



Mathematical Induction Ex 12.2 Q33

$P(n): \frac{n^{11}}{11} + \frac{n^5}{5} + \frac{n^3}{3} + \frac{62}{165}n$ is a positive integer

For $n = 1$

$$\begin{aligned} & \frac{1}{11} + \frac{1}{5} + \frac{1}{3} + \frac{62}{165} \\ &= \frac{15 + 33 + 55 + 62}{165} \\ &= \frac{165}{165} \end{aligned}$$

Which is a positive integer

Let $P(n)$ is true for $n = k$, so

$\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165}$ is a positive integer

$$\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} = \lambda \quad \text{---(i)}$$

For $n = k + 1$

$$\begin{aligned} & \frac{(k+1)^{11}}{11} + \frac{(k+1)^5}{5} + \frac{(k+1)^3}{3} + \frac{62}{165}(k+1) \\ &= \frac{1}{11} [k^{11} + 11k^{10} + 55k^9 + 165k^8 + 330k^7 + 462k^6 + 462k^5 + 330k^4 + 165k^3 + 55k^2 + 11k + 1] \\ & \quad + \frac{1}{5} [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1] + \frac{1}{3} [k^3 + 3k^2 + 3k + 1] + \frac{62}{165} [k + 1] \\ &= \left[\frac{k^{11}}{11} + \frac{k^5}{5} + \frac{k^3}{3} + \frac{62k}{165} \right] + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 30k^4 + 15k^3 + 5k^2 + 1 + \frac{1}{11} \\ & \quad + k^4 + 2k^3 + 2k^2 + k + \frac{1}{5} + k^2 + k + \frac{1}{3} + \frac{62}{165} \\ &= \lambda + k^{10} + 5k^9 + 15k^8 + 30k^7 + 42k^6 + 42k^5 + 31k^4 + 17k^3 + 8k^2 + 2k + 1 \\ &= \text{An integer} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in \mathbb{N}$ by PMI

Mathematical Induction Ex 12.2 Q34

$$\text{Let } P(n): \frac{1}{2} \tan\left(\frac{x}{2}\right) + \frac{1}{4} \tan\left(\frac{x}{4}\right) + \dots + \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n} \cot\left(\frac{x}{2^n}\right) - \cot x$$

For $n = 1$

$$\frac{1}{2} \tan \frac{x}{2} = \frac{1}{2} \cot\left(\frac{x}{2}\right) - \cot x$$

$$\begin{aligned} &= \frac{1}{2} \frac{1}{\tan \frac{x}{2}} - \frac{1}{\tan x} \\ &= \frac{1}{2 \tan \frac{x}{2}} - \frac{1}{\left(\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} \right)} \\ &= \frac{1}{2 \tan \frac{x}{2}} - \frac{1 - \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{1 - 1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{\tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} \\ &= \frac{1}{2} \tan \frac{x}{2} \end{aligned}$$

$\Rightarrow P(n)$ is true for $n = 1$

Let $P(n)$ is true for $n = k$, so

$$\frac{1}{2} \tan\left(\frac{x}{2}\right) + \frac{1}{4} \tan\left(\frac{x}{4}\right) + \dots + \frac{1}{2^k} \tan\left(\frac{x}{2^k}\right) = \frac{1}{2^k} \cot\left(\frac{x}{2^k}\right) - \cot x \quad \text{----(1)}$$

We have to show that,

$$\frac{1}{2} \tan \frac{x}{2} + \frac{1}{4} \tan \left(\frac{x}{4} \right) + \dots + \frac{1}{2^k} \tan \left(\frac{x}{2^k} \right) + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) = \frac{1}{2^{k+1}} \cot \left(\frac{x}{2^{k+1}} \right) - \cot x$$

Now,

$$\begin{aligned} & \left\{ \frac{1}{2} \tan \frac{x}{2} + \frac{1}{4} \tan \left(\frac{x}{4} \right) + \dots + \frac{1}{2^k} \tan \left(\frac{x}{2^k} \right) \right\} + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) \\ &= \frac{1}{2^k} \cot \left(\frac{x}{2^k} \right) - \cot x + \frac{1}{2^{k+1}} \tan \left(\frac{x}{2^{k+1}} \right) \\ &= \frac{1}{2^k} \cot \left(\frac{x}{2^k} \right) - \cot x + \frac{1}{2 \cdot 2^k} \frac{1}{\cot \left(\frac{x}{2^k} \cdot \frac{1}{2} \right)} \end{aligned}$$

$$= \frac{1}{2^k} \left[\frac{1}{\tan \left(\frac{x}{2^k} \right)} + \frac{1}{2} \cdot \tan \left\{ \left(\frac{x}{2^k} \right) \cdot \frac{1}{2} \right\} \right] - \cot x$$

$$= \frac{1}{2^k} \left[\frac{1 - \tan^2 \left(\frac{x}{2^{k+1}} \right)}{2 \tan \left(\frac{x}{2^{k+1}} \right)} + \frac{1}{2} \tan \left(\frac{x}{2 \cdot 2^k} \right) \right] - \cot x$$

$$= \frac{1}{2^k} \left[\frac{1 - \tan^2 \left(\frac{x}{2^{k+1}} \right) + \tan^2 \left(\frac{x}{2^{k+1}} \right)}{2 \tan \left(\frac{x}{2^{k+1}} \right)} \right] - \cot x$$

$$= \frac{1}{2^{k+1}} \left[\frac{1}{\tan \left(\frac{x}{2^{k+1}} \right)} \right] - \cot x$$

$$= \frac{1}{2^{k+1}} \cot \left(\frac{x}{2^{k+1}} \right) - \cot x$$

$\Rightarrow P(n)$ is true for $n = k + 1$

$\Rightarrow P(n)$ is true for all $n \in N$ by *PMI*

Mathematical Induction Ex 12.2 Q35

$$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots \left(1 - \frac{1}{n^2}\right)$$

Above can be written as

$$\begin{aligned} &= \left(\frac{2^2-1}{2^2}\right) \left(\frac{3^2-1}{3^2}\right) \left(\frac{4^2-1}{4^2}\right) \dots \left(\frac{n^2-1}{n^2}\right) \\ &= \left(\frac{(2+1)(2-1)}{2^2}\right) \left(\frac{(3+1)(3-1)}{3^2}\right) \\ &\quad \left(\frac{(4+1)(4-1)}{4^2}\right) \dots \left(\frac{(n+1)(n-1)}{n^2}\right) \\ &= \left(\frac{3 \cdot 1}{2^2}\right) \left(\frac{4 \cdot 2}{3^2}\right) \left(\frac{5 \cdot 3}{4^2}\right) \dots \left(\frac{(n+1) \cdot (n-1)}{n^2}\right) \end{aligned}$$

In the above product, there are two series in numerator

3.4.5.....(n+1) and 1.2.3.....(n-1)

All numbers from 3 to (n-1) are repeated twice

and 1, 2, n are appeared once in numerator

So after cancelling like terms we get

$$= \frac{(n+1)}{2n}$$

***** END *****