Complex Numbers Ex 13.4 Q3(ii)

Let $z = \tan\alpha - i$

$\tan\alpha$ is periodic function with period $\pi$.

So, let us take $\alpha$ lying in the interval $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$.

Case – I : When $\alpha \in \left[0, \frac{\pi}{2}\right)$

$$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = \sec\alpha$$

Let $\beta$ be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = \cot\alpha = \tan\left(\frac{\pi}{2} - \alpha\right)$$

$$\Rightarrow \beta = \frac{\pi}{2} - \alpha$$

As z is represented by a point in fourth quadrant.

$$\therefore \arg(z) = -\beta = \alpha - \frac{\pi}{2}.$$

So polar form of z is $\sec\alpha\left(\cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right)\right)$

Case – II : When $\alpha \in \left(\frac{\pi}{2}, \pi\right]$

$$|z| = \sqrt{\tan^2\alpha + 1} = \sqrt{\sec^2\alpha} = |\sec\alpha| = -\sec\alpha$$

Let $\beta$ be acute angle given by $\tan\beta = \frac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{1}{|\tan\alpha|} = |\cot\alpha| = -\cot\alpha = \tan\left(\alpha - \frac{\pi}{2}\right)$$

$$\Rightarrow \beta = \alpha - \frac{\pi}{2}$$

As z is represented by a point in third quadrant.

$$\therefore \arg(z) = \pi + \beta = \frac{\pi}{2} + \alpha.$$

So polar form of z is $-\sec\alpha\left(\cos\left(\frac{\pi}{2} + \alpha\right) + i\sin\left(\frac{\pi}{2} + \alpha\right)\right)$.

Complex Numbers Ex 13.4 Q3(iii)

Let $z = (1 - \sin\alpha) + i\cos\alpha$

Since sine and cosine are periodic functions with period $2\pi$.

So, let us take $\alpha$ lying in the interval $[0, 2\pi]$.

Now, $z = (1 - \sin\alpha) + i\cos\alpha$

$$\Rightarrow |z| = \sqrt{(1-\sin\alpha)^2 + \cos^2\alpha} = \sqrt{2-2\sin\alpha} = \sqrt{2}\sqrt{1-\sin\alpha}$$

$$\Rightarrow |z| = \sqrt{2}\sqrt{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2} = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right|$$

Let $\beta$ be acute angle given by $\tan\beta = \dfrac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{|\cos\alpha|}{|1-\sin\alpha|} = \left|\frac{\cos\alpha}{1-\sin\alpha}\right| = \left|\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)^2}\right| = \left|\frac{\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}}\right|$$

$$\Rightarrow \tan\beta = \left|\frac{1+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}}\right| = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right|$$

Following cases arise:

Case I: When $0 \le \alpha < \dfrac{\pi}{2}$

$\cos\dfrac{\alpha}{2} > \sin\dfrac{\alpha}{2}$ and $\dfrac{\pi}{4} + \dfrac{\alpha}{2} \in \left[\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

$\therefore\ |z| = \sqrt{2}\left(\cos\dfrac{\alpha}{2} - \sin\dfrac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)\right| = \tan\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right) \Rightarrow \beta = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

Clearly, z lies in the first quadrant.

$\therefore\ \arg(z) = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$

So polar form of z is $\sqrt{2}\left(\cos\dfrac{\alpha}{2} - \sin\dfrac{\alpha}{2}\right)\left(\cos\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right) + i\sin\left(\dfrac{\pi}{4} + \dfrac{\alpha}{2}\right)\right)$

Case II: When $\frac{\pi}{2} < \alpha < \frac{3\pi}{2}$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\frac{\pi}{2}, \pi\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = -\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = \tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{3\pi}{4} - \frac{\alpha}{2}\right)$

$\Rightarrow \beta = \frac{3\pi}{4} - \frac{\alpha}{2}$

Since $1 - \sin\alpha > 0$ and $\cos\alpha < 0$.

Clearly, z lies in the fourth quadrant.

$\therefore\ \arg(z) = -\beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$

Case III: When $\frac{3\pi}{2} < \alpha < 2\pi$

$\cos\frac{\alpha}{2} < \sin\frac{\alpha}{2}$ and $\frac{\pi}{4} + \frac{\alpha}{2} \in \left(\pi, \frac{5\pi}{4}\right)$

$\therefore\ |z| = \sqrt{2}\left|\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right| = -\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$

and, $\tan\beta = \left|\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right| = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) = -\tan\left\{\pi - \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right\} = \tan\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)$

$\Rightarrow \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

Clearly, $Re(z) < 0$ and $Im(z) > 0$.

So, z lies in the first quadrant.

$\therefore\ \arg(z) = \beta = \frac{\alpha}{2} - \frac{3\pi}{4}$

So polar form of z is $-\sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right) + i\sin\left(\frac{\alpha}{2} - \frac{3\pi}{4}\right)\right)$.

Complex Numbers Ex 13.4 Q3(iv)

Let $z = \dfrac{1-i}{\cos\frac{\pi}{3} + i\ \sin\frac{\pi}{3}} = \dfrac{1-i}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \dfrac{2-2i}{1+i\sqrt{3}} = \dfrac{(2-2i)\left(1-i\sqrt{3}\right)}{\left(1+i\sqrt{3}\right)\left(1-i\sqrt{3}\right)} =$

$$\frac{\left(2-2\sqrt{3}\right)-i\left(2\sqrt{3}+2\right)}{4} = \frac{\left(1-\sqrt{3}\right)}{2} - i\frac{\left(\sqrt{3}+1\right)}{2}$$

$$|z| = \sqrt{\frac{\left(1-\sqrt{3}\right)^2}{4} + \frac{\left(\sqrt{3}+1\right)^2}{4}} = \sqrt{\frac{8}{4}} = \sqrt{2}$$

Let $\beta$ be acute angle given by $\tan\beta = \dfrac{|\mathrm{Im}(z)|}{|\mathrm{Re}(z)|}$.

$$\tan\beta = \frac{\left|-\frac{\left(\sqrt{3}+1\right)}{2}\right|}{\left|\frac{\left(1-\sqrt{3}\right)}{2}\right|} = \left|\frac{-\left(\sqrt{3}+1\right)}{\left(1-\sqrt{3}\right)}\right| = \left|2+\sqrt{3}\right| = \tan\left(\frac{7\pi}{12}\right)$$

$$\Rightarrow \beta = \frac{7\pi}{12}$$

Z is represented by a point in second quadrant.

So polar form of z is $\sqrt{2}\left(\cos\dfrac{7\pi}{12} - i\sin\dfrac{7\pi}{12}\right)$.

*********** END **********