Maxima and Minima 18.5 Q19

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

$$V=\frac{1}{3\pi}\pi r^2 h \Rightarrow h=\frac{3V}{r^2}$$

The surface area (S) of the cone is given by,

$S=\pi rl$ (where l is the slant height)

$$=\pi r\sqrt{r^2+h^2}$$

$$=\pi r\sqrt{r^2+\frac{9V^2}{\pi^2 r^4}}=\pi\frac{r\sqrt{\pi^2 r^6+9V^2}}{\pi r^2}$$

$$=\frac{1}{r}\sqrt{\pi^2 r^6+9V^2}$$

$$\therefore \frac{dS}{dr}=\frac{r\cdot\frac{6\pi^2 r^5}{2\sqrt{\pi^2 r^6+9V^2}}-\sqrt{\pi^2 r^6+9V^2}}{r^2}$$

$$=\frac{3\pi^2 r^6-\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$=\frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

$$=\frac{2\pi^2 r^6-9V^2}{r^2\sqrt{\pi^2 r^6+9V^2}}$$

Now, $\frac{dS}{dr}=0 \Rightarrow 2\pi^2 r^6=9V^2 \Rightarrow r^6=\frac{9V^2}{2\pi^2}$

Thus, it can be easily verified that when $r^6=\frac{9V^2}{2\pi^2}$, $\frac{d^2S}{dr^2}>0.$

$\therefore$ By second derivative test, the surface area of the cone is the least when $r^6=\frac{9V^2}{2\pi^2}$.

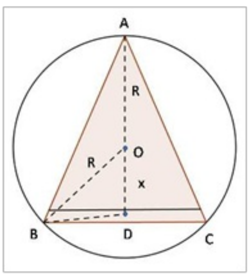
When $r^6=\frac{9V^2}{2\pi^2}$, $h=\frac{3V}{\pi r^2}=\frac{3}{\pi r^2}\left(\frac{2\pi^2 r^6}{9}\right)^{\frac{1}{2}}=\frac{3}{\pi r^2}\cdot\frac{\sqrt{2}\pi r^3}{3}=\sqrt{2}r.$

Hence, for a given volume, the right circular cone of the least curved surface has an altitude equal to $\sqrt{2}$ times the radius of the base.

Maxima and Minima 18.5 Q20

We have a cone, which is inscribed in a sphere.

Let v be the volume of greatest cone ABC. If is obvious that, for maximum volume the axis of the cone must be along the diameter of sphere.



Let $OD = x$ and $AO = OB = R$

$\Rightarrow \quad BD = \sqrt{R^2 - x^2}$ and $AD = R + x$

Now,

$$v = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi BD^2 \times AD$$
$$= \frac{1}{3}\pi \left(R^2 - x^2\right) \times \left(R + x\right)$$

$$\therefore \quad \frac{dv}{dx} = \frac{\pi}{3}\left[-2x\left(R + x\right) + R^2 - x^2\right]$$
$$= \frac{\pi}{3}\left[R^2 - 2xR - 3x^2\right]$$

For maximum and minimum

$$\frac{dv}{dx} = 0$$
$$\Rightarrow \quad \frac{\pi}{3}\left[R^2 - 2xR - 3x^2\right] = 0$$
$$\Rightarrow \quad \frac{\pi}{3}\left[\left(R - 3x\right)\left(R + x\right)\right] = 0$$
$$\Rightarrow \quad R - 3x = 0 \text{ or } x = -R$$
$$\Rightarrow \quad x = \frac{R}{3} \qquad \left[\because x = -R \text{ is not possible as, } x = -R \text{ will make the altitude } 0\right]$$

Now,

$$\frac{d^2v}{dx^2} = \frac{\pi}{3}\left[-2R - 6x\right]$$

At $\quad x = \frac{R}{3}, \quad \frac{d^2v}{dx^2} = \frac{\pi}{3}\left[-2R - 2R\right]$

$$= \frac{-4\pi R}{3} < 0$$

$\therefore \quad x = \frac{R}{3}$ is the point of local maxima.

*********** END **********