(*) <u>Residue at a Pole:→</u>

Let $z=a$ be a pole of order m of a single valued funcⁿ $f(z)$ and C any circle of radius r with centre at $z=a$ which does not contain any other singularities except $z=a$, Then $f(z)$ is analytic within the region $0<|z-a|<r$ then by Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} \longrightarrow ①$$

where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$ & $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^{-n+1}}dz$

In particular for $n=1$, $\boxed{b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz}$

∴ The coefficient b_1 is called residue of $f(z)$ at the pole $z=a$.

① <u>Residue at a pole→</u>

If $f(z)$ has a simple pole (i.e pole of order 1) at $z=a$, then

$$\boxed{Res\{f(z)\} = \lim_{z\to a}(z-a)f(z)}$$

② <u>Residue at a pole of order m:→</u>

If $f(z)$ has a pole of order m at $z=a$, then

$$\boxed{Res\{f(z)\} = \frac{1}{\lfloor m-1}\lim_{z\to a}\left[\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}\right]}$$

③ <u>Residue at infinity→</u>

$$\boxed{Res(z=\infty) = \lim_{z\to\infty}[-z f(z)]}$$

or = − [Coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ in the nbd of $z=\infty$]

(*) <u>Cauchy's Residue Theorem or The theorem of Residues→</u>

Let $f(z)$ be one valued and analytic within and on a closed curve C except at a finite number of poles $z_1, z_2, z_3, \ldots, z_n$ and let $R_1, R_2, \ldots R_n$ be respectively the residues of $f(z)$ at those poles, then

$$\boxed{\oint_C f(z)\,dz = 2\pi i(R_1+R_2+\cdots+R_n) = 2\pi i(\text{Sum of residues at poles within } C).}$$

<u>Ex-1</u>→ Find residue of $f(z) = \frac{\cos z}{z(z+5)}$ at $z=0$. (AKTU-2019)

<u>Solⁿ</u>→ Here $f(z)$ has simple pole at $z=0$ & $z=-5$.

∴ Residue of $f(z)$ at $z=0$ $= \lim_{z\to 0}(z-0)f(z)$

$$= \lim_{z\to 0} z\cdot\frac{\cos z}{z(z+5)}$$

$$= \lim_{z\to 0}\frac{\cos z}{z+5} = \frac{\cos 0}{5} = \frac{1}{5} \text{ Ans.}$$

Ex-2 Determine the poles of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole. Hence evaluate $\int_C \frac{z^2}{(z-1)^2(z+2)} dz$ where $C \equiv |z| = 3$

(UPTU-2015)

Solⁿ Given $f(z) = \frac{z^2}{(z-1)^2(z+2)}$.

Then ~~f(z)~~ $z=-2$ is a simple pole of $f(z)$

and $z=1$ is pole of order 2 of $f(z)$.

Now

Residue of $f(z)$ at $z=-2 \to$

$$R_1 = \lim_{z\to -2} [(z+2) f(z)] = \lim_{z\to-2} (z+2) \frac{z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}.$$

Residue of $f(z)$ at $z=1 \to$ (As $z=1$ is of order 2)

$$R_2 = \lim_{z\to 1} \left[\frac{d^{2-1}}{dz^{2-1}} \{(z-1)^2 f(z)\}\right] \times \frac{1}{\lfloor 2-1}$$

$$= \lim_{z\to 1} \left[\frac{d}{dz} \left\{(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z+2)}\right\}\right]$$

$$= \lim_{z\to 1} \left[\frac{d}{dz} \frac{z^2}{(z+2)}\right] = \lim_{z\to 1} \left[\frac{(z+2)\cdot 2z - z^2 \cdot 1}{(z+2)^2}\right] = \frac{5}{9}.$$

Since both the poles lies inside the given curve $C \equiv |z| \equiv 3$.

$$\therefore \int_C f(z)\, dz = \int_C \frac{z^2}{(z-1)^2(z+2)} dz = 2\pi i (R_1 + R_2) = 2\pi i \left(\frac{5}{9} + \frac{4}{9}\right) = 2\pi i$$

Ex-3 Determine the poles of the following funcⁿ and residues at each pole: $f(z) = \frac{z-1}{(z+1)^2(z-2)}$ & hence evaluate $\oint_C f(z) dz$ where C is the circle $|z-i| = 2$ [AKTU-2011, 2013]

Hint Both poles $z=-1$ & $z=2$ lies inside $|z-i| = 2$. [Ans $-\frac{2\pi i}{9}$].

Ex-4 Evaluate $\oint_C \frac{e^z}{(z+1)^2} dz$ where C is the circle $|z-1| = 3$.

[Ans $= \frac{2\pi i}{e}$].

Ex-5 Determine the poles of the following functions and the residue at each pole:

(a) $\frac{2z+1}{z^2-z-2}$ (b) $\frac{z+1}{z^2(z-2)}$ (c) $\frac{e^z}{z^2+\pi^2}$ (d) $z \cos\frac{1}{z}$ at $z=0$.

Ex-6 Find the residue at $z=0$ of the funcⁿ $f(z) = \frac{1+e^z}{\sin z + z\cos z}$

Solⁿ $z=0$ is a pole of order 1.

$$\text{Res}\{f(z)\} = \lim_{z\to 0} z f(z) = \lim_{z\to 0} \frac{z(1+e^z)}{\sin z + z\cos z} = \lim_{z\to 0} \frac{1+e^z}{\frac{\sin z}{z} + \cos z} = \frac{1+1}{1+1} = 1.$$

Ex-① find the residue of $\frac{z^2}{z^2+a^2}$ at $z=ia$.

Ex-② find the residue of $\frac{1}{(z^2+a^2)^2}$ at $z=ia$.

Solⁿ: Let $f(z)=\frac{1}{(z^2+a^2)^2}=\frac{1}{(z+ia)^2(z-ia)^2}$

$\therefore$ $z=ia$ is pole of order 2 of $f(z)$.

Then Residue of $f(z)$ at $z=ia$ is

$$R=\lim_{z\to ia}\left[\frac{d}{dz}\{(z-ia)^2 f(z)\}\right]\times\frac{1}{\lfloor 2-1}$$

$$=\lim_{z\to ia}\left[\frac{d}{dz}(z-ia)^2\frac{1}{(z+ia)^2(z-ia)^2}\right]$$

$$=\lim_{z\to ia}\left[\frac{d}{dz}\frac{1}{(z+ia)^2}\right]$$

$$=\lim_{z\to ia}\frac{(-2)}{(z+ia)^3}=\frac{-2}{(ia+ia)^3}=\frac{-2}{i^3a^3 2^3}$$

$$=\frac{-i}{4a^3}.$$

Ex-3: Find the residues of $f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles & $\int_C f(z)\,dz$ where $C\equiv |z|=4$.

Ex-4: By Residue theorem $\int_C \frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$ where C is the circle $|z|=3$.

Solⁿ: Let $f(z)=\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2(z-2)}$.

Then $z=2$ is a simple pole and $z=1$ is pole of order 2 of $f(z)$

Now Residue at $z=2$:

$$R_1=\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}\frac{\sin\pi z^2+\cos\pi z^2}{(z-1)^2}=\frac{\sin 4\pi+\cos 4\pi}{(2-1)^2}=1$$

Residue at $z=1$:

$$R_2=\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right]=\lim_{z\to 1}\frac{d}{dz}\left[\frac{\sin\pi z^2+\cos\pi z^2}{z-2}\right]$$

$$=\lim_{z\to 1}\frac{[(z-2)[\pi\cos\pi z^2-\pi\sin\pi z^2]\times 2z-[\sin\pi z^2+\cos\pi z^2].1}{(z-2)^2}$$

$$=(1-2)(\pi\cos\pi-0).2-\sin\pi-\cos\pi=2\pi+1.$$

$\therefore$ By residue theorem, we have

$$\int_C f(z)\,dz=2\pi i\,[R_1+R_2]=2\pi i\,[1+(2\pi+1)]=4\pi i\,(\pi+1)$$

Evaluate by Cauchy's Residue theorem :-

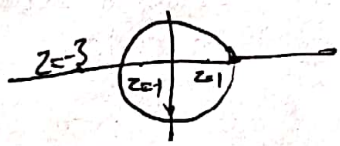
① $\oint_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z+1)(z+2)} dz$; $C \equiv |z| = 3$. ② $\oint_C \frac{5z-2}{z(z-1)} dz$; $C \equiv |z| = 2$.

③ $\oint_C \frac{24z-7}{(z-1)^2(2z+3)}$ where C is the circle of radius 2 & with centre at origin. [2013]

④ $\oint_C \frac{z^2+2z-2}{z-4} dz$ where C is a closed curve containing the point $z=4$ in its interior.

⑤ $\oint_C \frac{(3z^2+z+1)}{(z^2-1)(z+3)} dz$ where C is the circle $|z|=2$.

Ans 5:- Here $f(z) = \frac{3z^2+z+1}{(z-1)(z+1)(z+3)}$ & $z=1, z=-1, z=-3$ are simple pole of $f(z)$. But only $z=1$ & $z=-1$ lies inside $C, |z|=2$.



∴ Residue at $z=1$ →

$$R_1 = \lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{3z^2+z+1}{(z+1)(z+3)} = \frac{5}{8}$$

& Residue at $z=-1$ →

$$R_2 = \lim_{z\to -1} (z+1) f(z) = \lim_{z\to -1} \frac{3z^2+z+1}{(z-1)(z+3)} = \frac{3}{-4}$$

Then by Residue theorem for $C \equiv |z| = 2$,

$$\oint_C f(z) dz = \oint_C \frac{3z^2+z+1}{(z^2-1)(z+3)} dz = 2\pi i [R_1 + R_2]$$
$$= 2\pi i \left[\frac{5}{8} - \frac{3}{4}\right] = -\frac{\pi i}{4}.$$

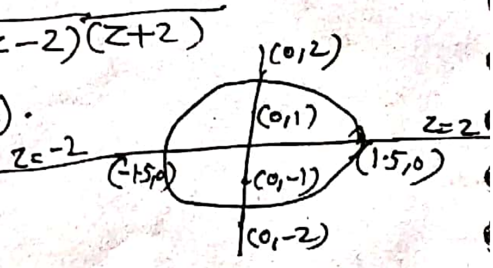
⑥ $\oint_C \frac{1-2z}{z(z-1)(z-2)} dz$, where $C \equiv |z| = 1.5$.

⑦ $\oint_C \frac{e^z - 1}{z(z-i)^2(z-1)} dz$; $C \equiv |z| = \frac{1}{2}$.

⑧ $\oint_C \frac{dz}{(z^2+1)(z^2-4)} dz$ where $C \equiv |z| = 1.5$.

Soln:- Here $f(z) = \frac{1}{(z^2+1)(z^2-4)} = \frac{1}{(z+i)(z-i)(z-2)(z+2)}$

∴ $z = i, -i, 2, -2$ are simple pole of $f(z)$.
But $z=i$ & $z=-i$ lies inside C.



Residue at $z=i$ →

$$R_1 = \lim_{z\to i} (z-i) f(z) = \lim_{z\to i} \frac{1}{(z+i)(z^2-4)} = \frac{1}{-10i}$$

$$R_2 = \lim_{z\to -i} (z+i) f(z) = \lim_{z\to -i} \frac{1}{(z-i)(z^2-4)} = \frac{1}{10i}$$

Then by C.R.T $\oint_C f(z) dz = 2\pi i [R_1 + R_2] = 2\pi i \left[-\frac{1}{10i} + \frac{1}{10i}\right] = 0$

(*) Ex-9 → $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$ where $C \equiv |z-2|=2$. (20)

(*) Ex-10 → Evaluate $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}$ where C is $|z|=10$.

Hint → $z=-1, z=\pm 2$ are pole lies inside C. [Ans → 0]

Ex-11 → $\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$ where C is $|z|=2$ (2011).

Ex-12 → Evaluate $\oint_C \frac{z-3}{z^2+2z+5}\,dz$ where C is the circle

(i) $|z|=1$ (Ans 0) (ii) $|z+(1-i)|=2$ (2013) (Ans $\pi(i-2)$) (iii) $|z+(1+i)|=2$. Ans $\pi(i+2)$

Ex-13 → find the poles (with its order) and residue at each pole of the following $f(z)=\frac{1-2z}{z(z-1)(z-2)^2}$ [AKTU-2017]

Ex-14 → Determine the poles and residues at each pole of $f(z)=\frac{z}{z^2-3z+2}$ and hence evaluate $\oint_C f(z)\,dz$ where C is the circle $|z-2|=\frac{1}{2}$. (2011, 2015) [Ans $-4\pi i$]

Ex-1 → find the residue of $\frac{z^2}{(z-a)(z-b)(z-c)}$ at infinity.

Soln →

Res of $f(z)$ at $z=\infty$ is

$$R=\lim_{z\to\infty}[-z f(z)] = \lim_{z\to\infty} \frac{-z^3}{(z-a)(z-b)(z-c)}$$

$$= \lim_{z\to\infty} \frac{-1}{(1-\frac{a}{z})(1-\frac{b}{z})(1-\frac{c}{z})} = -1.$$

Ex-2 → find the residue of

$\frac{z^3}{z^2-1}$ at $z=\infty$.

Soln → $f(z)=\frac{z^3}{z^2-1} = z\left(1-\frac{1}{z^2}\right)^{-1} = z\left(1+\frac{1}{z^2}+\frac{1}{z^4}+\cdots\right)$

$$= z+\frac{1}{z}+\frac{1}{z^3}+\cdots$$

∴ Residue at infinity

$= -$ [Coefficient of $\frac{1}{z}$ in expansion of $f(z)$]

$= -1$.

V. imp

Ex-3 → Evaluate the residues of $\frac{z^3}{(z-1)(z-2)(z-3)}$ at $z=1,2,3$ & ∞ & show that their sum is zero. (2010).

Lec-7 (Assignment) (Module-5)

Q-1: State Cauchy's Residue theorem. find residue of $f(z) = \frac{\cos 9z}{z(z+10)}$ at $z=0$.

Q-2: Determine the poles of the following funcⁿ and residue at each pole : $f(z) = \frac{z-1}{(z+1)^2(z-2)}$

and hence evaluate $\oint_C f(z)\,dz$ where C is the circle $|z-i| = 2$.

Q-3: By residue theorem find

(i) $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz$ where $C \equiv |z-2| = 2$.

(ii) $\oint_C \frac{z}{z^2-3z+2}\,dz$ where $C \equiv |z-2| = \frac{1}{2}$.