



$\therefore f$ is onto, thereby range $f = [-5, \infty)$.

Let us define $g: [-5, \infty) \rightarrow \mathbf{R}_+$ as $g(y) = \frac{\sqrt{y+6}-1}{3}$.

We now have:

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) = g(9x^2 + 6x - 5) \\ &= g((3x+1)^2 - 6) \\ &= \frac{\sqrt{(3x+1)^2 - 6 + 6} - 1}{3} \\ &= \frac{3x+1-1}{3} = x \end{aligned}$$

$$\begin{aligned} \text{And, } (f \circ g)(y) &= f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right) \\ &= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right) + 1\right]^2 - 6 \\ &= (\sqrt{y+6})^2 - 6 = y + 6 - 6 = y \end{aligned}$$

$\therefore g \circ f = I_{\mathbf{R}_+}$ and $f \circ g = I_{[-5, \infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let $f: X \rightarrow Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f . Then for all $y \in Y$,

$$f \circ g_1(y) = I_Y(y) = f \circ g_2(y). \text{ Use one-one ness of } f).$$

Answer

Let $f: X \rightarrow Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$$\begin{aligned} f \circ g_1(y) &= I_Y(y) = f \circ g_2(y) \\ \Rightarrow f(g_1(y)) &= f(g_2(y)) \\ \Rightarrow g_1(y) &= g_2(y) \quad [f \text{ is invertible} \Rightarrow f \text{ is one-one}] \\ \Rightarrow g_1 &= g_2 \quad [g \text{ is one-one}] \end{aligned}$$

Hence, f has a unique inverse.

Question 11:

Consider $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer

Function $f: \{1, 2, 3\} \rightarrow \{a, b, c\}$ is given by,

$$f(1) = a, f(2) = b, \text{ and } f(3) = c$$

If we define $g: \{a, b, c\} \rightarrow \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, then we have:

$$\begin{aligned} (f \circ g)(a) &= f(g(a)) = f(1) = a \\ (f \circ g)(b) &= f(g(b)) = f(2) = b \\ (f \circ g)(c) &= f(g(c)) = f(3) = c \end{aligned}$$

And,

$$\begin{aligned} (g \circ f)(1) &= g(f(1)) = g(a) = 1 \\ (g \circ f)(2) &= g(f(2)) = g(b) = 2 \\ (g \circ f)(3) &= g(f(3)) = g(c) = 3 \end{aligned}$$

$\therefore g \circ f = I_X$ and $f \circ g = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of f exists and $f^{-1} = g$.

$\therefore f^{-1}: \{a, b, c\} \rightarrow \{1, 2, 3\}$ is given by,

$$f^{-1}(a) = 1, f^{-1}(b) = 2, f^{-1}(c) = 3$$

Let us now find the inverse of f^{-1} i.e., find the inverse of g .

If we define $h: \{1, 2, 3\} \rightarrow \{a, b, c\}$ as
 $h(1) = a, h(2) = b, h(3) = c$, then we have:

$$\begin{aligned}(g \circ h)(1) &= g(h(1)) = g(a) = 1 \\ (g \circ h)(2) &= g(h(2)) = g(b) = 2 \\ (g \circ h)(3) &= g(h(3)) = g(c) = 3\end{aligned}$$

And,

$$\begin{aligned}(h \circ g)(a) &= h(g(a)) = h(1) = a \\ (h \circ g)(b) &= h(g(b)) = h(2) = b \\ (h \circ g)(c) &= h(g(c)) = h(3) = c\end{aligned}$$

$\therefore g \circ h = I_X$ and $h \circ g = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let $f: X \rightarrow Y$ be an invertible function. Show that the inverse of f^{-1} is f , i.e.,
 $(f^{-1})^{-1} = f$.

Answer

Let $f: X \rightarrow Y$ be an invertible function.

Then, there exists a function $g: Y \rightarrow X$ such that $g \circ f = I_X$ and $f \circ g = I_Y$.

Here, $f^{-1} = g$.

Now, $g \circ f = I_X$ and $f \circ g = I_Y$

$\Rightarrow f^{-1} \circ f = I_X$ and $f \circ f^{-1} = I_Y$

Hence, $f^{-1}: Y \rightarrow X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

Question 13:

If $f: \mathbf{R} \rightarrow \mathbf{R}$ be given by $f(x) = (3 - x^3)^{\frac{1}{3}}$, then $f \circ f(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer

$f: \mathbf{R} \rightarrow \mathbf{R}$ is given as $f(x) = (3 - x^3)^{\frac{1}{3}}$.

$$f(x) = (3 - x^3)^{\frac{1}{3}}$$

$$\begin{aligned}\therefore f \circ f(x) &= f(f(x)) = f\left((3 - x^3)^{\frac{1}{3}}\right) = \left[3 - \left((3 - x^3)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}} \\ &= \left[3 - (3 - x^3)\right]^{\frac{1}{3}} = (x^3)^{\frac{1}{3}} = x\end{aligned}$$

$$\therefore f \circ f(x) = x$$

The correct answer is C.

Question 14:

Let $f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is map g :

$f \rightarrow \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$

Answer

$f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbf{R}$ is defined as $f(x) = \frac{4x}{3x+4}$.

It is given that

Let y be an arbitrary element of Range f .

Then, there exists $x \in \mathbf{R} - \left\{-\frac{4}{3}\right\}$ such that $y = f(x)$.

$$\Rightarrow y = \frac{4x}{3x+4}$$

$$\Rightarrow 3xy + 4y = 4x$$

$$\Rightarrow x(4-3y) = 4y$$

$$\Rightarrow x = \frac{4y}{4-3y}$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$$

Now,

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)}$$

$$= \frac{4 - 3 \left(\frac{4x}{3x+4} \right)}{4 - 3 \left(\frac{4x}{3x+4} \right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\begin{aligned} \text{And, } (f \circ g)(y) &= f(g(y)) = f\left(\frac{4y}{4-3y}\right) \\ &= \frac{4 \left(\frac{4y}{4-3y} \right)}{3 \left(\frac{4y}{4-3y} \right) + 4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y \end{aligned}$$

$$\therefore g \circ f = I_{\mathbf{R} - \left\{ -\frac{4}{3} \right\}} \text{ and } f \circ g = I_{\text{Range } f}$$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map g : $\text{Range } f \rightarrow \mathbf{R} - \left\{ -\frac{4}{3} \right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.

Exercise 1.4

Question 1:

Determine whether or not each of the definition of given below gives a binary operation.

In the event that $*$ is not a binary operation, give justification for this.

(i) On \mathbf{Z}^+ , define $*$ by $a * b = a - b$

(ii) On \mathbf{Z}^+ , define $*$ by $a * b = ab$

(iii) On \mathbf{R} , define $*$ by $a * b = ab^2$

(iv) On \mathbf{Z}^+ , define $*$ by $a * b = |a - b|$

(v) On \mathbf{Z}^+ , define $*$ by $a * b = a$

Answer

(i) On \mathbf{Z}^+ , $*$ is defined by $a * b = a - b$.

It is not a binary operation as the image of $(1, 2)$ under $*$ is $1 * 2 = 1 - 2$

$= -1 \notin \mathbf{Z}^+$.

(ii) On \mathbf{Z}^+ , $*$ is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in \mathbf{Z}^+ .

This means that $*$ carries each pair (a, b) to a unique element $a * b = ab$ in \mathbf{Z}^+ .

Therefore, $*$ is a binary operation.

(iii) On \mathbf{R} , $*$ is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in \mathbf{R} .

This means that $*$ carries each pair (a, b) to a unique element $a * b = ab^2$ in \mathbf{R} .

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