Exercise 5.8 : Solutions of Questions on Page Number : **186**

Q1 : Verify Rolle's Theorem for the function $f(x)=x^2+2x-8, x\in[-4,2]$

Answer :

The given function, $f(x)=x^2+2x-8$, being a polynomial function, is continuous in [- 4, 2] and is differentiable in (- 4, 2).

$$f(-4)=(-4)^2+2\times(-4)-8=16-8-8=0$$
$$f(2)=(2)^2+2\times2-8=4+4-8=0$$

$\therefore$ f(- 4) = f(2) = 0

⇒ The value of $f(x)$ at - 4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in$ (- 4, 2) such that $f'(c)=0$

$$f(x)=x^2+2x-8$$
$$\Rightarrow f'(x)=2x+2$$
$$\therefore f'(c)=0$$
$$\Rightarrow 2c+2=0$$
$$\Rightarrow c=-1, \text{ where } c=-1\in(-4,2)$$

Hence, Rolle's Theorem is verified for the given function.

Answer needs Correction? **Click Here**

Q2 : Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

Answer :

By Rolle's Theorem, for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) $f(a)$ = $f(b)$

then, there exists some $c \in$ (a, b) such that $f'(c)=0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at x = 5 and x = 9

⇒ $f(x)$ is not continuous in [5, 9].

$$\text{Also, } f(5)=[5]=5 \text{ and } f(9)=[9]=9$$
$$\therefore f(5)\neq f(9)$$

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in$ (5, 9).

The left hand limit of f at $x=n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of f at x = n are not equal, f is not differentiable at x = n

$\therefore$ f is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$
$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h\to 0^-} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^-} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^-} \frac{n-1-n}{h} = \lim_{h\to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h\to 0^+} \frac{f(n+h)-f(n)}{h} = \lim_{h\to 0^+} \frac{[n+h]-[n]}{h} = \lim_{h\to 0^+} \frac{n-n}{h} = \lim_{h\to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Q3 : If $f : [-5, 5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer :

It is given that $f : [-5, 5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$
$$\Rightarrow 10 f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$$\therefore f'(c) \neq 0$$
$$\Rightarrow 10 f'(c) \neq 0$$
$$\Rightarrow f(5) - f(-5) \neq 0$$
$$\Rightarrow f(5) \neq f(-5)$$

Hence, proved.

Q4 : Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer :

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in [1, 4] and is differentiable in (1, 4) whose derivative is $2x - 4$.

$$f(1) = 1^2 - 4\times 1 - 3 = -6,\ f(4) = 4^2 - 4\times 4 - 3 = -3$$
$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$$f'(c) = 1$$
$$\Rightarrow 2c - 4 = 1$$
$$\Rightarrow c = \frac{5}{2}, \text{ where } c = \frac{5}{2} \in (1, 4)$$

Hence, Mean Value Theorem is verified for the given function.

Q5 : Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$

Answer :

The given function *f* is $f(x)=x^3-5x^2-3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2 - 10x - 3$.

$$f(1)=1^3-5\times1^2-3\times1=-7,\ f(3)=3^3-5\times3^2-3\times3=-27$$
$$\therefore \frac{f(b)-f(a)}{b-a}=\frac{f(3)-f(1)}{3-1}=\frac{-27-(-7)}{3-1}=-10$$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c)=-10$

$$f'(c)=-10$$
$$\Rightarrow 3c^2-10c-3=10$$
$$\Rightarrow 3c^2-10c+7=0$$
$$\Rightarrow 3c^2-3c-7c+7=0$$
$$\Rightarrow 3c(c-1)-7(c-1)=0$$
$$\Rightarrow (c-1)(3c-7)=0$$
$$\Rightarrow c=1,\ \frac{7}{3},\ \text{where } c=\frac{7}{3}\in(1,\ 3)$$

Hence, Mean Value Theorem is verified for the given function and $c=\frac{7}{3}\in(1,\ 3)$ is the only point for which $f'(c)=0$

Answer needs Correction? **Click Here**

Q6 : Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer :

Mean Value Theorem states that for a function $f:[a,\ b]\to\mathbf{R}$, if

(a) *f* is continuous on [*a*, *b*]

(b) *f* is differentiable on (*a*, *b*)

then, there exists some $c \in (a, b)$ such that $f'(c)=\frac{f(b)-f(a)}{b-a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x)=[x]$ for $x\in[5,\ 9]$

It is evident that the given function *f*(*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = 5 and *x* = 9

⇒ *f*(*x*) is not continuous in [5, 9].

The differentiability of *f* in (5, 9) is checked as follows.

Let *n* be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of *f* at *x* = *n* are not equal, *f* is not differentiable at *x* = *n*

∴ *f* is not differentiable in (5, 9).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[5,\ 9]$.

(ii) $f(x)=[x]$ for $x\in[-2,\ 2]$

It is evident that the given function *f*(*x*) is not continuous at every integral point.

In particular, *f*(*x*) is not continuous at *x* = - 2 and *x* = 2

⇒ *f*(*x*) is not continuous in [- 2, 2].

The differentiability of *f* in (- 2, 2) is checked as follows.

Let *n* be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x=n$ is,

$$\lim_{h\to0^-}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^-}\frac{[n+h]-[n]}{h}=\lim_{h\to0^-}\frac{n-1-n}{h}=\lim_{h\to0^-}\frac{-1}{h}=\infty$$

The right hand limit of f at $x=n$ is,

$$\lim_{h\to0^+}\frac{f(n+h)-f(n)}{h}=\lim_{h\to0^+}\frac{[n+h]-[n]}{h}=\lim_{h\to0^+}\frac{n-n}{h}=\lim_{h\to0^+}0=0$$

Since the left and right hand limits of *f* at *x* = *n* are not equal, *f* is not differentiable at *x* = *n*

∴ *f* is not differentiable in (- 2, 2).

It is observed that *f* does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x)=[x]$ for $x\in[-2,\ 2]$.

(iii) $f(x)=x^2-1$ for $x\in[1,\ 2]$

It is evident that *f*, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1,\ 2]$.

It can be proved as follows.

$$f(1) = 1^2 - 1 = 0,\ \ f(2) = 2^2 - 1 = 3$$

$$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(2) - f(1)}{2 - 1} = \frac{3 - 0}{1} = 3$$

$$f'(x) = 2x$$

$$\therefore f'(c) = 3$$

$$\Rightarrow 2c = 3$$

$$\Rightarrow c = \frac{3}{2} = 1.5,\ \text{where } 1.5 \in [1, 2]$$

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