$$= a^2b^2c^2(-1)\begin{vmatrix}0 & 2\\ 2 & 0\end{vmatrix}$$

$$= -a^2b^2c^2(0-4) = 4a^2b^2c^2$$

Question 8:

By using properties of determinants, show that:

(i) $$\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix} = (a-b)(b-c)(c-a)$$

(ii) $$\begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Answer

(i) $$\text{Let } \Delta = \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}.$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have:

$$\Delta = \begin{vmatrix}0 & a-c & a^2-c^2\\ 0 & b-c & b^2-c^2\\ 1 & c & c^2\end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix}0 & -1 & -a-c\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = (b-c)(c-a)\begin{vmatrix}0 & 0 & -a+b\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix}0 & 0 & -1\\ 0 & 1 & b+c\\ 1 & c & c^2\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = (a-b)(b-c)(c-a)\begin{vmatrix}0 & -1\\ 1 & b+c\end{vmatrix} = (a-b)(b-c)(c-a)$$

Hence, the given result is proved.

(ii) Let $$\Delta = \begin{vmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{vmatrix}.$$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta = \begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ a^3-c^3 & b^3-c^3 & c^3\end{vmatrix}$$

$$= \begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3\end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ -1 & 1 & c\\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta = (c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$= (b-c)(c-a)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -(a+b+c) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\end{vmatrix}$$

$$\begin{vmatrix}-1 & \left(b^2+bc+c^2\right) & c^3\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix}0 & 1\\ 1 & c\end{vmatrix}$$
$$=(a-b)(b-c)(c-a)(a+b+c)$$

Hence, the given result is proved.

Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Answer

$$\text{Let } \Delta=\begin{vmatrix}x & x^2 & yz\\ y & y^2 & zx\\ z & z^2 & xy\end{vmatrix}.$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=\begin{vmatrix}x & x^2 & yz\\ y-x & y^2-x^2 & zx-yz\\ z-x & z^2-x^2 & xy-yz\end{vmatrix}$$
$$=\begin{vmatrix}x & x^2 & yz\\ -(x-y) & -(x-y)(x+y) & z(x-y)\\ (z-x) & (z-x)(z+x) & -y(z-x)\end{vmatrix}$$
$$=(x-y)(z-x)\begin{vmatrix}x & x^2 & yz\\ -1 & -x-y & z\\ 1 & z+x & -y\end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta=(x-y)(z-x)\begin{vmatrix}x & x^2 & yz\\ -1 & -x-y & z\\ 0 & z-y & z-y\end{vmatrix}$$
$$=(x-y)(z-x)(z-y)\begin{vmatrix}x & x^2 & yz\\ -1 & -x-y & z\\ 0 & 1 & 1\end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=\left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix}x & yz\\ -1 & z\end{vmatrix}+1\begin{vmatrix}x & x^2\\ -1 & -x-y\end{vmatrix}\right]$$
$$=(x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$
$$=-(x-y)(z-x)(z-y)(xy+yz+zx)$$
$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, the given result is proved.

Question 10:

By using properties of determinants, show that:

(i) $$\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}=(5x+4)(4-x)^2$$

(ii) $$\begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}=k^2(3y+k)$$

Answer

(i) $$\Delta=\begin{vmatrix}x+4 & 2x & 2x\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta=\begin{vmatrix}5x+4 & 5x+4 & 5x+4\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$$
$$=(5x+4)\begin{vmatrix}1 & 1 & 1\\ 2x & x+4 & 2x\\ 2x & 2x & x+4\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta=(5x+4)\begin{vmatrix}1 & 0 & 0\\ 2x & -x+4 & 0\\ 2x & 0 & -x+4\end{vmatrix}$$
$$=(5x+4)(4-x)(4-x)\begin{vmatrix}1 & 0 & 0\\ 2x & 1 & 0\\ 2x & 0 & 1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (5x+4)(4-x)^2 \begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

Hence, the given result is proved.

(ii) $$\Delta = \begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 3y+k & 3y+k & 3y+k \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

$$= (3y+k) \begin{vmatrix} 1 & 1 & 1 \\ y & y+k & y \\ y & y & y+k \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & k & 0 \\ y & 0 & k \end{vmatrix}$$

$$= k^2 (3y+k) \begin{vmatrix} 1 & 0 & 0 \\ y & 1 & 0 \\ y & 0 & 1 \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = k^2 (3y+k) \begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix} = k^2 (3y+k)$$

Hence, the given result is proved.

*********** END **********