$= \text{R.H.S.}$

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$$
$$= \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$$
$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$
$$= 1$$

Question 4:

If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$,

Show that either $a + b + c = 0$ or $a = b = c$.

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$
$$= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$
$$= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]$$

It is given that $\Delta = 0$.

$$(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

$\Rightarrow$ Either $a+b+c=0$, or $ab+bc+ca-a^2-b^2-c^2=0$.

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$
$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$
$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$
$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \qquad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$
$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$

$\Rightarrow (a-b)=(b-c)=(c-a)=0$

$\Rightarrow a=b=c$

Hence, if $\Delta = 0$, then either $a + b + c = 0$ or $a = b = c$.

Question 5:

Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$(3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along R_1, we have:

$$(3x+a)\left[1 \times a^2\right] = 0$$

$$\Rightarrow a^2(3x+a) = 0$$

But $a \neq 0$.

Therefore, we have:

$$3x + a = 0$$

$$\Rightarrow x = -\frac{a}{3}$$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Answer

$$\Delta = \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b, and c from C_1, C_2, and C_3, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$= 2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we have:

$$\Delta = 2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2ab^2c\left[a(c-a)+a(a+c)\right]$$
$$= 2ab^2c\left[ac-a^2+a^2+ac\right]$$
$$= 2ab^2c(2ac)$$
$$= 4a^2b^2c^2$$

Hence, the given result is proved.

Question 8:

Let $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$ verify that

(i) $\left[adjA\right]^{-1} = adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1} = A$

Answer

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$$

$$\therefore |A| = 1(15-1)+2(-10-1)+1(-2-3)=14-22-5=-13$$

Now, $A_{11}=14, A_{12}=11, A_{13}=-5$

$A_{21}=11, A_{22}=4, A_{23}=-3$

$A_{31}=-5, A_{32}=-3, A_{13}=-1$

$$\therefore adjA=\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)$$

$$=-\frac{1}{13}\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

(i)

$$|adjA| = 14(-4-9)-11(-11-15)-5(-33+20)$$
$$=14(-13)-11(-26)-5(-13)$$
$$=-182+286+65=169$$

We have,

*********** END **********