EXERCISE.13.1

Question-1

Evaluate the Given limit: $\lim_{x \to 3} x + 3$

Ans.

Evaluate the Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7} \right)$

Question-2

Evaluate the Given limit: $\lim_{x \to \pi} \left(x - \frac{22}{7} \right)$

Ans.

$$\lim_{x \to \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$$

Question-3

Evaluate the Given limit: $\lim_{r \to 1} \pi r^2$

Ans.

$$\lim_{r \to 1} \pi r^2 = \pi (1)^2 = \pi$$

Question-4

Evaluate the Given limit: $\lim_{x \to 4} \frac{4x+3}{x-2}$

Ans.

$$\lim_{x \to 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$$

Question-5

Evaluate the Given limit: $\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1}$

Ans.

$$\lim_{x \to -1} \frac{x^{10}+x^5+1}{x-1} = \frac{(-1)^{10}+(-1)^5+1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$$

Question-6

Evaluate the Given limit: $\lim_{x \to 0} \frac{(x+1)^5-1}{x}$

Ans.

$$\lim_{x \to 0} \frac{(x+1)^5-1}{x}$$

Put $x + 1 = y$ so that $y \to 1$ as $x \to 0$.

$$\text{Accordingly, } \lim_{x \to 0} \frac{(x+1)^5-1}{x} = \lim_{y \to 1} \frac{y^5-1}{y-1}$$

$$= \lim_{y \to 1} \frac{y^5-1^5}{y-1}$$

$$= 5.1^{5-1} \qquad \left[\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 5$$

$$\therefore \lim_{x \to 0} \frac{(x+5)^5-1}{x} = 5$$

Question-7

Evaluate the Given limit: $\lim_{x\to 2}\frac{3x^2-x-10}{x^2-4}$

Ans.

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\begin{aligned}\therefore \lim_{x\to 2}\frac{3x^2-x-10}{x^2-4} &= \lim_{x\to 2}\frac{(x-2)(3x+5)}{(x-2)(x+2)}\\ &= \lim_{x\to 2}\frac{3x+5}{x+2}\\ &= \frac{3(2)+5}{2+2}\\ &= \frac{11}{4}\end{aligned}$$

Question-8

Evaluate the Given limit: $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

Ans.

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\begin{aligned}\therefore \lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3} &= \lim_{x\to 3}\frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}\\ &= \lim_{x\to 3}\frac{(x+3)(x^2+9)}{2x+1}\\ &= \frac{(3+3)(3^2+9)}{2(3)+1}\\ &= \frac{6\times 18}{7}\\ &= \frac{108}{7}\end{aligned}$$

Question-9

Evaluate the Given limit: $\lim_{x\to 0}\frac{ax+b}{cx+1}$

Ans.

$$\lim_{x\to 0}\frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Question-10

Evaluate the Given limit: $\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$

Ans.

$$\lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1}$$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}} = x$ so that $z \to 1$ as $x \to 1$.

$$\text{Accordingly, } \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = \lim_{x\to 1}\frac{x^2-1}{x-1}$$

$$= \lim_{x\to 1}\frac{x^2-1^2}{x-1}$$

$$= 2.1^{2-1} \qquad \left[\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= 2$$

$$\therefore \lim_{z\to 1}\frac{z^{\frac{1}{3}}-1}{z^{\frac{1}{6}}-1} = 2$$

Question-11

Evaluate the Given limit: $\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a}, a+b+c \neq 0$

Ans.

$$\lim_{x\to 1}\frac{ax^2+bx+c}{cx^2+bx+a} = \frac{a(1)^2+b(1)+c}{c(1)^2+b(1)+a}$$

$$= \frac{a+b+c}{a+b+c}$$

$$= 1 \qquad [a+b+c \neq 0]$$

Question-12

Evaluate the Given limit: $\lim_{x\to -2}\frac{\frac{1}{x}+\frac{1}{2}}{x+2}$

Ans.

$$\lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2}$$

At $x = -2$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x \to -2} \frac{\frac{1}{x}+\frac{1}{2}}{x+2} = \lim_{x \to -2} \frac{\left(\frac{2+x}{2x}\right)}{x+2}$$

$$= \lim_{x \to -2} \frac{1}{2x}$$

$$= \frac{1}{2(-2)} = \frac{-1}{4}$$

Question-13

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{bx}$

Ans.

$$\lim_{x \to 0} \frac{\sin ax}{bx}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, } \lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{\sin ax}{ax} \times \frac{ax}{bx}$$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right) \times \left(\frac{a}{b}\right)$$

$$= \frac{a}{b} \lim_{ax \to 0} \left(\frac{\sin ax}{ax}\right) \qquad [x \to 0 \Rightarrow ax \to 0]$$

$$= \frac{a}{b} \times 1 \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= \frac{a}{b}$$

Question-14

Evaluate the Given limit: $\lim_{x \to 0} \frac{\sin ax}{\sin bx}, \ a, \ b \neq 0$

Ans.

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx},\ a,\ b\neq 0$$

At x = 0, the value of the given function takes the form $\frac{0}{0}$.

$$\text{Now, }\lim_{x\to 0}\frac{\sin ax}{\sin bx}=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)\times ax}{\left(\frac{\sin bx}{bx}\right)\times bx}$$

$$=\left(\frac{a}{b}\right)\times\frac{\lim_{ax\to 0}\left(\frac{\sin ax}{ax}\right)}{\lim_{bx\to 0}\left(\frac{\sin bx}{bx}\right)} \qquad \left[\begin{array}{l} x\to 0\Rightarrow ax\to 0 \\ \text{and } x\to 0\Rightarrow bx\to 0\end{array}\right]$$

$$=\left(\frac{a}{b}\right)\times\frac{1}{1} \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{a}{b}$$

Question-15

Evaluate the Given limit: $\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$

Ans.

$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$

It is seen that $x \to п$ □ $(п - x) \to 0$

$$\therefore \lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}=\frac{1}{\pi}\lim_{(\pi-x)\to 0}\frac{\sin(\pi-x)}{(\pi-x)}$$

$$=\frac{1}{\pi}\times 1 \qquad \left[\lim_{y\to 0}\frac{\sin y}{y}=1\right]$$

$$=\frac{1}{\pi}$$

Question-16

Evaluate the given limit: $\lim_{x\to 0}\frac{\cos x}{\pi-x}$

Ans.

$$\lim_{x\to 0}\frac{\cos x}{\pi-x}=\frac{\cos 0}{\pi-0}=\frac{1}{\pi}$$

Question-17

Evaluate the Given limit: $\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$

Ans.

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \to 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1} \qquad \left[\cos x = 1 - 2\sin^2 \frac{x}{2}\right]$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \to 0} \frac{\left(\frac{\sin^2 x}{x^2}\right) \times x^2}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right) \times \frac{x^2}{4}}$$

$$= 4 \frac{\lim_{x \to 0}\left(\frac{\sin^2 x}{x^2}\right)}{\lim_{x \to 0}\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}\right)}$$

$$= 4 \frac{\left(\lim_{x \to 0} \frac{\sin x}{x}\right)^2}{\left(\lim_{\frac{x}{2} \to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2} \qquad \left[x \to 0 \Rightarrow \frac{x}{2} \to 0\right]$$

$$= 4 \frac{1^2}{1^2} \qquad \left[\lim_{y \to 0} \frac{\sin y}{y} = 1\right]$$

$$= 4$$

Question-18

Evaluate the Given limit: $\lim_{x \to 0} \frac{ax + x\cos x}{b \sin x}$

Ans.

$$\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x}$$

At $x=0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned}
\lim_{x\to 0}\frac{ax+x\cos x}{b\sin x} &= \frac{1}{b}\lim_{x\to 0}\frac{x(a+\cos x)}{\sin x}\\
&= \frac{1}{b}\lim_{x\to 0}\left(\frac{x}{\sin x}\right)\times\lim_{x\to 0}(a+\cos x)\\
&= \frac{1}{b}\times\frac{1}{\left(\lim_{x\to 0}\frac{\sin x}{x}\right)}\times\lim_{x\to 0}(a+\cos x)\\
&= \frac{1}{b}\times(a+\cos 0) \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]\\
&= \frac{a+1}{b}
\end{aligned}$$

Question-19

Evaluate the Given limit: $\lim_{x\to 0} x\sec x$

Ans.

$$\lim_{x\to 0} x\sec x = \lim_{x\to 0}\frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

Question-20

Evaluate the Given limit: $\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$ $a,b,a+b\neq 0$

Ans.

At $x=0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x\to 0}\frac{\sin ax+bx}{ax+\sin bx}$$

$$=\lim_{x\to 0}\frac{\left(\frac{\sin ax}{ax}\right)ax+bx}{ax+bx\left(\frac{\sin bx}{bx}\right)}$$

$$=\frac{\left(\lim_{ax\to 0}\frac{\sin ax}{ax}\right)\times\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx\left(\lim_{bx\to 0}\frac{\sin bx}{bx}\right)} \qquad [\text{As } x\to 0 \Rightarrow ax\to 0 \text{ and } bx\to 0]$$

$$=\frac{\lim_{x\to 0}(ax)+\lim_{x\to 0}bx}{\lim_{x\to 0}ax+\lim_{x\to 0}bx} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=\frac{\lim_{x\to 0}(ax+bx)}{\lim_{x\to 0}(ax+bx)}$$

$$=\lim_{x\to 0}(1)$$

$$=1$$

Question-21

Evaluate the Given limit: $\lim_{x \to 0}(\operatorname{cosec} x - \cot x)$

Ans.

At $x = 0$, the value of the given function takes the form $\infty - \infty$.

Now,

$$\lim_{x \to 0}(\operatorname{cosec} x - \cot x)$$
$$= \lim_{x \to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$
$$= \lim_{x \to 0}\left(\frac{1 - \cos x}{\sin x}\right)$$
$$= \lim_{x \to 0}\frac{\left(\frac{1 - \cos x}{x}\right)}{\left(\frac{\sin x}{x}\right)}$$
$$= \frac{\lim_{x \to 0}\frac{1 - \cos x}{x}}{\lim_{x \to 0}\frac{\sin x}{x}}$$
$$= \frac{0}{1} \qquad \left[\lim_{x \to 0}\frac{1 - \cos x}{x} = 0 \text{ and } \lim_{x \to 0}\frac{\sin x}{x} = 1\right]$$
$$= 0$$

Question-22

$$\lim_{x \to \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

Ans.

$$\lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}$$

At $x=\frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x-\frac{\pi}{2}=y$ so that $x\to\frac{\pi}{2},\ y\to 0$.

$$\therefore \lim_{x\to\frac{\pi}{2}}\frac{\tan 2x}{x-\frac{\pi}{2}}=\lim_{y\to 0}\frac{\tan 2\left(y+\frac{\pi}{2}\right)}{y}$$

$$=\lim_{y\to 0}\frac{\tan(\pi+2y)}{y}$$

$$=\lim_{y\to 0}\frac{\tan 2y}{y} \qquad [\tan(\pi+2y)=\tan 2y]$$

$$=\lim_{y\to 0}\frac{\sin 2y}{y\cos 2y}$$

$$=\lim_{y\to 0}\left(\frac{\sin 2y}{2y}\times\frac{2}{\cos 2y}\right)$$

$$=\left(\lim_{2y\to 0}\frac{\sin 2y}{2y}\right)\times\lim_{y\to 0}\left(\frac{2}{\cos 2y}\right) \qquad [y\to 0\Rightarrow 2y\to 0]$$

$$=1\times\frac{2}{\cos 0} \qquad \left[\lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

$$=1\times\frac{2}{1}$$

$$=2$$

Question-23

Find $\lim_{x\to 0} f(x)$ and $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$

Ans.

The given function is

$$f(x)=\begin{cases}2x+3, & x\le 0\\ 3(x+1), & x>0\end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}[2x+3]=2(0)+3=3$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}3(x+1)=3(0+1)=3$$

$$\therefore \lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)=\lim_{x\to 0} f(x)=3$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}3(x+1)=3(1+1)=6$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}3(x+1)=3(1+1)=6$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x)=6$$

Question-24

Find $\lim_{x\to 1} f(x)$, where $f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$

Ans.

The given function is

$$f(x) = \begin{cases} x^2-1, & x \le 1 \\ -x^2-1, & x > 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}\left[x^2-1\right] = 1^2-1 = 1-1 = 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}\left[-x^2-1\right] = -1^2-1 = -1-1 = -2$$

It is observed that $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$.

Hence, $\lim_{x\to 1} f(x)$ does not exist.

Question-25

Evaluate $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ans.

$f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-}\left[\frac{|x|}{x}\right] \\ &= \lim_{x\to 0}\left(\frac{-x}{x}\right) && \left[\text{When } x \text{ is negaitve, } |x| = -x\right] \\ &= \lim_{x\to 0}(-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+}\left[\frac{|x|}{x}\right] \\ &= \lim_{x\to 0}\left[\frac{x}{x}\right] && \left[\text{When } x \text{ is positive, } |x| = x\right] \\ &= \lim_{x\to 0}(1) \\ &= 1 \end{aligned}$$

It is observed that $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$.

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Question-26

Find $\lim\limits_{x \to 0} f(x)$, where $f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Ans.

The given function is

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0^-} \left[\frac{x}{|x|}\right] \\ &= \lim_{x \to 0} \left[\frac{x}{-x}\right] \qquad \left[\text{When } x < 0,\ |x| = -x\right] \\ &= \lim_{x \to 0} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \to 0^+} f(x) &= \lim_{x \to 0^+} \left[\frac{x}{|x|}\right] \\ &= \lim_{x \to 0} \left[\frac{x}{x}\right] \qquad \left[\text{When } x > 0,\ |x| = x\right] \\ &= \lim_{x \to 0} (1) \\ &= 1 \end{aligned}$$

It is observed that $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$.

Hence, $\lim\limits_{x \to 0} f(x)$ does not exist.

Question-27

Find $\lim\limits_{x \to 5} f(x)$, where $f(x) = |x| - 5$

Ans.

The given function is $f(x) = |x| - 5$.

$$\begin{aligned}\lim_{x\to 5^-} f(x) &= \lim_{x\to 5^-}\left[|x|-5\right]\\ &= \lim_{x\to 5}(x-5) \qquad \left[\text{When } x>0,\ |x|=x\right]\\ &= 5-5\\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x\to 5^+} f(x) &= \lim_{x\to 5^+}\left(|x|-5\right)\\ &= \lim_{x\to 5}(x-5) \qquad \left[\text{When } x>0,\ |x|=x\right]\\ &= 5-5\\ &= 0\end{aligned}$$

$$\therefore\ \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = 0$$

Hence, $\lim_{x\to 5} f(x) = 0$

Question-28

Suppose $f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$ and if $\lim_{x\to 1} f(x) = f(1)$ what are possible values of a and b?

Ans.

The given function is

$$f(x) = \begin{cases} a+bx, & x<1 \\ 4, & x=1 \\ b-ax & x>1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1}(a+bx) = a+b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1}(b-ax) = b-a$$

$$f(1) = 4$$

It is given that $\lim_{x\to 1} f(x) = f(1)$.

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1)$$

$\Rightarrow a+b=4$ and $b-a=4$

On solving these two equations, we obtain $a=0$ and $b=4$.

Thus, the respective possible values of a and b are 0 and 4.

Question-29

Let $a_1,\ a_2,\ \ldots,\ a_n$ be fixed real numbers and define a function

$$f(x) = (x-a_1)(x-a_2)\ldots(x-a_n)$$

What is $\lim_{x\to a_1} f(x)$? For some $a \neq a_1,\ a_2 \ldots,\ a_n,$ compute $\lim_{x\to a} f(x)$

Ans.

The given function is $f(x)=(x-a_1)(x-a_2)...(x-a_n)$.

$$\lim_{x\to a_1} f(x)=\lim_{x\to a_1}\left[(x-a_1)(x-a_2)...(x-a_n)\right]$$
$$=\left[\lim_{x\to a_1}(x-a_1)\right]\left[\lim_{x\to a_1}(x-a_2)\right]...\left[\lim_{x\to a_1}(x-a_n)\right]$$
$$=(a_1-a_1)(a_1-a_2)...(a_1-a_n)=0$$
$$\therefore \lim_{x\to a_1} f(x)=0$$
$$\text{Now, } \lim_{x\to a} f(x)=\lim_{x\to a}\left[(x-a_1)(x-a_2)...(x-a_n)\right]$$
$$=\left[\lim_{x\to a}(x-a_1)\right]\left[\lim_{x\to a}(x-a_2)\right]...\left[\lim_{x\to a}(x-a_n)\right]$$
$$=(a-a_1)(a-a_2).....(a-a_n)$$
$$\therefore \lim_{x\to a} f(x)=(a-a_1)(a-a_2)...(a-a_n)$$

Question-30

If $f(x) = \begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$.

For what value (s) of a does $\lim_{x\to a} f(x)$ exists?

Ans.

The given function is

$$f(x)=\begin{cases} |x|+1, & x<0 \\ 0, & x=0 \\ |x|-1, & x>0 \end{cases}$$

When $a=0$,

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(|x|+1)$$
$$=\lim_{x\to 0}(-x+1) \qquad [\text{If } x<0,\ |x|=-x]$$
$$=-0+1$$
$$=1$$
$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(|x|-1)$$
$$=\lim_{x\to 0}(x-1) \qquad [\text{If } x>0,\ |x|=x]$$
$$=0-1$$
$$=-1$$

Here, it is observed that $\lim_{x\to 0^-} f(x)\neq \lim_{x\to 0^+} f(x)$.

$\therefore \lim_{x\to 0} f(x)$ does not exist.

When $a < 0$,

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (|x|+1)$$
$$= \lim_{x\to a} (-x+1) \qquad [x < a < 0 \Rightarrow |x| = -x]$$
$$= -a+1$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (|x|+1)$$
$$= \lim_{x\to a} (-x+1) \qquad [a < x < 0 \Rightarrow |x| = -x]$$
$$= -a+1$$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = -a+1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a < 0$.

When $a > 0$

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < x < a \Rightarrow |x| = x]$$
$$= a-1$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (|x|-1)$$
$$= \lim_{x\to a} (x-1) \qquad [0 < a < x \Rightarrow |x| = x]$$
$$= a-1$$

$$\therefore \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = a-1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x\to a} f(x)$ exists for all $a \neq 0$.

Question-31

If the function $f(x)$ satisfies $\lim_{x\to 1} \frac{f(x)-2}{x^2-1} = \pi$, evaluate $\lim_{x\to 1} f(x)$.

Ans.

$$\lim_{x\to 1}\frac{f(x)-2}{x^2-1}=\pi$$

$$\Rightarrow \frac{\lim\limits_{x\to 1}\left(f(x)-2\right)}{\lim\limits_{x\to 1}\left(x^2-1\right)}=\pi$$

$$\Rightarrow \lim_{x\to 1}\left(f(x)-2\right)=\pi\lim_{x\to 1}\left(x^2-1\right)$$

$$\Rightarrow \lim_{x\to 1}\left(f(x)-2\right)=\pi\left(1^2-1\right)$$

$$\Rightarrow \lim_{x\to 1}\left(f(x)-2\right)=0$$

$$\Rightarrow \lim_{x\to 1}f(x)-\lim_{x\to 1}2=0$$

$$\Rightarrow \lim_{x\to 1}f(x)-2=0$$

$$\therefore \lim_{x\to 1}f(x)=2$$

Question-32

If $f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$. For what integers m and n does $\lim\limits_{x\to 0}f(x)$ and $\lim\limits_{x\to 1}f(x)$ exist?

Ans.

The given function is

$$f(x)=\begin{cases} mx^2+n, & x<0 \\ nx+m, & 0\le x\le 1 \\ nx^3+m, & x>1 \end{cases}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}\left(mx^2+n\right)$$

$$=m(0)^2+n$$

$$=n$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0}(nx+m)$$

$$=n(0)+m$$

$$=m.$$

Thus, $\lim_{x\to 0} f(x)$ exists if $m = n$.

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(nx+m)$$

$$=n(1)+m$$

$$=m+n$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}\left(nx^3+m\right)$$

$$=n(1)^3+m$$

$$=m+n$$

$$\therefore \lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=\lim_{x\to 1} f(x).$$

Thus, $\lim_{x\to 1} f(x)$ exists for any integral value of m and n.

*********** END **********