



EXERCISE 9.4

Question 1:

Find the sum to n terms of the series $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$

Ans:

The given series is $1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + \dots$

n^{th} term, $a_n = n (n + 1)$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n k(k+1) \\ &= \sum_{k=1}^n k^2 + \sum_{k=1}^n k \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2} \left(\frac{2n+1}{3} + 1 \right) \\ &= \frac{n(n+1)}{2} \left(\frac{2n+4}{3} \right) \\ &= \frac{n(n+1)(n+2)}{3}\end{aligned}$$

Question 2:

Find the sum to n terms of the series $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$

Ans:

The given series is $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \dots$

n^{th} term, $a_n = n (n + 1) (n + 2)$

$$= (n^2 + n) (n + 2)$$

$$= n^3 + 3n^2 + 2n$$

$$\begin{aligned}
\therefore S_n &= \sum_{k=1}^n a_k \\
&= \sum_{k=1}^n k^3 + 3 \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k \\
&= \left[\frac{n(n+1)}{2} \right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\
&= \left[\frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1) \\
&= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + 2n+1+2 \right] \\
&= \frac{n(n+1)}{2} \left[\frac{n^2+n+4n+6}{2} \right] \\
&= \frac{n(n+1)}{4} (n^2+5n+6) \\
&= \frac{n(n+1)}{4} (n^2+2n+3n+6) \\
&= \frac{n(n+1)[n(n+2)+3(n+2)]}{4} \\
&= \frac{n(n+1)(n+2)(n+3)}{4}
\end{aligned}$$

Question 3:

Find the sum to n terms of the series $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$

Ans:

The given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \dots$

n^{th} term, $a_n = (2n + 1) n^2 = 2n^3 + n^2$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k \\ &= \sum_{k=1}^n (2k^3 + k^2) = 2 \sum_{k=1}^n k^3 + \sum_{k=1}^n k^2 \\ &= 2 \left[\frac{n(n+1)}{2} \right]^2 + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{2} \left[n(n+1) + \frac{2n+1}{3} \right] \\ &= \frac{n(n+1)}{2} \left[\frac{3n^2 + 3n + 2n + 1}{3} \right] \\ &= \frac{n(n+1)}{2} \left[\frac{3n^2 + 5n + 1}{3} \right] \\ &= \frac{n(n+1)(3n^2 + 5n + 1)}{6}\end{aligned}$$

Question 4:

Find the sum to n terms of the series $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

Ans:

The given series is $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \dots$

n^{th} term, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (By partial fractions)

$$a_1 = \frac{1}{1} - \frac{1}{2}$$

$$a_2 = \frac{1}{2} - \frac{1}{3}$$

$$a_3 = \frac{1}{3} - \frac{1}{4} \dots$$

$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

Adding the above terms column wise, we obtain

$$a_1 + a_2 + \dots + a_n = \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n+1} \right]$$

$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

Question 5:

Find the sum to n terms of the series $5^2 + 6^2 + 7^2 + \dots + 20^2$

Ans:

The given series is $5^2 + 6^2 + 7^2 + \dots + 20^2$

n^{th} term, $a_n = (n + 4)^2 = n^2 + 8n + 16$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (k^2 + 8k + 16) \\ &= \sum_{k=1}^n k^2 + 8 \sum_{k=1}^n k + \sum_{k=1}^n 16 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n\end{aligned}$$

16th term is $(16 + 4)^2 = 20^2$

$$\begin{aligned}\therefore S_{16} &= \frac{16(16+1)(2 \times 16+1)}{6} + \frac{8 \times 16 \times (16+1)}{2} + 16 \times 16 \\ &= \frac{(16)(17)(33)}{6} + \frac{(8) \times 16 \times (16+1)}{2} + 16 \times 16 \\ &= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256 \\ &= 1496 + 1088 + 256 \\ &= 2840\end{aligned}$$

$$\therefore 5^2 + 6^2 + 7^2 + \dots + 20^2 = 2840$$

Question 6:

Find the sum to n terms of the series $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$

Ans:

The given series is $3 \times 8 + 6 \times 11 + 9 \times 14 + \dots$

$$a_n = (n^{\text{th}} \text{ term of } 3, 6, 9 \dots) \times (n^{\text{th}} \text{ term of } 8, 11, 14, \dots)$$

$$= (3n) (3n + 5)$$

$$= 9n^2 + 15n$$

$$\begin{aligned}\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (9k^2 + 15k) \\&= 9 \sum_{k=1}^n k^2 + 15 \sum_{k=1}^n k \\&= 9 \times \frac{n(n+1)(2n+1)}{6} + 15 \times \frac{n(n+1)}{2} \\&= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2} \\&= \frac{3n(n+1)}{2} (2n+1+5) \\&= \frac{3n(n+1)}{2} (2n+6) \\&= 3n(n+1)(n+3)\end{aligned}$$

Question 7:

Find the sum to n terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

Ans:

The given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$

$$a_n = (1^2 + 2^2 + 3^2 + \dots + n^2)$$

$$\begin{aligned}&= \frac{n(n+1)(2n+1)}{6} \\&= \frac{n(2n^2 + 3n + 1)}{6} = \frac{2^3 + 3n^2 + n}{6} \\&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n\end{aligned}$$

$$\begin{aligned}
\therefore S_n &= \sum_{k=1}^n a_k \\
&= \sum_{k=1}^n \left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k \right) \\
&= \frac{1}{3} \sum_{k=1}^n k^3 + \frac{1}{2} \sum_{k=1}^n k^2 + \frac{1}{6} \sum_{k=1}^n k \\
&= \frac{1}{3} \frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \times \frac{n(n+1)}{2} \\
&= \frac{n(n+1)}{6} \left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2} \right] \\
&= \frac{n(n+1)}{6} \left[\frac{n^2 + n + 2n + 1 + 1}{2} \right] \\
&= \frac{n(n+1)}{6} \left[\frac{n^2 + n + 2n + 2}{2} \right] \\
&= \frac{n(n+1)}{6} \left[\frac{n(n+1) + 2(n+1)}{2} \right] \\
&= \frac{n(n+1)}{6} \left[\frac{(n+1)(n+2)}{2} \right] \\
&= \frac{n(n+1)^2(n+2)}{12}
\end{aligned}$$

Question 8:

Find the sum to n terms of the series whose n^{th} term is given by $n(n+1)(n+4)$.

Ans:

$$a_n = n(n+1)(n+4) = n(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$$

$$\begin{aligned}
\therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n k^3 + 5 \sum_{k=1}^n k^2 + 4 \sum_{k=1}^n k \\
&= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} \\
&= \frac{n(n+1)}{2} \left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4 \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2 + 3n + 20n + 10 + 24}{6} \right] \\
&= \frac{n(n+1)}{2} \left[\frac{3n^2 + 23n + 34}{6} \right] \\
&= \frac{n(n+1)(3n^2 + 23n + 34)}{12}
\end{aligned}$$

Question 9:

Find the sum to n terms of the series whose n^{th} term is given by $n^2 + 2^n$

Ans:

$$a_n = n^2 + 2^n$$

$$\therefore S_n = \sum_{k=1}^n k^2 + 2^k = \sum_{k=1}^n k^2 + \sum_{k=1}^n 2^k \quad (1)$$

$$\text{Consider } \sum_{k=1}^n 2^k = 2^1 + 2^2 + 2^3 + \dots$$

The above series $2, 2^2, 2^3, \dots$ is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^n 2^k = \frac{(2) \left[(2)^n - 1 \right]}{2 - 1} = 2(2^n - 1) \quad (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^n k^2 + 2(2^n - 1) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

Question 10:

Find the sum to n terms of the series whose n^{th} term is given by $(2n - 1)^2$

Ans:

$$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$$

$$\begin{aligned}
 \therefore S_n &= \sum_{k=1}^n a_k = \sum_{k=1}^n (4k^2 - 4k + 1) \\
 &= 4 \sum_{k=1}^n k^2 - 4 \sum_{k=1}^n k + \sum_{k=1}^n 1 \\
 &= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n \\
 &= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n \\
 &= n \left[\frac{2(2n^2 + 3n + 1)}{3} - 2(n+1) + 1 \right] \\
 &= n \left[\frac{4n^2 + 6n + 2 - 6n - 6 + 3}{3} \right] \\
 &= n \left[\frac{4n^2 - 1}{3} \right] \\
 &= \frac{n(2n+1)(2n-1)}{3}
 \end{aligned}$$

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