Question 16:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n):\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$, we have

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad [\text{Using } (1)]$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n

Question 17:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

For n = 1, we have

$P(1): \frac{1}{3.5}=\frac{1}{3(2.1+3)}=\frac{1}{3.5}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k): \frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k}{3(2k+3)} \qquad ...(1)$$

We shall now prove that P(k + 1) is true.

$$\left[\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2k+1)(2k+3)}\right]+\frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$=\frac{k}{3(2k+3)}+\frac{1}{(2k+3)(2k+5)} \qquad [\text{Using } (1)]$$

$$=\frac{1}{(2k+3)}\left[\frac{k}{3}+\frac{1}{(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$=\frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$=\frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$=\frac{(k+1)}{3\{2(k+1)+3\}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 18:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+2+3+...+n<\frac{1}{8}(2n+1)^2$$

Ans :

Let P(k) be true for some positive integer k, i.e.,

$$1+2+\ldots+k<\frac{1}{8}(2k+1)^2 \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider inequality (1)

$$1+2+\ldots+k<\frac{1}{8}(2k+1)^2$$

Adding (k+1) on both the sides of the inequality, we have,

$$(1+2+\ldots+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$$

$$(1+2+\ldots+k)+(k+1) \ <\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$$

$$(1+2+\ldots+k)+(k+1) \ <\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$$

$$(1+2+\ldots+k)+(k+1) \ <\frac{1}{8}\left\{4k^2+12k+9\right\}$$

$$(1+2+\ldots+k)+(k+1) \ <\frac{1}{8}(2k+3)^2$$

$$(1+2+\ldots+k)+(k+1) \ <\frac{1}{8}\left\{2(k+1)+1\right\}^2$$

Hence, $(1+2+3+\ldots+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 19:

Prove the following by using the principle of mathematical induction for all $n \in N$: n (n + 1) (n + 5) is a multiple of 3.

Ans :

Let the given statement be P(n), i.e.,

P(n): n (n + 1) (n + 5), which is a multiple of 3.

It can be noted that P(n) is true for n = 1 since 1 (1 + 1) (1 + 5) = 12, which is a multiple of 3.

Let P(k) be true for some positive integer k, i.e.,

k (k + 1) (k + 5) is a multiple of 3.

$\therefore k(k+1)(k+5)=3m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$(k+1)\{(k+1)+1\}\{(k+1)+5\}$$
$$=(k+1)(k+2)\{(k+5)+1\}$$
$$=(k+1)(k+2)(k+5)+(k+1)(k+2)$$
$$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$$
$$=3m+(k+1)\{2(k+5)+(k+2)\}$$

$= 3m + (k+1)\{2k+10+k+2\}$

$= 3m + (k+1)(3k+12)$

$= 3m + 3(k+1)(k+4)$

$= 3\{m + (k+1)(k+4)\} = 3 \times q$, where $q = \{m + (k+1)(k+4)\}$ is some natural number

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 20:

Prove the following by using the principle of mathematical induction for all $n \in N$: $10^{2n-1} + 1$ is divisible by 11.

Ans :

Let the given statement be P(n), i.e.,

P(n): $10^{2n-1} + 1$ is divisible by 11.

It can be observed that P(n) is true for $n = 1$ since $P(1) = 10^{2.1-1} + 1 = 11$, which is divisible by 11.

Let P(k) be true for some positive integer k, i.e.,

$10^{2k-1} + 1$ is divisible by 11.

$\therefore 10^{2k-1} + 1 = 11m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$10^{2(k+1)-1} + 1$

$= 10^{2k+2-1} + 1$

$= 10^{2k+1} + 1$

$= 10^2\left(10^{2k-1} + 1 - 1\right) + 1$

$= 10^2\left(10^{2k-1} + 1\right) - 10^2 + 1$

$= 10^2.11m - 100 + 1$ $\quad$ [Using (1)]

$= 100 \times 11m - 99$

$= 11(100m - 9)$

$= 11r$, where $r = (100m - 9)$ is some natural number

Therefore, $10^{2(k+1)-1} + 1$ is divisible by 11.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 21:

Prove the following by using the principle of mathematical induction for all $n \in N$: $x^{2n} - y^{2n}$ is divisible by $x + y$.

Ans :

Let the given statement be P(n), i.e.,

P(n): $x^{2n} - y^{2n}$ is divisible by $x + y$.

It can be observed that P(n) is true for $n = 1$.

This is so because $x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y)$ is divisible by $(x + y)$.

Let P(k) be true for some positive integer k, i.e.,

$x^{2k} - y^{2k}$ is divisible by $x + y$.

$\therefore x^{2k} - y^{2k} = m(x + y)$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$x^{2(k+1)} - y^{2(k+1)}$$
$$= x^{2k} \cdot x^2 - y^{2k} \cdot y^2$$
$$= x^2\left(x^{2k} - y^{2k} + y^{2k}\right) - y^{2k} \cdot y^2$$
$$= x^2\left\{m\left(x+y\right) + y^{2k}\right\} - y^{2k} \cdot y^2 \qquad \left[\text{Using (1)}\right]$$
$$= m\left(x+y\right)x^2 + y^{2k} \cdot x^2 - y^{2k} \cdot y^2$$
$$= m\left(x+y\right)x^2 + y^{2k}\left(x^2 - y^2\right)$$
$$= m\left(x+y\right)x^2 + y^{2k}\left(x+y\right)\left(x-y\right)$$
$$= \left(x+y\right)\left\{mx^2 + y^{2k}\left(x-y\right)\right\}, \text{ which is a factor of } \left(x+y\right).$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 22:

Prove the following by using the principle of mathematical induction for all $n \in N$: $3^{2n+2} - 8n - 9$ is divisible by 8.

Ans :

Let the given statement be P(n), i.e.,

P(n): $3^{2n+2} - 8n - 9$ is divisible by 8.

It can be observed that P(n) is true for $n = 1$ since $3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8.

Let P(k) be true for some positive integer k, i.e.,

$3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore 3^{2k+2} - 8k - 9 = 8m$; where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$3^{2(k+1)+2} - 8\left(k+1\right) - 9$$
$$= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9$$
$$= 3^2\left(3^{2k+2} - 8k - 9 + 8k + 9\right) - 8k - 17$$
$$= 3^2\left(3^{2k+2} - 8k - 9\right) + 3^2\left(8k + 9\right) - 8k - 17$$
$$= 9.8m + 9\left(8k+9\right) - 8k - 17$$
$$= 9.8m + 72k + 81 - 8k - 17$$
$$= 9.8m + 64k + 64$$
$$= 8\left(9m + 8k + 8\right)$$

Therefore, $3^{2(k+1)+2} - 8\left(k+1\right) - 9$ is divisible by 8.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 23:

Prove the following by using the principle of mathematical induction for all $n \in N$: $41^n - 14^n$ is a multiple of 27.

Ans :

Let the given statement be P(n), i.e.,

P(n): $41^n - 14^n$ is a multiple of 27.

It can be observed that P(n) is true for $n = 1$ since $41^1 - 14^1 = 27$, which is a multiple of 27.

Let P(k) be true for some positive integer k, i.e.,

$41^k - 14^k$ is a multiple of 27

$\therefore 41^k - 14^k = 27m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$
\begin{aligned}
&41^{k+1} - 14^{k+1} \\
&= 41^k \cdot 41 - 14^k \cdot 14 \\
&= 41\left(41^k - 14^k + 14^k\right) - 14^k \cdot 14 \\
&= 41\left(41^k - 14^k\right) + 41.14^k - 14^k \cdot 14 \\
&= 41.27m + 14^k(41 - 14) \\
&= 41.27m + 27.14^k \\
&= 27\left(41m - 14^k\right) \\
&= 27 \times r, \text{ where } r = \left(41m - 14^k\right) \text{ is a natural number}
\end{aligned}
$$

Therefore, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 24:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

$(2n + 7) < (n + 3)^2$

Ans :

Let the given statement be P(n), i.e.,

P(n): $(2n + 7) < (n + 3)^2$

It can be observed that P(n) is true for $n = 1$ since $2.1 + 7 = 9 < (1 + 3)^2 = 16$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$(2k + 7) < (k + 3)^2$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$
\begin{aligned}
&\{2(k+1)+7\} = (2k+7)+2 \\
&\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2 + 2 \qquad [\text{using } (1)] \\
&2(k+1)+7 < k^2+6k+9+2 \\
&2(k+1)+7 < k^2+6k+11 \\
&\text{Now, } k^2+6k+11 < k^2+8k+16 \\
&\therefore 2(k+1)+7 < (k+4)^2 \\
&2(k+1)+7 < \{(k+1)+3\}^2
\end{aligned}
$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

*********** END **********