$\Rightarrow \lambda = \frac{4}{6} = \frac{2}{3}$

Hence, point B divides AC in the ratio $2:3$.

Question 9:

Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $(2\vec{a}+\vec{b})$ and $(\vec{a}-3\vec{b})$ externally in the ratio 1: 2. Also, show that P is the mid point of the line segment RQ.

Answer

It is given that $\overrightarrow{OP} = 2\vec{a}+\vec{b},\ \overrightarrow{OQ} = \vec{a}-3\vec{b}$.

It is given that point R divides a line segment joining two points P and Q externally in the ratio 1: 2. Then, on using the section formula, we get:

$$\overrightarrow{OR} = \frac{2(2\vec{a}+\vec{b})-(\vec{a}-3\vec{b})}{2-1} = \frac{4\vec{a}+2\vec{b}-\vec{a}+3\vec{b}}{1} = 3\vec{a}+5\vec{b}$$

Therefore, the position vector of point R is $3\vec{a}+5\vec{b}$.

Position vector of the mid-point of RQ = $\frac{\overrightarrow{OQ}+\overrightarrow{OR}}{2}$

$$= \frac{(\vec{a}-3\vec{b})+(3\vec{a}+5\vec{b})}{2}$$

$$= 2\vec{a}+\vec{b}$$

$$= \overrightarrow{OP}$$

Hence, P is the mid-point of the line segment RQ.

Question 10:

The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$.

Find the unit vector parallel to its diagonal. Also, find its area.

Answer

Adjacent sides of a parallelogram are given as: $\vec{a} = 2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}-3\hat{k}$

Then, the diagonal of a parallelogram is given by $\vec{a}+\vec{b}$.

$$\vec{a}+\vec{b} = (2+1)\hat{i}+(-4-2)\hat{j}+(5-3)\hat{k} = 3\hat{i}-6\hat{j}+2\hat{k}$$

Thus, the unit vector parallel to the diagonal is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{3^2+(-6)^2+2^2}} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{\sqrt{9+36+4}} = \frac{3\hat{i}-6\hat{j}+2\hat{k}}{7} = \frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}.$$

$\therefore$ Area of parallelogram ABCD = $|\vec{a}\times\vec{b}|$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$= \hat{i}(12+10)-\hat{j}(-6-5)+\hat{k}(-4+4)$$

$$= 22\hat{i}+11\hat{j}$$

$$= 11(2\hat{i}+\hat{j})$$

$$\therefore |\vec{a}\times\vec{b}| = 11\sqrt{2^2+1^2} = 11\sqrt{5}$$

Hence, the area of the parallelogram is $11\sqrt{5}$ square units.

Question 11:

Show that the direction cosines of a vector equally inclined to the axes OX, OY and OZ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Answer

Let a vector be equally inclined to axes OX, OY, and OZ at angle *α*.

Then, the direction cosines of the vector are cos *α*, cos *α*, and cos *α*.

Now,

$\cos^2\alpha+\cos^2\alpha+\cos^2\alpha=1$

$\Rightarrow 3\cos^2\alpha=1$

$\Rightarrow \cos\alpha=\frac{1}{\sqrt{3}}$

Hence, the direction cosines of the vector which are equally inclined to the axes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Question 12:

Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k},\ \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$, and $\vec{c}.\vec{d}=15$.

Answer

Let $\vec{d}=d_1\hat{i}+d_2\hat{j}+d_3\hat{k}$.

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, we have:

$\vec{d}\cdot\vec{a}=0$

$\Rightarrow d_1+4d_2+2d_3=0 \qquad \ldots\text{(i)}$

And,

$\vec{d}\cdot\vec{b}=0$

$\Rightarrow 3d_1-2d_2+7d_3=0 \qquad \ldots\text{(ii)}$

Also, it is given that:

$\vec{c}\cdot\vec{d}=15$

$\Rightarrow 2d_1-d_2+4d_3=15 \qquad \ldots\text{(iii)}$

On solving (i), (ii), and (iii), we get:

$d_1=\frac{160}{3}, d_2=-\frac{5}{3}$ and $d_3=-\frac{70}{3}$

$\therefore \vec{d}=\frac{160}{3}\hat{i}-\frac{5}{3}\hat{j}-\frac{70}{3}\hat{k}=\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$

Hence, the required vector is $\frac{1}{3}\left(160\hat{i}-5\hat{j}-70\hat{k}\right)$.

Question 13:

The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Answer

$\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$

$=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

Therefore, unit vector along $\left(2\hat{i}+4\hat{j}-5\hat{k}\right)+\left(\lambda\hat{i}+2\hat{j}+3\hat{k}\right)$ is given as:

$$\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{4+4\lambda+\lambda^2+36+4}}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$$

Scalar product of $\left(\hat{i}+\hat{j}+\hat{k}\right)$ with this unit vector is 1.

$\Rightarrow \left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow \sqrt{\lambda^2+4\lambda+44}=\lambda+6$

$\Rightarrow \lambda^2+4\lambda+44=(\lambda+6)^2$

$\Rightarrow \lambda^2+4\lambda+44=\lambda^2+12\lambda+36$

$\Rightarrow 8\lambda=8$

$\Rightarrow \lambda=1$

Hence, the value of λ is 1.

Question 14:

If $\vec{a},\vec{b},\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a},\vec{b}$ and $\vec{c}$.

Answer

Since $\vec{a},\vec{b},$ and $\vec{c}$ are mutually perpendicular vectors, we have

$\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{c}=\vec{c}\cdot\vec{a}=0.$

It is given that:

$|\vec{a}|=|\vec{b}|=|\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a},\vec{b},$ and $\vec{c}$ at angles $\theta_1,\ \theta_2,$ and θ_3 respectively.

Then, we have:

$$\cos\theta_1 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} \qquad [\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}=0]$$

$$= \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_2 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|}$$

$$= \frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}|\cdot|\vec{b}|} \qquad [\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=0]$$

$$= \frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\theta_3 = \frac{(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} = \frac{\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|}$$

$$= \frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} \qquad [\vec{a}\cdot\vec{c}=\vec{b}\cdot\vec{c}=0]$$

$$= \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now, as $|\vec{a}|=|\vec{b}|=|\vec{c}|$, $\cos\theta_1=\cos\theta_2=\cos\theta_3$.

$\therefore \theta_1=\theta_2=\theta_3$

Hence, the vector $(\vec{a}+\vec{b}+\vec{c})$ is equally inclined to $\vec{a}, \vec{b}$, and $\vec{c}$.

Question 15:

Prove that $(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$, if and only if $\vec{a}, \vec{b}$ are perpendicular, given $\vec{a}\neq\vec{0}, \vec{b}\neq\vec{0}$.

Answer

$$(\vec{a}+\vec{b}).(\vec{a}+\vec{b})=|\vec{a}|^2+|\vec{b}|^2$$

$$\Leftrightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=|\vec{a}|^2+|\vec{b}|^2 \qquad [\text{Distributivity of scalar products over addition}]$$

$$\Leftrightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2 \qquad [\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}\ (\text{Scalar product is commutative})]$$

$$\Leftrightarrow 2\vec{a}\cdot\vec{b}=0$$

$$\Leftrightarrow \vec{a}\cdot\vec{b}=0$$

$\therefore \vec{a}$ and $\vec{b}$ are perpendicular. $\qquad [\vec{a}\neq\vec{0},\ \vec{b}\neq\vec{0}\ (\text{Given})]$

Question 16:

If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b}\geq 0$ only when

(A) $0<\theta<\frac{\pi}{2}$ (B) $0\leq\theta\leq\frac{\pi}{2}$

(C) $0<\theta<\pi$ (D) $0\leq\theta\leq\pi$

Answer

Let θ be the angle between two vectors $\vec{a}$ and $\vec{b}$.

Then, without loss of generality, $\vec{a}$ and $\vec{b}$ are non-zero vectors so that $|\vec{a}|$ and $|\vec{b}|$ are positive.

*********** END **********