Exercise 10.3

Question 1:

Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.

Answer

It is given that,

$$|\vec{a}|=\sqrt{3},\ |\vec{b}|=2 \text{ and, } \vec{a}\cdot\vec{b}=\sqrt{6}$$

Now, we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$$

$$\Rightarrow \cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$\Rightarrow \cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

Answer

The given vectors are $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$.

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$

$$\begin{aligned}\text{Now, } \vec{a}\cdot\vec{b}&=\left(\hat{i}-2\hat{j}+3\hat{k}\right)\left(3\hat{i}-2\hat{j}+\hat{k}\right)\\&=1.3+(-2)(-2)+3.1\\&=3+4+3\\&=10\end{aligned}$$

Also, we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta=\frac{10}{14}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

Answer

Let $\vec{a}=\hat{i}-\hat{j}$ and $\vec{b}=\hat{i}+\hat{j}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}\left(\vec{a}.\vec{b}\right)=\frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\}=\frac{1}{\sqrt{2}}(1-1)=0$$

Hence, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

Question 4:

Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.

Answer

Let $\vec{a}=\hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b}=7\hat{i}-\hat{j}+8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b})=\frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\}=\frac{7-3+56}{\sqrt{49+1+64}}=\frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}),\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}),\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer

Let $\vec{a}=\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k})=\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k},$

$\vec{b}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$

$\vec{c}=\frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})=\frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$

$$|\vec{a}|=\sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2}=\sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}}=1$$

$$|\vec{b}|=\sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2}=\sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}}=1$$

$$|\vec{c}|=\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2}=\sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}}=1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a}\cdot\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$

$$\vec{b}\cdot\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$

$$\vec{c}\cdot\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$.

Answer

$$(\vec{a}\cdot\vec{b}).(\vec{a}-\vec{b})=8$$

$$\Rightarrow\vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$

$$\Rightarrow|\vec{a}|^2-|\vec{b}|^2=8$$

$$\Rightarrow\left(8|\vec{b}|\right)^2-|\vec{b}|^2=8 \qquad \left[|\vec{a}|=8|\vec{b}|\right]$$

$$\Rightarrow 64|\vec{b}|^2-|\vec{b}|^2=8$$

$$\Rightarrow 63|\vec{b}|^2=8$$

$$\Rightarrow|\vec{b}|^2=\frac{8}{63}$$

$$\Rightarrow|\vec{b}|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow|\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}|=8|\vec{b}|=\frac{8\times2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$.

Answer

$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$$

$$=3\vec{a}\cdot2\vec{a}+3\vec{a}\cdot7\vec{b}-5\vec{b}\cdot2\vec{a}-5\vec{b}\cdot7\vec{b}$$

$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$

$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Answer

Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.

It is given that $|\vec{a}|=|\vec{b}|,\ \vec{a}\cdot\vec{b}=\frac{1}{2}$, and $\theta=60°. \qquad ...(1)$

It is given that

We know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$\therefore \frac{1}{2}=|\vec{a}||\vec{a}|\cos 60^\circ \qquad [\text{Using }(1)]$

$\Rightarrow \frac{1}{2}=|\vec{a}|^2\times\frac{1}{2}$

$\Rightarrow |\vec{a}|^2=1$

$\Rightarrow |\vec{a}|=|\vec{b}|=1$

Question 9:

Find $|\vec{x}|$, if for a unit vector $\vec{a},\ (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$.

Answer

$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$

$\Rightarrow \vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=12$

$\Rightarrow |\vec{x}|^2-|\vec{a}|^2=12$

$\Rightarrow |\vec{x}|^2-1=12 \qquad [|\vec{a}|=1 \text{ as } \vec{a} \text{ is a unit vector}]$

$\Rightarrow |\vec{x}|^2=13$

$\therefore |\vec{x}|=\sqrt{13}$

Question 10:

If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.

Answer

The given vectors are $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$, and $\vec{c}=3\hat{i}+\hat{j}$.

Now,

$\vec{a}+\lambda\vec{b}=\left(2\hat{i}+2\hat{j}+3\hat{k}\right)+\lambda\left(-\hat{i}+2\hat{j}+\hat{k}\right)=(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}$

If $\left(\vec{a}+\lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$\left(\vec{a}+\lambda\vec{b}\right)\cdot\vec{c}=0.$

$\Rightarrow \left[(2-\lambda)\hat{i}+(2+2\lambda)\hat{j}+(3+\lambda)\hat{k}\right]\cdot\left(3\hat{i}+\hat{j}\right)=0$

$\Rightarrow (2-\lambda)3+(2+2\lambda)1+(3+\lambda)0=0$

$\Rightarrow 6-3\lambda+2+2\lambda=0$

$\Rightarrow -\lambda+8=0$

$\Rightarrow \lambda=8$

Hence, the required value of λ is 8.

Question 11:

Show that $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$

Answer

$\left(|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\right)$

$=|\vec{a}|^2\,\vec{b}\cdot\vec{b}-|\vec{a}||\vec{b}|\vec{b}\cdot\vec{a}+|\vec{b}||\vec{a}|\vec{a}\cdot\vec{b}-|\vec{b}|^2\,\vec{a}\cdot\vec{a}$

$=|\vec{a}|^2|\vec{b}|^2-|\vec{b}|^2|\vec{a}|^2$

$=0$

Hence, $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:

If $\vec{a}\cdot\vec{a}=0$ and $\vec{a}\cdot\vec{b}=0$, then what can be concluded about the vector $\vec{b}$?

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