$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad \text{...(i)}$$

$$\text{Now, } \cot^{-1}\frac{3}{2} = \tan^{-1}\frac{2}{3} \qquad \text{...(ii)} \qquad \left[\tan^{-1}\frac{1}{x} = \cot^{-1}x\right]$$

$$\text{Hence, } \tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{2}{3}\right) \qquad \text{[Using (i) and (ii)]}$$

$$= \tan\left(\tan^{-1}\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \cdot \frac{2}{3}}\right) \qquad \left[\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan\left(\tan^{-1}\frac{9+8}{12-6}\right)$$

$$= \tan\left(\tan^{-1}\frac{17}{6}\right) = \frac{17}{6}$$

Question 19:

Find the values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is equal to

(A) $\frac{7\pi}{6}$ (B) $\frac{5\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\frac{7\pi}{6} \notin x \in [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{-7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right] \quad \left[\cos(2\pi + x) = \cos x\right]$$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right] \text{ where } \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$$

The correct answer is B.

Question 20:

Find the values of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) 1

Answer

Let $\sin^{-1}\left(\frac{-1}{2}\right) = x$. Then, $\sin x = \frac{-1}{2} = -\sin\frac{\pi}{6} = \sin\left(\frac{-\pi}{6}\right)$.

We know that the range of the principal value branch of $\sin^{-1}$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore \sin^{-1}\left(\frac{-1}{2}\right) = \frac{-\pi}{6}$$

$$\therefore \sin\left(\frac{\pi}{3} - \sin^{-1}\left(\frac{-1}{2}\right)\right) = \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{3\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

The correct answer is D.

Miscellaneous Solutions

Question 1:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right)$$

Find the value of $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$

Answer

We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1}x$.

Here, $\dfrac{13\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right)$ can be written as:

$\cos^{-1}\left(\cos\dfrac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi + \dfrac{\pi}{6}\right)\right] = \cos^{-1}\left[\cos\left(\dfrac{\pi}{6}\right)\right]$, where $\dfrac{\pi}{6} \in [0, \pi]$.

$\therefore \cos^{-1}\left(\cos\dfrac{13\pi}{6}\right) = \cos^{-1}\left[\cos\left(\dfrac{\pi}{6}\right)\right] = \dfrac{\pi}{6}$

Question 2:

Find the value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$

Answer

We know that $\tan^{-1}(\tan x) = x$ if $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, which is the principal value branch of $\tan^{-1}x$.

Here, $\dfrac{7\pi}{6} \notin \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Now, $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$ can be written as:

$\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right) = \tan^{-1}\left[\tan\left(2\pi - \dfrac{5\pi}{6}\right)\right]$ $\left[\tan(2\pi - x) = -\tan x\right]$

$= \tan^{-1}\left[-\tan\left(\dfrac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(-\dfrac{5\pi}{6}\right)\right] = \tan^{-1}\left[\tan\left(\pi - \dfrac{5\pi}{6}\right)\right]$

$= \tan^{-1}\left[\tan\left(\dfrac{\pi}{6}\right)\right]$, where $\dfrac{\pi}{6} \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$\therefore \tan^{-1}\left(\tan\dfrac{7\pi}{6}\right) = \tan^{-1}\left(\tan\dfrac{\pi}{6}\right) = \dfrac{\pi}{6}$

Question 3:

Prove $2\sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{24}{7}$

Answer

Let $\sin^{-1}\dfrac{3}{5} = x$. Then, $\sin x = \dfrac{3}{5}$.

$\Rightarrow \cos x = \sqrt{1 - \left(\dfrac{3}{5}\right)^2} = \dfrac{4}{5}$

$\therefore \tan x = \dfrac{3}{4}$

$\therefore x = \tan^{-1}\dfrac{3}{4} \Rightarrow \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{3}{4}$

Now, we have:

$\text{L.H.S.} = 2\sin^{-1}\dfrac{3}{5} = 2\tan^{-1}\dfrac{3}{4}$

$= \tan^{-1}\left(\dfrac{2\times\dfrac{3}{4}}{1 - \left(\dfrac{3}{4}\right)^2}\right)$ $\left[2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}\right]$

$= \tan^{-1}\left(\dfrac{\dfrac{3}{2}}{\dfrac{16-9}{16}}\right) = \tan^{-1}\left(\dfrac{3}{2}\times\dfrac{16}{7}\right)$

$= \tan^{-1}\dfrac{24}{7} = \text{R.H.S.}$

Question 4:

Prove $\sin^{-1}\dfrac{8}{17} + \sin^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{77}{36}$

Answer

Let $\sin^{-1}\dfrac{8}{17} = x$. Then, $\sin x = \dfrac{8}{17} \Rightarrow \cos x = \sqrt{1 - \left(\dfrac{8}{17}\right)^2} = \sqrt{\dfrac{225}{289}} = \dfrac{15}{17}$.

$\therefore \tan x = \dfrac{8}{15} \Rightarrow x = \tan^{-1}\dfrac{8}{15}$

$$\therefore \sin^{-1}\frac{8}{17} = \tan^{-1}\frac{8}{15} \qquad \ldots(1)$$

Now, let $\sin^{-1}\frac{3}{5} = y$. Then, $\sin y = \frac{3}{5} \Rightarrow \cos y = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}.$

$$\therefore \tan y = \frac{3}{4} \Rightarrow y = \tan^{-1}\frac{3}{4}$$

$$\therefore \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(2)$$

Now, we have:

$$\begin{aligned}
\text{L.H.S.} &= \sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} \\
&= \tan^{-1}\frac{8}{15} + \tan^{-1}\frac{3}{4} \qquad \left[\text{Using (1) and (2)}\right] \\
&= \tan^{-1}\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}} \\
&= \tan^{-1}\left(\frac{32+45}{60-24}\right) \qquad \left[\tan^{-1}x+\tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{77}{36} = \text{R.H.S.}
\end{aligned}$$

Question 5:

$$\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$$

Prove

Answer

Let $\cos^{-1}\frac{4}{5} = x$. Then, $\cos x = \frac{4}{5} \Rightarrow \sin x = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}.$

$$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1}\frac{3}{4}$$

$$\therefore \cos^{-1}\frac{4}{5} = \tan^{-1}\frac{3}{4} \qquad \ldots(1)$$

Now, let $\cos^{-1}\frac{12}{13} = y$. Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}.$

$$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1}\frac{5}{12}$$

$$\therefore \cos^{-1}\frac{12}{13} = \tan^{-1}\frac{5}{12} \qquad \ldots(2)$$

Let $\cos^{-1}\frac{33}{65} = z$. Then, $\cos z = \frac{33}{65} \Rightarrow \sin z = \frac{56}{65}.$

$$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1}\frac{56}{33}$$

$$\therefore \cos^{-1}\frac{33}{65} = \tan^{-1}\frac{56}{33} \qquad \ldots(3)$$

Now, we will prove that:

$$\begin{aligned}
\text{L.H.S.} &= \cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} \\
&= \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{5}{12} \qquad \left[\text{Using (1) and (2)}\right] \\
&= \tan^{-1}\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}} \qquad \left[\tan^{-1}x+\tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right] \\
&= \tan^{-1}\frac{36+20}{48-15} \\
&= \tan^{-1}\frac{56}{33} \\
&= \tan^{-1}\frac{56}{33} \qquad \left[\text{by (3)}\right] \\
&= \text{R.H.S.}
\end{aligned}$$

********** END *********