Question 6:

Prove $\cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}=\sin^{-1}\frac{56}{65}$

Answer

Let $\sin^{-1}\frac{3}{5}=x$. Then, $\sin x=\frac{3}{5}\Rightarrow \cos x=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{\frac{16}{25}}=\frac{4}{5}$.

$\therefore \tan x=\frac{3}{4}\Rightarrow x=\tan^{-1}\frac{3}{4}$

$\therefore \sin^{-1}\frac{3}{5}=\tan^{-1}\frac{3}{4}$ ...(1)

Now, let $\cos^{-1}\frac{12}{13}=y$. Then, $\cos y=\frac{12}{13}\Rightarrow \sin y=\frac{5}{13}$.

$\therefore \tan y=\frac{5}{12}\Rightarrow y=\tan^{-1}\frac{5}{12}$

$\therefore \cos^{-1}\frac{12}{13}=\tan^{-1}\frac{5}{12}$ ...(2)

Let $\sin^{-1}\frac{56}{65}=z$. Then, $\sin z=\frac{56}{65}\Rightarrow \cos z=\frac{33}{65}$.

$\therefore \tan z=\frac{56}{33}\Rightarrow z=\tan^{-1}\frac{56}{33}$

$\therefore \sin^{-1}\frac{56}{65}=\tan^{-1}\frac{56}{33}$ ...(3)

Now, we have:

$$\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{12}{13}+\sin^{-1}\frac{3}{5}\\
&= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{3}{4} && [\text{Using (1) and (2)}]\\
&= \tan^{-1}\frac{\frac{5}{12}+\frac{3}{4}}{1-\frac{5}{12}\cdot\frac{3}{4}} && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\
&= \tan^{-1}\frac{20+36}{48-15}\\
&= \tan^{-1}\frac{56}{33}\\
&= \sin^{-1}\frac{56}{65}=\text{R.H.S.} && [\text{Using (3)}]
\end{aligned}$$

Question 7:

Prove $\tan^{-1}\frac{63}{16}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Answer

Let $\sin^{-1}\frac{5}{13}=x$. Then, $\sin x=\frac{5}{13}\Rightarrow \cos x=\frac{12}{13}$.

$\therefore \tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$ ...(1)

Let $\cos^{-1}\frac{3}{5}=y$. Then, $\cos y=\frac{3}{5}\Rightarrow \sin y=\frac{4}{5}$.

$\therefore \tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}$ ...(2)

Using (1) and (2), we have

$$\begin{aligned}
\text{R.H.S.} &= \sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\\
&= \tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\
&= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right) && \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\
&= \tan^{-1}\left(\frac{15+48}{36-20}\right)
\end{aligned}$$

$$= \tan^{-1}\frac{63}{16}$$
$$= \text{L.H.S.}$$

Question 8:

Prove $\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Answer

$$\text{L.H.S.} = \tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}$$
$$= \tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}\times\frac{1}{7}}\right)+\tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}\times\frac{1}{8}}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]$$
$$= \tan^{-1}\left(\frac{7+5}{35-1}\right)+\tan^{-1}\left(\frac{8+3}{24-1}\right)$$
$$= \tan^{-1}\frac{12}{34}+\tan^{-1}\frac{11}{23}$$
$$= \tan^{-1}\frac{6}{17}+\tan^{-1}\frac{11}{23}$$
$$= \tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}\times\frac{11}{23}}\right)$$
$$= \tan^{-1}\left(\frac{138+187}{391-66}\right)$$
$$= \tan^{-1}\left(\frac{325}{325}\right)=\tan^{-1}1$$
$$= \frac{\pi}{4} = \text{R.H.S.}$$

Question 9:

Prove $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right),\ x\in[0,\ 1]$

Answer

Let $x=\tan^2\theta$. Then, $\sqrt{x}=\tan\theta \Rightarrow \theta=\tan^{-1}\sqrt{x}$.

$$\therefore \frac{1-x}{1+x}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos 2\theta$$

Now, we have:

$$\text{R.H.S.}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\cos^{-1}(\cos 2\theta)=\frac{1}{2}\times 2\theta=\theta=\tan^{-1}\sqrt{x}=\text{L.H.S.}$$

Question 10:

Prove $\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2},\ x\in\left(0,\ \frac{\pi}{4}\right)$

Answer

Consider $\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$= \frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2} \qquad \text{(by rationalizing)}$$
$$= \frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x-1+\sin x}$$
$$= \frac{2\left(1+\sqrt{1-\sin^2 x}\right)}{2\sin x}=\frac{1+\cos x}{\sin x}=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$= \cot\frac{x}{2}$$
$$\therefore \text{L.H.S.}=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=\text{R.H.S.}$$

Question 11:

Prove $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}}\le x\le 1$ [**Hint:** put x = cos 2θ]

Answer

Put $x=\cos 2\theta$ so that $\theta=\frac{1}{2}\cos^{-1}x$. Then, we have:

$$\text{L.H.S.}=\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$$
$$= \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos^2\theta - \sqrt{2}\sin^2\theta}{\sqrt{2\cos^2\theta} + \sqrt{2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= \tan^{-1}1 - \tan^{-1}(\tan\theta) \qquad \left[\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y\right]$$

$$= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$

Question 12:

Prove $\frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Answer

$$\text{L.H.S.} = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\frac{1}{3}$$

$$= \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)$$

$$= \frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right) \qquad \text{....(1)} \quad \left[\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}\right]$$

Now, let $\cos^{-1}\frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1-\left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1}\frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1}\frac{1}{3} = \sin^{-1}\frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Answer

$$2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$\Rightarrow \frac{2\cos x}{1-\cos^2 x} = 2\operatorname{cosec} x$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \sin x$$

$$\Rightarrow \tan x = 1$$

$$\therefore x = \frac{\pi}{4}$$

Question 14:

Solve $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x > 0)$

Answer

$$\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1 - \tan^{-1}x = \frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4} = \frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x = \frac{\pi}{6}$$

$$\Rightarrow x = \tan\frac{\pi}{6}$$

$$\therefore x = \frac{1}{\sqrt{3}}$$

*********** END **********