

# Numerical Methods

ESC101: Fundamentals of Computing

Purushottam Kar

# Agenda

- Finding the square root of a number
- Finding the root (zero) of an arbitrary function
- Learn a generalization of the binary search method

# Finding Square Roots

The Babylonian Method



# The trick of the Babylonians

- Square roots: studied since antiquity – will study a method called the “Babylonian method” – known for over 3000 years!!
- We have a real number  $v > 0$  and we wish to find  $\sqrt{v}$
- But  $\sqrt{v}$  may be irrational so it may be impossible to represent it using 32 bit floating point numbers ☹
- Change our goal to find a number  $r$  such that  $|\sqrt{v} - r| < \epsilon$
- Quantities like  $\epsilon$  are often called the *tolerance* of the algorithm
- Not unusual to have algorithms that offer  $\epsilon \approx 10^{-10}$  or so ☺

# The trick of the Babylonians

- Relies on a curious property of the square root
- Suppose we have an estimate  $x > 0$  of the true square root  $\sqrt{v}$ 
  - If  $x$  is an *overestimate* i.e.  $x > \sqrt{v}$ , then  $v/x < \sqrt{v}$  - gives us a cute result  
*For any  $x > 0, v > 0$  we always have  $\sqrt{v} \in [v/x, x]$*
  - A similar result if  $x$  is an *underestimate*: if  $x < \sqrt{v}$ , then  $v/x > \sqrt{v}$
- The Babylonian method exploits this to set up an *active region* over the entire positive real line  $\mathbb{R}_+$
- Will maintain invariant that the active region always contains  $\sqrt{v}$ 
  - The above cute result will help us keep our promise ☺
- Will halve the length of this active region at every step!

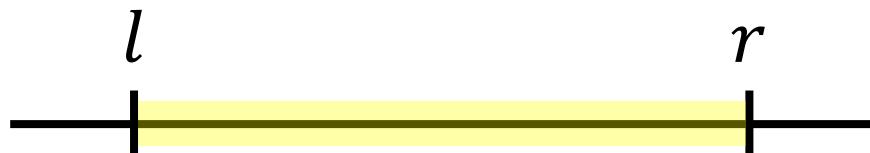
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- Exercise: prove that we have  $r \geq r^{\text{new}} \geq \sqrt{v}$  and  $l^{\text{new}} \geq l$
- This means that length of active region always shrinks by half ☺



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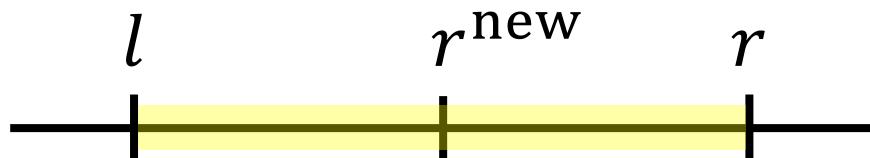
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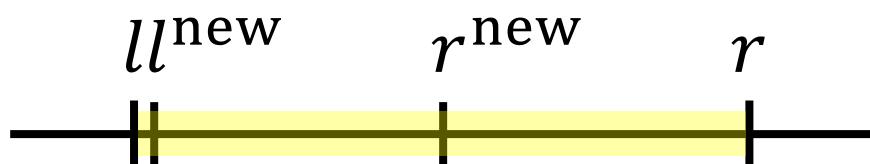
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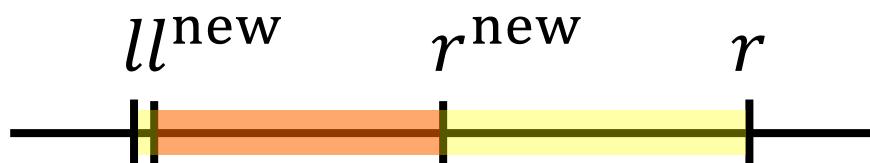
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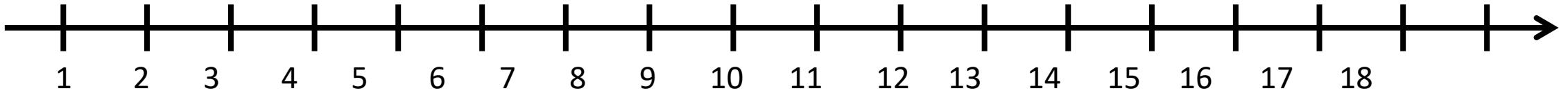
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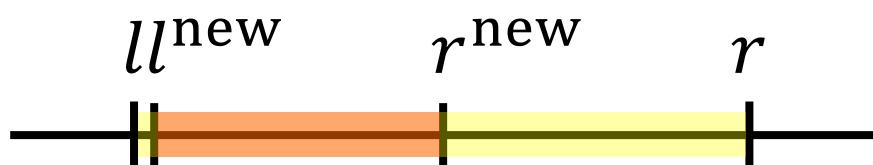


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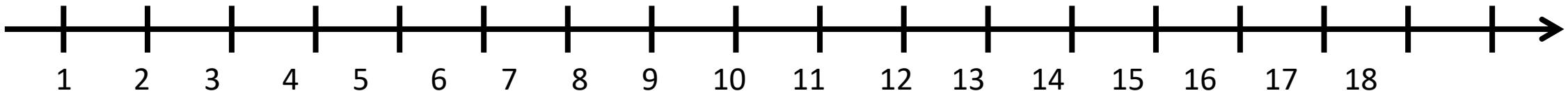
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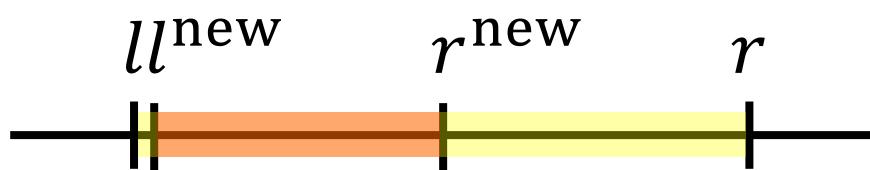
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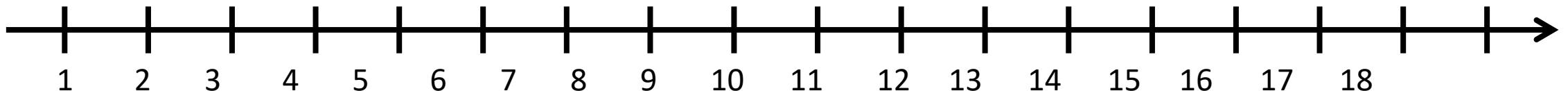
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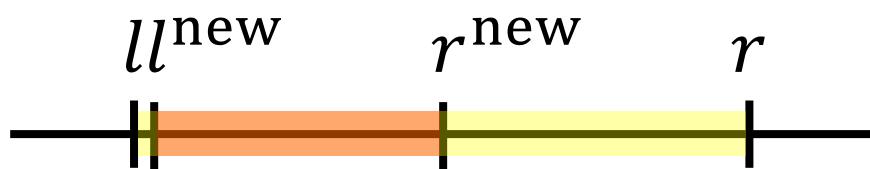
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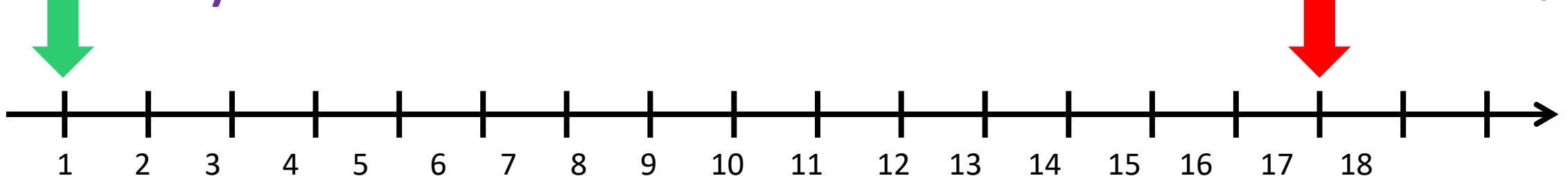


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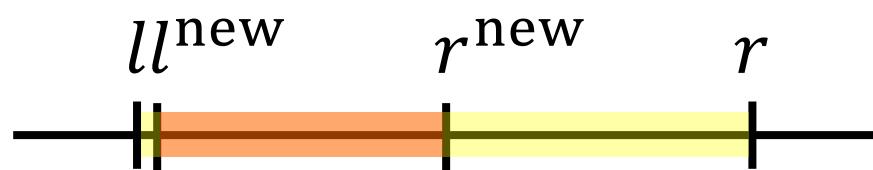


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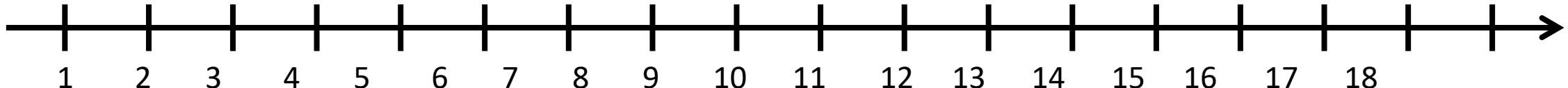
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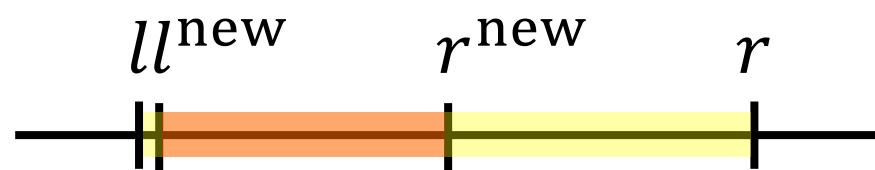
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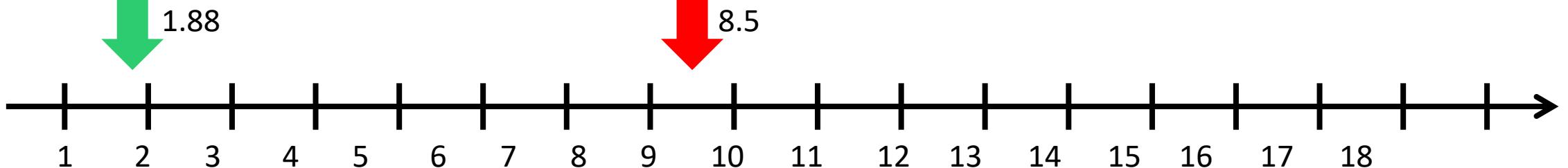
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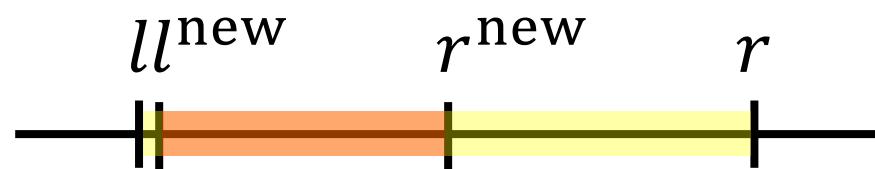
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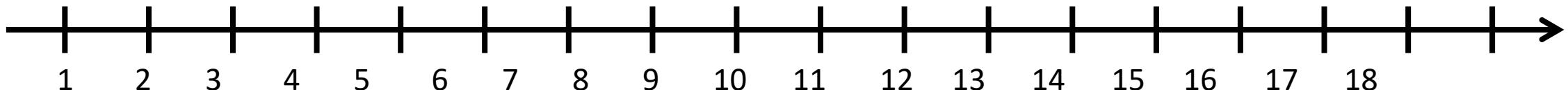
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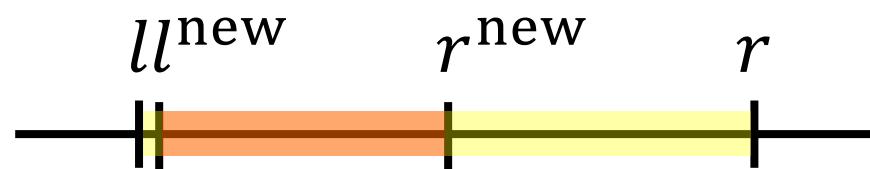
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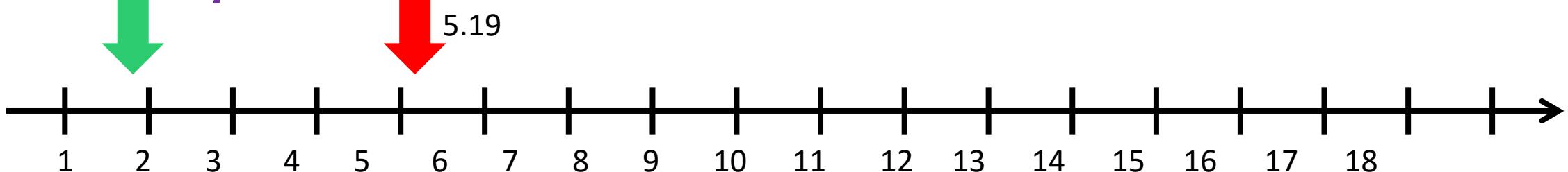
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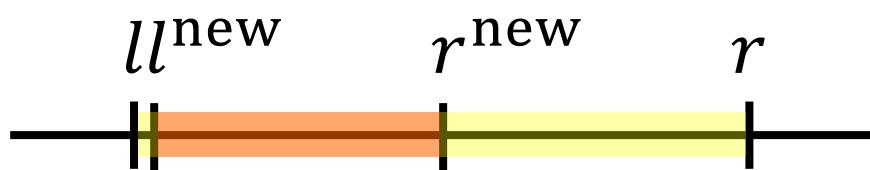
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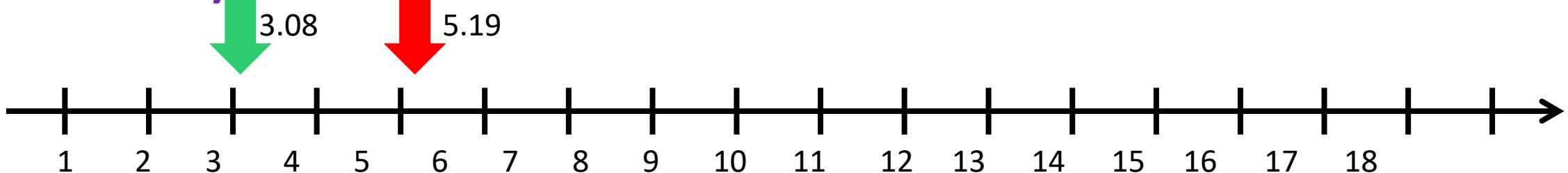
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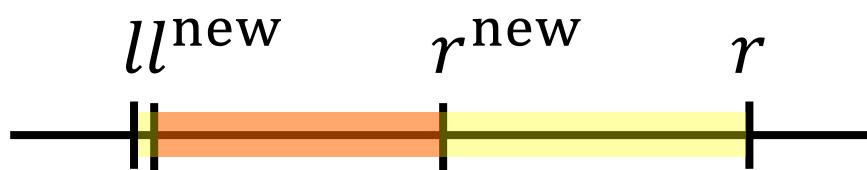
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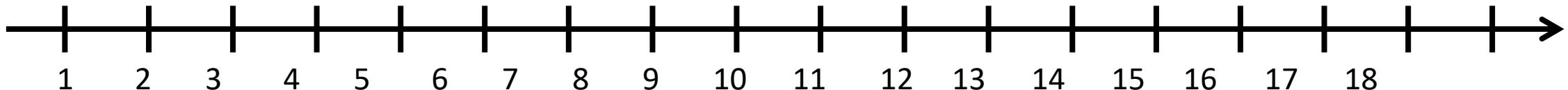
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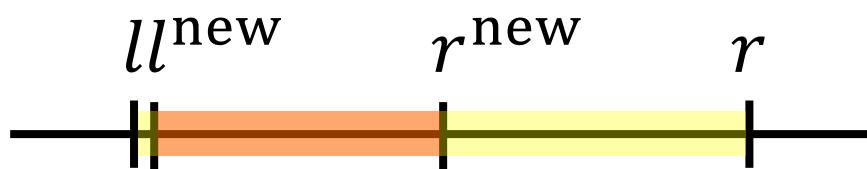
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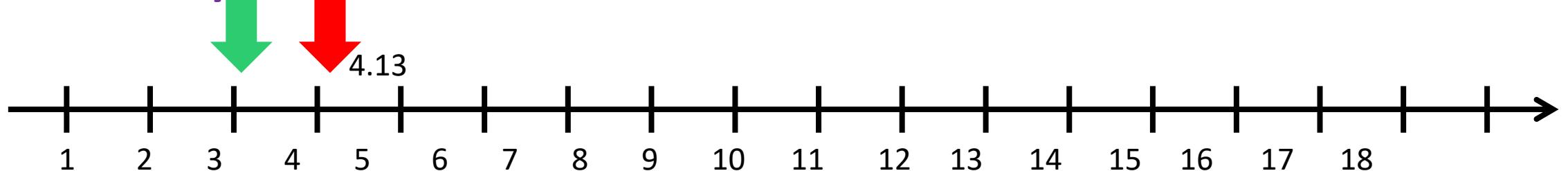
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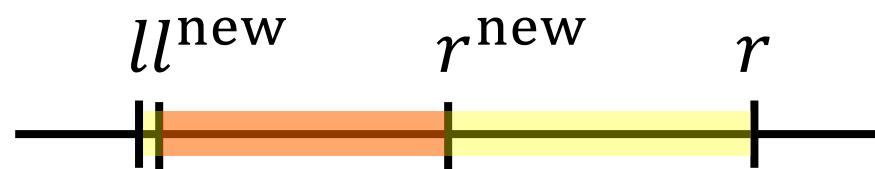
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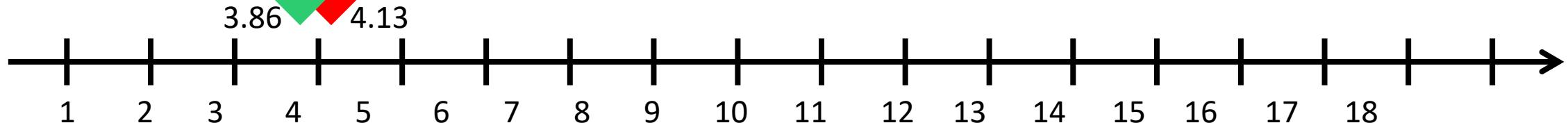
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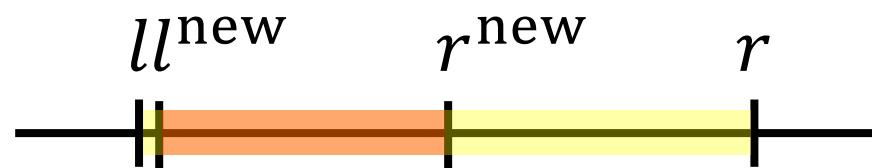
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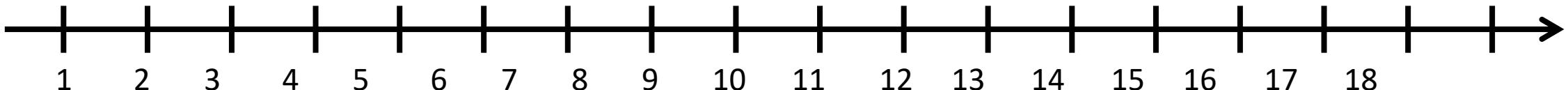
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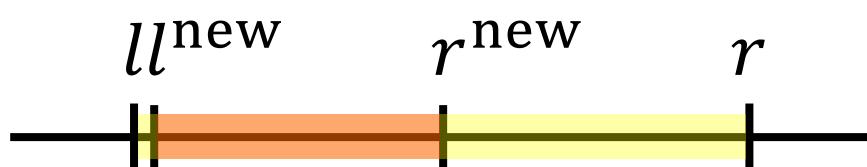
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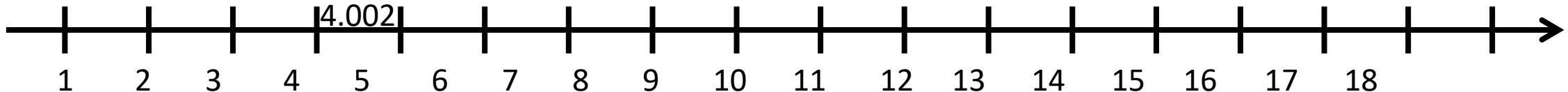
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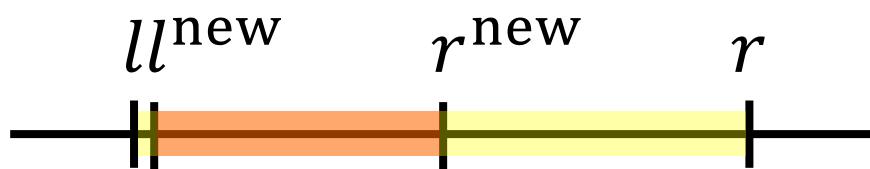
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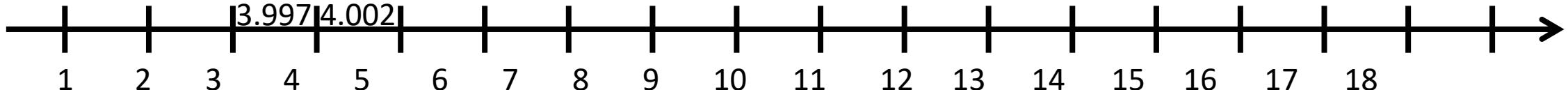
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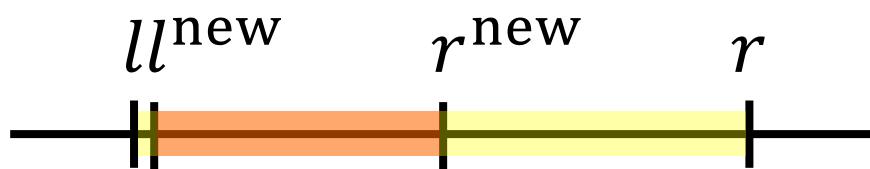
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## THE BABYLONIAN METHOD

1. Given: positive number  $v > 0$ , tolerance  $\epsilon > 0$
2. Initialize  $r \geq \sqrt{v}$  *//Initial overestimate*
3. Repeat
  1. Let  $l \leftarrow v/r$  *//Update left limit of active region*
  2. If  $r - l < \epsilon$ , return  $r$  *//If active region is tiny, we're done*
  3. Set  $r \leftarrow \frac{1}{2}(l + r)$  *//Shrink active region by half*

- At every step we are ensured  $\sqrt{v} \in [l, r]$ . If  $r - l < \epsilon$  then we must have  $r - \sqrt{v} < \epsilon$  as well as  $r \geq \sqrt{v}$  i.e.  $|r - \sqrt{v}| < \epsilon$  i.e. we are done!
- As active region halves every time, we will exit loop within  $\mathcal{O}\left(\log_2 \frac{1}{\epsilon}\right)$  iterations. For  $\epsilon = 10^{-10}$  this means only around 33 iterations 😊

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  3. Set  $r \leftarrow \frac{1}{2}(l + r)$  *re region by half*

Actually, a more careful analysis can show that the method takes only  $\mathcal{O}(\log \log \frac{1}{\epsilon})$  iterations – wow!

- At every step we are ensured  $\sqrt{v} \in [l, r]$ . If  $\sqrt{v} < l$  then we must have  $r - \sqrt{v} < \epsilon$  as well as  $r \geq \sqrt{v}$  i.e.  $|r - \sqrt{v}| < \epsilon$  i.e. we are done!
- As active region halves every time, we will exit loop within  $\mathcal{O}(\log_2 \frac{1}{\epsilon})$  iterations. For  $\epsilon = 10^{-10}$  this means only around 33 iterations 😊

# The Babylonian Method

## THE BABYLONIAN METHOD

1. Given: positive number  $v$

2. Initialize  $r \geq 0$

3. Repeat

1. Let  $l \leftarrow v/r$

2. If  $r - l < \epsilon$ , return  $r$

3. Set  $r \leftarrow \frac{1}{2}(l + r)$

Several algorithms exist for finding square roots. If you are interested, check out

[www.youtube.com/watch?v=Bwt5EZEb1Ns](https://www.youtube.com/watch?v=Bwt5EZEb1Ns)

- At every step we are ensured  $\sqrt{v} \in [l, r]$ . If  $\sqrt{v} > r$  then we must have  $r - \sqrt{v} < \epsilon$  as well as  $r \geq \sqrt{v}$  i.e.  $|r - \sqrt{v}| < \epsilon$  i.e. we are done!
- As active region halves every time, we will exit loop within  $\mathcal{O}\left(\log_2 \frac{1}{\epsilon}\right)$  iterations. For  $\epsilon = 10^{-10}$  this means only around 33 iterations 😊

Actually, a more careful analysis can show that the method takes only  $\mathcal{O}\left(\log \log \frac{1}{\epsilon}\right)$  iterations – wow!



# Finding Roots of Functions

The Bisection Method



# The Bisection Method

- Consider a continuous function  $f: \mathbb{R} \rightarrow \mathbb{R}$  (say a polynomial) – wish to find its root i.e. some  $x_0 \in \mathbb{R}$  such that  $f(x_0) = 0$
- Useful in finding eigenvalues of matrices (characteristic poly)
- Useful in optimization algorithms – wait a bit
- Suppose we are given an interval  $[a, b]$  so that  $f(a) < 0, f(b) > 0$
- Intermediate value theorem: there must lie a root of  $f$  in  $[a, b]$

# The Bisection Method

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Wait – this only tells us that there exist an *odd* number of roots in this interval – there may be more than one root

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Okay so suppose we want to find any one of those many roots ☺

# The Bisection Method

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- Intermediate value theorem: there must lie a root of  $f$  in  $[a, b]$
- Can we find that root? Or a good approximation to it?
- The bisection method does this by generalizing binary search
- Maintains an active region and is careful that the region always contains at least one root
- Will halve the size of that region at each step!



# The Bisection Method

- The secret sauce of bisection method is the intermediate value theorem combined with the binary search intuition
- Suppose we are ensured that  $f(a) < 0, f(b) > 0$ 
  - Think about what you would do if instead  $f(a) > 0, f(b) < 0$
- Suppose we calculate  $f(c)$  for some  $c \in (a, b)$  - three cases
  - $f(c) = 0$  Yay – we have found the root – go home and rest!
  - $f(c) < 0$  Apply IVT to  $[c, b]$  - there must lie a root in the interval  $[c, b]$

# The Bisection Method

- The secret sauce of bisection method is the intermediate value theorem combined with the fact that if  $f$  is continuous on  $[a, b]$  and  $f(a) > 0$  and  $f(b) < 0$ , then there is at least one root in  $(a, b)$ .  
Actually all this tells us that  $[c, b]$  contains an odd number of roots and  $[a, c]$  contains an even (possibly 0) number of roots
- Suppose we calculate  $f(c)$  for some  $c \in (a, b)$ . There are three cases
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  - $f(c) > 0$  Apply IVT to  $[a, c]$  - there must lie a root in the interval  $[a, c]$
- If we choose  $c = \frac{1}{2}(a + b)$  then no matter what the case we will halve the active region or else discover a root
- Once active region is tiny, we have found an approximate root



# The Bisection Method

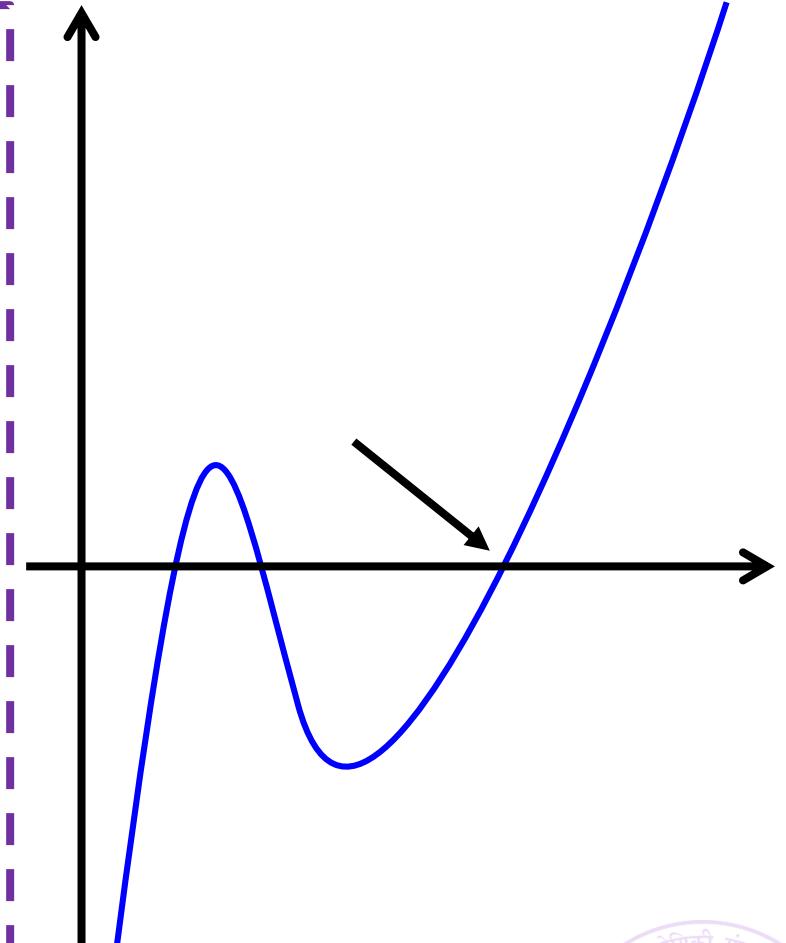
## THE BISECTION METHOD

1. Given: continuous function  $f$ , active region  $[a, b]$ ,  
tolerance  $\epsilon > 0$
2. Check  $f(a) \cdot f(b) < 0$       *//To apply IVT*
3. Repeat
  1. Let  $c = \frac{1}{2}(a + b)$       *//Mid point*
  2. If  $b - a < \epsilon$  return  $c$       *//Approx root*
  3. If  $f(c) = 0$  return  $c$       *//Found it!*
  4. If  $f(c) \cdot f(a) < 0$ ,  $b \leftarrow c$
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# The Bisection Method

## THE BISECTION METHOD

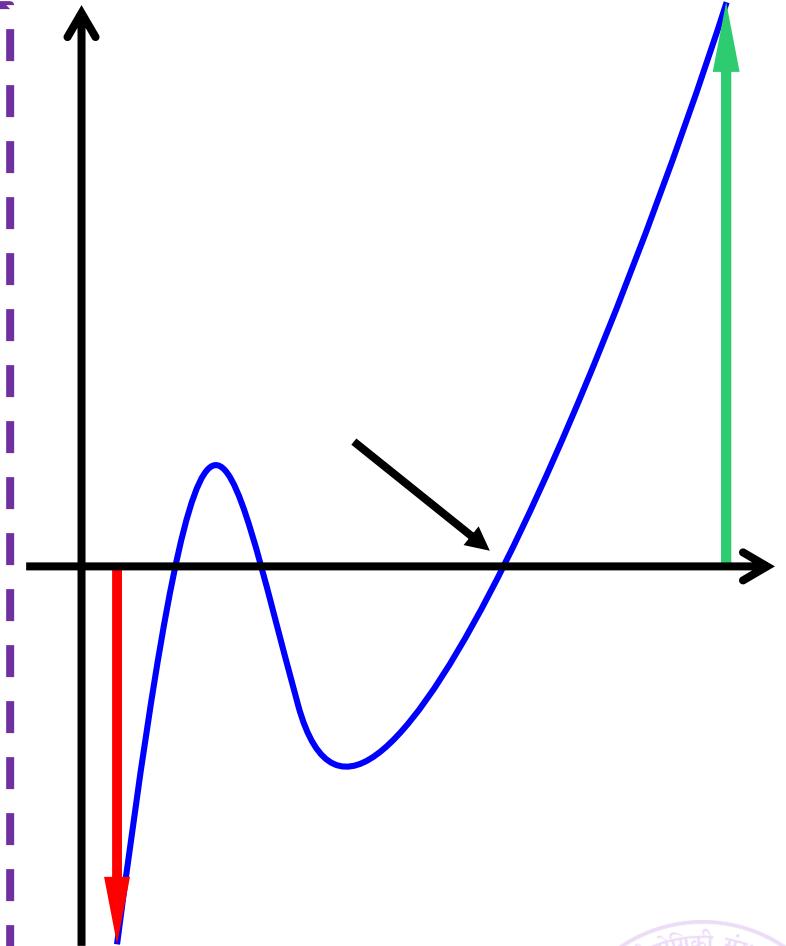
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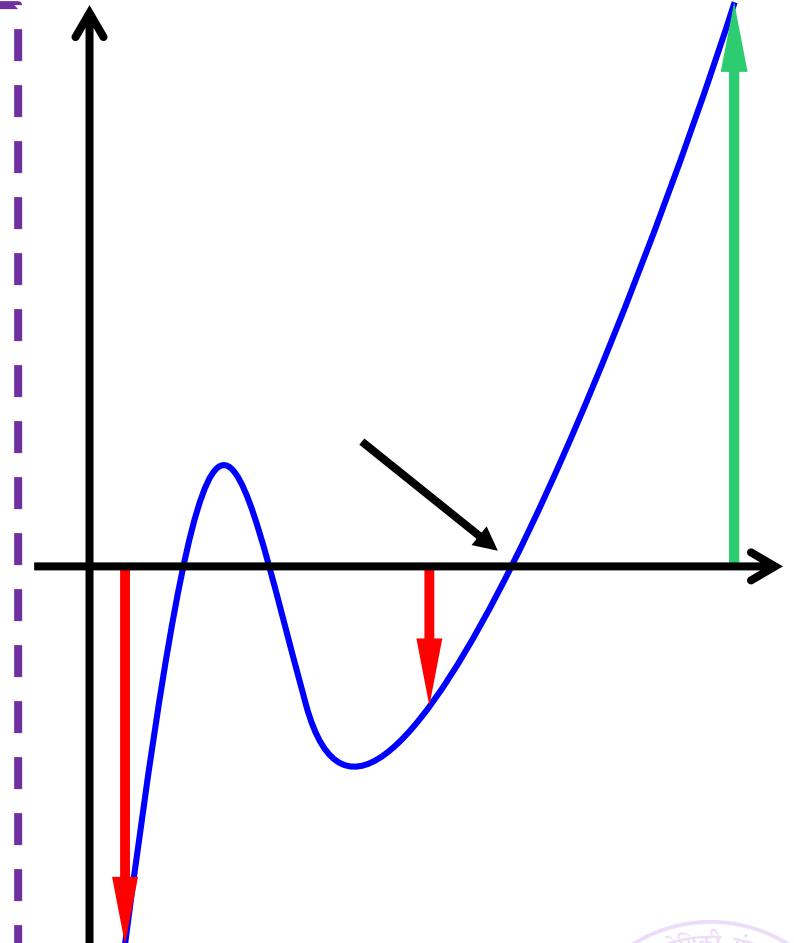
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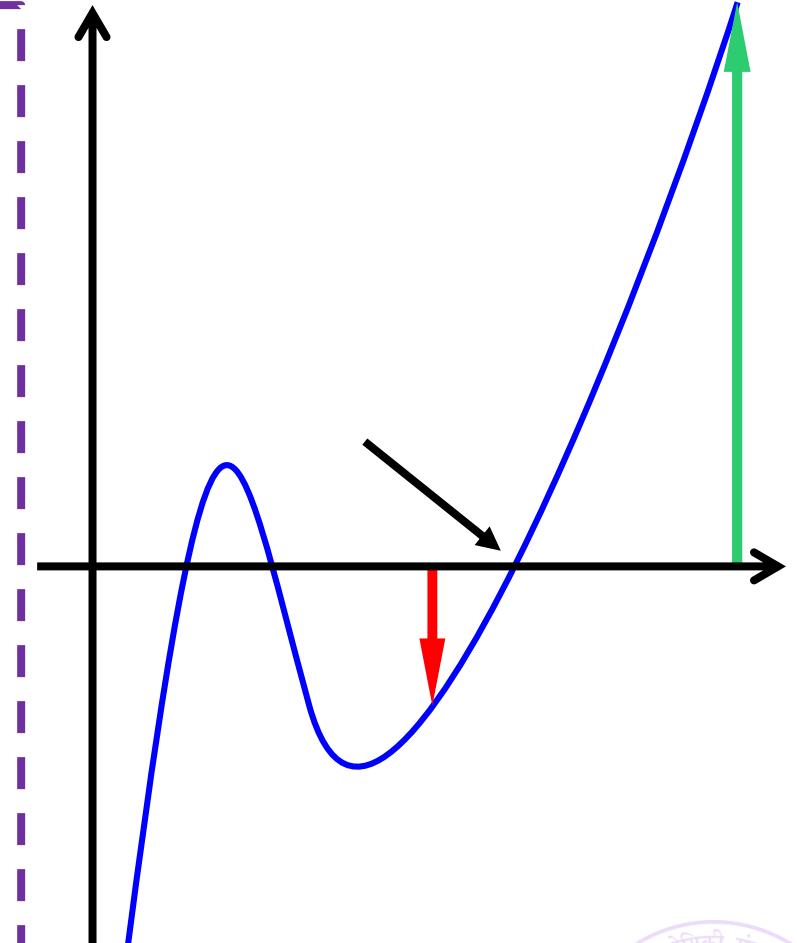
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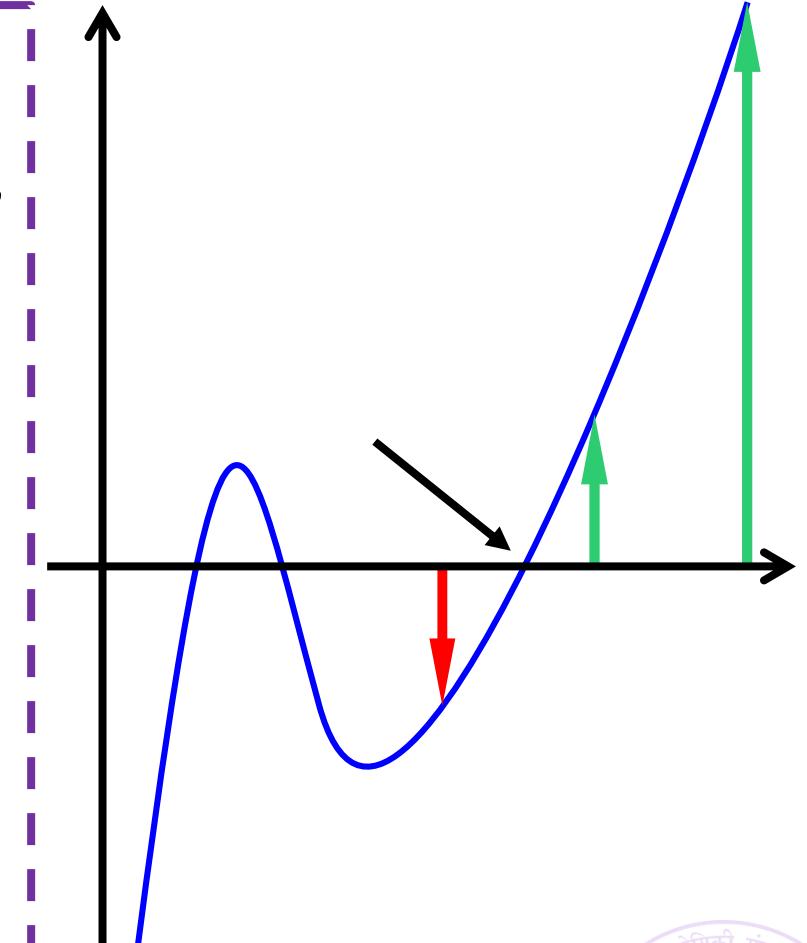
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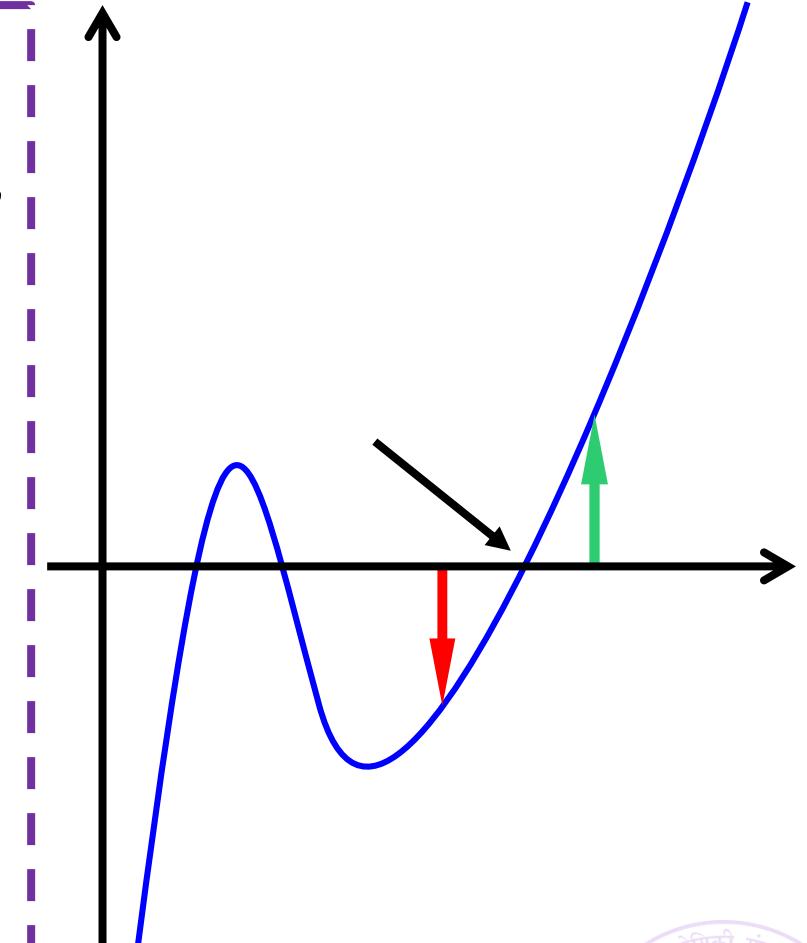
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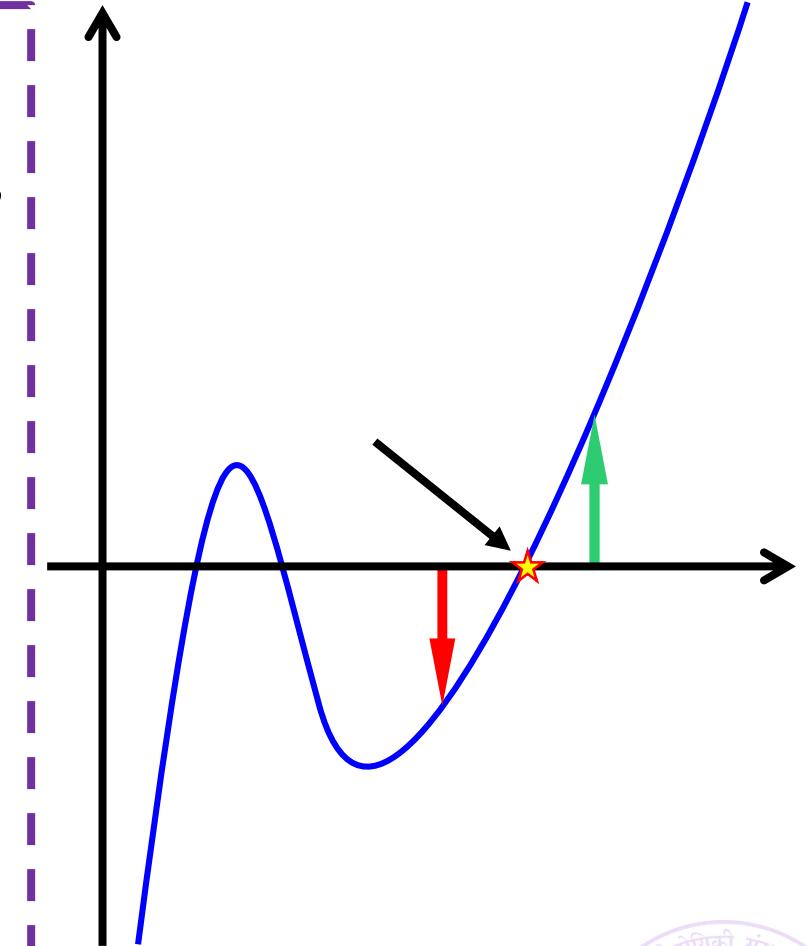
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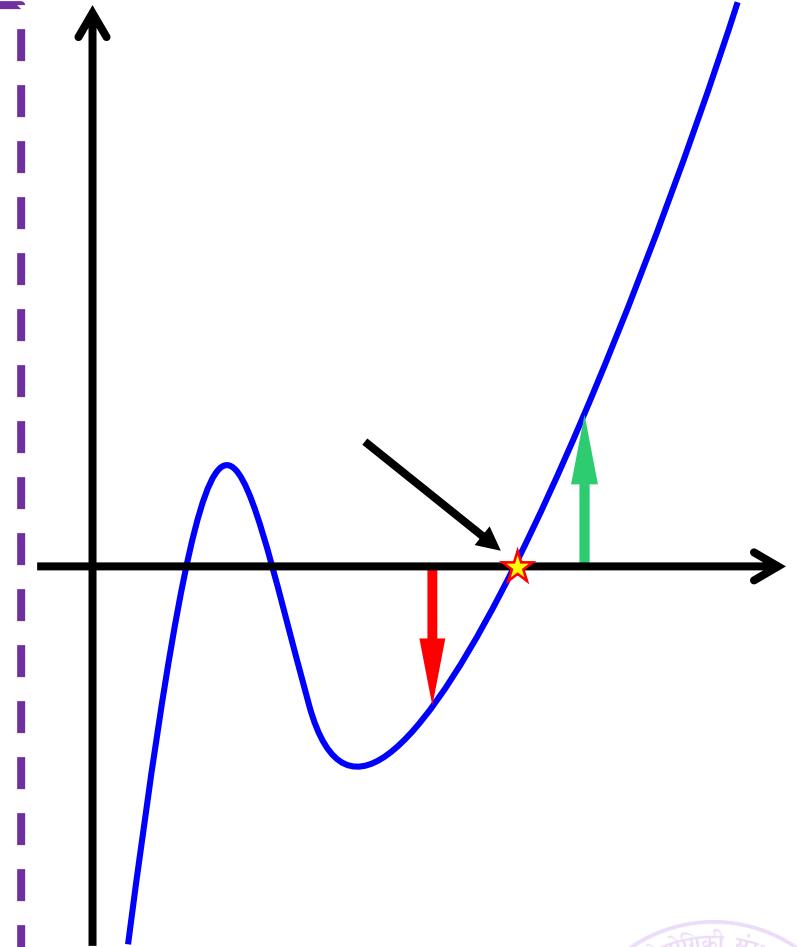


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Can you show that this method stops in  $\mathcal{O}\left(\log\frac{1}{\epsilon}\right)$  iterations?



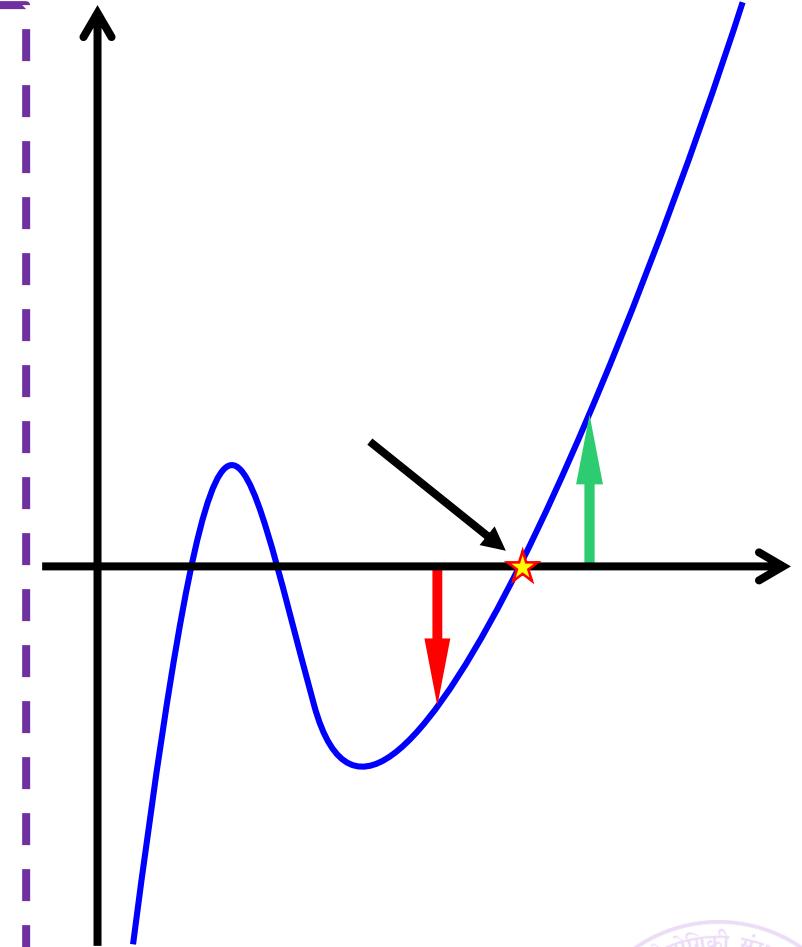
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Can you show that this method stops in  $\mathcal{O}\left(\log\frac{1}{\epsilon}\right)$  iterations?

Can you find the square root of numbers using this method? Hint  $f(x) = x^2 - v$

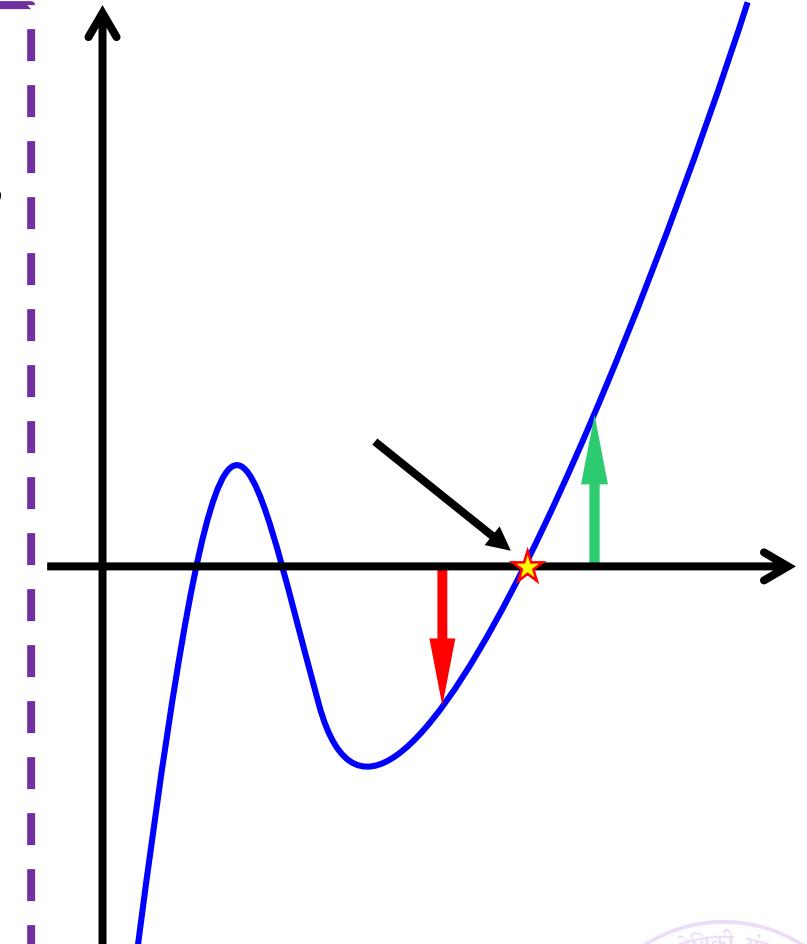


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Some of these intuitions extend to higher dimensions. If you are interested, check out [www.youtube.com/watch?v=b7FxPsqfkOY](https://www.youtube.com/watch?v=b7FxPsqfkOY)
3. If  $f(c) = 0$  return  $c$  *//Found it!*
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# Finding Roots of Functions

The Newton Raphson Method



# The Newton-Raphson Method

## NEWTON-RAPHSON METHOD

1. Initialize  $x$
2. Repeat
  1. Approximate  $f$  by  $g(y) = f(x) + f'(x) \cdot (y - x)$
  2. Update  $x \leftarrow \text{ROOT}(g) = x - \frac{f(x)}{f'(x)}$



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Tangent to  $f$  at  $x$



# The Newton-Raphson Method

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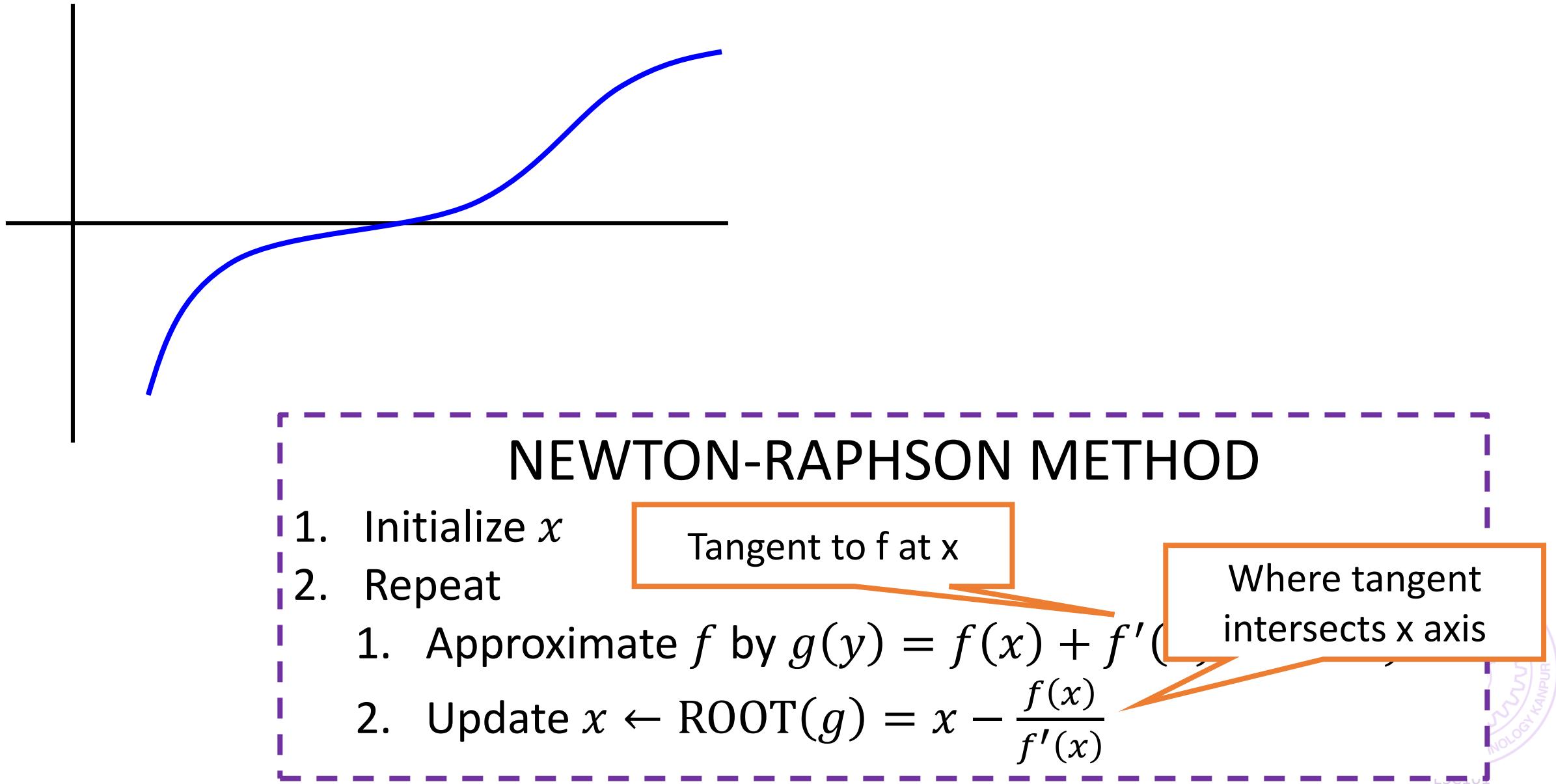
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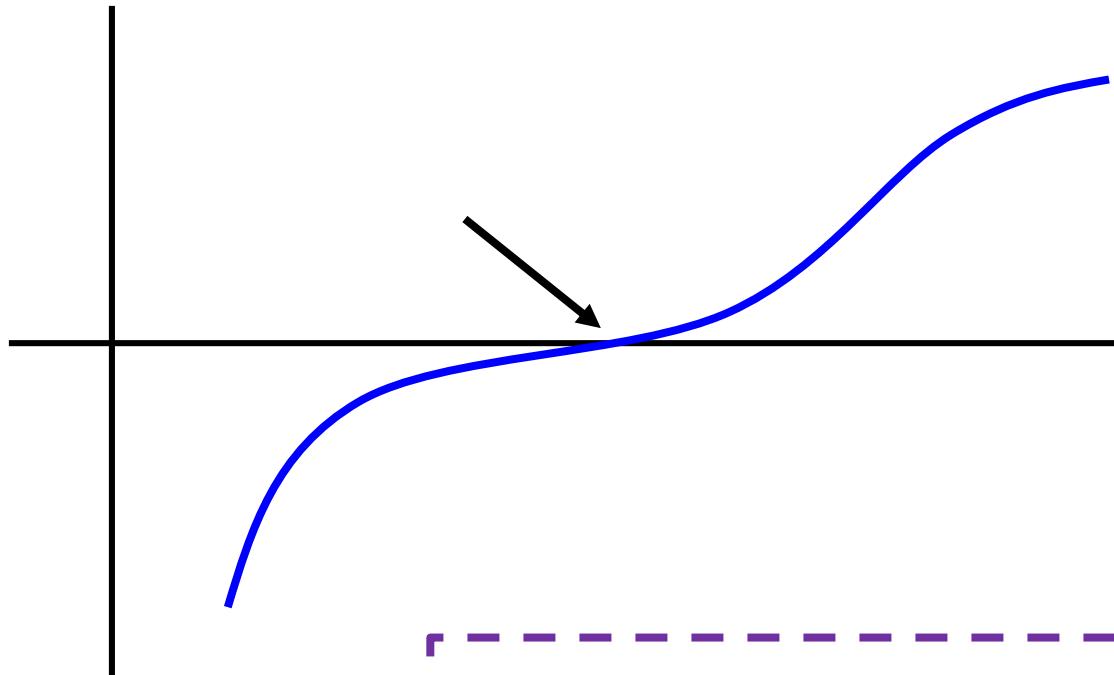
Where tangent intersects x axis

# The Newton-Raphson Method



# The Newton-Raphson Method

$$x : f(x) = 0$$



## NEWTON-RAPHSON METHOD

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Tangent to  $f$  at  $x$

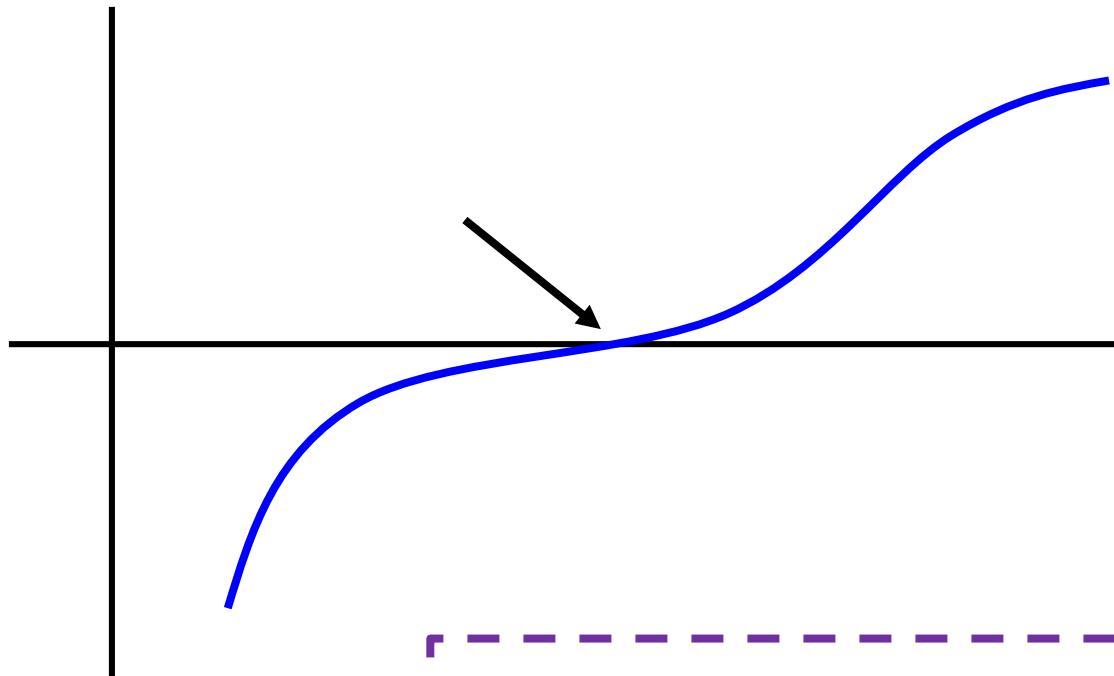
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# The Newton-Raphson Method



$$x : f(x) = 0$$

Finding roots of linear functions is easy  
 $f(x) = ax + b, x_0 = \frac{-b}{a}$

## NEWTON-RAPHSON METHOD

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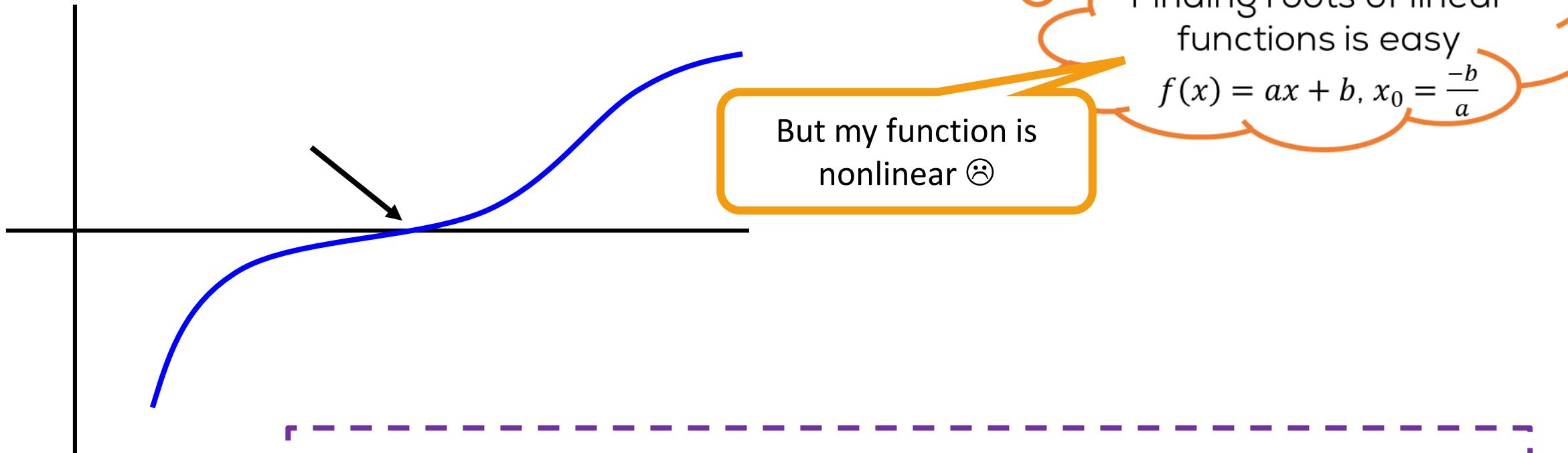
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# The Newton-Raphson Method



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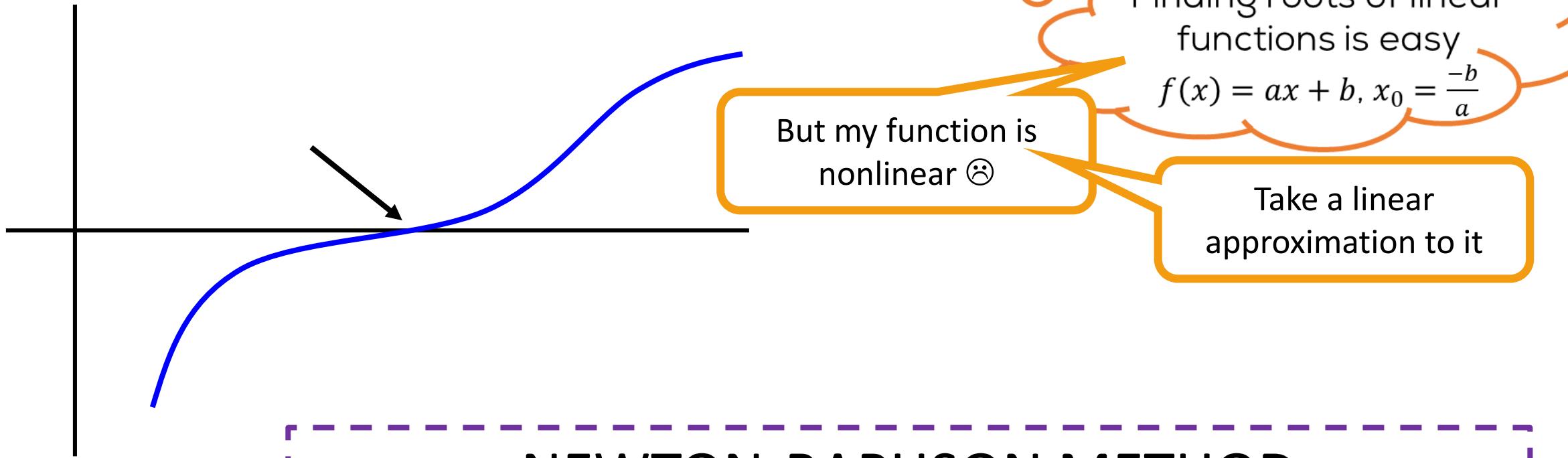
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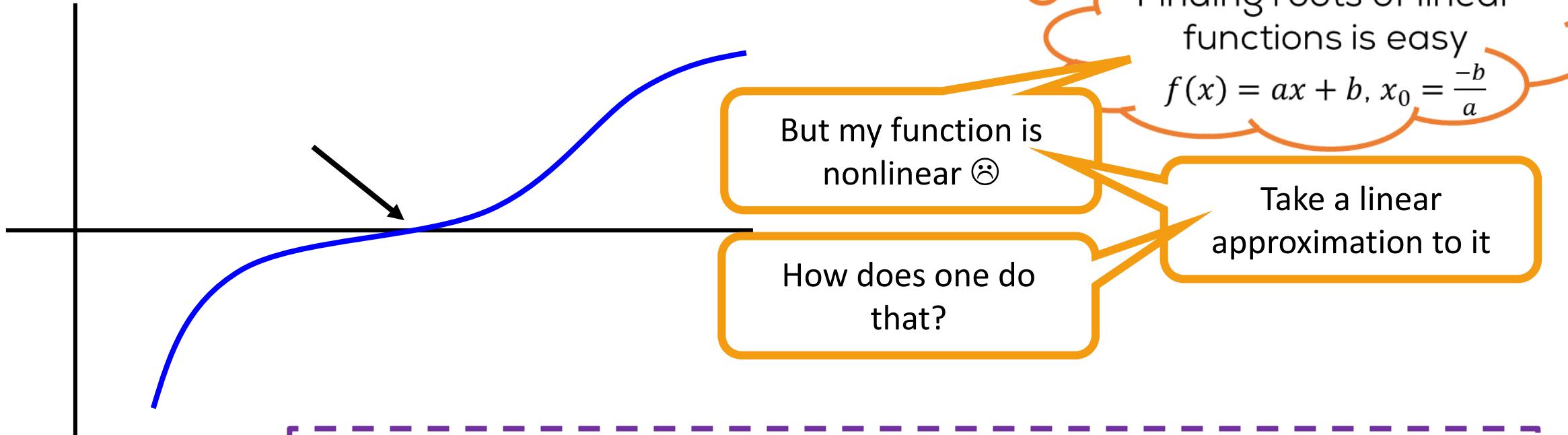
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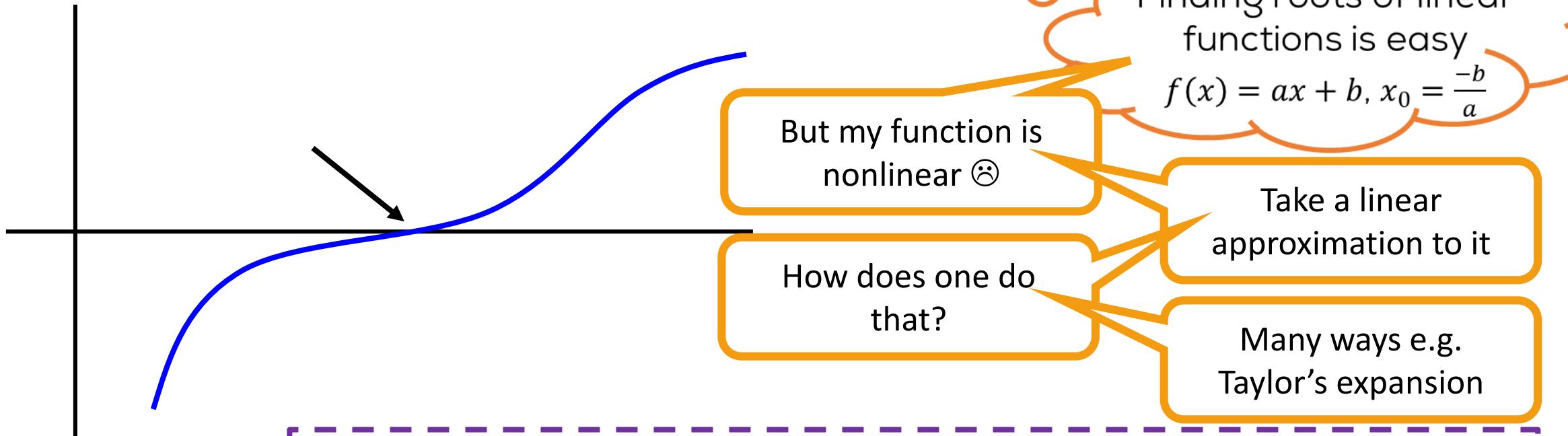
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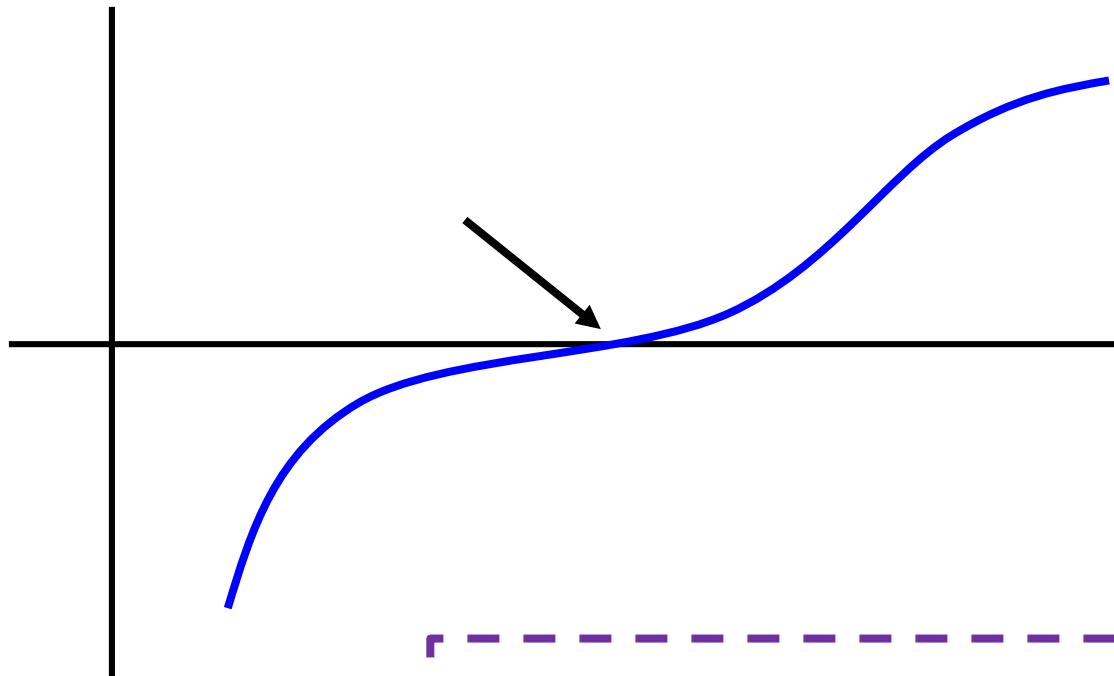
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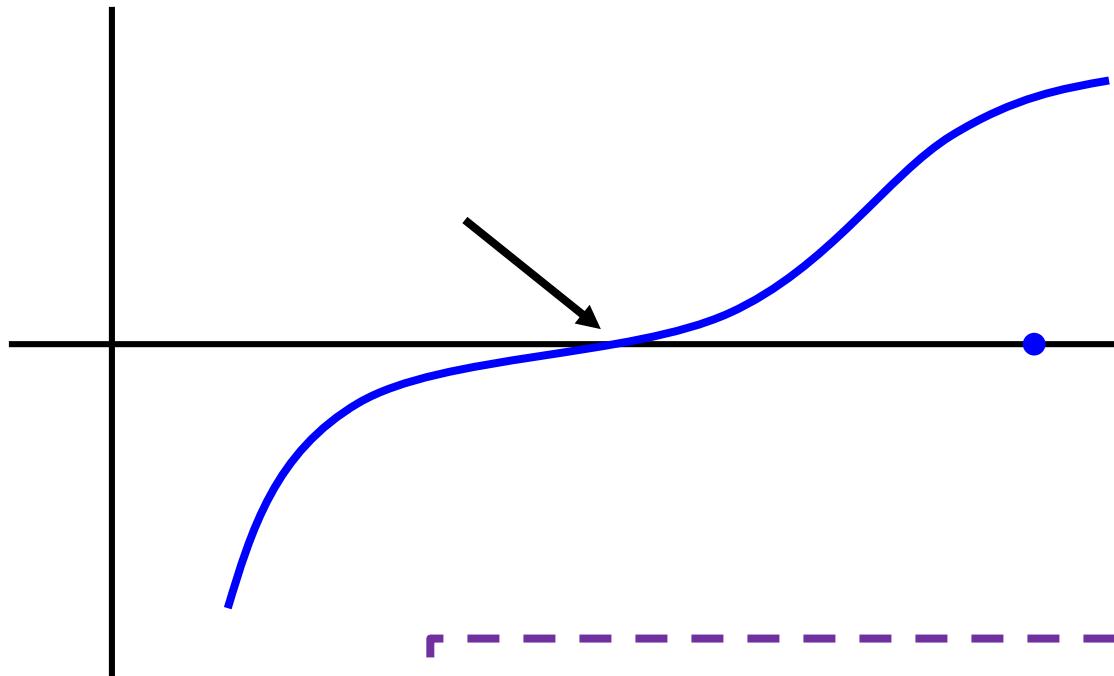
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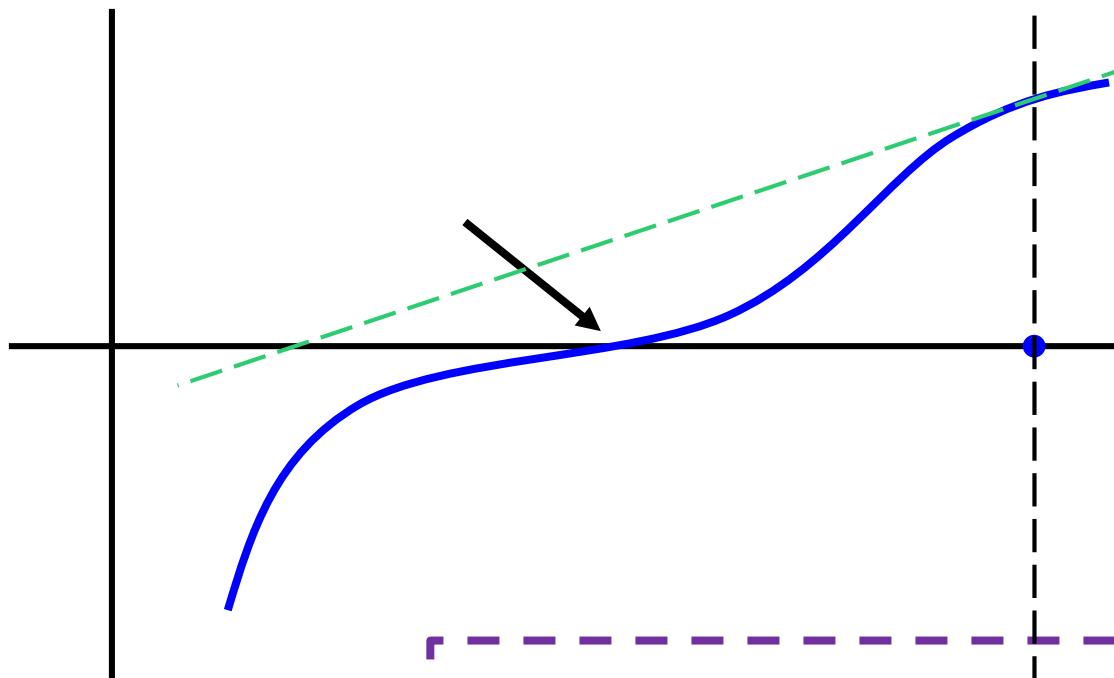
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Where tangent intersects x axis

# The Newton-Raphson Method



$$x : f(x) = 0$$

Finding roots of linear functions is easy  
 $f(x) = ax + b, x_0 = \frac{-b}{a}$

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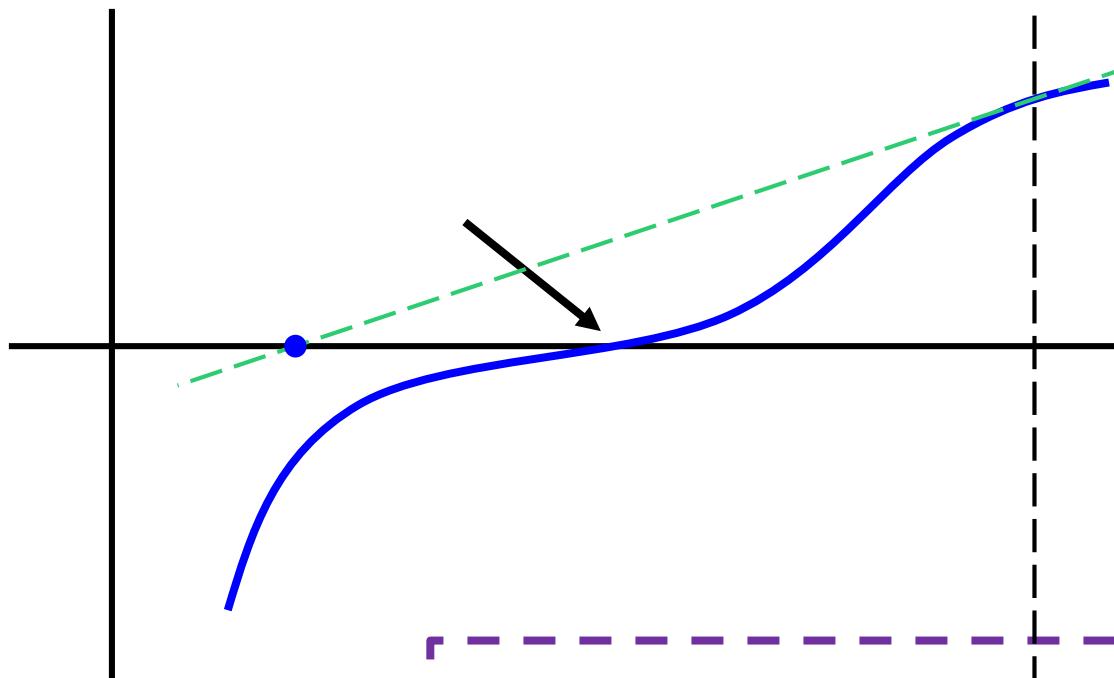
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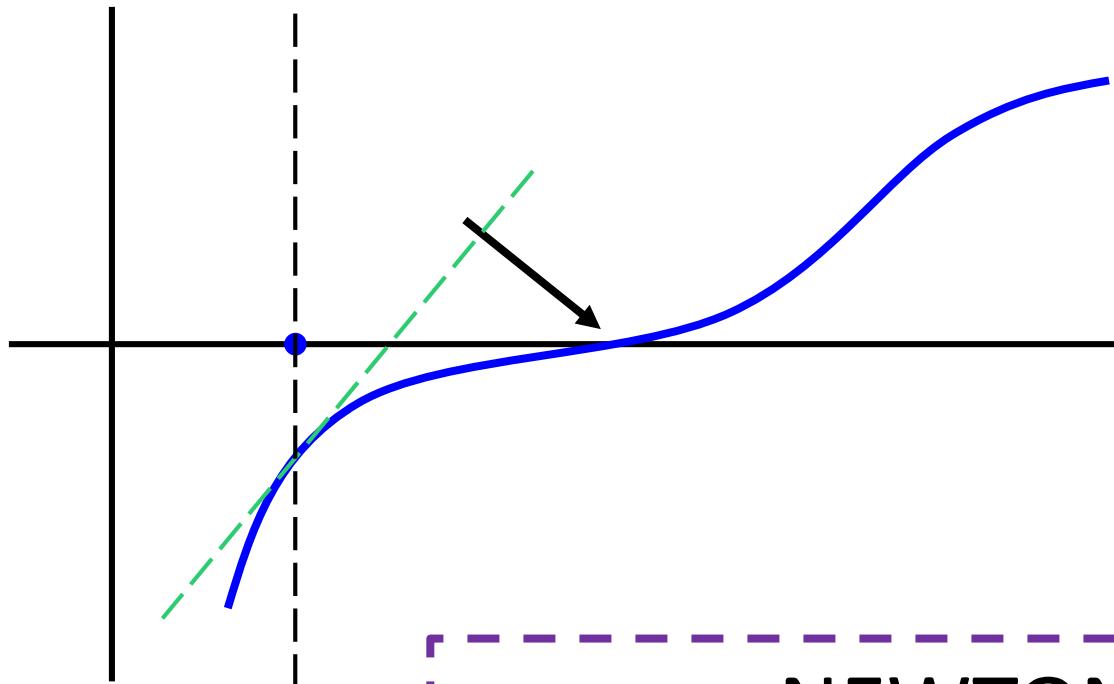
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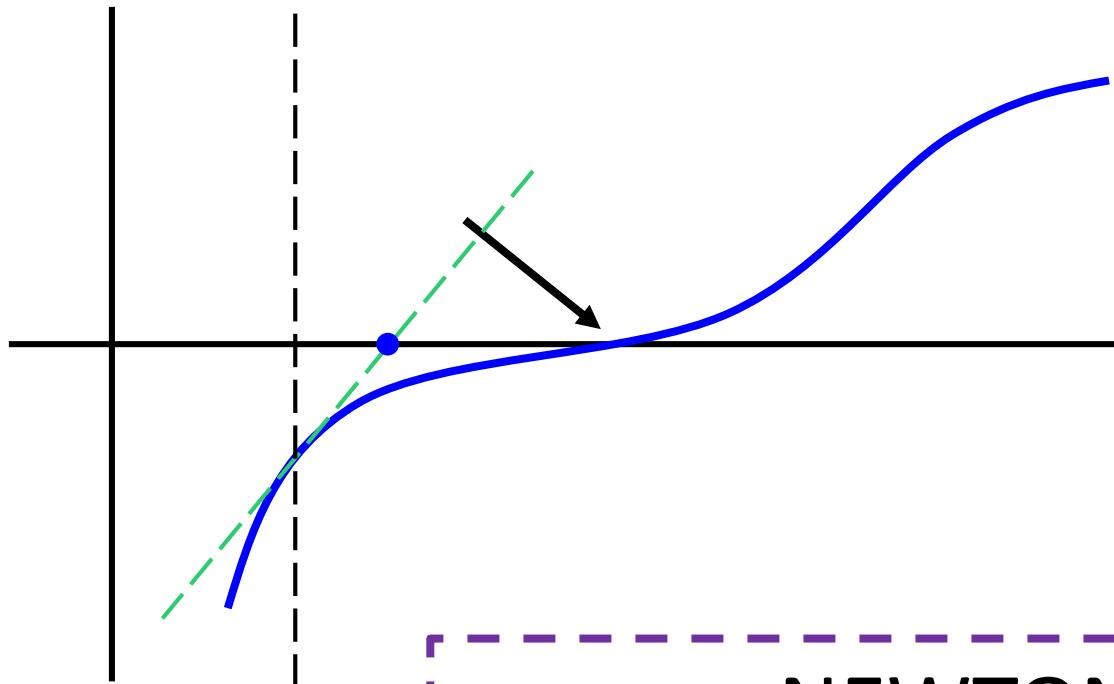
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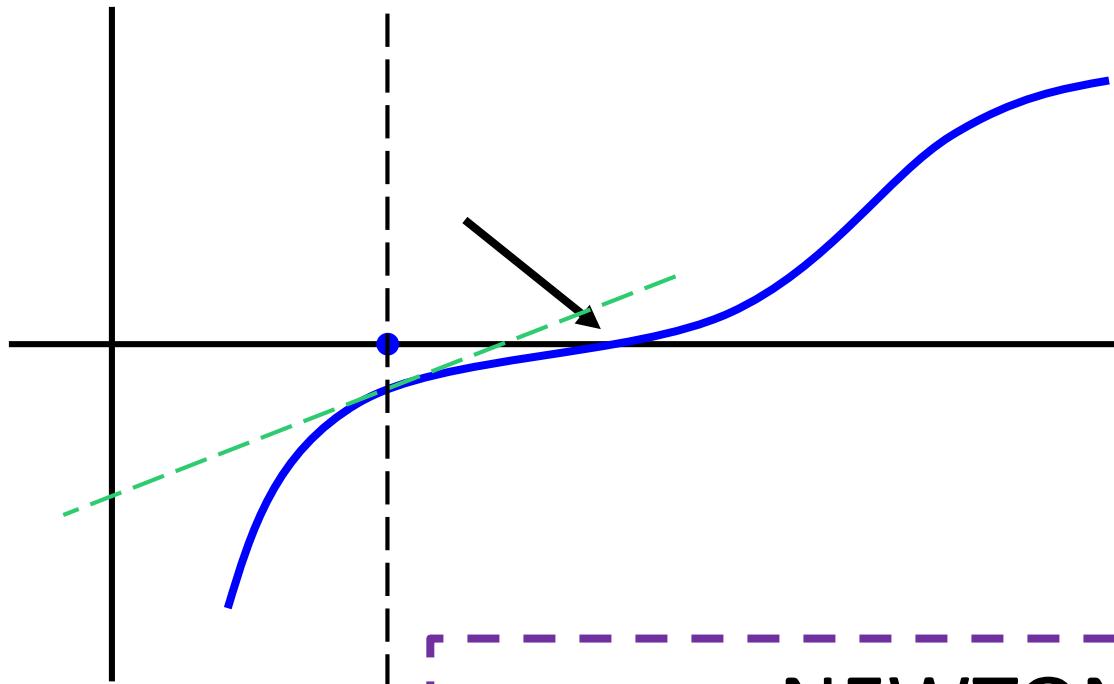
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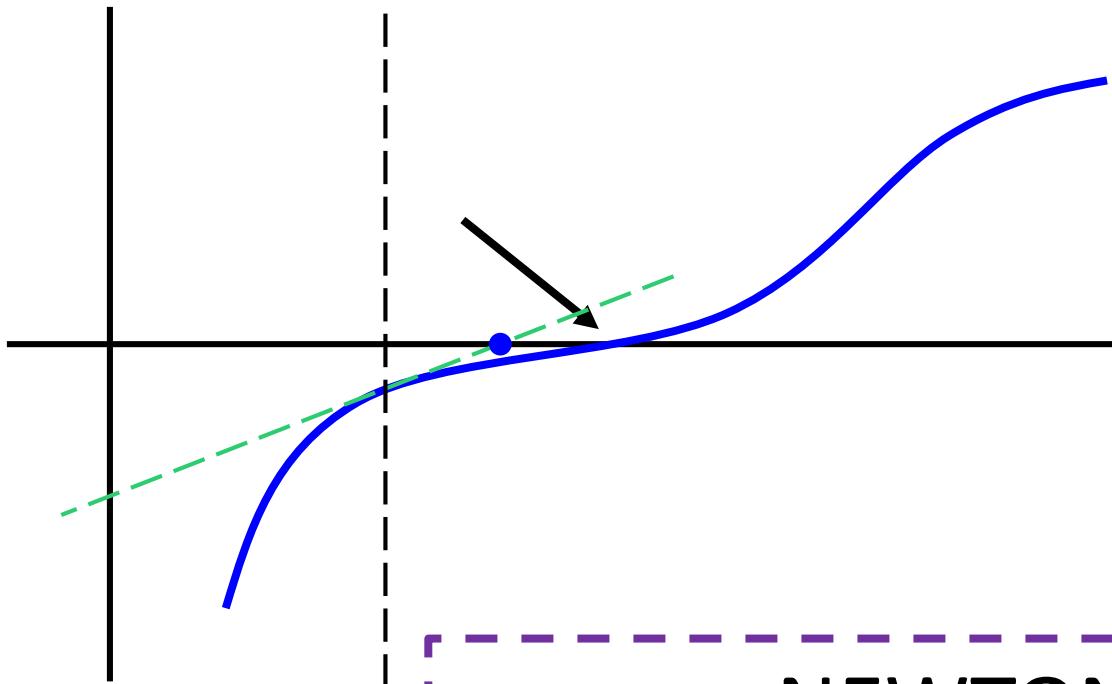
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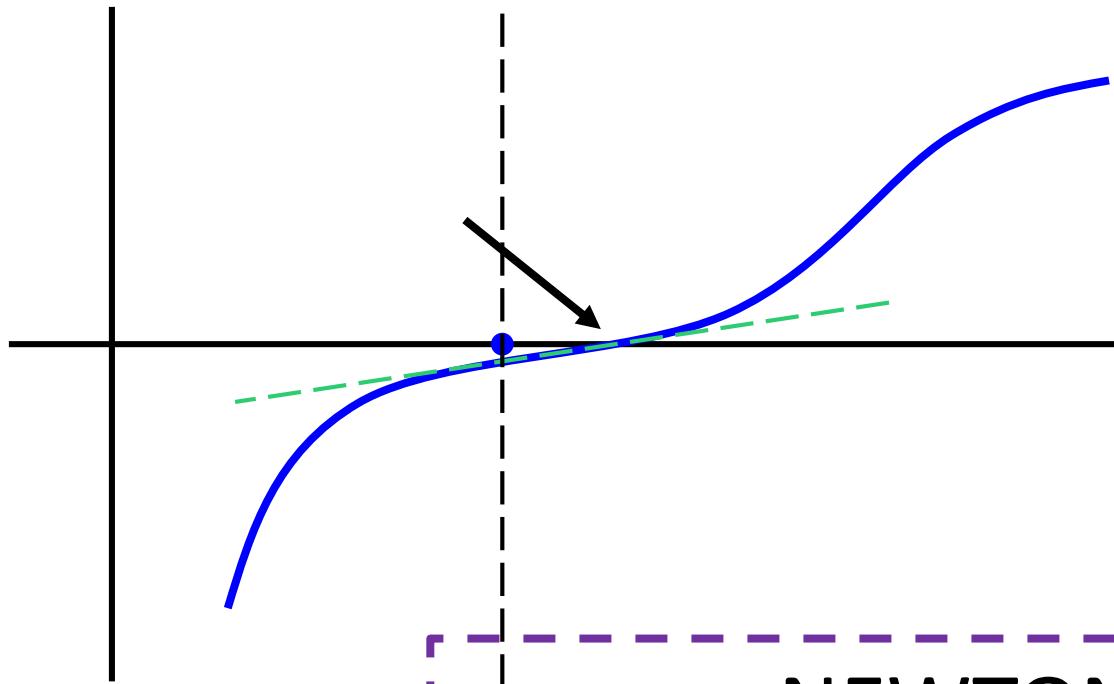
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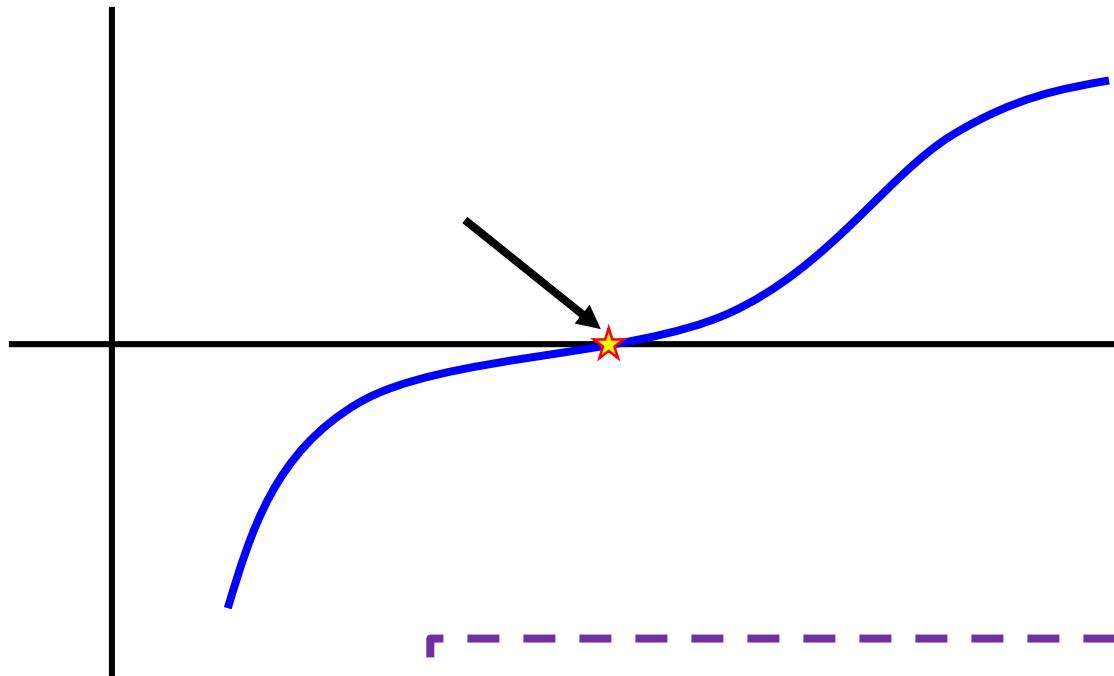
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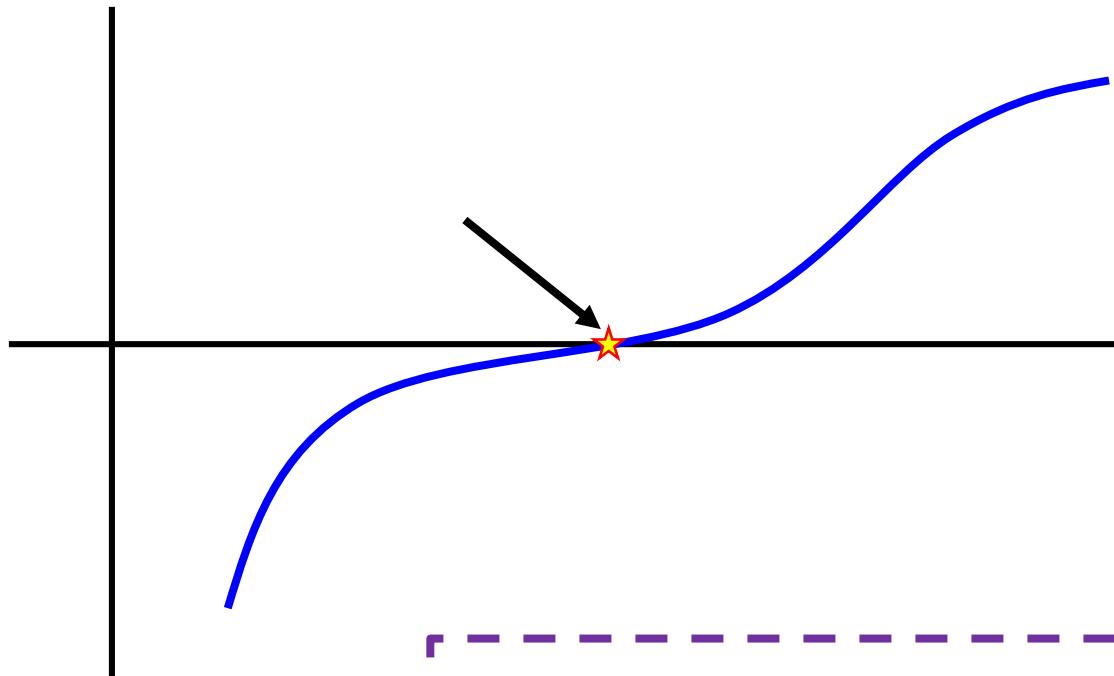
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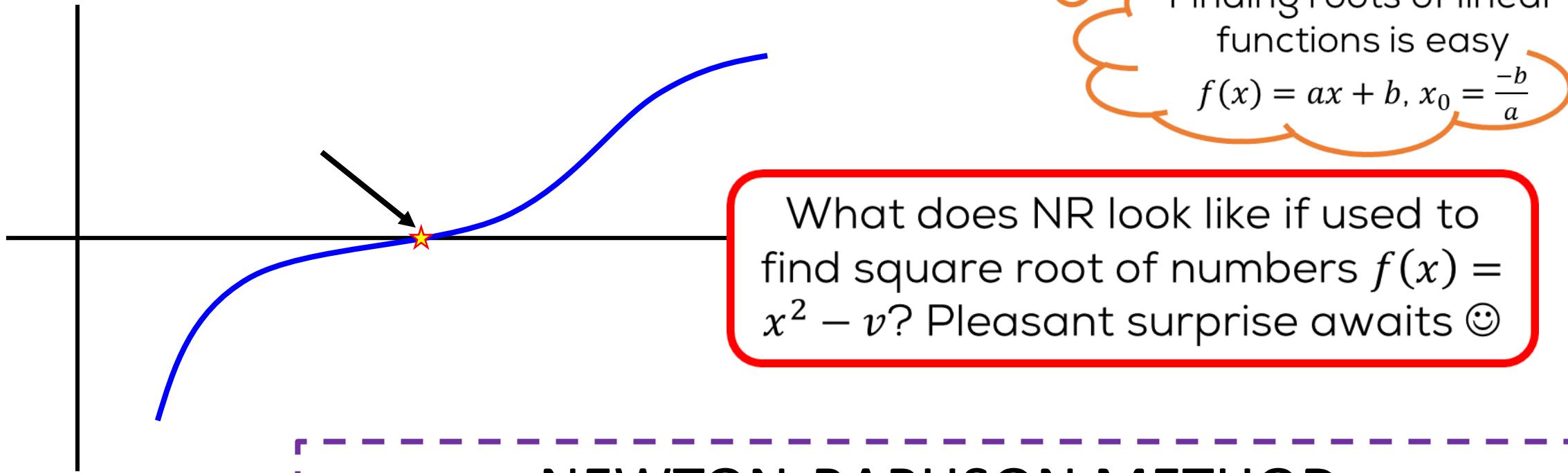
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Also check out the secant method and the Regula Falsi method (do not require derivative calculation)

What does NR look like if used to find square root of numbers  $f(x) = x^2 - v$ ? Pleasant surprise awaits 😊

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