1. Brief Review of Set

1.1 Definition of Set

Sets are represented as a collection of well-defined objects or elements. A set is represented by a capital letter. The number of elements in the finite set is known as the cardinal number of a set.

1.2 Elements of a Set

Let us take an example:

A = {1, 2, 3, 4, 5}

Since a set is usually represented by the capital letter. Thus, A is the set and 1, 2, 3, 4, 5 are the elements of the set or members of the set. The elements that are written in the set can be in any order but cannot be repeated. All the set elements are represented in small letter in case of alphabets. The cardinal number of the set is 5. Some commonly used sets are as follows:

N: Set of all natural numbers
Z: Set of all integers
Q: Set of all rational numbers
R: Set of all real numbers
Z+: Set of all positive integers

1.3 Order of Sets

The order of a set defines the number of elements a set is having. It describes the size of a set. The order of set is also known as the cardinality.

1.4 Representation of Sets

The sets are represented in curly braces, {}. For example, {2,3,4} or {a,b,c} or {Bat, Ball, Wickets}. The elements in the sets are depicted in either the Statement form, Roster Form or Set Builder Form.

1. **Statement Form**
   In statement form, the well-defined descriptions of a member of a set are written and enclosed in the curly brackets.
   For example, the set of even numbers less than 15.
   In statement form, it can be written as {even numbers less than 15}.

2. **Roster Form**
   In Roster form, all the elements of a set are listed.
   For example, the set of natural numbers less than 5.
   Natural Number = 1, 2, 3, 4, 5, 6, 7, 8,……….
   Natural Number less than 5 = 1, 2, 3, 4
   Therefore, the set is N = { 1, 2, 3, 4 }

3. **Set Builder Form**
   The general form is, A = { x : property }
   Example: Write the following sets in set builder form: A={2, 4, 6, 8}
   The set builder form is A = {x: x=2n, n ∈ N and 1 ≤ n ≤ 4}

1.5 Types of Sets

We have several types of sets in math. They are empty set, finite and infinite sets, proper set, equal sets, etc. Let us go through the classification of sets here.

Empty Set	A set which does not contain any element is called an empty set or void set or null set. It is denoted by { } or Ø.
Singleton Set	A set which contains a single element is called a singleton set. Example: There is only one apple in a basket of grapes.
Finite set	A set which consists of a definite number of elements is called a finite set. Example: A set of natural numbers up to 10. A = {1,2,3,4,5,6,7,8,9,10}
Infinite set	A set which is not finite is called an infinite set. Example: A set of all natural numbers. A = {1,2,3,4,5,6,7,8,9……}

Equivalent set	If the number of elements is the same for two different sets, then they are called equivalent sets. The order of sets does not matter here. It is represented as: $n(A) = n(B)$ where A and B are two different sets with the same number of elements. Example: If A = {1,2,3,4} and B = {Red, Blue, Green, Black} In set A, there are four elements and in set B also there are four elements. Therefore, set A and set B are equivalent
Equal sets	The two sets A and B are said to be equal if they have exactly the same elements, the order of elements do not matter. Example: A = {1,2,3,4} and B = {4,3,2,1} Here, A = B
Disjoint Sets	The two sets A and B are said to be disjoint if the set does not contain any common element. Example: Set A = {1,2,3,4} and set B = {5,6,7,8} are disjoint sets, because there is no common element between them.
Subsets	A set 'A' is said to be a subset of B if every element of A is also an element of B, denoted as $A \subseteq B$. Even the null set is considered to be the subset of another set. In general, a subset is a part of another set. Example: A = {1,2,3} Then $\{1,2\} \subseteq A$. Similarly, other subsets of set A are: {1},{2},{3},{1,2},{2,3},{1,3},{1,2,3},{}. If A is not a subset of B, then it is denoted as $A \not\subseteq B$.
Proper Subset	If $A \subseteq B$ and $A \neq B$, then A is called the proper subset of B and it can be written as $A \subset B$. Example: If A = {2,5,7} is a subset of B = {2,5,7} then it is not a proper subset of B = {2,5,7} But, A = {2,5} is a subset of B = {2,5,7} and is a proper subset also.
Superset	Set A is said to be the superset of B if all the elements of set B are the elements of set A. It is represented as $A \supset B$.
Universal Set	For example, if set A = {1, 2, 3, 4} and set B = {1, 3, 4}, then set A is the superset of B. A set which contains all the sets relevant to a certain condition is called the universal set. It is the set of all possible values. Example: If A = {1,2,3} and B {2,3,4,5}, then universal set here will be: U = {1,2,3,4,5}
Power set	The power set is a set which includes all the subsets including the empty set and the original set itself. If set A = {x, y, z} is a set, then all its subsets {x}, {y}, {z}, {x, y}, {y, z}, {x, z}, {x, y, z} and {} are the elements of power set, such as: Power set of A, P(A) = { {x}, {y}, {z}, {x, y}, {y, z}, {x, z}, {x, y, z}, {} }

1.6 Operations on Sets

In set theory, the operations of the sets are carried when two or more sets combine to form a single set under some of the given conditions. The basic operations on sets are:

- Union of sets
- Intersection of sets
- A complement of a set
- Cartesian product of sets.
- Set difference

Union of Sets	If set A and set B are two sets, then A union B is the set that contains all the elements of set A and set B. It is denoted as $A \cup B$. Example: Set A = {1,2,3} and B = {4,5,6}, then A union B is: $A \cup B$ = {1,2,3,4,5,6}

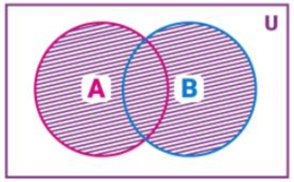
Intersection of Sets	If set A and set B are two sets, then A intersection B is the set that contains only the common elements between set A and set B. It is denoted as A ∩ B. Example: Set A = {1,2,3} and B = {4,5,6}, then A intersection B is: A ∩ B = { } or Ø
Complement of Sets	The complement of any set, say P, is the set of all elements in the universal set that are not in set P. It is denoted by P'. Properties of Complement sets 1. P ∪ P' = U 2. P ∩ P' = Φ 3. Law of double complement : (P')' = P 4. Laws of empty/null set(Φ) and universal set(U), Φ' = U and U' = Φ.
Cartesian Product of sets	If set A and set B are two sets then the cartesian product of set A and set B is a set of all ordered pairs (a,b), such that a is an element of A and b is an element of B. It is denoted by A × B. We can represent it in set-builder form, such as: A × B = {(a, b) : a ∈ A and b ∈ B} Example: set A = {1,2,3} and set B = {4,5}, then; A × B = {(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)} B × A = {(4,1),(4,2),(4,3),(5,1),(5,2),(5,3)}
Difference of Sets	If set A and set B are two sets, then set A difference set B is a set which has elements of A but no elements of B. It is denoted as A – B. Example: A = {1,2,3} and B = {2,3,4} A – B = {1}

1.7 Properties of Sets

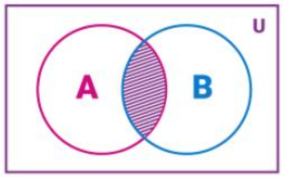
Commutative Property :	• AUB = BUA • A∩B = B∩A
Associative Property :	• A ∪ (B ∪ C) = (A ∪ B) ∪ C • A ∩ (B ∩ C) = (A ∩ B) ∩ C
Distributive Property :	• A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C) • A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)
De morgan's Law :	• Law of union : (A ∪ B)' = A' ∩ B' • Law of intersection : (A ∩ B)' = A' ∪ B'
Complement Law :	• A ∪ A' = A' ∪ A =U • A ∩ A' = Ø
Idempotent Law And Law of a null and universal set :	For any finite set A • A ∪ A = A • A ∩ A = A • Ø' = U • Ø = U'

1.8 Venn diagram

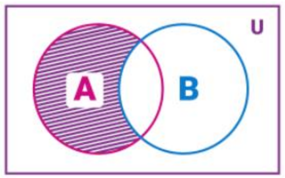
Venn diagrams are the diagrams that are used to represent the sets, relation between the sets and operation performed on them, in a pictorial way.



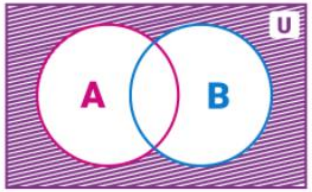
AUB



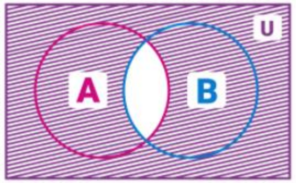
A∩B



A-B



(A∪B)’



(A∩B)’

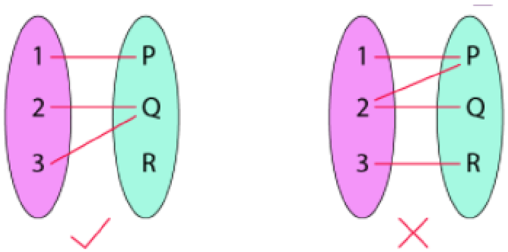
A’

2. Function and Relation

2.1 Function

A function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. Let A & B be any two non-empty sets, mapping from A to B will be a function only when every element in set A has one end only one image in set B.

Example:



In case of function: for (x, y) , each x has only one y.

In case of relation: for (x, y) , one, some or all x can have more than y.

2.2 Domain and Range

Domain:

For a function, y = f(x), the set of all the values of x is called the domain of the function. It refers to the set of possible input values.

Range:

Range of y = f(x) is a collection of all outputs f(x) corresponding to each real number in the domain. Range is the set of all the values of y. It refers to the set of possible output values.

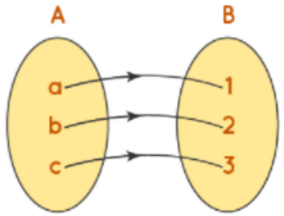
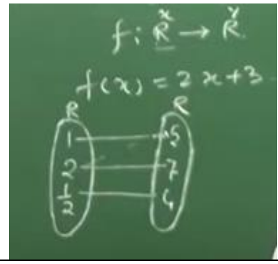
For example, consider the following relation.

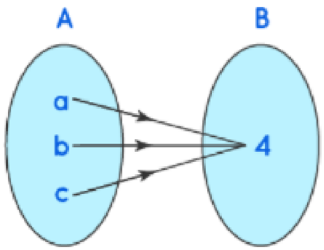
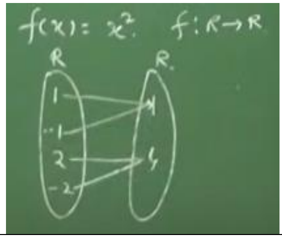
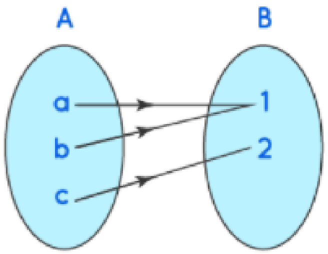
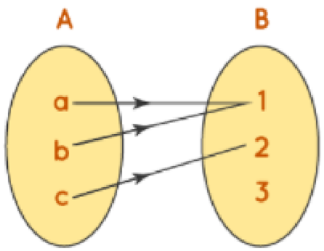
{(2, 3), (4, 5), (6, 7)}

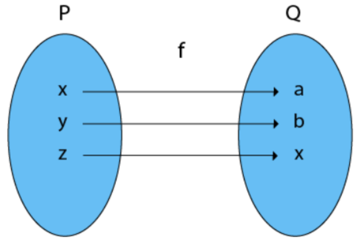
Here Domain = {2, 4, 6}

Range = { 3, 5, 7}

2.3 Types of functions

Types	Details	Mapping
One-to-One Functions (Injective)	A one-to-one function is defined by f: A → B such that every element of set A is connected to a distinct element in set B.	 

Many-to-One function	A many to one function is defined by the function f: A → B, such that more than one element of the set A are connected to the same element in the set B. In a many to one function, more than one element has the same co-domain or image.	
Onto Function (Surjective)	In an onto function, every codomain element is related to the domain element. For a function defined by f: A → B, such that every element in set B has a pre-image in set A. The onto function is also called a subjective function.	
Into function	The into function is exactly opposite in properties to an onto function. Here there are certain elements in the co-domain that do not have any pre-image. The elements in the set B are excess and are not connected to any elements in the set A.	

One-to-One Onto Function (Bijective)	A function which is both injective (one to - one) and surjective (onto) is called bijective (One-to-One Onto) Function.	

2.4 Relation

A relation R, from a non-empty set P to another non-empty set Q, is a subset of P X Q.

For example, Let P = {a, b, c} and Q = {3, 4} and

Let R = {(a, 3), (a, 4), (b, 3), (b, 4), (c, 3), (c, 4)}

Here R is a subset of A x B. Therefore, R is a relation from P to Q.

2.5 Properties of relation

Let R be a relation on A, and let x,y,z ∈ A.

A relation R is ...	**if ...**
reflexive	xRx
symmetric	xRy implies yRx
transitive	xRy and yRz implies xRz
Irreflexive	xRy implies x≠y
antisymmetric	xRy and yRx implies x=y

2.6 Relation's types

Reflexive relation	Reflexive relation is a relation of elements of a set A such that each element of the set is related to itself. Let A={1,2,3}

	Then R={(1,1), (2,2) , (3,3)} is reflexive relation defined on set A.
Symmetric relation	Relation R is symmetric for all a and b in A if (a,b) ∈ R, then also (b,a) ∈ R Let A={1,2,3} Then R={(1,2)(2,1) , (1,3) (3,1)} is symmetric relation defined on set A.
Transitive relation	Relation R is transitive if for all a,b and c in A(a,b) ∈ R and (b,c) ∈ R, then (a,c) ∈ R Let A={1,2,3} Then ={(1,2), (2,3) , (1,3)} is transitive relation defined on set A.
Equivalence relation	A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive. i.e. relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is equivalence relation as it is reflexive, symmetric, and transitive.
Partial order relation	Let R be a relation defined on set A, then R is partial order relation if it is reflexive, anti-symmetric and transitive. Let A={1,2,3} Then R={(1,1)(2,2) , (3,3) (1,2), (2,3), (1,3)} is partial order relation defined on set A.
Total order relation	A partial order relation is Total order relation if for all a and b in A , either (a,b) ∈ R or (b,a) ∈ R. Let A={1,2,3} Then R={(1,1)(2,2) , (3,3) (2,1), (2,3), (1,3)} is total order relation defined on set A.

3. Alphabets and Languages

3.1 Alphabets

Theory of computation is entirely based on symbols. These symbols are generally letters and digits.

Alphabets are defined as *a finite set of symbols.*

Examples:

∑ = {0, 1} is an alphabet of binary digits

∑ = {A, B, C, …., Z} is an alphabet.

3.2 String

A string is a finite sequence of symbols selected from some alphabet. It is generally denoted as w. For example, for alphabet ∑ = {0, 1} w = 010101 is a string.

Length of a string is denoted as |w| and is defined as the number of positions for the symbol in the string. For the above example length is 6.

The empty string is the string with zero occurrence of symbols. This string is represented as ϵ or λ.

The set of strings, including the empty string, over an alphabet ∑ is denoted by ∑*.

For ∑ = {0, 1} we have set of strings as ∑* = {ϵ, 0, 1, 01, 10, 00, 11, 10101,…}. and ∑1 = {0, 1} , ∑2 = {00, 01, 10, 11} and so on.

3.3 Power of alphabet

If Σ is an alphabet, the set of all strings can be expressed as a certain length from that alphabet by using exponential notation. The power of an alphabet is denoted by Σk and is the set of strings of length k.

For example,

- Σ ={0,1}
- Σ1= {0,1}
- Σ2= {00,01,10,11}
- Σ3= {000,001,010,011,100,101,110,111}

3.4 Concatenation of strings

Let w1 and w2 be two strings then w1w2 denotes their concatenation w. The concatenation is formed by making a copy of w1 and followed by a copy of w2.
For example, w1 = 001, w2 = 101
then w = w1w2 = 001101

3.5 Kleen closure

If S is a set of words then by S* we mean the set of all finite strings formed by concatenating words from S, where any word may be used as often we like, and where the null string is also included.
S* is the Kleen closure for S. We can think of kleen star (S*) as an operation that makes an infinite language of strings of letters out of an alphabet

For example, for ∑ = {a}
∑* = {ϵ, a, aa, aaa, ….}

3.6 Positive closure

The set of all strings over an alphabet ∑ except the empty string is called positive closure. It is denoted by ∑+.

$\sum^+ = \sum^1 \cup \sum^2 \cup \sum^3 \cup \ldots\ldots$

For example, for ∑ = {a}
∑+ = {a, aa, aaa, ….}

3.7 Reversal of string

For any string W, its reversal denoted by W^R us a sting spelled backward.
Eg. W=1010
W^R=0101

3.8 Language

A language is a set of string all of which are chosen from some ∑*, where ∑ is a particular alphabet. This means that language L is subset of ∑*.
Given : L={w ∈ {a, b}* : w has odd number of a}
This means : L={ a, ab, aaba, ……..}

The relation between Regular expression and the language they represent is established by a function L, such that if α is any regular expression, then L(α) is language represented by α.

The function L is defined as follows:

1. L(Φ)= Φ and L (a) ={a} for each a∈∑
2. If α and β are regular expressions, then L(α U β) =L(α) U L(β)
3. If α and β are regular expressions, then L(αβ) =L(α). L(β)
4. If α is regular expression, then L(α*) =L(α)*

What language is represented by following RE: L(((aUb)*a))

Here , L(((aUb)*a))	=	L((aUb)*) L(a)	From Rule 3
	=	L((aUb)*) {a}	From Rule 1
	=	L((aUb))* {a}	From Rule 4
	=	(L(a) U L(b))* {a}	From Rule 2
	=	({a} U {b})* {a}	From Rule 1
	=	(a,b)*{a}	
	=	{w∈{a,b}* : w ends with an a}	

3.9 Regular expressions

Regular expression is a formula representing a language in terms of a form that uses three operations : concatenation, union and kleen closure. Regular Expressions are used to denote regular languages. An expression is regular if:

- ϕ is a regular expression for regular language ϕ.
- ε is a regular expression for regular language {ε}.

- If a ∈ Σ (Σ represents the input alphabet), a is regular expression with language {a}.
- If a and b are regular expression, a + b is also a regular expression with language {a,b}.
- If a and b are regular expression, ab (concatenation of a and b) is also regular.
- If a is regular expression, a* (0 or more times a) is also regular.

Regular Expressions	Regular Set
(0 + 10*)	L = { 0, 1, 10, 100, 1000, 10000, … }
(0*10*)	L = {1, 01, 10, 010, 0010, …}
(0 + ε)(1 + ε)	L = {ε, 0, 1, 01}
(a+b)*	Set of strings of a's and b's of any length including the null string. So L = { ε, a, b, aa , ab , bb , ba, aaa…….}
(a+b)*abb	Set of strings of a's and b's ending with the string abb. So L = {abb, aabb, babb, aaabb, ababb, ………….}
(11)*	Set consisting of even number of 1's including empty string, So L= {ε, 11, 1111, 111111, ……….}
(aa)*(bb)*b	Set of strings consisting of even number of a's followed by odd number of b's , so L = {b, aab, aabbb, aabbbbb, aaaab, aaaabbb, ………….}
(aa + ab + ba + bb)*	String of a's and b's of even length can be obtained by concatenating any combination of the strings aa, ab, ba and bb including null, so L = {aa, ab, ba, bb, aaab, aaba, ..}
(aa)* (bb)*b	L={ w ∈{a,b}* : w has even no. of a followed by odd no. of b} i.e. L= {b, aab, aabbb, aaaab, ……}
(0U1)*00 Or (0+1)*00	L={ w ∈{0, 1}* : w has strings of 0 and 1 ending in 00} L= {0100, 110100, 100 , …….}
0(0U1)*1 or 0(0+1)*1	L={ w ∈{0, 1}* : w has strings of 0 and 1 beginning with 0 and ending with 1} L= {01 ,001, 0011…….}
(((00)*1*) + (01 – 0))*	Set of strings of 0 and 1 with even numbers of 0 i.e. 00,001, 00 , 0001, 1010, ….
0*(10*10*)*10*	Language containing odd no. of 1 i.e. 1,01, 01101, 0111, 111, ….

Given alphabet Σ=(a,b), Write Regular expressions for below:

S.No.	language	Regular Expression
1.	Start with ab	ab(a+b)*
2.	Start with bba	bba(a+b)*
3.	Ends with abb	(a+b)*abb
4.	Contains a substring aab	(a+b)*abb (a+b)*
5.	Start and ends with a	a+a(a+b)*a
6.	Starts and ends with same symbol	a+a(a+b)*a + b + b(a+b)*b
7.	Starts and ends with different symbol	a(a+b)*b+b(a+b)*a

8.	$\|w\|=3$	(a+b) (a+b) (a+b) OR $(a+b)^3$
9.	$\|w\|\geq 3$	$(a+b)^3$ (a+b)*
10.	$\|w\|\leq 3$	ε +(a+b) + $(a+b)^2$ +$(a+b)^3$ OR $(a+b+\varepsilon)^3$
11.	$\|w\|_a=2$	b*ab*ab*
12.	$\|w\|_a\geq 2$	(a+b)*a(a+b)* (a+b)*
13.	$\|w\|_a\leq 2$	b*(a+ ε)b*(a+ ε)b* OR b*+b*ab*+b*ab*ab*
14.	3rd symbol from left end is b	$(a+b)^2$ b (a+b)*
15.	28th symbol from right end is a	(a+b)*a$(a+b)^{27}$
16.	$\|w\|$= 0 (mod 3) i.e. length of string divided by 3 is 0. 0,3,6,9,……	$((a+b)^3)$*
17.	$\|w\|$= 2 (mod 3)	a+b)^2($(a+b)^3$)*
18.	$\|w\|_b$ = 0 (mod 2) i.e. no of b divisible by 2	a* + (a*ba*ba*)*
19.	$\|w\|_a$ = 1 (mod 3)	b*ab*(b*ab*ab*ab*)*
20.	$\|w\|_b$ = 2 (mod 3)	a*ba*ba*(a*ba*ba*ba*)*

Describe the following sets by Regular expressions.

1.	{101}	101
2.	{abba}	Abba
3.	{01, 10}	01+10
4.	{^ ,ab}	^+ab
5.	{abb, a,b, bba}	abb+ a+ b+ bba
6.	{^, 0,00,000,….}	0*
7.	{1,11,111,….}	11* OR 1^+

End of Chapter 1