Answer

It is given that *: **R** ×**R** → and o: **R** × **R** → **R** isdefined as

$a*b=|a-b|$ and a o b = a, □a, b ∈ **R**.

For a, b ∈ **R**, we have:

$a*b=|a-b|$

$b*a=|b-a|=|-(a-b)|=|a-b|$

∴a * b = b * a

∴ The operation * is commutative.

It can be observed that,

$(1*2)*3=(|1-2|)*3=1*3=|1-3|=2$

$1*(2*3)=1*(|2-3|)=1*1=|1-1|=0$

$\therefore (1*2)*3\neq 1*(2*3)$ (where $1, 2, 3\in \mathbf{R}$)

∴The operation * is not associative.

Now, consider the operation o:

It can be observed that 1 o 2 = 1 and 2 o 1 = 2.

∴1 o 2 ≠ 2 o 1 (where 1, 2 ∈ **R**)

∴The operation o is not commutative.

Let a, b, c ∈ **R**. Then, we have:

(a o b) o c = a o c = a

a o (b o c) = a o b = a

⇒ a o b) o c = a o (b o c)

∴ The operation o is associative.

Now, let a, b, c ∈ **R**, then we have:

a * (b o c) = a * b = $|a-b|$

(a * b) o (a * c) = $(|a-b|)\text{o}(|a-c|)=|a-b|$

Hence, a * (b o c) = (a * b) o (a * c).

Now,

1 o (2 * 3) = $1\ \text{o}(|2-3|)=1\ \text{o}\ 1=1$

(1 o 2) * (1 o 3) = 1 * 1 = $|1-1|=0$

∴1 o (2 * 3) ≠ (1 o 2) * (1 o 3) (where 1, 2, 3 ∈ **R**)

∴ The operation o does not distribute over *.

Question 13:

Given a non-empty set X, let *: P(X) × P(X) → P(X) be defined as A * B = (A − B) ∪ (B − A), □ A, B ∈ P(X). Show that the empty set Φ is the identity for the operation * and all the elements A of P(X) are invertible with A^{-1} = A. (Hint: (A − Φ) ∪ (Φ − A) = A and (A − A) ∪ (A − A) = A * A = Φ).

Answer

It is given that *: P(X) × P(X) → P(X) is defined as

A * B = (A − B) ∪ (B − A) □ A, B ∈ P(X).

Let A ∈ P(X). Then, we have:

A * Φ = (A − Φ) ∪ (Φ − A) = A ∪ Φ = A

Φ * A = (Φ − A) ∪ (A − Φ) = Φ ∪ A = A

∴A * Φ = A = Φ * A. □ A ∈ P(X)

Thus, Φ is the identity element for the given operation*.

Now, an element A ∈ P(X) will be invertible if there exists B ∈ P(X) such that

A * B = Φ = B * A. (As Φ is the identity element)

Now, we observed that $A*A=(A-A)\cup(A-A)=\phi\cup\phi=\phi\ \forall A\in \mathrm{P}(X)$.

Hence, all the elements A of P(X) are invertible with A^{-1} = A.

Question 14:

Define a binary operation *on the set {0, 1, 2, 3, 4, 5} as

$$a*b=\begin{cases}a+b, & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

Show that zero is the identity for this operation and each element a ≠ 0 of the set is invertible with 6 − a being the inverse of a.

invertible with 6 − a being the inverse of a.

Answer

Let X = {0, 1, 2, 3, 4, 5}.

The operation * on X is defined as:

$$a*b=\begin{cases}a+b & \text{if } a+b<6\\ a+b-6 & \text{if } a+b\geq 6\end{cases}$$

An element $e \in X$ is the identity element for the operation *, if $a*e=a=e*a\ \forall a\in X$.

For $a\in X$, we observed that:

$$a*0=a+0=a \qquad [a\in X\Rightarrow a+0<6]$$

$$0*a=0+a=a \qquad [a\in X\Rightarrow 0+a<6]$$

$$\therefore a*0=a=0*a\ \forall a\in X$$

Thus, 0 is the identity element for the given operation *.

An element $a \in X$ is invertible if there exists $b\in X$ such that $a * b = 0 = b * a$.

$$\text{i.e.,}\begin{cases}a+b=0=b+a, & \text{if } a+b<6\\ a+b-6=0=b+a-6, & \text{if } a+b\geq 6\end{cases}$$

i.e.,

$a = -b$ or $b = 6 - a$

But, X = {0, 1, 2, 3, 4, 5} and $a, b \in X$. Then, $a \neq -b$.

$\therefore b = 6 - a$ is the inverse of a □ $a \in X$.

Hence, the inverse of an element $a \in X$, $a \neq 0$ is $6 - a$ i.e., $a^{-1} = 6 - a$.

Question 15:

Let A = {−1, 0, 1, 2}, B = {−4, −2, 0, 2} and f, g: $A \to B$ be functions defined by $f(x) = x^2 - x$, $x \in$ A and $g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A$. Are f and g equal?

Justify your answer. (Hint: One may note that two function f: $A \to B$ and g: $A \to$ B such that $f(a)$ = g(a) □$a \in A$, are called equal functions).

Answer

It is given that A = {−1, 0, 1, 2}, B = {−4, −2, 0, 2}.

Also, it is given that f, g: $A \to B$ are defined by $f(x) = x^2 - x$, $x \in A$ and

$$g(x)=2\left|x-\frac{1}{2}\right|-1,\ x\in A.$$

It is observed that:

$$f(-1)=(-1)^2-(-1)=1+1=2$$

$$g(-1)=2\left|(-1)-\frac{1}{2}\right|-1=2\left(\frac{3}{2}\right)-1=3-1=2$$

$$\Rightarrow f(-1)=g(-1)$$

$$f(0)=(0)^2-0=0$$

$$g(0)=2\left|0-\frac{1}{2}\right|-1=2\left(\frac{1}{2}\right)-1=1-1=0$$

$$\Rightarrow f(0)=g(0)$$

$$f(1)=(1)^2-1=1-1=0$$

$$g(1)=2\left|1-\frac{1}{2}\right|-1=2\left(\frac{1}{2}\right)-1=1-1=0$$

$$\Rightarrow f(1)=g(1)$$

$$f(2)=(2)^2-2=4-2=2$$

$$g(2)=2\left|2-\frac{1}{2}\right|-1=2\left(\frac{3}{2}\right)-1=3-1=2$$

$$\Rightarrow f(2)=g(2)$$

$$\therefore f(a)=g(a)\ \forall a\in A$$

Hence, the functions f and g are equal.

Question 16:

Let A = {1, 2, 3}. Then number of relations containing (1, 2) and (1, 3) which are reflexive and symmetric but not transitive is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

The given set is A = {1, 2, 3}.

The smallest relation containing (1, 2) and (1, 3) which is reflexive and symmetric, but not transitive is given by:

R = {(1, 1), (2, 2), (3, 3), (1, 2), (1, 3), (2, 1), (3, 1)}

This is because relation R is reflexive as (1, 1), (2, 2), (3, 3) ∈ R.

Relation R is symmetric since (1, 2), (2, 1) ∈R and (1, 3), (3, 1) ∈R.

But relation R is not transitive as (3, 1), (1, 2) ∈ R, but (3, 2) ∉ R.

Now, if we add any two pairs (3, 2) and (2, 3) (or both) to relation R, then relation R will become transitive.

Hence, the total number of desired relations is one.

The correct answer is A.

Question 17:

Let $A = \{1, 2, 3\}$. Then number of equivalence relations containing (1, 2) is

(A) 1 (B) 2 (C) 3 (D) 4

Answer

It is given that $A = \{1, 2, 3\}$.

The smallest equivalence relation containing (1, 2) is given by,

$R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

Now, we are left with only four pairs i.e., (2, 3), (3, 2), (1, 3), and (3, 1).

If we odd any one pair [say (2, 3)] to R_1, then for symmetry we must add (3, 2). Also, for transitivity we are required to add (1, 3) and (3, 1).

Hence, the only equivalence relation (bigger than R_1) is the universal relation.

This shows that the total number of equivalence relations containing (1, 2) is two.

The correct answer is B.

Question 18:

Let f: **R** → **R** be the Signum Function defined as

$$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

and g: **R** → **R** be the Greatest Integer Function given by $g(x) = [x]$, where $[x]$ is greatest integer less than or equal to x. Then does *fog* and *gof* coincide in (0, 1]?

Answer

It is given that,

f: **R** → **R** is defined as $$f(x)=\begin{cases}1, & x>0\\ 0, & x=0\\ -1, & x<0\end{cases}$$

Also, g: **R** → **R** is defined as $g(x) = [x]$, where $[x]$ is the greatest integer less than or equal to x.

Now, let $x \in (0, 1]$.

Then, we have:

$[x] = 1$ if $x = 1$ and $[x] = 0$ if $0 < x < 1$.

$$\therefore fog(x)=f(g(x))=f([x])=\begin{cases}f(1), & \text{if } x=1\\ f(0), & \text{if } x\in(0,1)\end{cases}=\begin{cases}1, & \text{if } x=1\\ 0, & \text{if } x\in(0,1)\end{cases}$$

$$\begin{aligned}gof(x)&=g(f(x))\\ &=g(1) \qquad [x>0]\\ &=[1]=1\end{aligned}$$

Thus, when $x \in (0, 1)$, we have $fog(x) = 0$and $gof(x) = 1$.

Hence, *fog* and *gof* do not coincide in (0, 1].

Question 19:

Number of binary operations on the set $\{a, b\}$ are

(A) 10 (B) 16 (C) 20 (D) 8

Answer

A binary operation * on $\{a, b\}$ is a function from $\{a, b\} \times \{a, b\} \to \{a, b\}$

i.e., * is a function from $\{(a, a), (a, b), (b, a), (b, b)\} \to \{a, b\}$.

Hence, the total number of binary operations on the set $\{a, b\}$ is 2^4 i.e., 16.

The correct answer is B.

*********** END **********