Question 15:

Find the shortest distance between the lines $\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$ and $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$

Answer

The given lines are $\dfrac{x+1}{7} = \dfrac{y+1}{-6} = \dfrac{z+1}{1}$ and $\dfrac{x-3}{1} = \dfrac{y-5}{-2} = \dfrac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$, is given by,

$$d = \dfrac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2}} \qquad \ldots(1)$$

Comparing the given equations, we obtain

$x_1 = -1, \ y_1 = -1, \ z_1 = -1$

$a_1 = 7, \ b_1 = -6, \ c_1 = 1$

$x_2 = 3, \ y_2 = 5, \ z_2 = 7$

$a_2 = 1, \ b_2 = -2, \ c_2 = 1$

Then, $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$

$= 4(-6+2) - 6(7-1) + 8(-14+6)$

$= -16 - 36 - 64$

$= -116$

$\Rightarrow \sqrt{(b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2} = \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2}$

$= \sqrt{16 + 36 + 64}$

$= \sqrt{116}$

$= 2\sqrt{29}$

Substituting all the values in equation (1), we obtain

$d = \dfrac{-116}{2\sqrt{29}} = \dfrac{-58}{\sqrt{29}} = \dfrac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$

Since distance is always non-negative, the distance between the given lines is

$2\sqrt{29}$ units.

Question 16:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$

and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

Answer

The given lines are $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left| \dfrac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \qquad \ldots(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we obtain

$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$

$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k})$$
$$= -9 \times 3 + 3 \times 3 + 9 \times 3$$
$$= 9$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{9}{3\sqrt{19}} \right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$
$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

Answer

The given lines are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$
$$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \qquad ...(1)$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$
$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \qquad ...(2)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right| \qquad ...(3)$$

For the given equations,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$
$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$
$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$
$$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k}) = -4 + 12 = 8$$

Substituting all the values in equation (3), we obtain

$$d = \left| \frac{8}{\sqrt{29}} \right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise 11.3

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) z = 2 (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d) 5y + 8 = 0

Answer

(a) The equation of the plane is z = 2 or 0x + 0y + z = 2 ... (1)
The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$$

Dividing both sides of equation (1) by 1, we obtain

$0.x + 0.y + 1.z = 2$

This is of the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) x + y + z = 1 ... (1)

The direction ratios of normal are 1, 1, and 1.

$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \qquad \text{...(2)}$$

This equation is of the form lx + my + nz = d, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and $\frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) 2x + 3y − z = 5 ... (1)

The direction ratios of normal are 2, 3, and −1.

$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

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