Let $\tan^{-1}\sqrt{3} = x$. Then, $\tan x = \sqrt{3} = \tan\frac{\pi}{3}$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = -\sec\left(\frac{\pi}{3}\right) = \sec\left(\pi - \frac{\pi}{3}\right) = \sec\frac{2\pi}{3}$.

We know that the range of the principal value branch of $\sec^{-1}$ is $[0,\pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

Hence, $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$

Exercise 2.2

Question 1:

Prove $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Answer

To prove: $3\sin^{-1}x = \sin^{-1}\left(3x - 4x^3\right),\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Let $x = \sin\theta$. Then, $\sin^{-1}x = \theta$.

We have,

R.H.S. = $\sin^{-1}\left(3x - 4x^3\right) = \sin^{-1}\left(3\sin\theta - 4\sin^3\theta\right)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1}x$

= L.H.S.

Question 2:

Prove $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2}, 1\right]$

Answer

To prove: $3\cos^{-1}x = \cos^{-1}\left(4x^3 - 3x\right),\ x \in \left[\frac{1}{2}, 1\right]$

Let $x = \cos\theta$. Then, $\cos^{-1}x = \theta$.

We have,

$$\begin{aligned}
\text{R.H.S.} &= \cos^{-1}\left(4x^3 - 3x\right) \\
&= \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right) \\
&= \cos^{-1}(\cos 3\theta) \\
&= 3\theta \\
&= 3\cos^{-1}x \\
&= \text{L.H.S.}
\end{aligned}$$

Question 3:

Prove $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

Answer

To prove: $\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$

$$\begin{aligned}
\text{L.H.S.} &= \tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} \\
&= \tan^{-1}\frac{\frac{2}{11} + \frac{7}{24}}{1 - \frac{2}{11}\cdot\frac{7}{24}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]
\end{aligned}$$

$$= \tan^{-1}\frac{\dfrac{48+77}{11\times 24}}{\dfrac{11\times 24-14}{11\times 24}}$$

$$= \tan^{-1}\frac{48+77}{264-14} = \tan^{-1}\frac{125}{250} = \tan^{-1}\frac{1}{2} = \text{R.H.S.}$$

Question 4:

Prove $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Answer

To prove: $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

$$\text{L.H.S.} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{1}{\left(\frac{3}{4}\right)} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}} \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\frac{\left(\frac{28+3}{21}\right)}{\left(\frac{21-4}{21}\right)}$$

$$= \tan^{-1}\frac{31}{17} = \text{R.H.S.}$$

Question 5:

Write the function in the simplest form:

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},\ x \neq 0$$

Answer

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$$

Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$\therefore \tan^{-1}\frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

Question 6:

Write the function in the simplest form:

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Answer

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}},\ |x|>1$$

Put $x = \operatorname{cosec}\theta \Rightarrow \theta = \operatorname{cosec}^{-1}x$

$$\therefore \tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$$

$$= \tan^{-1}\left(\frac{1}{\cot\theta}\right) = \tan^{-1}(\tan\theta)$$

$$= \theta = \operatorname{cosec}^{-1}x = \frac{\pi}{2} - \sec^{-1}x \qquad \left[\operatorname{cosec}^{-1}x + \sec^{-1}x = \frac{\pi}{2}\right]$$

Question 7:

Write the function in the simplest form:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x<\pi$$

Answer

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right), x<\pi$$

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)=\tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$=\tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

$$=\frac{x}{2}$$

Question 8:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right),\ 0<x<\pi$$

Answer

$$\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$$

$$=\tan^{-1}\left(\frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$=\tan^{-1}(1)-\tan^{-1}(\tan x) \qquad \left[\tan^{-1}\frac{x-y}{1-xy}=\tan^{-1}x-\tan^{-1}y\right]$$

$$=\frac{\pi}{4}-x$$

Question 9:

Write the function in the simplest form:

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}},\ |x|<a$$

Answer

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}$$

$$\text{Put } x=a\sin\theta\Rightarrow\frac{x}{a}=\sin\theta\Rightarrow\theta=\sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}=\tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$=\tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)=\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

$$=\tan^{-1}(\tan\theta)=\theta=\sin^{-1}\frac{x}{a}$$

Question 10:

Write the function in the simplest form:

$$\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right),\ a>0;\ \frac{-a}{\sqrt{3}}\le x\le\frac{a}{\sqrt{3}}$$

Answer

*********** END **********