Question 14. 14. The piston in the cylinder head of a locomotive has a stroke (twice the amplitude) of 1.0 m. If the piston moves with simple harmonic motion with an angular frequency of 200 rev/min, what its maximum speed?

Answer:

Stroke of piston = 2 times the amplitude

Let A = amplitude, stroke = 1 m

$$\therefore \quad \Rightarrow A = \frac{1}{2}\text{m.}$$

Angular frequency, $\omega = 200$ rad/min.

$V_{max} = ?$

We know that the maximum speed of the block when the amplitude is A,

$$V_{max} = \omega A = 200 \times \frac{1}{2} = 100 \text{ m/min.}$$

$$= \frac{100}{60} = \frac{5}{3}\text{ms}^{-1} = 1.67 \text{ ms}^{-1}.$$

Question 14. 15. The acceleration due to gravity on the surface of moon is 1.7 ms^{-2}. What is the time period of a simple pendulum on the surface of moon if its time-period on the surface of Earth is 3.5 s? (g on the surface of Earth is 9.8 ms^{-2} .)

Answer:

Here, $g_m = 1.7 \text{ ms}^{-2}$; $g_e = 9.8 \text{ ms}^{-2}$; $T_m = ?$; $T_e = 3.5 \text{ s}^{-1}$

Since, $T_e = 2\pi\sqrt{\frac{1}{g_e}}$ and $T_m = 2\pi\sqrt{\frac{1}{g_m}}$

$$\therefore \quad \frac{T_m}{T_e} = \sqrt{\frac{g_e}{g_m}} \Rightarrow T_m = T_e = \sqrt{\frac{g_e}{g_m}}$$

$$= 3.5\sqrt{\frac{9.8}{1.7}} = 8.4 \text{ s.}$$

Question 14. 16.

Answer the following questions:

(a) *Time period of a particle in SHM depends on the force constant k and mass m of the particle:*

$$T = 2\pi\sqrt{\frac{m}{k}}$$

A simple pendulum executes SHM approximately.

Why then is the time period of a pendulum independent of the mass of the pendulum?

(b) *The motion of a simple pendulum is approximately simple harmonic for small angle oscillations.*

For larger angles of oscillation, a more involved analysis shows that T is greater than $2\pi\sqrt{\frac{l}{g}}$.

Think of a qualitative argument to appreciate this result.

(c) *A man with a wrist watch on his hand falls from the top of a tower. Does the watch give correct time during the free fall?*

(d) *What is the frequency of oscillation of a simple pendulum mounted in a cabin that is freely falling under gravity?*

Answer:

(a) In case of a spring, k does not depend upon m. However, in case of a simple pendulum, k is directly proportional to m and hence the ratio $\frac{m}{k}$ is a constant quantity.

(b) The restoring force for the bob of the pendulum is given by

$$F = -mg \sin\theta$$

If θ is small, then $\sin\theta = \theta = \frac{y}{l}$ $\therefore$ $F = -\frac{mg}{l}y$

i.e., the motion is simple harmonic and time period is $T = 2\pi\sqrt{\frac{l}{g}}$.

Cleraly, the above formula is obtained only if we apply the approximation $\sin\theta \approx \theta$.

For large angles, this approximation is not valid and T is greater than $2\pi\sqrt{\frac{l}{g}}$.

(c) The wrist watch uses an electronic system or spring system to give the time, which does not change with acceleration due to gravity. Therefore, watch gives the correct time.
(d) During free fall of the cabin, the acceleration due to gravity is zero. Therefore, the frequency of oscillations is also zero i.e., the pendulum will not vibrate at all.

Question 14. 17. A simple pendulum of length l and having a bob of mass M is suspended in a car. The car is moving on a circular track of radius R with a uniform speed v. If the pendulum makes small oscillations in a radial direction about its equilibrium position, what will be its time period?
Answer: In this case, the bob of the pendulum is under the action of two accelerations.

(i) Acceleration due to gravity 'g' acting vertically downwards.

(ii) Centripetal acceleration $a_c = \frac{v^2}{R}$ acting along the horizontal direction.

$\therefore$ Effective acceleration, $g' = \sqrt{g^2 + a_c^2}$

or $$g' = \sqrt{g^2 + \frac{v^4}{R^2}}$$

Now time period, $T' = 2\pi\sqrt{\frac{1}{g}} = 2\pi\sqrt{\frac{l}{\sqrt{g^2 + \frac{v^4}{R^2}}}}$.

Question 14. 18. A cylindrical piece of cork of density of base area A and height h floats in a liquid of density ρ_1. The cork is depressed slightly and then released. Show that the cork oscillates up and down simple harmonically with a period

$$T = 2\pi\sqrt{\frac{h\rho}{\rho_l g}}$$

where ρ is the density of cork. (Ignore damping due to viscosity of the liquid).
Answer:

Say, initially in equilibrium, y height of cylinder is inside the liquid. Then,
Weight of the cylinder = upthrust due to liquid displaced

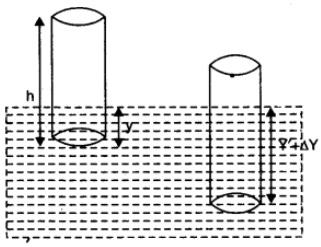
$\therefore$ $$Ah\rho g = Ay\rho_l g$$

When the cork cylinder is depressed slightly by Δy and released, a restoring force, equal to additional upthrust, acts on it. The restoring force is

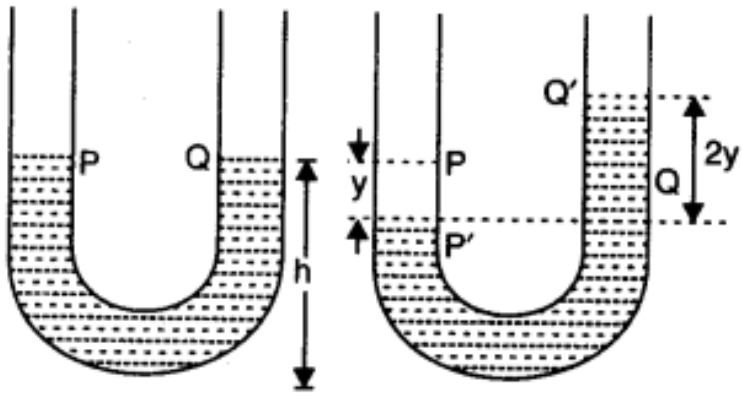
$$F = A(y + \Delta y)\rho_l g - Ay\rho_l g = A\rho_l g\Delta y$$

$\therefore$ Acceleration, $a = \frac{F}{m} = \frac{A\rho_l g\Delta y}{Ah\rho} = \frac{\rho_l g}{h\rho}\cdot\Delta y$ and the acceleration is directed in a direction opposite to Δy: Obviously, as $a \propto -\Delta y$, the motion of cork cylinder is SHM, whose time period is given by

$$T = 2\pi\sqrt{\frac{\text{displacement}}{\text{acceleration}}}$$
$$= 2\pi\sqrt{\frac{\Delta y}{a}}$$
$$= 2\pi\sqrt{\frac{h\rho}{\rho_l g}}.$$



Question 14. 19. One end of a U-tube containing mercury is connected to a suction pump and the other end to atmosphere.



A small pressure difference is maintained between the two columns. Show that, when the suction pump is removed, the column of mercury in the U-tube executes simple harmonic motion.
Answer: The suction pump creates the pressure difference, thus mercury rises in one limb of the U-tube. When it is removed, a net force acts on the liquid column due to the difference in levels of mercury in the two limbs and hence the liquid column executes S.H.M. which can be expressed as:
Consider the mercury contained in a vertical U-tube upto the level P and Q in its two limbs.
Let P = density of the mercury.
L = Total length of the mercury column in both the limbs.
A = internal cross-sectional area of U-tube. m = mass of mercury in U-tube = LAP.
Assume, the mercury be depressed in left limb to F by a small distance y, then it rises by the same amount in the right limb to position Q′.
∴ Difference in levels in the two limbs = P′Q′ = 2y.
∴ Volume of mercury contained in the column of length 2y = A X 2y
∴ m - A x 2y x ρ.
If W = weight of liquid contained in the column of length 2y.
Then W = mg = A x 2y x ρ x g
This weight produces the restoring force (F) which tends to bring back the mercury to its equilibrium position.

∴ $F = -2Ay\rho g = -(2A\rho g)y$

If a = acceleration produced in the liquid column, Then

$$a = \frac{F}{m}$$

$$= -\frac{(2A\rho g)y}{LA\rho} = -\frac{2A\rho g}{LA}$$

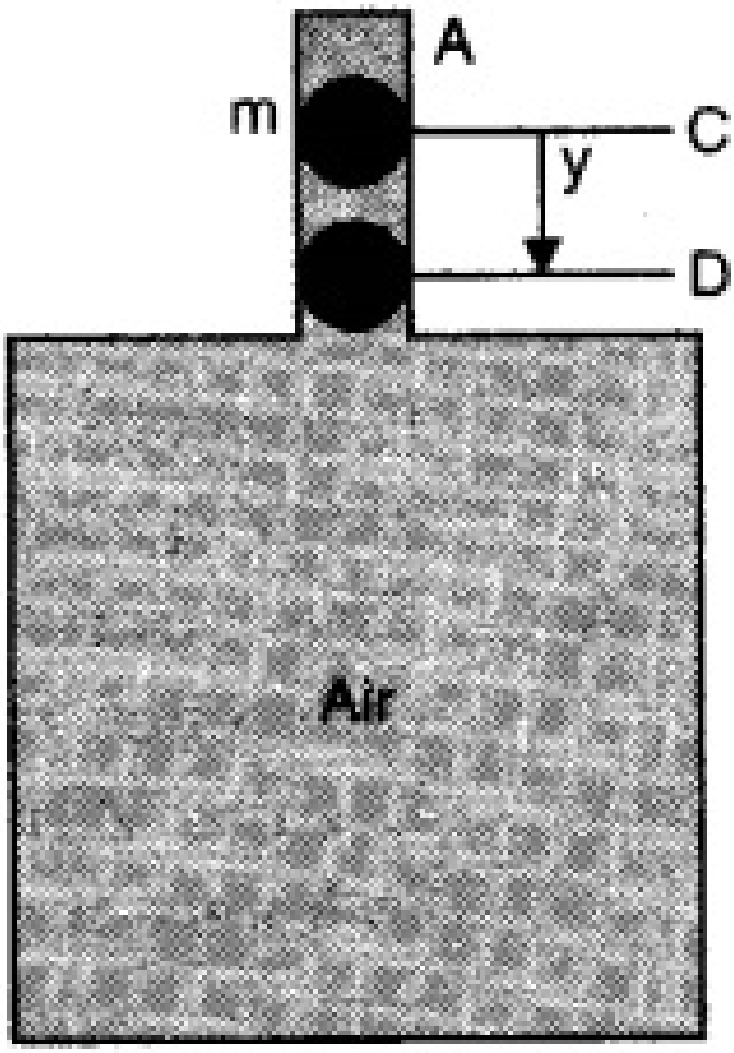
$$= -\frac{2\rho g}{2h\rho}y \quad ...(i) \quad (\because L = 2h)$$

where h = height of mercury in each limb. Now from eqn. (i), it is clear that $a \propto y$ and $-ve$ sign shows that it acts opposite to y, so the motion of mercury in u-tube is simple harmonic in nature having time period (T) given by

$$T = 2\pi\sqrt{\frac{y}{a}} = 2\pi\sqrt{\frac{2h\rho}{2\rho g}} = 2\pi\sqrt{\frac{h\rho}{\rho g}}$$

$$T = 2\pi\sqrt{\frac{h}{g}}$$

Question 14. 20. An air chamber of volume V has a neck area of cross section a into which a ball of mass m just fits and can move up and down without any friction (Fig.). Shaw that when the ball is pressed down a little and released, it executes SHM. Obtain an expression for the time period of oscillations assuming pressure-volume variations of air to be isothermal.



Answer: Consider an air chamber of volume V with a long neck of uniform area of cross-section A, and a frictionless ball of mass m fitted smoothly in the neck at position C, Fig. The pressure of air below the ball inside the chamber is equal to the atmospheric pressure.

Increase the pressure on the ball by a little amount p, so that the ball is depressed to position D, where CD = y.

There will be decrease in volume and hence increase in pressure of air inside the chamber. The decrease in volume of the air inside the chamber, $\Delta V = Ay$

$$\text{Volumetric strain} = \frac{\text{change in volume}}{\text{original volume}}$$

$$= \frac{\Delta V}{V} = \frac{Ay}{V}$$

$\therefore$ Bulk Modulus of elasticity E, will be

$$E = \frac{\text{stress (or increase in pressure)}}{\text{volumetric strain}}$$

$$= \frac{-p}{Ay/V} = \frac{-pV}{Ay}$$

Here, negative sign shows that the increase in pressure will decrease the volume of air in the chamber.

Now, $$p = \frac{-E\,Ay}{V}$$

Due to this excess pressure, the restoring force acting on the ball is

$$F = p \times A = \frac{-E\,Ay}{V} \cdot A = \frac{-E\,A^2}{V} y \quad ...(i)$$

Since $F \propto y$ and negative sign shows that the force is directed towards equilibrium position. If the applied increased pressure is removed from the ball, the ball will start executing linear SHM in the neck of chamber with C as mean position.

In S.H.M., the restoring force,

$$F = -ky \quad ...(ii)$$

Comparing (*i*) and (*ii*), we have

Spring factor, $k = EA^2/V$

Here, inertia factor = mass of ball = m.

$$\text{Period, } T = 2\pi\sqrt{\frac{\text{inertia factor}}{\text{spring factor}}}$$

$$= 2\pi\sqrt{\frac{m}{EA^2/V}} = \frac{2\pi}{A}\sqrt{\frac{mV}{E}}$$

$$\therefore \text{ Frequency, } v = \frac{1}{T} = \frac{A}{2\pi}\sqrt{\frac{E}{mV}}$$

Note: If the ball oscillates in the neck of chamber under isothermal conditions, thru E = P = picture of air inside the chamber, when ball is at equilibrium position. If the ball oscillate in the neck of chamber under adiabatic conditions, then E = gP. where g = C_p/C_v.

Question 14. 21. You are riding an automobile of mass 3000 kg. Assuming that you are examining the oscillation characteristics of its suspension system. The suspension sags 15 cm when the entire automobile is placed on it. Also, the amplitude of oscillation decreases by 50% during one complete oscillation. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming that each wheel supports 750 kg. g = 10 m/s^2.

Answer:

(*a*) Here, mass, M = 300 kg, displacement, x = 15 cm = 0.15 m, g = 10 m/s^2. There are four spring systems. If k is the spring constant of each spring, then total spring constant of all the four springs in parallel is

$$K_p = 4k \quad \therefore \quad M_g = k_p x = 4kx$$

$$\Rightarrow \quad K = \frac{Mg}{4x} = \frac{3000 \times 10}{4 \times 0.15} = 5 \times 10^4 \text{ N.}$$

(*b*) For each spring system supporting 750 kg of weight,

$$t = 2\pi\sqrt{\frac{m}{k}} = 2 \times 3.14 \times \sqrt{\frac{750}{5 \times 10^4}} = 0.77 \text{ sec.}$$

$\therefore$ Using $$x = x_0\, e^{-\frac{bt}{2m}}, \text{ we get}$$

$$\frac{50}{100} x_0 = x_0\, e^{-\frac{b \times 0.77}{2 \times 750}} \quad \text{or} \quad e^{\frac{0.77b}{1500}} = 2$$

Taking logarithm of both sides,

$$\frac{0.77b}{1500} = \ln 2 = 2.303 \log 2$$

$$\therefore \quad b = \frac{1500}{0.77} \times 2.303 \times 0.3010$$

$$= 1350.4 \text{ kg s}^{-1}$$

Question 14. 22. Show that for a particle in linear SHM the average kinetic energy over a period of oscillation equals the average potential energy over the same period.

Answer: Let the particle executing SHM starts oscillating from its mean position. Then displacement equation is

$$x = A \sin \omega t$$

$\therefore$ Particle velocity, $v = A\omega \cos \omega t$

$\therefore$ Instantaneous K.E., $K = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2 \cos^2 \omega t$

$\therefore$ Average value of K.E. over one complete cycle

$$K_{av} = \frac{1}{T}\int_0^T \frac{1}{2}mA^2\omega^2 \cos^2 \omega t \, dt = \frac{mA^2\omega^2}{2T}\int_0^T \cos^2 \omega t \, dt$$

$$= \frac{mA^2\omega^2}{2T}\int_0^T \frac{(1+\cos 2\omega t)}{2} dt$$

$$= \frac{mA^2\omega^2}{4T}\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{mA^2\omega^2}{4T}\left[(T-0) + \left(\frac{\sin 2\omega t - \sin 0}{2\omega}\right)\right]$$

$$= \frac{1}{4}mA^2w^2 \quad ...(i)$$

Again instantaneous P.E., $U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2 = \frac{1}{2}m\omega^2A^2 \sin^2 \omega t$

$\therefore$ Average value of P.E. over one complete cycle

$$U_{av} = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2A^2 \sin^2 \omega t = \frac{m\omega^2A^2}{2T}\int_0^T \sin^2 \omega t \, dt$$

$$= \frac{m\omega^2A^2}{2T}\int_0^T \frac{(1-\cos 2\omega t)}{2} dt$$

$$= \frac{m\omega^2A^2}{4T}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^T$$

$$= \frac{m\omega^2A^2}{4T}\left[(T-0) - \frac{(\sin 2\omega t - \sin 0)}{2\omega}\right]$$

$$= \frac{1}{4}m\omega^2A^2 \quad ...(ii)$$

Simple comparison of (*i*) and (*ii*), shows that

$$K_{av} = U_{av} = \frac{1}{4}m\omega^2A^2$$

Question 14. 23. A circular disc, of mass 10 kg, is suspended by a wire attached to its centre. The wire is twisted by rotating the disc and released. The period of torsional oscillations of found to be 1.5 s. The radius of the disc is 15 cm. Determine the torsional spring constant of the wire. (Torsional spring constant) a is defined by the relation J = - αθ, where J is the restoring couple and θ the angle of twist).

Answer:

$$T = 2\pi\sqrt{\frac{I}{\alpha}} \quad \text{or} \quad T^2 = \frac{4\pi^2 I}{\alpha}$$

or $$\alpha = \frac{4\pi^2 I}{T^2} \quad \text{or} \quad \alpha = \frac{4\pi^2}{T^2}\left(\frac{1}{2}MR^2\right)$$

or $$\alpha = \frac{2\pi^2 MR^2}{T^2}$$

or $$\alpha = \frac{2(3.14)^2 \times 10 \times (0.15)^2}{(1.5)^2} \text{ Nm rad}^{-1}$$

$$= \mathbf{1.97 \text{ Nm rad}^{-1}}.$$

Question 14. 24. A body describes simple harmonic motion with an amplitude of 5 cm and a period of 0.2 s. Find the acceleration and velocity of the body when the displacement is (a) 5 cm (b) 3 cm (c) 0 cm

Answer:

Here, $r = 5$ cm $= 0.05$ m; $T = 0.2$ s; $\omega = \frac{2\pi}{T} = \frac{2\pi}{0.2} = 10\pi$ rad/s

When displacement is y, then

$$\text{acceleration, } A = -\omega^2 y$$

$$\text{velocity, } V = \omega\sqrt{r^2 - y^2}$$

Case (*a*) When

$$y = 5 \text{ cm} = 0.05 \text{ m}$$
$$A = -(10\pi)^2 \times 0.05 = -5\pi^2 \text{ m/s}^2$$
$$V = 10\pi\sqrt{(0.05)^2 - (0.05)^2} = 0.$$

Case (*b*) When

$$y = 3 \text{ cm} = 0.03 \text{ m}$$
$$A = -(10\pi)^2 \times 0.03 = -3\pi^2 \text{ m/s}^2$$
$$V = 10\pi\sqrt{(0.05)^2 - (0.03)^2} = 10\pi \times 0.04 = 0.4\pi \text{ m/s}$$

Case (c) When

$$y = 0, \quad A = -(10\pi)^2 \times 0 = 0$$
$$V = 10\pi\sqrt{(0.05)^2 - 0^2} = 10\pi \times 0.05 = 0.5\pi \text{ m/s}.$$

Question 14. 25. A mass attached to a spring is free to oscillate, with angular velocity w, in a horizontal plane without friction or damping. It is pulled to a distance x_0 and pushed towards the centre with a velocity v_0 at time t = 0. Determine the amplitude of the resulting oscillations in terms of the parameters w, x_0 and v_0.

Answer:

$$x = \alpha \cos(\omega t + \theta)$$
$$v = \frac{dx}{dt} = -a\omega \sin(\omega t + \theta)$$

When $\quad t = 0, \quad x = x_0 \quad \text{and} \quad \frac{dx}{dt} = -v_0$

$\therefore \quad x_0 = a \cos\theta \qquad \ldots(i)$

and $\quad -v_0 = -a\,\omega \sin\theta \quad \text{or} \quad a \sin\theta = \frac{v_0}{\omega} \qquad \ldots(ii)$

Squaring and adding (*i*) and (*ii*), we get

$$a^2(\cos^2\theta + \sin^2\theta) = x_0^2 + \frac{v_0^2}{\omega^2}$$

or

$$a = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}.$$

********** END **********