Engineering Mathematics-II

Lec No-24

* By: Dr Anuj Kumar

Application of Laplace Transform to solve ordinary diff. eqn →

Step-1 → Take Laplace Transform of both sides of the given Differential eqn and use the formulae of L.T.

& Use

$$L(y') = pL(y) - y(0) \quad \text{(for derivatives)}$$
$$L(y'') = p^2L(y) - p\,y(0) - y'(0)$$
$$L(y''') = p^3L(y) - p^2y(0) - py'(0) - y''(0)$$
etc.

Step-2 → Solve the above eqn and find $L(y)$ in terms of p.

Step-3 → Take inverse Laplace Transform of both sides. Find the solution y.

Ex-1 → Solve $\frac{d^2y}{dt^2} + y = 0$ under the conditions that $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$. (AKTU-2008)

Soln → Given $y'' + y = 0 \rightarrow (1)$ & $y(0) = 1,\ y'(0) = 0 \rightarrow (2)$

Take Laplace Transform both side of (1),

$$L(y'') + L(y) = L(0)$$
$$\Rightarrow [p^2L(y) - p\,y(0) - y'(0)] + L(y) = 0$$
$$\Rightarrow p^2L(y) - p - 0 + L(y) = 0 \quad \text{(From (2))}$$
$$\Rightarrow (p^2+1)L(y) = p$$
$$\Rightarrow L(y) = \frac{p}{p^2+1}$$
$$\Rightarrow y = L^{-1}\left\{\frac{p}{p^2+1}\right\} = \cos t$$
$$\Rightarrow \boxed{y = \cos t}$$

By $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$

Ex-2: Using Laplace Transform, find the solution of the initial value problem $\frac{d^2y}{dt^2} + 9y = 6\cos 3t$; $y(0) = 2$, $y'(0) = 0$.

(AKTU-2007, 2015)

Solⁿ: Given $y'' + 9y = 6\cos 3t \rightarrow (1)$ & $y(0) = 2,\ y'(0) = 0 \rightarrow (2)$

Taking Laplace Transform on both side of eqⁿ (1), we get

$$L(y'') + 9L(y) = 6L\{\cos 3t\}$$

By $L\cos at = \frac{p}{p^2+a^2}$

$$\Rightarrow [p^2L(y) - py(0) - y'(0)] + 9L(y) = 6\frac{p}{p^2+9}$$

$$\Rightarrow p^2L(y) - 2p + 9L(y) = \frac{6p}{p^2+9}$$

$$\Rightarrow (p^2+9)L(y) = \frac{6p}{p^2+9} + 2p$$

$$\Rightarrow L(y) = \frac{6p}{(p^2+9)^2} + \frac{2p}{p^2+9}$$

$$\Rightarrow y = L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} + L^{-1}\left\{\frac{2p}{p^2+9}\right\} \longrightarrow (3)$$

Now we have

$$L^{-1}\left\{\frac{1}{p^2+9}\right\} = \frac{\sin 3t}{3}$$

By $L\left\{\frac{1}{p^2+a^2}\right\} = \frac{\sin at}{a}$

$$\therefore L^{-1}\left[\frac{d}{dp}\frac{1}{p^2+9}\right] = (-1)^1 t^1 \frac{\sin 3t}{3}$$

By $L^{-1}\{f'(p)\} = (-1)^n t^1 F(t)$

$$\Rightarrow L^{-1}\left\{\frac{(-1)\times 2p}{(p^2+9)^2}\right\} = -\frac{t\sin 3t}{3}$$

$$\Rightarrow L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} = t\sin 3t$$

Then from (3),

$$y = L^{-1}\left\{\frac{6p}{(p^2+9)^2}\right\} + 2L^{-1}\left\{\frac{p}{p^2+9}\right\}$$

$$\boxed{y = t\sin 3t + 2\cos 3t}$$

Ex-3 → Solve the following diff. eqⁿ using Laplace Transform

$$\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - y = t^2e^t \text{ where } y(0)=1,\ \left(\frac{dy}{dt}\right)_{t=0}=0,\ \left(\frac{d^2y}{dt^2}\right)_{t=0}=-2.$$

(AKTU-2007, 2018)

Solⁿ → Given

$$y''' - 3y'' + 3y' - y = t^2e^t \rightarrow (1) \quad \& \quad \left.\begin{aligned} y(0)&=1 \\ y'(0)&=0 \\ y''(0)&=-2 \end{aligned}\right\} \rightarrow (2)$$

Taking L.T on both side of eqⁿ (1),

$$L(y''') - 3L(y'') + 3L(y') - L(y) = L\{e^t.t^2\}$$

$$\Rightarrow \left[p^3L(y) - p^2y(0) - py'(0) - y''(0)\right] - 3\left[p^2L(y) - py(0) - y'(0)\right] + 3\left[pL(y) - y(0)\right] - L(y) = \frac{2}{(p-1)^3}.$$

$$\Rightarrow \left[p^3L(y) - p^2 + 2\right] - 3\left[p^2L(y) - p\right] + 3\left[pL(y) - 1\right] - L(y) = \frac{2}{(p-1)^3}.$$

By $L\{t^n\} = \frac{n!}{p^{n+1}}$ & $L\{e^{at}t^n\} = \frac{n!}{(p-a)^{n+1}}$

$$\Rightarrow (p^3 - 3p^2 + 3p - 1)L(y) - p^2 + 2 + 3p - 3 = \frac{2}{(p-1)^3}.$$

$$\Rightarrow (p-1)^3L(y) = p^2 - 3p + 1 + \frac{2}{(p-1)^3}.$$

$$\Rightarrow L(y) = \frac{p^2 - 3p + 1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow L(y) = \frac{(p-1)^2 - p}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$= \frac{1}{(p-1)} - \frac{p}{(p-1)^3} + \frac{2}{(p-1)^6}.$$

$$= \frac{1}{(p-1)} - \frac{(p-1)+1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow L(y) = \frac{1}{p-1} - \frac{1}{(p-1)^2} - \frac{1}{(p-1)^3} + \frac{2}{(p-1)^6}$$

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p-1}\right\} - L^{-1}\left\{\frac{1}{(p-1)^2}\right\} - L^{-1}\left\{\frac{1}{(p-1)^3}\right\} + 2L^{-1}\left\{\frac{1}{(p-1)^6}\right\}.$$

$$= e^tL^{-1}\left\{\frac{1}{p}\right\} - e^tL^{-1}\left\{\frac{1}{p^2}\right\} - e^tL^{-1}\left\{\frac{1}{p^3}\right\} + 2e^tL^{-1}\left\{\frac{1}{p^6}\right\}.$$

$$= e^t.1 - e^t\frac{t}{1!} - e^t\frac{t^2}{2!} + 2e^t\frac{t^5}{5!}$$

By $L^{-1}\{f(p-a)\} = e^{at}L^{-1}f(p)$

$$\Rightarrow \boxed{y = e^t\left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right)}$$

By $L^{-1}\frac{1}{p^n} = \frac{t^{n-1}}{(n-1)!}$

Ex-4) Using Laplace Transform, solve the following D.E:

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = e^{-t}\sin t, \text{ where } x(0)=0 \text{ \& } x'(0)=1$$

(AKTU-2005, 2011)

Solⁿ) Give $x'' + 2x' + 5x = e^{-t}\sin t \rightarrow (1)$ & $\left.\begin{matrix} x(0)=0 \\ x'(0)=1 \end{matrix}\right\} \rightarrow (2)$

Taking L.T of both side in eqⁿ (1),

$$L\{x''\} + 2L\{x'\} + 5L\{x\} = L\{e^{-t}\sin t\}$$

$$\Rightarrow [p^2L(x) - px(0) - x'(0)] + 2[pL(x) - x(0)] + 5L(x) = \frac{1}{(p+1)^2+1}$$

By $L\{\sin t\} = \frac{1}{p^2+1}$ & $L\{e^{at}\sin t\} = \frac{1}{(p-a)^2+1}$

$$\Rightarrow (p^2+2p+5)L(x) - 1 = \frac{1}{p^2+2p+2}$$

$$\Rightarrow (p^2+2p+5)L(x) = \frac{1}{(p^2+2p+2)} + 1$$

$$\Rightarrow L(x) = \frac{1}{(p^2+2p+2)(p^2+2p+5)} + \frac{1}{p^2+2p+5}$$

$$= \frac{1}{3}\left[\frac{(p^2+2p+5) - (p^2+2p+2)}{(p^2+2p+2)(p^2+2p+5)}\right] + \frac{1}{p^2+2p+5}$$

$$= \frac{1}{3}\frac{1}{p^2+2p+2} - \frac{1}{3}\frac{1}{p^2+2p+5} + \frac{1}{p^2+2p+5}$$

$$\Rightarrow L(x) = \frac{1}{3}\frac{1}{p^2+2p+2} + \frac{2}{3}\frac{1}{p^2+2p+5}$$

$$\Rightarrow x = \frac{1}{3}L^{-1}\left\{\frac{1}{p^2+2p+2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{p^2+2p+5}\right\}.$$

$$\Rightarrow x = \frac{1}{3}L^{-1}\left\{\frac{1}{(p+1)^2+1^2}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{(p+1)^2+2^2}\right\}$$

$$\Rightarrow x = \frac{1}{3}e^{-t}L^{-1}\left\{\frac{1}{p^2+1^2}\right\} + \frac{2}{3}e^{-t}L^{-1}\left\{\frac{1}{p^2+2^2}\right\}$$

$$\Rightarrow x = \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\frac{\sin 2t}{2}$$

By $L^{-1} f(p-a) = e^{at}L^{-1}f(p)$

$$\Rightarrow \boxed{x = \frac{1}{3}e^{-t}(\sin t + \sin 2t)}$$

By $L^{-1}\frac{a}{p^2+a^2} = \frac{\sin at}{a}$

Ex-5: Using Laplace Transformation, solve the D.E

$$\frac{d^2x}{dt^2} + 9x = \cos 2t \quad \text{if } x(0) = 1, \; x\left(\frac{\pi}{2}\right) = -1.$$ (AKTU-2017).

Sol^n: Given $x'' + 9x = \cos 2t \rightarrow (1)$ & $\left.\begin{matrix} x(0)=1 \\ x(\frac{\pi}{2})=-1 \end{matrix}\right\} \rightarrow (2)$.

Taking the Laplace Transform of both side, we get

$$L\{x''\} + 9L\{x\} = L\{\cos 2t\}.$$

$$\Rightarrow [p^2 L(x) - p\,x(0) - x'(0)] + 9L(x) = \frac{p}{p^2+4}$$

$$\Rightarrow (p^2+9)L(x) - p - A = \frac{p}{p^2+4}$$

$\boxed{\text{from (2) \& } x'(0) = A}$

$$\Rightarrow (p^2+9)L(x) = \frac{p}{p^2+4} + p + A$$

$$\Rightarrow L(x) = \frac{p}{(p^2+4)(p^2+9)} + \frac{p}{p^2+9} + \frac{A}{p^2+9} \longrightarrow (3)$$

Now $\frac{1}{(p^2+4)(p^2+9)}$ take $p^2 = m$

$$\Rightarrow \frac{1}{(m+4)(m+9)} = \frac{1/5}{m+4} - \frac{1/5}{m+9}$$ By Partial fraction

$$\Rightarrow \frac{1}{(p^2+4)(p^2+9)} = \frac{1}{5}\frac{1}{p^2+4} - \frac{1}{5}\frac{1}{p^2+9}$$

Then (3) becomes,

$$L(x) = \frac{1}{5}\frac{p}{p^2+4} - \frac{1}{5}\frac{p}{p^2+9} + \frac{p}{p^2+9} + \frac{A}{p^2+9}$$

$$\Rightarrow L(x) = \frac{1}{5}\frac{p}{p^2+4} + \frac{4}{5}\frac{p}{p^2+9} + \frac{A}{p^2+9}.$$

$$\Rightarrow x = \frac{1}{5}L^{-1}\left\{\frac{p}{p^2+4}\right\} + \frac{4}{5}L^{-1}\left\{\frac{p}{p^2+9}\right\} + A\,L^{-1}\left\{\frac{1}{p^2+9}\right\}$$

$$\Rightarrow \boxed{x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + A\frac{\sin 3t}{3}} \rightarrow (4)$$

By
$L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$
$L^{-1}\left\{\frac{a}{p^2+a^2}\right\} = \sin at$

Put $t = \frac{\pi}{2}$ in (4), $-1 = -\frac{1}{5} + 0 + \frac{A}{3}\cdot(-1) \Rightarrow \boxed{A = \frac{12}{5}}$

Hence

$$\boxed{x(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t}$$

Ex-6: Solve by Laplace Transform: (AKTU-2008, 2016)

$\frac{d^2y}{dt^2} + y = t\cos 2t$, $t > 0$ given that $y = \frac{dy}{dt} = 0$ for $t = 0$.

Sol^n: Given $y'' + y = t\cos 2t \rightarrow ①$ & $y(0) = 0,\ y'(0) = 0] \rightarrow ②$

Then $L(y'') + L(y) = L(t\cos 2t)$

$\Rightarrow [p^2 L(y) - p\,y(0) - y'(0)] + L(y) = (-1)^1 \frac{d}{dp} L(\cos 2t)$

$\Rightarrow p^2 L(y) + L(y) = (-1)\frac{d}{dp}\left(\frac{p}{p^2+4}\right)$

[by $L\{t\,F(t)\} = (-1)^1 \frac{d}{dp} L\{F(t)\}$]

$\Rightarrow (p^2+1)L(y) = (-1)\left[\frac{(p^2+4)\cdot 1 - p\cdot 2p}{(p^2+4)^2}\right]$

$\Rightarrow (p^2+1)L(y) = \frac{(-1)(4-p^2)}{(p^2+4)^2}$

$\Rightarrow (p^2+1)L(y) = \frac{p^2-4}{(p^2+4)^2}$

$\Rightarrow L(y) = \frac{(p^2-4)}{(p^2+1)(p^2+4)^2} \longrightarrow ①$

Now $\frac{p^2-4}{(p^2+1)(p^2+4)^2}$ (Put $p^2 = m$)

$\Rightarrow \frac{m-4}{(m+1)(m+4)^2} = \frac{A}{(m+1)} + \frac{B}{(m+4)} + \frac{C}{(m+4)^2} \rightarrow ②$

$\therefore \frac{m-4}{(m+1)(m+4)^2} = \frac{-5/9}{m+1} + \frac{B}{m+4} + \frac{8/3}{(m+4)^2} \rightarrow ③$

Put $m = 0$ in ③,

$\frac{-4}{16} = -\frac{5}{9} + \frac{B}{4} + \frac{8}{3\times 16}$

$\Rightarrow -\frac{1}{4} + \frac{5}{9} - \frac{1}{6} = \frac{B}{4} \Rightarrow \boxed{B = \frac{5}{9}}$

Hence ② gives,

$L(y) = \frac{p^2-4}{(p^2+1)(p^2+4)^2} = \frac{-5/9}{p^2+1} + \frac{5/9}{p^2+4} + \frac{8/3}{(p^2+4)^2}$.

$\Rightarrow y = -\frac{5}{9}L^{-1}\left\{\frac{1}{p^2+1}\right\} + \frac{5}{9}L^{-1}\left\{\frac{1}{p^2+4}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{(p^2+4)^2}\right\}$.

$\Rightarrow y = -\frac{5}{9}\sin t + \frac{5}{9}\cdot\frac{\sin 2t}{2} + \frac{8}{3}\cdot\frac{1}{16}(\sin 2t - 2t\cos 2t)$

[By C.T. $L^{-1}\left\{\frac{1}{(p^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at)$]

Q-7) Determine the response of damped mass-spring system under the ~~eq~~ square wave given by (AKTU - 2007, 2013, 2017)

$$y'' + 3y' + 2y = U(t-1) - U(t-2), \quad y(0)=0, \; y'(0)=0.$$

Solⁿ Given

$$y'' + 3y' + 2y = U(t-1) - U(t-2) \rightarrow (1) \quad \& \quad \left.\begin{matrix} y(0)=0 \\ y'(0)=0 \end{matrix}\right\} \rightarrow (2)$$

Taking Laplace both side of eqⁿ (1),

$$L(y'') + 3L(y') + 2L(y) = L\{U(t-1)\} - L\{U(t-2)\}$$

$$\Rightarrow [p^2 L(y) - py(0) - y'(0)] + 3[pL(y) - y(0)] + 2L(y) = \frac{e^{-p}}{p} - \frac{e^{-2p}}{p}$$

$$\Rightarrow (p^2+3p+2)\, L(y) = \frac{e^{-p}}{p} - \frac{e^{-2p}}{p}$$

By $L\{U(t-a)\} = \frac{e^{-ap}}{p}$

$$\Rightarrow L(y) = \frac{e^{-p}}{p(p^2+3p+2)} - \frac{e^{-2p}}{p(p^2+3p+2)} \longrightarrow (3)$$

$$\Rightarrow y = L^{-1}\left\{e^{-p}\frac{1}{p(p^2+3p+2)}\right\} - L^{-1}\left\{e^{-2p}\frac{1}{p(p^2+3p+2)}\right\} \longrightarrow (3)$$

Let

$$f(p) = \frac{1}{p(p^2+3p+2)}$$

$$\Rightarrow f(p) = \frac{1}{p(p+1)(p+2)} = \frac{1/2}{p} - \frac{1}{p+1} + \frac{1/2}{p+2}$$

(by Partial fraction)

Then

$$L^{-1}\{f(p)\} = \frac{1}{2}L^{-1}\left\{\frac{1}{p}\right\} - L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p+2}\right\}$$

$$= \frac{1}{2}\cdot 1 - e^{-t} + \frac{1}{2}e^{-2t} = F(t) \text{ (say)} \longrightarrow (4)$$

Now from (3),

$$y = L^{-1}\{e^{-p}F(p)\} - L^{-1}\{e^{-2p}f(p)\}$$

$$= F(t-1)\,U(t-1) - F(t-2)\,U(t-2)$$

By $L^{-1}\{e^{-ap}f(p)\} = F(t-a)\,U(t-a)$

$$y = \left[\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right]U(t-1) - \left[\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right]U(t-2)$$

(From (4))

Ex-8: Solve by Laplace Transform

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t, \quad y(0)=1.$$

(AKTU-2015, 2017)

Sol^n: Given $y' + 2y + \int_0^t y\,dt = \sin t \rightarrow (1)$ $\quad y(0)=1 \rightarrow (2)$

Then $L(y') + 2L(y) + L\left\{\int_0^t y\,dt\right\} = L\{\sin t\}$

By $L\int_0^t F(t)\,dt = \frac{f(p)}{p} = \frac{L\{F(t)\}}{p}$

$$\Rightarrow [pL(y) - y(0)] + 2L(y) + \frac{L(y)}{p} = \frac{1}{p^2+1}$$

$$\Rightarrow \left[p + 2 + \frac{1}{p}\right]L(y) - 1 = \frac{1}{p^2+1}$$

$$\Rightarrow \frac{(p^2+2p+1)}{p}L(y) = \frac{1}{p^2+1} + 1 = \frac{p^2+2}{p^2+1}$$

$$\Rightarrow L(y) = \frac{p(p^2+2)}{(p^2+1)(p^2+2p+1)}$$

$$\Rightarrow L(y) = \frac{(p^3+2p)}{(p+1)^2(p^2+1)} = \frac{A}{p+1} + \frac{B}{(p+1)^2} + \frac{Cp+D}{p^2+1}$$

$$\Rightarrow L(y) = \frac{1}{p+1} - \frac{3}{2}\frac{1}{(p+1)^2} + \frac{1}{2}\frac{1}{p^2+1} \longrightarrow (3)$$

by Partial Fraction

$$\Rightarrow y = L^{-1}\left\{\frac{1}{p+1}\right\} - \frac{3}{2}L^{-1}\left\{\frac{1}{(p+1)^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{p^2+1}\right\}.$$

$$\Rightarrow y = e^{-t} - \frac{3}{2}e^{-t}L^{-1}\frac{1}{p^2} + \frac{1}{2}\sin t.$$

$$\Rightarrow \boxed{y = e^{-t} - \frac{3}{2}e^{-t}t + \frac{1}{2}\sin t}$$

Home Assignment

Ex-1→ Solve by using Laplace Transform

$y''(t) + 4y'(t) + 4y(t) = 6e^{-t}$; $y(0) = -2$, $y'(0) = 8$.

Ans $\boxed{y = 6e^{-t} - 8e^{-2t} - 2t\,e^{-2t}}$

Ex-2→ $y''' - 2y'' + 5y' = 0$, $y(0)=0$, $y'=1$ at $t=0$ and $y=1$ at $t=\frac{\pi}{8}$.

Ex-3→ $(D^2+n^2)x = a\sin(nt+\alpha)$, $x = Dx = 0$ at $t=0$.

Ex-4→ $y''' + 2y'' - y' - 2y = 0$ where $y=1$, $\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at $t=0$

Hint

Ans $L(y) = \frac{5}{3}\frac{1}{p-1} - \frac{1}{p+1} + \frac{1}{3(p+2)}$.

& $\boxed{y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}}$

Ex-5→ $(D^3 - D^2 - D + 1)y = 8t\,e^{-t}$, $y(0)=0$, $y'(0)=1$, $y''(0)=0$. (AKTU-2014)

Ans $\boxed{y = e^{-t}(1+2t+t^2) - e^{t}(1-t)}$

Ex-6→ $\frac{d^2x}{dt^2} + 16x = 2\sin 4t$; $x(0) = -\frac{1}{2}$, $x'(0) = 0$. (AKTU-2007, 2014)

Ans - $\boxed{x = -\left(\frac{1}{2} + \frac{t}{4}\right)\cos 4t + \frac{1}{16}\sin 4t}$

Ex-7→ $y'' + 2y' + y = t\,e^{-t}$; $y(0) = 1$, $y'(0) = -2$. (AKTU-2015)

Ans: $\boxed{y = \left(1 - t + \frac{t^3}{6}\right)e^{-t}}$

Ex-8→ $y'' + 9y = \sin 3t$, $y=0$, $\frac{dy}{dt} = 0$ at $t=0$ (AKTU-2012)

$\boxed{y = \frac{1}{18}(\sin 3t - 3t\cos 3t)}$

Ex-9→ $y'' + y = \sin t \cdot \sin 2t$; $y(0) = 1$, $y'(0) = 0$.

Ans $\boxed{y = \frac{15}{16}\cos t + \frac{t}{4}\sin t + \frac{1}{16}\cos 3t}$

Ex-10→ $y'' - 3y' + 2y = 4t + e^{3t}$, $y(0)=1$, $y'(0)=-1$.

Ans→ $\boxed{y = 3 + 2t - \frac{1}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}}$