Now, we prove that the result is true for $n = k + 1$.

Now, $A^{k+1} = A \cdot A^k$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \\ 3^{k-1} & 3^{k-1} & 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \\ 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} & 3 \cdot 3^{k-1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \\ 3^{(k+1)-1} & 3^{(k+1)-1} & 3^{(k+1)-1} \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$.

Thus by the principle of mathematical induction, we have:

$$A^n = \begin{bmatrix} 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \\ 3^{n-1} & 3^{n-1} & 3^{n-1} \end{bmatrix}, n \in \mathbf{N}$$

Question 3:

If $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then prove $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer

Answer

It is given that $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

To prove: $\mathrm{P}(n): A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, n \in \mathbf{N}$

We shall prove the result by using the principle of mathematical induction.

For $n = 1$, we have:

$$P(1): A^1 = \begin{bmatrix} 1+2 & -4 \\ 1 & 1-2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$$

Therefore, the result is true for $n = 1$.

Let the result be true for $n = k$.

That is,

$$P(k): A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}, n \in \mathbf{N}$$

Now, we prove that the result is true for $n = k + 1$.

Consider

$$A^{k+1} = A^k \cdot A$$

$$= \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3(1+2k)-4k & -4(1+2k)+4k \\ 3k+1-2k & -4k-1(1-2k) \end{bmatrix}$$

$$= \begin{bmatrix} 3+6k-4k & -4-8k+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$$

$$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ 1+k & 1-2(k+1) \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$.

Thus, by the principle of mathematical induction, we have:

$$A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}, n \in \mathbf{N}$$

Question 4:

If A and B are symmetric matrices, prove that $AB - BA$ is a skew symmetric matrix.

Answer

It is given that A and B are symmetric matrices. Therefore, we have:

$A' = A$ and $B' = B$ ...(1)

$$\begin{aligned}
\text{Now, } (AB-BA)' &= (AB)' - (BA)' && \left[(A-B)' = A' - B'\right] \\
&= B'A' - A'B' && \left[(AB)' = B'A'\right] \\
&= BA - AB && \left[\text{Using (1)}\right] \\
&= -(AB-BA)
\end{aligned}$$

$$\therefore (AB-BA)' = -(AB-BA)$$

Thus, $(AB - BA)$ is a skew-symmetric matrix.

Question 5:

Show that the matrix $B'AB$ is symmetric or skew symmetric according as A is symmetric or skew symmetric.

Answer

We suppose that A is a symmetric matrix, then $A' = A$ ... (1)

Consider

$$\begin{aligned}
(B'AB)' &= \{B'(AB)\}' \\
&= (AB)'(B')' && \left[(AB)' = B'A'\right] \\
&= B'A'(B) && \left[(B')' = B\right] \\
&= B'(A'B) \\
&= B'(AB) && \left[\text{Using (1)}\right]
\end{aligned}$$

$$\therefore (B'AB)' = B'AB$$

Thus, if A is a symmetric matrix, then $B'AB$ is a symmetric matrix.

Now, we suppose that A is a skew-symmetric matrix.

Then, $A' = -A$

Consider

$$\begin{aligned}
(B'AB)' &= \left[B'(AB)\right]' = (AB)'(B')' \\
&= (B'A')B = B'(-A)B \\
&= -B'AB
\end{aligned}$$

$$\therefore (B'AB)' = -B'AB$$

Thus, if A is a skew-symmetric matrix, then $B'AB$ is a skew-symmetric matrix.

Hence, if A is a symmetric or skew-symmetric matrix, then $B'AB$ is a symmetric or skew-symmetric matrix accordingly.

Question 6:

Solve system of linear equations, using matrix method.

$2x - y = -2$

$3x + 4y = 3$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Now,

$|A| = 8 + 3 = 11 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|}adjA = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Hence, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$.

Question 7:

For what values of x, $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

Answer

We have:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 1+4+1 & 2+0+0 & 0+2+2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 6 & 2 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

$$\Rightarrow \left[6(0)+2(2)+4(x)\right] = O$$

$$\Rightarrow [4+4x] = [0]$$

$\therefore 4 + 4x = 0$

$\Rightarrow x = -1$

Thus, the required value of x is -1.

Question 8:

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$

Answer

It is given that $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)+1(-1) & 3(1)+1(2) \\ -1(3)+2(-1) & -1(1)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 9-1 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = A^2 - 5A + 7I$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

*********** END **********