Probability and Statistics: To p, or not to p?

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6.4 Linear programming

Linear programming is probably one of the most-used type of **quantitative business model**. It can be applied in any environment where **finite resources** must be allocated to competing activities or processes for maximum benefit, for example:

- selecting an investment portfolio of stocks to maximise return
- allocating a fixed budget between competing departments
- allocating lorries to routes to minimise the transportation costs incurred by a distribution company.

Optimisation models

All optimisation models have several common elements.

- **Decision variables**, or the variables whose values the decision-maker is allowed to choose. These are the variables which a company must know to function properly – they determine everything else.
- **Objective function** to be optimised – either maximised or minimised.
- **Constraints** which must be satisfied – physical, logical or economic restrictions, depending on the nature of the problem.

Microsoft Excel can be used to solve linear programming problems using *Solver*. Excel has its own terminology for optimisation.

- The **changing cells** contain values of the decision variables.
- The **objective cell** contains the objective function to be minimised or maximised.

- The **constraints** impose restrictions on the values in the changing cells.
- **Non-negativity** constraints imply that the changing cells must contain non-negative numbers.

Solving optimisation problems

The first step is the **model development step**. You must decide:

- the decision variables, the objective and the constraints
- how everything fits together, i.e. develop correct algebraic expressions and relate all variables with appropriate formulae.

The second step is to **optimise**.

- A *feasible solution* is a solution which satisfies all of the constraints.
- The *feasible region* is the set of all feasible solutions.
- An *infeasible solution* violates at least one of the constraints.
- The *optimal solution* is the feasible solution which optimises the objective.

The third step is to perform a **sensitivity analysis** – to what extent is the final solution *sensitive* to parameter values used in the model. We omit this stage in our simple example below.

Example

A frequent problem in business is the **product mix problem**. Suppose a company must decide on a product mix (how much of each product to introduce) to maximise profit.

Suppose a firm produces two types of chocolate bar – type A bars and type B bars. A type A bar requires 10 grammes of cocoa and 1 minute of machine time. A type B bar requires 5 grammes of cocoa and 4 minutes of machine time. So type A bars are more cocoa-intensive, while type B bars are more intricate requiring a longer production time.

Altogether 2,000 grammes of cocoa and 480 minutes of machine time are available each day. Assume no other resources are required.

The manufacturer makes 10 pence profit from each type A bar and 20 pence profit from each type B bar. Assume all chocolate bars produced are sold.

Define the **decision variables** as follows:

- $\boldsymbol{x}$ **= quantity of type A bars**
- $\boldsymbol{y}$ **= quantity of type B bars.**

The **objective function** is:

$$\mathbf{10}\boldsymbol{x} + \mathbf{20}\boldsymbol{y}$$

which should be maximised subject to the **constraints**:

$$\mathbf{10}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{2000} \quad \textbf{(cocoa)}$$

$$\boldsymbol{x} + \mathbf{4}\boldsymbol{y} = \mathbf{480} \quad \textbf{(machine time).}$$

We also require **non-negativity** for the solution to be economically meaningful, so:

$$\boldsymbol{x} \geq \mathbf{0} \quad \textbf{and} \quad \boldsymbol{y} \geq \mathbf{0}.$$

The maximum (and minimum) values of the objective function lie at a corner of the feasible region. The two lines **intersect** when both constraint equations are true, hence:

$$\mathbf{10}\boldsymbol{x} + \mathbf{5}\boldsymbol{y} = \mathbf{2000} \quad \textbf{and} \quad \boldsymbol{x} + \mathbf{4}\boldsymbol{y} = \mathbf{480}.$$

This happens at the point (160, 80). We can find the values of the objective function at each corner by substitution into $10x + 20y$. The four corners of the feasible region are:

- (0, 0), with profit $10x + 20y = 0$
- (0, 120), with profit $10x + 20y = 2400$
- (200, 0), with profit $10x + 20y = 2000$
- (160, 80), with profit $10x + 20y = 3200$.

The **optimal solution** is £32 (= 3200 pence) which occurs by making 160 type A bars and 80 type B bars.

In the accompanying Excel file, the problem is solved using *Solver*. Cells `A1` and `A2` are the changing cells (where the solution $x = 160$ and $y = 80$ is returned). Cell `B1` is the objective cell, with formula `=10*A1+20*A2` representing the objective function $10x+20y$. Cells `C1` and `C2` contain the constraints, with formulae `=10*A1+5*A2` and `=A1+4*A2`, respectively, representing the left-hand sides of the constraints $10x + 5y = 2000$ and $x + 4y = 480$.

With the *Solver* Add-in loaded, opening it shows that:

- 'Set Target Cell' is set to `$B$1`
- 'Equal To' is set to 'Max' (since we want to maximise the objective function)
- 'By Changing Cells' is set to `$A$1:$A$2` identifying the cells where the solution should be returned
- 'Subject to the Constraints' lists `$C$1 <= 2000` and `$C$2 <= 480` as the full constraints
- 'Options' allows for non-negativity to be stipulated by checking 'Assume Non-Negative'.