EXERCISE.13.2

Question-1

Find the derivative of $x^2 - 2$ at $x = 10$.

Ans.

Let $f(x) = x^2 - 2$. Accordingly,

$$f'(10) = \lim_{h \to 0} \frac{f(10+h) - f(10)}{h}$$

$$= \lim_{h \to 0} \frac{\left[(10+h)^2 - 2\right] - \left(10^2 - 2\right)}{h}$$

$$= \lim_{h \to 0} \frac{10^2 + 2.10.h + h^2 - 2 - 10^2 + 2}{h}$$

$$= \lim_{h \to 0} \frac{20h + h^2}{h}$$

$$= \lim_{h \to 0} (20 + h) = (20 + 0) = 20$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question-2

Find the derivative of $99x$ at $x = 100$.

Ans.

Let $f(x) = 99x$. Accordingly,

$$f'(100) = \lim_{h \to 0} \frac{f(100+h) - f(100)}{h}$$
$$= \lim_{h \to 0} \frac{99(100+h) - 99(100)}{h}$$
$$= \lim_{h \to 0} \frac{99 \times 100 + 99h - 99 \times 100}{h}$$
$$= \lim_{h \to 0} \frac{99h}{h}$$
$$= \lim_{h \to 0} (99) = 99$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question-3

Find the derivative of x at $x = 1$.

Ans.

Let $f(x) = x$. Accordingly,

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$$
$$= \lim_{h \to 0} \frac{h}{h}$$
$$= \lim_{h \to 0} (1)$$
$$= 1$$

Thus, the derivative of x at $x = 1$ is 1.

Question-4

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Ans.

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{\left[(x+h)^3-27\right]-(x^3-27)}{h} \\
&= \lim_{h\to 0}\frac{x^3+h^3+3x^2h+3xh^2-x^3}{h} \\
&= \lim_{h\to 0}\frac{h^3+3x^2h+3xh^2}{h} \\
&= \lim_{h\to 0}\left(h^2+3x^2+3xh\right) \\
&= 0+3x^2+0=3x^2
\end{aligned}
$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0}\frac{(x+h-1)(x+h-2)-(x-1)(x-2)}{h} \\
&= \lim_{h\to 0}\frac{\left(x^2+hx-2x+hx+h^2-2h-x-h+2\right)-\left(x^2-2x-x+2\right)}{h} \\
&= \lim_{h\to 0}\frac{\left(hx+hx+h^2-2h-h\right)}{h} \\
&= \lim_{h\to 0}\frac{2hx+h^2-3h}{h} \\
&= \lim_{h\to 0}(2x+h-3) \\
&= (2x+0-3) \\
&= 2x-3
\end{aligned}
$$

(iii) Let $f(x)=\frac{1}{x^2}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-x^2-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-h^2-2hx}{x^2(x+h)^2}\right]$$

$$=\lim_{h\to 0}\left[\frac{-h-2x}{x^2(x+h)^2}\right]$$

$$=\frac{0-2x}{x^2(x+0)^2}=\frac{-2}{x^3}$$

(iv) Let $f(x)=\frac{x+1}{x-1}$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(\frac{x+h+1}{x+h-1}-\frac{x+1}{x-1}\right)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x-1)(x+h+1)-(x+1)(x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{(x^2+hx+x-x-h-1)-(x^2+hx-x+x+h-1)}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{-2h}{(x-1)(x+h-1)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2}{(x-1)(x+h-1)}\right]$$

$$=\frac{-2}{(x-1)(x-1)}=\frac{-2}{(x-1)^2}$$

Question-5

For the function

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$$

Prove that $f'(1)=100f'(0)$

Ans.

The given function is

$$f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left[\frac{x^{100}}{100}+\frac{x^{99}}{99}+...+\frac{x^2}{2}+x+1\right]$$

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(\frac{x^{100}}{100}\right)+\frac{d}{dx}\left(\frac{x^{99}}{99}\right)+...+\frac{d}{dx}\left(\frac{x^2}{2}\right)+\frac{d}{dx}(x)+\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n)=nx^{n-1}$, we obtain

$$\frac{d}{dx}f(x)=\frac{100x^{99}}{100}+\frac{99x^{98}}{99}+...+\frac{2x}{2}+1+0$$

$$=x^{99}+x^{98}+...+x+1$$

$$\therefore f'(x)=x^{99}+x^{98}+...+x+1$$

At $x=0$,

$f'(0)=1$

At $x=1$,

$$f'(1)=1^{99}+1^{98}+...+1+1=[1+1+...+1+1]_{100\text{ terms}}=1\times100=100$$

Thus, $f'(1)=100\times f^1(0)$

Question-6

Find the derivative of $x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$ for some fixed real number a.

Ans.

Let $f(x)=x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n$

$$\therefore f'(x)=\frac{d}{dx}\left(x^n+ax^{n-1}+a^2x^{n-2}+...+a^{n-1}x+a^n\right)$$

$$=\frac{d}{dx}(x^n)+a\frac{d}{dx}(x^{n-1})+a^2\frac{d}{dx}(x^{n-2})+...+a^{n-1}\frac{d}{dx}(x)+a^n\frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}x^n=nx^{n-1}$, we obtain

$$f'(x)=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}+a^n(0)$$

$$=nx^{n-1}+a(n-1)x^{n-2}+a^2(n-2)x^{n-3}+...+a^{n-1}$$

Question-7

For some constants a and b, find the derivative of

(i) $(x - a)(x - b)$ (ii) $(ax^2 + b)^2$ (iii) $\dfrac{x-a}{x-b}$

Ans.

(i) Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}\left(x^2 - (a+b)x + ab\right)$$

$$= \frac{d}{dx}\left(x^2\right) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let $f(x) = \left(ax^2 + b\right)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}\left(a^2x^4 + 2abx^2 + b^2\right) = a^2\frac{d}{dx}\left(x^4\right) + 2ab\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(b^2\right)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$f'(x) = a^2\left(4x^3\right) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax\left(ax^2 + b\right)$$

(iii) Let $f(x) = \dfrac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

$$= \frac{x - b - x + a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question-8

Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a.

Ans.

$$\text{Let } f(x) = \frac{x^n - a^n}{x - a}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x^n - a^n}{x - a}\right)$$

By quotient rule,

$$f'(x) = \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question-9

Find the derivative of

(i) $2x - \frac{3}{4}$ (ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$ (iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$ (vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Ans.

(i) Let $f(x) = 2x - \frac{3}{4}$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(2x - \frac{3}{4}\right) \\ &= 2\frac{d}{dx}(x) - \frac{d}{dx}\left(\frac{3}{4}\right) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= \left(5x^3 + 3x - 1\right)\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}\left(5x^3 + 3x - 1\right) \\ &= \left(5x^3 + 3x - 1\right)(1) + (x-1)\left(5.3x^2 + 3 - 0\right) \\ &= \left(5x^3 + 3x - 1\right) + (x-1)\left(15x^2 + 3\right) \\ &= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\ &= 20x^3 - 15x^2 + 6x - 4 \end{aligned}$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{-3}\frac{d}{dx}(5+3x)+(5+3x)\frac{d}{dx}\left(x^{-3}\right) \\
&= x^{-3}(0+3)+(5+3x)\left(-3x^{-3-1}\right) \\
&= x^{-3}(3)+(5+3x)\left(-3x^{-4}\right) \\
&= 3x^{-3}-15x^{-4}-9x^{-3} \\
&= -6x^{-3}-15x^{-4} \\
&= -3x^{-3}\left(2+\frac{5}{x}\right) \\
&= \frac{-3x^{-3}}{x}(2x+5) \\
&= \frac{-3}{x^{4}}(5+2x)
\end{aligned}
$$

(iv) Let $f(x) = x^{5}(3 - 6x^{-9})$

By Leibnitz product rule,

$$
\begin{aligned}
f'(x) &= x^{5}\frac{d}{dx}\left(3-6x^{-9}\right)+\left(3-6x^{-9}\right)\frac{d}{dx}\left(x^{5}\right) \\
&= x^{5}\left\{0-6(-9)x^{-9-1}\right\}+\left(3-6x^{-9}\right)\left(5x^{4}\right) \\
&= x^{5}\left(54x^{-10}\right)+15x^{4}-30x^{-5} \\
&= 54x^{-5}+15x^{4}-30x^{-5} \\
&= 24x^{-5}+15x^{4} \\
&= 15x^{4}+\frac{24}{x^{5}}
\end{aligned}
$$

(v) Let $f(x) = x^{-4}(3 - 4x^{-5})$

By Leibnitz product rule,

$$f'(x) = x^{-4}\frac{d}{dx}\left(3-4x^{-5}\right)+\left(3-4x^{-5}\right)\frac{d}{dx}\left(x^{-4}\right)$$
$$= x^{-4}\left\{0-4(-5)x^{-5-1}\right\}+\left(3-4x^{-5}\right)(-4)x^{-4-1}$$
$$= x^{-4}\left(20x^{-6}\right)+\left(3-4x^{-5}\right)\left(-4x^{-5}\right)$$
$$= 20x^{-10}-12x^{-5}+16x^{-10}$$
$$= 36x^{-10}-12x^{-5}$$
$$= -\frac{12}{x^5}+\frac{36}{x^{10}}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx}\left(\frac{2}{x+1}\right)-\frac{d}{dx}\left(\frac{x^2}{3x-1}\right)$$

By quotient rule,

$$f'(x) = \left[\frac{(x+1)\frac{d}{dx}(2)-2\frac{d}{dx}(x+1)}{(x+1)^2}\right]-\left[\frac{(3x-1)\frac{d}{dx}(x^2)-x^2\frac{d}{dx}(3x-1)}{(3x-1)^2}\right]$$
$$= \left[\frac{(x+1)(0)-2(1)}{(x+1)^2}\right]-\left[\frac{(3x-1)(2x)-(x^2)(3)}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\left[\frac{6x^2-2x-3x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\left[\frac{3x^2-2x^2}{(3x-1)^2}\right]$$
$$= \frac{-2}{(x+1)^2}-\frac{x(3x-2)}{(3x-1)^2}$$

Question-10

Find the derivative of $\cos x$ from first principle.

Ans.

Let $f(x) = \cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$
$$= \lim_{h\to 0}\left[\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)-\sin x\sin h}{h}\right]$$
$$= \lim_{h\to 0}\left[\frac{-\cos x(1-\cos h)}{h}-\frac{\sin x\sin h}{h}\right]$$
$$= -\cos x\left(\lim_{h\to 0}\frac{1-\cos h}{h}\right)-\sin x\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$
$$= -\cos x(0)-\sin x(1) \qquad \left[\lim_{h\to 0}\frac{1-\cos h}{h}=0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h}=1\right]$$
$$= -\sin x$$
$$\therefore f'(x) = -\sin x$$

Question-11

Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5 \sec x + 4 \cos x$

(iv) $\operatorname{cosec} x$ (v) $3\cot x + 5\operatorname{cosec} x$

(vi) $5\sin x - 6\cos x + 7$ (vii) $2\tan x - 7\sec x$

Ans.

(i) Let $f(x) = \sin x \cos x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x+h)\cos(x+h)-\sin x\cos x}{h}$$
$$= \lim_{h\to 0}\frac{1}{2h}\left[2\sin(x+h)\cos(x+h)-2\sin x\cos x\right]$$
$$= \lim_{h\to 0}\frac{1}{2h}\left[\sin 2(x+h)-\sin 2x\right]$$
$$= \lim_{h\to 0}\frac{1}{2h}\left[2\cos\frac{2x+2h+2x}{2}\cdot\sin\frac{2x+2h-2x}{2}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\cos\frac{4x+2h}{2}\sin\frac{2h}{2}\right]$$
$$= \lim_{h\to 0}\frac{1}{h}\left[\cos(2x+h)\sin h\right]$$
$$= \lim_{h\to 0}\cos(2x+h).\lim_{h\to 0}\frac{\sin h}{h}$$
$$= \cos(2x+0).1$$
$$= \cos 2x$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$=\frac{1}{\cos x}.\lim_{h\to 0}\frac{\left[\sin\left(\frac{2x+h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]}{\cos(x+h)}$$

$$=\frac{1}{\cos x}.\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}.\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}$$

$$=\frac{1}{\cos x}.1.\frac{\sin x}{\cos x}$$

$$=\sec x\tan x$$

(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{5\sec(x+h)+4\cos(x+h)-[5\sec x+4\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{[\sec(x+h)-\sec x]}{h}+4\lim_{h\to 0}\frac{[\cos(x+h)-\cos x]}{h}$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\cos(x+h)}-\frac{1}{\cos x}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos(x+h)-\cos x]$$

$$=5\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos x-\cos(x+h)}{\cos x\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}[\cos x\cos h-\sin x\sin h-\cos x]$$

$$= \frac{5}{\cos x}\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)}\right]+4\lim_{h\to 0}\frac{1}{h}\left[-\cos x(1-\cos h)-\sin x\sin h\right]$$

$$= \frac{5}{\cos x}.\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos(x+h)}\right]+4\left[-\cos x\lim_{h\to 0}\frac{(1-\cos h)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}\right]$$

$$= \frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4\left[(-\cos x).(0)-(\sin x).1\right]$$

$$= \frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$= \frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$= 5\sec x\tan x.-4\sin x$$

$$= \frac{5}{\cos x}.\lim_{h\to 0}\left[\frac{\sin\left(\frac{2x+h}{2}\right).\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)}\right]+4\left[(-\cos x).(0)-(\sin x).1\right]$$

$$= \frac{5}{\cos x}.\left[\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)}.\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]-4\sin x$$

$$= \frac{5}{\cos x}.\frac{\sin x}{\cos x}.1-4\sin x$$

$$= 5\sec x\tan x.-4\sin x$$

(iv) Let $f(x)$ = cosec x. Accordingly, from the first principle,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h\to 0}\frac{-\cos\left(\frac{2x+h}{2}\right)\cdot\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$= \lim_{h\to 0}\left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x}\right).\lim_{\frac{h}{2}\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x\sin x}\right).1$$

$$= -\operatorname{cosec}x\cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{3\cot(x+h)+5\operatorname{cosec}(x+h)-3\cot x-5\operatorname{cosec} x}{h}$$

$$= 3\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]+5\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec} x\right] \qquad ...(1)$$

Now, $\lim_{h\to 0}\frac{1}{h}\left[\cot(x+h)-\cot x\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)}{\sin(x+h)}-\frac{\cos x}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\cos(x+h)\sin x-\cos x\sin(x+h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(x-x-h)}{\sin x\sin(x+h)}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin(-h)}{\sin x\sin(x+h)}\right]$$

$$=-\left(\lim_{h\to 0}\frac{\sin h}{h}\right).\left(\lim_{h\to 0}\frac{1}{\sin x\cdot\sin(x+h)}\right)$$

$$=-1.\frac{1}{\sin x\cdot\sin(x+0)}=\frac{-1}{\sin^2 x}=-\operatorname{cosec}^2 x \qquad ...(2)$$

$\lim_{h\to 0}\frac{1}{h}\left[\operatorname{cosec}(x+h)-\operatorname{cosec}x\right]$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{1}{\sin(x+h)}-\frac{1}{\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{\sin x-\sin(x+h)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{x+x+h}{2}\right)\cdot\sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x}\right]$$

$$= \lim_{h \to 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x}$$

$$= \lim_{h \to 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) . \lim_{\frac{h}{2} \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) .1$$

$$= -\operatorname{cosec} x \cot x \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7 \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[5\{\sin(x+h) - \sin x\} - 6\{\cos(x+h) - \cos x\} \right]$$

$$= 5\lim_{h \to 0} \frac{1}{h} \left[\sin(x+h) - \sin x \right] - 6\lim_{h \to 0} \frac{1}{h} \left[\cos(x+h) - \cos x \right]$$

$$= 5\lim_{h \to 0} \frac{1}{h} \left[2\cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) \right] - 6\lim_{h \to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= 5\lim_{h \to 0} \left(\cos\left(\frac{2x+h}{2}\right) \frac{\sin\frac{h}{2}}{\frac{h}{2}} \right) - 6\lim_{h \to 0} \left[\frac{-\cos x(1-\cos h)}{h} - \frac{\sin x \sin h}{h} \right]$$

$$= 5\left[\lim_{h \to 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{\frac{h}{2} \to 0} \frac{\sin\frac{h}{2}}{\frac{h}{2}} \right] - 6\left[(-\cos x)\left(\lim_{h \to 0} \frac{1-\cos h}{h} \right) - \sin x \lim_{h \to 0}\left(\frac{\sin h}{h} \right) \right]$$

$$= 5\cos x.1 - 6\left[(-\cos x).(0) - \sin x.1 \right]$$

$$= 5\cos x + 6\sin x$$

(vii) Let $f(x) = 2\tan x - 7\sec x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[2\tan(x+h) - 7\sec(x+h) - 2\tan x + 7\sec x \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[2\{\tan(x+h) - \tan x\} - 7\{\sec(x+h) - \sec x\} \right]$$

$$= 2\lim_{h \to 0} \frac{1}{h} \left[\tan(x+h) - \tan x \right] - 7\lim_{h \to 0} \frac{1}{h} \left[\sec(x+h) - \sec x \right]$$

$$= 2\lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7\lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= 2\lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7\lim_{h \to 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right]$$

$$= 2\lim_{h \to 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7\lim_{h \to 0} \frac{1}{h} \left[\frac{-2\sin\left(\frac{x+x+h}{2}\right)\sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\lim_{h\to 0}\left[\left(\frac{\sin h}{h}\right)\frac{1}{\cos x\cos(x+h)}\right]-7\lim_{h\to 0}\frac{1}{h}\left[\frac{-2\sin\left(\frac{2x+h}{2}\right)\sin\left(-\frac{h}{2}\right)}{\cos x\cos(x+h)}\right]$$

$$=2\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\left(\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}\right)-7\left(\lim_{\frac{h}{2}\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right)\left(\lim_{h\to 0}\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x\cos(x+h)}\right)$$

$$=2.1.\frac{1}{\cos x\cos x}-7.1\left(\frac{\sin x}{\cos x\cos x}\right)$$

$$=2\sec^2 x-7\sec x\tan x$$

*********** END **********