



(i) $f: \{1, 2, 3, 4\} \rightarrow \{10\}$ defined as:

$$f = \{(1, 10), (2, 10), (3, 10), (4, 10)\}$$

From the given definition of f , we can see that f is a many one function as: $f(1) = f(2) =$

$$f(3) = f(4) = 10$$

$\therefore f$ is not one-one.

Hence, function f does not have an inverse.

(ii) $g: \{5, 6, 7, 8\} \rightarrow \{1, 2, 3, 4\}$ defined as:

$$g = \{(5, 4), (6, 3), (7, 4), (8, 2)\}$$

From the given definition of g , it is seen that g is a many one function as: $g(5) = g(7) =$

$$4.$$

$\therefore g$ is not one-one,

Hence, function g does not have an inverse.

(iii) $h: \{2, 3, 4, 5\} \rightarrow \{7, 9, 11, 13\}$ defined as:

$$h = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$$

It is seen that all distinct elements of the set $\{2, 3, 4, 5\}$ have distinct images under h .

\therefore Function h is one-one.

Also, h is onto since for every element y of the set $\{7, 9, 11, 13\}$, there exists an element x in the set $\{2, 3, 4, 5\}$ such that $h(x) = y$.

Thus, h is a one-one and onto function. Hence, h has an inverse.

Question 6:

$$f(x) = \frac{x}{(x+2)}$$

Show that $f: [-1, 1] \rightarrow \mathbf{R}$, given by $f(x) = \frac{x}{(x+2)}$ is one-one. Find the inverse of the function $f: [-1, 1] \rightarrow \text{Range } f$.

(Hint: For $y \in \text{Range } f$, $y = \frac{x}{x+2}$, for some x in $[-1, 1]$, i.e., $x = \frac{2y}{(1-y)}$)

Answer

$$f(x) = \frac{x}{(x+2)}$$

$f: [-1, 1] \rightarrow \mathbf{R}$ is given as

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2}$$

$$\Rightarrow xy + 2x = xy + 2y$$

$$\Rightarrow 2x = 2y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one-one function.

It is clear that $f: [-1, 1] \rightarrow \text{Range } f$ is onto.

$\therefore f: [-1, 1] \rightarrow \text{Range } f$ is one-one and onto and therefore, the inverse of the function:

$f: [-1, 1] \rightarrow \text{Range } f$ exists.

Let $g: \text{Range } f \rightarrow [-1, 1]$ be the inverse of f .

Let y be an arbitrary element of range f .

Since $f: [-1, 1] \rightarrow \text{Range } f$ is onto, we have:

$$y = f(x) \text{ for some } x \in [-1, 1]$$

$$\Rightarrow y = \frac{x}{x+2}$$

$$\Rightarrow xy + 2y = x$$

$$\Rightarrow x(1-y) = 2y$$

$$\Rightarrow x = \frac{2y}{1-y}, y \neq 1$$

Now, let us define $g: \text{Range } f \rightarrow [-1, 1]$ as

$$g(y) = \frac{2y}{1-y}, y \neq 1.$$

$$\text{Now, } (g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x+2}\right) = \frac{2\left(\frac{x}{x+2}\right)}{1 - \frac{x}{x+2}} = \frac{2x}{x+2-x} = \frac{2x}{2} = x$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{2y}{1-y}\right) = \frac{\frac{2y}{1-y}}{\frac{2y}{1-y} + 2} = \frac{2y}{2y + 2 - 2y} = \frac{2y}{2} = y$$

$$\therefore g \circ f = I_{[-1,1]} \text{ and } f \circ g = I_{\text{Range } f}$$

$$\therefore f^{-1} = g$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{1-y}, y \neq 1$$

Question 7:

Consider $f: \mathbf{R} \rightarrow \mathbf{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f .

Answer

$f: \mathbf{R} \rightarrow \mathbf{R}$ is given by,

$$f(x) = 4x + 3$$

One-one:

$$\text{Let } f(x) = f(y).$$

$$\Rightarrow 4x + 3 = 4y + 3$$

$$\Rightarrow 4x = 4y$$

$$\Rightarrow x = y$$

$\therefore f$ is a one-one function.

Onto:

For $y \in \mathbf{R}$, let $y = 4x + 3$.

$$\Rightarrow x = \frac{y-3}{4} \in \mathbf{R}$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \frac{y-3}{4} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y.$$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: \mathbf{R} \rightarrow \mathbf{R}$ by $g(x) = \frac{y-3}{4}$.

$$\text{Now, } (g \circ f)(x) = g(f(x)) = g(4x + 3) = \frac{(4x + 3) - 3}{4} = x$$

$$(f \circ g)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

$$\therefore g \circ f = f \circ g = I_{\mathbf{R}}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{y-3}{4}.$$

Question 8:

Consider $f: \mathbf{R}_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse

f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where \mathbf{R}_+ is the set of all non-negative real numbers.

Answer

$f: \mathbf{R}_+ \rightarrow [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

$$\text{Let } f(x) = f(y).$$

$$\Rightarrow x^2 + 4 = y^2 + 4$$

$$\Rightarrow x^2 = y^2$$

$$\Rightarrow x = y \quad [\text{as } x = y \in \mathbf{R}_+]$$

$\therefore f$ is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$$\Rightarrow x^2 = y - 4 \geq 0 \quad [\text{as } y \geq 4]$$

$$\Rightarrow x = \sqrt{y-4} \geq 0$$

Therefore, for any $y \in \mathbf{R}$, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$$f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = y - 4 + 4 = y.$$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define $g: [4, \infty) \rightarrow \mathbf{R}_+$ by,

$$g(y) = \sqrt{y-4}$$

$$\text{Now, } g \circ f(x) = g(f(x)) = g(x^2 + 4) = \sqrt{(x^2 + 4) - 4} = \sqrt{x^2} = x$$

$$\text{And, } f \circ g(y) = f(g(y)) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$$

$$\therefore g \circ f = f \circ g = I_{\mathbf{R}_+}$$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \sqrt{y-4}.$$

Question 21:

Find the values of $\tan^{-1} \sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

(A) $-\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) 0 (D) $2\sqrt{3}$

Answer

Let $\tan^{-1} \sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan \frac{\pi}{3}$ where $\frac{\pi}{3} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

We know that the range of the principal value branch of \tan^{-1} is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$\therefore \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

Let $\cot^{-1}(-\sqrt{3}) = y$.

Then, $\cot y = -\sqrt{3} = -\cot\left(\frac{\pi}{6}\right) = \cot\left(\pi - \frac{\pi}{6}\right) = \cot \frac{5\pi}{6}$ where $\frac{5\pi}{6} \in (0, \pi)$.

The range of the principal value branch of \cot^{-1} is $(0, \pi)$.

$$\therefore \cot^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$$

$$\therefore \tan^{-1} \sqrt{3} - \cot^{-1}(-\sqrt{3}) = \frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\frac{\pi}{2}$$

The correct answer is B.

Question 9:

Consider $f: \mathbf{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$$f^{-1}(y) = \left(\frac{(\sqrt{y+6})-1}{3} \right).$$

Answer

$f: \mathbf{R}_+ \rightarrow [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$$\Rightarrow y = (3x+1)^2 - 1 - 5 = (3x+1)^2 - 6$$

$$\Rightarrow (3x+1)^2 = y+6$$

$$\Rightarrow 3x+1 = \sqrt{y+6} \quad [\text{as } y \geq -5 \Rightarrow y+6 > 0]$$

$$\Rightarrow x = \frac{\sqrt{y+6}-1}{3}$$

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