Algebra of Matrices Ex 5.3 Q60

Given,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To prove, $A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$, we will use the principle of mathematical induction.

Step 1: Put $n = 1$

$$A^1 = \begin{bmatrix} 1 & 1 & \frac{1(1+1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So, A^n is true for $n = 1$

Step 2: Let, A^n be true for $n = k$, so,

$$A^k = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3: We will prove that A^n be true for $n = k+1$

Now,

$$A^{k+1} = A^k \times A$$

$$= \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{\text{using equation (i) and given}\}$$

$$= \begin{bmatrix} 1+0+0 & 1+k+0 & 1+k+\frac{k(k+1)}{2} \\ 0+0+0 & 0+1+0 & 0+1+k \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, A^n is true for $n = k+1$ whenever it is true for $n = k$.

So, by principle of mathematical induction A^n is true for all positive integer n.

Algebra of Matrices Ex 5.3 Q61

We will prove P(n): $A^{n+1} = B^n\left[B + (n+1)C\right]$ is true for all natural numbers using mathematical induction.

Given,

$A = B + C,\ BC = CB,\ C^2 = 0$

$A = B + C$

Squaring both the sides, so

$A^2 = (B + C)^2$

$\Rightarrow\quad A^2 = (B + C)(B + C)$

$\Rightarrow\quad A^2 = B \times B + BC + CB + C \times C$ {using distributive property}

$\Rightarrow\quad A^2 = B^2 + BC + BC + C^2$ {using $BC = CB$ given}

$\Rightarrow\quad A^2 = B^2 + 2BC + 0$ {since, given $C^2 = 0$}

$\Rightarrow\quad A^2 = B^2 + 2BC$ ---(1)

$A^2 = B(B + 2C)$

Now, consider

$$P(n):\ A^{n+1} = B^n\left[B + (n+1)C\right]$$

Step 1: Tp prove P(1) is true, put $n = 1$

$$A^{1+1} = B^1\left[B + (1+1)C\right]$$
$$A^2 = B\left[B + 2C\right]$$
$$A^2 = B^2 + 2BC$$

From equation (i), P(1) is true.

Step 2: Suppose P(k) is true.

$$\therefore A^{k+1} = B^k\left[B + (k+1)C\right] \qquad \text{---}(2)$$

Step 3: Now, we have to show that P(k+1) is true.

That is we need to prove that,

$$A^{k+2} = B^{k+1}\left[B + (k+2)C\right]$$

Now,

$A^{k+2} = A^k \times A^2$

$= B^{(k-1)}\left[B + kC\right] \times \left[B(B + 2C)\right]$

$= B^k\left[B + kC\right] \times \left[B + 2C\right]$

$= B^k\left[B \times B + B \times 2C + kC \times B + 2kC^2\right]$

$= B^k\left[B^2 + 2BC + kBC + 2k \times 0\right]$ {since $BC = CB,\ C^2 = 0$}

$= B^k\left[B^2 + BC(2 + k)\right]$

$= B^k \times B\left[B + (k+2)C\right]$

$= B^{k+1}\left[B + (k+2)C\right]$

So, $P(n)$ is true for $n = k + 1$ whenever $P(n)$ is true for $n = k$

Therefore by principle of mathematical induction $P(n)$ is true for all natural number.

*********** END **********