFL and $L_2$ is CFL and $G$ is context-free grammar so $L_1 L_2$ is also context-free language.

**Kleene-Star.** Let $L$ be context free language, generated by the following grammar

$$G = (V_n, V_t, P, S)$$

Now let Kleene star of language $L$ that is $L^*$ is generated by a grammar $G'$

$$G' = (V'_n, V'_t, P', S')$$

where

$$V'_n = V_n$$

$$V'_t = V_t$$

$S'$ is the starting non terminal.

and set of production is given by

$$P' = P \cup \{S' \to S, S' \to \epsilon, S' \to SS\}$$

When we carefully analyse $P'$ then it is very clear that if follow all the properties of CFG since $P$ is production of given CFG and $S' \to S$, $S' \to \epsilon$, $S' \to SS$ also fulfill the requirement so we can say that $G'$ is a context-free grammar, then $L^*$ will be context free.

**Theorem 9.2. The family of context free language is not closed under intersection and complementation.**

**Proof. Intersection :**

We know that languages

$$L_1 = \{a^m b^m c^n / n, m \geq 0\}$$

and

$$L_2 = \{a^n b^m c^m / n, m \geq 0\}$$

are context free.

Now let see $L_1 \cap L_2 = \{a^n b^n c^n / n \geq 0\}$ is not context-free (we will prove it in the next section by pumping lemma that $a^n b^n c^n / n \geq 0$ is not CFL). So it is proved that family of context-free languages is not closed under intersection.

**Complementation.** Let $L_1$ and $L_2$ be two context free languages, their union $L_1 \cup L_2$ also is context-free (by the theorem).

Now let assume that family of context-free language is closed under complementation, that is complement of a context-free language is also context free.

So according to our assumption $\bar{L}_1$ and $\bar{L}_2$ are context free and their union $\bar{L}_1 \cup \bar{L}_2$ is also CFL (by theorem). Again $\overline{\bar{L}_1 \cup \bar{L}_2}$ will be CFL (by the assumption).

We know that Demorgan's law is

$$L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$$

We have right now prove that $\overline{\overline{L_1} \cup \overline{L_2}}$ is context-free. So $L_1 \cap L_2$ should also be context free but it is not true, we have already seen it so our assumption that complement of context-free language is also context-free is wrong.

So we can say that, family of context-free languages is not closed under complementation.

**Theorem 9.3. The intersection of a context-free language and a regular language is a context-free language.**

**Proof.** Here we give a proof based on finite and push down automata; which would be simpler than one based on grammars. Let $L$ be context-free language, and $R$ is a regular set. So three exists a PDA $A_1 = (Q_1, \Sigma_1, \Gamma_1, \delta_1, S_1, F_1)$ and a finite automata $A_2 = (Q_2, \Sigma_2, \delta_2, S_2, F_2)$ for these languages respectively. Now we will combine these machines into a single PDA $P$ that carries out computations by $A_1$ and $A_2$ in parallel and accepts a string if both would have accepted. Let combined PDA be

$$A = (Q, \Sigma, \Gamma, \delta, S, F)$$

where $Q = Q_1 \times Q_2$

(Cartesian product of the states of $A_1$ and $A_2$)

$\Sigma = \Sigma_1 \cup \Sigma_2$

$\Gamma = \Gamma_1$

$F = F_1 \times F_2$ and $\delta$ is defined as follows.

$(((P_1, P_2), W, \alpha), ((q_1, q_2), \beta))) \in \delta$, it and only if, $((P_1, W, \alpha), (q_1, \beta)) \in \delta_1$ and $(P_2, W) \vdash_{A_2}^{*} (q_2, \in)$

It means that combined PDA '$P$' goes from state $(P_1, P_2)$ to $(q_1, q_2)$ in the same way that $A_1$ passes from state $p_1$ and $q_1$ and also keeps track that $A_2$ changes its state on the same input. It is easy to see that indeed $W \in L(A)$ if and only if $W \in L(A_1) \cap L(A_2)$.

## 9.2. PERIODICITY PROPERTIES

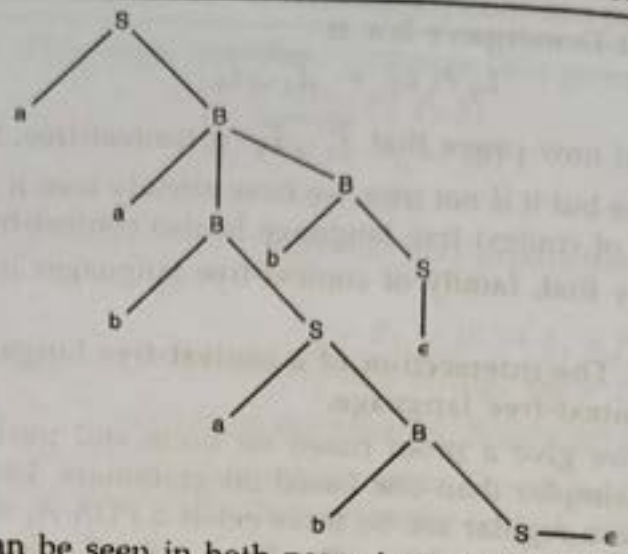
CFL's also exhibit periodicity that is paths (substring) may be repeated periodically is the derivation. It can be seen is parse trees also :

Let G is a grammar

$$S \rightarrow aB/bA/\in$$
$$A \rightarrow aS/bAA$$
$$B \rightarrow bS/aBB$$

**Derivations :**

$$\begin{aligned} S &\Rightarrow a\underline{B} \Rightarrow aa\underline{B}B \\ &\Rightarrow aabS\underline{B} \\ &\Rightarrow aabSb\underline{S} \\ &\Rightarrow aab\underline{S}b \\ &\Rightarrow aaba\underline{B}b \\ &\Rightarrow aababS\underline{b} \\ &\Rightarrow aababb \end{aligned}$$




So repeations can be seen in both parse tree as well as derivations.

## 9.3. THE PUMPING LEMMA FOR CFL'S

Let $G$ be any CFG, then there is a constant, depending only on $G$, such that if $W$ is in $L(G)$ and $|W| \geq n$ then we may write $W = \mu vxyz$ such that

(a) $|vy| \geq 1$ that is either $v$ or $y$ is non-empty

(b) $|vxy| \leq n$ then for all $i \geq 0$

$\mu v^i x y^i z$ is in $L(G)$.

**Proof.** Let our grammar be $G = (V_n, V_t, P, S)$. Now in order to prove the theorem it is sufficient to show that there is a constant $n$ such that any $W \in L(G)$ of length greater than or equal to $n$ has a derivation of the form.

$$S \overset{*}{\Rightarrow} \mu Xz$$

$$X \overset{*}{\Rightarrow} vXy$$

$$X \Rightarrow x$$

Now if we use above productions as :

$$S \overset{*}{\Rightarrow} \mu Xz \overset{*}{\Rightarrow} \mu vXyz \overset{i-1}{\Rightarrow} \mu v^i Xy^i z \Rightarrow \mu v^i Xy^i z$$

where $\mu, v, x, y, z \in \Sigma^*$ and $X, S \in V_n$ and $|vy| \geq 1$.

Then derivation $X \overset{*}{\Rightarrow} vXy$ can be repeated any number of times to obtain $\mu v^i x y^i z$.

So Now we get the line to prove the theorem, Now we will concentrated to prove above discussed terminology.

Let $P$ be the largest number of symbols on the right hand side of any Rule $X \to \alpha$ in $G$. Now we know it by the theorem that a parse tree of height $m$ can have at most $P^m$ leaves where $m \geq 1$, and can therefore have yield of length at most $P^m$.

Now let $m$ = number of non-terminals and $n = P^m$ and suppose $|W| \geq n$. Let $T$ be a parse tree with root $S$ and which derives the string $W$. Then $T$ has at least one path with more than $(m + 1)$ nodes, and therefore, at least one path including two nodes labeled with the same number of $X$ of $V_n$. Now let us see this concept picturally.

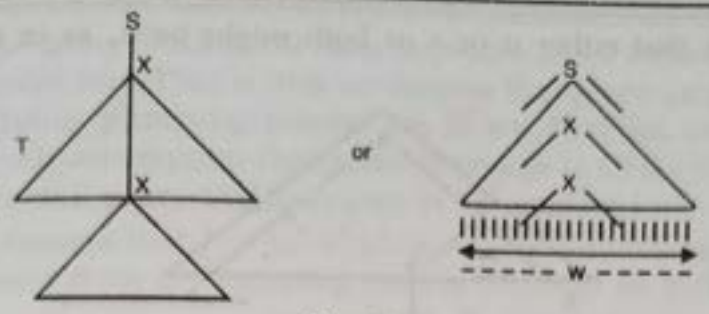



**Fig. 9.1.**

The triangle indicated encloses the whole part of the tree generated from the first $X$ down to where second $X$ is produced :

Let us divide $W$ in to these five parts :

$\mu$ = the substring of all the letters of $W$ generated to the left of the triangle above (this may be $\epsilon$).

$v$ = the substring of all the letters of $W$ descended from the first $X$ but to the left of the letters generated by the second $X$ (this may be $\epsilon$).

$x$ = the substring of $W$ descended from the lower $X$ (this may not be $\epsilon$ because this non-terminal must turn in to some terminal).

$y$ = the substring of $W$ of all letters generated by the first $X$ but the right of the letters descending from the second $X$ (this may be $\epsilon$, but as we shall see, not if $v = \epsilon$).

$z$ = the substring of all the letters of $w$ generated to the right of the triangle (this may be $\epsilon$).

Picturally,

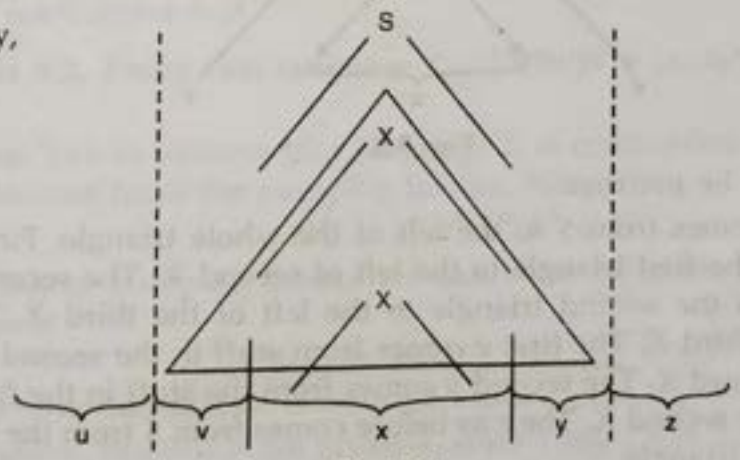



**Fig. 9.2**

For example, the following is a complete tree in an unspecified grammar :

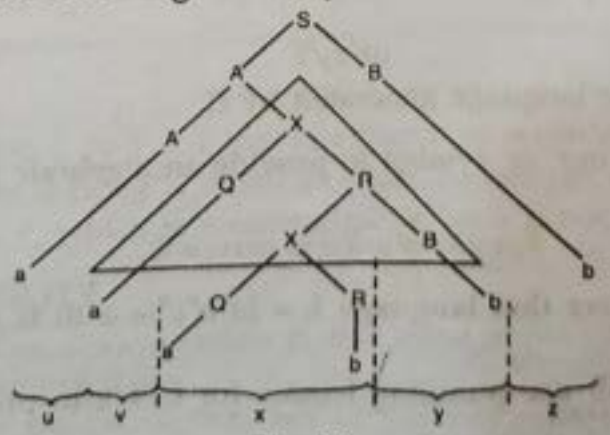
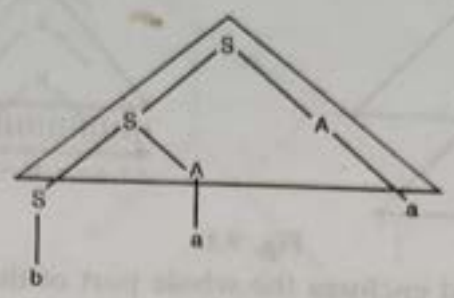



**Fig. 9.3.**

It is possible that either μ or z or both might be ∈, as in the following example




$\mu = \epsilon$ $\quad v = \epsilon$ $\quad x = ba$ $\quad y = a$ $\quad z = \epsilon$

**Fig. 9.4**

Now let us see the situation picturally it we change the derivation tree by reiterating productions inside the triangle :

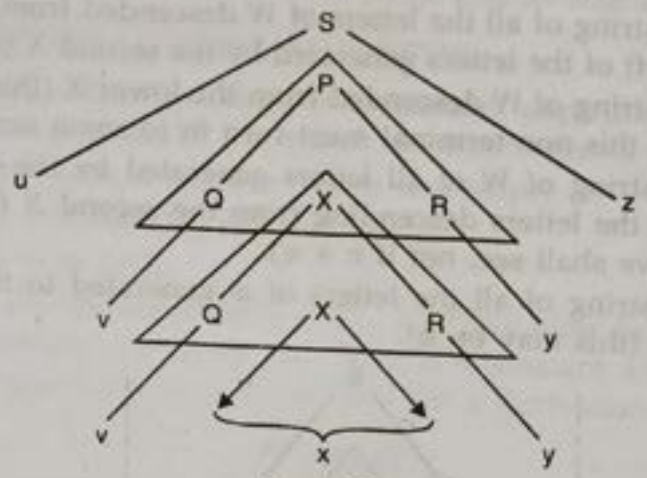



**Fig. 9.5.**

that is yield will be $\mu vvxyyz$

The part μ comes from $S$ to the left of the whole triangle. First $v$ comes from the inside the first triangle to the left of second $X$. The second $v$ comes from the stuff in the second triangle to the left of the third $X$. The $x$ part comes from the third $X$. The first $y$ comes from stuff in the second triangle to the right of the third $X$. The second $y$ comes from the stuff in the first triangle to the right of the second $X$. The $z$ as before comes from $S$ from the stuff to the right of the first triangle.

In general if we repeat the triangle $i$ times, we get a derivation for the word

$$\mu v^i x y^i z$$

which must be in the language generated by $G$.

We can also use our $\overset{*}{\Rightarrow}$ symbol to provide an algebraic proof.

$$S \overset{*}{\Rightarrow} \mu X z \overset{*}{\Rightarrow} \mu v X y z \overset{*}{\Rightarrow} \mu v x y z = W.$$

**Example 9.1. Prove that language $L = \{a^n b^n c^n / n \geq 0\}$ is not context-free language.**

**Solution.** We will use pumping lemma for CFL's to prove the $L$ is not CFL.

Here we are going to adopt the same approach as we follow in the pumping lemma for regular sets. That is first we assume that given language is regular then by applying pumping lemma on it we find an contradiction so automatically our assumption that given language is CFL will be wrong and in this way we will prove that language is not context-free.

So let us assume that $L = \{a^n\ b^n\ c^n / n \geq 0\}$ is context-free and $m$ be the constant obtained from the pumping lemma. Now let us pick the string

$$W = a^m\ b^m\ c^m,\ |\ W\ | > n$$

and according to pumping lemma let us break $W$ in substrings as

$$W = \mu vxyz,\ |\ vy\ | \geq 1,\ |\ vxy\ | \leq m$$

then lemma says that $\mu v^i x y^i z \in L$ for all $i \geq 0$.

**Observations :**

(*a*) But this is not possible; because if $v$ or $y$ contains two symbols from $\{a, b, c\}$ then $\mu v^2 x y^2 z$ contains a '$b$' before an '$a$' or a '$c$' before '$b$' then $\mu v^2\ x y^2 z$ will not be in $L$.

(*b*) In other case if $v$ and $y$ each contains only $a$'s, $b$'s or only $c$'s, then $\mu v^i\ x y^i z$ can not contain equal number of $a$'s, $b$'s and $c$'s. So $\mu v^i x y^i z$ will not be $L$.

(*c*) If both $v$ and $y$ contains all three symbols $a$, $b$ and $c$ then $\mu v^2 x y^2 z$ will not be in $L$ by the same logic as in observation *a*.

So it is a contradiction to our statement (we assume that $L$ is CFL).

So $L$ is not context-free.

**Example 9.2. Prove that language $L = \{WW/W \in \{a, b\}^*\}$ is not context-free.**

**Solution.** Let us assume that language $L$ is context-free and $m$ must be constant obtained from the pumping lemma. Now

$$W = a^m\ b^m\ a^m\ b^m \in L$$

and according to pumping lemma we can write $W$ in the following in the following form

$$W = \mu vxyz,\ |\ vy\ | \geq 1,\ |\ vxy\ | \leq m$$

then lemma says that $\mu v^i x y^i z \in L$ for all $i \geq 0$.

**Observation.** But this is not possible, since $|\ vxy\ | \leq m$ that is $vxy$ contains only solid $a$'s as follows

$$\underbrace{a_1, a_2, \ldots, a_m}_{m} \qquad \underbrace{b_1, b_2, \ldots, b_m}_{m} \qquad \underbrace{c_1, c_2, \ldots, c_m}_{m}$$

$$\longleftarrow (vxy) \longrightarrow$$

and we know that $|\ vy\ | \geq 1$ that is either $v$ or $y$ contains at least one $a$ so consider $i = 2$ that is $\mu v^2 x y^2 z$, it will increase the number of $a$'s but not $b$'s and $c$'s. So $\mu v^2 x y^2 z \notin L$, which is a contradiction, so $L$ is not context-free.

## 9.4. DECIDIABILITY

Some questions always comes in our mind about context-free grammar; let us discuss some of them.

(*a*) How can we decide whether or not two different CFG's define the same language ?

(*b*) Given a particular CFG, how can we decide whether or not it is ambigious?

(*c*) Given a particular CFG, which is ambiguous, can we tell whether or not there is a different CFG that generate the same language but is not ambiguous?

(*d*) How can we tell whether or not the complement of a given CFL is also context-free ?

(*e*) Given two CFG, how can we tell whether or not they have a word in common ?

(*f*) Given a CFG, how can we tell whether or not there are any words that it does not generate ?

There are however, some other fundamental questions about CFG that we can answer :

(*a*) Given a CFG, can G tell whether or not it generates any words at all? This is the question of emptiness.

(*b*) Given a CFG, we can tell whether or not the language it generate is finite or infinite ? This is the question of finiteness.

(*c*) Given a CFG and a particular string of letter *W*, can be tell whether or not *W* can be generated by the CFG ? This is the question of membership.

### 9.4.1. Emptiness

Prove that for a given CFG, there is an algorithm to determine whether or not it can generate any word or not, or it is empty or not.

**Proof.** We already know about nullable non-terminals, and how to decide whether the start symbol *S* is nullable :

$$S \overset{*}{\Rightarrow} \in ?$$

We also know that every CFG that does not generate $\in$ can be written without $\in$-production.

Now let us assume that $\in$ is not a word generated by CFG. In this case we can convert the CFG to CNF, preserving the entire language.

If there is a production of the form

$$S \rightarrow a$$

where *a* is terminal, then *a* is word in the language if there is no such production, we then propose the following algorithm :

**Step 1.** For each non-terminal *N* that has some productions of the form

$$V_n \rightarrow a$$

where *a* is terminal or string of terminals, we choose one of them and put *a* in all other productions having $V_n$ at right side, this eliminating non-terminal $V_n$ altogether.

We have changed the grammar so that it no longer accepts the same language. It may no longer be in CNF.

That is fine with us. Every word that can be generated from the new grammar could have been generated by the old CFG. If old CFG generated any words, then the new one does also.

**Step 2.** Repeat step-1 until it eliminates *S* or it eliminates no new non-terminals. If *S* has been eliminated, then the CFG produces some words, if not then not.

**Example 9.3. Consider this CFG :**

$$S \rightarrow XY$$
$$X \rightarrow AX$$

$$X \to AA$$
$$A \to a$$
$$Y \to BY$$
$$Y \to BB$$
$$B \to b$$

**Solution.**

**Step 1.** Replace all $A$'s by $a$ and $B$'s by $b$.

$$S \to XY$$
$$X \to aX$$
$$X \to aa$$
$$Y \to bY$$
$$Y \to bb$$

Now replace $X$'s by $aa$ and all $Y$'s by $bb$

$$S \to aabb$$

replace $S$'s by $aabb$.

**Step 2.** Terminate step 1 and discover that $S$ has been eliminated so CFG produces at least one word.

**Example 9.4. Consider the CFG :**

$$S \to XY$$
$$X \to AX$$
$$Y \to BY$$
$$A \to a$$
$$Y \to BB$$
$$B \to b$$

**Solution.**

**Step 1.** Replace all $A$'s by $a$ and all $B$'s by $b$, we get

$$S \to XY$$
$$X \to aX$$
$$Y \to bY$$
$$Y \to bb$$

Replace all $Y$'s by $bb$ this gives

$$S \to Xbb$$
$$X \to aX$$

**Step 2.** Terminate step 1 and discover that $X$ is still there. So CFG generates no such words.

### 9.4.2. Finiteness

There is an algorithm to decide whether a given CFG generates an infinite language or a finite language.

**Proof.** We assume that grammar contains no $\epsilon$-productions (if there is any useless or $\epsilon$-production in the grammar then we have to remove those productions) or no unit-production, and no useless symbols.

Now let as assume that grammar has a production like

$$X \overset{*}{\Rightarrow} xXy$$

Since grammar is assumed to have no $\epsilon$-production an no unit production, $x$ and $y$ cannot both be simultaneously empty. Since $X$ is neither nullable nor a useless symbol, we have :

$$S \overset{*}{\Rightarrow} uXv \overset{*}{\Rightarrow} w$$

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and $$X \overset{*}{\Rightarrow} z$$

where $u$, $v$ and $z$ are in $V_t^*$. But then

$$S \overset{*}{\Rightarrow} uXv \overset{*}{\Rightarrow} ux^nXy^nv \overset{*}{\Rightarrow} ux^nzy^nv$$

is possible for all $n$, so that $L(G)$ is infinite.

If no variable can ever repeat, then the length of any derivation is bounded by $|V_n|$. In that case, $L(G)$ is finite.

Thus, to get an algorithm for determining whether $L(G)$ is finite, we need only to determine whether the grammar has some repeating variables. This can be done simply by drawing a dependency graph for the variables in such a way that there is an edge $(X, Y)$ whenever there is a corresponding production

$$X \rightarrow xYy$$

Then any variable which is at the base of a cycle is a repeating one. Consequently, the grammar has a repeating variable if and only if the dependency graph has a cycle.

Since we have an algorithm for deciding whether a grammar has a repeating variable, we have an algorithm for determining whether or not $L(G)$ is infinite.

### 9.4.3. Membership—The CYK Algorithm

Given a CFG and a string $w$ in the some alphabet, we can decide whether or not $w$ can be generated by the CFG.

#### Proof by CYK Algorithm

Algorithm is named CYK because it was invented by John Cocke and subsequently also published by Tadao Kasami (1965) and Daniel H. Younger (1967).

First let us change the grammar from CFG to CNF form and let us make a list of all the non terminals in the grammar $S, X_1, X_2, X_3, \ldots$ and let the string we are examining for membership in the language be denoted by

$$w = a_1, a_2, a_3, a_4, \ldots, a_n$$

In general, it may be that the letters are not all different, but what we are interested in here in the position of every possible substring of $w$. We shall be answering the questions of which substring of $w$ are producible from which non-terminal $S$. For example, if we already know that the substring $a_3 \ldots a_7$ can be derived from the non-terminal $X_8$, the substring $a_8, \ldots a_{11}$ can be derived from the non-terminal $X_2$, and we have the CNF production

$$X_4 \rightarrow X_8X_2,$$

then we can conclude that the total substring $a_3 \ldots a_{11}$ can be derived from the non-terminal $X_4$, that is :

$$X_8 \overset{*}{\Rightarrow} a_3 \ldots a_7 \text{ and } X_2 \overset{*}{\Rightarrow} x_8 \ldots x_{11} \text{ and } X_4 \overset{*}{\Rightarrow} X_8X_2$$

We can colude that $X_4 \overset{*}{\Rightarrow} x_3 \ldots x_{11}$.

Let us see this whole procedure step by step :

**Step 1.** Change the given CFG into CNF form provided it is not already in CNF form.

**Step 2.** Start our list with all the substring of length 1 (the single letter) of $w$ and for such derivations come immediately from the CNF production non terminal $\rightarrow$ terminal.

| Substring | All producing non-terminals |
|---|---|
| $a_1$ | $X_{this}$, $X_{that}$ |
| $a_2$ | ⋮ |
| ⋮ | ⋮ |
| $a_n$ | $X_{something}$. |

**Step 3.** Now we look at all substrings of length 2, such that as $a_6a_7$. This can only be produced by a non-terminal $X_p$ if the first half can be produced by some non-terminal $X_q$ and the second half by some non-terminal $X_r$, and there is a rule of production in the grammar that says $X_p \rightarrow X_qX_r$. We can systimatically check all the rules of production of our list above to determine whether the length–2 substring can be produced :

| Substring | All producing non-terminals |
|---|---|
| $a_1a_2$ | X... |
| $a_2a_3$ | X... |
| ⋮ | ⋮ |
| ⋮ | ⋮ |
| $a_{n-1}a_n$ | X... |

It may be the case that some of these substrings cannot be derived from any non-terminal, but it is also may be case that some can be derived in serveral ways.

**Step 4.** We now move onto substrings of length three for example $a_5a_6a_7$

| Substrings | All producing nonterminals |
|---|---|
| $a_1a_2a_3$ | X... |
| $a_2a_3a_4$ | X... |
| ⋮ | ⋮ |
| $a_{n-2}a_{n-1}a_n$ | X... |

Our list keeps growing. Next, we examine all substrings of length 4. They can be broken into halves in three different : the first three letters and the last letter, the first two letters and the last two letters, the first letter and the last three letters. For all these possibilities, we check the list to see what non-terminals produce these halves and whether the two non-terminals can be merged into one by a rule of production.

$$X \rightarrow XX$$

| Substring | All producing Non-terminals |
|---|---|
| $a_1a_2a_3a_4$ | X |
| $a_2a_3a_4a_5$ | X |
| ... ... | ... |
| ... ... | ... |

We continue this same process with substring of length 5, length 6 and so on. The whole process terminates when we have all of $w$ as the length of the substring :

| Substring | All producing non-terminals |
|---|---|
| $a_1, a_2, \ldots, a_n$ | N... |

We now examine the set of producing non-terminals, and if $S$ is among them, then $w$ can be poduced, and if $S$ is not among them, then simply $w$ cannot be produced by this CFG.

This algorithm is finite and decisive.

**Example 9.4.** Let us consider the following CFG in CNF

$$S \rightarrow AA$$
$$A \rightarrow AA$$
$$A \rightarrow a$$

**Solution.** The list starts off easily enough :

| Substring | All producing non-terminals |
|---|---|
| $a_1 = a$ | $A$ |
| $a_2 = a$ | $A$ |
| $a_3 = a$ | $A$ |

We can see now that both substrings of length 2 are the same, aa, and are factorable into exactly $AA$. This is the right side of two productions whose left sides are $S$ and $A$ therefore the list contrues.

| Substring | All producing non-terminals |
|---|---|
| $a_1a_2$ | $S, A$ |
| $a_2a_3$ | $S, A$ |

There is only one length-3 substring, $x$ itself and it can be factored into $a_1$ times $a_2a_3$ or $a_1a_2$ times $a_3$. The first case gives the non-terminal possibilites $AS$ or $AA$, and the second gives the possibilities $SA$ or $AA$. Of these, only $AA$ is the right side of a production. The left sides are $S$ and $A$. Therefore, the list concludes with the following :

| Substring | All producing non-terminals |
|---|---|
| $w = a_1a_2a_3$ | $S, A$ |

From the list, we see that the word $w$ can be derived from the start symbol $S$ and so it is in the language. It should also be clear that similarly any string of more than three $a$'s can also be produced by this CFG from non-terminal $S$ and $A$.

## EXERCISE

1. Show that following languages on $\Sigma = \{a, b, c\}$ are not context-free.
   (*i*) $L = \{a^n b^j c^k : k = j^n\}$
   (*ii*) $L = \{a^i b^j : i \geq (j-1)^3\}$
   (*iii*) $L = \{a^i b^j c^k : k > i, k > j\}$
2. Show that following are not context-free languages :
   (*i*) The set of all strings over $\{a, b, c\}$ in which the number of occurrences of $a, b, c$ is the same
   (*ii*) $L = \{a^i b^j c^j / i \leq j \leq 2i\}$
3. For each of the following grammars, decide whether the language they generate is finite or infinite.
   (*i*) $S \rightarrow XS/b$
   $X \rightarrow YZ$
   (*ii*) $S \rightarrow XY$
   $X \rightarrow AA/YY/b$

$Z \to XY$
$Y \to ab$

$A \to BC$
$B \to AC$
$C \to BA$
$Y \to a$

(iii) $S \to XS/b$
$X \to YZ$
$Z \to XY$
$X \to ab$

(iv) $S \to XY/bb$
$X \to YY$
$Y \to XY/SS$

(v) $S \to XY/bb$
$X \to YY$
$Y \to XY/SS$

(vi) $S \to SS/b$
$X \to SS/SX/a$

(vii) $S \to XX$
$X \to SS/a$

4. For the following grammars and target strings, decide whether or not the word is generated by the grammars using the CYK algorithm :

(i) $S \to SS$ $\quad w = abba$
$S \to a$
$S \to bb$

(ii) $S \to XS$ $\quad w = baab$
$X \to XX$
$X \to a$
$S \to b$

(iii) $S \to AB$
$A \to BB/a$ $\quad w = bbaab$
$B \to AB/b$

(iv) $S \to XY$ $\quad w = abbaa$
$X \to XY$
$Y \to SS$
$X \to a/bb$
$Y \to aa$

(v) $S \to AB/CD/a/b$ $\quad w = bababab$
$A \to a$
$B \to SA$
$C \to DS$
$D \to b$

5. For each of the following languages, determine whether the language is context-free and prove your answer :

(i) $L = \{a^i b^j / i \neq j, i \neq 2j\}$

(ii) $L = \{ww^R w / w \in \{a, b\}^*\}$

(iii) Prove that following language is CFL or not ?

$$L = \{a, b\}^+ - \{(a^n b^n)^n / n > 1\}.$$

*****

# 10

CHAPTER

# Turing Machine

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## 10.1. INTRODUCTION

We have seen that neither deterministic finite automata nor pushdown automata can be regarded as truly general models for computers, since they are not capable of recognising even such simple languages as $\{a^n b^n c^n / n >= 0\}$. Turing machine is the next model of computation under machine based approach and context-sensitive or phrase-structure grammar in the corresponding model under grammatical approach.

In the 1930's (before advent of digital computers) several mathematicians began to think about what it means to be able to compute a function. Alenzo-Church and Alan turing independently arrived at equivalent conclusion. As we might phrase their common definition now. **"A function is computable if it can be computed by a Turing Machine".**

A Turing machine is a very simple machine, but logically speaking, has all the power of any digital computer. The Turing Machine (TM) is an abstract model created by Alan Turing (1912-1954), who was mathematician by trade and the inspiration behind the computers as we know them today. The impact of the turing machine and the theories around them were revolutionary in his time. Turing machine were used to show significant results since the foundation of mathematics and are also highly relevant to computer science. Turing introduced Turing Machine to demonstrate the indecision of the halting problem.

## 10.2. TURING MACHINE AS PHYSICAL COMPUTING DEVICE

In order to help our understanding of the subject-matter of TMs, we can visualize a TM as a physical computing device that can be represented as a diagram as shown in Fig. 10.1 below :

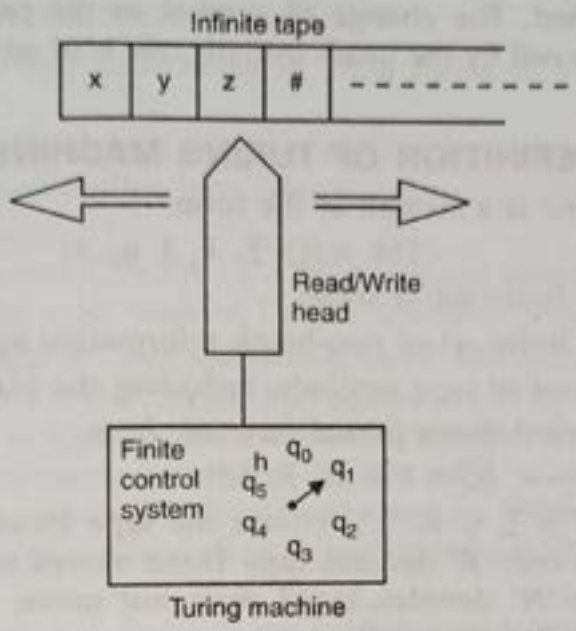



**Fig. 10.1.**

According to this view of TM, it consists of

(*i*) *A Tape*, with an end on the left but infinite on the right side. The tape is divided into squares or cells, with each cell capable of holding one of the tape symbols *including the blank symbol #*. At any time, there can be only finitely many cells of the tape that can contain non-blank symbols. The set of tape symbols is denoted by Γ.

As the very first step in the sequence of operations of a TM, *the input, as a finite sequence of input symbols is placed in the left-most cells of the tape.* ***The set of the input symbols*** **denoted by Σ, does not contain the blank symbol #. However, during operation of a TM, a cell may contain a *tape symbol* which is not necessarily an input symbol.**

(*ii*) *A Finite Control*, which can be in any one of the finite number of states. The states in TM can be divided in three categories that is :

(*a*) The initial state, the state of the control just at the time when TM starts its operations. The initial state is denoted by $q_0$.

(*b*) The *Halt state*, which is the state in which TM stops all further operations. The halt sate is *generally denoted by 'h'*. The halt state is distinct from the initial state. Thus a *TM has atleast two states.*

(*c*) Other intermediate states.

(*iii*) A tape head is always stationed at one of the tape cells and provides communication for interaction between the tape and the finite control. The head can read or scan the symbol in the cell under it. The symbol is communicated to the finite control. The control taking into consideration the symbol and its current state decides for further course of action including :

(*a*) The change of the symbol in the cell being scanned and/or

(*b*) Change of its state and/or

(*c*) Moving the head to the left or to the right. The control may decide not to move the head.

(*iv*) The course of action is called a move of the Turing Machine. In other words, the move is a function of current state of the control and the tape

symbol being scanned. The change of symbol in the cell being scanned is called writing of the cell by the head. Initially, *the head scans the right-most cell of the tape.*

## 10.3. FORMAL DEFINITION OF TURING MACHINE

A Turing machine is a sixtape of the form

$$TM = (Q, \Sigma, \Gamma, \delta, q_0, h)$$

where (*i*) $Q$ is the finite set of states.

(*ii*) $\Sigma$ is the finite set of non-blank information symbols,

(*iii*) $\Gamma$ is the set of tape symbols, including the blank symbol #.

(*iv*) $\delta$ is the next-move partial function from

$(Q \times \Gamma) \longrightarrow (Q \times \Gamma \times \{L, R, N\})$

where $\Gamma = \Sigma \cup \#$, '$L$' denotes the tape Head moves to the left adjacent cell, '$R$' denotes tape Head moves to the right adjacent cell and '$N$' denotes Head does not move, that is continuously scanning the same cell.

(*v*) $q_0 \in Q$, is the initial state.

(*vi*) $h \in Q$ is the 'Halt State', in which the machine stops any further activity.

Let us discuss some move of the turing machine;

(*i*) if $q \in Q, c \in \Sigma$ and $\delta(q, c) = (q_1, b, R)$, then Turing machine (Tm) when in state $q$ and currently scanning symbol $c$, will enter in state $q_1$ and move towards right adjacent cell after replacing '$c$' with '$b$'.

(*ii*) $\delta(q_1, a) = (q_2, b, L)$ means that at present Tm is in state $q_1$ and scanning the symbol $a$, will enter in state $q_2$ and write '$b$' in place of '$a$' and move towards left adjacent cell.

(*iii*) $\delta(q_1, x) = (q_2, y, N)$ means that at present Tm is in state $q_1$, scanning the symbol $x$, will enter state $q_2$ and write '$y$' in the place of '$x$' and remain stick with current position that it does't move towards left or right.

Let us see some example for the better understanding.

**Example 10.1.** Consider the Turing Machine $(Q, \Sigma, \Gamma, \delta, q_0, h)$ defined below that erases all non-blank symbols on the tape, where the sequence of non-blank symbols does not contain any blank symbol # in-between :

$$Q = \{q_0, h\}$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, \#\}$$

and the next move function is defined as follows :

| $q$ : *state* | $\sigma$ : *input symbol* | $\delta(q, \sigma)$ |
|---|---|---|
| $q_0$ | $a$ | $\{q_0, \#, L\}$ |
| $q_0$ | $b$ | $\{q_0, \#, L\}$ |
| $q_0$ | # | $\{h, \#, N\}$ |
| $h$ | # | ACCEPT |

Here it is assumed that tape head is scanning right most non-blank symbol first, for example *#ababbaq#*.

## 10.4. INSTANTANEOUS DESCRIPTION

There are following differences in Tape Head of FA/PDA and Turing machine :

(*i*) The cells of the tape of an FA or a PDA are only read/scanned but are *never* changed/*written* into, whereas the cells of the tape of TM may be written also.

(*ii*) The tape head of an FA or a PDA always moves from left to right. However, the tape head of TM can move in both the directions.

As a consequence of facts mentioned in (*i*) and (*ii*) above, we conclude that in the case of FA and PDA the *information in the tape cells already scanned do not play any role in deciding future* moves of the automation, but in the case of a TM, the information contents of all the cells, *including the ones earlier scanned* also play a role in deciding future moves. This leads to the slightly different definitions of configuration or Instantaneous Description (ID) in the case of a TM.

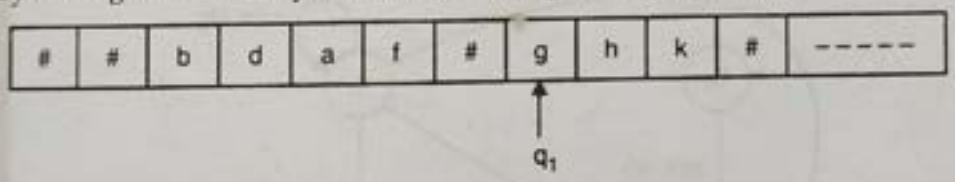
The ID of a Turing machine is the information in respect of :

(*i*) Contents of all the cells of the tape, starting from the right most cell upto atleast the last cell, containing a non-blank symbol and containing all cells upto the cell being scanned.

(*ii*) The cell currently being scanned by the machine and

(*iii*) The state of the machine.

For example let us assume that Turing machine TM is at state $q_1$ scanning the symbol $g$ with the symbols on the tape as follows :




or

We can show the situation as follows :

$$(q_1, \#\#bdaf\#\underline{g}hk\#)$$

## 10.5. TRANSITION DIAGRAM

In some situations, graphical representation of the next-move partial function $\delta$ of Turing machine may give better idea of the behaviour of TM in comparison to the tabular representation of $\delta$.

A Transition Diagram of the next-move functions $\delta$ of a TM is a graphical representation consisting of a finite number of nodes and directed-labelled arcs between the nodes. Each node represents a state of the TM and a label on an arc from one state (say $q_1$) to a state ($q_2$) represents the information about the required input symbol say '$a$' for the transition from $q_1$ to $q_2$ to take place and the action on the part of the control of TM. The action part consists of

(*i*) The symbol say '$b$' to be written in the current cell and

(*ii*) The movement of the tape head.

The label of an arc is generally written as $a/(b, m)$, where $m$ is $L$, $R$ or $N$.

During the designing procedure of turning machine Tm, any string/input is assumed to be written on input taple in following form :

$$\#abbabb\underline{\#}$$

that is Head is on extreme right-blank in the starting and when machine starts processing the string at $q_0$ (starting state) then it is supposed to read first input symbol of the string from right to left; as follows :

$$\#abbab\underline{b}\#$$
$$\uparrow$$
$$q_0 \text{ (initial state of the machine)}$$

**Example 10.2. Let TM = $\{Q, \Sigma, \Gamma, \delta, q_0, h\}$**

**Solution.**

where

$$Q = \{q_0, q_1, q_2, h\}$$
$$\Sigma = \{0, 1\}$$
$$\Gamma = \{0, 1, \#\}$$

and $\delta$ be given by the following table

| | 0 | 1 | # |
|---|---|---|---|
| $q_0$ | — | — | $(q_2, \#, R)$ |
| $q_1$ | $(q_2, 0, R)$ | $(q_1, \#, R)$ | $(h, \#, N)$ |
| $q_2$ | $(q_2, 0, L)$ | $(q_1, 1, R)$ | $(h, \#, N)$ |
| $h$ | — | — | — |

The transition diagram for above TM will be as shown below, where we assume that '$q_0$' is the initial state and '$h$' is a final state/half state.

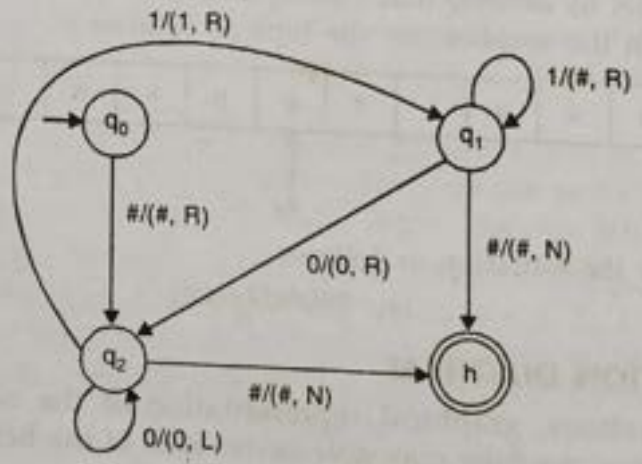



**Fig. 10.2.**

**Example 10.3. Design a TM that recognizes the language of all strings of even length over alphabet $\{a, b\}$.**

**Solution.** Let Turing machine is

$$Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, \#\}$$
$$Q = \{q_0, q_1, h\}$$

Here $h$ is half state when machine halts after accepting the language $q_0$ is initial state. Here next move partial function $\delta$ is given as follows :

| | $a$ | $b$ | # |
|---|---|---|---|
| $q_0$ | $(q_1, a, L)$ | $(q_1, b, L)$ | $(h, \#, N)$ |
| $q_1$ | $(q_0, a, L)$ | $(q_0, b, L)$ | * |
| $h$ | * | * | Accept |

Here '*' is for undefined move. We can construct the transition diagram as follows :

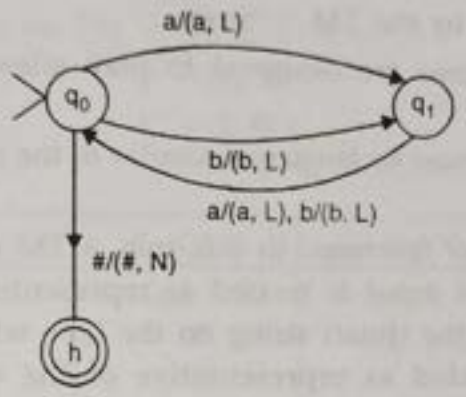



**Fig. 10.3.**

**Example 10.4. Design a Turing Machine that accepts the language of all strings which contain *aba* as a substring.**

**Solution.** Let Turing Machine is TM = $(Q, \Sigma, \Gamma, \delta, q_0, h)$

$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, \#\}$$
$$Q = \{q_0, q_1, q_2, h\}$$
$$q_0 \text{ is initial state}$$

and $\delta$ is defined as follows :

| | $a$ | $b$ | # |
|---|---|---|---|
| $q_0$ | $(q_1, a, L)$ | $(q_0, b, L)$ | * |
| $q_1$ | $(q_1, a, L)$ | $(q_2, b, L)$ | * |
| $q_2$ | $(h, a, L)$ | $(q_0, b, L)$ | $(h, \#, N)$ |
| $h$ | * | * | Accept |

Now we can also design transition diagram as follows :

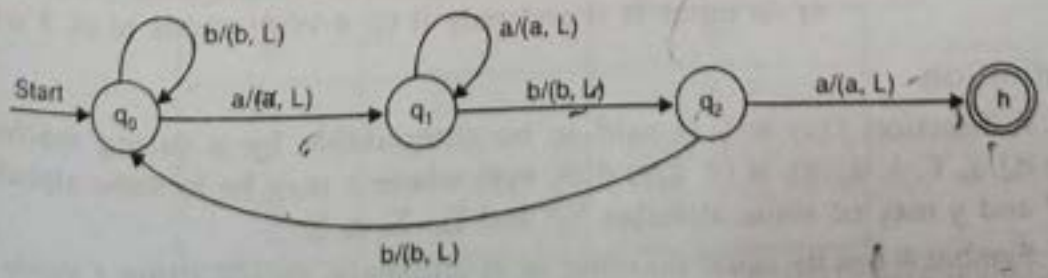



**Fig. 10.4.**

## 10.6. TURING THESIS

"The power of any computational process is captured within the class of Turing Machines."

It may be noted that Turing thesis is just a conjecture and not a theorem, hence, Turing thesis cannot be logically deduced from more elementary facts. However, the conjecture can be shown to be false, if a more powerful computational model is proposed that can recognize all the languages which are recognized by the TM model and also recognizes at least one more language that is not recognized by the TM.

The Turing Machines are designed to play atleast the following three different roles :

(*i*) *As accepting devices for languages,* similar to the role played by FAs and PDAs.

(*ii*) *As a computer of functions.* In this role, a TM represents a particular function. Initial input is treated as representing an *argument* of the function. And the (final) string on the tape when the TM enters the Halt state treated as representative of the value obtained by an application of the function to the argument represented by the initial string.

(*iii*) *As an enumerator of strings of a language,* that outputs the strings of a language, one at a time, in some systematic order that is as a list.

## 10.7. TURING MACHINE FOR COMPUTING FUNCTIONS

Turing machine can be used to compute functions. How is it possible ? First we see how to input a string to a turing machine. We adopt the following policy to input any string to a turing machine.

The string $W$ is presented in to the form of # $W$ #, that is string, $W$ is surrounded by blanks symbols from both sides and is placed on the left most square of tape, the head of turing machine is positioned at the right most blank symbol which immediately follows the string $W$.

This is shown by an underscore that is we use an underscore to show the current position of machine head in the tape. A turing machine is said to halt on input '$W$' if we can reach to a halting state '$h$', after performing some operations, that is if,

$T_m = (Q, \Sigma, \Gamma, \delta, q_0, h)$ is a turing machine then Tm is said to halt to on input $W$ if and only if $(q, \# W \underline{\#})$ yields to $(h, \# \mu \underline{\#})$.

### Definition

A function $f(x) = y$ is said to be computable by a turing machine Tm $(Q, \Sigma, \Gamma, \delta, q_0, S)$, if $(S, \#x\underline{\#}) \overset{*}{\vdash} (h, \#y\underline{\#})$ where $x$ may be in some alphabet $\Sigma_1^*$ and $y$ may be some alphabet $\Sigma_2^*$ and $\Sigma_1, \Sigma_2 \subseteq \Sigma$.

Symbol $\overset{*}{\vdash}$ has its usual meaning as in automata, means string $x$ yields to $y$ after a finite number of steps.

It means that if we give input $x$ to the turing machine Tm, turing machine gives output as a string if it computes the function $f(x) = y$.

**Example 10.5. Design a turing machine which compute the function $f(m) = m + 1$ for each $m$ that belongs to the set of natural numbers.**

**Solution.** Given function is $f(m) = m + 1$. Here we represents input $m$ on the tape by a number of $I$'s on the tape.

For example $m = 1$; input will be $\#I\#$,

For $n = 2$;

input will be $\#II\#$ and so on.

Similarly output can be seen by the number of $I$'s on the tape.

Let Turing machine be Tm $= (Q, \Sigma, \Gamma, \delta, q_0, h)$

where $Q = \{q_0, h\}$

$\Gamma = \{I, \#\}$

$q_0$ is initial state.

| | $\sigma = I$ | $\sigma = \#$ |
|---|---|---|
| $q_0$ | $(q_1, I, R)$ | $(q_1, \#, R)$ |
| $q_1$ | * | $(h, I, R)$ |
| $h$ | * | * |

Here $\sigma$ is input at any time.

Let input be $n = 3$ that is tape situation is $\#III\#$

Now let us process the given string.

(**Note :** We are processing string from right to left).

$$\#III\ q_0 \vdash \#III\#q_1$$
$$\vdash \#IIII\#h$$

**Example 10.6. Prove that following function is computable**

$$f(n) = n + 2$$

**Solution.** We know that if any function is computable then there exist a turing machine for it. "so it will be sufficient to construct a turing machine to prove any function computable".

Let Turing machine for given function be

$$\text{Tm} = (Q, \Sigma, \Gamma, \delta, q_0, h)$$
$$Q = \{q_0, q_1, q_2, h\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state and halting state is represented by '$h$'.

Transition relations are defined as follows :

| $\delta$ | $I$ | # |
|---|---|---|
| $q_0$ | $(q_1, I, R)$ | $(q_1, \#, R)$ |
| $q_1$ | * | $(q_2, I, R)$ |
| $q_2$ | * | $(h, I, R)$ |
| $h$ | * | * |

Initially turing machine is in $q_0$ state, when it reads $I$ or # then it changes it's state to $q_1$ and moves towards right (right symbol is only blank). At $q_1$ it encountered with # and replace it by $I$, change its state to $q_2$ and moves towards right. At $q_2$, # is replaced by $I$ and machine halts after moving right.

CamScanner

Let $n = 3$ that is #III#

then Tm, behaves as follows $L$

$$\#III q_0 \vdash \#III\# \, q_1$$
$$\vdash \#IIII\# q_2$$
$$\vdash \#IIIII\# \, h$$

**Example 10.7. Design a turing machine that replace every 0 and 1 with every 1 with 0 in a binary string.**

**Or**

**Prove that function $f(w) = \overline{w}$ is computable where $w \in \{0, 1\}^*$ and $\overline{w}$ the one's complement of $w$.**

**Solution.** We want to design a turing machine that takes input as #101# then its output would be #010#. This machines can be designed by the following idea; scan backwards through the input, replace every 0 with 1 and every 1 with 0, until the blank marking the left most sequence is found. In order to leave the tape in its proper format, the machine must then return its head to the blank marking the right end of the input.

Let Turing Machine is $\quad Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$

$$Q = \{q_0, q_1, h\}$$
$$\Sigma = \{0, 1\}$$
$$\Gamma = \{0, 1, \#\}$$

$q_0$ is initial state.

$\delta$ can be defined by following transition diagram.

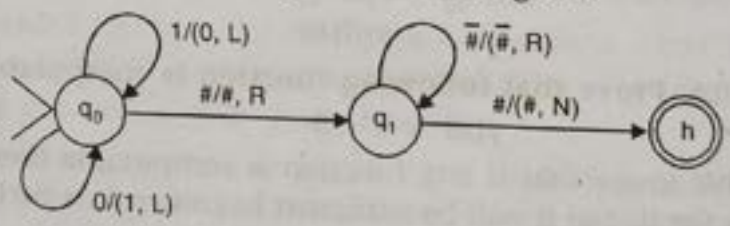



**Fig. 10.5.**

Here $\overline{\#}$ represent non-blank symbol (in this case either 0 or 1).

Let input string be $w$ = #101100#

$$\#101100 \, q_0 \vdash \#10110 q_0 1$$
$$\vdash \#1011 q_0 11$$
$$\vdash \#101 q_0 11$$
$$\vdash \#10 q_0 0011$$
$$\vdash \#1 q_0 10011$$
$$\vdash \# q_0 010011$$
$$\vdash \# q_1 010011$$
$$\vdash \#0 q_1 10011$$
$$\vdash \#01 q_1 0011$$
$$\vdash \#010 q_1 011 \vdash \#0100 q_1 11$$
$$\vdash \#01001 q_1 1 \vdash \#010011 q_1 \# \vdash \#010011 \# h$$

## 10.8. TURING MACHINE AS LANGUAGE ACCEPTORS

A turing machine works as language acceptor for a language $L$ if it is able to tell us whether a string '$w$' belongs to the language $L$ or not. In this way it gives the exact information that an automata can given. But remember, an automata is only able to accept regular languages and a turing machine can be used for some difficult structures also. We will use symbol '$y$' and '$n$' to show that a string is accepted by Turing machine or not. If Tm accepts a string $w$ then it will leave symbol $y$ on the tape and halts, that is

$$(S, \# w \underline{\#}) \vdash^{*} (h, \# y \underline{\#})$$

and it will leave symbol $\eta$ if $w$ is not accepted, that is

$$(S, \# w \underline{\#}) \vdash^{*} (h, \# n \underline{\#})$$

Now at this level we can compare, the language accepted by the regular expression, context free grammar and type-0 grammar ($I$ will introduced it letter); as follows in Fig. 10.6.

| *Language defined by* | *Corresponding Acceptor* | *Non-determinism = Determinson* | *Language closed under* | *What can be decided* | *Example of application* |
|---|---|---|---|---|---|
| Regular Expression | Finite Automata | Yes | Under union, Product, kleene star, intersection complement | Equivalence, emptiness, finiteness, membership | Text editors, sequential circuit |
| Context-free grammar | Pushdown automata | No | Union, product, kleene star | Emptiness finiteness membership | Programming language statements compilers |
| Type-0 grammar | Turing machine, post machine DPDA, $n$PDA | Yes | Union, product, Intersection, Kleene star. | Not much | Computers |

**Fig. 10.6.**

**Example 10.8. Design a turing machine that recognises the set of all strings of 0's and 1's containing at least one 1.**

**Solution.** Designing such a turing machine is very simple. For the designing this turing machine we will move towards left until 1 is found, and then halt.

Let Turing machine be $\text{Tm} = (Q, \Sigma, \Gamma, \delta, q_0, h)$

where

$$Q = \{q_0, q_1, h\}$$
$$\Gamma = \{0, 1, \#\}$$
$$\Sigma = \{0, 1\}$$

$q_0$ is initial state

and transition relation $\delta$ can be represented by following transition diagram :

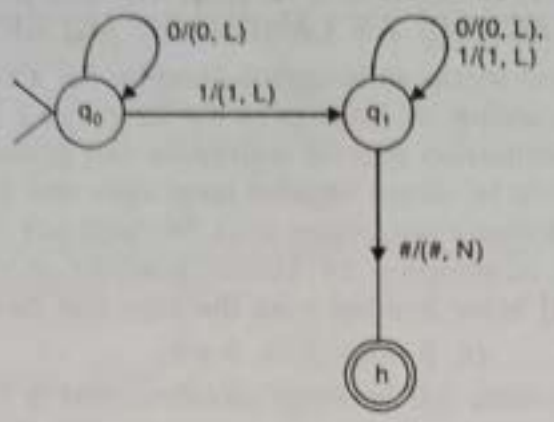



Fig. 10.7.

Here halt state $h$ is reached only when at least one 1 is encountered and $h$ works as accepted state.

Let us process the string : #00011#

$$\begin{aligned} \#00011q_0 &\vdash \#001q_1 1 \\ &\vdash \#00q_1 11 \\ &\vdash \#0q_1 011 \\ &\vdash \#0q_1 0011 \\ &\vdash h\#000011\# \end{aligned}$$

## 10.9. COMBINING TURING MACHINE

Till now we have seen only simple examples of turing machine that do not impress us, because we have said that turing machine are ultimate computers. that is turing machines are capable of carrying out complex calculations, in spite of their limited mode of operation.

In this section we present a method to build complex turing machines by combining simple ones. In this respect a Simple Turing Machine can work as "function" for other complex turing machines. Now we see how to combine turing machine. Suppose $Tm_1$ is a turing machine that is not known to hang. Then $Tm_1$ can be made part of a large turing machine Tm as follows : Tm prepares some string as input to $Tm_1$, placing it near the right end of non blank portion of the tape, passes control to $Tm_1$ with read/write head just beyond the end of that string and finally retrieves control from $Tm_1$ when $Tm_1$ has finished its computing. It is guaranteed that $Tm_1$ will never interfere with the operational computation of Tm. This is like a function, when we call a function (say $Tm_1$) in the main function of a simple 'C' program the control simply returns to main program after function $Tm_1$, performs its calculations.

### 10.9.1. Rules for Combining Turing Machines

Now we adopt a graphical representation to show the combining of turing machines. In fact this representation is the same as that of a finite automata. Individual machines are like the states of the automata and these machines are connected in the same way as the states of automata. The connection from one machine to another will not be pursued until the first machine halts; the other machine is then started from its initial state with the tape and head position as they were left by the first machine. This can be shown by the following figure :

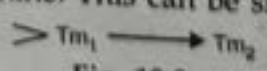



Fig. 10.8.

Here $Tm_1$ machine starts computing (that is initial state in finite automata) and when $Tm_1$ halts, control, now goes to machine $Tm_2$.

Now consider the following machine

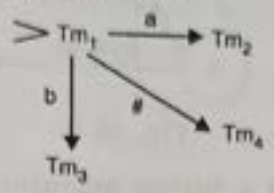



**Fig. 10.9.**

This machine starts computing with machine $Tm_1$. Now when $Tm_1$ halts, the symbol on the tape under the current head position can be *a*, *b* or #. Then the figure 10.9 shows when $Tm_1$ halts, the control goes to machine $Tm_2$, $Tm_3$ or $Tm_4$ respectively, according to the symbol under current head position which is *a* or *b* or #.

Here we are going to discuss certain symbols to represent simple machines. These symbols will be used further.

(*i*) Fig. 10.9 is the basic machine which moves towards left by one cell during scanning any input (σ) symbol. State change is also possible.

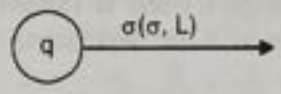



**Fig. 10.10.**

(*ii*) Fig. 10.10 is the basic machine moves towards right by one cell, when current input symbol is blank or non-blank (σ).

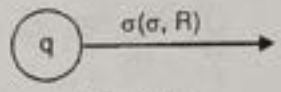



**Fig. 10.11.**

(*iii*) Fig. 10.11 is the basic machine, which moves towards left till the first blank symbol is encountered.

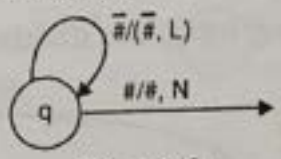



**Fig. 10.12.**

(*iv*) Fig. 10.13 is the basic machine, which moves towards right until the first blank symbol is encountered.

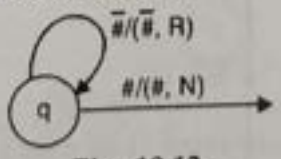



**Fig. 10.13.**

(*v*) Fig. 10.13 is the basic machine, which moves towards left until the first non-blank symbol is encountered.

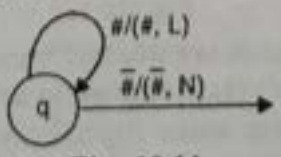



**Fig. 10.14.**

(*vi*) Fig. 10.14 is the basic machine, which moves towards right until the first non-blank symbol is encountered.

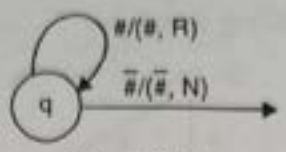



**Fig. 10.15.**

**Example 10.9. Construct a turing machine that computes the following function.**

$$f(n, m) = n + m$$

**Solution.** Let us decide the policy to show that input numbers : *I* represents 1, #*II*# represents 2 and so on. We assume that input numbers (that is *n* and *m*) are written on the tape separated by #. That is if we want to add 2 and 3, the input string will be #*II*#*III*#.

Now we want to such a turing machine which gives output as follows $f(2, 3) = 2 + 3 = 5$.

$$\#II\#III\# \vdash \#IIIII\#$$

When we analyse the output then it becomes clear that for constructing turing machine for $f(n, m) = n + m$, we have to remove #, which separates *n* and *m*.

Infact resulting turing machine works as a **left-shift machine** (say SL). It shift the each symbol of m portion of input by one cell towards left.

Let turing machine is $Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$

where

$$Q = \{q_0, q_1, q_2, q_3, q_4, h\}$$
$$\Sigma = \{I\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state

$h$ is halt state

and $\delta$ is defined by following transition diagram.

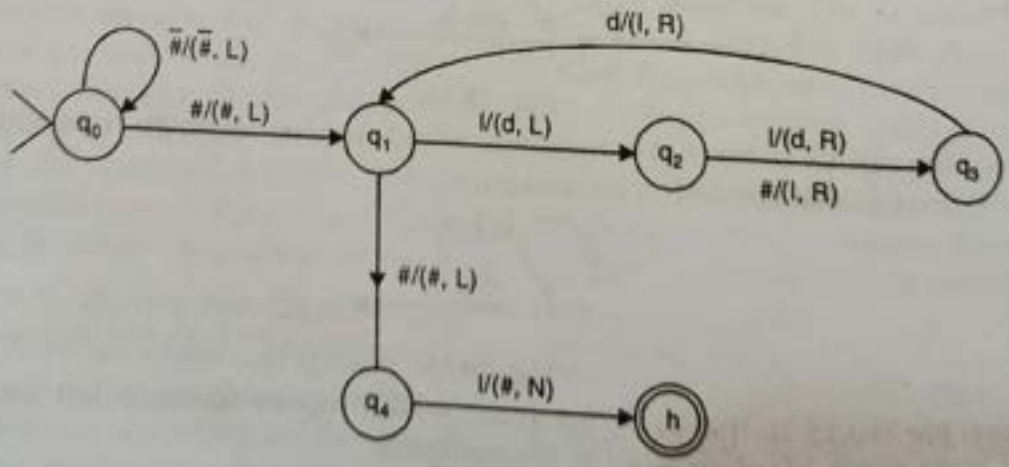



**Fig. 10.16.**

In the above turing machine we are not showing all the moves from every state and those move for certain input symbols are considered as, path (transition) is not available for some inputs and machine rejects the input or not processed if it is not halting at state '*h*'.

Let us process a string #II#II#

$$\#II\#IIq_0 \vdash \#II\#Iq_0I$$
$$\vdash \#II\#q_0II$$
$$\vdash \#II\#q_1II$$
$$\vdash \#II\#Iq_1I$$
$$\vdash \#II\#q_2dI$$
$$\vdash \#IIIdq_3I$$
$$\vdash \#IIIIIq_1$$
$$\vdash \#IIIIq_2d$$
$$\vdash \#IIIIdq_3$$
$$\vdash \#IIIII\#q_1$$
$$\vdash \#IIIIIq_4\#$$
$$\vdash \#IIII\#h$$

**Example 10.10. Design a Turing Machine which works as eraser.**

**Solution.** Let us first analysed the problem, here we want to design a turing machine which works as eraser, that is if we pass input #abb# then output should be ##, that is null string.

Let the Turing machine be Tm = $(Q, \Sigma, \Gamma, \delta, q_0, h)$

where $Q = \{q_0, q_1, h\}$

$\Sigma = \bar{\#}$ symbols

$\Gamma = \bar{\#} \cup \#$

$q_0$ is initial state and

Transition relation $\delta$ is defined as follows :

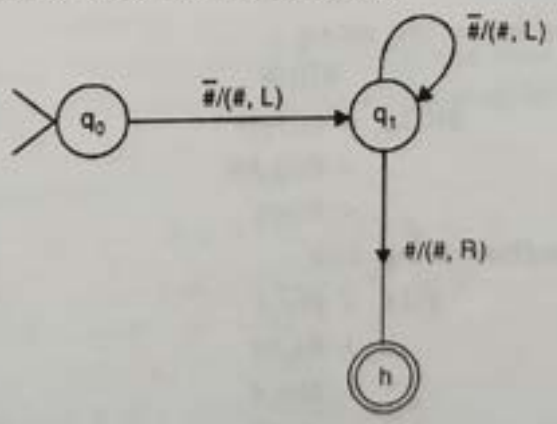



**Fig. 10.17.**

**Here it is preassumed that there is at least one non-blank symbol in the string.**

Now let us process the following string :

$$\#ababaq_0 \vdash \#ababq_1\#$$
$$\vdash \#abaq_1\#\#$$
$$\vdash \#abq_1\#\#\#$$
$$\vdash \#aq_1\#$$
$$\vdash \#q_1\# \vdash \#\#h$$

**Example 10.11. Prove that following function is turing computable.**

$$f(m) = \begin{cases} m-2, \text{ if } m > 2 \\ 1 \quad , \text{if } m \le 2 \end{cases}$$

**Solution.** We know that if any function is turing computable then there exist a turing machine for it, so we have to design a turing machine of the $f(m)$ function. For the input strings $\#I\#$ $(m = 1)$, $\#\#(m = 0)$ and $\#II\#$ $(m = 2)$ output will be $\#I\#$ $(m = 1)$. For the input string $\#IIII\#$ $(m = 4)$, output will be $\#II\#$ $(m = 4 - 2 = 2$, since $m > 2)$.

Let Turing machine be $\quad Tm = (Q, \Sigma, \delta, \Gamma, q_0, h)$

where

$$Q = \{q_0, q_1, q_2, q_3, h\}$$
$$\Sigma = \{I\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state and $h$ is halting state.

Transition relation $\delta$ is defined as follows :

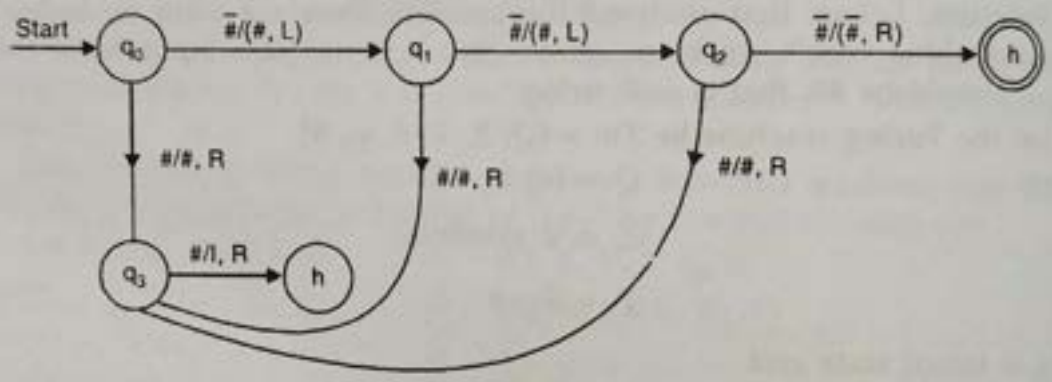



**Fig. 10.18.**

Let us process the following string :

$$\#IIII\#$$
$$\#IIIIq_0 \vdash \#IIIq_1\#$$
$$\vdash \#IIq_2\#\#$$
$$\vdash \#II\#h$$

Let us process another string $\#II\#$

$$\#IIq_0 \vdash \#Iq_1\#$$
$$\vdash \#q_2\#\#$$
$$\vdash \#\#q_3\#$$
$$\vdash \#I\#h$$

**Example 10.12. Prove that following function is turing computable.**

$$f(n) = \begin{cases} n-1, n > 0 \\ 0 \quad , n = 0 \end{cases}$$

**Solution.** If function $f(n)$ is turing computable then there exist a turing machine for it, so basically we have to design a turing machine for function $f(x)$. This turing machine should give output $\#\#$ for $\#\#$ $(n = 0)$ and $\#I\#$ for $\#III\#$ $(n = 3)$.

Let Turing machine be

$$Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$$

where

$$Q = \{q_0, h\}$$
$$\Sigma = \{I\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state and $h$ is halting state

Transition relation $\delta$ is defined by following transition diagram :

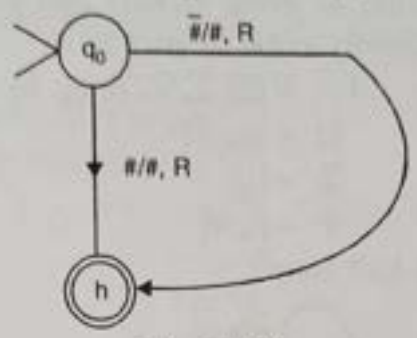



**Fig. 10.19.**

Let us process the following string :

$$w = \#IIII\#$$

$$\#IIIIq_0 \vdash \#III\#h$$

For string $\#q_0\# \vdash \#\#h.$

**Example 10.13. Design a Turing machine for the following function :**

$$f(x, y) = y.$$

**Solution.** Let us understand the meaning of $f(x, y) = y$, here we want to design a turing machine which takes input $x = \#II\#$ and $y = \#III\#$ as string $w = xy = \#II\#III\#$ on the input tape and output of the required turing machine is $\#III\#$ that is $y$ part of string.

Now let us discuss the logic for this turing machine, consider the string $\#II\#III\#$ move at extreme left and erase the $x$ part and leave the tape head at extreme right.

Let turing machine be :

$$Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$$

where

$$Q = \{q_0, q_1, q_2, q_3, h\}$$
$$\Sigma = \{I\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state and $h$ is halting state.

$\delta$ is defined by the following transition diagram :

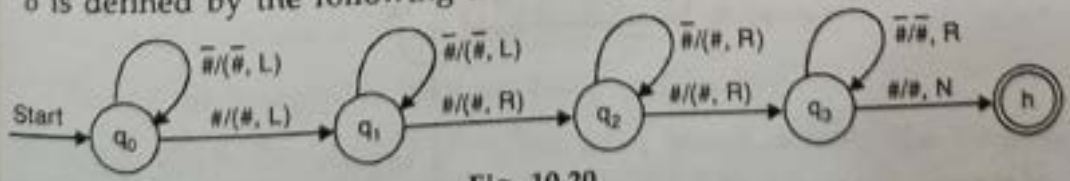



**Fig. 10.20.**

$$\#II\#Iq_0 \vdash \#II\#q_0I$$
$$\vdash \#IIq_1\#I$$
$$\vdash \#Iq_1I\#I$$

$$\vdash \#q_1 II\#I \vdash \#Iq_2 I\#I$$
$$\vdash \#\#Iq_2\#I \vdash \#\#\#\#q_2 I$$
$$\vdash \#\#\#\#Iq_3 \vdash \#\#\#\#I\#q_3 \vdash \#\#\#\#I\#h.$$

**Example 10.14. Design a Turing machine for the following function $f(x, y) = x$.**

**Solution.** We can solve this problem in similar fashion as we have solved example 10.13. Here we have to design a turing which takes input $w = xy$ (where $x = \#II\#$ and $y = \#III\#$) as string $\#II\#III\#$ and gives output $\#II\#$ that is only $x$ part of $w$ and completely erase of part of $w$.

Let Turing machine be $\quad Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$

$$Q = \{q_0, h\}$$
$$\Sigma = \{I\}$$
$$\Gamma = \{I, \#\}$$

$q_0$ is initial state and $h$ halt state

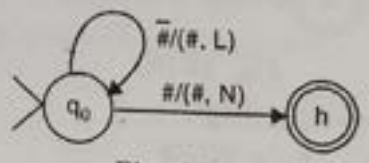



**Fig. 10.21.**

Let us process the string $\#II\#III\#$

$$\#II\#IIIq_0 \vdash \#II\#IIq_0\# \vdash \#II\#Iq_0\#\#$$
$$\vdash \#II\#q_0\#\#\# \vdash \#II\#h.$$

**Example 10.15. Design a Turing machine for the regular expression $r = aa^*$.**

**Solution.** We have to design a turing machine which accepts strings which are in regular expression $r$.

$$r = aa^*$$

It is collection of strings starts with '$a$' followed by any number of $a$'s and contain only $a$'s in the strings.

Let turing machine is $\quad Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$

$$Q = \{$$
$$\Sigma = \{a\}$$
$$\Gamma = \{a, \#\}$$

$q_0$ is initial state and $h$ is halt state

$\delta$ is defined as follows :

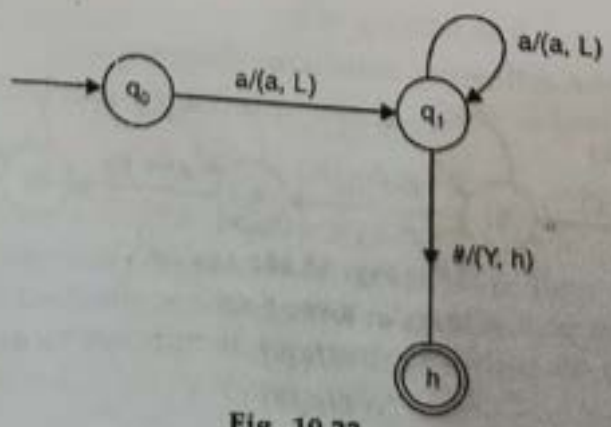



**Fig. 10.22.**

Y stand for yes string is accepted here halt state $Y_h$ signifies as accepting state. Consider the string #aaa#

$$\begin{aligned} \#aaaq_0 &\vdash \#aaq_1a \\ &\vdash \#aq_1aa \\ &\vdash \#q_1aaa \\ &\vdash Yhaaa\# \end{aligned}$$

when we get Y at input cell, means that string is accepted.

**Example 10.16. Design a Turing machine for the language**

$$L = \{(ab)^n / n \geq 0\}$$

**or for the regular expression $r = (ab)^*$.**

**Solution.** Let Turing machine be

$$\begin{aligned} Tm &= (Q, \Sigma, \Gamma, \delta, q_0, h) \\ Q &= \{q_0, q_1, h\} \\ \Sigma &= \{a, b\} \\ \Gamma &= \{a, b, \#\} \end{aligned}$$

$q_0$ is initial state and $h$ is halt state.

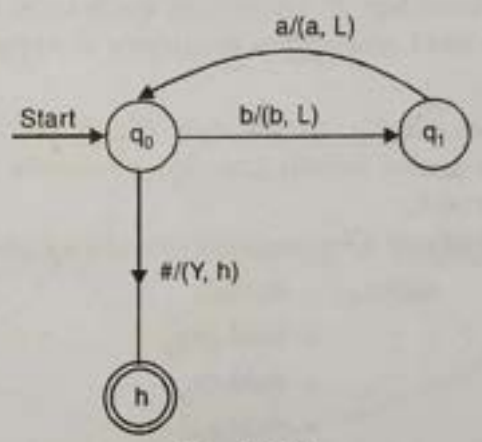



**Fig. 10.23.**

**Note : All the move which are not shown, leads towards not accepting states.**

Let us pass a string $w = \#ab\underset{q_0}{\underline{a}}b\#$

$$\begin{aligned} \#ababq_0 &\vdash \#abaq_1b \\ &\vdash \#abq_0ab \\ &\vdash \#aq_1bab \\ &\vdash \#q_0abab \\ &\vdash Yhabab \end{aligned}$$

Y : Stands for accept, Tm accepts string and halts.

**Example 10.17. Design a Turing machine which accepts the language $L = \{w \in (a, b)^* / w$ has equal number of $a$'s and $b$'s$\}$.**

**Solution.** The language $L$ is the set of strings (included null string) in which every strings either start by $a$ or $b$ but number of $a$'s and $b$'s in the string are always equal. That is turing machine should accept strings like

*ab, aaabbb, baba, abab*, and so on and it must reject strings like *a, b, abb, baa, abbb, abbab* and so on.

The Turing machine that accepts $L$ is shown below :

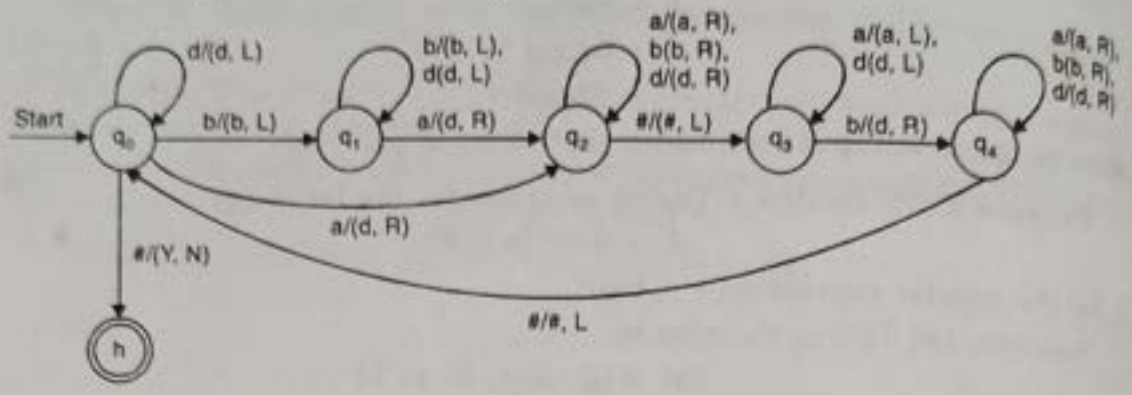



**Fig. 10.24.**

Y : Stands for Accept

In one main cycle, this machine changes one *a* and one *b* in two *d*'s. It does so by scanning the the non-blank part of take two times from right to left, once for an '*a*' and once for '*b*', returning each time to the right end of the tape before starting the next scan. This sequence is repeated until the supply of *a* or *b* is exhausted.

Now if machine sees all *d*'s on the non-blank part then it halts that is, it accept the string. The move which are not available in the machine leads towards not accepting state.

Following is the sequence of operation on string abba.

$$\begin{aligned}
\#abbaq_0 &\vdash \#abbdq_2 \\
&\vdash \#abbd\#q_2 \\
&\vdash \#abbdq_3 \\
&\vdash \#abbq_3d \\
&\vdash \#abddq_4 \\
&\vdash \#abdd\#q_4 \\
&\vdash \#abddq_0 \qquad \text{(one cycle is completed)} \\
&\vdash \#abdq_0d \\
&\vdash \#abq_0dd \\
&\vdash \#aq_1bdd \\
&\vdash \#dbq_2dd \\
&\vdash \#dbdq_2d \\
&\vdash \#dbddq_2 \\
&\vdash \#dbdd\#q_2 \\
&\vdash \#dbddq_3 \\
&\vdash \#dbdq_3d \\
&\vdash \#dbq_3dd \\
&\vdash \#dddq_4d \\
&\vdash \#ddddq_4 \\
&\vdash \#dddd\#q_4
\end{aligned}$$

$$\vdash \#dddd q_0$$
$$\vdash \#ddd q_0 d$$
$$\vdash \#dd q_0 dd$$
$$\vdash \#d q_0 ddd$$
$$\vdash \#q_0 dddd$$
$$\vdash Yhdddd$$

**Example 10.18. Design a Turing machine for the following languages :**

$$L = \{a^n b^n / n \geq 1\}$$

**Solution.** We are going to design a turing machine which accepts a set of strings in which every string starts with '$a$', followed by any number of $a$'s, every string ends in equal number of $b$'s as $a$'s. No '$a$' is encountered after first '$b$' is read. It will not accepts the strings like $\in$, $a$, $aa$, $bb$, $abb$, $aabbb$, $aab$ and so on.

Let required turing machine is :

$$\text{Tm} = (Q, \Sigma, \Gamma, \delta, q_0, h)$$
$$Q = \{q_0, q_1, q_2, q_3, h\}$$
$$\Sigma = \{a, b\}$$
$$\Gamma = \{a, b, \#\}$$

$q_0$ is initial state and $h$ is halt state.

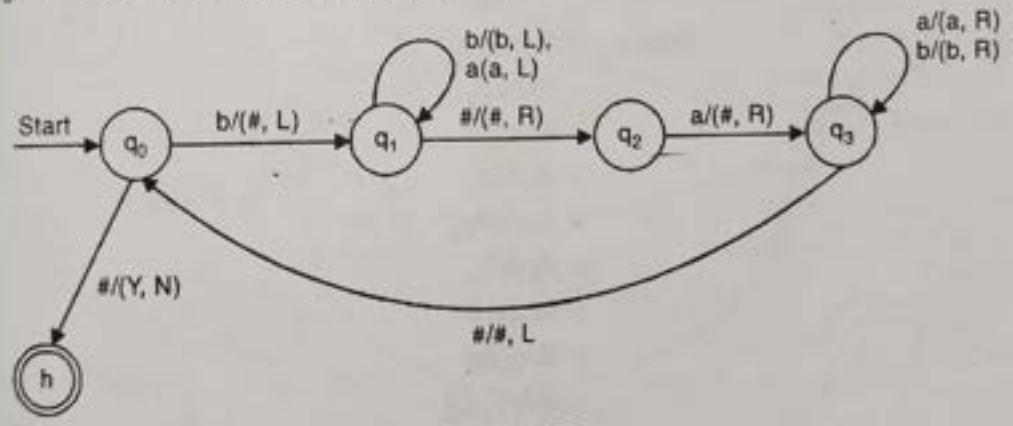



**Fig. 10.25.**

The moves which are unaddressed, leads towards rejecting states.

Let us pass the string $w = \#a\underset{q_0}{\underline{b}}\#$

$$\#abq_0 \vdash \#aq_1\#$$
$$\vdash \#q_1a\# \vdash \#aq_2\#$$
$$\vdash \#\#\#q_3 \vdash \#\#q_0\#$$
$$\vdash \#Yh\#$$

**Example 10.19. Design a Turing machine for the following languages :**

$$L = \{w \in \{a, b, c\}^* / w \text{ has equal number of } a\text{'s}, b\text{'s and } c\text{'s}\}.$$

**Solution.** we can solve this problem in similar fashion as example 10.17. Here turing machine start searching from right and first search '$a$' then move to extreme right and this '$a$' is repalced by $d$. Now machine search for '$b$' and if find replace it by '$d$' and again move on extreme right and search

corresponding '$c$' and if find replace it by $d$, this completes one cycle. This process reepated and if all symbols are replaced by $d$'s then string is accepted and rejected in any other case.

Turing machine is :

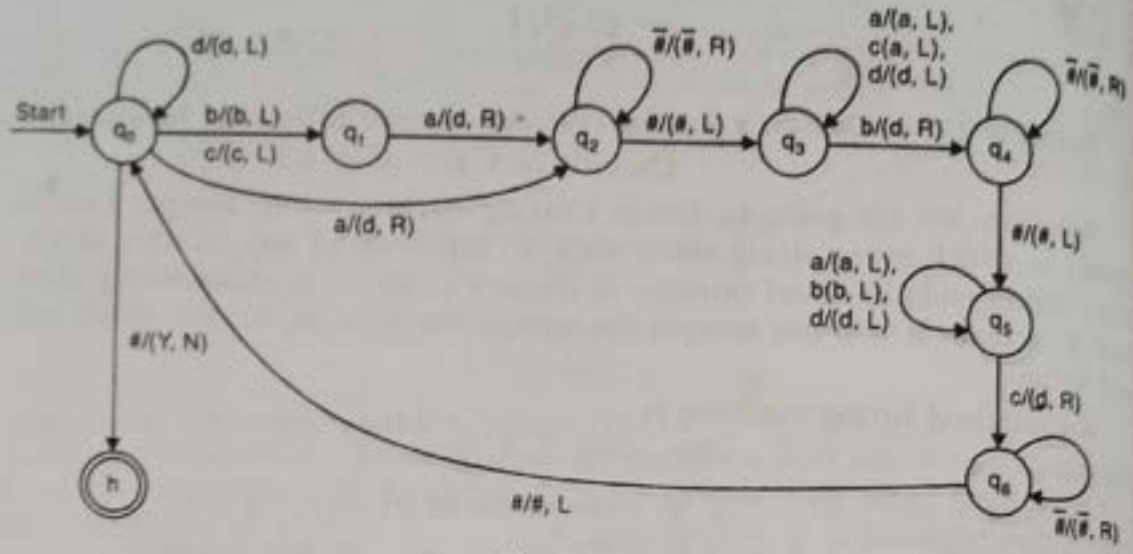



Fig. 10.26.

Let

$$w = \#cb\underline{a}\#$$
$$q_0$$

$$\begin{aligned}
\#cbaq_0 &\vdash \#cbd\#q_2 \\
&\vdash \#cbdq_3 \\
&\vdash \#cbq_3d \\
&\vdash \#cddq_4 \\
&\vdash \#cdd\#q_4 \\
&\vdash \#cddq_5 \\
&\vdash \#cdq_5d \\
&\vdash \#cq_5dd \\
&\vdash \#ddq_6dd \\
&\vdash \#dddq_6d \\
&\vdash \#ddddq_6 \\
&\vdash \#dddd\#q_6 \\
&\vdash \#ddddq_0 \\
&\vdash \#dddq_0d \\
&\vdash \#ddq_0dd \\
&\vdash \#dq_0ddd \\
&\vdash \#q_0dddd \\
&\vdash Yhdddd
\end{aligned}$$

**Example 10.20. Design a Turing machine for the following language :**

$$L = \{ab^n / n > 0\}$$

**Solution.** Here we want to design a Turing machine which accepts the language $L$ which consist a set of strings in which every string $w$ begins with single '$a$' and followed by one or more $b$'s.

Clearly Turing machine will accept strings like *abbbb*, *ab*, and rejects strings like $\in$, *a*, *b*, aaabb and so on.

Here is required turing machine

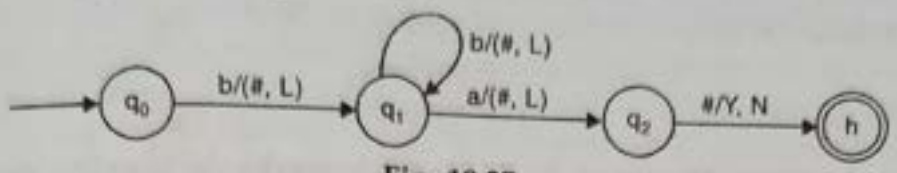



**Fig. 10.27.**

The moves which are not shown are considered as rejecting states.

$$
\begin{aligned}
w &= \#abb\# \\
\#abbq_0 &\vdash \#abq_1\# \\
&\vdash \#aq_1\#\# \\
&\vdash \#q_2\#\#\# \\
&\vdash Yh\#
\end{aligned}
$$

**Example 10.21. Design a Turing machine which computes following function.**

$f(w) = ww^R$, **where** $w^R$ **is the reverse of string** $w$ . $(w \in (a, b)^*)$.

**Solution.** Let input is *#ab#* then Tm should give output as *#abba#*.

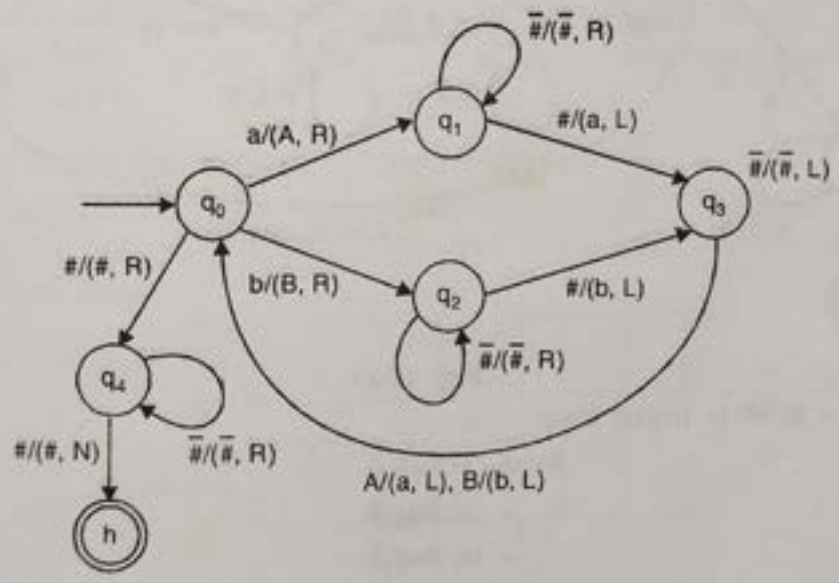



**Fig. 10.28.**

$$
\begin{aligned}
\#abq_0 &\vdash \#aB\#q_2 \\
&\vdash \#aBq_3b \\
&\vdash \#aq_0bb \\
&\vdash \#Abq_1b \\
&\vdash \#Abbq_1 \\
&\vdash \#Abb\#q_1 \\
&\vdash \#Abbq_3a \\
&\vdash \#Abq_3ba \\
&\vdash \#Aq_3bba \\
&\vdash \#q_0abba
\end{aligned}
$$

$\vdash \#aq_4bba$
$\vdash \#abq_4ba$
$\vdash abbq_4a$
$\vdash \#abbaq_4$
$\vdash \#abba\#q_4$
$\vdash \#abba\#h$

**Example 10.22. Design a Turing which works as copying machine (c), for $w \in (a, b)^+$.**

**Solution.** We want to design a turing machine which gives output for input $w = \#ab\#$ as $\#ab\#ab\#$ that is

$$\#w\# \vdash \text{Tm } \#w\#w\#$$

The transition diagram of turing machine will be as follows :

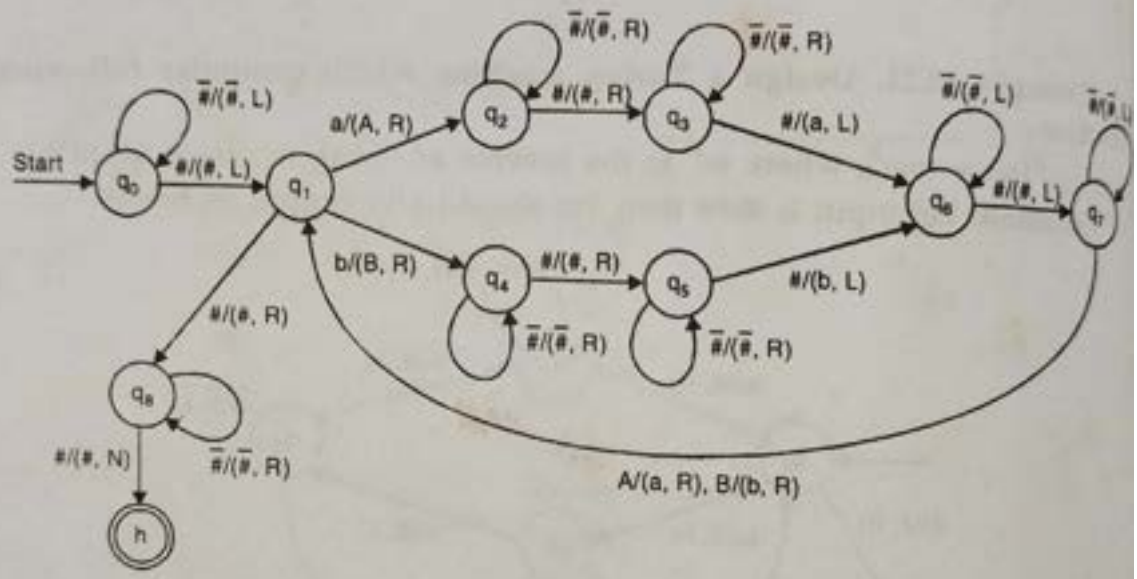



**Fig. 10.29.**

Let $w = \#ab\#$ is input then :

$\#abq_0 \vdash \#aq_0b$
$\vdash \#q_0ab$
$\vdash \#aq_1b$
$\vdash \#Abq_2$
$\vdash \#Ab\#q_2$
$\vdash \#Ab\#\#q_3$
$\vdash \#Ab\#q_6a$
$\vdash \#Abq_7\#a \vdash \#Aq_7b\#a$
$\vdash \#abq_1\#a \vdash aB\#q_4a$
$\vdash \#aB\#aq_5 \vdash aB\#a\#q_5$
$\vdash \#aB\#a\#q_5 \vdash \#aB\#aq_6b$
$\vdash \#aB\#q_6ab \vdash \#aBq_7\#ab$
$\vdash \#ab\#q_1ab \vdash \#ab\#aq_8b$
$\vdash \#ab\#abq_8 \vdash \#ab\#ab\#q_8$
$\vdash \#ab\#ab\#h.$

## 10.10. EXTENSION OF TURING MACHINES

Now it becomes clear that turing machine can perform fairly powerful computation. In order to better understand their surprising power, we shall consider the effect of extending the turing machine model in various directions. The "new, improved models" of the turing machine can in each instance be simulated by the standard model. Such results increase our confidence that the turing machine is indeed the ultimate computational device, the end of our progression to move and move powerful automata. A side benefit by these results is that we shall feel free subsequently to use the additional features when designing turing machines to solve particular problems.

The extensions of Turing machine considered are :

(*i*) There may be several tapes instead of one only, each tape having its own independent head.

(*ii*) The tape may be allowed to be infinite in both the directions.

(*iii*) There may be more than one head scanning various cells of the tape. Two or more heads simultaneously read the same cell or may attempt to write in the same cell.

(*iv*) The tape may be *K*-dimensional, $K \geq 2$, instead of only one-dimensional.

(*v*) For a given pair of current state and symbol under the Head, instead of atmost one possible move, there may be any finite, possible zero, number, of next moves. This is called **Non-deterministic Turing Machine.**

### 10.10.1. Multiple Tapes Turing Machine

One can think of turing machines that have several tapes (see figure 10.29). Each tape is connected to the finite control by means of a read/write head (one on each tape).

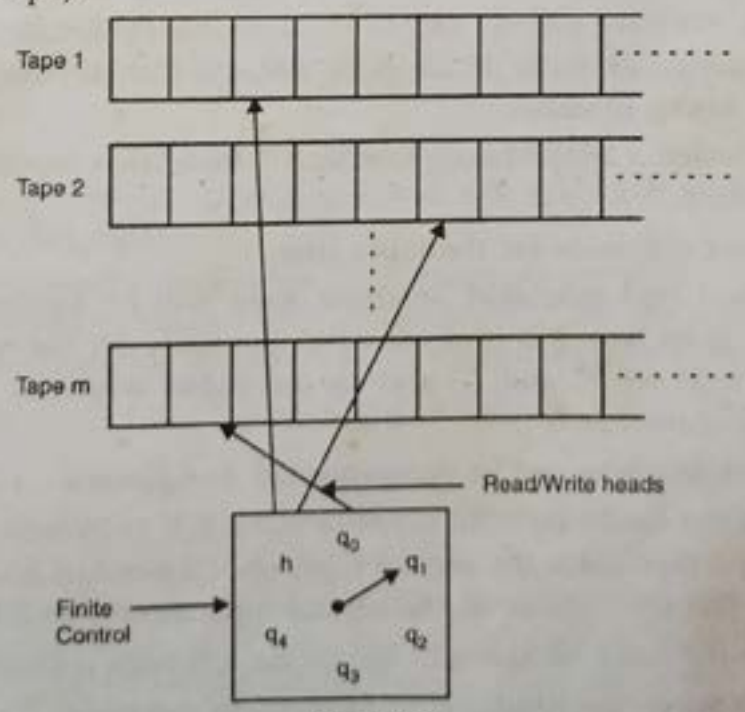



**Fig. 10.30.**

The machine can in one step read the symbols scanned by all its heads and then, depending on these symbols in current state, rewrite some of those

scanned squares and move some of the heads to the left or right, in addition to changing the state.

For any fixed integer $m \geq 1$, an $m$-tape turing machine is a turing machine equipped with $m$ tapes and corresponding heads. Thus standard turing machine studied so far in this chapter is just an $m$-tape turing machine, with $m = 1$.

## Definition

Let $m \geq 1$ be an integer.

An $m$-tape turing machine is a six tuple machine as follows :

$$Tm = (Q, \Sigma, \Gamma, \delta, q_0, h)$$

where
- $Q$ : the finite set of states of the finite control
- $\Sigma$ : The finite set of input symbols
- $\Gamma$ : The tape symbol where $\Gamma = \Sigma \cup \#$
- $h$ : halt state $h \in Q$
- $q_0$ : The initial state, $q_0 \in Q$
- $\delta$ : The transition function is a function which maps :

$$(Q \times \Sigma^m) \rightarrow Q \times (\Sigma \cup \{L, R, N\}^m).$$

That is, for each state-$q$, and each $m$-tapes of tape symbols $(a_1, a_2, ..., a_k)$, $\delta(a_1, a_2, ..., a_k) = (P, (b_1, b_2 ...b_k))$, where $P$ is, as before, the new state, and $b_j$ is the action taken by Tm at tape $j$.

Computation takes place in all $m$-tapes of a $m$-tape Turing machine. Accordingly, a configuration of such a machine must include information about all tapes.

**Example 10.23. Design an $m$-tape turing machine which works as copying machine.**

**Solution.** We have already solved this problem by the standard one-tape turing machine in example 10.22. Now we are free to use $m$-tape in the designing of turing machine.

We will design a 2-tape turing machine, which takes input $\#w\underline{\#}$ and gives output as $\#w\#w\underline{\#}$.

Let us clear notations for the tapes first.

For the first tape machines notations used will be written as $R'$ and $L'$ input symbol from tape-1 is read-out as $\sigma'$. Similarly for the tape-2 notations of machines will be $R^2$ and $L^2$ and so on. Input which is read-out from tape-2 will be shown as $\sigma^2$.

This turing machine can be accomplished as follows :

(1) **Move** the heads on both tapes to the right, copying each symbol on the first tape on to the second tape, until a blank is found on the first tape. The first square of the second tape should be left blank.

(2) **Move** the head on second tape to the left until a blank is found.

(3) Again **move** the heads on both tapes to the right, this time copying symbols from the second tape on to the first tape. Halt when a blank is found on the second tape.

The required 2-tape turing machine is shown in Fig. 10.30 as follows :

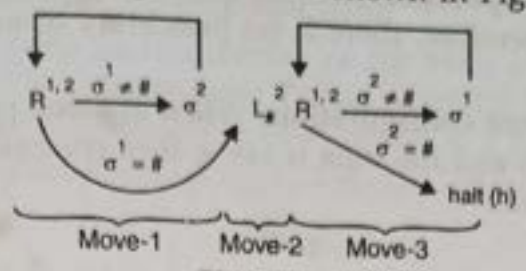
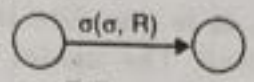
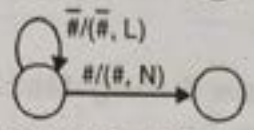



Fig. 10.31.

Here : $R \simeq$ 


and $L_{\#} \simeq$ 


The sequence of actions can be pictured as follows :

First tape $\#w\underline{\#}$

Second tape $\#\underline{\#}$

After (move-1) : First tape $\#w\underline{\#}$

Second tape $\#w\underline{\#}$

After (move-2) : First tape $\#w\underline{\#}$

Second tape $\underline{\#}w\#$

After (move-3) : First tape $\#w\#w\underline{\#}$

Second tape $\#w\underline{\#}$

### 10.10.2. Two-way Infinite Tape

The extensions are distinguished from each other and from the standard Tm through different definitions of next-move relation δ and of configurations for each of the extension. So here we discuss the extension only in terms of definitions of δ and of configuration.

Like standard Tm, in this case also, the next-move is given by δ as a partial function from

$$(Q \times \Gamma) \text{ to } (Q \times \Gamma \times \{L, R, N\})$$

The following three points need to be noted in respect of configurations of Two-way Turing Machine.

(*i*) **Configuration/Instantaneous Description.** In standard Tm, if there are a number of left-most positions which contain blanks, then those are included in the configuration that is if the one-way configuration tape is of the form

$$\#ab\#c\underset{\substack{\uparrow \\ q_2}}{d}ef\#\#\ldots\#$$

then the configuration in the standard Tm is written as :

$$(q_2, \#ab\#c\underline{d}ef\#\#\ \ldots\ \#)$$

However, in the two-way infinite tape Tm, both left-hand and right-hand parts of the tape are symmetrical in the sense that there is an infinite continuous sequence of blanks on each of the right hand and left hand of the sequence of non-blanks. Therefore, in case of two-way infinite tape, if the above string is on the tape then it will be in the form.

$$(q_2, \#\ \ldots\ \#\#\ ab\ \#\ c\underline{d}ef\ \#\#\ \ldots\ \#)$$

**(ii) No Ceasing of operation without Halting.** In this case, as there is no left end of the tape, therefore, there is no possibility of jumping off the left-end of the tape.

**(iii) The empty Tape configuration.** When at some point of time all the cells of the Tape are #'s and the state is say $q$, then configuration in Two-way tape may be denoted as :

$$(q, \#)$$

where only the current cell containing # is shown in configuration.

**Despite the fact that, it is possible in the new model of computer to move left as far as required. The model does not provide any additional computational capability.**

### 10.10.3. Multiple Heads Turing Machines

In order to simplify the discussion, we assume that there are only two Heads on the tape. The tape is assumed to be one-way infinite. We explain the involved concepts with the help of an example. Let the content of the tape and the position of the two heads that is $H_1$ and $H_2$, be as given below :

$$\#\#a\ \underset{\substack{\uparrow \\ H_2}}{b}\ c\#d\ \underset{\substack{\uparrow \\ H_1}}{e}\ f\#\#\ldots$$

Further, let the state of the Tm be $q$.

The move function of the two-head one-way turing may be defined as

δ(state, symbol under Head1, symbol under Head2)

= (New state, $(S_1, M_1)$, $(S_2, M_2)$)

where $S_i$ is the symbol to be written in the cell under $H_i$ (the $i$th Head) and $M_i$ denotes the movement of $H_i$, where the movement may be $L$, $R$ or $N$ and further $L$ denotes movement to the left, $R$ denotes movement to the right of the current cell and $N$ denotes 'no movement of the Head'.

Two special cases of the δ function defined above, need to be considered :

(i) What should be written in the current cell when both Heads are scanning the same cell at a particular time and the next moves $(S_1, M_1)$, $(S_2, M_2)$ for the two heads, are such that $S_1 \neq S_2$ that is symbol to be written in current cell by $H_1$ is not same as symbol to be written in current cell by $H_2$.

In such a situation, a general rule may be defined, say, as whatever is to be done by $H_1$ will take precedence over whatever is to be done by $H_2$.

(ii) For two-head one way tape, a configuration shall be called hanging if δ($q$, symbol under $H_1$, symbol under $H_2$) = $(q_1, (S_1, M_1), (S_2, M_2))$ is such that either

(a) Symbol under $H_1$ is in the left-most cell and $M_1$ is $L$, that is movement of $H_1$ is to be to the left,

or

(b) Symbol under $H_2$ is in the left-most cell and $M_2$ is $L$, that is movement of $H_2$ is to be to the left.

**The above discussion can be further extended easily to the case when number of Heads is more than two. The power of Tm is not enhanced by the use of extra heads.**

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### 10.10.4. K-dimensional Turing Machine

Again to facilitate the understanding of the basic ideas involved, let us discuss initially only Two-dimensional Turing Machine. Then these ideas can be easily generated to K-dimensional case, where $K > 2$.

In the case of two-dimensional tape as shown below, we assume that the tape is bounded on the left and the bottom.

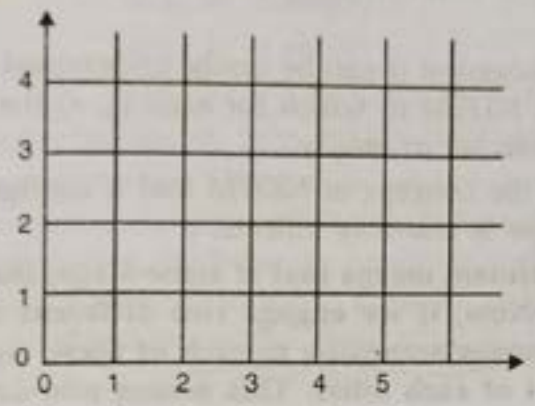


Fig. 10.32.

Each cell is given an address say $(i, j)$ where $i$ is the row-number of the cell and $j$ is the column number of the cell.

A configuration of a two-dimensional Tm at a particular time may be described in terms of finitely many of the triplets of the form, $(i_1, i_2, c)$ where for each such triplets of the form, $(i_1, i_2, c)$ where for each such triplet $(i_1, i_2)$ is the address of a cell and $c$ denotes the contents of the cell. Only these cells are included in an ID, for which, the contents, are non-blank symbols.

In the configuration or ID, order of the cell which are included in an ID, **Row Major Ordering** is to be followed, that is first all the elements in the row with least index are included in the ID, followed by the elements of the row with next least index and so on. Within cells of each row, the cell with non # contents and having least column number is included first followed by the non blank cell with next least column number and so on.

Let $q \in Q$, $C_K \in \Gamma - \{\#\}$, that is $C_K$ is a non-blank tape symbol.

The configuration at a particular instant is denoted by

$$(q, (H_1, H_2)\ (i_1, j_1, C_{i_1j_1}), (i_2, j_2, C_{i_2j_2}) \ldots (i_k, j_k, C_{i_kj_k}) \ldots))$$

where each of $C_{i_1j_1}, C_{i_2j_2} \ldots$ is non-blank and these are the only non-blank on the tape.

Also $(H_1, H_2)$ denotes the location of the cell currently being scanned that is the cell under the Head.

### 10.10.5. Non-determinstic Turing Machine

We have already seen that when finite automata are allowed to act non-deterministically no increase in the computational power, but that non-determinism in push down automata are more powerful than deterministic ones.

We can also imagine turing machine that acts non-deterministically. In standard Tm, to each pair of current state (except the halt-state) and the symbol being scanned, there is a unique triplet comprising of the next state, unique action in terms of writing a symbol in the cell being scanned and the motion,

if any, to the right or left. However, in case of NDTM (non deterministic turing machine), to each pair $(q, a)$ with $q$ as current state and '$a$' as symbol being scanned, there may be a **finite set of triplets** $\{(q_i, a_i, m_i); i = 1, 2, ...\}$ of possible moves. The set of triplets may be empty that is for some particular $(q, a)$ the Tm may not have any next move. Or alternatively the set $\{(q_i, a_i, m_i)\}$ may have more than one triplet, meaning thereby that the NDTM in the state $q$ and scanning symbol '$a$' and can choose next move from the set $\{(q_i, a_i, m_i)\}$ of next moves.

From the above discussion it can be easily understood that standard Tm is a special case of the NDTM in which for each $(q, a)$ the set $\{(q_i, a_i, m_i)\}$ in next moves is a singleton set or empty.

In order to define the concept of NDTM and a configuration in NDTM, we assume that the tape is one-way infinite.

**Proper non-determinism means that at some stage, there are atleast two next possible moves. Now, if we engage two different machines to work out further possible moves according to each of these two moves, the two can work independent of each other. This means non-determinism allows parallel computation.**

**Definition.** A non-deterministc Turing machine is a six tuple $(Q, \Sigma, \Gamma, \delta, q_0, h)$ where

$Q$ : set of states
$\Sigma$ : set of input symbols
$\Gamma$ : set of tape symbols
$\delta$ : $(Q \times \Gamma) \rightarrow$ power set of $(Q \times \Gamma \times \{L, R, N\})$
$q_0$ : initial state and
$h$ : hut state, $h \in Q$

The concept of a configuration is same as in case of standard Tm. But the concept of 'yield in one step' denoted by Tm has different meaning. Here one configuration may yield more than one configuration.

**Example 10.24. Construct an NDTM which accepts the language $\{a^n b^m : n \geq 1, m \geq 1\}$, that is language of all strings over $\{a, b\}$, in which there is at least one '$a$' and one '$b$' and all $a$'s precede all $b$'s.**

**Solution.** The diagrammatic representation of the required NDTM is as given below : (initially tape head is at leftmost symbol, that is '$a$')

In the proposed NDTM, as the motion of the head is always to the right except in the Halt state, therefore, $R$ is not mentioned in the labels in the diagram below :

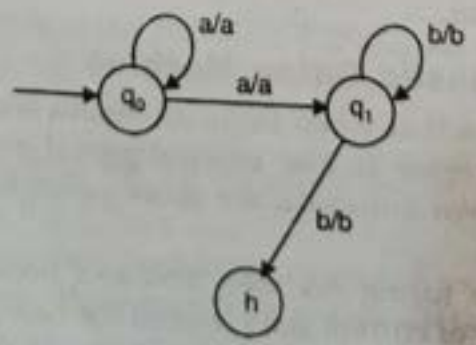


**Fig. 10.33.**

where the label $x/y$ on an arc denotes that if symbol in the current cell is '$x$' then contents of the cell are to be replaced by '$y$'. Formally the proposed NDTM may be defined as

$$M = \{\{q_0, q_1\}, \{a, b\}, \{a, b, \#\}, \delta, q_0, h\}$$

where $\delta$ is defined as follows :

$$\delta(q_0, a) = \{(q_0, a, R), (q_1, a, R)\}$$
$$\delta(q_0, b) = \text{empty}$$
$$\delta(q_1, a) = \text{empty}$$
$$\delta(q_1, b) = \{(q_1, b, R), (h, b, N)\}$$

If the machine has no move, then it halts without accepting the string.

When turing machines are allowed to act non-deterministically, is there any increase in computational power ? We must first define what it means for a non-deterministic turing machine to compute some thing. Since a non-deterministic machine could produce two different outputs or final states from the same input, we have to be careful about what is considered to be the end result of a computation by such a machine.

For a non-deterministic machine to decide a language and compute a function, we require that all of its computations halt; we achieve this by assuming that there is no computation containing after '$m$' steps, where $m$ is an "upperbound" depending on the machine and the input. For Tm to decide a language, we only require that at least one of its possible computations end up accepting the input.

For a non-deterministic turing machine to compute a function, we require that all possible computations agree on the output. If not, we would not be able to decide which one is the right value of the function.

### 10.10.6. Recursively Enumerable and Recursive Languages

When a Turing machine executes an input, there are four possible outcomes of execution. Then Tm

(*i*) Halts and accept the input

(*ii*) Halts and rejects the input

(*iii*) Never halts (fall into loop), or

(*iv*) Crash.

We also know that the language accepted by Tm is known as recursively enumerable and the set of recursive languages is subset of recursive enumerable set.

The set of recursively enumerable languages are precisely those languages that can be listed (enumerated) by a turing machine. Basically, if a Tm can generate all the strings in a language, another Tm can accept all the strings that are not in the given language. (Strings that are not in the language may be rejected or may cause the turing machine to go into an infinite loop).

A language is recursive if there exists a Turing machine that accepts every string of the language and rejects every string that is not in the language. So, we are sure about the rejection of the strings with are not in the given recursive language.

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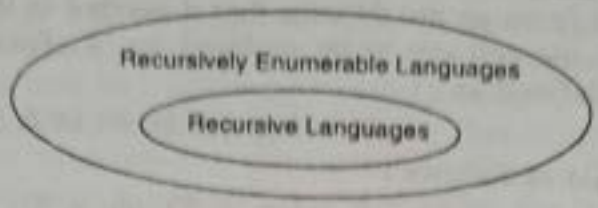


Fig. 10.34.

**Theorem 10.2.** If a language $L$ is recursive, then it is recursively enumerable languages.

**Proof.** A language is recursively enumerable if there exists a Turing machine that accepts every string of the language and a language is recursive if there is a Tm that accepts every string of the language and does not accept strings that are not in the language.

So, every recursive language is also recursively enumerable. Hence, statement of the theorem is true.

### 10.10.7. Turing Machine as Enumerators

We have viewed turing machines as recognizers of languages and as computers of functions on the non-negative integers. There is a third useful view of turing machines, as generating devices. Consider a multi-tape turing machine Tm that uses one tape as an output tape, on which a symbol, once written, can never be changed, and whose tape head never moves left. Suppose also that on the output tape, Tm writes strings over some alphabet $\Sigma$, separated by a marker symbol #. We can define $L$(Tm), the language generated by Tm, to be set of $w$ in $\Sigma^*$ such that $w$ is eventually printed between a pair of #'s on the output tape.

Note that unless Tm runs forever, $L$(Tm) is finite. Also, we do not require that words be generated in any particular order, or that any particular word be generated only once. **If $L$ is $L$(Tm) for some turing machine Tm, then $L$ is an regular set and conversely.** The recursive sets also have a characterization in terms of generators.

So finally we can say that :

**"To enumerate a set means to list the elements one at a time. To stay that a set is enumerable should mean that there is an algorithm for enumerating it, so far, we have seen Turing machines as recognizers of languages and as computers of functions on the positive integers. There is another purpose a Tm can serve. It can be a generating device.**

**A Turing machine Tm is said to be generating or enumerating if we are able to produce each word of any languae $L$ separated by blank symbol (initially tape is empty). The order of the strings is not important and any string may be repeated indefiitely.**

**Theorem 10.2.** If any language $L$ is generated by the Turing machine, then $L$ is accepted by some other turing machine.

or

If any language is generated by the Tm then $L$ is recursively enumerable.

**Proof.** Let $Tm_1$ be a turing machine which generates $L$. Now let us construct another turing machine $Tm_2$ which accepts $L$ and $Tm_2$ is having two tapes.

The $Tm_2$ saves any input on the extra tape before beginning its generation process. The $Tm_2$ simulates $Tm_1$, except that every time # is written on output tape the simulation on $Tm_1$ pauses while $Tm_2$ compare its input string with the string listed just before #. If they are same, $Tm_2$ accepts otherwise $Tm_2$ continues to simulate $Tm_1$.

Clearly $Tm_2$ accepts an input string if and only if it is generated by $Tm_1$.

So $$L(Tm_2) = \underbrace{L(Tm_1)}_{\text{generating language by } Tm_1}$$

**Theorem 10.3.** If the language $L$ is accepted by some Turing machine (*i.e.* $L$ is recursive enumerable) then $L$ is generated by some Turing machine.

**Proof.** Let $Tm_1$ is a Turing machine which accepts the language $L$. We may define another Turing machine $Tm_2$ which generate the words over $\Sigma$. After each new word is generated, we run a simulation of it on the machine $Tm_1$. If $Tm_1$ accepts, we print out the word on the tape inside #'s. If $Tm_1$ does not accepts, we skip it and go on to the next possibility from the word generator $Tm_2$. Again, we apply the same method for the next word and we get list of words separated by # on the tape of $Tm_2$.

Unfortunately, this method works if $Tm_1$ is guaranteed to halt on all inputs (either in final or non final states). However, it may possible that $Tm_1$ will loop forever for some words, in such case we shall stuck waiting forever, and we cannot go on to test the next possible input string.

To overcome this situation, we adopt the canonical order on $\Sigma^*$. The canonical order (or lexicographical order or alphabetical order) over $\Sigma = \{a, b\}$ is $\{\in, a, b, aa, ab, ba, bb, aaa, aab, ab, ...\}$ and over $\Sigma = \{0, 1\}$, we define canonical order as

$$\{\in, 0, 1, 00, 01, 10, 11, 000, 001, 010, ...\}$$

Let us number the possible input strings as $w_1, w_2, ...$ in the usual canonical order. Let us assume that $Tm_2$ has four tracks. In second track it generates, in order, all the integers. Let at some point in the operation of $Tm_2$, track 2 has the number $n$ on it. Now on track 3 we generate, one by one, all possible strings from $w_1$ to $w_n$. Each time we generate another input string, we copy the string from track 3 to track 4 and simulate the running of $Tm_1$ on it. But we run simulation for exactly $n$ steps unless $Tm_1$ halts before. If $n$ steps have not been enough to halt $Tm_1$ we do not waste more effort on this input string at this iteration. We erase track 4 and we go back to track 3 and generate the next input string to be tested. However, if the input string is accepted within $n$ steps, then we print the string on track between #'s. we still erase the track 4 and go back to track 3 for the next input string to be tested.

When we go back to track 3 to get the next string, we have to be sure that we have not already tried all the input strings up to $w_n$. In order to be sure of this we must keep a counter on track 2 telling us how many strings we have indeed produced. If we have not gone up to $n$ yet, then we do produce next string and repeat the process. However, if we find that we have already gone up to our limit $w_n$ then what we must do is erase this track and increment track 2. Track 2 now has the contents $n + 1$ on it. We begin again to generate strings on track 3 from $w_1$ to $w_{n+1}$ and one by one simulate on track 4 the running of them on $Tm_1$ for we generate all the strings on track 3 exactly $n + 1$

steps this lie. Again, if they are neither accepted nor rejected, they are abandoned temporarily. If they are accepted they are printed on track if they have been printed on track 1 already. The simulation of $Tm_1$ on a particular input string begins at the start State of $Tm_1$ even though we have once before already simulated the first $n$ steps of the processing. May be $n$ steps were not enough, but the $n + 1$ steps will do the trick. If $Tm_1$ neither accepted nor rejected in $n + 1$ steps, then we erase back 4 and get the next input text case from tracks 3, unless we have already generated up to $w_{n+1}$, in which case we erase tracks 3 and increment tracks 2 to $n + 2$.

Clearly, the only strings that appear on track 1 are the words that have been discovered to already be in $L$ by having been accepted $Tm_1$.

**Theorem 10.4.** $L$ is recursive if and only if there is a Turning machine that enumerates $L$ in canonical order.

**Proof.** First we prove if $L$ is recursive then there is a Turning machine that enumerates $L$ is canonical order. We prove this by constructive algorithm. Let $L$ is the recursive language accepted by the Turning machine $Tm_1$, clearly $Tm_1$ halts on every input.

Now, we construct $Tm_2$ such a way that $Tm_2$ generates the words in canonical order, one at a time, on one of its track. Then $Tm_2$ simulates $Tm_1$ on that word. If $Tm_1$ accepts that word, $Tm_2$ print that word on the other track, separated by #'s It should be clear that $Tm_1$ will finish processing each word after a finite time because $Tm_1$ never enters into infinite loop ($L$ is recursive and $Tm_1$ halts for every words). It is clear that $Tm_2$ enumerate the words in $L$ is canonical order.

Now, we prove the converse *i.e.* if $L$ is a language enumerated by some turing machine in canonical order then $L$ is recursive. Let $Tm_1$ be Turing machine that enumerates the words of $L$ in canonical order. We make a Turing machine $Tm_2$ that accepts $L$ and rejects $L'$. Into $Tm_2$ we input the strings $w$ to be tested. We then simulate the running of $Tm_1$ until its output (words of $L$) are longer than $w$. This will takes a finite amount of time. Now, we simply check to see whether $w$ is among the words generated thus far by $Tm_2$. If it is, we accept it; if not, we reject it. It is clear that all strings are either accepted or rejected *i.e.* there is a complete decision procedure and hence $L$ is recessive.

### 10.10.7.1. Non recursively Enumerable Languages

As we have seen that there are pumping lemma's to show that certain language is not regular or context free language. Unfortunately, there is no such pumping lemma for recursively enumerable languages because according to church's thesis any conceivable algorithm can be implemented by Turing machines. But there are languages to which no turing machine can accept. We show that such language exist by the certain theorems only and those languages are called non-recursively enumerable languages.

## 10.11. CHURCH'S THESIS

What function can not be computed by turing machines ? This is very surprising. It is believed that there are no functions that can be defined by humans, whose calculation can be described by any well-defined mathematical algorithm that people can be taught to perform, that can not be computed by Tm's. The Tm is believed to be the ultimate calculating mechanism.

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In other words *"No computational procedure will be considered an algorithm unless it can be represented as a turing machine"*. This statement is called Church's thesis because Alonzo Church (1936) gave many sophisticated reasons for believing it. Church's original statement was slightly different because his thesis was presented slightly before the turing invented his machines.

Church actually said that any machine that can do certain list of operations will be able to perform all conceivable algorithms. He tied together what logicians had called recursive functions and computable functions. Tm's can do all that Church asked, so they are one possible model of the universal algorithm machine Church described.

Unfortunately, Church's thesis can not be a theorem in mathematics because ideas such as *"can ever be defined by humans"* and *"algorithm that people can taught to perform"* are not part of any known mathematics. *There are no axioms that deal with "people"*.

## 10.12. UNIVERSAL TURING MACHINES

We can consider turing machine in both ways : The turing machine is an "un programmable" piece of hardware, specialized at solving one particular problem, with instructions that are *"hard-wired at the factory"*.

We shall now take the opposite point of view. We shall argue that turing machines are also software. That is, we shall show that there is a certain "generic" turing machine that can be programmed, about the same way that a general purpose computer can, to solve any problem that can be solved by turing machine. The "program" that makes this generic machine behave like a specific machine Tm will have to be a description of Tm. In other words, we shall be thinking of the formalism of turing machines as a programming language, in which we can write programs. Programs written in this language can then be **interpreted by universal machine** that is to say another program in same language.

To begin, we must present a general way of specifying turing machines, so that their descriptions can be used as input to other turing machines.

Universal turing machines 'U' takes two arguments, a description of a machine Tm, "Tm", and a description of an input string $w$, "$w$". We want U to have the following property : U halts on input "Tm" "$w$" if and only if Tm halts on input $w$.

$$U\ (\text{“}m\text{”}\ \text{“}w\text{”}) = \text{“}m\ (w)\text{”}.$$

It is the functional notation of universal turing machine.

To justify this we have to define a language whose strings are all legal representation of turing machines. We adopt the following convention. A string representing a turing machine state is of the form $\{q\}\{0, 1\}^*$; that is, the letter $q$ followed by binary string. Similarly, a tape symbol is always represented as a string in $\{q\}\ \{0, 1\}^*$.

Let Tm = $(Q, \Sigma, \delta, S)$ be a turing machine, and $k$ and $l$ be smallest integers such that $2^k \geq |Q|$, and $2^l \geq |\Sigma| + 2$. Then each state in $Q$ will be represented by $q$ followed by a binary string of length $k$; each symbol in $\Sigma$ will be likewise represented as letter a followed by a binary string of $l$ bits. The head directions that left ($L$) and right ($R$) are included in input tape symbols to justify "+2" term in the definition of $l$.

**Example 10.25.** Consider the turing machine $m = (Q, \Sigma, \delta, S)$ where

$Q = \{S, q, h\}$

$\Sigma = \{\#, b, a\}$, and $\delta$ is given in following table.

| States | Symbol | $\delta$ |
|---|---|---|
| $S$ | $a$ | $(q, \#)$ |
| $S$ | $\#$ | $(h, \#)$ |
| $S$ | $b$ | $(S, R)$ |
| $q$ | $a$ | $(S, a)$ |
| $q$ | $\#$ | $(S, R)$ |
| $q$ | $b$ | $(q, R)$ |

Since there are three states, three symbols in $\Sigma$, we have $k = 2$ and $l = 3$. These are the smallest integers such that $2^k \geq 3$ and $2^l \geq 3 + 2$. The states and symbols are represented as follows :

| State/Symbol | Representation |
|---|---|
| $S$ | $q00$ |
| $q$ | $q01$ |
| $h$ | $q11$ |
| $\#$ | $a000$ |
| $b$ | $a001$ |
| $L$ | $a010$ |
| $R$ | $a011$ |
| $a$ | $a100$ |

Thus representation of $baa\#a$ is

$$\text{"}baa\#a\text{"} = a001a100a100a000a100.$$

The representing "Tm" of the turing machine Tm is the following strings

$$\text{"Tm"} = (q00, a100, q01, a000), (q00, a000, q11, a000), (q00, a001, q00, a011), (q01, a100, q00, a011), (q01, a000, q00, a011), (q01, a001, q01, 011)$$

In the last let us see pictorial representation of universal turing machine

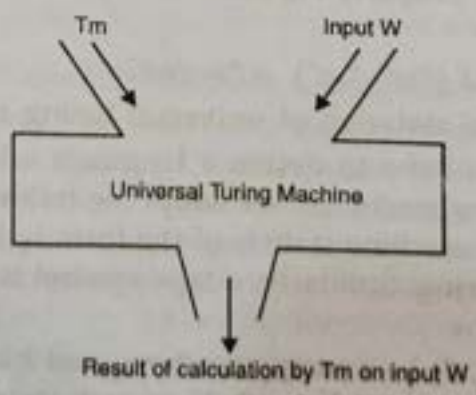


**Fig. 10.35. Result of calculation by Tm on input $w$.**

After discussing of above example we can have following observation :

**Observation 1.** A Turing machine $M$ designed to solve a particular problem $P$ consists, apart from the description of the set of possible states and the set of possible input etc. of mainly the description of the process in some coded form of a sequence of steps required to solve the problem in the form of the

move function δ. In case of universal turing machine, the **process part** involving δ of the turing machine $M$, and the inputs are expressed in the code (that is language) of the Universal turing machine. This code of the process alongwith the code of the input, is stored in the memory of the UTM. And just on the lines of the control unit of a general purpose computer the control unit of UTM, reads the codes for steps, one step at a time, decodes and execute the code for each step, until the code for the final result is stored on the tape of UTM.

**Observation 2.** A Turing machine Tm designed to solve a particular problem $P$ can easily be specified by

(*i*) The initial state say $q_{0\text{Tm}}$ of the Turing machine $M$.

(*ii*) The next-move function $\delta_m$ of Tm, which can be described by the rules of the form : if the current state of Tm is $q_i$ and contents of cell being scanned are $a_j$ then the next state of Tm is $q_k$, the symbol to be written in the current cell is $a_1$ and move $m_f$ of the Tape Head may be : To-left, To-right or None.

Thus, each of these rules for a particular Tm can be specified by **quintupoles** of the form $(q_i, a_i, a_1, m_f)$. And hence the next-move function $\delta_{\text{Tm}}$ for machine Tm is completely specified by the set.

$$\{(q_i, a_j, q_k, a_1, m_f) : q_i, q_j \in Q_{\text{Tm}}; a_j, a_1, \in \Gamma_{\text{Tm}}; m_f \in \{\text{To-left, To-right, None}\}\}$$

**Observation 3.** Next question that arises in the context of the construction of universal Turing machine, is about the number of distinct states in UTM and number of distinct inputs/tape symbols required in the UTM, so that it can solve any solvable problem.

**As UTM should be able to simulate each Turing Machine**, therefore, it may appear that number of distinct states and number of distinct tape symbols in the UTM, should be at least as much as is possible in any Tm, because UTM may be required to accomplish the task of any Tm. However, by proper coding techniques we may use only two symbols to represent set of symbols.

## 10.13. THE HALTING PROBLEM

Let assume that we have given the description of turing machine Tm and an input $w$, when started in the initial configuration $q_0w$, perform a computation that eventually halts? Using an abbreviated way of talking about the problem, we ask wheather Tm applied to $w$, or simply (Tm, $w$) halts or does not halt. The domain of this problem is to be taken as the set of all turing machines and all $w$; that is, we are looking for a single turing machine that, given the description of an arbitrary Tm and $w$, will predict whether or not the computation of Tm applied to $w$ will halt.

We can not find the answer by simulating the action of Tm on $w$, say by performing it on universal turing machine, because there is no limit on the length of the computation. If Tm enters an infinite loop, then no matter how long we wait, we can never be sure that Tm is in fact in a loop. It may be simple case of very long computation. What we need is an algorithm that can determine the correct answer for any Tm and $w$ by performing some analysis on the machine's description and the input. But it is clear that no such algorithm exists.

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In short halting problem is :

To determine for an arbitrary given Turing machine Tm and input $w$,

**Whether Tm will eventually halt on input $w$.**

## 10.14. UNDECIDABILITY/DECIDABILITY

We know that recursive languages are those languages which are accepted by atleast one turing machine and these sets of recursive languages are subclass of regular sets, called the recursive sets.

*"A problem whose language is recursive is said to be decidable"*. Otherwise problem is undecidable. That is, a problem is undecidable if there exist no algorithm that takes as input an instance of the problem and determine whether the answer to that instance is "yes" or "no".

### 10.14.1. Facts about Turing-Decidable and Turing Acceptable Languages

1. If $L$ is turing-decidable then $L$ is turing-acceptable

**Proof.** If $m$ decides $L$, then the turing machine

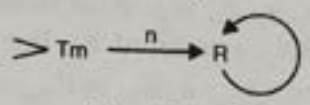


**Fig. 10.36.**

accepts/semidecides $L$.

That if Tm decides $L$, we enter into either state 'Y' or 'n'.

- If Tm halts in 'Y', this machine halts (because if $w \in L$, Tm goes to state $Y$ and halts).

So clearly if $L$ is decided by some turing machine Tm then $L$ is also accepted by some turing machine.

2. If $L$ is turing-decidable then so is $\overline{L}$.

**Proof.** Let Tm be a turing machine such as :

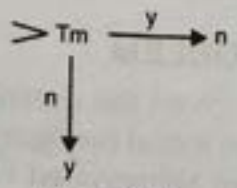


**Fig. 10.37.**

This machine goes to a no ($n$) state when Tm ended up in a yes ($Y$) state and vice-versa.

Therefore, there is also a turing machine which decides $\overline{L}$ (complement of the language $L$).

### 10.14.2. Turing Acceptable

We know that a turing machine Tm accepts $w \in \Sigma^*$ if Tm halts on input $w$, language $L$ is turing-acceptable if there is some turing machine that accepts it.

"Let assume, given a countably infinite number of turing machines $Tm_1$, $Tm_2$, $Tm_3$, ..., $Tm_i$ ... and language $L$ is turing acceptable if at least one of the $Tm_i$ accepts (halts on) any $w \in L$".

But the question is how do we know or discover which $Tm_i$ (if any) accepts $L$. If we start testing $Tm_1$, $Tm_2$, etc. If $Tm_1$ doesn't accept $L$ it will never halt. Yet, how do we know that if we run $Tm_1$ long enough it would not eventually halt on $w$ ? So the issue comes down to : we would like to be able to accurately predict whether a machine will eventually halt on input $w$. If it were possible to predict for any turing machine and any input string whether or not that turing machine would eventually halt on that input then every turing acceptable language would also be turing-decidable. Some authors call **Turing Acceptable Language** as **Recursively Enumerable Language also.**

### 10.14.3. Turing-acceptable Vs Turing-decidable

Consider the language $L_0$ = {"Tm" "$w$" : Turing machine Tm accepts $w$}

Then let us see some points :

(1) $L_0$ is turing-acceptable (by definition) :

There is a machine, $Tm_0$ (a variant of U)

Which accepts input = "Tm" "$w$" such that Tm accepts input $w$.

(2) So if Tm accepts $w \in L$ then $Tm_0$ also accepts "Tm" "$w$" $\in L_0$.

(3) $L_0$ is like infinite dictionary :

$$L_0 = \{\text{"Tm}_1\text{" "}w_{11}\text{", "Tm}_1\text{" "}w_{12}\text{", "Tm}_1\text{", "}w_{13}\text{", ... "Tm}_1\text{" "}w_n\text{", ...}$$
$$\text{"Tm}_2\text{" "}w_{21}\text{", "Tm}_2\text{" "}w_{22}\text{", ...}\}$$

a dictionary which contains answer to questions :

Does arbitrary turing machine Tm accepts input $w$ ?

Because to answer the question we

(*a*) look in the dictionary to see if Tm is paired with input $w$.

(*b*) If Tm does appear with $w$ then Tm accepts $w$, else Tm doesn't accept $w$.

(4) So $L_0$ is formulized version of the halting problem because for any arbitrary Tm and $w$, we could just examine $L_0$ to see if "Tm" "$w$" $\in L_0$. But since $L_0$ is infinite, if we don't find "Tm" "$w$", we will never know if we have searched long enough.

(5) However, if we could answer the question in general is "Tm" "$w$" $\in L_0$.

(6) But $L_0$ is not turing-decidable $\Rightarrow$ the halting problem is not solvable.

(7) For specific Tm, $w$ we could predict whether Tm will halt on input $w$ that is :

(*a*) could see if Tm has any halt state

(*b*) for real simple Tm, $w$.

But there is no completely general method that correctly decides all cases.

Now on the basis of above discussion we can understand that in the case of a Language $L$ which is Turing Acceptable Language but which is not Turing Decidable, there may be a Tm which halts on a large number of strings $W$, where $W \notin L$, but there must be at least one string $W \notin L$ on which Tm does not halt.

Similarly, in the case of a language $L$ which is not Turing Acceptable (and hence cannot be turing decidable), it may happen that there is a Turing machine

Tm which may halt for a large number of inputs W which belong to $L$. But there must be atleast one string $W \in L$ for which $M$ does not halt.

Now we can make following observations :

(*i*) Each Turing Decidable Language $L$ is necessarily Turing Acceptable.

(*ii*) However, there may be languages which are Turing Acceptable but not Turing Decidable.

(*iii*) Further, there may be languages $L \subseteq \Sigma^*$ which may neither be Turing Acceptable and hence not Turing Decidable. For a language $L$ which is not Turing Acceptable, there cannot be any Turing Machine Tm which halts for every string W of $L$.

### 10.14.4. Reducibility

A reduction (reducibility) is a way of converging one problem into another problem in such a way that a solution to the second problem can be used to solve first problem. Such "reducibilities" come up often in every day life, even if we don't usually refer to them this way.

- The problem of understanding in French, the answer to the question : "where is the nearest hotel" reduces to the general problem of understanding French at all.
- Suppose you want to find your way around a new city. You know this would be easy if you have a map. Thus you can reduce the problem of finding your way around the city to the problem of obtaining a map of the city.

Reduction doesn't mean to make smaller, it means to transform or convert a problem $X$ to another problem $Y$ that is at least as hard as $X$. Usually $Y$ is at least as hard as $X$, and so we express a reduction from $X$ to $Y$ as $X \leq Y$.

#### 10.14.4.1. Use of Reducibility to show that a problem is unsolvable.

Reducibility is the primary method for proving that a problem is computationally unsolvable. Here is how it works :

(*a*) Reduce an unsolvable problem $X$ to a more difficult or general problem, $Y$.

(*b*) Then show that if you could solve the **more general** problem, B, you could use the solution to $Y$ to solve the **more specific** problem, $X$, because $Y$ is at least as hard as $X$ ($X \leq Y$).

(*c*) Conclude that since $X$ is known to be unsolvable, and the solution to $Y$ would also solve problem $X$ then $Y$ must be unsolvable too.

(that is a solution to the more general problem $Y$, would also have to cover the more specific problem $X$, which of course, can't be solved)

**Note.** *Reducibility says nothing about solving X or Y alone; only about the solvability of X in the presence or context of Y.*

- For example problem of measuring the area of a rectangle to the problem of increasing its height and width.
- The problem of solving of linear equation reduces to the problem of inverting matrices.

#### 10.14.4.2. Reducibility and Decidability/Solvability

Reducibility also plays an important role in classifying problem as decidable or undecidable, and later on in complexity there too.

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- When $X$ is reducible to $Y$, solving $X$ can't be harder than solving $Y$, because the solution to $Y$ can also be used to solve $X$.
- In terms of the decidability/computability question, then
  (a) If $X$ is reducible to $Y$ and $Y$ is decidable, then $X$ is also decidable.
  (b) Equivalently, if $X$ is undecidable/unsolvable and $X$ reduces to $Y$, then $Y$ is also undecidable. This is the key to proving that problem are undecidable or unsolvable.

**Note.** In this case, where $X$ is undecidable, to reduce $Y$ to $X$ would prove nothing : It would simply show that $Y$ could be decidable if we could decide $X$–which we know we can not.

### 10.14.5. Properties of Recursive and Enumerable Languages

**We know that a Language $L$ is said to be recursive enumerable if there exist a Turing machine which accepts all words in $L$ and rejects for every words in the complement of $L$, that is $\bar{L}$.** We have already studied that number of problems can be solved by reducing one problem to another. This reduction involves combining several turing machines to form a composite machine. The state of the composite turing machine has a component for each individual component machine. Similarly the composite machine has separate tapes for each individual machine.

Given an algorithm, we can allow the composite turing machine to perform one action if the algorithm accepts and another if it does not accept. We could not do this if we were given an arbitrary turing machine than an algorithm, since if the turing machine did not accept, it might run forever, and the composite machine would never initiate the next task.

**Theorem 10.5. The complement of a recursive language is recursive.**

**Proof.** Let $L$ be a recursive language and Tm be turing machine that halts on all inputs and accepts $L$. Let us construct a turing machine Tm' from Tm so that if Tm enters a final state on input $w$, then Tm' halts without accepting. If Tm halts without accepting, Tm' enters a final state.

Since one of these two events occurs, Tm' is an algorithm. So clearly T(Tm') is the component of $L$ and thus the complement of $L$ is recursive language. Following figure shows construction of Tm' from Tm.

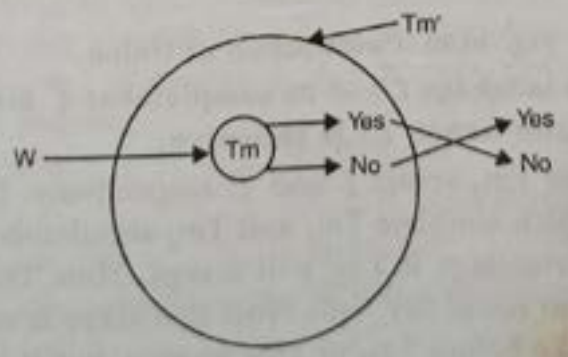


**Fig. 10.38.**

**Theorem 10.6. The union of two recursive languages is recursive.**

**Proof.** Let $L_1$ and $L_2$ be two recursive languages accepted by turing machines $Tm_1$ and $Tm_2$. We construct a turing machine Tm, which first simulate $Tm_1$. If $Tm_1$ accepts, then Tm accepts. If $Tm_1$ rejects then Tm simulates $Tm_2$ and accepts if and only if $Tm_2$ accepts. Since both $Tm_1$ and $Tm_2$ are algorithm so Tm is guaranteed to halt. Clearly Tm accepts $L_1 \cup L_2$.

So $L_1 \cup L_2$ is also recursive since there exist a turing machine Tm for it.

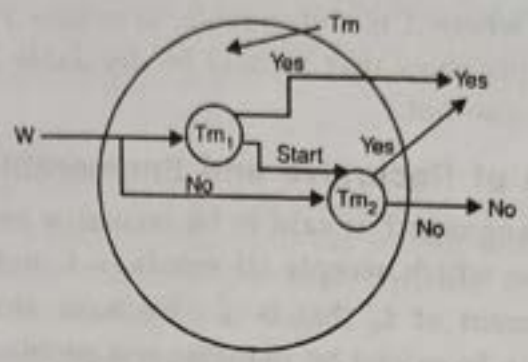


**Fig. 10.39. Construction for union.**

**Theorem 10.7. The union of two recursively enumerable languages is recursively enumerable.**

**Proof.** We have studied earlier that enumerating turing machine have one input tape and one output tape.

Let $L_1$ and $L_2$ are recursively enumerable languages and their enumerative turing machines are $Tm_1$ and $Tm_2$ respectively. Let us design a turing machine Tm which can simulate $Tm_1$ and $Tm_2$ simultaneously on separate tape. If either accepts, then Tm accepts as follows :

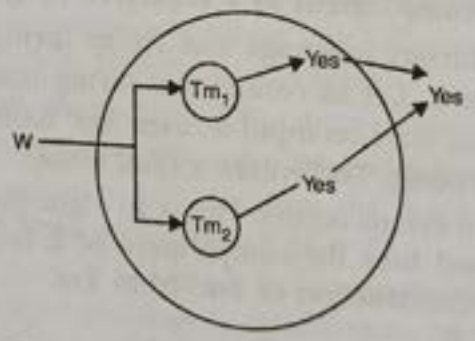


**Fig. 10.40. Construction of Union.**

**Theorem 10.8. If a language $L$ and its complement $\overline{L}$ are both recursively enumerable, then $L$ (and hence $\overline{L}$) is recursive.**

**Proof.** Let $Tm_1$ and $Tm_2$ accept $L$ and $\overline{L}$ respectively. Let us construct a turing machine Tm which simulate $Tm_1$ and $Tm_2$ simultaneously. Tm accepts $w$ if $Tm_1$ accepts and rejects $w$ if $Tm_2$ will accept. Thus Tm will always say either "Yes" or "No", but never say both. Note that there is not a priority limit on how long it may take before $Tm_1$ or $Tm_2$ accepts, but it is certain that one or the other will do so. Since Tm is algorithm that accepts $L$, if follows that $L$ is recursive.

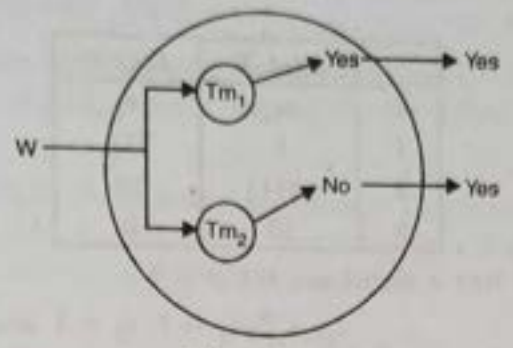


Fig. 10.41.

**Theorem 10.9. If $L$ is a recursive language than $\Sigma^* - L$ is recursive.**

**Proof.** The required Turing machine Tm-complement can be represented by following diagram.

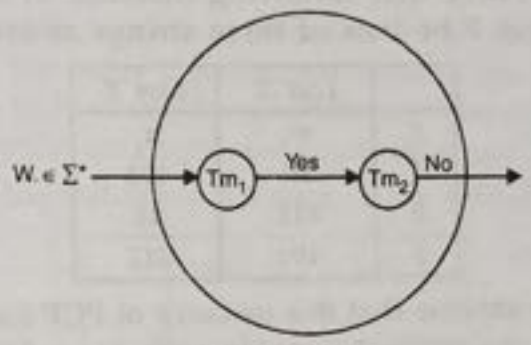


Fig. 10.42.

The machine Tm-complement functions as follows : when a string $W \in \Sigma^*$ is given an input to Tm-complement, its control passes the string to $Tm_1$ as input to $Tm_1$. As $Tm_1$ decides the language $L$, therefore, for $W \in L$ after a finite number of moves, $Tm_1$ outputs "Yes", which is the given as input to $Tm_2$, which in turn returns "No".

Similarly for $W \notin L$, $Tm_2$ returns "Yes". Hence there exist a turing machine Tm-complement for $\Sigma^* - L$. So it is turing-decidable, that is recursive.

### 10.14.6. The Post Correspondence Problem

The post correspondence problem is another undecidable problem that turns out to be a very helpful tool for proving problems in logic or in formal language theory to be undecidable.

Let $\Sigma$ be an alphabet with at least two letters. An instance of the post corresponding problem (for short PCP) is given by two sequences $U = (u_1, u_2, ..., u_m)$ and $V = (v_1, v_2, ..., v_m)$ of strings $u_i, v_i \in \Sigma^*$. The problem is to find whether there is a (finite) sequence

$(i_1, i_2, ..., i_p)$, with $i_j \in \{1, 2, ..., m\}$ for

$$i_j = 1, 2, ... p \text{ so that, } P \geq 1$$

$$u_{i_1}, u_{i_2} u_{i_3} ... u_{i_p} = v_{i_1} v_{i_2} ... v_{i_p}.$$

Equivalently, on instance of the PCP is a sequence of pairs

$$\begin{pmatrix} \mu_1 \\ \vdots \\ v_1 \end{pmatrix}, ..., \begin{pmatrix} \mu_m \\ \vdots \\ v_m \end{pmatrix}$$

The sequence $i_1, i_2 ... i_p$ is said to be solution to this instance of PCP.

**Example 10.26. Let $\Sigma = \{0, 1\}$. Let $X$ and $Y$ be lists of three strings each, defined as follows :**

| | List X | List Y |
|---|---|---|
| $i$ | $w_i$ | $x_i$ |
| 1 | 1 | 111 |
| 2 | 10111 | 10 |
| 3 | 10 | 0 |

In this case PCP has a solution, let $P = 4$

$$i_1 = 2,\ i_2 = 1,\ i_3 = 1 \text{ and } i_4 = 3 \text{ then}$$

$$w_2\, w_1\, w_1\, w_3 = x_2\, x_1\, x_1\, x_3$$
$$= 101111110$$

which is the solution of instance of PCP.

**Example 10.27. Prove that following instance of PCP has no solution over $\Sigma = \{0, 1\}$, $X$ and $Y$ be lists of three strings as follows :**

| | List X | List Y |
|---|---|---|
| $i$ | $w_i$ | $x_i$ |
| 1 | 10 | 101 |
| 2 | 011 | 11 |
| 3 | 101 | 011 |

**Solution.** Let us assume that this instance of PCP has solution $i_1, i_2, \ldots, i_p$. Clearly $i_1 = 1$ since no string beginning with $w_2 = 011$ can equal a string beginning with $x_2 = 11$; no string beginning with $w_3 = 101$ can equal a string beginning with $x_3 = 011$.

We write the string from list $X$ the corresponding string from $Y$. So for we have

10
101

The next selection from $X$ must begin with a 1. Thus $i_2 = 1$ or $i_2 = 3$. But $i_2 = 1$ will not do, since no string beginning with $w_1\, w_1 = 1010$ can equal a string beginning with $x_1\, x_1 = 101101$. with $i_2 = 3$ we have

10101
101011

Since the string from list $Y$ again exceeds the string from list $X$ by the single symbol 1, a similar argument shows that $i_3 = i_4 = \ldots = 3$. Thus there is only one sequence of choices that generates compatible strings, and for this sequence string $Y$ is always one character longer. Thus this instance of PCP has no solution.

**Example 10.28. Find the solution of following instance of PCP**

$$\begin{pmatrix} abab \\ ababaaa \end{pmatrix}, \begin{pmatrix} aaabbb \\ bb \end{pmatrix}, \begin{pmatrix} aab \\ baab \end{pmatrix}, \begin{pmatrix} ba \\ baa \end{pmatrix}, \begin{pmatrix} ab \\ ba \end{pmatrix}, \begin{pmatrix} aa \\ a \end{pmatrix}.$$

**Solution.** The solution for this instance is

$i_1 = 1,\ i_2 = 2,\ i_3 = 3,\ i_4 = 4,\ i_5 = 5,\ i_6 = 5,\ i_7 = 6$

$$ababaaabbbaabbaababaa = ababaaabbbaabbaababaa.$$

### 10.14.6.1. The "Modified" PCP

It is easy to reduce undecidable language $L$ to PCP if we first introduce an intermediate version of PCP, which we call the modified post's correspondence problem, or MPCP.

In the modified PCP, there is the additional requirement on a solution that the first pair on the list $X$ and list $Y$ must be the first pair in the solution. More formally, an instance of MPCP is two lists

$$X = w_1, w_2, \ldots, w_k$$

and

$$Y = x_1, x_2, x_3, \ldots, x_k,$$

and a solution is a list of 0 or more integers $i_1, i_2, i_3, \ldots i_p$ such that

$$w_1, w_{i_1}, w_{i_2}, \ldots, w_{i_m} = x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_p}$$

Notice that the pair $(w_1, x_1)$ is forced to be at the beginning of the two strings, even though the index 1 is not mentioned at the front of the list that is solution. Also unlike PCP, where the solution has to have at least one integer on the solution list, in MPCP, the empty, the empty list could be a solution if $w_1 = x_1$.

Let us consider the following instance of the MPCP.

| | *List X* | *List Y* |
|---|---|---|
| $i$ | $w_i$ | $x_i$ |
| 1 | 1 | 111 |
| 2 | 10111 | 10 |
| 3 | 10 | 0 |

Then as an instance of MPCP it has no solution. In proof, observe that any partial solution has to begin with index 1, so the two strings of a solution would begin :

$$X : 1 \ldots$$

$$Y : 111 \ldots$$

The next integer could be 2 or 3, since both $w_2$ and $w_3$ begin with 10 and thus would produce a mismatch at the third position. Thus the next index would have to be 1, yielding :

$$X : 11$$

$$Y : 111111$$

We can argue this way indefinitely. Only another 1 in the solution can avoid a mismatch, but if we can only pick index 1, the string remains three times as long as the $X$ string, and the two strings can never become equal.

## 10.15. PUSHDOWN AUTOMATA WITH TWO STACKS

We define the machine as a six tuple where

$$M = (Q, \Sigma, \Gamma, \delta, S, F)$$

where $K$ : set of states

$\Sigma$ : alphabet of input tape

$\Gamma$ : alphabet of the stacks (assume same alphabet for both the stacks)

$S$ : the starting state ($S \in Q$)

$F$ : set of accepting states ($F \subseteq Q$) and

$\delta$ : the transition relation, a finite subset of $(Q \times \Sigma \times \Gamma^* \times \Gamma^*) \times (Q \times \Gamma^* \times \Gamma^*)$ or we can say that $\delta$ is transition function which maps

$$(Q \times \Sigma \times \Gamma^* \times \Gamma^*) \longrightarrow (Q \times \Gamma^* \times \Gamma^*)$$

Intuitively if $((q, a, x, y), (q_1, x_1, y_1)) \in \delta$ then $M$, whenever it is in state $q$ with $x$ at top of stack$_1$ and $y$ at the top of stack$_2$ (here two stacks are said to be as stack$_1$ and stack$_2$), may read '$a$' from input tape, replace $x$ by $x_1$ on stack$_1$ and replace $y$ with $y_1$ on stack$_2$ and then enter state $q_1$. Such a pair $((q, a, x, y), (q_1, x_1, y_1))$ is called a transition of $M$ where :

$q$ : current state

$a$ : input on the tape yet to read

$x$ : top of stack$_1$ (can be $\in$)

$y$ : top of stack$_2$ (can be $\in$)

$q_1$ : new state

$x_1$ : new top of the stack$_1$ (can be $\in$)

$y_1$ : new top of the stack$_2$ (can be $\in$)

Since to push a symbol is to add it to the top of the stack; to pop a symbol is to remove it from the top of the stack, we must define these functions for our machine.

**Note.** Any time the input string or stack top is represented by $\in$, it denotes that it is not to be consulted for the transition. For example, the transition :

$((P, \in, x, b), (P_1, \in, b))$ Pops $x$ from stack$_1$

$((P, x, a, \in), (P_1, a, x))$ Push $x$ on the stack$_2$

Pushes and Pops are permitted between the two stacks as well :

$((q, \in, a\ \eta, \in), (q_1, r, a))$ where $\eta \in \Gamma$ and is the next symbol on the stack. Here '$a$' moves from stack$_1$ to stack$_2$.

We define a configuration of $M$ to be member of $Q \times \Sigma^* \times \Gamma^* \times \Gamma^*$ where if $(q, x, a_1, a_2)$ and $(q, y, \alpha_1, \alpha_2)$ are configuration of $M$, we say that $(q, x, a_1, a_2)$ yields in one step $(q_1, y, \alpha_1, \alpha_2)$ (notation : $(q_1, x, a_1, a_2) \vdash_m (q_1, y, \alpha_1, \alpha_2)$ if there is a transition :

$((q, x, b_1, b_2), (q_1, P_1, P_2)) \in \delta$ such that

$$x = ay$$
$$a_1 = b_1\eta_1$$
$$a_2 = b_2\eta_2$$
$$\alpha_1 = b_1\eta_1$$
$$\alpha_2 = b_2\eta_2 \qquad \text{(for some } \eta_1\eta_2 \in \Gamma^*\text{)}.$$

We say that $M$ accepts a string $W \in \Sigma^*$ if and only if $(S, W, \in, \in) \vdash_m^* (q, \in, \in, \in)$ for some $q \in F$. $M$ accepts string $W$ if and only if there is a

sequence of configuration $C_0, C_1, ..., C_n$ $(n > 0)$ such that $C_0 \vdash_m C_1 \vdash_m ... \vdash_m C_n$, $C_0 = (S, W, \epsilon, \epsilon)$ and $C_n = (P, \epsilon, \epsilon, \epsilon)$ for some $P \in F$. Any sequence of configurations $C_0, C_1, ..., C_n$ such that $C_i \vdash_m C_{i+1}$ for $i = 0, ..., n-1$, will be called a computation by $M$;

The Language accepted by $M$, denoted $L(M)$, is the set of all strings accepted by $M$.

### 10.15.1. Equivalence of Turing Machine and Two Stack Pushdown Automata

Our Two stack PDA (let us refer as TSPDA) is provable equivalent to a Tm. Using the two stack to represent a Tm's ability to move left and right along a tape, we can POP/COPY the top of one stack and push it onto the top of other. **We can use the first stack of our PDA to represent the portion of the Tm's tape to the right of the Tm's read/write Head, and the second stack of our PDA to represent data on the left side of Tm's read/write head.**

To represent the boundaries of our stacks, we initially push a special character onto the bottom of both stacks; this could be any character so long as it is not a member of the input or stack alphabets. For example, we will use \$ to represent the bottom of the stack. After these pushes are made, we can set up the stacks so that they can simulate the Tm tape. Symbol by symbol, the input tape is pushed to the second stack. When the input tape is exhausted, that is to say it is empty, the input data is stored on the second stack in reverse order (since we push each character on top of the $stack_2$, that had preceded it on the input tape). To correct the reversal problem, we then push each symbol (again, one at a time) to first stack. Now, since the top of the first stack holds the character that had been at the beginning of the input tape, our corner state is $(P, \epsilon, W\$, \$)$ where $W$ is the input string from the input tape. Now we can begin passing the string. Here is how the movement for our TSPDA is choreographed :

(*a*) To simulate moving right on Tm tape : The TSPDA POPS the top of the first stack and pushes it on the top of second stack.

(*b*) To simulate moving left on a TM tape : POP the top of the second stack and push it onto the top of first stack.

(*c*) To simulate the unbounded right-hand side of a TM tape ; whenever TSPDA tries to POP a character off of the empty first stack—meaning the machine is in a state $(q, \epsilon, \$, \alpha\$)$ and the second stack is left untouched and a space is pushed onto the top of the first stack.

(*d*) To simulate the bounded left-hand side of a TM tape : whenever our TSPDA tries to POP a character off of the empty second stack—meaning the machine is in a state $(q, \epsilon, \alpha\$, \$)$, the second stack is again not modified and the top of the first stack is overwritten with a space.

(*e*) To simulate how to write to the tape : whenever you want to write a character to the simulated tape, simply push it onto a stack.

**This shows our TSPDA to be atleast as powerful as a TM, our machine occupies the same space in the Chomsky Language Hierachy that TM does.**

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**We can also show that our TSPDA is no more powerful than a TM. Because a TM can only solve every solvable problem, only an oracle machine could be more powerful than a Tm. Since the TSPDA is not more powerful and no less powerful than a Tm, we can say it is provably equivalent.**

### 10.15.2. Two Stack Push-down Automata as an Algorithm

It is reasonable to consider our TSPDA an algorithm, as we have proven above the TSPDA is equivalent to a TM. As stated by Lewis and Papadimitriou :

"A finite automation always halts on a final or a non-final state. In this sense it is a useful computational device, an algorithm from which we can reliably obtain answers as to whether an input belongs in the accepted language".

Since our TSPDA is equivalent to a TM and TMs are algorithm, we can consider our TSPDA to be an algorithm.

**Example 10.29. Construct a two stack pushdown automata, which creates $ww^R$ where $w \in (a, b)^*$.**

**Solution.** The solution of this problem, we have already seen by Turing machine.

Here input tape will have $w \in (a, b)^*$ and we are going to design a TSPDA which will left $ww^R$ on stack$_2$, this can be done in three steps :

(*i*) Read input, pushing each character to both the stacks.

(*ii*) POP everything from stack$_1$ to stack$_2$ until stack$_1$ is empty.

(*iii*) You now have $ww^R$ on stack$_2$.

Now let us write transition relation as follows :

(*i*) Let us put each character of string $W \in (a, b)^*$, to both the stack. This can be done by following two moves. Let us assume that initial state of TSPDA is $q_0$.

$$((q_0, a, \in, \in), (q_0, a, a)) \quad ...(1)$$

$$((q_0, b, \in, \in), (q_0, b, b)) \quad ...(2)$$

transition relations one and two can push '*a*' and '*b*' on the stacks respectively.

(*ii*) Now let us write the transition relations to POP W from stack$_1$ to stack$_2$ as follows :

$$(q_0, \in, a, \in), (q_0, \in, a) \quad ...(3)$$

$$(q_0, \in, b, \in), (q_0, \in, b) \quad ...(4)$$

moves '3' and '4' will POP '*a*' and '*b*' respectively from stack top of stack$_1$ to stack$_2$.

(*iii*) $(q_0, \in, \in, a), (f, \in, ww^R)$ ...(5)

$(q_0, \in, \in, b), (f, \in, ww^R)$ ...(6)

Either move '5' or move '6' will ensure that input tape is empty, stack$_1$ is also empty and stack$_2$ contains $ww^R$, state is changed from $q_0$ to $f$ which is accepted state.

So here TSPDA is $M = (Q, \Sigma, \Gamma, \delta, S, F)$

$Q = \{q_0, f\}$

$S = \{q_0\}$

$F = \{f\}$

$\Sigma = \{a, b\}$

$\Gamma = \{a, b, \#\}$

and $\alpha$ will have move 1, 2, 3, 4, 5 and 6.

**Example 10.30. Design a TSPDA for $L = \{a^{n^2} : n \geq 0\}$.**

**Solution.** Let TSPDA be $M = (Q, \Sigma, \Gamma, \delta, S, F)$

where $Q = \{q_0, q_1, q_2, f\}$

$\Sigma = \{a\}$

$\Gamma = \{\$, A, B, a, C, b, d, t, J, \in\}$

$S = \{q_0\}$

$F = \{f\}$

$\delta$ is defined as follows :

(*i*) Load the input string into stack$_2$

(*ii*) Move stack$_2$ to stack$_1$ using the following rules :

$((q_0, \in, \$, a), (q_0, A, \in))$ (\$ is symbol to show the stack bottom)

$((q_0, \in, A, a), (q_0, B, \in))$

$((q_0, \in, B, a), (q_0, A, \in))$

(*iii*) Once you reach a state : $(q_0, \in, X, \$)$ where $X$ is either $A$ or $B$, you must then use the following rule to determine the acceptability of the string or the next state.

$((q_0, \in, A, \$), (q_0, C, \in))$

$((q_0, \in, C, \$), (q_0, \in, b))$

$((q_0, \in, A, b), (q_0, \in, d))$

$((q_0, \in, \$, d), (f, \in, t)) \rightarrow$ This is final acceptable state.

$((q_0, \in, B, d), (q_2, \in, t)) \rightarrow$ This indicates an unacceptable state

$((q_0, \in, B, \$), (q_0, J, J)) \rightarrow$ Signal empty stack$_2$ to change the state

$((q_0, \in, J, J), (q_1, \in, \in) \rightarrow$ Changes states and removes $J$ off both stacks.

Here $q_0$ : is state for moving off stack$_2$

$q_1$ : is for moving off stack$_1$

$f$ : for finish

$q_2$ : failed state

(*iv*) Once the machine has changed to state $q_1$, you must then transfer the stack$_1$ to stack$_2$ using the following rules :

$((q_1, \in, B, \$), (q_1, \in, \in))$

$((q_1, \in, A, \$), (q_1, \in, C))$

$((q_1, \in, \$, C), (f, \in, \in))$

$((q_1, \in, B, C), (q_1, C, \in)$

$((q_1, \in, C, \$), (q_1, \in, a))$

$((q_1, \in, A, a), (q_1, \in, a))$

$((q_1, \in, B, a), (q_1, \in, \in))$

$((q_1, \in, \$, a), (q_1, f, f))$

$((q_1, \in, f, f), (q_0, \in, \in))$

(*v*) Repeat steps (*ii*), (*iii*) and (*iv*).

## EXERCISE

1. Let Tm be the Turing machine $(Q, \Sigma, \delta, S)$, where

$$Q = \{q_0, q_1, q_2\}$$
$$\Sigma = \{a, \#\} \text{ and}$$
$$S = q_0 \text{ and } \delta \text{ is as follows :}$$

| $q$ | $\sigma$ | $\delta(q, \sigma)$ |
|---|---|---|
| $q_0$ | $a$ | $(q_1, L)$ |
| $q_0$ | # | $(q_0, \#)$ |
| $q_1$ | $a$ | $(q_2, \#)$ |
| $q_1$ | # | $(h, \#)$ |
| $q_2$ | $a$ | $(q_2, a)$ |
| $q_2$ | # | $(q_0, L)$ |

Describe the operation of Tm when started as $(q_0, \# \underline{aaaaa})$

2. Describe what the following Turing machine does.

$$>R \xrightarrow{\sigma_1 \neq \#} R \xrightarrow{\sigma_2 \neq \#} R_{\#}\sigma_1 R_{\#}\sigma_2$$

**Fig. 10.43.**

3. (*i*) Build a Turing machine that accepts the language

$$L = \{a^n b^{n+1}\}$$

(*ii*) Build a Turing machine that accepts the language

$$L = \{a^n b^{2n}\}$$

4. Build a Turing machine that accepts the language of all words that contain the substring *bbb*.

5. Build a turing machine that accepts the language of all words that do not contain the substring *bbb*.

6. Build a Turing machine that accepts the language ODD PALINDROME.

7. Build a Turing Machine that accepts all strings with more *a*'s than *b*'s, the language MORE.

8. Construct the Turing machine for the following languages :

(*i*) $L = L(aba^*b)$

(*ii*) $L = \{w : |w| \text{ is even}\}$

(iii) $L = \{w : |w| \text{ is a multiple of } 3\}$

(iv) $L = \{a^n b^m a^{n+m} : n \geq 0, m \geq 1\}$

(v) $L = \{a^n b^n a^n b^n : n \geq 0\}$

(vi) $L = \{a^n b^{2n} : n \geq 1\}$.

9. Prove that following functions are computable functions :

(i) $f(x) = 3x$

(ii) $f(a, b) = \begin{cases} a - b & a > b \\ 0 & a \leq b \end{cases}$

(iii) $f(a, b) = 2a + 3b$

(iv) $f(a) = a \bmod 5$

10. Construct the Turing machine for the language :

$$L = \{0^n 1^n / n \geq 1\}.$$

11. Prove that PCP with $\{(01, 011), (1, 10), (1, 11)\}$ has no solution.

12. Does the PCP with $x = (b^3, ab^2)$ and $y = (b^3, bab^3)$ have a solution.

13. Find atleast three solutions to PCP defined by the dominoes :

$$\begin{pmatrix} 1 \\ 111 \end{pmatrix}, \begin{pmatrix} 10 \\ 0 \end{pmatrix}, \begin{pmatrix} 10111 \\ 10 \end{pmatrix}$$

14. Design a TSPDA for

$$L = \{ww : w \in \{a, b\}^*\}.$$

15. Implement a binary incrementer on turing machine. Use general terms to describe action taken. Would your solution be deterministic or non-deterministic ? Why ?

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# 11 CHAPTER

# Recursive Function Theory

**INSIDE THIS CHAPTER**

**11.1.** Introduction **11.2.** Partial, Total Recursive and Primitive Recursive Functions **11.3.** Godel Numbering.

## 11.1. INTRODUCTION

In the previous units, we have discussed the automata or machine models of the computational phenomenon. The automata approach to computation is **Operational** in nature, *i.e.* automata approach is concerned with the computational aspect of '**how** the computation is to be performed'

In this unit, we will be concerned with Recursive Function Theory, which is a **functional or declarative approach to computation. Under this approach, computation is described in terms of 'what is to be accomplished' instead of 'how to accomplish'.**

Each computational theory (rather each theory about any other phenomenon also) starts with some *assumptions*, for example, *about basic (undefined) concepts, operational capabilities and a set of statements, called axioms and postulates, which are assumed to be fundamentally true* (*i.e.* assumed to be true without any argument). In Automata Theory, the concepts like 'state' 'initial state', 'final state' and 'input' etc. are assumed to be understood, without any elaboration. Further, *the capabilities of an automata* to accept an input from the environment; to change its state on some, or even on no input; to give signal about acceptability/unacceptability of a string; are assumed.

In Recursive Function Theory, *to begin with*, it is **assumed** that **three types of functions** and **three structuring rules** (that is combination, composition and primitive recursion) for constructing more complex functions out of the already constructed or assumed to be constructible functions **are** so simple that ***our ability*** to construct machines to realize these functions and the structuring rules is taken as *acceptable without any argument*. **The functions, obtained by applying a finite sequence of the structuring rules to the initial functions, are called Primitive Recursive Functions.** However, with these simple functions and elementary structuring rules, though it is possible to construct very complex functions yet, even some simple functions like **division** are not constructible by the above mechanism.



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## 11.2. PARTIAL, TOTAL RECURSIVE AND PRIMITIVE RECURSIVE FUNCTIONS

We know that **partial recursive functions** are Turing Computable, it means that there exist a turing machine Tm for every partial recursive function.

All those partial recursive functions for which Turing machine halts are called **total recursive functions.** From the definition of both it is clear that total recursive function is the subset of partial recursive function.

**Primitive Recursive function** is another subclass of partial recursive function and can be defined using initial functions. Initial functions are defined over a set of natural numbers $N = \{0, 1, 2, ...\}$ and some alphabet of symbols.

### 11.2.1. Initial Functions For Natural Numbers

It is clear that all elementary functions are all functions of natural numbers. hence, they may take zero as input, but not a negative number, and not any rational or irrational fraction.

Let $N = \{0, 1, 2, ...\}$ be a set of natural numbers. We have three initial functions over $N$ defined below :

1. **Zero Function** is denoted by $Z$ and defined as follows

$$Z(n) = 0 \quad \text{for} \quad \forall\, n \in N$$

Clearly the zero function returns zero regardless of its argument.

2. **Successor Function(s)** is defined as follows :

$$S(n) = n + 1 \quad \text{for } \forall\, n \in N$$

Clearly successor function takes one argument and returns the succeeding number.

For example : $S(2) = 2 + 1 = 3$ where $2, 3 \in N$

$$\left.\begin{aligned} Z(0) &= 0 \\ Z(2) &= 0 \end{aligned}\right\} \quad \text{where } 0, 2 \in N$$

3. **Projection Function ($P_i^n$)** is defined as follows :

$$P_i^n(a_1, a_2, ..., a_n) = a_i, \text{ where } a_i \in N \text{ for } i = 1, 2, ..., n \text{ and } i \le n.$$

Here projection function takes $n$ arguments and returns their $i^{th}$ argument. When it takes one argument it returns its argument as its value.

### 11.2.2. Initial Functions for Symbols

We know that set of alphabet is represented by $\Sigma$. Two initial functions are defined over $\Sigma$ :

1. **Null Function** is defined as follows :

$$\text{NULL}(n) = \epsilon \quad \text{for } \forall\, n \in \Sigma^*$$

For example : If $\Sigma = \{a, b\}$

then $\text{NULL}(a) = \epsilon$

and $\text{NULL}(b) = \epsilon$

2. **Concatenation Function is defined as follows :**

$$\text{CONCATE } W(W_1) = WW_1 \text{ for } W, W_1 \in \Sigma^* \text{ and } |\, W_1 \,| \ge 0.$$

For example :

CONCATE K(IET) = KIET

CONCATE $a$(CONCATE $b$ (NULL ($abc$))

= CONCATE $a$(CONCATE $b(\in)$)

= CONCATE $a(b)$

= $ab$

### 11.2.3. Composition of Functions

We can define a new function by the combination of two or more functions. Such defined functions are called composition function.

For example :

$$S(Z(a)) = S(t) = 1$$

and $S(S(Z(n))) = 2$ and so on.

### 11.2.4. Recursion

We know that recursion is widely used term in computer science. "A function which calls itself directly or indirectly and terminates after finite number of steps is known as recursive function."

In recursive functions, terminating point is also known as base point. A recursive function will be good if it has terminating point and it call to itself should be nearer to its terminating point. The terminating point is also known as base point.

Let $g$ be a function of $n + 1$ arguments, $g'$ be a function of $n$ arguments, and $h$ be a function of $m$ arguments where $m \le n + 2$. We define the function $g$ recursively using the $g'$ and $h$ function as follows :

$$\text{If } (x_1, x_2, \ldots, x_n, 0) = g'(y_1, y_2, \ldots, y_n) \text{ then } g(x_1, x_2, \ldots, x_n, x_{n+1} + 1) = h(x_1, x_2, \ldots, x_n, x_{n+1}, g(x_1, x_2, \ldots, x_n, x_{n+1})),$$

where $g'$ and $h$ are defined functions.

**Example 11.1. Define $m!$ recursively.**

**Solution.** Let $g(m) = m! = m * (m - 1) * (m - 2) \ldots * 1$

We know that $0! = 1! = 1$,

So $g(0) = 1$(it is a terminating point)

$$h(x, y) = x * y$$

$$g(n + 1) = h(n + 1, g(n)) \quad \text{(Indirect call)}$$

**Example 11.2. Define addition of two positive integers $m$ and $n$ recursively.**

**Solution.**

Let $f$ is the addition function of two positive integers, so

$$f(m, n) = m + n$$

We define $f$ to increment the argument $m$, $n$ times one at a time.

So,

$$f(m, 0) = m \quad \text{(it is terminating point)}$$

$$f(m, n + 1) = f(m, n) + 1 \quad \text{(direct call)}$$

$$f(5, 2) = f(5, 1 + 1)$$

$$= f(5, 0 + 1) + 1$$
$$= 5 + 1 + 1$$
$$= 7$$

### 11.2.5. Primitive Recursive Function

A function is primitive if it follows the condition :

(*i*) It is an initial function, or

(*ii*) It is obtained from recursion or composition of initial functions.

Most popular total functions are primitive recursive functions; however some total functions are not. For example, the ackerman function is total function but not primitive recursive.

**Example 11.3. Show that addition function of two positive integers is primitive recursive.**

**Solution.** In example 11.2, we have defined the addition function $f$ for $m$ and $n$ as $f(m, 0) = m$, $f(m, n + 1) = f(m, n) + 1$. We define $f$ using the initial functions as follows :

$$f(m, 0) = m = P_1^1(m),$$

and $$f(m, n + 1) = f(m, n) + 1 = S(f(m, n))$$

For example, let us consider $f(7, 2)$

$$\begin{aligned} f(7, 2) &= S(f(7, 1)) \\ &= f(7, 1) + 1 \\ &= S(f(7, 0)) + 1 \\ &= f(7, 0) + 1 + 1 \\ &= 7 + 1 + 1 \\ &= 9 \end{aligned}$$

Hence, $f$ is a primitive recursive function.

**Theorem 11.1.** Every primitive recursive function is turing computable.

**Proof.** We know that primitive functions are defined using initial functions with sometimes applying composition and recursion. So if it is possible to design turing machine (Tm) for every initial function, their composition and recursion then we can say that every primitive recursive function is turing computable. Let us see following cases :

**Case 1.** Turing machine for zero function $Z(x) = 0$, for $x \in N$. The Tm for zero function should work as follows :

$$\# I I I I \# \xrightarrow{\text{Tm}} \#\#$$

Here $I I I I$ means that $x$ is 4. So, here we are supposed to design an eraser turing machine :

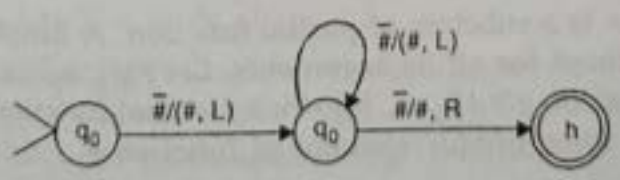


**Fig. 11.1. Eraser Turing machine for zero function.**

**Case 2.** Turing Machine for **successor function** $S(n) = n + 1$, for $n \in N$.
The Tm for successor function will work as follows :

$$\#\,I\,I\,I\,\# \xrightarrow{\text{Tm}} \#\,I\,I\,I\,I\,\#$$

input $\Rightarrow n = 3$ output $\Rightarrow n + 1 = 4$

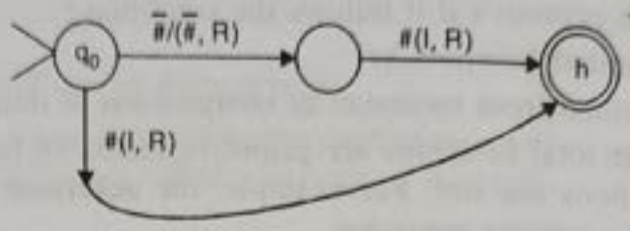


**Fig. 11.2. Tm for S(n) = n + 1.**

**Case 3.** Turing machine for **Projection function**

$$P_i^n(x_1, x_2, \ldots, x_n) = x_i \text{ for } x_1, x_2, \ldots, x_n \in N.$$

So we can easily design a Turing machine Tm for projection function :

$$\#\,x_1\,\#\,x_2\,\#\ldots\#\,x_i\,\#\ldots\#\,x_n\,\# \xrightarrow{\text{Tm}} \#\,x_i\,\#$$

**Case 4.** We can also design Turing machine for **composition function** by using the concept of combination of Turing machine.

**Case 5. For recursion,** we can design a Tm using multiple calls to a simple Tm designated for initial function with some fixed termination point.

**Case 6.** We can also combine our both cases 4 and 5 into one to design a Turing machine Tm for compsition and recursion together.

So, our theorem statement is true.

### 11.2.6. Minimalization

If $g$ is the function of $(n \geq 1)$ arguments, then minimalization of $G$ is defined as follows :

$$G_y(x, y) = \begin{cases} Y, & \text{the least } Y \text{ such that } G(x, y) = 0 \\ 0, & \text{otherwise} \end{cases}$$

For example, consider $g(x, y) = x + y - 2$,
So putting $g(x, y) = 0$, we get
$y = 2 - x$ if $x \leq 2$ is the result of minimalization.
So, minimalization,

$$(g(x, y)) = 2 - x \text{ for } x \leq 2$$
$$= 0 \text{ for } n > 2$$

### 11.2.7. Total Recursive Function

A total function is a subclass of partial function. A function is called total function if it is defined for all its arguments. Let $f(a_1, a_2, \ldots, a_n)$ be a function and defined on function $g(b_1, b_2, \ldots, b_m)$, then $f$ is total function if every element of $f$ is assigned to some unique element of function $g$.

**Example 11.4. Addition of two positive integer is a total function.**

**Solution.** Let $f(m, n) = m + n$ is the addition function from $N \times N$ to $N$ of two positive integers $m$ and $n$. This function can be defined recursively as

$$f(m, 0) = 1, \quad f(m, n+1) = f(m, n) + 1.$$

Since, $f$ is defined for all values of $m, n \in N$. So $f$ is a total function.

**Example 11.5. Multiplication of two positive integers is a total function.**

**Solution.** Let $g(m, n)$ is multiplication function from $N \times N$ to $N$ of two positive integers $m$ and $n$, defined recursively as : $g(m, 0) = 0$, $g(m, n+1) = m + g(m, n)$. Since, $g$ is defined for all $m, n \in N$. So $g$ is a total recursive function.

### 11.2.8. Partial Recursive Function

A function is called partial recursive if it is defined for some of its arguments. Let $f(a_1, a_2, ..., a_n)$ is a function and defined on function $g(b_1, b_2, ..., b_m)$, then $f$ is partial function if some elements of $f$ is assigned to almost one element of function $g$.

A partial function is recursive if

(i) It is an initial function over $N$, or

(ii) It is obtained by applying recursion or composition or minimalization on initial function over $N$.

## 11.3. GODEL NUMBERING

We have already seen the primitive recursive functions. These functions were taking natural numbers as arguments and producing natural numbers as output. However those functions can be extended for strings also. For this, we require to convert strings into unique numbers. Such an encoding system is suggested by Kurt Godel (1906–1978).

"Godelization is an encoding technique which encodes a string as a number". This is known as **Godel numbering.**

Let $\Sigma = \{a_1, a_2, ..., a_{r-1}\}$ be any alphabet. Any string $W \in \Sigma^*$ can be represented by an integer in radix $r$. Let us define a function gnum from $\Sigma^*$ to $N$ as follows :

If $W = a_{i1}, a_{i2}, ..., a_{ik}$, where $k \geq 0$ and $1 \leq ij \leq r - 1$ for $j = 1, 2, ..., k$

then $$\text{GNUM}(W) = r^{k-1} i_1 + r^{k-2} i_2 + ... + r^1 i_{k-1} + i_k$$

$$\text{GNUM}(\in) = 0$$

The function GNUM (W) represents the godel numbering of the string W. This function converts each string into a unique integer and each integer corresponds to at most one string.

**Example 11.6. Let $\Sigma = \{a, b, c\}$**

**Solution.** We can rewrite $\Sigma$ as $\{a_1, a_2, a_3\}$

Here $a_1 = a$, $a_2 = b$, $a_3 = c$ and $r = 4$

Now we will calculate godel numbering for strings $\in$, $a$, $ab$, $abc$

$$\text{GNUM}(\in) = 0$$

$$\text{GNUM}(a) = 1$$

$$\begin{aligned} \text{GNUM } (ab) &= 4 \,.\, 1 + 2 = 6 \\ \text{GNUM } (abc) &= 4^2 \,.\, 1 + 4^1 \,.\, 2 + 4^0 \,.\, 3 \\ &= 16 + 8 + 3 \\ &= 27 \end{aligned}$$

### 11.3.1. Digression: Penrose's attack on 'strong' AI

Gödel's theorem says that if we are given a sound fonnal proof system for arithmetic, then we can construct a statement of arithmetic (the Godel formula of the system) which, though true, is not a theorem of the system.

Penrose deduces from this that

**"Human mathematicians are not using a knowably sound algorithm in order to ascertain mathematical truth."** ***(Shadows of the Mind,*** **§2.5)**

More exactly, we can prove the following theorem:

**Theorem 11.2. It is not the case that we can correctly know, of some sound formal proof system whose details are known to me, that it constitutes the totality of processes by which we can come to accept arithmetical statements as true.**

**Proof.** For any given sound formal proof system $S$ whose details I know, I can construct its Godel formula $G(S)$, whose truth I thereby come to accept. Since $G(S)$ is not a theorem of 5, the theorems of 5 do not exhaust the totality of arithmetical statements I can come to accept as true.

So far, there is nothing to quarrel with here. But now Penrose further infers that the principles underlying the human mind are not computational in nature; and since they are presumed to be physical, the laws of physics must include some as yet unknown non-computational component.

What does Penrose mean by 'use'? Sometimes he refers to explicit procedures used by mathematicians, sometimes to low-level mechanisms in the brain. It seems obvious that the explicit procedures cannot exhaust the processes by which I come to believe arithmetical statements; while there is no reason to believe that the lower-level processes (which must include influences on the brain from outside) are, in detail, knowable to us.

Suppose that, against Penrose, we believe that all physical processes are computational in the sense that they can be simulated to an arbitrary degree of precision by means of digital computation (and hence expressible by means of a formal system of the kind to which Godel's theorem applies). If in addition we believe with Penrose that all mental processes are, at some level of description, physical processes, then what we can actually deduce from Godel's theorem is that *the totality of processes by which I can come to accept arithmetical statements as true is either unknowable to me, or unsound.* Penrose's argument depends on his finding this statement incredible; whereas to me it seems quite plausible. Deadlock!

Of course, Penrose's real target, which is Strong AI, doesn't come out of this too well either. We can hardly simulate what is unknowable to us!

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### 11.3.2. Postscript on the Arithmetic of the Real Numbers

The set of real numbers $R$ consists of all numbers expressible by means of finite or infinite decimal expansions. It includes the integers $Z$ and the rational numbers $Q$ and a whole lot else besides.

Many problems are *easier* to solve for this larger set than for the integers; indeed they can become trivial. Consider for example the following problem (a special case of Fermat's Last Theorem) :

Do there exist numbers $x$, $y$, $z$ such that $x^7 + y^7 = z^7$?

If $x$, $y$, and $z$ are required to be positive integers, this is quite a hard problem, and the answer turns out to be negative: there do not exist integers with this property. If, on the other hand, $x$, $y$, and $z$ are allowed to be any real numbers, the problem is quite trivial. I can choose, say $x = y = 2$, and then put $z = \sqrt[7]{256}$, and *I* have a solution. Existence problems are easier to solve for real numbers than for integers because there are many *more* real numbers than integers. It is easier to see whether or not there is a real number with a specified property.

The first-order theory of the real numbers has the following non-logical vocabulary (with the usual interpretations) :

Constants : 0, 1

Unary functions : $-$, $^{-1}$

Binary function : $+$, $*$

Binary predicate : $\leq$

The Polish logician Alfred Tarski (1902-1983) proved that this is a decidable theory. The following axioms are complete for this theory :

1. $z + (y + z) = (x + y) + z$
2. $z + y = y + z$
3. $z + 0 = x$
4. $z + (-z) = 0$
5. $x * (y * z) = (x * y) * z$
6. $x * y = y * x$
7. $x * 1 = x$
8. $z \neq 0 \rightarrow z*z^{-1} = 1$
9. $z * (y + z) = (z * y) + (x * z)$
10. $0 \neq 1$
11. $0^{-1} = 0$
12. $0 \leq x\ x \vee 0 \leq (-x)$
13. $0 \leq x \wedge 0 \leq (-x) \rightarrow x = 0$
14. $0 \leq x \wedge 0 \leq y \rightarrow x + y$
15. $0 \leq x \wedge 0 \leq y \rightarrow x * y$
16. $x \leq y \leftrightarrow 0 \leq y + (-x)$
17. $\exists x\Phi(x) \wedge \exists y\forall x(\Phi(x) \rightarrow x \leq y) \rightarrow \exists z\forall y\ (\forall x(\Phi(x) \rightarrow x \leq y) \leftrightarrow z \leq y)$

Note that all function symbols have to be defined as total, and therefore $0^{-1}$ has to be assigned a meaning even though in reality the number 0 has no reciprocal; axioms 8 and 11 in effect define $z^{-1}$ as the reciprocal of $z$ when

$x \neq 0$, and 0 otherwise. This does not interfere with the correct functioning of the system.

Axiom 17 (which is actually an axiom schema) expresses the least upper bound property of the real numbers: if a set of real numbers has an upper bound, then it has a least upper bound. As stated, the axiom does not cover all possible sets, since $\Phi$ can only be instantiated to first-order predicates constructible in the language; but since we are only considering the first-order theory of the reals, it does not matter that sets only expressible in higher-order logic are not handled.

## EXERCISE

1. Prove that the proper subtraction of two integers is a primitive recursive function. Proper subtraction is defined as

$$\begin{aligned} \text{sub}(x, y) &= x - y \text{ if } x > y \\ &= 0, \text{ otherwise} \end{aligned}$$

2. Prove that absolute function over positive integer is primitive recursive function, where $N = \{0, 1, 2, \ldots\}$.
3. Prove that the function, which finds the minimum of two positive integers is primitive recursive function.
4. Prove that $f(n)$ is primitive recursive function where $f(n)$ is defined as :

$$f(n) = \begin{cases} x/2, & \text{if } x \text{ is even} \\ (x-2)/2, & \text{if } x \text{ is odd} \end{cases}$$

5. Show that exponentiation function $\exp(x, y) = x^y$ is primitive recursive function.
6. Find the Godel numbering for $a$, $ab$, $abc$, $ccc$, $abcd$ where $\Sigma = \{a, b, c, d\}$.

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# 12

CHAPTER

# The Chomsky Hierachy

**INSIDE THIS CHAPTER**

**12.1.** Phrase-Structure Grammars **12.2.** Chomsky Hierachy **12.3.** Unrestricted (type-0) Grammars **12.4.** Context Sensitive Grammar **12.5.** Kuroda Normal Form **12.6.** One-Sided Context-Sensitive Grammars **12.7.** Normal Form for Unrestricted Phase Structure Grammars **12.8.** Derivation Graph **12.9.** Linear Bounded Automata **12.10.** Relation between Classes of Languages.

## 12.1. PHRASE-STRUCTURE GRAMMARS

A phrase-structure grammar is a collection of three things :

1. A finite alphabet $\Sigma$ of letters called terminals.
2. A finite set of symbols called non-terminals that includes the start symbol $S$.
3. A finite set of production of the form

$$\text{String 1} \rightarrow \text{String 2}$$

where string 1 can be any string of terminals and non-terminals that contains at least one non-terminals and where string 2 is any string of terminals and non-terminals what so ever.

A derivation in a phrase-structure grammar is a series of working string beginning with the start symbol $S$, which by making substitution according to the productions, arrives at a string of all terminals, at which point generation must stop.

The language generated by a phrase structure grammar is the set of all strings of terminals that can be derived starting at $S$.

**Example 12.1. The following is a phrase structure grammar over**

$$\Sigma = \{a, b\} \text{ with nonterminals } A \text{ an } B :$$

$$A \rightarrow BA/\epsilon \qquad (P_1, \text{ say})$$

$$B \rightarrow aB/a \qquad (P_2, \text{ say})$$

$$aaaB \rightarrow ba \qquad (P_3, \text{ say})$$

**Solution.** The first production $P_1$ says that we can start with $A$ and derive number of symbols of the type $B$, for example

$$\begin{aligned} A &\Rightarrow BA \\ &\Rightarrow BBA \\ &\Rightarrow BBBA \\ &\Rightarrow BBBBA \\ &\Rightarrow BBBB \end{aligned}$$

The second production shows us that each $B$ can be any string of $a$'s (with at least one $a$) :

$$\begin{aligned} B &\Rightarrow aB \\ &\Rightarrow aaB \\ &\Rightarrow aaaB \\ &\Rightarrow aaaaB \\ &\Rightarrow aaaaa \end{aligned}$$

The third production ($P_3$) says that any time we find three $a$'s and $B$, we can replace these four symbols with the two-terminal string $ba$.

The following is a summary of one possible derivation in this grammar :

$$\begin{aligned} A &\overset{*}{\Rightarrow} BBBBBB \\ &\overset{*}{\Rightarrow} aaaaBBBBB \\ &\Rightarrow aabaBBBB \\ &\Rightarrow aabaaaBB \\ &\Rightarrow aabbaBB \\ &\Rightarrow aabbaaaB \\ &\Rightarrow aabbba \end{aligned}$$

All CFG's are phrase-structure grammars in which we restrict ourselves as to what we put on the left side of productions. So, all CFLs can be generated by phrase-structure grammars.

## 12.2. CHOMSKY HIERACHY

We can exhibit the relationship between grammars by the Chomsky Hierarchy. Noum Chomsky, a founder of formal language theory, provided an initial classification in to four languages types :

Type – 0 (*Unrestricted grammar*)
Type – 1 (*Context sensitive grammar*)
Type – 2 (*Context free grammar*)
Type – 3 (*Regular grammar*)

Type 0 languages are those generated by unrestricted grammars, that is, the recursively enumerable languages. Type 1 consist of the context-sentive languages, Type 2 consists of the context-free languages and Type 3 consists of the regular languages. Each language family of type $k$ is a proper subset of the family of type $k - 1$. Following diagram shows the original Chomsky Hierachy.

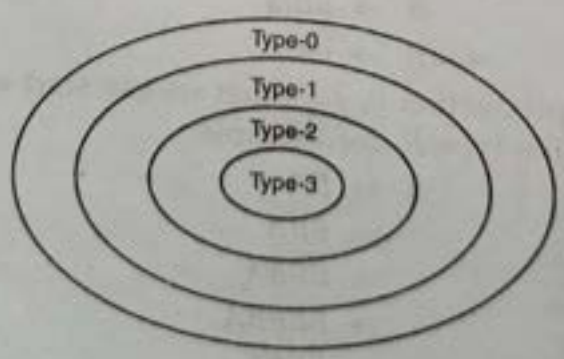


**Fig. 12.1. The original Chomsky Hierachy.**

We have also met several other language families that can be fitted in to this picture. Including the families of deterministic context-free languages ($L_{DCF}$), and recursive languages ($L_{REC}$). The modified chomsky Hierachy can be seen in Fig. 12.2

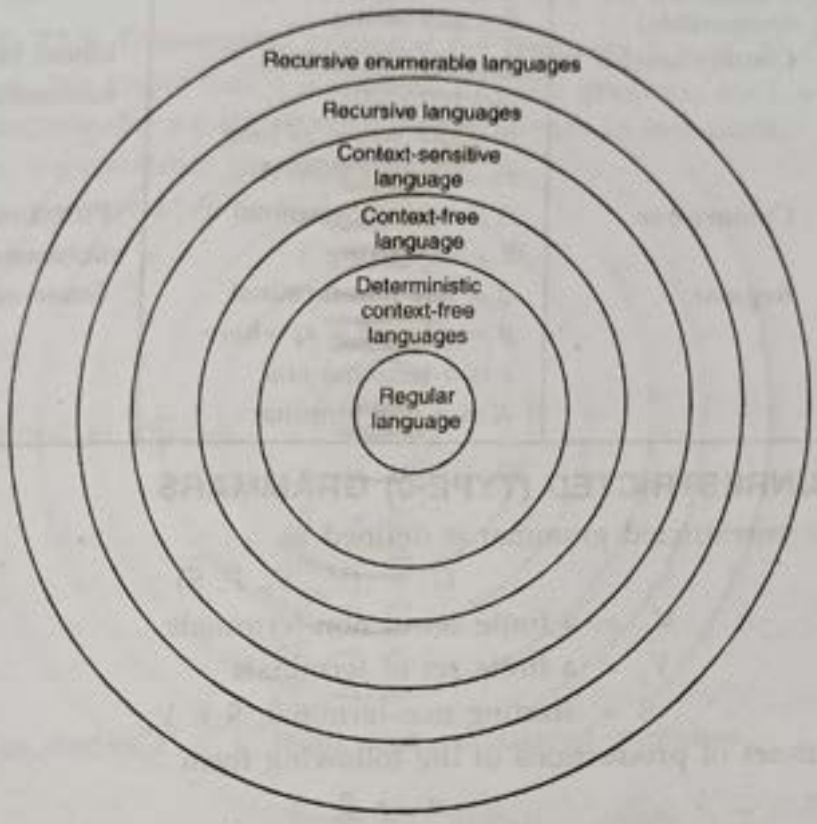


**Fig. 12.2.**

We know that $L = \{w : n_a(w) = n_b(w)\}$ is deterministic, but not linear. On the other hand, the language

$$L = \{a^n b^n\} \cup \{a^n b^{2n}\}$$

is linear, but not determinstic. The relationship between, linear, determinsitic context-free and non-determinstic context-free language is shown in Fig. 12.3.

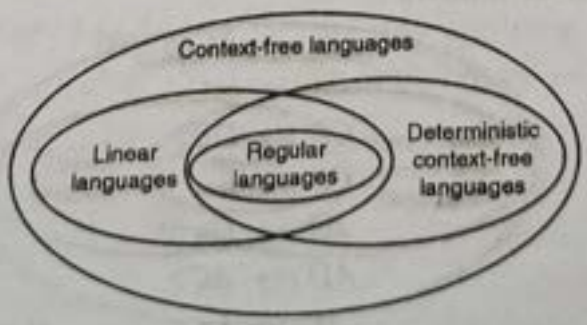


**Fig. 12.3.**

In the last we can summarise our discussion as follows :

| *Type* | *Name of languages generated* | *Production Restrictions* $A \to B$ | *Acceptor* |
|---|---|---|---|
| 0 | Unrestricted (reversively-enumerable) | $A$ = any string with non-terminal<br>$B$ = any string | Turing machine |
| 1 | Context-sentive | $A$ = any string with non-terminals<br>$B$ = any string as long as or longer than $A$ | Linear bound automata |
| 2 | Context-free | $A$ = one non-terminal<br>$B$ = any string | Pushdown automata |
| 3 | Regular | $A$ = one non-terminal<br>$B = aX$ or $B = a$, where $a$ is a terminal and $X$ is a non-terminal. | Finite automata |

## 12.3. UNRESTRICTED (TYPE-0) GRAMMARS

The unrestricted grammar is defined as

$$G = (V_n, V_t, P, S)$$

where $V_n$ = a finite set of non-terminals

$V_t$ = a finite set of terminals

$S$ = starting non-terminal, $S \in V_n$

and $P$ is set of productions of the following form

$$\alpha \to \beta$$

where $\alpha$ and $\beta$ are arbitrary string of grammar symbols with $\alpha \neq \epsilon$. These grammars are known as type-0, phase-structure or unrestricted grammars.

**Example 12.2. Write a grammar generating $\{a^i / i$ is a positive power of 2$\}$.**

**Solution.** Let grammar be $G = (V_n, V_t, P, S)$

where $V_n = \{S, A, C, B, D, E\}$

$V_t = \{a, \epsilon\}$

$S = \{S\}$

and $P$ is defined as follows :

$$S \to ACaB \quad (P_1)$$
$$Ca \to aaC \quad (P_2)$$
$$CB \to DB \quad (P_3)$$
$$CB \to E \quad (P_4)$$
$$aD \to Da \quad (P_5)$$
$$AD \to AC \quad (P_6)$$
$$aE \to Ea \quad (P_7)$$
$$AE \to \epsilon \quad (P_8)$$

$A$ and $B$ serve as left and right end marks for sentential forms; $C$ is marker that moves through the string of $a$'s between $A$ and $B$, doubling the

number by production $P_2$. When $C$ hits the right end marker $B$, it becomes a $D$ or $E$ by production $P_3$ or $P_4$. If a $D$ is choosen, that $D$ migrates left by production $P_5$ until the left end marker $A$ is reached. At the point the $D$ becomes a $C$ again by production $P_6$, and the process starts over. If an $E$ is chosen, the right endmarker is consumed. The $E$ migrates left by production $P_7$ and consumer the left endmarker, leaving a string of $2^i$ $a$'s for some $i > 0$.

**Example 12.3. Construct a grammar for language $L = \{a^n b^n c^n / n \geq 0\}$.**

**Solution.** We know that it is difficult to write grammar for $L = \{a^n b^n c^n / n \geq 0\}$ recursively. So we try to write this grammar in two parts.

(*i*) First we construct grammar for $a^n x^n$

(*ii*) Then convert $x^n$ in to $b^n c^n$.

Let grammar be $G = (V_n, V_t, P, S)$

where
$$V_n = \{S, B, C\}$$
$$V_t = \{a, b, c\}$$
$$S = \{S\}$$

and $P$ is defined as follows :

$$S \rightarrow aSBC/aBC$$
$$CB \rightarrow BC$$
$$aB \rightarrow ab$$
$$bB \rightarrow bb$$
$$bC \rightarrow bc$$
$$cC \rightarrow cc$$

Now let us derive $a^n b^n c^n$ from above discussed grammar

$$S \overset{*}{\Rightarrow} a^{n-1} S(BC)^{n-1} \quad \text{(Use } S \rightarrow aSBC \ (n-1) \text{ times)}$$
$$\Rightarrow a^{n-1} a\, BC\, (BC)^{n-1} \quad \text{(Use } S \rightarrow aBC)$$
$$\overset{*}{\Rightarrow} a^n B^n C^n$$

(By applying $CB \rightarrow BC$ several times since $CB \rightarrow BC$ interchange $B$ and $C$)

$$\Rightarrow a^{n-1} abB^{n-1} c^n \quad \text{(By applying } aB \rightarrow ab)$$
$$\overset{*}{\Rightarrow} a^n b^n C^n \quad \text{(By applying } bB \rightarrow bb \text{ several times)}$$
$$\Rightarrow a^n b^{n-1} bcc^{n-1} \quad \text{(By applying } bC \rightarrow bc \text{ by once)}$$
$$\Rightarrow a^n b^n c^n \quad \text{(By applying } cC \rightarrow Cc \text{ several times)}$$

**Example 12.4. Let $G = (V_n, V_t, P, S)$**

**where**
$$V_n = \{S, X, Y\}$$
$$V_t = \{a, b\}$$
$$S = \{S\}$$

**and $P$ is defined as follows :**

$$S \rightarrow aXYa$$
$$X \rightarrow baXYb$$
$$Y \rightarrow Xab$$

$$aX \rightarrow baa$$
$$bYb \rightarrow abab$$

**Test whether $w$ =*baabbabaaabbaba* is in $L(G)$.**

**Solution.** Here I am underline the substring to be replaced by the use of a production :

$$\begin{aligned} S &\Rightarrow a\underline{XY}a \\ &\Rightarrow baa\underline{Y}a \\ &\Rightarrow baa\underline{X}aba \\ &\Rightarrow baa\underline{baXY}baba \\ &\Rightarrow baabbaa\underline{Y}baba \\ &\Rightarrow baabbaa\underline{X}abbaba \\ &\Rightarrow baabbabaaabbaba \\ &\Rightarrow w \end{aligned}$$

So $w \in L(G)$.

## 12.4. CONTEXT SENSITIVE GRAMMAR

Let $G = (V_n, V_t, P, S)$ be context sentive grammar

where

$V_n$ = finite set of non-terminals

$V_t$ = finite set of terminals

$S$ = starting non-terminals $S \in V_n$

and $P$ is the set of rules called productions defined as

$$\alpha \rightarrow \beta$$

where $\beta$ is at least as long as $\alpha$ that is clearly

$$|\alpha| \leq |\beta|$$

The term "Context-sentive" comes from a normal form for these grammars where each production is of the form $\alpha_1 A \alpha_2 \rightarrow \alpha_1 \beta \alpha_2$, with $\beta \neq \epsilon$. Replacement of variable $A$ by string $\beta$ is permitted in the "Context" of $\alpha_1$ and $\alpha_2$.

For example, language $L = \{a^n b^n c^n : n \geq 0\}$ is context sensitive. We can show this exhibiting a context sensitive grammar for the language $L_1$ grammar as

$$\begin{aligned} S &\rightarrow abc/aAbc \\ Ab &\rightarrow bA \\ Ac &\rightarrow Bbcc \\ bB &\rightarrow Bb \\ aB &\rightarrow aa/aaA \end{aligned}$$

"A type 0 grammer is called length-increasing if for all rules $\alpha \rightarrow \beta$ in we have $|\alpha| \leq |\beta|$" (same as context sensitive grammar).

So we can say that $\epsilon$ free type-0 grammar is said to be length increasing grammar.

## 12.5. KURODA NORMAL FORM

A length-increasing grammar is said to be in Kuroda normal form if each of its rules has any of the following four forms :

(1) $A \rightarrow a$

(2) $A \rightarrow B$

(3) $A \rightarrow BC$

(4) $AB \rightarrow CD$

**Theorem 12.1. For every length-increasing grammar, we can give an equivalent grammar in Kuroda normal form.**

**Proof.** Let us assume that terminal symbols occur only in $\alpha \to a$ form rules. If $\alpha \to \beta \in P$ such that $|\alpha| = 1$ and $|\beta| > 2$ then we can replace it by form '3' rules in the same way as we have done it for the Chomsky Normal form. If $|\alpha| = 2$ and $|\beta| = 2$ then $\alpha \to \beta$ is of form '4'. So now we have to deal only with the case $|\alpha| \geq 2$ and $|\beta| > 2$. A rule of this form

$$A_1 \ldots A_m \to B_1 \ldots B_n \ (2 \leq m \leq n)$$

with $A_i, B_i \in V_N$ can be replaced by the set of rules.

$$\begin{array}{ll} A_1A_2 \to B_1C_2 & C_m \to B_mC_{m+1} \\ C_2A_3 \to B_2C_3 & C_{m+1} \to B_{m+1}C_{m+2} \\ \vdots & \vdots \\ C_{m-1}A_m \to B_{m-1}C_m & C_{n-1} \to B_{n-1}B_n \end{array}$$

where $C_i \notin V_N$ $(i = 1 \ldots n-1)$ are newly introduced variables. Resulting grammar is in Kuroda Normal form.

**Note :** Each $AB \to CD$ type rule in Kuroda Normal form can be repalced by four context-sensitive rules as follows :

$$\begin{array}{ll} AB \to AB' & A'B' \to A'D \\ AB' \to A'B' & A'D \to CD \end{array}$$

where $A'$ and $B'$ are new variables, above rules are not enough to replace the rules $AB \to CD$. For example if we have the rules $S \to AB$, $B \to DE$, $AB \to CD$ in the original grammar, then this replacement makes the derivation.

$$S \to AB \to A'B \to A'DE \to CDE$$

possible, though $CDE$ is not derivable in the original grammar.

## 12.6. ONE-SIDED CONTEXT-SENSITIVE GRAMMARS

It can be observed that in the Context-Sensitive refinement of Kuroda normal form we have only a one-sided context in each rule, but both versions that is left-context and right context, occur in same rule. Every context-sensitive language can not be generated by only one sided context rules.

According to *M*. Pentonnen, left-context sensitive rules plus inverting rules with form $AB \to BA$ are sufficient to generate every context-sensitive language.

"For every length increasing grammar we can give an equivalent grammar whose rules have the forms".

1. $A \to a$
2. $A \to B$
3. $A \to BC$
4. $AB \to AC$ (left sensitive rule)
5. $AB \to BA$ (inverting rule)

**Example 12.5. Let** $V_N = (S, A, B, C, D, E, H, K)$

**Solution.** $V_t = \{a, +, -, \times, /, (, )\}$ and rules are following :

$$\begin{array}{lll} S \to A & (\ ) \to D) & B \to CA \\ A \to B) & \underline{CH \to DH} & H \to + \\ A \to a & D \to HA & H \to - \\ B \to (C & D \to EA & K \to \times \\ B \to BC & E \to DK & K \to / \end{array}$$

This is right-sensitive grammar that generates the closed arithmetic expression where the operands are all denoted by $a$.

The above grammar is in normal form with four kinds of rules :

1. $A \to a$
2. $A \to B$
3. $A \to BC$
4. $AB \to CB$

**"So finally we can say that one-sided context-sensitive grammars are the grammars which have only either left-sensitive rule ($\underline{A}B \to \underline{A}C$) or right-sensitive rule ($AB \to C\underline{B}$).**

## 12.7. NORMAL FORM FOR UNRESTRICTED PHASE STRUCTURE GRAMMARS

A generalization of Kuroda normal form can be developed for type-0 grammars. This normal form will be very useful to establish the relationship between automata and grammar.

"A type-0 grammar is said to be in normal form if every rule has any of the following forms :

1. $S \to \epsilon$
2. $A \to a$
3. $A \to B$
4. $A \to BC$
5. $\underline{A}B \to \underline{A}C$ (Left sensitive)
6. $A\underline{B} \to C\underline{B}$ (Right sensitive)
7. $AB \to B$

where the initial non-terminal S may occur only on the left-hand sides of the rules.

**Theorem 12.2. Every type-0 grammar has an equivalent normal form in which every rule has any of the above written rules.**

**Proof.** First of all to get the grammar $G'$ for which $L(G') = L(G) - \{\epsilon\}$, we repalce every rule of the form $\alpha \to \epsilon$ by the set of rules $\alpha Y \to Y$ and $Y\alpha \to Y$ for all $Y \in (V_N \cup V_T)$. Now if $L(G)$ contains empty string then we will add the rule $S' \to \epsilon$. Next we transform $G'$ such that terminal symbols will occur in form '2' rules only. All the other rules will then have form $\alpha \to \beta$ with $\alpha, \beta \in V_N'$. For a length increasing rules of this form with $|\alpha| \le |\beta|$ we apply the same method, as we have used for the Kuroda normal form.

Now we will consider only length-decreasing rules of the form.

$$A_1 \dots A_m \to B_1 \dots B_n \qquad (m > n \ge 1)$$

where $A_i, B_i \in V_N$. Such a rule will be replaced by the set of rules.

$$\begin{array}{ll}
A_{m-1}A_m \to C_{m-1}D_m & C_mD_m \to D_m \\
A_{m-2}D_m \to C_{m-1}D_{m-1} & C_{m-1}D_{m-1} \to D_{m-1} \\
\vdots & \vdots \\
A_nD_{n+2} \to C_{n+1}D_{n+1} & C_{n+1}D_{n+1} \to D_nB_n \\
A_{n-1}D_n \to D_{n-1}B_{n-1} &
\end{array}$$

$$A_1D_2 \rightarrow D_1B_1$$
$$D_1B_1 \rightarrow B_1$$

where $D_1 \ldots D_m$ and $C_{n+1} \ldots C_m$ are newly introduced variables. This way we can obtain a grammar in the above normal form which generates the same language.

## 12.8. DERIVATION GRAPH

We have already seen the representation of context-free derivations by directed trees. The same represented cannot be used for derivations in type-1 or type-0 grammar because here we have more than one symbol on the left-hand side of the rules.

Given a phase-structure grammar

$G = (V_N, V_t, P, S)$, we assign a unique name to each production rule in $P$; in symbols

$$p : \alpha \rightarrow \beta \in P$$

where $\alpha, \beta \in (V_N \cup V_t)^*$ while $p \notin (V_N \cup V_t)$ is the name of the rule.

Hence, the set of rules can be represented by the set of names

$$V_P = \langle p_1, p_2, \ldots, p_n \rangle$$

where each $p_1$ stands for a specific production rule assigned to it. The graph of a derivation for the following rules

$$p_1 : S \rightarrow SAb$$
$$p_2 : bAb \rightarrow aBa$$
$$p_3 : Aa \rightarrow AB$$

can be represented as follows :

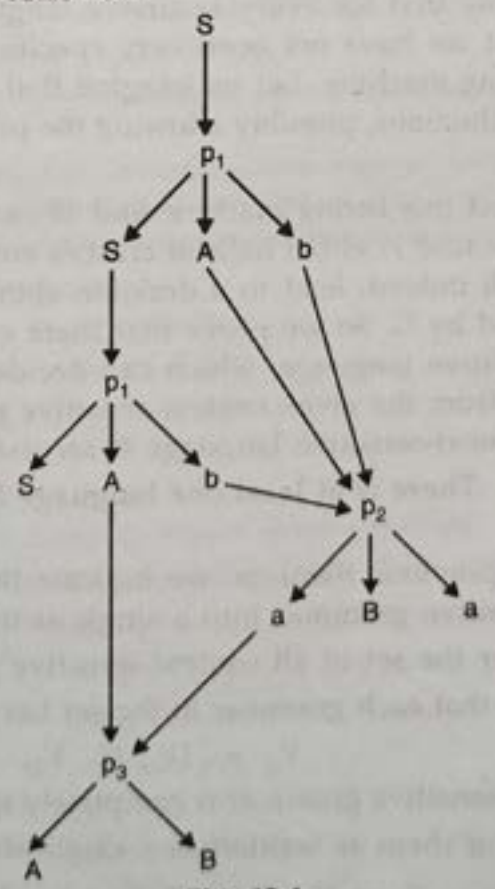


Fig. 12.4.

## 12.9. LINEAR BOUNDED AUTOMATA

A Linear Bounded Automata (LBA) is a non-deterministic turing machine satisfying the following two conditions.

(*a*) Its input alphabet includes two special symbols $\$_1$ and $\$_2$, the left and right end marker, respectively.

(*b*) The LBA has no more left from $\$_1$ and no right from $\$_2$, nor may it print another symbol over $\$_1$ or $\$_2$.

The linear bound automata is simply a turing machine which instead of having potentially infinite tape on which to compute, is restricted to the portion of the tape containing the input plus the two tape squares holding the end markers.

LBA will be denoted as

$T_m = (Q, \Sigma, \Gamma, \delta, q_0, h, \$_1, \$_2)$ where $Q, \Sigma, \delta, S$ are as for non-deterministic turing machine; $\$_1$ and $\$_2$ are symbols in $\Gamma$, the left and right end markers.

The $L(T_m)$ the languages accepted by turing machine, is

$\{W/W$ is in $\Sigma^*$ and $(q_0, \$_1\ W\ \$_2) \vdash_{T_m}^{*} (h, \$_1\ y\ \$_2)\}$

Here $y$ is for yes and $h$ shows halting of LBA after accepting the string $W$.

## 12.10. RELATION BETWEEN CLASSES OF LANGUAGES

The four classes of languages—recursive sets, context sensitive languages, context-free languages and regular sets are often referred to as language of types 0, 1, 2 and 3 respectively.

We can find the relation between all these sets of languages through following theorems :

**Theorem 12.3. Every context sensitive language is recursive.**

**Proof.** We know that for every recursive language there exists a turing machine for it. But we have not been very specific about how one inputs a grammar into turing machine. Let us imagine that some string of delimiters separating the productions, possibly allowing the production arrow to be tape character as well.

Now let us feed this turing machine and $W$ (input string) into universal turing machine. Because $W$ either halts or crashes on the coded turing machine, this procedure will indeed, lead to a decision about $W$'s membership in the language generated by $G$. So we prove that there exists a turing machine for every context-sensitive language, which can decide that any arbitrary string $W$ can be derived from the given context-sensitive grammar or not, so we can say that every context-sensitive language is recursive.

**Theorem 12.4. There is at least one language $L$ that is recursive but not context-sensitive.**

**Proof.** In the previous theorem, we indicate that there was encoding an entire context-sensitive grammar into a single string of symbols.

Let us consider the set of all context-sensitive grammars on $V_t = \{a, b\}$.

Let us assume that each grammar in the set has variable in following form

$$V_n = \{V_0, V_1, V_2, \ldots\}$$

Every context-sensitive grammar is completely specified by its productions.

We can think of them as written as a single string

$$x_1 \to y_1 ; x_2 \to y_2 ; \ldots, x_m \to y_m$$

We can represent these string in homomorphism as follows

$$h(a) = 0\,1\,0$$
$$h(b) = 0\,1^2\,0$$
$$h(R) = 0\,1^3\,0$$
$$h(i) = 0\,1^4\,0$$
$$h(V_i) = 0\,1^{i+5}\,0$$

Thus, any context-sensitive grammar can be represented uniquely by a string of the form $L((011^*0)^*)$. Further more, the representation is invertible in the sense that, given any such string, there is at most one context-sensitive grammar corresponding to it.

Let us introduce a proper ordering on $\{0, 1\}^+$, so we can write strings in the order $w_1$, $w_2$ etc. A given string $w_i$ may not define a context-sensitive grammar; if it does, call the grammar $G_i$.

Now let us define a language.

$$L = \{W_i : W_i \text{ defines a CSG } G_i \text{ and } W_i \notin L(G_i)\}$$

**Observation 1.** $L$ is recursive. To see this we construct membership algorithms. Given $W_i$, we check it to see if it defines a sensitive-grammar $G_i$. If not, then $W_i \notin L$. If string does not define a grammar then $L(G_i)$ is recursive and we can use membership algorithm to find out if $W_i \in L(G_i)$. If it is not then $W_i \in L$. So $L$ is recursive.

**Observation 2.** $L$ is not a context sensitive. If it were there would exist some $W_j$ such that $L = L(G_j)$. We can ask if $W_j$ is in $L(G_j)$. If we assume that $W_j \in L(G_j)$, then by definition $W_j$ is not in $L$. But $L = L(G_j)$, so we have a contradiction. Conversely if we assume that $W_j \notin L(G_j)$, then by definition $W_j \in L$ and we have another contradiction.

Taking two observations in consideration we conclude that $L$ is not context-sensitive.

## EXERCISE

1. Consider the following grammar :

$$S \to ABS/\epsilon$$
$$AB \to BA$$
$$BA \to AB$$
$$A \to a$$
$$B \to b$$

(i) Derive the *abba* and *babaabbbaa* from above grammar.

(ii) Prove that all words with an equal number of *a*'s and *b*'s can be generated by this grammar.

2. Consider the grammar

$$S \to ABCS/\epsilon$$
$$AB \to BA$$
$$BC \to CB$$
$$AC \to CA$$
$$BA \to AB$$
$$CB \to BC$$
$$CA \to AC$$

$$A \to a$$
$$B \to b$$
$$C \to c$$

Derive the *ababcc* and *cbaabccba* from this grammar.

3. What language is generated by the grammar

$$S \to aXYba$$
$$XY \to XYbZ / \epsilon$$
$$Zb \to bZ$$
$$Za \to aa$$

4. Consider the following grammar

$$S \to UVX$$
$$UV \to aUX$$
$$UV \to bUZ$$
$$YX \to VaX$$
$$ZX \to VbX$$
$$Ya \to aY$$
$$Yb \to bY$$
$$Za \to aZ$$
$$Zb \to bZ$$
$$UV \to \epsilon$$
$$X \to \epsilon$$
$$aV \to Va$$
$$bV \to Vb$$

Derive the following words from this grammar :

(*i*) $\epsilon$ (*ii*) *aa*

(*iii*) *bb* (*iv*) *abab*

5. Construct a grammar $G$ generating

$$\{WW / W \in \{a, b\}^*\}$$

6. Show that $\{b^{m^2} / m \geq 1\}$ is generated by the grammar

$$S \to b$$
$$S \to X_3X_4$$
$$X_3 \to X_1X_3X_2$$
$$X_3 \to X_1X_2$$
$$X_1X_2 \to bX_2X_1$$
$$X_1b \to bX_1$$
$$X_2b \to bX_2$$
$$X_1X_4 \to X_4b$$
$$X_2X_4 \to X_5b$$
$$X_2X_5 \to X_5b$$
$$X_5 \to b$$

7. Let $G = (\{A, B, S\}, \{0, 1\}, P, S)$ where $P$ consists of

$$S \to 0AB$$
$$A0 \to S0B$$
$$A1 \to SB1$$
$$B \to SA$$
$$B \to 01$$

Show that $L(G) = \phi$.

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# 13 CHAPTER

# Lexical and Syntax Analysis

**INSIDE THIS CHAPTER**

13.1. Introduction 13.2. Lexical Analysis 13.3. Ambiguity and Formal Power Series 13.4. Syntax Analysis

## 13.1. INTRODUCTION

The process of compilation is very complex hence it comes out to be customary from the logical as well as implementation point of view to partition the compilation process into several phases. A phase is nothing but a logically cohesive operation that takes source program as input and gives object code as output.

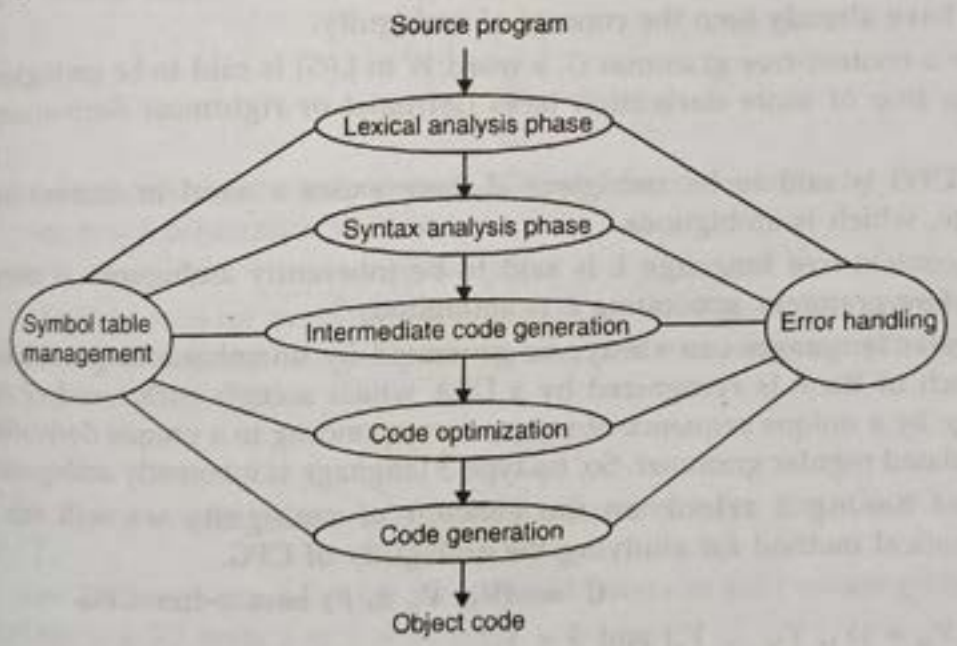


**Fig. 13.1. Phase diagram of compiler.**

## 13.2. LEXICAL ANALYSIS

In Lexical analysis phase, compiler separates characters of source language into groups that logically belong together, these groups are called tokens. This part of compiler is called lexical analyzer or scanner. In order to perform this task, lexical analyzer must know what are the tokens of the language to be implemented, that is, it must know what are keywords, identifiers, operators, delimiters, punctuation symbols etc. so that when it scans the source program it will be able to return a suitable token, whenever it encounters a lexeme

(sequence of characters) for a token. There are basically two main issues involved in the design of lexical analyzer.

(*a*) Specification of the token of the source language is required.

(*b*) A recognizer is required for recognizing the tokens.

We cannot specify the tokens of the language by enumerating each and every identifier, operator, keyword, delimeter, and punctuation symbol, because then our specification will run in several pages, and sometimes it will be never ending, especially for those languages, which does not put any limit on the number of characters that you can have in an identifier. Therefore, it is required that specification of tokens should be done generatively, by specifying the rules, which tell us in what way one can combine the symbols belonging to the alphabet of the language, so that the result of the combination will be a token of a language. This requires the use of suitable language notation. **There exist a notation called as regular expression notation** that can be used for specification of tokens, because tokens of the language constitute a regular set. There always exist a finite automata (DFA), accepting the language specified by the regular expression, which can be used as a recognizer of the language specified by the regular expression, and this makes automatic construction of recognizer of tokens possible.

## 13.3. AMBIGUITY AND FORMAL POWER SERIES

We have already seen the concept of ambiguity.

"For a context-free grammar $G$, a word $W$ in $L(G)$ is said to be ambigious if $W$ has two or more derivation trees (leftmost or rightmost derivations) in $G$."

"A CFG is said to be ambigious if there exists a word in context free language, which is ambiguous."

"A context-free language $L$ is said to be inherently ambiguous if every context-free grammar generating $L$ is ambiguous."

Regular languages can always be generated by unambiguous grammars, since each of them is recognized by a DFA which accepts each word of the language by a unique sequence of moves corresponding to a unique derivation in the related regular grammar. So, no type-3 language is inherently ambiguous.

After having a relook on the concept of ambiguity we will see a mathematical method for studying the ambiguity of CFG.

Let $G = (V_N, V_t, S, P)$ be a $\epsilon$-free CFG.

where $V_N = \{Y_1, Y_2, \ldots, Y_n\}$ and $S = Y_1$.

Let us assume that there is no $Y_i \to Y_j$ type rules. For all words $w \in V_t^*$ we define the integer number $d(w, G) \geq 0$ to be the **degree of ambiguity** of '$w$' with respect to $G$, which is infact number of the different derivation trees of $w$.

If $w \notin L(G)$ then $d(w, G) = 0$ obviously.

Now let us define a **formal power series** as follows :

$$p(G) = \sum_{w \in V_t^*} d(w, G) w$$

where summation is extended over entire $V_t^*$. Now let us consider two formal power series $p_1$ and $p_2$ as follows :

$$p_1 = \sum_{i=1}^{\infty} a_i w_i$$

and

$$p_2 = \sum_{i=1}^{\infty} b_i w_i$$

where $w_i \in V_t^*$ and $a_i$, $b_i$ are numeric factors for which addition and multiplication have their usual meaning. Then we adopt the definitions :

$$p_1 + p_2 = \sum_{i=1}^{\infty} (a_i + b_i) w_i$$

$$p_1 p_2 = \sum_{i=1}^{\infty} \left( \sum_{w_j w_k = W_i} (a_j b_k) w_i \right)$$

Our aim is to compute the formal power series $p(G)$ from the grammar $G$. Now we will compute a sequence of formal polynomials (finite power series) approximately $p(G)$.

Let $P_{i,1}, \ldots, P_{i,m}$ $(i = 1, \ldots, n)$ denote the right hand sides of productions whose left hand side is $Y_i$.

From the recursion we can define formal polynomials $p_i^K$ for $i = 1, \ldots, n$ and $K \geq 0$, as follows :

$$p_i^0 = 0 \ (i = 1, \ldots, n) \text{ and for } K > 0$$

$$p_i^K = \sum_{j=1}^{m_i} p^{K-1}(w_{i,j}) \qquad (i = 1, \ldots, n)$$

where $p_i^K(w)$ denotes the polynomial obtained from the word $w$ by substituting $p_i^{K-1}$ for every occurrence of $Y_i$ in $w$.

So $$p_i^1 = p_i^0(w_{i,1}) + p_i^0(w_{i,2}) + \ldots + p_i^0(w_{i,m})$$

Substitute zero for every non-terminal in $w_{i,j}$ will result in zero unless $w_{ij}$ contains only terminal letters. By continuing this reasoning we get the result which also implies that

$$p(G) = \lim_{K \to \infty} p_1^K$$

For every unambiguous grammar $G$ we must have $p(w, G) \leq 1$ for all $w \in V_t^*$.

Now let us substitute '$a$' for all terminal letters in $p(G)$ which gives us $a^K$ for every $w \in V_t^*$ with $|w| = K$, then

$$p(w) = \sum_{K=0}^{\infty} \left( \sum_{|w|=K} d(w, G) \right) a^K$$

Let us say $$p(K, G) = \sum_{|w|=K} p(w, G)$$

then $$p(w) = \sum_{K=0}^{\infty} p(K, G)\, a^K$$

The values of $p(K, G)$ may also be computed from the equations obtained from the rules of $G$ by substituting '$a$' for each terminal symbol. In some cases we can do even better by solving the above system of simultaneous equation

for $Y_i$, which yields an algebric expression of '$a$' for each $Y_i$. The algebric expression obtained in this fashion for $Y_i$ must be equal to the power series $p(w)$.

**Example 13.1. Consider the grammar with following rules**

$Y_1 \to aY_3$ $\quad Y_1 \to bY_2$ $\quad Y_1 \to aY_3Y_1$

$Y_1 \to bY_2Y_1$ $\quad Y_2 \to a$ $\quad Y_2 \to bY_2Y_2$

$Y_3 \to b$ $\quad Y_3 \to aY_3Y_3$

**Check the ambiguity of the grammar.**

**Solution.** The approximating polynomial of $p(G)$ will be :

$$p_1^0 = p_2^0 = p_3^0 = 0$$
$$p_1^1 = 0$$
$$p_2^1 = a$$
$$p_3^1 = b$$
$$p_1^2 = ap_3^1 + bp_2^1 + ap_3^1p_1^1 + bp_2^1p_1^1 = ab + ba$$
$$p_2^2 = a + bp_2^1p_2^1 = a + baa$$
$$p_3^2 = b + ap_3^1p_3^1 = b + abb$$
$$p_1^3 = ap_3^2 + bp_2^2 + ap_3^2p_1^2 + bp_2^2p_1^2 = a(b + abb) + b(a + baa) + a(b + abb)(ab + ba) + b(a + baa)(ab + ba)$$

Now we show with aid of $p(a)$ that grammar is unambiguous. The algebraic equations obtained from the rules are as follows :

$$Y_1 = aY_3 + aY_2 + aY_3Y_1 + aY_2Y_1 \quad ...(1)$$
$$Y_2 = a + aY_2Y_2 \quad ...(2)$$
$$Y_3 = a + aY_3Y_3 \quad ...(3)$$

The solution of equations is

$$Y_2 = \frac{1-\sqrt{1-4a^2}}{2a}$$

$$Y_3 = \frac{1-\sqrt{1-4a^2}}{2a}$$

$$Y_1 = \frac{a(Y_2+Y_3)}{1-a(Y_2+Y_3)} = (1-4a^2)^{-1/2} - 1$$

Let us put $\quad 4a^2 = Z$

So $(1 - Z)^{-1/2}$ from binomial series :

$$= \sum_{K=0}^{\infty}(-1)^K \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2K-1}{2}\right)}{K!} Z^K$$

$$= 1 + \sum_{K=0}^{\infty} \frac{1.3\ldots(2K-1)}{2^K K!} = 1 + \sum_{K=1}^{\infty} \frac{(2K)!}{2^K K! 2^K K!} Z^K$$

Hence $\quad (1 - Z)^{-1/2} - 1 = \sum_{K=1}^{\infty} \frac{(2K)!}{2^K K! 2^K K!} Z^K$

which means $$p(2K, G) = \frac{(2K)!}{(K!)^2}$$

But a simple combinatorial argument shows that the number of words in $L(G)$ containing precisely $K$ occurrences of both $a$'s and $b$'s equal to

$$\frac{(2K)!}{(K!)^2}$$

Hence, each of these words can have only one left most derivation in $G$ because the equation

$$\sum_{|W|=2K} d(W, G) = \frac{(2K)!}{(K!)^2}$$

then implies $d(W, G) = 1$ for all $W \in L(G)$. Therefore, grammar is unambiguous.

## 13.4. SYNTAX ANALYSIS

Syntax analysis is performed by a special program, called syntax analyzer or parser. A parser obtains a string of tokens from the lexica analyzer and verify, whether the string is a valid construct of the source language. Parser performs this task by verifying whether the string of tokens obtained from the lexical analyzer, can be generated by the grammar for the source language, and for this a parser either attempts to derive the string of tokens $w$ from the start symbol '$S$', or attempts to reduce $w$ to start symbol '$S$' of the grammar.

In this section we discuss briefly some of the useful subclass of context-free grammars which have fast parsing algorithm. The most common parsing algorithms fall into two categories : **top down or bottom up,** meaning the way of constructing the derivation tree for a given input word. the input word is always supposed to be read from left to right, therefore, the step-by-step development of derivation trees represents **leftmost derivations** in case of top-down construction whereas it produces a **rightmost derivation in reverse sense** when bottom up construction is used.

### 13.4.1. Top-down Parsing

Top-down parsing is an attempt to find left most drivations for an input string $w$, which is equivalent to an attempt to construct a parse tree for the input string $w$, starting from the root and creating the nodes of parse tree in pre-order. The reason why top-down parsing is an attempt to find left most derivations for an input string $w$, and not an attempt to find the right most derivations for the input string $w$, is that the input string $w$ is scanned by the parser left to right one symbol at a time, and the left most derivations generate the leaves of the parse tree in left to right order, which matches against order of scanning of the input.

In an attempt to obtain the leftmost derivation of the input string $w$, a parser may encounter a situation in which, a non-terminal $B$ is required to be derived next, and there are multiple $B$-productions, $B \to \beta_1/\beta_2/ \ldots \beta_n$, in such a situation deciding which $B$-production to use for the derivation of $B$ is a problem, therefore, a parser will select one of the $B$-production, if this derivation finally leads to the derivation of $w$, then parser announces successful completion of parsing, otherwise parser resets the pointer in the input to where it was when non terminal was derived, and try another $B$-production. The parser will continue this until either it announces successful completion of the

parsing or it reports failure after trying all the alternates. This problem is called **backtracking problem.**

Let us consider a grammar :

$$S \rightarrow aBb$$
$$b \rightarrow cd/c$$

Let the input string is $w = acb$, the parser first use the production $S \rightarrow aBb$ and expand the tree,

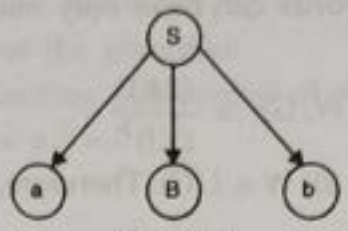


**Fig. 13.2.**

The left most leaf labelled a matches the first input symbol of $w$. Hence pointer will be advanced to '$c$', the second symbol of string $w$ and consider the next leaf labelled $B$ and then expand B using the first alternative for $B$ to obtain the following tree.

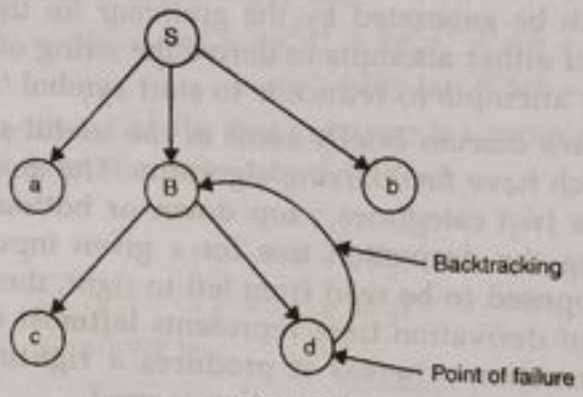


**Fig. 13.3.**

When input pointer is advanced to '$d$' input string does not match against $d$, and hence it reports failure and go back to $B$ and parser will reset the input pointer to second input symbol, the position it had when parser came to $B$, and will try second alternative of $B$ to obtain the tree as follows :

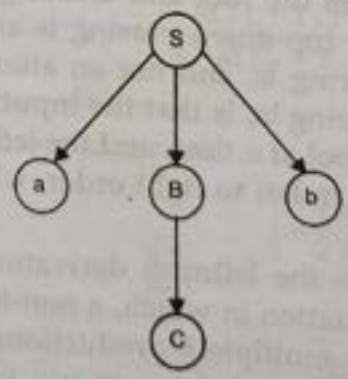


**Fig. 13.3.**

Now $W = acb$ is derived and parser announces successful completion of parsing.

**"To avoid problem of backtracking we make a deterministic recognizer, if a top-down parser is made capable of predicting or detecting which alternative**

**comes out to be a right choice for the expansion of a non-terminal during the parsing of input $w$, if that non terminal derives to more than one alternatives."**

Let us consider

If $B \to \beta_1/\beta_2/\beta_3 \ldots/\beta_n$ are the productions in the grammar, then if anyhow we are able to know the first character of each $\beta_1, \beta_2, \ldots, \beta_n$, the problem of backtracking can be removed.

This information can be easily computed using the production of the grammar. To compute this information we will define a term FIRST ($\beta$), where $\beta \in (V_N \cup V_t)^*$ as follows :

"FIRST ($\beta$) = set of those terminals with which the string derivable from $\beta$ starts"

If $\beta = PQR$ then

(*i*) FIRST ($\beta$) = FIRST ($PQR$) = $\{P\}$ if $P$ is terminal

(*ii*) FIRST ($\beta$) = FIRST ($PQR$) = FIRST ($P$) if $P$ does not derive to an empty string, that is if FIRST ($P$) does not contain $\epsilon$.

(*iii*) FIRST ($\beta$) = FIRST ($PQR$) = FIRST ($P$) $- \{\epsilon\} \cup$ FIRST ($QR$) if FIRST ($P$) contains $\epsilon$.

FIRST ($QR$) can be calculated in the same manner.

**Example 13.2. Consider the following grammar**

$$S \to ACB/CbB/Ba$$
$$A \to da/BC$$
$$B \to g/\epsilon$$
$$C \to h/\epsilon$$

**Find the FIRST of $S$, $A$, $B$ and $C$.**

**Solution.**

$$\text{FIRST}(S) = \text{FIRST}(ACB) \cup \text{FIRST}(CbB) \cup \text{FIRST}(Ba)$$
$$\text{FIRST}(A) = \text{FIRST}(da) \cup \text{FIRST}(BC) = \{d\} \cup \text{FIRST}(BC)$$
$$\text{FIRST}(B) = \text{FIRST}(g) \cup \text{FIRST}(\epsilon) = \{g, \epsilon\}$$
$$\text{FIRST}(C) = \text{FIRST}(h) \cup \text{FIRST}(\epsilon) = \{h, \epsilon\}$$
$$\text{FIRST}(BC) = \text{FIRST}(B) - \{\epsilon\} \cup \text{FIRST}(C)$$
$$= \{g, \epsilon\} - \{\epsilon\} \cup \{h, \epsilon\} = \{g, h, \epsilon\}$$

So $\text{FIRST}(A) = \{d\} \cup \{g, h, \epsilon\} = \{d, g, h, \epsilon\}$

$$\text{FIRST}(ACB) = \text{FIRST}(A) - \{\epsilon\} \cup \text{FIRST}(CB)$$
$$= \{d, g, h, \epsilon\} - \{\epsilon\} \cup \text{FIRST}(CB)$$
$$= \{d, g, h\} \cup \text{FIRST}(CB)$$
$$\text{FIRST}(CB) = \text{FIRST}(C) - \{\epsilon\} \cup \text{FIRST}(B)$$
$$= \{h, \epsilon\} - \{\epsilon\} \cup \{g, \epsilon\} = \{h, g, \epsilon\}$$

So $\text{FIRST}(ACB) = \{d, g, h\} \cup \{h, g, \epsilon\}$

$$= \{d, g, h, \epsilon\}$$

In the same manner we can find :

$$\text{FIRST}(CbB) = \text{FIRST}(C) - \{\epsilon\} \cup \text{FIRST}(bB)$$
$$= \{h, \epsilon\} - \{\epsilon\} \cup \{b\} = \{b, h\}$$
$$\text{FIRST}(Ba) = \text{FIRST}(B) - \{\epsilon\} \cup \{a\} = \{a, g\}$$

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So No $\quad$ FIRST $(S) = \{d, g, h, \in\} \cup \{b, h\} \cup \{a, g\}$

$$= \{a, b, d, g, h, \in\}$$

### 13.4.2. LL(K) Grammar

**"A context free grammar is called *LL(K)*, for left to right scan, producing a leftmost derivation, with *K* symbol lookahead, if we can always make a correct decision by checking at most the first *K* symbols of *W*."**

*LL*(1) grammars have been used successfully for constructing compilers for many programming languages. The parser for *LL*(1) grammar can act as deterministic recognizer for *L*(*G*). A parsing table constructed for a given grammar *G*, will have multiple enteries, if the grammar contains multiple productions deriving the same non-terminal, that is the grammar contains productions $B \to X/Y$ and both *X* and *Y*, derive to a string starting with the same terminal symbol. Therefore, one of the basic requirement for a grammar to be *LL*(1), when the grammar contains a multiple productions deriving the same non-terminal is :

For every pair of productions

$$B \to X/Y$$

$$\text{FIRST}(X) \cap \text{FIRST}(Y) = \phi$$

The satisfaction of the above condition is necessary as well sufficient, for a grammar to be *LL*(1), if the grammar is $\in$-free.

Another way to decide the *LL*(1) grammar is :

*W* : *K* meaning the *K* letters long initial segment of *W*, that is

$$W : K = \begin{cases} R, \text{ if } W = RQ \text{ for some } R, Q \text{ with } |R| = K \\ W, \text{ if } |W| < K \end{cases}$$

"A context-free grammar is *LL*(*K*) if and only if for every pair of leftmost derivation of the form

$$S \overset{*}{\Rightarrow} RAW' \Rightarrow RXW' \overset{*}{\Rightarrow} RQ$$

and $\quad S \overset{*}{\Rightarrow} RAW' \Rightarrow RYW' \overset{*}{\Rightarrow} RT$

with $R, Q, T \in V_t^*$ the equality $Q : K = T : K \Rightarrow Q = T$

A context-free language is called *LL*(*K*) if it is generated by an *LL*(*K*) grammar.

**Example 13.3. Check that following grammar :**

$$S \to AaAb/BbBa$$
$$A \to \in$$
$$B \to \in$$

**is *LL*(1) or not.**

**Solution.** Since grammar contains a pair of productions $S \to AaAb/BbBa$, for a grammar to be *LL*(1) it is required that :

$$\text{FIRST}(AaAb) \cap \text{FIRST}(BbBa) = \phi$$
$$\text{FIRST}(AaAb) = \text{FIRST}(A) - \{\in\} \cup \text{FIRST}(aAb) = \{a\}$$
$$\text{FIRST}(BbBa) = \text{FIRST}(B) - \{\in\} \cup \text{FIRST}(bBa) = \{b\}$$

So
$$\text{FIRST}(AaAb) \cap \text{FIRST}(BbBa)$$
$$= \{a\} \cap \{b\}$$
$$= \phi$$

So given grammar is $LL(1)$.

**Example 13.4. Prove that following grammar is not $LL(K)$ for any $K$.**

$$G = (V_N, V_t, P, S)$$

where
$$V_N = \{S, A, B\}$$
$$V_t = \{a, b, c\}$$
$S = \{S\}$ and $P$ is defined as follows :
$$S \to A,\ A \to aAb,\ B \to aBc$$
$$S \to B,\ A \to ab,\ B \to ac$$

**Solution.** If grammar is not $LL(K)$ for any $K$ then we can find left most derivation for every $K$.

$$S \overset{*}{\Rightarrow} S \Rightarrow A \overset{*}{\Rightarrow} a^K b^K$$

and $$S \overset{*}{\Rightarrow} S \Rightarrow B \overset{*}{\Rightarrow} a^K c^K$$

where $a^K b^K : K = a^K c^K$

But $A \neq B$

So grammar is not $LL(K)$ for every $K$.

**Theorem 13.1. Every $LL(K)$ grammar non-ambiguous.**

**Proof.** We know that every ambiguous grammer has more than one leftmost derivation for some word(s), (By the definition of ambiguity). But then $LL(K)$ condition is obviously violated for two derivations of the same word. So clearly every $LL(K)$ grammar is unambiguous.

### 13.4.3. Handling LL(K) grammar with moves like $A \to \epsilon$

We have said that if $LL(K)$ grammar contains productions like $A \to \epsilon$ then for every production

$$B \to X/Y$$
$$\text{FIRST}(X) \cap \text{FIRST}(Y) = \phi$$

Condition is satisfactory but not sufficient, because it may happen that FIRST $(X)$ or FIRST $(Y)$ may contain $\epsilon$ but not in both. To counter this problem we will introduce another term FOLLOW $(A)$.

FOLLOW (A) = Set of terminals, which immediately follow $A$ in any string occurring on the right side of production of the grammar.

Let us consider a production $A \to xB\beta$, then

FOLLOW $(B)$ can be computed as follows :

(i) FOLLOW $(B)$ = FIRST $(\beta)$ if FIRST $(\beta)$ does not contain $\epsilon$

(ii) FOLLOW $(B)$ = FIRST $(\beta) - \{\epsilon\} \cup$ FOLLOW $(A)$, when FIRST$(\beta)$ contains $\epsilon$.

(because when $\beta$ derives to $\epsilon$, that time the terminal symbol immediately following $A$, will follow $B$).

**Example 13.5. Find FIRST for $S$, $A$, $B$, $C$, $D$ and FOLLOW for the $S$, $A$, $B$ and $C$ in following given grammar.**

$$S \to xABC$$

$$A \rightarrow a/bbD$$
$$B \rightarrow a/\epsilon$$
$$C \rightarrow b/\epsilon$$
$$D \rightarrow c/\epsilon$$

**Solution.**

$$\text{FIRST}(S) = \text{FIRST}(xABC) = \{x\}$$
$$\text{FIRST}(A) = \text{FIRST}(a) \cup \text{FIRST}(bbD)$$
$$= \{a\} \cup \{b\} = \{a, b\}$$
$$\text{FIRST}(B) = \{a, \epsilon\}$$
$$\text{FIRST}(C) = \{b, \epsilon\}$$
$$\text{FIRST}(D) = \{c, \epsilon\}$$

Now $\text{FOLLOW}(S) = \{\#\}$

(where # is the symbol at the bottom of the stack in parser to mark the end of stack).

$$\text{FOLLOW}(A) = \text{FIRST}(BC) - \{\epsilon\} \cup \text{FOLLOW}(S)$$
$$= \{\text{FIRST}(B) - \{\epsilon\} \cup \text{FIRST}(C)\} - \{\epsilon\} \cup \text{FOLLOW}(S)$$
$$= \{(a, \epsilon) - \{\epsilon\} \cup \{b\ \epsilon\}\} - \{\epsilon\} \cup \{\#\}$$
$$= \{a, b, \epsilon\} - \{\epsilon\} \cup \{\#\}$$
$$= \{a, b, \#\}$$
$$\text{FOLLOW}(B) = \text{FIRST}(C) - \{\epsilon\} \cup \text{FOLLOW}(S)$$
$$= \{b, \epsilon\} - \{\epsilon\} \cup \{\#\} = \{b, \#\}$$
$$\text{FOLLOW}(C) = \text{FOLLOW}(S) = \{\#\}$$

### 13.4.4. Bottom-up Parsing

Bottom-up parsing can be defined as an attempt to reduce input string $W$ to the start symbol of the grammar, by tracing out the rightmost derivations of $W$ in reverse. This is equivalent to an attempt to construct a parse tree for the input string $W$, by starting with leaves and proceeding towards the root, that is, attempting to construct parse tree bottom up. This involves searching for the substring matching against the right side of any of the productions of the grammar and replacing it by the left hand side of the produciton, if this replacement leads to the generation of the sentential form which comes one step before in the rightmost derivation. **This process of replacing the right side of the production by the left side non terminal is called reduction.**

Let the rightmost derivation sequence of some input word $W$ is :

$S \rightarrow \beta_1 \rightarrow \beta_2 \rightarrow \beta_3 \rightarrow \ldots \rightarrow b_{n-1} \rightarrow W$, then bottom-up parser starts with $W$, searches for the occurrence of a substring of $W$, which matches against right side of some production, such that replacement of left hand side of production by right hand side will lead to generation of $\beta_{n-1}$ and this process continues till it reduces the entire $W$ to $S$ or encounters an error.

The parser decides in each step whether a **handle** occurs on top of the stack, and, if so it determines the production to be used for the reduction **(a handle of a right sentential form $\gamma$ is a production $A \rightarrow \beta$, and position of $\beta$ in $\gamma$, where the string $\beta$ will be found and replaced by $A$ to produce right sentential form in the right most derivation of $\gamma$, that is if $S \rightarrow \alpha A\beta \rightarrow \alpha\gamma\beta$ then $A \rightarrow \gamma$ is handle of $\alpha\gamma\beta$)**

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### 13.4.5. LR(K) Grammar

Now it became clear that for making the correct and fast decision, parser needs some extra information on the contents of the stack and on the first $K$ symbol of the remaining input. **In this situation, grammar which support this concept is called *LR(K)* grammar.**

The rule selected for application by an $LR(K)$ parser is called the handle-production and its right hand side is called handle for the corresponding reduction.

"A CFG $G = (V_N, V_t, P, S)$ is $LR(K)$, if

(i) it does not have any move like $S \to S$

(ii) for every pair of rightmost derivations of the form

$$S \overset{*}{\Rightarrow} XAW \overset{*}{\Rightarrow} XQW$$

and $$S \overset{*}{\Rightarrow} UBW' \overset{*}{\Rightarrow} URW'$$

where $A$, $B$ are variables $X$, $U$, $Q$ and $R \in (V_N \cup V_t)^*$ and $W$ and $W' \in V_t^*$.

Then equality can be defined as follows :

$$XQW : (|\ XQ\ | + K) = URW' : (|\ XQ\ | + K)$$

$\Rightarrow X = U$, $A = B$ and $Q = R$.

Now for the better understanding let us see an example :

A given CFG $\quad G = (V_N, V_t, P, S)$

where production rules $P$ is defined as follows :

$$S \to A$$
$$S \to S + A$$
$$A \to A * A$$
$$A \to x$$
$$V_N = \{S, A\}$$
$$V_t = \{a, +, *\}$$

Initial state is $S$.

Let input string $W = x + x * x$

Now let us do replacement as follows :

$$\begin{aligned} x + x * x &\Rightarrow A + x * x \\ &\Rightarrow S + x * x \\ &\Rightarrow S + A * x \\ &\Rightarrow S + A * A \\ &\Rightarrow S + A \\ &\Rightarrow S \end{aligned}$$

So clearly this grammar is $LR(1)$ grammar.

**Theorem 13.2. Every *LR(K)* grammar is unambiguous.**

**Proof.** We know that, if $W \in L(G)$ for which two different right most derivations exist, then grammar is ambiguous.

Let the last step of these derivations be

$$XAW \Rightarrow XQU = W' \quad ...(1)$$
$$UBW'' \Rightarrow URW'' = W' \quad ...(2)$$

then by the definition of $LR(K)$ grammar

$X = U$, $A = B$, and $Q = R$.

So both (1) and (2) rules are same/identical, so every $LR(K)$ grammar is unambiguous.

## EXERCISE

1. Consider the following grammar and test whether the grammar is $LL(1)$ or not.

$$S \rightarrow 1XY/\epsilon$$
$$X \rightarrow 1XZ/0Z$$
$$Y \rightarrow 0S$$
$$C \rightarrow 1.$$

2. Test whether the grammar is $LL(1)$ or not.

$$S \rightarrow AaAb/BbBa$$
$$A \rightarrow \epsilon$$
$$B \rightarrow \epsilon$$

3. For the grammar

$$S \rightarrow aBDh$$
$$B \rightarrow cC$$
$$C \rightarrow bC/\epsilon$$
$$D \rightarrow EF$$
$$E \rightarrow g/\epsilon$$
$$F \rightarrow f/\epsilon$$

compute FIRST for $S$, $B$, $C$, $D$, $E$ and $F$, also calculate follow for $B$, $D$, $C$, $E$ and $F$.

4. Consider the following grammar, and show that handle of each right sentential for the string $(a, (a, a))$.

$$S \rightarrow (L)/a$$
$$L \rightarrow L, S/S$$

5. Show that Grammar $G = \{S\}, \{a\}, S, \{S \rightarrow aSa, S \rightarrow a\})$ is not $LR(K)$ for any $K$.
6. Prove that $E \rightarrow E + E/E * E/id$ Grammar is ambiguous.

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# 14 CHAPTER

# Introduction to Intractable Problems

**INSIDE THIS CHAPTER**

14.1. Introduction 14.2. The Classes *P* 14.3. The Class NP 14.4. NP Completeness

## 14.1. INTRODUCTION

In this chapter we will focus on the problems that are decidable, and which of them can be computed by turing machine that run in an amount of time that is polynomial in the size of the input.

Following figure 14.1 helps us to develop the view about the complexity of the problems.

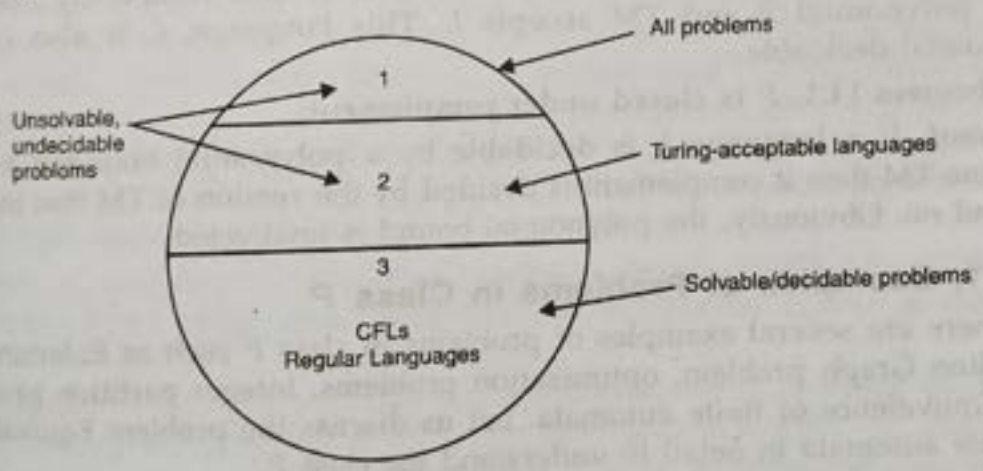


**Fig. 14.1.**

1. Strictly unsolvable problems.
2. Turing-acceptable but not turing-decidable problems.
3. Solvable/decidable problems.

In this chapter we will introduce the theory of "interactability", that is techniques for showing problems not to be solvable in polynomial time.

## 14.2. THE CLASSES *P*

We have already seen that a solvable problem is one that can be solved by a particular algorithm. In this case problem is solvable "in principle" that there is a certain algorithm to solve this problem. But in practice algorithm may require a lot of space and time. When the space and time required for implementing the steps of the particular algorithm are reasonable, we can say that the problem is tractable, that is solvable in practice.

"A decision problem is tractable if there is an algorithm to solve the given problem and time required is expressed as a polynomial $P(n)$, $n$ being the length of the input string."

"Usually problems are intractable if the time required for any of the algorithm (which can solve the problem) is at least $f(n)$, where $f$ is an exponential function of $n$.

We know that if a turing machine of any kind, either multiple tape, multidimensional or random access model, halts after a polynomial number of steps, then there is an equivalent turing machine of any other kind which also halts after a polynomial number of steps—only polynomials may differ. So we might as well settle on the simplest model, the standard turing machine.

### Polynomially Bounded Turing Machine

A turing machine TM = $(Q, \Sigma, \delta, S)$ is said to be polynomially bounded turing machine if there is a polynomial $P(n)$ such machine always halts after atmost $P(n)$ steps, where $n$ is the length of the input.

Now we can define class $P$ among the class of all languages :

### Definition of Class *P*

A language $L$ is said to be in class $P$ if there exist a polynomial bounded turing machine (determinstic) such that TM is of time complexity $P(n)$ for some polynomial $P$ and TM accepts $L$. This language, $L$ is also called polynomial decidable.

**Theorem 14.1.** **$P$ is closed under complement.**

**Proof.** If a language $L$ is decidable by a polynomial bounded turing machine TM then it complement is decided by the version of TM that inverts yes and no. Obviously, the polynomial bound is unaffected.

### 14.2.1. Examples of Problems in Class *P*

There are several examples of problems in class $P$ such as Eulerian and Hamilton Graph problem, optimization problems, Integer partition problem and Equivalence of finite automata. Let us discuss the problem Equivalence of finite automata in detail to understand the class $P$.

**Equivalence of determinstic finite automata.** Let we are given two determinstic finite automata $m_1$ and $m_2$ and problem is that :

$$L(m_1) = L(m_2) \ ?$$

We only know that how to test non-determinstic finite automata equivalence in exponential time.

That is, we do not know whether either of the following two problems in $P$.

(*i*) **Equivalence of non-determinstic finite automata.** We have given two ndfa $m_1$ and $m_2$ and is

$$L(m_1) = L(m_2) \ ?$$

(*ii*) Equivalence of Regular expressions : Given two regular expression $r_1$ and $r_2$ is

$$L(r_1) = L(r_2) \ ?$$

One could solve either problem by transforming the two given non-determinstic automata (or regular expressions) to determinstic ones, and then

checking the resulting determinstic automata for equivalence. The problem is, of course, that the transformation from regular expression or non-determinstic finite automata to determinstic finite automata may increase exponentially the number of the states of automation. Hence this approach fails to establish that either one of equivalence of non-determinstic finite automata and equivalence of regular expressions is in $P$.

**14.2.1.1. An Example. Kruskal's Algorithm.** Let us discuss the problem of finding a minimum weight spanning free for a graph.

Let us consider a graph; shown in Fig. 14.2 as follows :

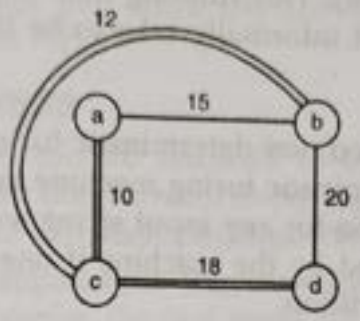


**Fig. 14.2. A graph; its minimum-weight spanning tree is indicated by double lines.**

In above graph there are four nodes, which are numbered a-d in this example graph, and there are edges between some pairs of nodes. Each edge has a weight, which is an integer. A spanning tree is a subset of the edges such that all nodes are connected through these edges, yet there are no cycles. An example of a spanning tree appears in Fig. 14.2; it is the three edges with double lines. A minimum-weight spanning tree has the least possible total edge weight of all spanning tree.

There is a well-known "greedy" algorithm, called Kruskal's Algorithm, for finding minimum weight spanning tree. Here is informal outline of the key ideas :

(a) Maintain for each node the connected component in which the node appears, using whatever of the tree have been selected so far. Initially, no edge are selected, so every node is in its a connected component by itself.

(b) Consider the lowest-weight edge that has not yet been considered; break ties any way you like. If this edge connects two nodes that are currently in different connected components then :

(i) Select the edge for the spanning tree, and

(ii) Merge the two connected components involve, by changing the component of all nodes in one of the two components to be the same as the component number of the other.

If, on the other hand, the selected edge connects two nodes of the same component, then this edge does not belong in the spanning tree; it would create a cycle.

(c) Continue considering edges untill either all edges have been considered, or the number of edges selected for the spanning tree is one less than the number of nodes. Note that in the latter case, all

nodes must be in one connected component, and we can stop considering edges.

It is possible to implement this algorthim (using a computer, not a turing machine) on a graph with $m$ nodes and $x$ edges in time $O(m + x \log x)$. A simpler easier to follow implementation proceeds in $x$ rounds. A table gives the current component of each node. We pick the lowest-weight remaining edge in $O(x)$ time, and find the components of the two nodes connected by the edges in $O(m)$ time. If they are in different components, merge all nodes with those numbers in $O(m)$ time, by scanning the table of nodes. The total time taken by this algorithm is $O(x\ (x + m)$. This running time is polynomial in the "size" of the input, which we might informally take to be the sum of $x$ and $m$.

## 14.3. THE CLASS NP

We have already studied non-determinstic turing machine. On reading an input symbol, a non-determinstic turing machine moves from one state to one of several possible states. So for any input string $w$ there are several sequence of moves and $w$ is accepted by the machine if one of the sequence of moves leads to an accepting condition.

"A non-determinstic turing machine TM = $(Q, \Sigma, \delta, S)$ is said to be **polynomially bounded** if there is a polynomial $P(n)$ such that no computation of this machine continues for more than polynomially many steps."

Now we can define class NP.

**"A language $L$ is in class NP if there is polynomially bounded non-determinstic turing machine TM such that TM is of time complexity $P(n)$ for some polynomial $P$ and TM accepts $L$."**

Although the definition of $P$ and NP are seems similar, but there is a vast difference between them. When $L$ is in $P$, the number of moves to test whether any string of length $n$ is less than or equal to $P(n)$.

When $L$ is NP, the number of moves for testing is less than or equal to $n$ only for string accepted by TM. Thus in case of NP, the bound $P(n)$ for the number of moves is useful only when we are able to find a string $w$ in $L$. But it is very difficult to find a string $w$ in $L$. In short, members of $P$ admits 'easy' solutions whereas of NP admits 'easy' solutions to verify that some string is a solutions.

### 14.3.1. An NP Example : The Travelling Salesman Problem

Let us consider an example of a problem to get a feel for the power of NP but not in $P$ : the travelling salesman problem (TSP).

The input to TSP is same as to minimum weight spanning tree, a graph with integer weights on the edges such as Fig. 14.2, and a weight limit $w$. The question asked is whether the graph has a "Hamilton circuit" of total weight at most $w$. A Hamilton circuit is a set of edges that connect the nodes into a single cycle, with each node appearing exactly once. Note that the number of edges on a Hamilton circuit must equal the number of nodes in the graph.

It appears that all ways to solve the TSP involve trying essentially all cycles and computing their total weight. By being clever, we can eliminate some obviously bad choices. But it seems that no matter what we do, we must examine an exponential number of cycles before we can conclude that there is

none with the desired weight limit $w$, or to find one if we are unlucky in the order in which we consider the cycles.

On the other hand, if we had non-determinstic computer, we could guess a permutation of the nodes, and computer the total weight for the cycle of nodes in that order.

If there were a real computer that was non-determinstic, no branch would use more than $O(n)$ steps if the input was of length $n$. On a multiple tape non-determinstic turing machine, we can guess a permutation in $O(n^2)$ steps and check its total weight in a similar amount of time. Thus, a single-tape non-determinstic turing machine can solve the TSP in $O(n^4)$ time at most. We conclude that the TSP is in NP.

## 14.4. NP COMPLETENESS

"All problems in NP can be reduced to them via polynomial-time reductions, in the same way that all recursively enumerable languages reduce to the halting problem. We call such problems **NP-Complete**."

The concept of *NP*-completeness was introduced by S.A. Cook to solve the problem "$P = NP$". Despite the fact these *NP*-complete problems fail to provide a proof that $P \neq NP$, they do occur an important place in our study of complexity. If we assume that "$P = NP$" is a conjecture that is very widely accepted, although still for from proved, then all *NP*-complete problems are indeed not in $P$. This some what indirect evidence of difficulty is the most we can expect for a problem in $NP$, short of proving that $P \neq NP$.

Let $L$ be a problem in $NP$. We say that $L$ is *NP*-complete if the following statesments are true about $L$ :

(*a*) $L$ is in $NP$

(*b*) For every language $L_1$ in $NP$ there exist a polynomial time reduction of $L_1$ to $L$.

An example of an *NP*-complete problem is the travelling salesman problem. Since it appears that $P \neq NP$, and in particular, all the *NP*-complete problem are in $NP$-$P$.

Once we have some *NP*-complete problems, we can prove a new problem to be *NP*-complete by reducing some known *NP*-complete problem to it, using a polynomial-time reduction.

Now letus discuss some properties of *NP*-complete problems.

1. No polynomial time algorithm has been found for any one of them.
2. It is not established that polynomial time algorithm for these problems do not exist.
3. If a polynomial-time algorithm is found for one of them, there will be polynomial-time algorithm for all of them.
4. If it can be proved that no polynomial time algorithm exists for any one of them, then it will not exist for every one of them.

There are several example of *NP*-complete problems such as travelling salesman problem, zero-one programming problem, satisfiability problem and vertex-cover problem here we are not discussing detail of these example.

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# 15 CHAPTER

# Introduction to Derivation Languages

**INSIDE THIS CHAPTER**

15.1. Rewriting System 15.2. Derivation Words 15.3. Derivation Languages
15.4. Canonical Derivations

## 15.1. REWRITING SYSTEM

Let us consider a CFG $G = (V_N, V_t, P, S)$ then if we consider $V_N$, $V_t$ and S without any difference then grammar can be written as

$$R_G = (V_N \cup V_t, P)$$

$R_G$ is obtained by rewriting system $G$ and called **Rewritten system.**

Where $V = V_N \cup V_t$ and $P$ is ordered pair $<Q, R>$ such that $Q \in V^*$ and $R \in V^*$. Here let us define another term $V_p$, which represents production names or production labels.

Two basic operations can be performed on the Rewriting System.

(*i*) Composition

(*ii*) Juxtaposition.

### 15.1.1. Composition

Let $P_1$ and $P_2$ are two derivations of any rewriting system $R$ such that

$$P_1 : X \Rightarrow X_0 \Rightarrow X_1 \Rightarrow X_2 \Rightarrow \ldots \Rightarrow X_m = Y$$
$$P_2 : Y \Rightarrow Y_0 \Rightarrow Y_1 \Rightarrow Y_2 \Rightarrow \ldots \Rightarrow Y_m = T$$

then composition of $P_1$ and $P_2$ can be written as $P_1 \circ P_2$.

$$P_1 \circ P_2 : X \Rightarrow Y \Rightarrow T$$

So $P_0 \circ P_2$ is a derivation of $T$ from $X$.

The composition is defined only when derived word of the first derivation is the start prefix of the second derivation. It is associative, non-commutative on the derivations. Note that if

$$P : X \Rightarrow Y$$

then $$P \circ id_X = id_Y \circ P = P$$

where $id_X$ is a rule like $X \to X$

and $id_Y$ is a rule like $Y \to Y$.

### 15.1.2. Juxtaposition

Consider the derivation $P_1$ and $P_2$ defined as follows :

$$P_1 : X \Rightarrow X_0 \Rightarrow X_1 \Rightarrow \ldots \Rightarrow X_m = Q$$
$$P_2 : Y \Rightarrow Y_0 \Rightarrow Y_1 \Rightarrow \ldots \Rightarrow Y_n = R$$

The juxtaposition of $P_1$ with $P_2$, written as $P_1 \times P_2$ is the derivation.

$$P_1 \times P_2 : XY \Rightarrow X_0Y_0 \Rightarrow X_1Y_0 \Rightarrow \ldots \Rightarrow X_mY_0 \Rightarrow X_mY_1 \Rightarrow X_mY_2 \ldots \Rightarrow$$
$$\ldots \Rightarrow X_mY_n = QR$$

which is a derivation from $XY$ to $QR$.

The derivation graph of $P_1 \times P_2$ is obtained by placing the graph of $P_1$ and $P_2$ side by side.

## 15.2. DERIVATION WORDS

We have already defined $V = V_N \cup V_t$ and $V_p$ as names of productions, then words $w \in (V \cup V_p)$ are called derivation words. All derivation words show composition and juxtaposition properties of canonical derivations.

"A set of all derivation words for a given rewriting system $R$ is called derivation language and denoted by $L(R)$."

Two functions are defined over $L(R)$

(*a*) Domain (*b*) Co-domain

**Domain.** For a rewriting system $R = (V, P)$, the domain of the word $w \in (V \cup V_p)^*$, $dm\ (w)$ is defined as follows :

(*i*) if $w \in V^*$, then domain of $w$ that is $dm\ (w) = w$

(*ii*) if $dm\ (ww')$ is defined and $p : R \to Q$ is a rule in $P$, then $dm(wpQw') = dm(ww')$. (Here $w$ and $w' \in (V \cup V_p)^*$)

**Co-domain.** For the word $w \in (V \cup V_p)^*$ that is $Cd(w)$ is defined as follows :

(*i*) If $w \in V^*$ then $Cd(w) = w$

(*ii*) If $Cd(ww')$ is defined and $p : R \to Q$ is a rule in $P$ then

$$Cd(wRpw') = Cd(ww')$$

It is relevant to note that $dm(w)$ and $Cd(w)$ are in $V^*$ whenever they are defined.

**Theorem 15.1. If $w$, $w'$, and $w''$ are in $(V_N \cup V_t \cup V_p)^*$ then**

$$dm(ww'w'') = dm(w\ dm(w')w'') \quad \ldots(i)$$

and

$$Cd(ww'w'') = Cd(w\ Cd(w')w'') \quad \ldots(ii)$$

**provided right hand sides are defined.**

**Proof.** Let us see proof for $dm\ (ww'w'') = dm(wdm(w')w'')$ by induction approach.

Let us assume that $w' \in (V_N \cup V_t)^*$, and induction step suppose that $w'$ contain $m$ production names ($m \geq 1$) and $dm(w\ dm(w')w'')$ is defined. Then $dm$ $(w')$ also must be defined, hence $w' = XpQY$ for $X, Y \in (V \cup V_p)^*$ and

$$p : R \to Q \in V_p \text{ with } dm(w') = dm(XpQY)$$
$$= dm(XY)$$

then

$$dm(ww'w'') = dm(wXpQYw'') = dm(wXYw'')$$

But by induction hypothesis

$$dm(wXYw'') = dm(w\ dm(XY)w'')$$

No let us see for $Cd(ww'w'') = Cd(w\ Cd(w')w'')$ again we will adopt the same approach as follows :

Basic : $w' \in V^*$

**Induction step :** Suppose $w'$ contains $m$ production names ($m \geq 1$) and $Cd(w\ Cd(w')\ w'')$ is defined. Then $Cd(w')$ must be defined, hence $w'' = XRpY$ for some $X, Y \in (V \cup V_p)^*$ and $p : R \to Q \in P$ with $Cd(w') = Cd(wXRpYw'')$

$$= Cd(wXYw'')$$

and $\quad Cd(wXYw'') = Cd(w\ Cd(XY)w'')$ by induction hypothesis.

This complete the proof.

## 15.3. DERIVATION LANGUAGES

For a given rewriting system $R = (V, P)$, the derivation language of $R$, $L(R)$ is defined inductively as follows :

(*i*) If $w \in V^*$ then $w \in L(R)$.

(*ii*) If $YZ \in L(R)$ with $Cd(Y)$ and $Cd(Z)$ being defined and $p : Q \to R \in V_p$ such that $Cd(Y) = LQ$ for some $L \in V^*$ then $Yp\ RZ \in L(R)$.

(*iii*) Other then above two conditions nothing belongs to the $L(R)$.

Let us consider the following derivation graph :

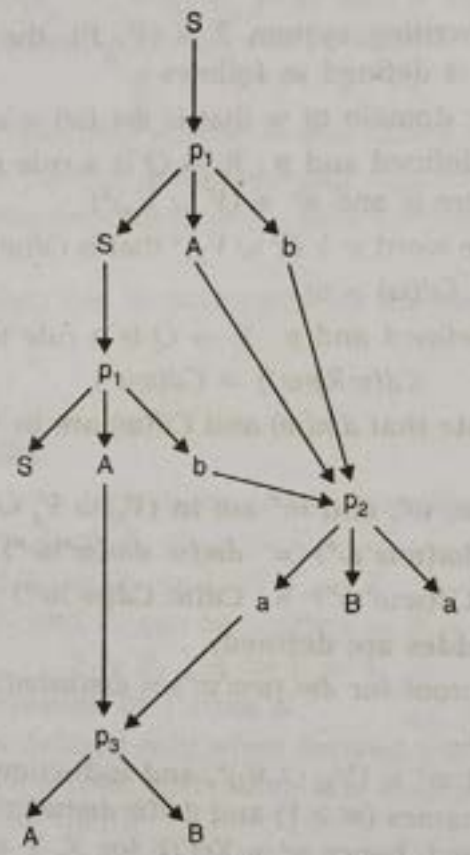


**Fig. 15.1.**

This graph will be represented by the word

$$Sp_1Sp_1SAbp_2aP_3ABBa$$

**Domain :** $S(p_1S(p_1SAb)\ Ab)\ (p_2a(p_3AB)Ba)$

**Co-domain :** $(sp_1)\ (sp_1)\ S(A(bAbp_2)ap_3)ABBa)$

### 15.3.1. Composition of Derivation Words

If $w'$ and $w'' \in (V \cup V_p)^*$ such that $Cd(w')$ and $dm(w'')$ is defined, $Cd(w') = dm(w'')$ and $Cd(w') \neq \in$ if $w' \neq \in$.

The composition of $w'$ and $w''$ is defined recursively as follows :

(i) If $w' = \epsilon$ then $w' \circ w'' = w''$

(ii) If there is some $w \in V^*$ and there are some $w_1, w_2, w_3, w_4 \in (V \cup V_p)^*$ such that

$$w' = w_1 w w_3 \text{ with } Cd(w') = w\, Cd(w_3)$$

and

$$w'' = w_2 w w_4 \text{ with } dm(w_2) = \epsilon$$

then

$$w'' \circ w' = w_2 w_1 w\, (w_4 \circ w_3)$$

The composition shows a fundamental property that is for $X, Y \in L(R)$ and their $Y \circ X$ is defined, then

$$dm(Y \circ X) = dm(X)$$

and

$$Cd(Y \circ X) = Cd(Y)$$

**Note.** For every $w_1, w_2, w_3, w_4 \in (V \cup V_p)^*$ the identity

$$(w_2 \times w_4) \circ (w_1 \times w_3) = (w_2 \circ w_1) \times (w_4 \circ w_3)$$

**Example 15.1. Let $G = (V_N, V_t, P, S)$ is phrase structure grammar with**

$$V_N = \{S, A, B\}$$

$$V_t = \{x, y\} \text{ and } P \text{ is defined as}$$

$p_1 : S \to ABB$ $\quad$ $p_2 : S \to BA$

$p_3 : AB \to SAB$ $\quad$ $p_4 : BAS \to AA$

$p_5 : A \to a$ $\quad$ $p_6 : B \to b$

**and two derivation words are given as**

$$w_1 = ABp_3SABp_3Sp_2BAAB$$

$$w_2 = Sp_2BABp_3BAABp_3Sp_4AAAB$$

**Find their composition by derivation graph method.**

**Solution.** The derivation graph for $w_1$ will be

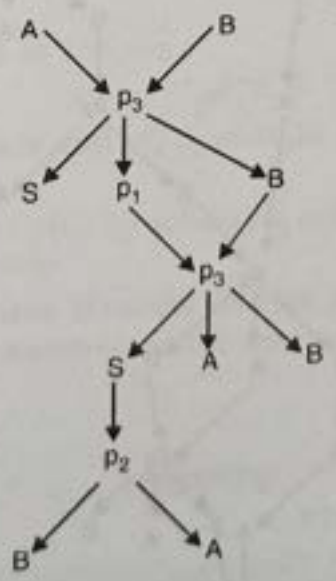


**Fig. 15.2.**

The derivation for $w_2$ is as follows :

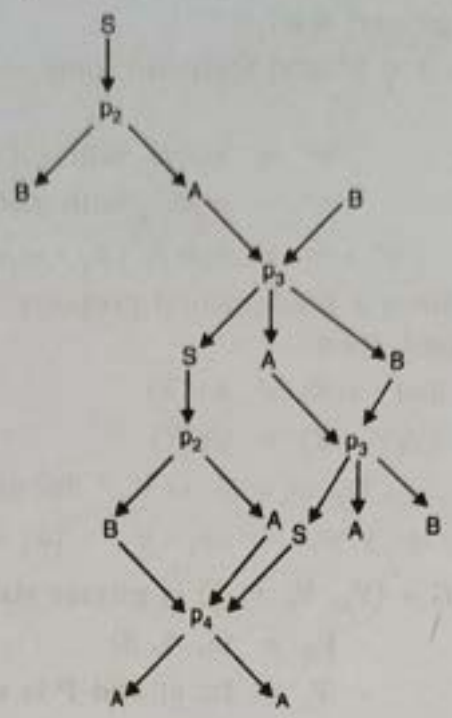


**Fig. 15.3.**

The derivation graph for the composition will be :

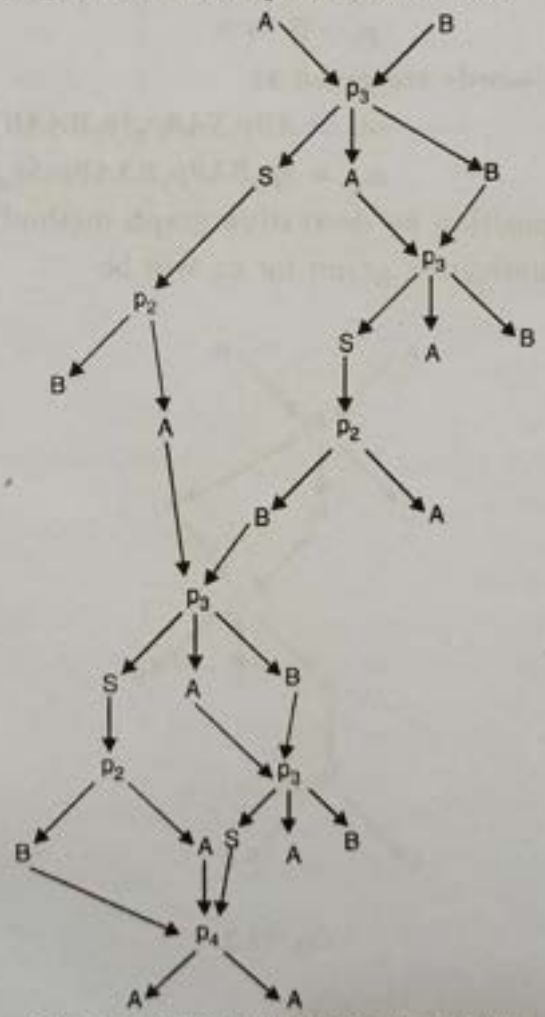


**Fig. 15.4. Composition of $w_1$ and $w_2$.**

**Here the ending node of derivation graph for $w_1$ are same as starting node of derivation graph for $w_2$.**

## 15.4. CANONICAL DERIVATIONS

For every production $X \overset{*}{\Rightarrow} Y$ in a rewriting system, there is a leftmost derivation $X \overset{*}{\Rightarrow} Y$ with the same number of steps, **also called left canonical derivation.** We know that every derivation is represented by a unique derivation graph, and each derivation graph represents a unique left most derivation. As a result of this we can say that two derivations are equivalent if they have the same derivation graph or left most derivation or canonical derivation. The two equivalent derivations differ only in the order of application of some productions.

**"Let us assume a rewriting system $R = (V, P)$ with $w \in (V \cup V_p)^*$. Let $w = Y_1pQY_2$ where $p : R \to Q \in P$, $Y_1 \in (V \cup V_p)^*$, and $Y_2 \in V^*$**

**We say that $w$ right reduces $Y_1Y_2$, written as**

$$w = Y_1pQY_2 \vdash Y_1Y_2$$

The right reduction of a word, if it exists is unique.

The canonical or leftmost derivation sequence can now be obtained from the sequence of right reductions of a derivation word.

## 15.5. THE CONTEXT-SENSITIVITY OF DERIVATION LANGUAGES

The Context-sensitivity of Derivation Languages can be understood by following theorem.

**Theorem 15.2. Let $R = (V, P)$ be a rewriting system. Then there is a deterministic linear bounded automata $A$, with input $(V \cup V_p)$ such that $L(R) = L(A)$.**

**Proof.** According to the theorem we can say that $w \in L(R)$ if both $dm(w)$ and $Cd(w)$ are defined. The linear bound automata is constructed so that it uses two pushdown stack in two separate steps :

1. Find $dm(w)$ if exists.

We can write canonical as follows :

$$w = Y_1pQY_2 \vdash Y_1Y_2 \qquad \text{(from section 15.4)}$$

Now with $Y_1$ at left stack and $pQY_2$ at right stack, it is easy to erase $pQ$ and $Y_1Y_2$ on the stack.

2. We can also compute $Cd(w)$ by following entirely symmetrical operation, starting from the left end of $w$.

**"So on the basis of above theorem and its proof we can say that every Derivation Language is context-sensitive and can be recognized by Linear Bound Automata."**

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# 16 CHAPTER

# Introduction to Cellular Automata

INSIDE THIS CHAPTER

16.1. Introduction 16.2. Cellular Automata 16.3. Formal Languages and Cellular Automata
16.4. Algebraic Properties of Cellular Automata 16.5. Invariants in Cellular Automaton Evolution
16.6. Computational Universality and Undecidability in Cellular Automata

## 16.1. INTRODUCTION

It appears that the basic Laws of Physics relevant to everyday phenomena are known. Yet there are many everyday natural systems whose complex structure and behaviour have not so far been defined. For example, the laws that govern the freezing of water and the conduction of heat are known, but analyzing their consequences for the intricate pattern for snowlake growth has not been possible. While many complex systems may be broken down into identical component, each obeying simple laws, the huge number of components that make up the whole system act together to yield very complex behaviour.

In some cases, this complex behaviour may be simulated numerically with just a few components. But in most cases the simulation requires too many components and this direct approach fails. One most instead attempt to distill the mathematical essence of the process by which complex behaviour is generated. The hope in such an approach is to identify fundamental mathematical mechanisms that are common to many different natural systems. Such commonality would correspond to universal features in the behavior of very different complex natural systems. To discover and analyze the mathematical basis for the generation of complexity, one must identify simple mathematical system that capture the essence of the process. Cellular automata are a candidate class of such systems. Cellular automata promise to provide mathematical models for a wide variety of complex phenomena, from turbulence in fluids to pattern in biological growth.

## 16.2. CELLULAR AUTOMATA

Cellular Automata were originally conceived by Ulam and Von Neumann in the 1940s to provide a formal framework for investigating the behaviour of complex and extended systems.

Cellular automata are simple mathematical idealizations of natural system. They consist of a lattice of discrete identical sites, each site taking on a finite set of, say integer values. The values of the sites evolve in discrete time steps according to deterministic rules that specify the value of each site in terms of the values of neighbouring sites.

**"Cellular Automata are—by definition—dynamical systems which are discrete in space and time, operate on a uniform, regular lattice and are characterised by "local interaction" ".**

**"Cellular automata may be considered as discrete idealization of the partial differential equations often used to describe natural system".**

**Their discrete nature also allows an important analogy with digital computer : Cellular automata may be viewed as parallel processing computers of simple construction.**

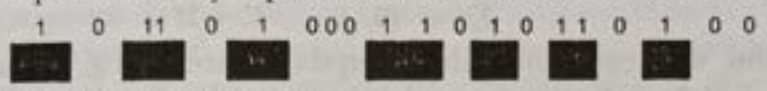
As a first example of cellular automation, consider a line of sites, each with value 0 or 1. Take the value of a site at position '$i$' on time step '$t$' to be $a_i^{(t)}$. One very simple rule for the time evolution of these site value is

$$a_i^{(t+1)} = (a_{i-1}^{(t)} + a_{i+1}^{(t)}) \bmod 2 \qquad ...(i)$$

where mod 2 indicates that 0 or 1 reminder after division by 2 is taken.

According to this rule, the value of a particular site is given by the sum modulo 2 (or, equivalently, the Boolean algebra "exclusive or") of the values of its left and right-hand nearest neighbour sites on the previous time step. The rule is implemented simultaneously at each site. Even with this very simple rule quite complicated behaviour is nevertheless found.

A typical configuration in the simple cellular automation described by equation-1 consisting of a sequence of sites with values '0' or '1'. Sites with value '1' are represented by squares; those with value '0' are blank.




Finally we can conclude that :

**"A cellular automation consists of a regular grid of cells, each of which can be in one of a finite number of K possible states, updated synchronously in discrete time steps according to a local, identical interaction rule. The state of a cell is determined by the previous states of a surrounding neighbourhood of cells.**

The infinite or finite cellular array (grid) is n-dimensional, where $n = 1, 2, 3$ is used in practice. The identical rule contained in each cell is essentially a finite state machine, usually specified in the form of a rule table (also known as transition function), with an entry for every possible neighbourhood configuration of states. The neighbourhood of a cell consists of the surrounding (adjacent cells). For one-dimensional CA's a cell is connected to '$r$' local neighbours (cells) on either side, where '$r$' is a parameter referred to as the radius (thus, each cell has $2r + 1$ neighbours, including itself). For two-dimensional CA's two types of cellular neighbourhoods are usually considered : 5 cells, consisting of the cell alongwith its four immediate non-diagonal neighbours, and 9 cells, consisting of the cell alongwith its eight surrounding neighbours. The term **configuration** refers to an assignment of states to cell in the grid. A one-dimensional CA is illustrated as follows :

**Rule table :**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Neighbourhood : | 111 | 110 | 101 | 100 | 011 | 010 | 001 | 000 |
| Output bit : | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |

Grid :

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| t = 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |
| t = 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 1 |

It is the illustration of a one-dimensional, 2-state CA. Each cell can be in one of two states, denoted 0 and 1. The connectivity radius is $r = 1$, meaning that each cell has two neighbours, one to its immediate left and one to its immediate right. Grid size is $N = 15$. The rule table for updating the grid is shown on the top. The grid configuration over one time step is shown at the bottom. Spatially periodic boundary conditions are applied, meaning that the grid is viewed as a circle, with the leftmost and rightmost cells each acting as the other's neighbour.

## 16.3. FORMAL LANGUAGES AND CELLULAR AUTOMATA

The evolution of cellular automata is viewed as a computation which processes information specified as the initial state. The structure of the output from such information processing is then described using the mathematical theory of formal languages.

A one dimensional cellular automation consists of a line of sites, with each site taking on a finite set of possible values, updated in discrete time steps according to a deterministic rule involving a local neighbourhood of sites around it. The value of site $i$ at time step $t$ is denoted by $a_i^t$ and is a symbol chosen from the alphabet

$$\Sigma = \{0, 1, ..., K - 1\}$$

At each time step each site value is updated according to the values of a neighbourhood of $2r + 1$ sites around it by a local rule.

$$\phi : S^{2r+1} \rightarrow S$$

and the value

$$a_i^t = \phi\left[a_{i-r}^{(t-1)}, a_{i-r+1}^{(t-1)}, a_{i+r}^{(t-1)}\right]$$

This local rule leads to a global mapping as follows :

$$\phi : S \rightarrow S$$

on complete cellular automation configuration **where $S$ is the set of cellular automation configuration.**

In general

$$\Omega^{(t+1)} = \phi\Omega^{(t)} \subseteq \Omega^{(t)}$$

where $\Omega^{(t)} = \phi^t$, S is the set of configuration generated after iterated application of $\phi$.

We know that Formal Languages consist of set of words formed from strings of symbols in a finite alphabet '$\Sigma$' according to definite grammatical rules. **Sets of cellular automation configuration may thus be considered as formal languages, with each word in the language representing a cellular automata configuration.** Such infinite set of configuration are then completely specified by finite sets of grammatical rules.

In order of increasing apparent complexity, qualitative characterization of classes of cellular automata is as follows :

1. Tends to a spatially homogeneous state.
2. Yields a sequence of simple stable or periodic structures.
3. Exhibits chaotic aperiodic behaviour.
4. Yields complicated localized structures, some propagating.

The classes are also distinguished by the effects of small changes in initial configuration.

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1. No change in final state
2. Changes only in a region of finite size.
3. Changes over a region of ever-increasing size.
4. Irregular changes.

Now we will try to establish the relation between various classes of cellular automation and various languages like unrestricted languages, context sensitive languages, context free languages and regular languages.

In the next section we will see that sets of configuration $\Omega^{(t)}$ generated by any finite number of steps in the evolution of a cellular automation from a regular language. For some cellular automata, the complexities of the regular language obtained tend to a fixed after a few time steps, yielding a large time limiting set of configuration corresponding to a regular language. In general, **it appears that the limit set for all cellular automata that exhibit only class 1 or 2 behaviour are given by regular languages.**

**Unrestricted (type 0) languages** are associated with universal computers. A system is considered capable of "universal computation" if, with some particular input, it can simulate the behaviour of any other computational system. A universal Turing machine, which has an infinite memory, and a central processing unit with a particular "instruction set", can be considered as universal computer. Some cellular automata have also been proved capable of universal computation. For example, a one-dimensional cellular automation with $K = 18$ and $r = 1$ is equivalent to the simplest known universal Turing machine. **It is conjectured that all cellular automata in the fourth class capable of universal computation.** For a cellular automata, an analogue of the halting problem is to determine whether a particular finite initial configuration will ultimately evolve to the null configuration.

**Any problem that depends on the result of infinite information processing may potentially be undecidable. Class-3 and class-4 cellular automata concerns about undecidable problems.** For example, the entropies of the invariant sets for class-3 and 4 cellular automata may in general be non-computable numbers. **This would be the case if the languages corresponding to these limit sets were of type 0 and 1.**

**The problems, which can be solved by a deterministic procedure in a time given by some polynomial function of the size of their input,** are in class '*P*'. Since most universal computers can simulate any other computer in a polynomial time, the times required on different computers usually differ at most by a polynomial transformation, and the set of problems in class P is defined almost independent of computer. Non-deterministic polynomial time problems (*NP*) form a second class. Solution to such problems may not necessarily be obtained in a polynomial time by a systematic procedure, but the correctness of a candidate solution, can be tested in a polynomial time. Clearly $P \subseteq NP$, and there is considerable circumstantial evidence that $P \neq NP$. **The problem of finding a pre-image for a sequence under cellular automata is in the class *NP*.** This would imply that no simple description exists even for some finite time properties of cellular automata; result may be found essentially only by explicit simulation of all possibilities.

### 16.3.1. Construction of Finite Time Sets

In this section, we will discuss the construction of the set of configuration $\Omega^{(t)}$ generated after a finite number of time steps '*t*' of cellular automation

evolution, starting from the set $\Omega^{(0)} = S$ of all possible configuration. It is shown that $\Omega^{(t)}$ may be represented as a regular language.

To describe the construction we begin with a simple example. The procedure followed may be generalized directly.

Consider the construction of the set $\Omega^{(1)}$ generated by one time step in the evolution of the $K = 2$, $r = 1$ cellular automation with a local rule $\phi$ given by

In transition graph :

(*a*) Possible sequence of site values are represented by possible paths through the graph.

(*b*) The nodes in the graph are labelled by pairs of initial site values;

(*c*) Each arc corresponds to triples of initial site values, each such triple is mapped under given rule to a partial site value.

Note that the basic graph is same for all $K = 2$, $r = 1$ cellular automata; only the images of the initial site value triples change from one rule to another.

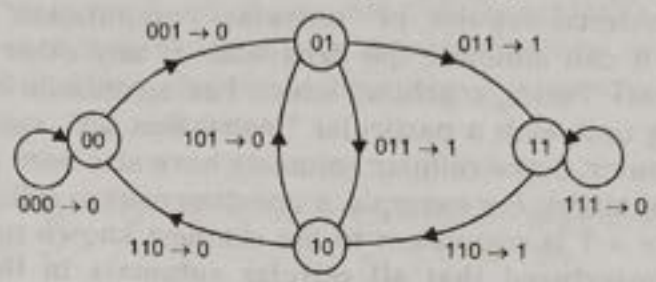


Fig. 16.1.

**The graph in Fig. 16.1 may be considered as the state transition graph for a finite automation which generates the formal language $\Omega^{(1)}$. Each node of graph corresponds to a state of the finite automation, and each arc to a transition in the finite automata or equivalently to a production rule in the grammar represented by the finite automation. The set $\Omega^{(t)}$ thus forms a regular language.**

$111 \to 0$, $110 \to 1$, $101 \to 0$, $100 \to 0$, $011 \to 1$, $010 \to 1$, $001 \to 0$, $000 \to 0$

The value $a_i^{(1)}$ of a site at position '$i$' in a configuration $A^{(1)} = \phi A^{(1)} \in \Omega^{(1)}$ depends on a neighbourhood of three sites $\left\{a_{i-1}^{(0)}, a_i^{(0)}, a_{i+1}^{(0)}\right\}$ in the preceding configuration $A^0 = \Omega^{(0)}$. The adjacent site $a_{i+1}^{(1)}$ depends on the overlapping neighbourhood $\left\{a_i^{(0)}, a_{i+1}^{(0)}, a_{i+2}^{(0)}\right\}$. The dependence of $a_{i+1}^{(1)}$ on $a_i^{(0)}$ associated with this two-site overlap in neighbourhoods may be represented by the graph. **The Nodes in the graph represent the overlaps $[a_i^0, a_{i+1}^0]$. These nodes are joined by directed arcs corresponding to three-site neighbourhoods. The local cellular automation rule $\phi$ defines a transformation for each three-site neighbourhood and thus associates a symbol with each arc of graph.** Each possible path through graph corresponds to a particular initial configuration $A^{(0)}$. The successor $A^{(1)}$ of each initial configuration is given by the sequence of symbols associated with the arcs on the path. The sequence of symbols obtained by following all possible paths through graph, thus correspond to all possible configuration $A^{(1)}$ obtained after one time step in the evolution of the cellular automation.

The complete set $\Omega^{(1)}$ may thus be represented by the transition graph. It is clear that not all possible sequence of 0's and 1's can appear in the configuration of $\Omega^{(1)}$. For example, no path in scan includes the sequence 111, and thus no configuration in $\Omega^{(1)}$ can contain a block of sites 111.

The state transition graph for a non-deterministic finite automata (NDFA), Fig. 16.1, that generates configurations obtained after one time step in the evolution of the $K = 2$, $r = 1$, cellular automata with above given local rules.

Labelling the states in above transition graph as $x_0, x_1, x_2, x_3$, the production in the grammar for this language are :

$$x_0 \to 0x_0,\ x_0 \to 0x_1,\ x_1 \to 1x_2,$$
$$x_1 \to 1x_3,\ x_2 \to 0x_0,\ x_2 \to 0x_1,$$
$$x_0 \to 0x_0,\ x_0 \to 1x_2$$

This finite set of rules provides a complete specification of the finite set $\Omega^{(1)}$.

## 16.4. ALGEBRAIC PROPERTIES OF CELLULAR AUTOMATA

We consider first the formulation for one-dimensional cellular automata in which the evolution of a particular site depends on its own vlaue and those of its nearest neighbours.

Here we are taking the cellular automata to consist of $N$ sites arranged around a circle. The values of the sites at time step $t$ are denoted $a_0^{(t)}, \ldots a_{N-1}^{(t)}$. The possible site values are taken to be elements of a finite commutative ring $\mathcal{R}_K$ with $K$ elements, much of the discussion below concerns the case $\mathcal{R}_K = Z_K$, in which site values are conveniently represented as integers modulo $K$. For example consider that each site takes on a value 0 or 1.

The complete configuration of a cellular automation is specified by the values of its $N$ sites, and may be represented by a characteristic polynomial (generating function), as follows :

$$A^{(t)}(x) = \sum_{i=0}^{N-1} a_i^{(t)} x^i$$

where the value of site $i$ is the coefficient of $x^i$, and all coefficients are elements of the ring $\mathcal{R}_K$. We shall often refer to configuration by their corresponding characteristic polynomials.

It is often convenient to consider generalized polynomials containing both positive and negative powers of $x$; such objects will be formed "dipolynomials". In general $H(x)$ is a dipolynomial if there exist some integer in such that $x^m H(x)$ is an ordinary polynomial in $x$. We know that dipolynomials possess divisibility and congruence properties analogous to those ordinary polynomials.

Multiplication of a characteristic polynomial $A(x)$ by $x^{\pm i}$ yields a dipolynomial which represent a configuration in which the value of each site has been transferred to a site $i$ places to its right (left). Periodic boundary conditions in CA's are implemented by reducing the characteristic dipolynomial modulo the fixed length $x^N - 1$ at all stages, according to

$$\sum_i a_i x^i \bmod (x^N - 1) = \sum_{i=0}^{N-1} \left( \sum_j a_{i+j} + N \right) x^i$$

Note that any dipolynomial is congruent modulo $(x^N - 1)$ to a unique ordinary polynomial of degree less than $N$.

In general the value $a_i^{(t)}$ of a site in a cellular automation is taken to be an arbitrary function of the values $a_{i-1}^{(t-1)}$, $a_i^{(t-1)}$ and $a_{i+1}^{(t-1)}$ at the previous time step. Let us consider a special class of "additive" cellular automata which evolve with time according to simple linear combination rule of the form :

$$a_i^{(t)} = \alpha_{-1} a_{i-1}^{(t-1)} + \alpha_0 a_i^{(t-1)} + \alpha_{+1} a_{i+1}^{(t-1)}$$

where the $\alpha_j$ are fixed elements of $\mathcal{R}_K$, and all arithmetic is performed in $\mathcal{R}_K$. This time evolution may be represented by multiplication of the characteristic polynomial by a fixed dipolynomial in $x$, as follows :

$$\pi(x) = \alpha_{-1} x + \alpha_0 + \alpha_{+1} x^{-1}$$

according to $$A^{(t)}(x) = \pi(x)\, A^{(t-1)}(x) \bmod (x^N - 1)$$

where arithmetic is again performed in $\mathcal{R}_K$.

## 16.5. INVARIANTS IN CELLULAR AUTOMATON EVOLUTION

The existence of invariants or conservation laws in the evolution of a cellular automaton would imply a partitioning of its state space, much as energy provides a partitioning of the state space for Hamiltonian (energy-conserving) dynamical systems. For some class 1 and 2 cellular automata it is straightforward to identify invariants. In other cases, one can specifically construct cellular automaton rules that exhibit certain conservation laws. For example, the cellular automata may evolve as if on several disjoint spatial lattices. Or they may support a set of persistent structures or "particles" that interact in simple ways. But in general, the identification of numerical invariants in cellular automata will probably be as difficult as it is in other non-linear dynamical systems.

It is nevertheless often possible to find partitionings of the state space for a cellular automaton that are left invariant by its evolution. The partitionings may be formed for example from sets of configurations corresponding to particular regular formal languages. For example, the set of configurations with a particular period under a cellular automaton mapping is invariant and in one dimension forms a finite-complement regular language (or "subshift of finite type"). Different elements in such partitionings may be considered to carry different values of what is often an infinite set of conserved quantities.

A particular cellular automaton rule usually evolves to give qualitatively similar behaviour from almost all initial states (each site is chosen to have each of the k possible values with equal probabilities). Often there are sets of initial states that occur with probability zero (for example, states in which all sites have the same value) that evolve differently from the rest. Such states may be distinguished by invariant or conserved quantities. But most initial states evolve to configuration with the same statistical properties. This suggests that even if the possible states could be partitioned according to the value of some invariant, they would be essentially equivalent. It remains conceivable, however, that there exist cellular automata in which two sets of initial states that occur with non-zero probabilities could lead to two qualitatively different forms of behaviour.

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## 16.6. COMPUTATIONAL UNIVERSALITY AND UNDECIDABILITY IN CELLULAR AUTOMATA

If a system is capable of universal computation, then with appropriate initial conditions, its evolution can carry out any finite computational process. A computationally universal system can thus mimic the behaviour of any other system, and so can in a sense exhibit the most complicated possible behaviour.

Several specific cellular automata are known to be capable of universal computation. The two dimensional nearest neighbour cellular automaton with two possible values at each site known as the "Game of Life" has been proved computation universal. The proof was carried out by showing that the cellular automaton could support structures that correspond to all the components of an idealized digital electronic computer, and that these components could be connected so to implement any algorithm. Some one-dimensional nearest-neighbour cellular automata with $k = 18$ have been shown to be computationally equivalent to the simplest known universal Turing machines, and are thus capable of universal computation.

One speculates that cellular automata identified on statistical grounds as class 4 are in fact generically capable of universal computation. This would imply that there exist one-dimensional computationally universal cellular automata in cases as simple as $k = 2$, $r = 2$ or $k = 3$, $r = 1$. But it remains to prove the computational universality of any such particular rule. Several methods could be used for such a proof. One is to identify a set of persistent structures in the cellular automaton that could act as the components of a digital computer, or like combinations of symbols and internal states for Turing machine. Structures that remain fixed, propagate, and interact in various ways have been found. A structure that can act as a "clock", producing an infinite sequence of "signals", has not yet been found in such cellular automata. Another method of proving universality would be a direct demonstration that this cellular automaton rule could simulate any other cellular automaton rule with an appropriate encoding of initial states. Blocking transformations may provide the necessary encoding; so one must find out whether a particular cellular automaton rule is connected to all others in the simulation networks constructed from blocking transformations.

If class 4 cellular automata are indeed capable of universal computation, then the capability for universal computation is quite common among one-dimensional cellular automata. Class 4 behaviour is, however, much rarer in two dimensional cellular automata.

There may well be cellular automata whose behaviour is usually computationally simple, but which with very special initial states can perform arbitrary computations. It is certainly possible to construct cellular automata in which universal computation occurs only with initial states in which say every other site has zero value, a condition that occurs in disordered states with probability zero. Such phenomena may be common in class 3 cellular automata.

Any predictions about the behaviour of a cellular automaton must be made by performing some computation. But if the cellular automaton is capable of universal computation, then this computation must, in general, reduce to a direct simulation of the cellular automaton evolution. So questions about the

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infinite time limiting behaviour of cellular automata may require infinite computations, and therefore, be formally undecidable.

For example, one may consider the question of whether the patterns generated from particular finite initial seeds ever die out in the evolution of the cellular automaton. One may simulate the evolution explicitly to find out whether a pattern dies out after say a thousand time steps; but to determine its ultimate fate in general requires a computation of unbounded length. The question is, therefore, formally undecidable.

The set of finite configuration that evolve to the null configuration after a fixed finite time can be specified by a regular formal language. But there is no such finite specification for the set of finite configuration that evolve after any time to the null configuration. Even the fraction of configuration in this set is in general a non-computable number.

A similar problem is to determine whether a particular finite sequence of site values occurs in any configurations in the limit set for a cellular automaton. Again this problem is in general undecidable. An explicit finite calculation can show that a sequence is forbidden after say three time steps. But a particular sequence may only be forbidden after some arbitrarily large number of time steps. In a one-dimensional cellular automaton, the length $L^{(t)}$ of the shortest sequence newly excluded at a given time step in the evolution is bounded by $L^{(t)} \geq L^{(t-1)} - 2r$. In most actual examples $L^{(t)}$ seems to increase monotonically with time, so that the exclusion of a particular finite sequence must occur before some predictable finite time. But in some cases $L^{(t)}$ is not monotonic, and the occurrence of particular sequences may be undecidable.

The capability for universal computation can be used to establish the undecidability of question about the behaviour of a system. But undecidability can occur even in systems not capable of full universal computation. For example, one may arrange to disable all computations that give results of a certain form. In this way, the system fails to be able to perform arbitrary computations. Nevertheless, there may be undecidable questions about the class of computations that it still can perform. These may well occur in cellular automata. Proofs of undecidability usually use a diagonal argument based essentially on universal computation. To establish undecidability in a system not itself capable of universal computation, one must usually find another system that is capable of universal computation, and show that a reduction of its capabilities does not affect undecidability.

Rice's theorem states that almost all questions about an arbitrary recursively-enumerable set are undecidable. However, it may be noted that natural or simple questions, which can be stated in say a few logical symbols, are usually decidable. So for example the halting of all simple initial seeds in a particular cellular automaton might be easy to determine, and it might only be very large and specially chosen initial seeds whose halting was difficult to determine. There are certain examples in which the halting problem appears to be difficult to answer even for simple seeds. One must establish in general not only whether there are any undecidable propositions about the behaviour of a particular cellular automaton, but whether simple propositions about it are in fact undecidable.

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