$$\Rightarrow \cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b_1} = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b_2} = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$\therefore |\vec{b_1}| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b_2}| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\vec{b_1} \cdot \vec{b_2} = (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k})$$
$$= 1 \cdot 3 - 1(-5) - 2(-4)$$
$$= 3 + 5 + 8$$
$$= 16$$

$$\cos Q = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{\sqrt{2} \cdot \sqrt{3} \cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 11:

Find the angle between the following pairs of lines:

(i) $\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3}$ and $\dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}$

(ii) $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ and $\dfrac{x-5}{4} = \dfrac{y-2}{1} = \dfrac{z-3}{8}$

   i.   Answer

   ii.   Let $\vec{b_1}$ and $\vec{b_2}$ be the vectors parallel to the pair of lines,

$$\dfrac{x-2}{2} = \dfrac{y-1}{5} = \dfrac{z+3}{-3} \text{ and } \dfrac{x+2}{-1} = \dfrac{y-4}{8} = \dfrac{z-5}{4}, \text{ respectively.}$$

$$\therefore \vec{b_1} = 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } \vec{b_2} = -\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\vec{b_1}| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$|\vec{b_2}| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\vec{b_1} \cdot \vec{b_2} = (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k})$$
$$= 2(-1) + 5 \times 8 + (-3) \cdot 4$$
$$= -2 + 40 - 12$$
$$= 26$$

The angle, Q, between the given pair of lines is given by the relation,

$$\cos Q = \left|\frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}||\vec{b_2}|}\right|$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$$

(ii) Let $\vec{b_1}, \vec{b_2}$ be the vectors parallel to the given pair of lines, $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{z}{1}$ and

$$\dfrac{x-5}{4} = \dfrac{y-5}{1} = \dfrac{z-3}{8}, \text{ respectively.}$$

$$\vec{b_1} = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b_2} = 4\hat{i} + \hat{j} + 8\hat{k}$$

$\therefore \left|\vec{b_1}\right| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$

$\left|\vec{b_2}\right| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$

$\vec{b_1} \cdot \vec{b_2} = \left(2\hat{i} + 2\hat{j} + \hat{k}\right) \cdot \left(4\hat{i} + \hat{j} + 8\hat{k}\right)$

$= 2 \times 4 + 2 \times 1 + 1 \times 8$

$= 8 + 2 + 8$

$= 18$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\dfrac{\vec{b_1} \cdot \vec{b_2}}{\left|\vec{b_1}\right|\left|\vec{b_2}\right|}\right|$

$\Rightarrow \cos Q = \dfrac{18}{3 \times 9} = \dfrac{2}{3}$

$\Rightarrow Q = \cos^{-1}\left(\dfrac{2}{3}\right)$

Question 12:

Find the values of p so the line $\dfrac{1-x}{3} = \dfrac{7y-14}{2p} = \dfrac{z-3}{2}$ and

$\dfrac{7-7x}{3p} = \dfrac{y-5}{1} = \dfrac{6-z}{5}$ are at right angles.

Answer

The given equations can be written in the standard form as

$\dfrac{x-1}{-3} = \dfrac{y-2}{\frac{2p}{7}} = \dfrac{z-3}{2}$ and $\dfrac{x-1}{-3p\over7} = \dfrac{y-5}{1} = \dfrac{z-6}{-5}$

The direction ratios of the lines are -3, $\dfrac{2p}{7}$, 2 and $\dfrac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore (-3) \cdot \left(\dfrac{-3p}{7}\right) + \left(\dfrac{2p}{7}\right) \cdot (1) + 2 \cdot (-5) = 0$

$\Rightarrow \dfrac{9p}{7} + \dfrac{2p}{7} = 10$

$\Rightarrow 11p = 70$

$\Rightarrow p = \dfrac{70}{11}$

Thus, the value of p is $\dfrac{70}{11}$.

Question 13:

Show that the lines $\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\dfrac{x-5}{7} = \dfrac{y+2}{-5} = \dfrac{z}{1}$ and $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$

$= 7 - 10 + 3$

$= 0$

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$ and

$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

Answer

The equations of the given lines are

$\vec{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} + \hat{k}\right)$

$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, is given by,

$\dfrac{\left|\left(\vec{b_1} \times \vec{b_2}\right) \cdot \left(\vec{a_2} - \vec{a_1}\right)\right|}{}$

$$d = \left| \frac{(\overline{}) \cdot (\overline{})}{|\vec{b_1} \times \vec{b_2}|} \right| \qquad \text{...(1)}$$

Comparing the given equations, we obtain

$$\vec{a_1} = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b_1} = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{a_2} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b_2} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a_2} - \vec{a_1} = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b_1} \times \vec{b_2} = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow |\vec{b_1} \times \vec{b_2}| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{\left(-3\hat{i} + 3\hat{k}\right) \cdot \left(\hat{i} - 3\hat{j} - 2\hat{k}\right)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-3.1 + 3(-2)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\dfrac{3\sqrt{2}}{2}$ units.

********** END *********