

## CHAPTER FIFTEEN

# WAVES

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### 15.1 INTRODUCTION

In the previous Chapter, we studied the motion of objects oscillating in isolation. What happens in a system, which is a collection of such objects ? A material medium provides such an example. Here, elastic forces bind the constituents to each other and, therefore, the motion of one affects that of the other. If you drop a little pebble in a pond of still water, the water surface gets disturbed. The disturbance does not remain confined to one place, but propagates outward along a circle. If you continue dropping pebbles in the pond, you see circles rapidly moving outward from the point where the water surface is disturbed. It gives a feeling as if the water is moving outward from the point of disturbance. If you put some cork pieces on the disturbed surface, it is seen that the cork pieces move up and down but do not move away from the centre of disturbance. This shows that the water mass does not flow outward with the circles, but rather a moving disturbance is created. Similarly, when we speak, the sound moves outward from us, without any flow of air from one part of the medium to another. The disturbances produced in air are much less obvious and only our ears or a microphone can detect them. These patterns, which move without the actual physical transfer or flow of matter as a whole, are called **waves**. In this Chapter, we will study such waves.

In a wave, information and energy, in the form of signals, propagate from one point to another but no material object makes the journey. All our communications depend on the transmission of signals through waves. When we make a telephone call to a friend at a distant place, a sound wave carries the message from our vocal cords to the telephone. There, an electrical signal is generated which propagates along the copper wire. If the distance is too large, the electrical signal generated may be transformed into a light signal or

electromagnetic waves and transmitted through optical cables or the atmosphere, possibly by way of a communication satellite. At the receiving end, the electrical or light signal or the electromagnetic waves are transformed back into sound waves travelling from the telephone to the ear.

Not all waves require a medium for their propagation. We know that light waves can travel through vacuum. The light emitted by stars, which are hundreds of light years away, reaches us through inter-stellar space, which is practically a vacuum.

The waves we come across are mainly of three types: (a) mechanical waves, (b) electromagnetic waves and (c) matter waves. Mechanical waves are most familiar because we encounter them constantly; common examples include water waves, sound waves, seismic waves, etc. All these waves have certain central features: They are governed by Newton's laws, and can exist only within a material medium, such as water, air, and rock. The common examples of electromagnetic waves are visible and ultra-violet light, radio waves, microwaves, x-rays etc. All electromagnetic waves travel through vacuum at the same speed  $c$ , given by  $c = 299,792,458 \text{ m s}^{-1}$  (speed of light) (15.1)

Unlike the mechanical waves, the electromagnetic waves do not require any medium for their propagation. You would learn more about these waves later.

Matter waves are associated with moving electrons, protons, neutrons and other fundamental particles, and even atoms and molecules. Because we commonly think of these as constituting matter, such waves are called **matter waves**. They arise in quantum mechanical description of nature that you will learn in your later studies. Though conceptually more abstract than mechanical or electromagnetic waves, they have already found applications in several devices basic to modern technology; matter waves associated with electrons are employed in electron microscopes.

In this chapter we will study mechanical waves, which require a material medium for their propagation.

The aesthetic influence of waves on art and literature is seen from very early times; yet the first scientific analysis of wave motion dates back

to the seventeenth century. Some of the famous scientists associated with the physics of wave motion are Christiaan Huygens (1629-1695), Robert Hooke and Isaac Newton. The understanding of physics of waves followed the physics of oscillations of masses tied to springs and physics of the simple pendulum. Waves in elastic media are intimately connected with harmonic oscillations. (Stretched strings, coiled springs, air, etc., are examples of elastic media.) We shall illustrate this connection through simple examples.

Consider a collection of springs connected to one another as shown in Fig. 15.1. If the spring at one end is pulled suddenly and released, the disturbance travels to the other end. What has happened? The first spring is disturbed from its equilibrium length. Since the second spring is connected to the first, it is also stretched or compressed, and so on. The disturbance moves



**Fig. 15.1** A collection of springs connected to each other. The end A is pulled suddenly generating a disturbance, which then propagates to the other end.

from one end to the other; but each spring only executes small oscillations about its equilibrium position. As a practical example of this situation, consider a stationary train at a railway station. Different bogies of the train are coupled to each other through a spring coupling. When an engine is attached at one end, it gives a push to the bogie next to it; this push is transmitted from one bogie to another without the entire train being bodily displaced.

Now let us consider the propagation of sound waves in air. As the wave passes through air, it compresses or expands a small region of air. This causes a change in the density of that region, say  $\delta\rho$ , this change induces a change in pressure,  $\delta p$ , in that region. Pressure is force per unit area, so there is a **restoring force proportional** to the disturbance, just like in a spring. In this case, the quantity similar to extension or compression of the spring is the change in density. If a region is compressed, the molecules in that region are packed together, and they tend to move out to the adjoining region, thereby increasing the density or creating compression in the adjoining region. Consequently, the air

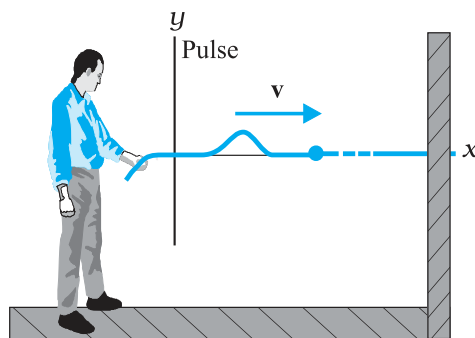
in the first region undergoes rarefaction. If a region is comparatively rarefied the surrounding air will rush in making the rarefaction move to the adjoining region. Thus, the compression or rarefaction moves from one region to another, making the propagation of a disturbance possible in air.

In solids, similar arguments can be made. In a crystalline solid, atoms or group of atoms are arranged in a periodic lattice. In these, each atom or group of atoms is in equilibrium, due to forces from the surrounding atoms. Displacing one atom, keeping the others fixed, leads to restoring forces, exactly as in a spring. So we can think of atoms in a lattice as end points, with springs between pairs of them.

In the subsequent sections of this chapter we are going to discuss various characteristic properties of waves.

## 15.2 TRANSVERSE AND LONGITUDINAL WAVES

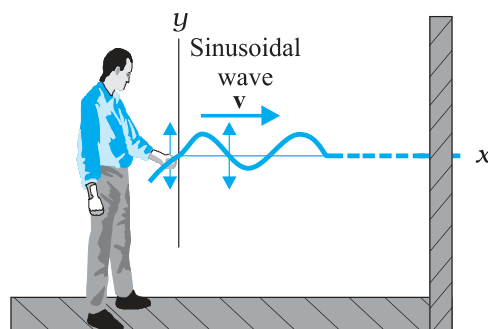
Mechanical waves can be transverse or longitudinal depending on the relationship between the directions of disturbance or displacement in the medium and that of the propagation of wave. To differentiate between them let us consider the response of a stretched string fixed at one end. If you give a single up-and-down jerk to the free end of this string, as shown in Fig. 15.2, a wave in the form of a single **pulse** travels along the string. We assume that the string is very long as compared to the size of the pulse, so that the pulse dissipates out by the time it reaches the other end and, therefore, its reflection from the other end may be ignored. The formation and propagation of this pulse is possible because the string is under tension. When you pull your end of the string upwards it begins to pull upwards on the adjacent section of the string, because of the tension between the two sections. As the adjacent section begins to move upwards, it begins to pull the next section upwards, and so on. In the meanwhile you have pulled down your end of the string. As each section moves upwards in turn, it begins to be pulled back downwards by neighbouring sections that are already on the way down. The net result is that a distortion in the shape of the string (the pulse) moves along the string with a certain velocity  $\mathbf{v}$ .



**Fig. 15.2** A single pulse is sent along a stretched string. A typical element of the string (such as that marked with a dot) moves up and then down as the pulse passes through. The element's motion is perpendicular to the direction in which the wave travels.

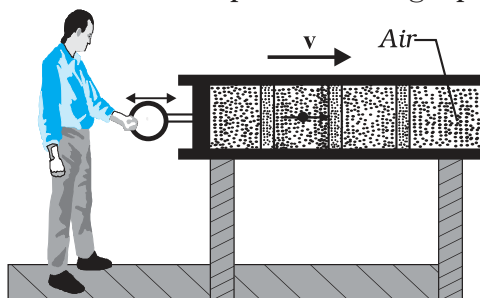
If you move your end up and down in a continuous manner, a continuous wave travels along the string with a velocity  $\mathbf{v}$ . However, if the motion of your hand is a sinusoidal function of time, at any given instant of time the wave will have a sinusoidal shape as shown in Fig. 15.3. The wave has the shape of a sine or cosine curve.

The waves shown in Fig. 15.3 can be studied in two ways. Oneway is to monitor the waveforms as they move to the right, i.e. take a 'snapshot' of the string at a given instant of time. Alternatively, we fix our attention to a particular position on the string and monitor the motion of an element at that point as it oscillates up and down while a wave passes through it. We would find that the displacement of every such oscillating string element is transverse (i.e. perpendicular) to the direction of travel of the wave as indicated in Fig. 15.3. Such a wave is said to be a **transverse wave**.



**Fig. 15.3** A sinusoidal wave is sent along the string. A typical element of the string moves up and down continuously as the wave passes. It is a transverse wave.

Now, let us consider the production of waves in a long air-filled pipe by the movement of a piston as shown in Fig. 15.4. If you suddenly move the piston to the right and then to the left, you are sending a pulse of pressure along the pipe. The motion of the piston to the right pushes



**Fig. 15.4** A sound wave is set up in an air filled pipe by moving a piston back and forth. As the oscillations of an element of air are parallel to the direction in which the wave travels, the wave is a longitudinal wave.

the elements of air next towards the right, changing the air pressure there. The increased pressure in this region then pushes on the elements of air somewhat farther along the pipe. Moving the piston to the left reduces the air pressure next to it. This causes the elements of air next to it move back to the left and then the farther elements follow. Thus, the motion of the air and the change in air pressure travel towards the right along the pipe as a pulse.

If you push and pull on the piston in a simple harmonic manner, a sinusoidal wave travels along the pipe. It may be noted that the motion of the elements of air is parallel to the direction of propagation of the wave. This motion is said to be **longitudinal** and the wave produced is, therefore, called a **longitudinal wave**. The sound waves produced in air are such pressure waves and are therefore of longitudinal character.

In short, **in transverse waves, the constituents of the medium oscillate perpendicular to the direction of wave propagation and in longitudinal waves they oscillate along the direction of wave propagation.**

A wave, transverse or longitudinal, is said to be **travelling or progressive** if it travels from one point of the medium to another. A progressive wave is to be distinguished from a standing or stationary wave (see Section 15.7). In Fig. 15.3 transverse waves travel from one end of the string to the other end while the

longitudinal waves in Fig. 15.4 travel from one end of the pipe to its other end. We note again that in both the cases, it is the wave or the disturbance that moves from end to end, not the material through which the waves propagate.

In transverse waves, the particle motion is normal to the direction of propagation of the wave. Therefore, as the wave propagates, each element of the medium undergoes a shearing strain. Transverse waves can, therefore, be propagated only in those media which can sustain shearing stress, such as solids and strings, and not in fluids. Fluids as well as solids can sustain compressive strain; therefore, longitudinal waves can propagate in all elastic media. For example, in medium like a steel bar, both transverse and longitudinal waves can propagate while air can sustain only longitudinal waves. The waves on the surface of water are of two kinds: **capillary waves** and **gravity waves**. The former are ripples of fairly short wavelength—no more than a few centimetres—and the restoring force that produces them is the surface tension of water. Gravity waves have wavelengths typically ranging from several metres to several hundred metres. The restoring force that produces these waves is the pull of gravity, which tends to keep the water surface at its lowest level. The oscillations of the particles in these waves are not confined to the surface only, but extend with diminishing amplitude to the very bottom. The particle motion in the water waves involves a complicated motion; they not only move up and down but also back and forth. The waves in an ocean are a combination of both longitudinal and transverse waves.

It is found that generally transverse and longitudinal waves travel with different speeds in the same medium.

► **Example 15.1** Given below are some examples of wave motion. State in each case if the wave motion is transverse, longitudinal or a combination of both:

- Motion of a kink in a longitudinal spring produced by displacing one end of the spring sideways.
- Waves produced in a cylinder containing a liquid by moving its piston back and forth.
- Waves produced by a motorboat sailing in water.
- Ultrasonic waves in air produced by a vibrating quartz crystal.

**Answer**

- (a) Transverse and longitudinal
- (b) Longitudinal
- (c) Transverse and longitudinal
- (d) Longitudinal

**15.3 DISPLACEMENT RELATION IN A PROGRESSIVE WAVE**

To describe the propagation of a wave in a medium (and the motion of any constituent of the medium), we need a function that completely gives the shape of the wave at every instant of time. For example, to completely describe the wave on a string (and the motion of any element along its length) we need a relation which describes the displacement of an element at a particular position as a function of time and also describes the state of vibration of various elements of the string along its length at a given instant of time. For a sinusoidal wave, as shown in Fig. 15.3, this function should be periodic in space as well as in time. Let  $y(x, t)$  denote the transverse displacement of the element at position  $x$  at time  $t$ . As the wave sweeps through succeeding elements of the string, the elements oscillate parallel to the  $y$ -axis. At any time  $t$ , the displacement  $y$  of the element located at position  $x$  is given by

$$y(x, t) = a \sin(kx - \omega t + \phi) \quad (15.2)$$

One can as well choose a cosine function or a linear combination of sine and cosine functions such as,

$$y(x, t) = A \sin(kx - \omega t) + B \cos(kx - \omega t), \quad (15.3)$$

then in Eq. (15.2),

$$a = \sqrt{A^2 + B^2} \text{ and } \phi = \tan^{-1}\left(\frac{B}{A}\right)$$

The function represented in Eq. (15.2) is periodic in position coordinate  $x$  and time  $t$ . It represents a transverse wave moving along the  $x$ -axis. At any time  $t$ , it gives the displacement of the elements of the string as a function of their position. It can tell us the shape of the wave at any given time and show how the wave progresses. Functions, such as that given in Eq. (15.2), represent a progressive wave travelling along the positive direction of the  $x$ -axis. On the other hand a function,

$$y(x, t) = a \sin(kx + \omega t + \phi), \quad (15.4)$$

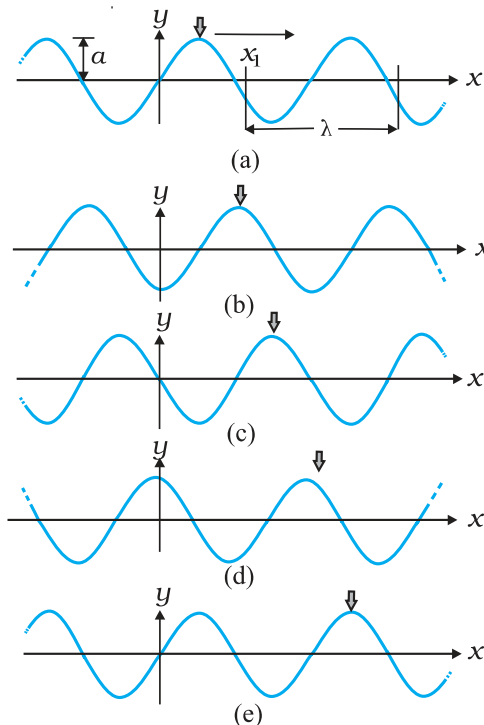
represents a wave travelling in the negative direction of  $x$ -axis (see Section 15.4). The set of

four parameters  $a$ ,  $\phi$ ,  $k$ , and  $\omega$  in Eq. (15.2) completely describe a harmonic wave. The names of these parameters are displayed in Fig. 15.5 and are defined later.

$$\begin{array}{ccccccc} \text{Displacement} & & \text{Amplitude} & & \text{Phase} & & \\ y(x, t) = & = & a & \sin(kx - \omega t + \phi) \\ & & & \uparrow & \uparrow & \uparrow & \\ & & & \text{Angular} & \text{Angular} & \text{Initial} & \\ & & & \text{Wave} & \text{Frequency} & \text{Phase} & \\ & & & \text{Number} & & \text{Angle} & \end{array}$$

**Fig. 15.5** The names of the quantities in Eq. (15.2) for a progressive wave.

To understand the definition of the quantities in Eq. (15.2), let us consider the graphs shown in Fig. 15.6. These graphs represent plots of Eq. (15.2) for five different values of time  $t$  as the wave travels in positive direction of  $x$ -axis. A point of maximum positive displacement in a wave, shown by the arrow, is called **crest**, and a point of maximum negative displacement is called **trough**. The progress of the wave is indicated by the progress of the short arrow pointing to a crest of the wave towards the right. As we move from one plot to another, the short arrow moves to the right with the wave shape,



**Fig. 15.6** Plots of Eq. (15.2) for a wave travelling in the positive direction of an  $x$ -axis at five different values of time  $t$ .



but the string moves only parallel to  $y$ -axis. It can be seen that as we go from plot (a) to (e), a particular element of the string has undergone one complete cycle of changes or completed one full oscillation. During this course of time the short arrow head or the wave has moved by a characteristic distance along the  $x$ -axis.

In the context of the above five plots, we will now define various quantities in Eq. (15.2) and shown in Fig. 15.5.

### 15.3.1 Amplitude and Phase

The **amplitude**  $a$  of a wave such as that in Figs. 15.5 and 15.6 is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them. It is depicted in Fig. 15.6 (a). Since  $a$  is a magnitude, it is a positive quantity, even if the displacement is negative.

The **phase** of the wave is the argument  $(kx - \omega t + \phi)$  of the oscillatory term  $\sin(kx - \omega t + \phi)$  in Eq. (15.2). It describes the state of motion as the wave sweeps through a string element at a particular position  $x$ . It changes linearly with time  $t$ . The sine function also changes with time, oscillating between  $+1$  and  $-1$ . Its extreme positive value  $+1$  corresponds to a peak of the wave moving through the element; then the value of  $y$  at position  $x$  is  $a$ . Its extreme negative value  $-1$  corresponds to a valley of the wave moving through the element, then the value of  $y$  at position  $x$  is  $-a$ . Thus, the sine function and the time dependent phase of a wave correspond to the oscillation of a string element, and the amplitude of the wave determines the extremes of the element's displacement. The constant  $\phi$  is called the **initial phase angle**. The value of  $\phi$  is determined by the initial ( $t = 0$ ) displacement and velocity of the element (say, at  $x = 0$ ).

It is always possible to choose origin ( $x = 0$ ) and the initial instant ( $t = 0$ ) such that  $\phi = 0$ . There is no loss of generality in working with Eq. (15.2) with  $\phi = 0$ .

### 15.3.2 Wavelength and Angular Wave Number

The **wavelength**  $\lambda$  of a wave is the distance (parallel to the direction of wave propagation) between the consecutive repetitions of the shape of the wave. It is the minimum distance between two consecutive troughs or crests or

two consecutive points in the same phase of wave motion. A typical wavelength is marked in Fig. 15.6(a), which is a plot of Eq. (15.2) for  $t = 0$  and  $\phi = 0$ . At this time Eq. (15.2) reduces to

$$y(x, 0) = a \sin kx \quad (15.5)$$

By definition, the displacement  $y$  is same at both ends of this wavelength, that is at  $x = x_1$  and at  $x = x_1 + \lambda$ . Thus, by Eq. (15.2),

$$\begin{aligned} a \sin kx_1 &= a \sin k(x_1 + \lambda) \\ &= a \sin (kx_1 + k\lambda) \end{aligned}$$

This condition can be satisfied only when,

$$k\lambda = 2\pi n$$

where  $n = 1, 2, 3, \dots$  Since  $\lambda$  is defined as the least distance between points with the same phase,  $n = 1$  and

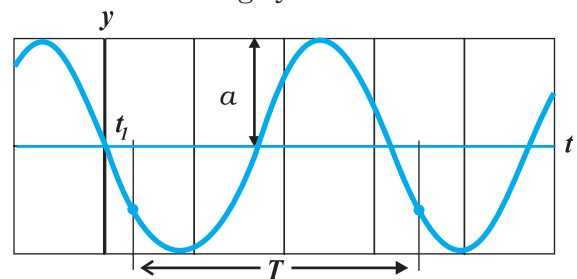
$$k = \frac{2\pi}{\lambda} \quad (15.6)$$

$k$  is called the **propagation constant** or the **angular wave number**; its SI unit is radian per metre or  $\text{rad m}^{-1}$ .\*

It may be noted that in Fig. 15.6, as we move from one plot to another, the wave moves to the right by a distance equal to  $\frac{1}{4}\lambda$ . Thus, by the fifth plot, it has moved to the right by a distance equal to  $\lambda$ .

### 15.3.3 Period, Angular Frequency and Frequency

Figure 15.7 shows a graph of the displacement  $y$ , of Eq. (15.2), versus time  $t$  at a certain position along the string, taken to be  $x = 0$ . If you were to monitor the string, you would see that the



**Fig. 15.7** A graph of the displacement of the string element at  $x = 0$  as a function of time, as the sinusoidal wave of Fig. 14.6 passes through it. The amplitude  $a$  is indicated. A typical period  $T$ , measured from an arbitrary time  $t_i$  is also indicated.

\* Here again, 'radian' could be dropped and the units could be written merely as  $\text{m}^{-1}$ . Thus,  $k$  represents  $2\pi$  times the number of waves (or the total phase difference) that can be accommodated per unit length, with SI units  $\text{m}^{-1}$ .

element of the string at that position moves up and down in simple harmonic motion given by Eq. (15.2) with  $x = 0$ ,

$$y(0, t) = a \sin(-\omega t) \\ = -a \sin \omega t$$

Figure 15.7 is a graph of this equation; it does not show the shape of the wave.

The **period** of oscillation  $T$  of a wave is defined as the time any string element takes to move through one complete oscillation. A typical period is marked on the Fig. 15.7. Applying Eq. (15.2) on both ends of this time interval, we get

$$-a \sin \omega t_1 = -a \sin \omega(t_1 + T) \\ = -a \sin(\omega t_1 + \omega T)$$

This can be true only if the least value of  $\omega T$  is  $2\pi$ , or if

$$\omega = 2\pi/T \quad (15.7)$$

$\omega$  is called the **angular frequency** of the wave, its SI unit is  $\text{rad s}^{-1}$ .

Look back at the five plots of a travelling wave in Fig. 15.6. The time between two consecutive plots is  $T/4$ . Thus, by the fifth plot, every string element has made one full oscillation.

The **frequency**  $\nu$  of a wave is defined as  $1/T$  and is related to the angular frequency  $\omega$  by

$$\nu = \frac{1}{T} = \frac{\omega}{2\pi} \quad (15.8)$$

It is the number of oscillations per unit time made by a string element as the wave passes through it. It is usually measured in hertz.

In the discussion above, reference has always been made to a wave travelling along a string or a transverse wave. In a longitudinal wave, the displacement of an element of the medium is parallel to the direction of propagation of the wave. In Eq. (15.2), the displacement function for a longitudinal wave is written as,

$$s(x, t) = a \sin(kx - \omega t + \phi) \quad (15.9)$$

where  $s(x, t)$  is the displacement of an element of the medium in the direction of propagation of the wave at position  $x$  and time  $t$ . In Eq. (15.9),  $a$  is the displacement amplitude; other quantities have the same meaning as in case of a transverse wave except that the displacement function  $y(x, t)$  is to be replaced by the function  $s(x, t)$ .

► **Example 15.2** A wave travelling along a string is described by,

$$y(x, t) = 0.005 \sin(80.0x - 3.0t),$$

in which the numerical constants are in SI units (0.005 m, 80.0  $\text{rad m}^{-1}$ , and 3.0  $\text{rad s}^{-1}$ ). Calculate (a) the amplitude, (b) the wavelength, and (c) the period and frequency of the wave. Also, calculate the displacement  $y$  of the wave at a distance  $x = 30.0$  cm and time  $t = 20$  s?

**Answer** On comparing this displacement equation with Eq. (15.2),

$$y(x, t) = a \sin(kx - \omega t),$$

we find

- (a) the amplitude of the wave is 0.005 m = 5 mm.  
(b) the angular wave number  $k$  and angular frequency  $\omega$  are

$$k = 80.0 \text{ m}^{-1} \text{ and } \omega = 3.0 \text{ s}^{-1}$$

We then relate the wavelength  $\lambda$  to  $k$  through Eq. (15.6),

$$\lambda = 2\pi/k \\ = \frac{2\pi}{80.0 \text{ m}^{-1}} \\ = 7.85 \text{ cm}$$

- (c) Now we relate  $T$  to  $\omega$  by the relation

$$T = 2\pi/\omega \\ = \frac{2\pi}{3.0 \text{ s}^{-1}} \\ = 2.09 \text{ s}$$

and frequency,  $\nu = 1/T = 0.48 \text{ Hz}$

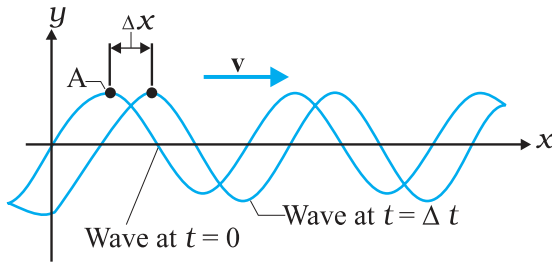
The displacement  $y$  at  $x = 30.0$  cm and time  $t = 20$  s is given by

$$y = (0.005 \text{ m}) \sin(80.0 \times 0.3 - 3.0 \times 20) \\ = (0.005 \text{ m}) \sin(-36 + 12\pi) \\ = (0.005 \text{ m}) \sin(1.699) \\ = (0.005 \text{ m}) \sin(97^\circ) \approx 5 \text{ mm}$$

## 15.4 THE SPEED OF A TRAVELLING WAVE

Let us monitor the propagation of a travelling wave represented by Eq. (15.2) along a string. The wave is travelling in the positive direction of  $x$ . We find that an element of string at a particular position  $x$  moves up and down as a function of time but the waveform advances to

the right. The displacement of various elements of the string at two different instants of time  $t$  differing by a small time interval  $\Delta t$  is depicted in Fig. 15.8 (the phase angle  $\phi$  has been taken to be zero). It is observed that during this interval of time the entire wave pattern moves by a distance  $\Delta x$  in the positive direction of  $x$ . Thus the wave is travelling to the right, in the positive direction of  $x$ . The ratio  $\Delta x/\Delta t$  is the wave speed  $v$ .



**Fig. 15.8** The plots of Eq.(15.2) at two instants of time differing by an interval  $\Delta t$ , at  $t = 0$  and then at  $t = \Delta t$ . As the wave moves to the right at velocity  $v$ , the entire curve shifts a distance  $\Delta x$  during  $\Delta t$ . The point A rides the waveform but the string element moves only up and down.

As the wave moves (see Fig. 15.8), each point of the moving waveform represents a particular phase of the wave and retains its displacement  $y$ . It may be noted that the points on the string do not retain their displacement, but the points on the waveform do. Let us consider a point like A marked on a peak of the waveform. If a point like A on the waveform retains its displacement as it moves, it follows from Eq. (15.2) that this is possible only when the argument is constant. It, therefore, follows that

$$kx - \omega t = \text{constant} \quad (15.10)$$

Note that in the argument both  $x$  and  $t$  are changing; therefore, to keep the argument constant, if  $t$  increases,  $x$  must also increase. This is possible only when the wave is moving in the positive direction of  $x$ .

To find the wave speed  $v$ , let us differentiate Eq. (15.10) with respect to time ;

$$\frac{d}{dt} (kx - \omega t) = 0$$

$$\text{or } k \frac{dx}{dt} - \omega = 0$$

$$\text{or } \frac{dx}{dt} = \frac{\omega}{k} = v \quad (15.11)$$

Making use of Eqs. (15.6)-(15.8), we can write,

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda \nu \quad (15.12)$$

Equation (15.11) is a general relation valid for all progressive waves. It merely states that the wave moves a distance of one wavelength in one period of oscillation. The speed of a wave is related to its wavelength and frequency by the Eq. (15.12), **but it is determined by the properties of the medium**. If a wave is to travel in a medium like air, water, steel, or a stretched string, it must cause the particles of that medium to oscillate as it passes through it. For this to happen, the medium must possess mass and elasticity. Therefore the linear mass density (or mass per unit length, in case of linear systems like a stretched string) and the elastic properties determine how fast the wave can travel in the medium. Conversely, it should be possible to calculate the speed of the wave through the medium in terms of these properties. In subsequent sub-sections of this chapter, we will obtain specific expressions for the speed of mechanical waves in some media.

#### 15.4.1 Speed of a Transverse Wave on Stretched String

The speed of transverse waves on a string is determined by two factors, (i) the linear mass density or mass per unit length,  $\mu$ , and (ii) the tension  $T$ . The mass is required so that there is kinetic energy and without tension no disturbance can be propagated in the string. The exact derivation of the relationship between the speed of wave in a stretched string and the two parameters mentioned above is outside the scope of this book. However, we take recourse to a simpler procedure. In dimensional analysis, we have already learnt (Chapter 2) how to get a relationship between different quantities which are interrelated. Such a relation is, however, uncertain to the extent of a constant factor.

The linear mass density,  $\mu$ , of a string is the mass  $m$  of the string divided by its length  $l$ . Therefore its dimension is  $[ML^{-1}]$ . The tension  $T$



has the dimension of force — namely,  $[MLT^{-2}]$ . Our goal is to combine  $\mu$  and  $T$  in such a way as to generate  $v$  [dimension  $[LT^{-1}]$ ]. If we examine the dimensions of these quantities, it can be seen that the ratio  $T/\mu$  has the dimension

$$\frac{[MLT^{-2}]}{[ML^{-1}]} = [L^2 T^{-2}]$$

Therefore, if  $v$  depends only on  $T$  and  $\mu$ , the relation between them must be

$$v = C \sqrt{\frac{T}{\mu}} \quad (15.13)$$

Here  $C$  is a dimensionless constant that cannot be determined by dimensional analysis. By adopting a more rigorous procedure it can be shown that the constant  $C$  is indeed equal to unity. The speed of transverse waves on a stretched string is, therefore, given by

$$v = \sqrt{\frac{T}{\mu}} \quad (15.14)$$

Equation (15.14) tells us :

**The speed of a wave along a stretched ideal string depends only on the tension and the linear mass density of the string and does not depend on the frequency of the wave.**

The **frequency** of the wave is determined by the source that generates the wave. The **wavelength** is then fixed by Eq. (15.12) in the form,

$$\lambda = \frac{v}{\nu} \quad (15.15)$$

► **Example 15.3** A steel wire 0.72 m long has a mass of  $5.0 \times 10^{-3}$  kg. If the wire is under a tension of 60 N, what is the speed of transverse waves on the wire ?

**Answer** Mass per unit length of the wire,

$$\mu = \frac{5.0 \times 10^{-3} \text{ kg}}{0.72 \text{ m}}$$

$$= 6.9 \times 10^{-3} \text{ kg m}^{-1}$$

Tension,  $T = 60 \text{ N}$

The speed of wave on the wire is given by

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{60 \text{ N}}{6.9 \times 10^{-3} \text{ kg m}^{-1}}} = 93 \text{ m s}^{-1}$$

### 15.4.2 Speed of a Longitudinal Wave Speed of Sound

In a longitudinal wave the constituents of the medium oscillate forward and backward in the direction of propagation of the wave. We have already seen that the sound waves travel in the form of compressions and rarefactions of small

#### Propagation of a pulse on a rope

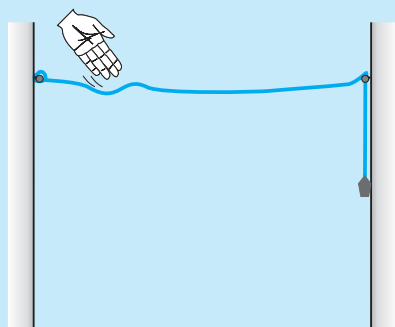
You can easily see the motion of a pulse on a rope. You can also see its reflection from a rigid boundary and measure its velocity of travel. You will need a rope of diameter 1 to 3 cm, two hooks and some weights. You can perform this experiment in your classroom or laboratory.

Take a long rope or thick string of diameter 1 to 3 cm, and tie it to hooks on opposite walls in a hall or laboratory. Let one end pass on a hook and hang some weight (about 1 to 5 kg) to it. The walls may be about 3 to 5 m apart.

Take a stick or a rod and strike the rope hard at a point near one end. This creates a pulse on the rope which now travels on it. You can see it reaching the end and reflecting back from it. You can check the phase relation between the incident pulse and reflected pulse. You can easily watch two or three reflections before the pulse dies out. You can take a stopwatch and find the time for the pulse to travel the distance between the walls, and thus measure its

velocity. Compare it with that obtained from Eq. (15.14).

This is also what happens with a thin metallic string of a musical instrument. The major difference is that the velocity on a string is fairly high because of low mass per unit length, as compared to that on a thick rope. The low velocity on a rope allows us to watch the motion and make measurements beautifully.



volume elements of air. The property that determines the extent to which the volume of an element of a medium changes when the pressure on it changes, is the **bulk modulus**  $B$ , defined (see Chapter 9) as,

$$B = -\frac{\Delta P}{\Delta V/V} \quad (15.16)$$

Here  $\Delta V/V$  is the fractional change in volume produced by a change in pressure  $\Delta P$ . The SI unit for pressure is  $\text{N m}^{-2}$  or pascal (Pa). Now since the longitudinal waves in a medium travel in the form of compressions and rarefactions or changes in density, the inertial property of the medium, which could be involved in the process, is the density  $\rho$ . The dimension of density is  $[\text{ML}^{-3}]$ . Thus, the dimension of the ratio  $B/\rho$  is,

$$\frac{[\text{M L}^{-1} \text{T}^{-2}]}{[\text{M L}^{-3}]} = [\text{L}^2 \text{T}^{-2}] \quad (15.17)$$

Therefore, on the basis of dimensional analysis the most appropriate expression for the speed of longitudinal waves in a medium is

$$v = C \sqrt{\frac{B}{\rho}} \quad (15.18)$$

where  $C$  is a dimensionless constant and can be shown to be unity. Thus the speed of longitudinal waves in a medium is given by,

$$v = \sqrt{\frac{B}{\rho}} \quad (15.19)$$

The speed of propagation of a longitudinal wave in a fluid therefore depends only on the bulk modulus and the density of the medium.

When a solid bar is struck a blow at one end, the situation is somewhat different from that of a fluid confined in a tube or cylinder of constant cross section. For this case, the relevant modulus of elasticity is the Young's modulus, since the sideways expansion of the bar is negligible and only longitudinal strain needs to be considered. It can be shown that the speed of a longitudinal wave in the bar is given by,

$$v = \sqrt{\frac{Y}{\rho}} \quad (15.20)$$

where  $Y$  is the Young's modulus of the material of the bar.

Table 15.1 gives the speed of sound in various media.

**Table 15.1 Speed of Sound in some Media**

Medium	Speed ( $\text{m s}^{-1}$ )
<b>Gases</b>	
Air ( $0^\circ\text{C}$ )	331
Air ( $20^\circ\text{C}$ )	343
Helium	965
Hydrogen	1284
<b>Liquids</b>	
Water ( $0^\circ\text{C}$ )	1402
Water ( $20^\circ\text{C}$ )	1482
Seawater	1522
<b>Solids</b>	
Aluminium	6420
Copper	3560
Steel	5941
Granite	6000
Vulcanised Rubber	54

It may be noted that although the densities of liquids and solids are much higher than those of the gases, the speed of sound in them is higher. It is because liquids and solids are less compressible than gases, i.e. have much greater bulk modulus.

In the case of an ideal gas, the relation between pressure  $P$  and volume  $V$  is given by (see Chapter 11)

$$PV = Nk_B T \quad (15.21)$$

where  $N$  is the number of molecules in volume  $V$ ,  $k_B$  is the Boltzmann constant and  $T$  the temperature of the gas (in Kelvin). Therefore, for an isothermal change it follows from Eq.(15.21) that

$$V\Delta P + P\Delta V = 0$$

$$\text{or } -\frac{\Delta P}{\Delta V/V} = P$$

Hence, substituting in Eq. (15.16), we have

$$B = P$$

Therefore, from Eq. (15.19) the speed of a longitudinal wave in an ideal gas is given by,

$$v = \sqrt{\frac{P}{\rho}} \quad (15.22)$$

This relation was first given by Newton and is known as Newton's formula.

► **Example 15.4** Estimate the speed of sound in air at standard temperature and pressure. The mass of 1 mole of air is  $29.0 \times 10^{-3}$  kg.

**Answer** We know that 1 mole of any gas occupies 22.4 litres at STP. Therefore, density of air at STP is :

$$\rho_o = (\text{mass of one mole of air}) / (\text{volume of one mole of air at STP})$$

$$\begin{aligned} & \frac{29.0 \times 10^{-3} \text{ kg}}{22.4 \times 10^{-3} \text{ m}^3} \\ &= 1.29 \text{ kg m}^{-3} \end{aligned}$$

According to Newton's formula for the speed of sound in a medium, we get for the speed of sound in air at STP,

$$v = \left[ \frac{1.01 \times 10^5 \text{ N m}^{-2}}{1.29 \text{ kg m}^{-3}} \right]^{1/2} = 280 \text{ m s}^{-1} \quad (15.23)$$

The result shown in Eq.(15.23) is about 15% smaller as compared to the experimental value of  $331 \text{ m s}^{-1}$  as given in Table 15.1. Where did we go wrong ? If we examine the basic assumption made by Newton that the pressure variations in a medium during propagation of sound are isothermal, we find that this is not correct. It was pointed out by Laplace that the pressure variations in the propagation of sound waves are so fast that there is little time for the heat flow to maintain constant temperature. These variations, therefore, are adiabatic and not isothermal. For adiabatic processes the ideal gas satisfies the relation,

$$PV^\gamma = \text{constant}$$

$$\text{i.e.} \quad \Delta(PV^\gamma) = 0$$

$$\text{or} \quad P\gamma V^{\gamma-1} \Delta V + V^\gamma \Delta P = 0$$

Thus for an ideal gas the adiabatic bulk modulus is given by,

$$\begin{aligned} B_{ad} &= -\frac{\Delta P}{\Delta V/V} \\ &= \gamma P \end{aligned}$$

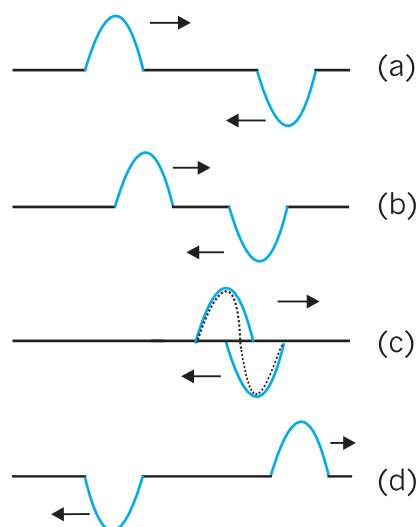
where  $\gamma$  is the ratio of two specific heats,  $C_p/C_v$ . The speed of sound is, therefore, given by,

$$v = \sqrt{\frac{\gamma P}{\rho}} \quad (15.24)$$

This modification of Newton's formula is referred to as the **Laplace correction**. For air  $\gamma = 7/5$ . Now using Eq. (15.24) to estimate the speed of sound in air at STP, we get a value  $331.3 \text{ m s}^{-1}$ , which agrees with the measured speed.

## 15.5 THE PRINCIPLE OF SUPERPOSITION OF WAVES

Let us consider that two waves are travelling simultaneously along the same stretched string in opposite directions. The sequence of pictures shown in Fig. 15.9 depicts the state of displacement of various elements of the string at different time instant. Each picture depicts the resultant waveform in the string at a given instant of time. It is observed that the **net displacement of any element of the string at a given time is the algebraic sum of the displacements due to each wave**. This way of



**Fig. 15.9** A sequence of pictures depicting two pulses travelling in opposite directions along a stretched string. They meet and pass through each other and move on independently as shown by the sequence of time snapshots (a) through (d). The total disturbance is the algebraic sum of the displacements due to each pulse. When the two disturbances overlap they give a complicated pattern as shown in (c). In region (d) they have passed each other and proceed unchanged.

addition of individual waveforms to determine the net waveform is called the **principle of superposition**. To put this rule in a mathematical form, let  $y_1(x, t)$  and  $y_2(x, t)$  be the displacements that any element of the string would experience if each wave travelled alone. The displacement  $y(x, t)$  of an element of the string when the waves overlap is then given by,

$$y(x, t) = y_1(x, t) + y_2(x, t) \quad (15.25)$$

The principle of superposition can also be expressed by stating that **overlapping waves algebraically add to produce a resultant wave (or a net wave)**. The principle implies that the overlapping waves do not, in any way, alter the travel of each other.

If we have two or more waves moving in the medium the resultant waveform is the sum of wave functions of individual waves. That is, if the wave functions of the moving waves are

$$y_1 = f_1(x - vt),$$

$$y_2 = f_2(x - vt),$$

.....

.....

$$y_n = f_n(x - vt)$$

then the wave function describing the disturbance in the medium is

$$\begin{aligned} y &= f_1(x - vt) + f_2(x - vt) + \dots + f_n(x - vt) \\ &= \sum_{i=1}^n f_i(x - vt) \end{aligned} \quad (15.26)$$

As illustrative examples of this principle we shall study the phenomena of interference and reflection of waves.

Let a wave travelling along a stretched string be given by,

$$y_1(x, t) = a \sin(kx - \omega t) \quad (15.27)$$

and another wave, shifted from the first by a phase  $\phi$

$$y_2(x, t) = a \sin(kx - \omega t + \phi) \quad (15.28)$$

Both the waves have the same angular frequency, same angular wave number  $k$  (same wavelength) and the same amplitude  $a$ . They travel in the positive direction of  $x$ -axis, with the same speed. Their phases at a given distance and time differ by a constant angle  $\phi$ . These waves are said to be out of phase by  $\phi$  or have a phase difference  $\phi$ .

Now, applying the superposition principle, the resultant wave is the algebraic sum of the two constituent waves and has displacement

$$y(x, t) = a \sin(kx - \omega t) + a \sin(kx - \omega t + \phi) \quad (15.29)$$

We now use the trigonometric relation

$$\sin \alpha + \sin \beta = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) \quad (15.30)$$

Applying this relation to Eq. (15.29) we have

$$y(x, t) = [2a \cos \frac{1}{2}\phi] \sin(kx - \omega t + \frac{1}{2}\phi) \quad (15.31)$$

Equation (15.31) shows that the resultant wave is also a sinusoidal wave, travelling in the positive direction of  $x$ -axis.

The resultant wave differs from the constituent waves in two respects: (1) its phase angle is  $(\frac{1}{2})\phi$  and (2) its amplitude is the quantity in brackets in Eq. (15.31) viz.,

$$A(\phi) = 2a \cos(\frac{1}{2}\phi) \quad (15.32)$$

If  $\phi = 0$ , i.e. the two waves are in phase, Eq. (15.31) reduces to

$$A(0) = 2a \sin(kx - \omega t) \quad (15.33)$$

The amplitude of the resultant wave is  $2a$ , which is the largest possible value of  $A(\phi)$ .

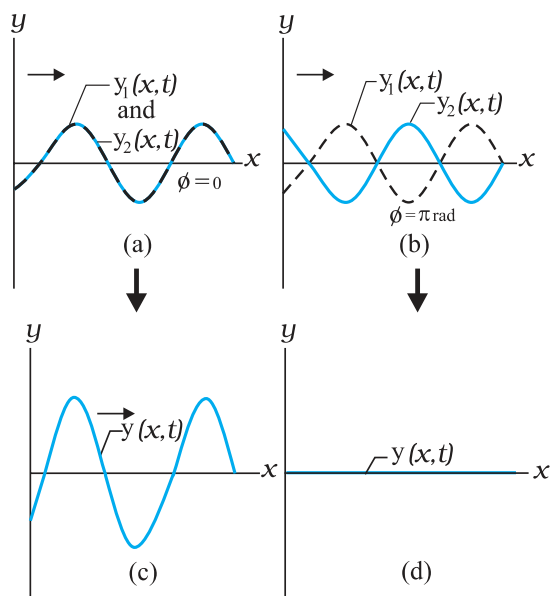
If  $\phi = \pi$ , the two waves are completely out of phase, the amplitude of the resultant wave given by Eq. (15.32) reduces to zero. We then have for all  $x$  and  $t$ ,

$$y(x, t) = 0 \quad (15.34)$$

These cases are shown in Fig. 15.10.

## 15.6 REFLECTION OF WAVES

In previous sections we have discussed wave propagation in unbounded media. What happens when a pulse or a travelling wave encounters a rigid boundary? It is a common experience that under such a situation the pulse or the wave gets reflected. An everyday example of the reflection of sound waves from a rigid boundary is the phenomenon of echo. If the boundary is not completely rigid or is an interface between two different elastic media, the effect of boundary conditions on



**Fig. 15.10** Two identical sinusoidal waves,  $y_1(x, t)$  and  $y_2(x, t)$ , travel along a stretched string in the positive direction of  $x$ -axis. They give rise to a resultant wave  $y(x, t)$ . The phase difference between the two waves is (a) 0 and (b)  $\pi$  or  $180^\circ$ . The corresponding resultant waves are shown in (c) and (d).

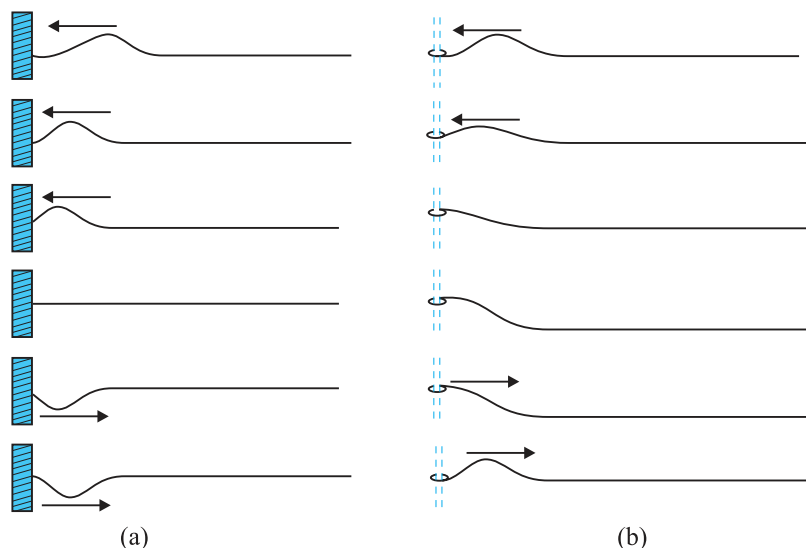
an incident pulse or a wave is somewhat complicated. A part of the wave is reflected and a part is transmitted into the second medium. If a wave is incident obliquely on the boundary between two different media the transmitted wave is called the **refracted wave**. The incident and refracted waves obey Snell's law of refraction, and the incident and reflected waves obey the usual laws of reflection.

To illustrate the reflection of waves at a boundary, we consider two situations. First, a string is fixed to a rigid wall at its left end, as shown in Fig. 15.11(a). Second, the left end of the string is tied to a ring, which slides up and down without any friction on a rod, as shown in Fig. 15.11(b). A pulse is allowed to propagate

in both these strings, the pulse, on reaching the left end, gets reflected; the state of disturbance in the string at various times is shown in Fig. 15.11.

In Fig. 15.11(a), the string is fixed to the wall at its left end. When the pulse arrives at that end, it exerts an upward force on the wall. By Newton's third law, the wall exerts an opposite force of equal magnitude on the string. This second force generates a pulse at the support (the wall), which travels back along the string in the direction opposite to that of the incident pulse. In a reflection of this kind, there must be no displacement at the support as the string is fixed there. The reflected and incident pulses must have opposite signs, so as to cancel each other at that point. Thus, in case of a travelling wave, the reflection at a rigid boundary will take place with a phase reversal or with a phase difference of  $\pi$  or  $180^\circ$ .

In Fig. 15.11(b), the string is fastened to a ring, which slides without friction on a rod. In this case, when the pulse arrives at the left end, the ring moves up the rod. As the ring moves, it pulls on the string, stretching the string and producing a reflected pulse with the same sign and amplitude as the incident pulse. Thus, in such a reflection, the incident and reflected pulses reinforce each other, creating the



**Fig. 15.11** (a) A pulse incident from the right is reflected at the left end of the string, which is tied to a wall. Note that the reflected pulse is inverted from the incident pulse. (b) Here the left end is tied to a ring that can slide up and down without friction on the rod. Now the reflected pulse is not inverted by reflection.



maximum displacement at the end of the string; the maximum displacement of the ring is twice the amplitude of either of the pulses. Thus, the reflection is without any additional phase shift. In case of a travelling wave the reflection at an open boundary, such as the open end of an organ pipe, the reflection takes place without any phase change.

We can thus, summarise the reflection of waves at a boundary or interface between two media as follows:

**A travelling wave, at a rigid boundary or a closed end, is reflected with a phase reversal but the reflection at an open boundary takes place without any phase change.**

To express the above statement mathematically, let the incident wave be represented by

$$y_i(x, t) = a \sin(kx - \omega t),$$

then, for reflection at a rigid boundary the reflected wave is represented by,

$$\begin{aligned} y_r(x, t) &= a \sin(kx + \omega t + \pi). \\ &= -a \sin(kx + \omega t) \end{aligned} \quad (15.35)$$

For reflection at an open boundary, the reflected wave is represented by

$$y_r(x, t) = a \sin(kx + \omega t). \quad (15.36)$$

### 15.6.1 Standing Waves and Normal Modes

In the previous section we have considered a system which is bounded at one end. Let us now consider a system which is bounded at both the ends such as a stretched string fixed at the ends or an air column of finite length. In such a system suppose that we send a continuous sinusoidal wave of a certain frequency, say, toward the right. When the wave reaches the right end, it gets reflected and begins to travel back. The left-going wave then overlaps the wave, travelling to the right. When the left-going wave reaches the left end, it gets reflected again and the newly reflected wave begins to travel to the right, overlapping the left-going wave. This process will continue and, therefore, very soon we have many overlapping waves, which interfere with one another. In such a system, at any point  $x$  and at any time  $t$ , there are always two waves, one moving to the left and another to the right. We, therefore, have

$$y_i(x, t) = a \sin(kx - \omega t) \quad \begin{array}{l} \text{(wave travelling} \\ \text{in the positive} \\ \text{direction of } x\text{-axis)} \end{array}$$

$$\text{and } y_2(x, t) = a \sin(kx + \omega t) \quad \begin{array}{l} \text{(wave travelling} \\ \text{in the negative} \\ \text{direction of} \\ x\text{-axis).} \end{array}$$

The principle of superposition gives, for the combined wave

$$\begin{aligned} y(x, t) &= y_1(x, t) + y_2(x, t) \\ &= a \sin(kx - \omega t) + a \sin(kx + \omega t) \\ &= (2a \sin kx) \cos \omega t \end{aligned} \quad (15.37)$$

The wave represented by Eq. (15.37) does not describe a travelling wave, as the waveform or the disturbance does not move to either side. Here, the quantity  $2a \sin kx$  within the brackets is the amplitude of oscillation of the element of the string located at the position  $x$ . In a travelling wave, in contrast, the amplitude of the wave is the same for all elements. Equation (15.37), therefore, represents a **standing wave**, a wave in which the waveform does not move. The formation of such waves is illustrated in Fig. 15.12.

It is seen that the points of maximum or minimum amplitude stay at one position.

The amplitude is zero for values of  $kx$  that give  $\sin kx = 0$ . Those values are given by

$$kx = n\pi, \text{ for } n = 0, 1, 2, 3, \dots$$

Substituting  $k = 2\pi/\lambda$  in this equation, we get

$$x = n \frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3, \dots \quad (15.38)$$

The positions of zero amplitude are called **nodes**. Note that a distance of  $\frac{\lambda}{2}$  or half a wavelength separates two consecutive nodes.

The amplitude has a maximum value of  $2a$ , which occurs for the values of  $kx$  that give  $|\sin kx| = 1$ . Those values are

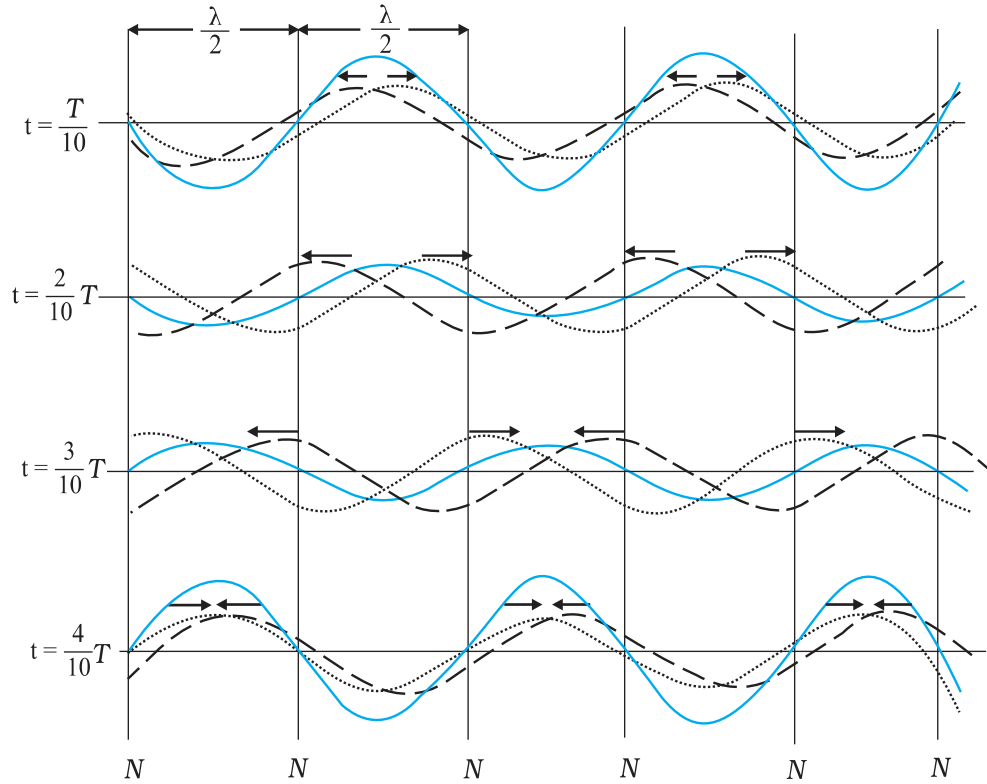
$$kx = (n + \frac{1}{2})\pi \text{ for } n = 0, 1, 2, 3, \dots$$

Substituting  $k = 2\pi/\lambda$  in this equation, we get

$$x = (n + \frac{1}{2}) \frac{\lambda}{2} \text{ for } n = 0, 1, 2, 3, \dots \quad (15.39)$$

as the positions of maximum amplitude. These are called the **antinodes**. The antinodes are separated by  $\lambda/2$  and are located half way between pairs of nodes.

For a stretched string of length  $L$ , fixed at both ends, the two ends of the string have to be nodes.



**Fig. 15.12** The formation of a standing wave in a stretched string. Two sinusoidal waves of same amplitude travel along the string in opposite directions. The set of pictures represent the state of displacements at four different times. The displacement at positions marked as  $N$  is zero at all times. These positions are called nodes.

If one of the ends is chosen as position  $x = 0$ , then the other end is  $x = L$ . In order that this end is a node; the length  $L$  must satisfy the condition

$$L = n \frac{\lambda}{2}, \text{ for } n = 1, 2, 3, \dots \quad (15.40)$$

This condition shows that standing waves on a string of length  $L$  have restricted wavelength given by

$$\lambda = \frac{2L}{n}, \text{ for } n = 1, 2, 3, \dots \text{ etc.} \quad (15.41)$$

The frequencies corresponding to these wavelengths follow from Eq. (15.12) as

$$\nu = n \frac{v}{2L}, \text{ for } n = 1, 2, 3, \dots \text{ etc.} \quad (15.42)$$

where  $v$  is the speed of travelling waves on the string. The set of frequencies given by Eq. (15.42)

are called the natural frequencies or **modes** of oscillation of the system. This equation tells us that the natural frequencies of a string are integral multiples of the lowest frequency

$$\nu = \frac{v}{2L}, \text{ which corresponds to } n = 1. \text{ The}$$

oscillation mode with that lowest frequency is called the **fundamental mode** or the **first harmonic**. The **second harmonic** is the oscillation mode with  $n = 2$ . The third harmonic corresponds to  $n = 3$  and so on. The frequencies associated with these modes are often labelled as  $\nu_1, \nu_2, \nu_3$  and so on. The collection of all possible modes is called the **harmonic series** and  $n$  is called the harmonic number.

Some of the harmonics of a stretched string fixed at both the ends are shown in Fig. 15.13. According to the principle of superposition, a stretched string tied at both ends can vibrate simultaneously in more than one modes. Which mode is strongly excited depends on where the

string is plucked or bowed. Musical instruments like sitar and violin are designed on this principle.

We now study the modes of vibration of a system closed at one end, with the other end being free. Air columns such as glass tubes partially filled with water provide examples of such systems. In these, the length of the air column can be adjusted by changing the water level in the tube. In such systems, the end of the air column in touch with the water suffers no displacement as the reflected and incident waves are exactly out of phase. For this reason the pressure changes here are the largest, since when the compressional part is reflected the pressure increase is doubled, and when the rarefaction is reflected the decrease in pressure is doubled. On the other hand, at the open end, there is maximum displacement and minimum pressure change. The two waves travelling in opposite directions are in phase here, so there are no pressure changes. Now if the length of the air column is  $L$ , then the open end,  $x = L$ , is an antinode and therefore, it follows from Eq. (15.39) that

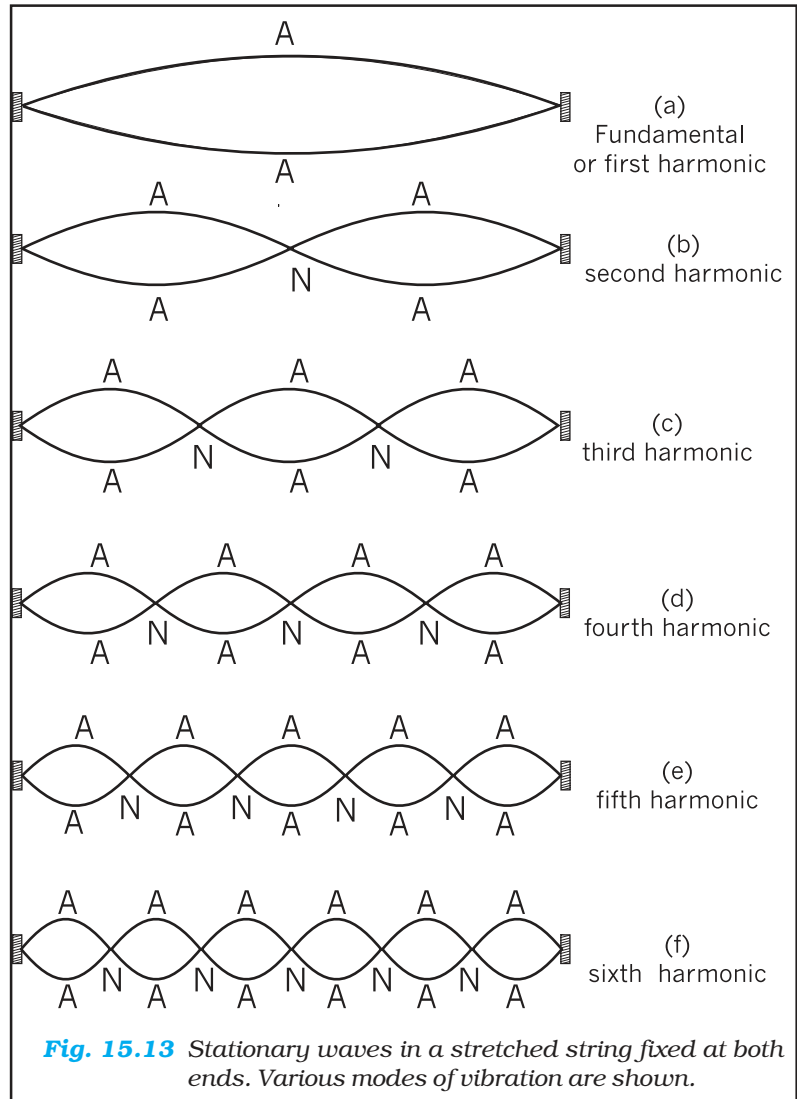
$$L = \left(n + \frac{1}{2}\right) \frac{\lambda}{2}, \text{ for } n = 0, 1, 2, 3, \dots$$

The modes, which satisfy the condition

$$\lambda = \frac{2L}{\left(n + \frac{1}{2}\right)}, \text{ for } n = 0, 1, 2, 3, \dots \quad (15.43)$$

are sustained in such an air column. The corresponding frequencies of various modes of such an air column are given by,

$$v = \left(n + \frac{1}{2}\right) \frac{v}{2L}, \text{ for } n = 0, 1, 2, 3, \dots \quad (15.44)$$



**Fig. 15.13** Stationary waves in a stretched string fixed at both ends. Various modes of vibration are shown.

Some of the normal modes in an air column with the open end are shown in Fig. 15.14. The

fundamental frequency is  $\frac{v}{4L}$  and the higher frequencies are **odd harmonics** of the fundamental frequency, i.e.  $3\frac{v}{4L}, 5\frac{v}{4L}$ , etc.

In the case of a pipe open at both ends, there will be antinodes at both ends, and **all harmonics** will be generated.

Normal modes of a circular membrane rigidly clamped to the circumference as in a tabla are determined by the boundary condition that no

point on the circumference of the membrane vibrates. Estimation of the frequencies of normal modes of this system is more complex. This problem involves wave propagation in two dimensions. However, the underlying physics is the same.

We have seen above that in a string, fixed at both ends, standing waves are produced only at certain frequencies as given by Eq. (15.42) or the system **resonates** at these frequencies. Similarly an air column open at one end resonates at frequencies given by Eq. (15.44).

**Example 15.5** A pipe, 30.0 cm long, is open at both ends. Which harmonic mode of the pipe resonates a 1.1 kHz source? Will resonance with the same source be observed if one end of the pipe is closed? Take the speed of sound in air as  $330 \text{ m s}^{-1}$ .

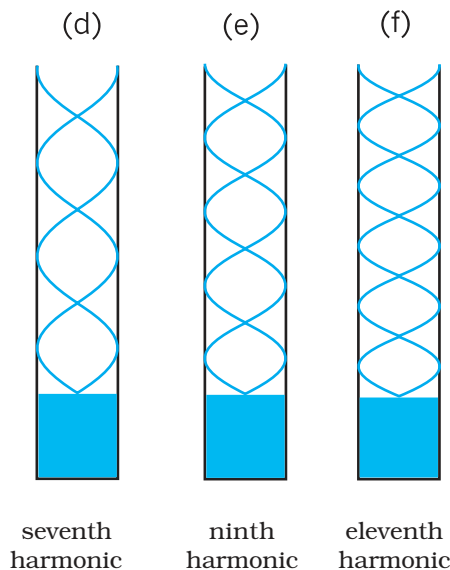
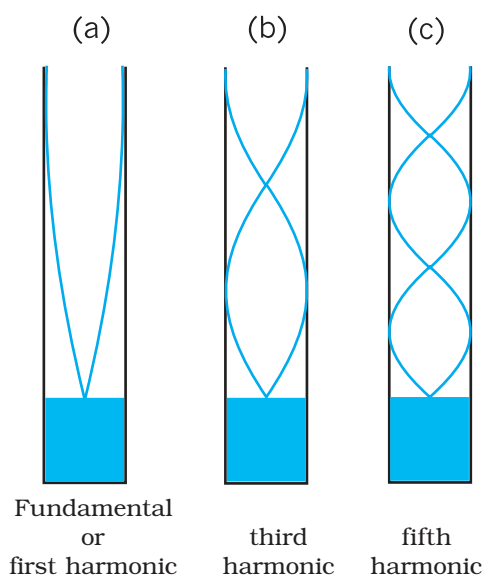
**Answer** The first harmonic frequency is given by

$$v_1 = \frac{v}{\lambda_1} = \frac{v}{2L} \quad (\text{open pipe})$$

where  $L$  is the length of the pipe. The frequency of its  $n$ th harmonic is:

$$v_n = \frac{nv}{2L}, \text{ for } n = 1, 2, 3, \dots (\text{open pipe})$$

First few modes of an open pipe are shown in Fig. 15.14.



**Fig. 15.14** Some of the normal modes of vibration of an air column open at one end.

For  $L = 30.0 \text{ cm}$ ,  $v = 330 \text{ m s}^{-1}$ ,

$$v_n = \frac{n \times 330 (\text{m s}^{-1})}{0.6 (\text{m})} = 550 n \text{ s}^{-1}$$

Clearly, a source of frequency 1.1 kHz will resonate at  $v_2$ , i.e. the **second harmonic**.

Now if one end of the pipe is closed (Fig. 15.15), it follows from Eq. (14.50) that the fundamental frequency is

$$v_1 = \frac{v}{\lambda_1} = \frac{v}{4L} \quad (\text{pipe closed at one end})$$

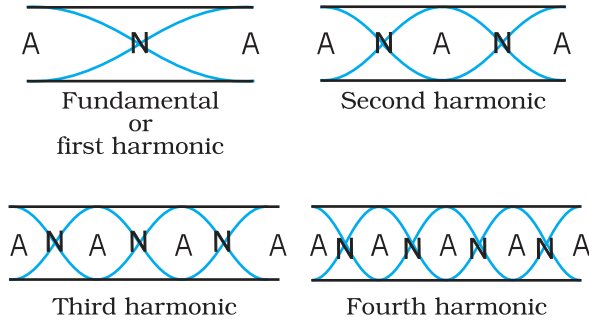
and only the odd numbered harmonics are present :

$$v_3 = \frac{3v}{4L}, \quad v_5 = \frac{5v}{4L}, \text{ and so on.}$$

For  $L = 30 \text{ cm}$  and  $v = 330 \text{ m s}^{-1}$ , the fundamental frequency of the pipe closed at one end is 275 Hz and the source frequency corresponds to its fourth harmonic. Since this harmonic is not a possible mode, no resonance will be observed with the source, the moment one end is closed. ◀

## 15.7 BEATS

If we listen, a few minutes apart, two sounds of very close frequencies, say 256 Hz and 260 Hz, we will not be able to discriminate between



**Fig. 15.15** Standing waves in an open pipe, first four harmonics are depicted.

them. However, if both these sounds reach our ears simultaneously, what we hear is a sound of frequency 258 Hz, the **average** of the two combining frequencies. In addition we hear a striking variation in the intensity of sound — it increases and decreases in slow, wavering **beats** that repeat at a frequency of 4 Hz, the **difference** between the frequencies of two incoming sounds. The phenomenon of wavering of sound intensity when two waves of nearly same frequencies and amplitudes travelling in the same direction, are superimposed on each other is called **beats**.

Let us find out what happens when two waves having slightly different frequencies are superposed on each other. Let the time dependent variations of the displacements due to two sound waves at a particular location be

$$s_1 = a \cos \omega_1 t \text{ and } s_2 = a \cos \omega_2 t \quad (15.45)$$

where  $\omega_1 > \omega_2$ . We have assumed, for simplicity, that the waves have same amplitude and phase. According to the superposition principle, the resultant displacement is

$$\begin{aligned} s &= s_1 + s_2 = a (\cos \omega_1 t + \cos \omega_2 t) \\ &= 2a \cos \frac{(\omega_1 - \omega_2)t}{2} \cos \frac{(\omega_1 + \omega_2)t}{2} \end{aligned} \quad (15.46)$$

If we write  $\omega_b = \frac{(\omega_1 - \omega_2)}{2}$  and  $\omega_a = \frac{(\omega_1 + \omega_2)}{2}$

then Eq. (15.46) can be written as

$$s = [2a \cos \omega_b t] \cos \omega_a t \quad (15.47)$$

If  $|\omega_1 - \omega_2| \ll \omega_1, \omega_2, \omega_a \gg \omega_b$ , then in Eq. (15.47) the main time dependence arises from cosine function whose angular frequency

### Musical Pillars

Temples often have some pillars portraying human figures playing musical instruments, but seldom do these pillars themselves produce music. At the Nellaiappar temple in Tamil Nadu, gentle taps on a



cluster of pillars carved out of a single piece of rock produce the basic notes of Indian classical music, viz. Sa, Re, Ga, Ma, Pa, Dha, Ni, Sa. Vibrations of these pillars depend on elasticity of the stone used, its density and shape.

Musical pillars are categorised into three types: The first is called the **Shruti Pillar**, as it can produce the basic notes — the “swaras”. The second type is the **Gana Thoongal**, which generates the basic tunes that make up the “ragas”. The third variety is the **Laya Thoongal** pillars that produce “taal” (beats) when tapped. The pillars at the Nellaiappar temple are a combination of the Shruti and Laya types.

Archaeologists date the Nelliappar temple to the 7th century and claim it was built by successive rulers of the Pandyan dynasty.

The musical pillars of Nelliappar and several other temples in southern India like those at Hampi (picture), Kanyakumari, and Thiruvananthapuram are unique to the country and have no parallel in any other part of the world.

is  $\omega_a$ . The quantity in the brackets can be regarded as the amplitude of this function (which is not a constant but, has a small variation of angular frequency  $\omega_b$ ). It becomes maximum whenever  $\cos \omega_b t$  has the value +1 or -1, which happens twice in each repetition of cosine function. Since  $\omega_1$  and  $\omega_2$  are very close,  $\omega_a$  cannot be differentiated easily from either of them. Thus, the result of superposition of two waves having nearly the same



### Reflection of sound in an open pipe



When a high pressure pulse of air traveling down an open pipe reaches the other end, its momentum drags the air out into the open, where pressure falls rapidly to the atmospheric pressure. As a

result the air following after it in the tube is pushed out. The low pressure at the end of the tube draws air from further up the tube. The air gets drawn towards the open end forcing the low pressure region to move upwards. As a result a pulse of high pressure air travelling down the tube turns into a pulse of low pressure air travelling up the tube. We say a pressure wave has been reflected at the open end with a change in phase of  $180^\circ$ . Standing waves in an open pipe organ like the flute is a result of this phenomenon.

Compare this with what happens when a pulse of high pressure air arrives at a closed end: it collides and as a result pushes the air back in the opposite direction. Here, we say that the pressure wave is reflected, with no change in phase.

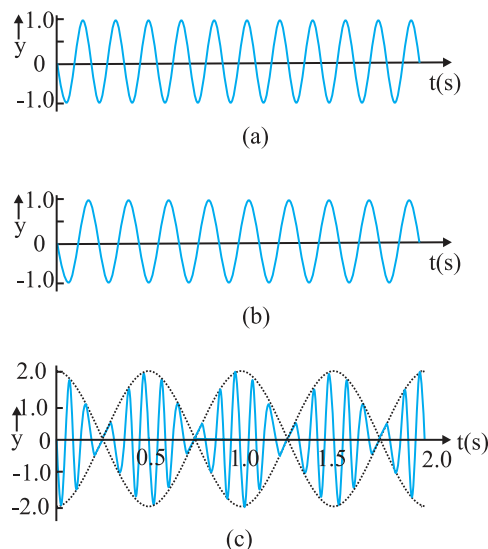
frequencies is a wave with nearly same angular frequency but its amplitude is not constant. Thus the intensity of resultant sound varies with an angular frequency  $\omega_{beat} = 2\omega_b = \omega_1 - \omega_2$ . Now using the relation,

$$\omega = 2\pi\nu$$

the beat frequency,  $\nu_{beat}$ , is given by

$$\nu_{beat} = \nu_1 - \nu_2 \quad (15.48)$$

Thus we hear a waxing and waning of sound with a frequency equal to the difference between the frequencies of the superposing waves. The time-displacement graphs of two waves of frequency 11 Hz and 9 Hz is shown in Figs. 15.16(a) and 15.16(b). The result of their 'superposition' is shown in Fig. 15.16(c).



**Fig. 15.16** (a) Plot of a harmonic wave of frequency 11 Hz. (b) Plot of a harmonic wave of frequency 9 Hz. (c) Superposition of (a) and (b), showing clearly the beats in the slow (2 Hz) of the total disturbance.

Musicians use the beat phenomenon in tuning their instruments. If an instrument is sounded against a standard frequency and tuned until the beat disappears, then the instrument is in tune with that standard.

► **Example 15.6** Two sitar strings A and B playing the note 'Dha' are slightly out of tune and produce beats of frequency 5 Hz. The tension of the string B is slightly increased and the beat frequency is found to decrease to 3 Hz. What is the original frequency of B if the frequency of A is 427 Hz?

**Answer** Increase in the tension of a string increases its frequency. If the original frequency of B ( $\nu_B$ ) were greater than that of A ( $\nu_A$ ), further increase in  $\nu_B$  should have resulted in an increase in the beat frequency. But the beat frequency is found to decrease. This shows that  $\nu_B < \nu_A$ . Since  $\nu_A - \nu_B = 5$  Hz, and  $\nu_A = 427$  Hz, we get  $\nu_B = 422$  Hz. ◀

### 15.8 DOPPLER EFFECT

It is an everyday experience that the pitch (or frequency) of the whistle of a fast moving train

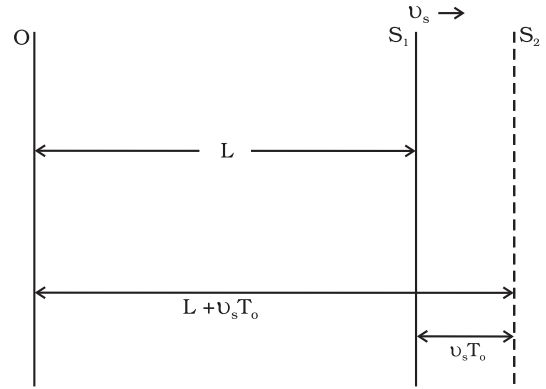
decreases as it recedes away. When we approach a stationary source of sound with high speed, the pitch of the sound heard appears to be higher than that of the source. As the observer recedes away from the source, the observed **pitch** (or frequency) becomes lower than that of the source. This motion-related frequency change is called **Doppler effect**. The Austrian physicist Johann Christian Doppler first proposed the effect in 1842. Buys Ballot in Holland tested it experimentally in 1845. Doppler effect is a wave phenomenon, it holds not only for sound waves but also for electromagnetic waves. However, here we shall consider only sound waves.

We shall analyse changes in frequency under three different situations: (1) observer is stationary but the source is moving, (2) observer is moving but the source is stationary, and (3) both the observer and the source are moving. The situations (1) and (2) differ from each other because of the absence or presence of relative motion between the observer and the medium. Most waves require a medium for their propagation; however, electromagnetic waves do not require any medium for propagation. If there is no medium present, the Doppler shifts are same irrespective of whether the source moves or the observer moves, since there is no way of distinction between the two situations.

### 15.8.1 Source Moving ; Observer Stationary

**Let us choose the convention to take the direction from the observer to the source as the positive direction of velocity.** Consider a source S moving with velocity  $v_s$  and an observer who is stationary in a frame in which the medium is also at rest. Let the speed of a wave of angular frequency  $\omega$  and period  $T_0$ , both measured by an observer at rest with respect to the medium, be  $v$ . We assume that the observer has a detector that counts every time a wave crest reaches it. As shown in Fig. 15.17, at time  $t = 0$  the source is at point  $S_1$ , located at a distance  $L$  from the observer, and emits a crest. This reaches the observer at time  $t_1 = L/v$ . At time  $t = T_0$  the source has moved a distance  $v_s T_0$  and is at point  $S_2$ , located at a distance  $(L + v_s T_0)$  from the observer. At  $S_2$ , the source emits a second crest. This reaches the observer at

$$t_2 = T_0 + \frac{(L + v_s T_0)}{v}$$



**Fig. 15.17** A source moving with velocity  $v_s$  emits a wave crest at the point  $S_1$ . It emits the next wave crest at  $S_2$  after moving a distance  $v_s T_0$ .

At time  $n T_0$ , the source emits its  $(n+1)^{\text{th}}$  crest and this reaches the observer at time

$$t_{n+1} = n T_0 + \frac{L + v_s T_0}{v}$$

Hence, in a time interval

$$n T_0 \quad \frac{L + v_s T_0}{v} \quad \frac{L}{v}$$

the observer's detector counts  $n$  crests and the observer records the period of the wave as  $T$  given by

$$\begin{aligned} T &= n T_0 + \frac{L + v_s T_0}{v} - \frac{L}{v} / n \\ &= T_0 + \frac{v_s T_0}{v} \\ &= T_0 \left( 1 + \frac{v_s}{v} \right) \end{aligned} \quad (15.49)$$

Equation (15.49) may be rewritten in terms of the frequency  $\nu_0$  that would be measured if the source and observer were stationary, and the frequency  $\nu$  observed when the source is moving, as

$$\nu = \nu_0 \left( 1 + \frac{v_s}{v} \right)^{-1} \quad (15.50)$$

If  $v_s$  is small compared with the wave speed  $v$ , taking binomial expansion to terms in first order in  $v_s/v$  and neglecting higher power, Eq. (15.50) may be approximated, giving

$$\nu = \nu_0 \left( 1 - \frac{v_s}{v} \right) \quad (15.51)$$

For a source approaching the observer, we replace  $v_s$  by  $-v_s$  to get

$$v = v_0 \left( 1 + \frac{v_s}{v} \right) \quad (15.52)$$

The observer thus measures a lower frequency when the source recedes from him than he does when it is at rest. He measures a higher frequency when the source approaches him.

### 15.8.2 Observer Moving; Source Stationary

Now to derive the Doppler shift when the observer is moving with velocity  $v_o$  towards the source and the source is at rest, we have to proceed in a different manner. We work in the reference frame of the moving observer. In this reference frame the source and medium are approaching at speed  $v_o$  and the speed with which the wave approaches is  $v_o + v$ . Following a similar procedure as in the previous case, we find that the time interval between the arrival of the first and the  $(n+1)$ th crests is

$$t_{n+1} - t_1 = n T_0 - \frac{nv_o T_0}{v_o + v}$$

The observer thus, measures the period of the wave to be

$$= T_0 \left( 1 - \frac{v_o}{v_o + v} \right)$$

$$T_0 \left( 1 - \frac{v_o}{v_o + v} \right)^{-1}$$

giving

$$v = v_0 \left( 1 + \frac{v_o}{v} \right) \quad (15.53)$$

If  $\frac{v_o}{v}$  is small, the Doppler shift is almost same whether it is the observer or the source moving since Eq. (15.53) and the approximate relation Eq. (15.51) are the same.

### 15.8.3 Both Source and Observer Moving

We will now derive a general expression for Doppler shift when both the source and the observer are moving. As before, let us take the direction from the observer to the source as the positive direction. Let the source and the observer be moving with velocities  $v_s$  and  $v_o$  respectively as shown in Fig. 15.18. Suppose at time  $t = 0$ , the observer is at  $O_1$  and the source is at  $S_1$ ,  $O_1$  being to the left of  $S_1$ . The source emits a wave of velocity  $v$ , of frequency  $\nu$  and

#### Application of Doppler effect

The change in frequency caused by a moving object due to Doppler effect is used to measure their velocities in diverse areas such as military, medical science, astrophysics, etc. It is also used by police to check over-speeding of vehicles.

A sound wave or electromagnetic wave of known frequency is sent towards a moving object. Some part of the wave is reflected from the object and its frequency is detected by the monitoring station. This change in frequency is called **Doppler shift**.

It is used at airports to guide aircraft, and in the military to detect enemy aircraft. Astrophysicists use it to measure the velocities of stars.

Doctors use it to study heart beats and blood flow in different part of the body. Here they use ultrasonic waves, and in common practice, it is called **sonography**. Ultrasonic waves enter the body of the person, some of them are reflected back, and give information about motion of blood and pulsation of heart valves, as well as pulsation of the heart of the foetus. In the case of heart, the picture generated is called **echocardiogram**.

period  $T_0$  all measured by an observer at rest with respect to the medium. Let  $L$  be the distance between  $O_1$  and  $S_1$  at  $t = 0$ , when the source emits the first crest. Now, since the observer is moving, the velocity of the wave relative to the observer is  $v + v_o$ . Therefore the first crest reaches the observer at time  $t_1 = L/(v + v_o)$ . At time  $t = T_0$ , both the observer and the source have moved to their new positions  $O_2$  and  $S_2$  respectively. The new distance between the observer and the source,  $O_2 S_2$ , would be  $L + (v_s - v_o) T_0$ . At  $S_2$ , the source emits a second crest. This reaches the observer at time.

$$t_2 = T_0 + [L + (v_s - v_o)T_0] / (v + v_o)$$

At time  $nT_0$  the source emits its  $(n+1)$ th crest and this reaches the observer at time

$$t_{n+1} = nT_0 + [L + n(v_s - v_o)T_0] / (v + v_o)$$

Hence, in a time interval  $t_{n+1} - t_1$ , i.e.,

$$nT_0 + [L + n(v_s - v_o)T_0] / (v + v_o) - L / (v + v_o),$$

the observer counts  $n$  crests and the observer records the period of the wave as equal to  $T$  given by

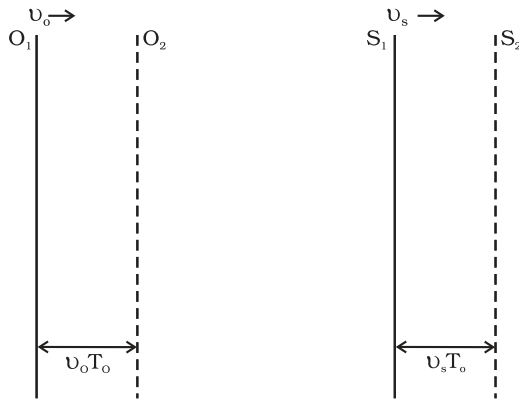
$$T = T_0 \left( 1 + \frac{v_s - v_o}{v + v_o} \right) = T_0 \left( \frac{v + v_s}{v + v_o} \right) \quad (15.54)$$

The frequency  $\nu$  observed by the observer is given by

$$\nu = \nu_0 \left( \frac{v + v_o}{v + v_s} \right) \quad (15.55)$$

Consider a passenger sitting in a train moving on a straight track. Suppose she hears a whistle sounded by the driver of the train. What frequency will she measure or hear? Here both the observer and the source are moving with the same velocity, so there will be no shift in frequency and the passenger will note the natural frequency. But an observer outside who is stationary with respect to the track will note a higher frequency if the train is approaching him and a lower frequency when it recedes from him.

Note that we have defined the direction from the observer to the source as the positive



**Fig. 15.18** The observer O and the source S, both moving respectively with velocities  $v_o$  and  $v_s$ . They are at position  $O_1$  and  $S_1$  at time  $t = 0$ , when the source emits the first crest of a sound, whose velocity is  $v$  with respect to the medium. After one period,  $t = T_0$ , they have moved to  $O_2$  and  $S_2$ , respectively through distances  $v_o T_0$  and  $v_s T_0$ , when the source emits the next crest.

direction. Therefore, if the observer is moving towards the source,  $v_o$  has a positive (numerical)

value whereas if O is moving away from S,  $v_o$  has a negative value. On the other hand, if S is moving away from O,  $v_s$  has a positive value whereas if it is moving towards O,  $v_s$  has a negative value. The sound emitted by the source travels in all directions. It is that part of sound coming towards the observer which the observer receives and detects. Therefore the relative velocity of sound with respect to the observer is  $v + v_o$  in all cases.

► **Example 15.7** A rocket is moving at a speed of  $200 \text{ m s}^{-1}$  towards a stationary target. While moving, it emits a wave of frequency  $1000 \text{ Hz}$ . Some of the sound reaching the target gets reflected back to the rocket as an echo. Calculate (1) the frequency of the sound as detected by the target and (2) the frequency of the echo as detected by the rocket.

**Answer** (1) The observer is at rest and the source is moving with a speed of  $200 \text{ m s}^{-1}$ . Since this is comparable with the velocity of sound,  $330 \text{ m s}^{-1}$ , we must use Eq. (15.50) and not the approximate Eq. (15.51). Since the source is approaching a stationary target,  $v_o = 0$ , and  $v_s$  must be replaced by  $-v_s$ . Thus, we have

$$\nu = \nu_0 \left( 1 - \frac{v_s}{v} \right)^{-1}$$

$$\nu = 1000 \text{ Hz} \times [1 - 200 \text{ m s}^{-1} / 330 \text{ m s}^{-1}]^{-1}$$

$$\approx 2540 \text{ Hz}$$

(2) The target is now the source (because it is the source of echo) and the rocket's detector is now the detector or observer (because it detects echo). Thus,  $v_s = 0$  and  $v_o$  has a positive value. The frequency of the sound emitted by the source (the target) is  $\nu$ , the frequency intercepted by the target and not  $\nu_0$ . Therefore, the frequency as registered by the rocket is

$$\begin{aligned} \nu' &= \nu \left( \frac{v + v_o}{v} \right) \\ &= 2540 \text{ Hz} \times \frac{200 \text{ m s}^{-1} + 330 \text{ m s}^{-1}}{330 \text{ m s}^{-1}} \\ &\approx 4080 \text{ Hz} \end{aligned}$$

## SUMMARY

1. *Mechanical waves* can exist in material media and are governed by Newton's Laws.
2. *Transverse waves* are waves in which the particles of the medium oscillate perpendicular to the direction of wave propagation.
3. *Longitudinal waves* are waves in which the particles of the medium oscillate along the direction of wave propagation.
4. *Progressive wave* is a wave that moves from one point of medium to another.
5. *The displacement* in a sinusoidal wave propagating in the positive  $x$  direction is given by

$$y(x, t) = a \sin(kx - \omega t + \phi)$$

where  $a$  is the amplitude of the wave,  $k$  is the angular wave number,  $\omega$  is the angular frequency,  $(kx - \omega t + \phi)$  is the phase, and  $\phi$  is the phase constant or phase angle.

6. *Wavelength  $\lambda$*  of a progressive wave is the distance between two consecutive points of the same phase at a given time. In a stationary wave, it is twice the distance between two consecutive nodes or anti nodes.
7. *Period  $T$*  of oscillation of a wave is defined as the time any element of the medium takes to move through one complete oscillation. It is related to the *angular frequency  $\omega$*  through the relation

$$T = \frac{2\pi}{\omega}$$

8. *Frequency  $\nu$*  of a wave is defined as  $1/T$  and is related to angular frequency by

$$f = \frac{\omega}{2\pi}$$

9. *Speed* of a progressive wave is given by  $v = \frac{\omega}{k} = \frac{T}{\lambda}$
10. *The speed of a transverse wave* on a stretched string is set by the properties of the string. The speed on a string with tension  $T$  and linear mass density  $\mu$  is

$$v = \sqrt{\frac{T}{\mu}}$$

11. *Sound waves* are longitudinal mechanical waves that can travel through solids, liquids, or gases. The speed  $v$  of sound wave in a fluid having *bulk modulus  $B$*  and density  $\rho$  is

$$v = \sqrt{\frac{B}{\rho}}$$

The speed of longitudinal waves in a metallic bar is

$$v = \sqrt{\frac{Y}{\rho}}$$

For gases, since  $B = \gamma P$ , the speed of sound is

$$v = \sqrt{\frac{\gamma P}{\rho}}$$



12. When two or more waves traverse the same medium, the displacement of any element of the medium is the algebraic sum of the displacements due to each wave. This is known as the *principle of superposition* of waves

$$y = \sum_{i=1}^n f_i(x - vt)$$

13. Two sinusoidal waves on the same string exhibit *interference*, adding or cancelling according to the principle of superposition. If the two are travelling in the same direction and have the same amplitude  $a$  and frequency but differ in phase by a *phase constant*  $\phi$  the result is a single wave with the same frequency  $\omega$ :

$$y(x, t) = 2a \cos \frac{\phi}{2} \sin \left( kx - \omega t + \frac{\phi}{2} \right)$$

If  $\phi = 0$  or an integral multiple of  $2\pi$ , the waves are exactly in phase and the interference is constructive; if  $\phi = \pi$ , they are exactly out of phase and the interference is destructive.

14. A travelling wave, at a rigid boundary or a closed end, is reflected with a phase reversal but the reflection at an open boundary takes place without any phase change.

For an incident wave

$$y_i(x, t) = a \sin(kx - \omega t)$$

the reflected wave at a rigid boundary is

$$y_r(x, t) = -a \sin(kx + \omega t)$$

For reflection at an open boundary

$$y_r(x, t) = a \sin(kx + \omega t)$$

15. The interference of two identical waves moving in opposite directions produces *standing waves*. For a string with fixed ends, the standing wave is given by

$$y(x, t) = [2a \sin kx] \cos \omega t$$

Standing waves are characterised by fixed locations of zero displacement called *nodes* and fixed locations of maximum displacements called *antinodes*. The separation between two consecutive nodes or antinodes is  $\lambda/2$ .

A stretched string of length  $L$  fixed at both the ends vibrates with frequencies given by

$$\nu = \frac{n}{2} \frac{v}{L}, \quad n = 1, 2, 3, \dots$$

The set of frequencies given by the above relation are called the *normal modes* of oscillation of the system. The oscillation mode with lowest frequency is called the *fundamental mode* or the *first harmonic*. The *second harmonic* is the oscillation mode with  $n = 2$  and so on.

A pipe of length  $L$  with one end closed and other end open (such as air columns) vibrates with frequencies given by

$$\nu = n \frac{v}{4L}, \quad n = 1, 2, 3, \dots$$

The set of frequencies represented by the above relation are the *normal modes* of oscillation of such a system. The lowest frequency given by  $v/4L$  is the fundamental mode or the first harmonic.

16. A string of length  $L$  fixed at both ends or an air column closed at one end and open at the other end, vibrates with frequencies called its normal modes. Each of these frequencies is a *resonant frequency* of the system.
17. *Beats* arise when two waves having slightly different frequencies,  $\nu_1$  and  $\nu_2$  and comparable amplitudes, are superposed. The beat frequency is

$$\nu_{\text{beat}} = \nu_1 - \nu_2$$

18. The *Doppler effect* is a change in the observed frequency of a wave when the source and the observer O moves relative to the medium. For sound the observed frequency  $\nu$  is given in terms of the source frequency  $\nu_o$  by

$$\nu = \nu_o \left( \frac{v + v_o}{v + v_s} \right)$$

here  $v$  is the speed of sound through the medium,  $v_o$  is the velocity of observer relative to the medium, and  $v_s$  is the source velocity relative to the medium. In using this formula, velocities in the direction OS should be treated as positive and those opposite to it should be taken to be negative.

Physical quantity	Symbol	Dimensions	Unit	Remarks
Wavelength	$\lambda$	[L]	m	Distance between two consecutive points with the same phase.
Propagation constant	$k$	[L <sup>-1</sup> ]	m <sup>-1</sup>	$k = \frac{2\pi}{\lambda}$
Wave speed	$v$	[LT <sup>-1</sup> ]	m s <sup>-1</sup>	$\nu = v\lambda$
Beat frequency	$\nu_{beat}$	[T <sup>-1</sup> ]	s <sup>-1</sup>	Difference of two close frequencies of superposing waves.

#### POINTS TO PONDER

1. A wave is not motion of matter as a whole in a medium. A wind is different from the sound wave in air. The former involves motion of air from one place to the other. The latter involves compressions and rarefactions of layers of air.
2. In a wave, energy and *not the matter* is transferred from one point to the other.
3. Energy transfer takes place because of the coupling through elastic forces between neighbouring oscillating parts of the medium.
4. Transverse waves can propagate only in medium with shear modulus of elasticity, Longitudinal waves need bulk modulus of elasticity and are therefore, possible in all media, solids, liquids and gases.
5. In a harmonic progressive wave of a given frequency all particles have the same amplitude but different phases at a given instant of time. In a stationary wave, all particles between two nodes have the same phase at a given instant but have different amplitudes.
6. Relative to an observer at rest in a medium the speed of a mechanical wave in that medium ( $v$ ) depends only on elastic and other properties (such as mass density) of the medium. It does not depend on the velocity of the source.
7. For an observer moving with velocity  $v_o$  relative to the medium, the speed of a wave is obviously different from  $v$  and is given by  $v \pm v_o$ .

## EXERCISES

- 15.1** A string of mass 2.50 kg is under a tension of 200 N. The length of the stretched string is 20.0 m. If the transverse jerk is struck at one end of the string, how long does the disturbance take to reach the other end?
- 15.2** A stone dropped from the top of a tower of height 300 m high splashes into the water of a pond near the base of the tower. When is the splash heard at the top given that the speed of sound in air is  $340 \text{ m s}^{-1}$ ? ( $g = 9.8 \text{ m s}^{-2}$ )
- 15.3** A steel wire has a length of 12.0 m and a mass of 2.10 kg. What should be the tension in the wire so that speed of a transverse wave on the wire equals the speed of sound in dry air at  $20^\circ\text{C} = 343 \text{ m s}^{-1}$ .
- 15.4** Use the formula  $v = \sqrt{\frac{\gamma P}{\rho}}$  to explain why the speed of sound in air
- is independent of pressure,
  - increases with temperature,
  - increases with humidity.
- 15.5** You have learnt that a travelling wave in one dimension is represented by a function  $y = f(x, t)$  where  $x$  and  $t$  must appear in the combination  $x - vt$  or  $x + vt$ , i.e.  $y = f(x \pm vt)$ . Is the converse true? Examine if the following functions for  $y$  can possibly represent a travelling wave :
- $(x - vt)^2$
  - $\log [(x + vt)/x_0]$
  - $1/(x + vt)$
- 15.6** A bat emits ultrasonic sound of frequency 1000 kHz in air. If the sound meets a water surface, what is the wavelength of (a) the reflected sound, (b) the transmitted sound? Speed of sound in air is  $340 \text{ m s}^{-1}$  and in water  $1486 \text{ m s}^{-1}$ .
- 15.7** A hospital uses an ultrasonic scanner to locate tumours in a tissue. What is the wavelength of sound in the tissue in which the speed of sound is  $1.7 \text{ km s}^{-1}$ ? The operating frequency of the scanner is 4.2 MHz.
- 15.8** A transverse harmonic wave on a string is described by
- $$y(x, t) = 3.0 \sin (36 \pi t + 0.018 \pi x + \pi/4)$$
- where  $x$  and  $y$  are in cm and  $t$  in s. The positive direction of  $x$  is from left to right.
- Is this a travelling wave or a stationary wave ?  
If it is travelling, what are the speed and direction of its propagation ?
  - What are its amplitude and frequency ?
  - What is the initial phase at the origin ?
  - What is the least distance between two successive crests in the wave ?
- 15.9** For the wave described in Exercise 15.8, plot the displacement ( $y$ ) versus ( $t$ ) graphs for  $x = 0, 2$  and  $4$  cm. What are the shapes of these graphs? In which aspects does the oscillatory motion in travelling wave differ from one point to another: amplitude, frequency or phase ?
- 15.10** For the travelling harmonic wave
- $$y(x, t) = 2.0 \cos 2\pi (10t - 0.0080 x + 0.35)$$
- where  $x$  and  $y$  are in cm and  $t$  in s. Calculate the phase difference between oscillatory motion of two points separated by a distance of
- 4 m,
  - 0.5 m,
  - $\lambda/2$ ,
  - $3\lambda/4$

- 15.11** The transverse displacement of a string (clamped at its both ends) is given by

$$y(x, t) = 0.06 \sin \frac{2}{3}x \cos (120 \pi t)$$

where  $x$  and  $y$  are in m and  $t$  in s. The length of the string is 1.5 m and its mass is  $3.0 \times 10^{-2}$  kg.

Answer the following :

- (a) Does the function represent a travelling wave or a stationary wave?
  - (b) Interpret the wave as a superposition of two waves travelling in opposite directions. What is the wavelength, frequency, and speed of each wave ?
  - (c) Determine the tension in the string.
- 15.12** (i) For the wave on a string described in Exercise 15.11, do all the points on the string oscillate with the same (a) frequency, (b) phase, (c) amplitude? Explain your answers. (ii) What is the amplitude of a point 0.375 m away from one end?
- 15.13** Given below are some functions of  $x$  and  $t$  to represent the displacement (transverse or longitudinal) of an elastic wave. State which of these represent (i) a travelling wave, (ii) a stationary wave or (iii) none at all:
- (a)  $y = 2 \cos (3x) \sin (10t)$
  - (b)  $y = 2\sqrt{x - vt}$
  - (c)  $y = 3 \sin (5x - 0.5t) + 4 \cos (5x - 0.5t)$
  - (d)  $y = \cos x \sin t + \cos 2x \sin 2t$
- 15.14** A wire stretched between two rigid supports vibrates in its fundamental mode with a frequency of 45 Hz. The mass of the wire is  $3.5 \times 10^{-2}$  kg and its linear mass density is  $4.0 \times 10^{-2}$  kg m<sup>-1</sup>. What is (a) the speed of a transverse wave on the string, and (b) the tension in the string?
- 15.15** A metre-long tube open at one end, with a movable piston at the other end, shows resonance with a fixed frequency source (a tuning fork of frequency 340 Hz) when the tube length is 25.5 cm or 79.3 cm. Estimate the speed of sound in air at the temperature of the experiment. The edge effects may be neglected.
- 15.16** A steel rod 100 cm long is clamped at its middle. The fundamental frequency of longitudinal vibrations of the rod are given to be 2.53 kHz. What is the speed of sound in steel?
- 15.17** A pipe 20 cm long is closed at one end. Which harmonic mode of the pipe is resonantly excited by a 430 Hz source ? Will the same source be in resonance with the pipe if both ends are open? (speed of sound in air is 340 m s<sup>-1</sup>).
- 15.18** Two sitar strings A and B playing the note 'Ga' are slightly out of tune and produce beats of frequency 6 Hz. The tension in the string A is slightly reduced and the beat frequency is found to reduce to 3 Hz. If the original frequency of A is 324 Hz, what is the frequency of B?
- 15.19** Explain why (or how):
- (a) in a sound wave, a displacement node is a pressure antinode and vice versa,
  - (b) bats can ascertain distances, directions, nature, and sizes of the obstacles without any "eyes",
  - (c) a violin note and sitar note may have the same frequency, yet we can distinguish between the two notes,
  - (d) solids can support both longitudinal and transverse waves, but only longitudinal waves can propagate in gases, and
  - (e) the shape of a pulse gets distorted during propagation in a dispersive medium.

- 15.20** A train, standing at the outer signal of a railway station blows a whistle of frequency 400 Hz in still air. (i) What is the frequency of the whistle for a platform observer when the train (a) approaches the platform with a speed of  $10 \text{ m s}^{-1}$ , (b) recedes from the platform with a speed of  $10 \text{ m s}^{-1}$ ? (ii) What is the speed of sound in each case? The speed of sound in still air can be taken as  $340 \text{ m s}^{-1}$ .
- 15.21** A train, standing in a station-yard, blows a whistle of frequency 400 Hz in still air. The wind starts blowing in the direction from the yard to the station with at a speed of  $10 \text{ m s}^{-1}$ . What are the frequency, wavelength, and speed of sound for an observer standing on the station's platform? Is the situation exactly identical to the case when the air is still and the observer runs towards the yard at a speed of  $10 \text{ m s}^{-1}$ ? The speed of sound in still air can be taken as  $340 \text{ m s}^{-1}$ .

### Additional Exercises

- 15.22** A travelling harmonic wave on a string is described by  

$$y(x, t) = 7.5 \sin(0.0050x + 12t + \pi/4)$$
 (a) what are the displacement and velocity of oscillation of a point at  $x = 1 \text{ cm}$ , and  $t = 1 \text{ s}$ ? Is this velocity equal to the velocity of wave propagation?  
 (b) Locate the points of the string which have the same transverse displacements and velocity as the  $x = 1 \text{ cm}$  point at  $t = 2 \text{ s}$ ,  $5 \text{ s}$  and  $11 \text{ s}$ .
- 15.23** A narrow sound pulse (for example, a short pip by a whistle) is sent across a medium. (a) Does the pulse have a definite (i) frequency, (ii) wavelength, (iii) speed of propagation? (b) If the pulse rate is 1 after every 20 s, (that is the whistle is blown for a split of second after every 20 s), is the frequency of the note produced by the whistle equal to  $1/20$  or  $0.05 \text{ Hz}$ ?
- 15.24** One end of a long string of linear mass density  $8.0 \times 10^{-3} \text{ kg m}^{-1}$  is connected to an electrically driven tuning fork of frequency 256 Hz. The other end passes over a pulley and is tied to a pan containing a mass of 90 kg. The pulley end absorbs all the incoming energy so that reflected waves at this end have negligible amplitude. At  $t = 0$ , the left end (fork end) of the string  $x = 0$  has zero transverse displacement ( $y = 0$ ) and is moving along positive  $y$ -direction. The amplitude of the wave is 5.0 cm. Write down the transverse displacement  $y$  as function of  $x$  and  $t$  that describes the wave on the string.
- 15.25** A SONAR system fixed in a submarine operates at a frequency 40.0 kHz. An enemy submarine moves towards the SONAR with a speed of  $360 \text{ km h}^{-1}$ . What is the frequency of sound reflected by the submarine? Take the speed of sound in water to be  $1450 \text{ m s}^{-1}$ .
- 15.26** Earthquakes generate sound waves inside the earth. Unlike a gas, the earth can experience both transverse (S) and longitudinal (P) sound waves. Typically the speed of S wave is about  $4.0 \text{ km s}^{-1}$ , and that of P wave is  $8.0 \text{ km s}^{-1}$ . A seismograph records P and S waves from an earthquake. The first P wave arrives 4 min before the first S wave. Assuming the waves travel in straight line, at what distance does the earthquake occur?
- 15.27** A bat is flitting about in a cave, navigating via ultrasonic beeps. Assume that the sound emission frequency of the bat is 40 kHz. During one fast swoop directly toward a flat wall surface, the bat is moving at 0.03 times the speed of sound in air. What frequency does the bat hear reflected off the wall?