Chapter 9 Continuity Ex 9.2 Q16

The given function is $f(x) = |x| - |x+1|$

The two functions, g and h, are defined as

$g(x) = |x|$ and $h(x) = |x+1|$

Then, $f = g - h$

The continuity of g and h is examined first.

$g(x) = |x|$ can be written as

$$g(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

Clearly, g is defined for all real numbers.

Let c be a real number.

Case I:

If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} (-x) = -c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x < 0$

Case II:

If $c > 0$, then $g(c) = c$ and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} x = c$

$\therefore \lim\limits_{x \to c} g(x) = g(c)$

Therefore, g is continuous at all points x, such that $x > 0$

Case III:

If $c = 0$, then $g(c) = g(0) = 0$

$$\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-} (-x) = 0$$
$$\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+} (x) = 0$$
$$\therefore \lim_{x \to 0^-} g(x) = \lim_{x \to 0^+} (x) = g(0)$$

Therefore, g is continuous at $x = 0$

From the above three observations, it can be concluded that g is continuous at all points.

$h(x) = |x+1|$ can be written as

$$h(x) = \begin{cases} -(x+1), & \text{if, } x < -1 \\ x+1, & \text{if } x \geq -1 \end{cases}$$

Clearly, h is defined for every real number.

Let c be a real number.

Case I:

If $c < -1$, then $h(c) = -(c+1)$ and $\lim_{x \to c} h(x) = \lim_{x \to c} [-(x+1)] = -(c+1)$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x < -1$

Case II:

If $c > -1$, then $h(c) = c+1$ and $\lim_{x \to c} h(x) = \lim_{x \to c} (x+1) = c+1$

$$\therefore \lim_{x \to c} h(x) = h(c)$$

Therefore, h is continuous at all points x, such that $x > -1$

Case III:

If $c = -1$, then $h(c) = h(-1) = -1+1 = 0$

$$\lim_{x \to -1^-} h(x) = \lim_{x \to -1^-} [-(x+1)] = -(-1+1) = 0$$

$$\lim_{x \to -1^+} h(x) = \lim_{x \to -1^+} (x+1) = (-1+1) = 0$$
$$\therefore \lim_{x \to -1^-} h(x) = \lim_{h \to -1^+} h(x) = h(-1)$$

Therefore, h is continuous at $x = -1$

From the above three observations, it can be concluded that h is continuous at all points of the real line.

g and h are continuous functions. Therefore, $f = g - h$ is also a continuous function.

Therefore, f has no point of discontinuity.

Chapter 9 Continuity Ex 9.2 Q17

He given function f is $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

It is evident that f is defined at all points of the real line.

Let c be a real number.

Case I:

$$\text{If } c \neq 0,\ \text{then } f(c) = c^2 \sin\frac{1}{c}$$

$$\lim_{x\to c} f(x) = \lim_{x\to c}\left(x^2 \sin\frac{1}{x}\right) = \left(\lim_{x\to c} x^2\right)\left(\lim_{x\to c} \sin\frac{1}{x}\right) = c^2 \sin\frac{1}{c}$$

$$\therefore \lim_{x\to c} f(x) = f(c)$$

Therefore, f is continuous at all points $x \neq 0$

Case II:

$$\text{If } c = 0,\ \text{then } f(0) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right)$$

$$\text{It is known that, } -1 \leq \sin\frac{1}{x} \leq 1,\ x \neq 0$$

$$\Rightarrow -x^2 \leq \sin\frac{1}{x} \leq x^2$$

$$\Rightarrow \lim_{x\to 0}\left(-x^2\right) \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \leq \lim_{x\to 0} x^2$$

$$\Rightarrow 0 \leq \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) \leq 0$$

$$\Rightarrow \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = 0$$

$$\text{Similarly, } \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(x^2 \sin\frac{1}{x}\right) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0$$

$$\therefore \lim_{x\to 0^-} f(x) = f(0) = \lim_{x\to 0^+} f(x)$$

Therefore, f is continuous at $x = 0$

From the above observations, it can be concluded that f is continuous at every point of the real line.

Thus, f is a continuous function.

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