Trigonometric Functions Ex 5.1 Q26

We have,

$$T_n = \sin^n\theta + \cos^n\theta \qquad \text{---(i)}$$

To show: $\dfrac{T_3 - T_5}{T_1} = \dfrac{T_5 - T_7}{T_3}$

LHS $= \dfrac{T_3 - T_5}{T_1}$

$= \dfrac{\left(\sin^3\theta + \cos^3\theta\right) - \left(\sin^5\theta + \cos^5\theta\right)}{\sin\theta + \cos\theta}$ $\left[\begin{array}{l}\text{Substituting the values of}\\ T_3,\ T_5 \text{ and } T_1 \text{ from (i)}\end{array}\right]$

$= \dfrac{\sin^3\theta - \sin^5\theta + \cos^3\theta - \cos^5\theta}{\sin\theta + \cos\theta}$

$= \dfrac{\sin^3\theta\left(1 - \sin^2\theta\right) + \cos^3\theta\left(1 - \cos^2\theta\right)}{\sin\theta + \cos\theta}$

$= \dfrac{\sin^3\theta\cos^2\theta + \cos^3\theta\sin^2\theta}{\sin\theta + \cos\theta}$ $\left[\begin{array}{l}\because 1 - \sin^2\theta = \cos^2\theta\\ \text{and } 1 - \cos^2\theta = \sin^2\theta\end{array}\right]$

$= \dfrac{\sin^2\theta\cos^2\theta + \left(\sin\theta + \cos\theta\right)}{\sin\theta + \cos\theta}$

$= \sin^2\theta\cos^2\theta$

RHS $= \dfrac{\sin^5\theta + \cos^5\theta - \left(\sin^7\theta + \cos^7\theta\right)}{\sin^3\theta + \cos^3\theta}$

$= \dfrac{\sin^5\theta - \sin^7\theta + \cos^5\theta - \cos^7\theta}{\sin^3\theta + \cos^3\theta}$

$= \dfrac{\sin^5\theta\left(1 - \sin^2\theta\right) + \cos^5\theta\left(1 - \cos^2\theta\right)}{\sin^3\theta + \cos^3\theta}$

$= \dfrac{\sin^5\theta\cos^2\theta + \cos^5\theta\sin^2\theta}{\sin^3\theta + \cos^3\theta}$

$= \dfrac{\sin^2\theta\cos^2\theta\left(\sin^3\theta + \cos^3\theta\right)}{\sin^3\theta + \cos^3\theta}$

$= \sin^2\theta\cos^2\theta$

$\therefore$ LHS = RHS Proved.

LHS $= 2T_6 - 3T_4 + 1$

$= 2\left(\sin^6\theta + \cos^6\theta\right) - 3\left(\sin^4\theta + \cos^4\theta\right) + 1$

$= 2\left(\left(\sin^2\theta\right)^3 + \left(\cos^2\theta\right)^3 - 3\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2\right) + 1$

$= 2\left(\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2 - \left(\sin^2\theta\cos^2\theta\right)\right) -$

$3\left(\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\right) + 1$

$\left[\begin{array}{l}\text{Using } a^3 + b^3 = (a+b)\left(a^2 + b^2 - ab\right) \text{ and adding and subtracting}\\ 2\sin^2\theta\cos^2\theta\end{array}\right]$

$= 2\left(\left(\sin^2\theta + \cos^2\theta\right)^2 - 3\sin^2\theta\cos^2\theta\right) - 3\left(1 - 2\sin^2\theta\cos^2\theta\right) + 1$

$= 2\left(1 - 3\sin^2\theta\cos^2\theta\right) - 3 + 6\sin^2\theta\cos^2\theta + 1$

$= 2 - 6\sin^2\theta\cos^2\theta - 2 + 6\sin^2\theta\cos^2\theta$

$= 0$

$=$ RHS Proved.

LHS $= 6T_{10} - 15T_8 + 10T_6 - 1$

$$= 6\left(\sin^{10}\theta + \cos^{10}\theta\right) - 15\left(\sin^8\theta + \cos^8\theta\right) + 10\left(\sin^6\theta + \cos^6\theta\right) - 1$$

$$= 6\sin^{10}\theta - 15\sin^8\theta + 10\sin^6\theta + 6\cos^{10}\theta - 15\cos^8\theta + 10\cos^6\theta - 1$$

$$= \sin^6\theta\left(6\sin^4\theta - 15\sin^2\theta + 10\right) + \cos^6\theta\left(6\cos^4\theta - 15\cos^2\theta + 10\right) - \left(\sin^2\theta + \cos^2\theta\right)^3$$

$\left[\because 1 = \sin^2\theta + \cos^2\theta\right]$

$$= \sin^6\theta\left(6\sin^4\theta - 15\sin^2\theta + 10\right) + \cos^6\theta\left(6\cos^4\theta - 15\cos^2\theta + 10\right) -$$
$$\left(\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta\left(\sin^2\theta + \cos^2\theta\right)\right)$$

$\left[\text{Using } (a+b)^3 = a^3 + b^3 + 3ab(a+b)\right]$

$$= \sin^6\theta\left(6\sin^4\theta - 15\sin^2\theta + 10 - 1\right) + \cos^6\theta\left(6\cos^4\theta - 15\cos^2\theta + 10 - 1\right) - 3\sin^2\theta\cos^2\theta \times 1$$

$\left[\because \cos^2\theta + \sin^2\theta = 1\right]$

$$= \sin^6\theta\left(6\sin^4\theta - 9\sin^2\theta - 6\sin^2\theta + 9\right) + \cos^6\theta\left(6\cos^4\theta - 9\cos^2\theta - 6\cos^2\theta + 9\right) - 3\sin^2\theta\cos^2\theta$$

[On splitting the middle term]

$$= \sin^6\theta\left(3\sin^2\theta\left(2\sin^2\theta - 3\right) - 3\left(2\sin^2\theta - 3\right)\right) + \cos^6\theta\left(3\cos^2\theta\left(2\cos^2\theta - 3\right) - 3\left(2\cos^2\theta - 3\right)\right)$$
$$-3\sin^2\theta\cos^2\theta$$

$$= \sin^6\theta\left(2\sin^2\theta - 3\right)\left(3\sin^2\theta - 3\right) + \cos^6\theta\left(2\cos^2\theta - 3\right)\left(3\cos^2\theta - 3\right) - 3\sin^2\theta\cos^2\theta$$

$$= \sin^6\theta \times (-3)\left(2\sin^2\theta - 3\right)\left(1 - \sin^2\theta\right) + \cos^6\theta \times (-3)\left(2\cos^2\theta - 3\right)\left(1 - \cos^2\theta\right) - 3\sin^2\theta\cos^2\theta$$

$$= -3\sin^6\theta\left(2\sin^2\theta - 3\right)\cos^2\theta - 3\cos^6\theta\left(2\cos^2\theta - 3\right)\sin^2\theta - 3\sin^2\theta\cos^2\theta$$

$$= 6\sin^8\theta + \cos^2\theta + 6\sin^6\theta\cos^2\theta - 6\cos^8\theta\sin^2\theta + 9\cos^6\theta + \sin^2\theta - 3\sin^2\theta\cos^2\theta$$

$$= -6\sin^2\theta + \cos^2\theta\left(\sin^6\theta + \cos^6\theta\right) + 9\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right) - 3\sin^2\theta\cos^2\theta$$

$$= -6\sin^2\theta\cos^2\theta\left(\left(\sin^2\theta\right)^3 + \left(\cos^2\theta\right)^3\right) + 9\sin^2\theta\cos^2\theta\left(\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2\right) - 3\sin^2\theta\cos^2\theta$$

$$= -6\sin^2\theta\cos^2\theta\left(\sin^2\theta + \cos^2\theta\right)\left(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta\right)$$
$$+ 9\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right) - 3\sin^2\theta\cos^2\theta$$

$\left(\text{Using } a^3 + b^3 (a+b)\left(a^2 + b^2 - ab\right)\right)$

$$= -6\sin^2\theta\cos^2\theta\left(\sin^4\theta\cos^4\theta - \sin^2\theta\cos^2\theta\right) + 9\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right)$$
$$- 3\sin^2\theta\cos^2\theta \quad \left(\because \cos^2\theta + \sin^2\theta = 1\right)$$

$$= -6\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right) + 6\sin^4\theta\cos^4\theta + 9\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right) - 3\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta\cos^2\theta\left(\sin^4\theta + \cos^4\theta\right) + 6\sin^4\theta\cos^4\theta - 3\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta\cos^2\theta\left(\left(\sin^2\theta\right)^2 + \left(\cos^2\theta\right)^2 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta\right)$$
$$+ 6\sin^4\theta\cos^4\theta - 3\sin^2\theta\cos^2\theta \qquad \left(\text{adding and subtracting } 2\sin^2\theta\cos^2\theta\right)$$

$$= 3\sin^2\theta\cos^2\theta\left(\left(\sin^2\theta + \cos^2\theta\right)^2 - 2\sin^2\theta\cos^2\theta\right) + 6\sin 4\theta\cos^4\theta - 3\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta\cos^2\theta\left(1 - 2\sin^2\theta\cos^2\theta\right) + 6\sin^4\theta\cos^4\theta - 3\sin^2\theta\cos^2\theta$$

$$= 3\sin^2\theta\cos^2\theta - 6\sin^4\theta\cos^4\theta + 6\sin^4\theta\cos^4\theta - 3\sin^2\theta\cos^2\theta$$

$= 0$

= RHS

Proved

*********** END **********