NCERT MISCELLANEOUS SOLUTIONS

Question 1:

Show that the sum of $(m + n)^{\text{th}}$ and $(m - n)^{\text{th}}$ terms of an A.P. is equal to twice the mth term.

Ans:

Let a and d be the first term and the common difference of the A.P. respectively.

It is known that the kth term of an A. P. is given by

$a_k = a + (k - 1) d$

$a_{m+n} = a + (m + n - 1) d$

$a_{m-n} = a + (m - n - 1) d$

$a_m = a + (m - 1) d$

$a_{m+n} + a_{m-n} = a + (m + n - 1) d + a + (m - n - 1) d$

$= 2a + (m + n - 1 + m - n - 1) d$

$= 2a + (2m - 2) d$

$= 2a + 2 (m - 1) d$

$= 2 [a + (m - 1) d]$

$= 2a_m$

Thus, the sum of $(m + n)^{\text{th}}$ and $(m - n)^{\text{th}}$ terms of an A.P. is equal to twice the mth term.

Question 2:

If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Ans:

Let the three numbers in A.P. be $a - d$, a, and $a + d$.

According to the given information,

$(a - d) + (a) + (a + d) = 24$ ... (1)

$3a = 24$

$a = 8$

$(a - d) a (a + d) = 440$ ... (2)

$(8 - d) (8) (8 + d) = 440$

$(8 - d) (8 + d) = 55$

$64 - d^2 = 55$

$d^2 = 64 - 55 = 9$

$d = \pm 3$

Therefore, when $d = 3$, the numbers are 5, 8, and 11 and when $d = -3$, the numbers are 11, 8, and 5.

Thus, the three numbers are 5, 8, and 11.

Question 3:

Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3 (S_2 - S_1)$

Ans:

Let a and b be the first term and the common difference of the A.P. respectively.

Therefore,

$$S_1 = \frac{n}{2}[2a + (n-1)d] \qquad \text{...(1)}$$

$$S_2 = \frac{2n}{2}\left[2a + (2n-1)d\right] = n\left[2a + (2n-1)d\right] \qquad \text{...(2)}$$

$$S_3 = \frac{3n}{2}\left[2a + (3n-1)d\right] \qquad \text{...(3)}$$

From (1) and (2), we obtain

$$S_2 - S_1 = n\left[2a + (2n-1)d\right] - \frac{n}{2}\left[2a + (n-1)d\right]$$

$$= n\left\{\frac{4a + 4nd - 2d - 2a - nd + d}{2}\right\}$$

$$= n\left[\frac{2a + 3nd - d}{2}\right]$$

$$= \frac{n}{2}\left[2a + (3n-1)d\right]$$

$$\therefore 3(S_2 - S_1) = \frac{3n}{2}\left[2a + (3n-1)d\right] = S_3 \qquad \text{[From (3)]}$$

Question 4:

Find the sum of all numbers between 200 and 400 which are divisible by 7.

Ans:

The numbers lying between 200 and 400, which are divisible by 7, are

203, 210, 217, ... 399

First term, $a = 203$

Last term, $l = 399$

Common difference, $d = 7$

Let the number of terms of the A.P. be n.

$a_n = 399 = a + (n - 1) d$

$399 = 203 + (n - 1) 7$

$7 (n - 1) = 196$

$n - 1 = 28$

$n = 29$

Question 5:

Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Ans:

The integers from 1 to 100, which are divisible by 2, are 2, 4, 6... 100.

This forms an A.P. with both the first term and common difference equal to 2.

$100 = 2 + (n - 1) 2$

$n = 50$

$$\therefore 2 + 4 + 6 + \ldots + 100 = \frac{50}{2} \left[2(2) + (50 - 1)(2) \right]$$
$$= \frac{50}{2} [4 + 98]$$
$$= (25)(102)$$
$$= 2550$$

The integers from 1 to 100, which are divisible by 5, are 5, 10... 100.

This forms an A.P. with both the first term and common difference equal to 5.

$100 = 5 + (n - 1) 5$

$5n = 100$

$n = 20$

$$\therefore 5+10+...+100 = \frac{20}{2}\left[2(5)+(20-1)5\right]$$
$$= 10\left[10+(19)5\right]$$
$$= 10\left[10+95\right] = 10 \times 105$$
$$= 1050$$

The integers, which are divisible by both 2 and 5, are 10, 20, ... 100.

This also forms an A.P. with both the first term and common difference equal to 10.

100 = 10 + (n −1) (10)

100 = 10n

n = 10

$$\therefore 10+20+...+100 = \frac{10}{2}\left[2(10)+(10-1)(10)\right]$$
$$= 5\left[20+90\right] = 5(110) = 550$$

Required sum = 2550 + 1050 – 550 = 3050

Thus, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

Question 6:

Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

Ans:

The two-digit numbers, which when divided by 4, yield 1 as remainder, are

13, 17, ... 97.

This series forms an A.P. with first term 13 and common difference 4.

Let n be the number of terms of the A.P.

It is known that the nth term of an A.P. is given by, $a_n = a + (n-1) d$

97 = 13 + (n −1) (4)

4 (n −1) = 84

n − 1 = 21

n = 22

Sum of n terms of an A.P. is given by,

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$\therefore S_{22} = \frac{22}{2}\left[22(13)+(22-1)(4)\right]$$

$$= 11\left[26+84\right]$$

$$= 1210$$

Thus, the required sum is 1210.

Question 7:

If f is a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.

Ans:

It is given that,

$f(x + y) = f(x) \times f(y)$ for all $x, y \in N$ ... (1)

$f(1) = 3$

Taking $x = y = 1$ in (1), we obtain

$f(1 + 1) = f(2) = f(1)f(1) = 3 \times 3 = 9$

Similarly,

$f(1 + 1 + 1) = f(3) = f(1 + 2) = f(1)f(2) = 3 \times 9 = 27$

$f(4) = f(1 + 3) = f(1)f(3) = 3 \times 27 = 81$

$f(1), f(2), f(3), ...,$ that is $3, 9, 27, ...,$ forms a G.P. with both the first term and common ratio equal to 3.

It is known that, $S_n = \dfrac{a(r^n - 1)}{r - 1}$

It is given that, $\displaystyle\sum_{x=1}^{n} f(x) = 120$

$\therefore 120 = \dfrac{3(3^n - 1)}{3 - 1}$

$$\Rightarrow 120 = \frac{3}{2}(3^n - 1)$$

$$\Rightarrow 3^n - 1 = 80$$

$$\Rightarrow 3^n = 81 = 3^4$$

$$\therefore n = 4$$

Thus, the value of n is 4.

Question 8:

The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

Ans:

Let the sum of n terms of the G.P. be 315.

It is known that, $S_n = \dfrac{a(r^n - 1)}{r - 1}$

It is given that the first term a is 5 and common ratio r is 2.

$\therefore 315 = \dfrac{5(2^n - 1)}{2 - 1}$

$\Rightarrow 2^n - 1 = 63$

$\Rightarrow 2^n = 64 = (2)^6$

$\Rightarrow n = 6$

Last term of the G.P = 6th term = $ar^{6-1} = (5)(2)^5 = (5)(32) = 160$

Thus, the last term of the G.P. is 160.

Question 9:

The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Ans:

Let a and r be the first term and the common ratio of the G.P. respectively.

$a = 1$

$a_3 = ar^2 = r^2$

$a_5 = ar^4 = r^4$

$r^2 + r^4 = 90$

$r^4 + r^2 - 90 = 0$

$$\Rightarrow r^2 = \frac{-1 + \sqrt{1 + 360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = \frac{-1 \pm 19}{2} = -10 \text{ or } 9$$

$$\therefore r = \pm 3 \qquad \qquad (\text{Taking real roots})$$

Thus, the common ratio of the G.P. is ±3.

Question 10:

The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Ans:

Let the three numbers in G.P. be a, ar, and ar^2.

From the given condition, $a + ar + ar^2 = 56$

$a (1 + r + r^2) = 56$

$$\Rightarrow a = \frac{56}{1 + r + r^2} \quad \dots (1)$$

$a - 1$, $ar - 7$, $ar^2 - 21$ forms an A.P.

$(ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$

$ar - a - 6 = ar^2 - ar - 14$

$ar^2 - 2ar + a = 8$

$ar^2 - ar - ar + a = 8$

$a(r^2 + 1 - 2r) = 8$

$a (r - 1)^2 = 8 \dots (2)$

$$\Rightarrow \frac{56}{1 + r + r^2}(r - 1)^2 = 8 \qquad \qquad [\text{Using (1)}]$$

$7(r^2 - 2r + 1) = 1 + r + r^2$

$7r^2 - 14r + 7 - 1 - r - r^2 = 0$

$6r^2 - 15r + 6 = 0$

$6r^2 - 12r - 3r + 6 = 0$

$6r\,(r-2) - 3\,(r-2) = 0$

$(6r-3)\,(r-2) = 0$

$\therefore r = 2, \dfrac{1}{2}$

When $r = 2$, $a = 8$

When $r = \dfrac{1}{2}, a = 32$

Therefore, when $r = 2$, the three numbers in G.P. are 8, 16, and 32.

When $r = \dfrac{1}{2}$, the three numbers in G.P. are 32, 16, and 8.

Thus, in either case, the three required numbers are 8, 16, and 32.

Question 11:
A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Ans:

Let the G.P. be $T_1, T_2, T_3, T_4, \ldots T_{2n}$.

Number of terms $= 2n$

According to the given condition,

$T_1 + T_2 + T_3 + \ldots + T_{2n} = 5\,[T_1 + T_3 + \ldots + T_{2n-1}]$

$T_1 + T_2 + T_3 + \ldots + T_{2n} - 5\,[T_1 + T_3 + \ldots + T_{2n-1}] = 0$

$T_2 + T_4 + \ldots + T_{2n} = 4\,[T_1 + T_3 + \ldots + T_{2n-1}]$

Let the G.P. be $a, ar, ar^2, ar^3, \ldots$

$\therefore \dfrac{ar\left(r^n - 1\right)}{r-1} = \dfrac{4 \times a\left(r^n - 1\right)}{r-1}$

$\Rightarrow ar = 4a$

$\Rightarrow r = 4$

Thus, the common ratio of the G.P. is 4.

Question 12:
The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Ans:

Let the A.P. be $a, a + d, a + 2d, a + 3d, \ldots a + (n - 2) d, a + (n - 1)d$.

Sum of first four terms $= a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d$

Sum of last four terms $= [a + (n - 4) d] + [a + (n - 3) d] + [a + (n - 2) d] + [a + n - 1) d]$

$= 4a + (4n - 10) d$

According to the given condition,

$4a + 6d = 56$

$4(11) + 6d = 56$ [Since $a = 11$ (given)]

$6d = 12$

$d = 2$

$4a + (4n - 10) d = 112$

$4(11) + (4n - 10)2 = 112$

$(4n - 10)2 = 68$

$4n - 10 = 34$

$4n = 44$

$n = 11$

Thus, the number of terms of the A.P. is 11.

Question 13:

If $\dfrac{a + bx}{a - bx} = \dfrac{b + cx}{b - cx} = \dfrac{c + dx}{c - dx} (x \neq 0)$, then show that a, b, c and d are in G.P.

Ans:

It is given that,

$$\frac{a + bx}{a - bx} = \frac{b + cx}{b - cx}$$

$$\Rightarrow (a + bx)(b - cx) = (b + cx)(a - bx)$$

$$\Rightarrow ab - acx + b^2x - bcx^2 = ab - b^2x + acx - bcx^2$$

$$\Rightarrow 2b^2x = 2acx$$

$$\Rightarrow b^2 = ac$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} \qquad \ldots(1)$$

Also, $\dfrac{b + cx}{b - cx} = \dfrac{c + dx}{c - dx}$

$$\Rightarrow (b + cx)(c - dx) = (b - cx)(c + dx)$$

$$\Rightarrow bc - bdx + c^2x - cdx^2 = bc + bdx - c^2x - cdx^2$$

$$\Rightarrow 2c^2x = 2bdx$$

$$\Rightarrow c^2 = bd$$

$$\Rightarrow \frac{c}{d} = \frac{d}{c} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Thus, a, b, c, and d are in G.P.

Question 14:
Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2 R^n = S^n$

Ans:

Let the G.P. be a, ar, ar^2, ar^3, ... ar^{m-1}...

According to the given information,

$$S = \frac{a(r^n - 1)}{r - 1}$$

$$P = a^n \times r^{1+2+...+n-1}$$

$$= a^n r^{\frac{n(n-1)}{2}} \qquad \left[\because \text{ Sum of first } n \text{ natural numbers is } n\frac{(n+1)}{2} \right]$$

$$R = \frac{1}{a} + \frac{1}{ar} + ... + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + ...r + 1}{ar^{n-1}}$$

$$= \frac{1(r^n - 1)}{(r - 1)} \times \frac{1}{ar^{n-1}} \qquad \left[\because 1, r, ... r^{n-1} \text{ forms a G.P} \right]$$

$$= \frac{r^n - 1}{ar^{n-1}(r - 1)}$$

$$\therefore P^2 R^n = a^{2n} r^{n(n-1)} \frac{(r^n - 1)^n}{a^n r^{n(n-1)} (r - 1)^n}$$

$$= \frac{a^n (r^n - 1)^n}{(r - 1)^n}$$

$$= \left[\frac{a(r^n - 1)}{(r - 1)} \right]^n$$

$$= S^n$$

Hence, $P^2 R^n = S^n$

Question 15:
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Ans:

Let t and d be the first term and the common difference of the A.P. respectively.

The nth term of an A.P. is given by, $a_n = t + (n - 1)d$

Therefore,

$a_p = t + (p - 1)d = a$ ... (1)

$a_q = t + (q - 1)d = b$ ... (2)

$a_r = t + (r - 1)d = c$ ... (3)

Subtracting equation (2) from (1), we obtain

$(p - 1 - q + 1)d = a - b$

$(p - q)d = a - b$

$$\therefore d = \frac{a - b}{p - q} \qquad\qquad ...(4)$$

Subtracting equation (3) from (2), we obtain

$(q - 1 - r + 1)d = b - c$

$(q - r)d = b - c$

$$\Rightarrow d = \frac{b - c}{q - r} \qquad\qquad ...(5)$$

Equating both the values of d obtained in (4) and (5), we obtain

$$\frac{a - b}{p - q} = \frac{b - c}{q - r}$$
$$\Rightarrow (a - b)(q - r) = (b - c)(p - q)$$
$$\Rightarrow aq - bq - ar + br = bp - bq - cp + cq$$
$$\Rightarrow bp - cp + cq - aq + ar - br = 0$$
$$\Rightarrow (-aq + ar) + (bp - br) + (-cp + cq) = 0 \qquad \text{(By rearranging terms)}$$
$$\Rightarrow -a(q - r) - b(r - p) - c(p - q) = 0$$
$$\Rightarrow a(q - r) + b(r - p) + c(p - q) = 0$$

Thus, the given result is proved.

Question 16:

If $a\left(\dfrac{1}{b} + \dfrac{1}{c}\right), b\left(\dfrac{1}{c} + \dfrac{1}{a}\right), c\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Ans:

It is given that $a\left(\dfrac{1}{b}+\dfrac{1}{c}\right), b\left(\dfrac{1}{c}+\dfrac{1}{a}\right), c\left(\dfrac{1}{a}+\dfrac{1}{b}\right)$ are in A.P.

$$\therefore b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$

$$\Rightarrow \frac{b(a+c)}{ac}-\frac{a(b+c)}{bc}=\frac{c(a+b)}{ab}-\frac{b(a+c)}{ac}$$

$$\Rightarrow \frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$

$$\Rightarrow b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$

$$\Rightarrow ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$

$$\Rightarrow ab(b-a)+c(b-a)(b+a)=a(c-b)(c+b)+bc(c-b)$$

$$\Rightarrow (b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$

$$\Rightarrow b-a=c-b$$

Thus, a, b, and c are in A.P.

Question 17:

If a, b, c, d are in G.P, prove that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P.

Ans:

It is given that $a, b, c,$ and d are in G.P.

$b^2 = ac$ ... (1)

$c^2 = bd$ ... (2)

$ad = bc$ ... (3)

It has to be proved that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P. i.e.,

$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$

Consider L.H.S.

$(b^n + c^n)^2 = b^{2n} + 2b^n c^n + c^{2n}$

$= (b^2)^n + 2b^n c^n + (c^2)^n$

$= (ac)^n + 2b^n c^n + (bd)^n$ [Using (1) and (2)]

$= a^n c^n + b^n c^n + b^n c^n + b^n d^n$

$= a^n c^n + b^n c^n + a^n d^n + b^n d^n$ [Using (3)]

$= c^n (a^n + b^n) + d^n (a^n + b^n)$

$= (a^n + b^n)(c^n + d^n)$

$=$ R.H.S.

$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$

Thus, $(a^n + b^n), (b^n + c^n),$ and $(c^n + d^n)$ are in G.P.

Question 18:

If a and b are the roots of $x^2-3x+p=0$ and c, d are roots of $x^2-12x+q=0$, where a, b, c, d, form a G.P. Prove that $(q+p):(q-p)=17:15$.

Ans:

It is given that a and b are the roots of $x^2 - 3x + p = 0$

$a + b = 3$ and $ab = p$ ... (1)

Also, c and d are the roots of $x^2 - 12x + q = 0$

$c + d = 12$ and $cd = q$ ... (2)

It is given that a, b, c, d are in G.P.

Let $a = x, b = xr, c = xr^2, d = xr^3$

From (1) and (2), we obtain

$x + xr = 3$

$x(1 + r) = 3$

$xr^2 + xr^3 = 12$

$xr^2(1 + r) = 12$

On dividing, we obtain

$$\frac{xr^2(1+r)}{x(1+r)} = \frac{12}{3}$$
$$\Rightarrow r^2 = 4$$
$$\Rightarrow r = \pm 2$$

When $r = 2, x = \dfrac{3}{1+2} = \dfrac{3}{3} = 1$

When $r = -2, x = \dfrac{3}{1-2} = \dfrac{3}{-1} = -3$

Case I:

When $r = 2$ and $x = 1$,

$ab = x^2r = 2$

$cd = x^2r^5 = 32$

$$\therefore \frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$

i.e., $(q+p):(q-p) = 17:15$

Case II:

When $r = -2$, $x = -3$,

$ab = x^2r = -18$

$cd = x^2r^5 = -288$

$$\therefore \frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$

i.e., $(q+p):(q-p) = 17:15$

Thus, in both the cases, we obtain $(q+p):(q-p) = 17:15$

Question 19:

The ratio of the A.M and G.M. of two positive numbers a and b, is $m:n$. Show that $a:b = \left(m + \sqrt{m^2 - n^2}\right):\left(m - \sqrt{m^2 - n^2}\right)$.

Ans:

Let the two numbers be a and b.

$$\text{A.M} = \frac{a+b}{2} \text{ and G.M.} = \sqrt{ab}$$

According to the given condition,

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

$$\Rightarrow \frac{(a+b)^2}{4(ab)} = \frac{m^2}{n^2}$$

$$\Rightarrow (a+b)^2 = \frac{4ab\,m^2}{n^2}$$

$$\Rightarrow (a+b) = \frac{2\sqrt{ab}\,m}{n} \qquad \ldots(1)$$

Using this in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$$(a-b)^2 = \frac{4ab\,m^2}{n^2} - 4ab = \frac{4ab\left(m^2 - n^2\right)}{n^2}$$

$$\Rightarrow (a-b) = \frac{2\sqrt{ab}\sqrt{m^2 - n^2}}{n} \qquad \ldots(2)$$

Adding (1) and (2), we obtain

$$2a = \frac{2\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

$$\Rightarrow a = \frac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

Substituting the value of a in (1), we obtain

$$b = \frac{2\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)$$

$$= \frac{\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\sqrt{m^2 - n^2}$$

$$= \frac{\sqrt{ab}}{n}\left(m - \sqrt{m^2 - n^2}\right)$$

$$\therefore a:b = \frac{a}{b} = \frac{\frac{\sqrt{ab}}{n}\left(m + \sqrt{m^2 - n^2}\right)}{\frac{\sqrt{ab}}{n}\left(m - \sqrt{m^2 - n^2}\right)} = \frac{\left(m + \sqrt{m^2 - n^2}\right)}{\left(m - \sqrt{m^2 - n^2}\right)}$$

Thus, $a:b = \left(m + \sqrt{m^2 - n^2}\right):\left(m - \sqrt{m^2 - n^2}\right)$

Question 20:
If a, b, c are in A.P.; b, c, d are in G.P and $\frac{1}{c}, \frac{1}{d}, \frac{1}{e}$ are in A.P. prove that a, c, e are in G.P.

Ans:

It is given that a, b, c are in A.P.

$b - a = c - b$ ... (1)

It is given that b, c, d, are in G.P.

$c^2 = bd$ ... (2)

Also, $\dfrac{1}{c}, \dfrac{1}{d}, \dfrac{1}{c}$ are in A.P.

$$\dfrac{1}{d} - \dfrac{1}{c} = \dfrac{1}{e} - \dfrac{1}{d}$$

$$\dfrac{2}{d} = \dfrac{1}{c} + \dfrac{1}{e} \qquad \qquad ...(3)$$

It has to be proved that a, c, e are in G.P. i.e., $c^2 = ae$

From (1), we obtain

$2b = a + c$

$\Rightarrow b = \dfrac{a+c}{2}$

From (2), we obtain

$d = \dfrac{c^2}{b}$

Substituting these values in (3), we obtain

$$\dfrac{2b}{c^2} = \dfrac{1}{c} + \dfrac{1}{e}$$

$$\Rightarrow \dfrac{2(a+c)}{2c^2} = \dfrac{1}{c} + \dfrac{1}{e}$$

$$\Rightarrow \dfrac{a+c}{c^2} = \dfrac{e+c}{ce}$$

$$\Rightarrow \dfrac{a+c}{c} = \dfrac{e+c}{e}$$

$$\Rightarrow (a+c)e = (e+c)c$$

$$\Rightarrow ae + ce = ec + c^2$$

$$\Rightarrow c^2 = ae$$

Thus, $a, c,$ and e are in G.P.

Question 21:

Find the sum of the following series up to n terms:

(i) $5 + 55 + 555 + \dots$ (ii) $.6 + .66 + .666 + \dots$

Ans:

(i) 5 + 55 + 555 + ...

Let S_n = 5 + 55 + 555 + ... to n terms

$$= \frac{5}{9}\left[9+99+999+...\text{to n terms}\right]$$

$$= \frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+...\text{to n terms}\right]$$

$$= \frac{5}{9}\left[(10+10^2+10^3+...\text{n terms})-(1+1+...\text{n terms})\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$= \frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$= \frac{50}{81}(10^n-1)-\frac{5n}{9}$$

(ii) .6 +.66 +. 666 +...

Let S_n = 06. + 0.66 + 0.666 + ... to n terms

$$= 6\left[0.1+0.11+0.111+...\text{to n terms}\right]$$

$$= \frac{6}{9}\left[0.9+0.99+0.999+...\text{to n terms}\right]$$

$$= \frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+...\text{to n terms}\right]$$

$$= \frac{2}{3}\left[(1+1+...\text{n terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+...\text{n terms}\right)\right]$$

$$= \frac{2}{3}\left[n - \frac{1}{10}\left(\frac{1 - \left(\frac{1}{10}\right)^n}{1 - \frac{1}{10}} \right) \right]$$

$$= \frac{2}{3}n - \frac{2}{30} \times \frac{10}{9}\left(1 - 10^{-n}\right)$$

$$= \frac{2}{3}n - \frac{2}{27}\left(1 - 10^{-n}\right)$$

Question 22:

Find the 20th term of the series $2 \times 4 + 4 \times 6 + 6 \times 8 + \ldots + n$ terms.

Ans:

The given series is $2 \times 4 + 4 \times 6 + 6 \times 8 + \ldots n$ terms

n^{th} term $= a_n = 2n \times (2n + 2) = 4n^2 + 4n$

$a_{20} = 4(20)^2 + 4(20) = 4(400) + 80 = 1600 + 80 = 1680$

Thus, the 20th term of the series is 1680.

Question 23:

Find the sum of the first n terms of the series: $3 + 7 + 13 + 21 + 31 + \ldots$

Ans:

The given series is 3 + 7 + 13 + 21 + 31 + ...

$S = 3 + 7 + 13 + 21 + 31 + ... + a_{n-1} + a_n$

$S = 3 + 7 + 13 + 21 + ... + a_{n-2} + a_{n-1} + a_n$

On subtracting both the equations, we obtain

$S - S = [3 + (7 + 13 + 21 + 31 + ... + a_{n-1} + a_n)] - [(3 + 7 + 13 + 21 + 31 + ... + a_{n-1}) + a_n]$

$S - S = 3 + [(7 - 3) + (13 - 7) + (21 - 13) + ... + (a_n - a_{n-1})] - a_n$

$0 = 3 + [4 + 6 + 8 + ... (n - 1) \text{ terms}] - a_n$

$a_n = 3 + [4 + 6 + 8 + ... (n - 1) \text{ terms}]$

$$\Rightarrow a_n = 3 + \left(\frac{n-1}{2}\right)\left[2 \times 4 + (n - 1 - 1)2\right]$$

$$= 3 + \left(\frac{n-1}{2}\right)\left[8 + (n - 2)2\right]$$

$$= 3 + \frac{(n-1)}{2}(2n + 4)$$

$$= 3 + (n - 1)(n + 2)$$

$$= 3 + (n^2 + n - 2)$$

$$= n^2 + n + 1$$

$$\therefore \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= n\left[\frac{(n+1)(2n+1) + 3(n+1) + 6}{6}\right]$$

$$= n\left[\frac{2n^2 + 3n + 1 + 3n + 3 + 6}{6}\right]$$

$$= n\left[\frac{2n^2 + 6n + 10}{6}\right]$$

$$= \frac{n}{3}\left(n^2 + 3n + 5\right)$$

Question 24:

If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$

Ans:

From the given information,

$$S_1 = \frac{n(n+1)}{2}$$

$$S_3 = \frac{n^2(n+1)^2}{4}$$

Here, $S_3(1 + 8S_1) = \frac{n^2(n+1)^2}{4}\left[1 + \frac{8n(n+1)}{2}\right]$

$$= \frac{n^2(n+1)^2}{4}\left[1 + 4n^2 + 4n\right]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad \ldots(1)$$

Also, $9S_2^2 = 9\dfrac{\left[n(n+1)(2n+1)\right]^2}{(6)^2}$

$$= \frac{9}{36}\left[n(n+1)(2n+1)\right]^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad \ldots(2)$$

Thus, from (1) and (2), we obtain $9S_2^2 = S_3(1 + 8S_1)$

Question 25:

Find the sum of the following series up to n terms: $\dfrac{1^3}{1} + \dfrac{1^3 + 2^3}{1+3} + \dfrac{1^3 + 2^3 + 3^3}{1+3+5} + \ldots$

Ans:

The nth term of the given series is $\dfrac{1^3 + 2^3 + 3^3 + \ldots + n^3}{1+3+5+\ldots+(2n-1)} = \dfrac{\left[\dfrac{n(n+1)}{2}\right]^2}{1+3+5+\ldots+(2n-1)}$

Here, $1, 3, 5, \ldots (2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms as n

$\therefore 1 + 3 + 5 + \ldots + (2n-1) = \dfrac{n}{2}\left[2\times 1 + (n-1)2\right] = n^2$

$\therefore a_n = \dfrac{n^2(n+1)^2}{4n^2} = \dfrac{(n+1)^2}{4} = \dfrac{1}{4}n^2 + \dfrac{1}{2}n + \dfrac{1}{4}$

$\therefore S_n = \sum_{K=1}^{n} a_K = \sum_{K=1}^{n}\left(\dfrac{1}{4}K^2 + \dfrac{1}{2}K + \dfrac{1}{4}\right)$

$$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{1}{4}n$$

$$= \frac{n\left[(n+1)(2n+1) + 6(n+1) + 6\right]}{24}$$

$$= \frac{n\left[2n^2 + 3n + 1 + 6n + 6 + 6\right]}{24}$$

$$= \frac{n(2n^2 + 9n + 13)}{24}$$

Question 26:

Show that $\dfrac{1\times 2^2 + 2\times 3^2 + \ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots + n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$

Ans:

n^{th} term of the numerator $= n(n + 1)^2 = n^3 + 2n^2 + n$

n^{th} term of the denominator $= n^2(n + 1) = n^3 + n^2$

$$\frac{1\times 2^2 + 2\times 3^2 + \ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots + n^2\times(n+1)} = \frac{\sum\limits_{K=1}^{n} a_K}{\sum\limits_{K=1}^{n} a_K} = \frac{\sum\limits_{K=1}^{n}\left(K^3 + 2K^2 + K\right)}{\sum\limits_{K=1}^{n}\left(K^3 + K^2\right)} \qquad \ldots(1)$$

Here, $\sum\limits_{K=1}^{n}\left(K^3 + 2K^2 + K\right)$

$$= \frac{n^2(n+1)^2}{4} + \frac{2\,n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n + 8n + 4 + 6}{6}\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 11n + 10\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 6n + 5n + 10\right]$$

$$= \frac{n(n+1)}{12}\left[3n(n+2) + 5(n+2)\right]$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \qquad \ldots(2)$$

Also, $\sum\limits_{K=1}^{n}\left(K^3 + K^2\right) = \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6}$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n + 4n + 2}{6}\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 7n + 2\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2 + 6n + n + 2\right]$$

$$= \frac{n(n+1)}{12}\left[3n(n+2) + 1(n+2)\right]$$

$$= \frac{n(n+1)(n+2)(3n+1)}{12} \qquad \ldots(3)$$

From (1), (2), and (3), we obtain

$$\frac{1\times 2^2 + 2\times 3^2 + \ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots + n^2\times(n+1)} = \frac{\dfrac{n(n+1)(n+2)(3n+5)}{12}}{\dfrac{n(n+1)(n+2)(3n+1)}{12}}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)} = \frac{3n+5}{3n+1}$$

Thus, the given result is proved.

Question 27:

A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual installments of Rs 500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

Ans:

It is given that the farmer pays Rs 6000 in cash.

Therefore, unpaid amount = Rs 12000 – Rs 6000 = Rs 6000

According to the given condition, the interest paid annually is

12% of 6000, 12% of 5500, 12% of 5000, ..., 12% of 500

Thus, total interest to be paid = 12% of 6000 + 12% of 5500 + 12% of 5000 + ... + 12% of 500

= 12% of (6000 + 5500 + 5000 + ... + 500)

= 12% of (500 + 1000 + 1500 + ... + 6000)

Now, the series 500, 1000, 1500 ... 6000 is an A.P. with both the first term and common difference equal to 500.

Let the number of terms of the A.P. be n.

6000 = 500 + (n – 1) 500

1 + (n – 1) = 12

n = 12

Sum of the A.P $= \dfrac{12}{2}\left[2(500)+(12-1)(500)\right]=6\left[1000+5500\right]=6(6500)=39000$

Thus, total interest to be paid = 12% of (500 + 1000 + 1500 + ... + 6000)

= 12% of 39000 = Rs 4680

Thus, cost of tractor = (Rs 12000 + Rs 4680) = Rs 16680

Question 28:

Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual installment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Ans:

It is given that Shamshad Ali buys a scooter for Rs 22000 and pays Rs 4000 in cash.

∴Unpaid amount = Rs 22000 – Rs 4000 = Rs 18000

According to the given condition, the interest paid annually is

10% of 18000, 10% of 17000, 10% of 16000 ... 10% of 1000

Thus, total interest to be paid = 10% of 18000 + 10% of 17000 + 10% of 16000 + ... + 10% of 1000

= 10% of (18000 + 17000 + 16000 + ... + 1000)

= 10% of (1000 + 2000 + 3000 + ... + 18000)

Here, 1000, 2000, 3000 ... 18000 forms an A.P. with first term and common difference both equal to 1000.

Let the number of terms be n.

18000 = 1000 + (n – 1) (1000)

n = 18

$$\therefore 1000+2000+....+18000 = \dfrac{18}{2}\left[2(1000)+(18-1)(1000)\right]$$
$$=9\left[2000+17000\right]$$
$$=171000$$

Total interest paid = 10% of (18000 + 17000 + 16000 + ... + 1000)

= 10% of Rs 171000 = Rs 17100

∴Cost of scooter = Rs 22000 + Rs 17100 = Rs 39100

Question 29:

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8th set of letter is mailed.

Ans:

The numbers of letters mailed forms a G.P.: $4, 4^2, \ldots 4^8$

First term = 4

Common ratio = 4

Number of terms = 8

It is known that the sum of n terms of a G.P. is given by

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore S_8 = \frac{4(4^8 - 1)}{4 - 1} = \frac{4(65536 - 1)}{3} = \frac{4(65535)}{3} = 4(21845) = 87380$$

It is given that the cost to mail one letter is 50 paisa.

Cost of mailing 87380 letters $= \text{Rs } 87380 \times \dfrac{50}{100} = \text{Rs } 43690$

Thus, the amount spent when 8th set of letter is mailed is Rs 43690.

Question 30:

A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15th year since he deposited the amount and also calculate the total amount after 20 years.

Ans:

It is given that the man deposited Rs 10000 in a bank at the rate of 5% simple interest annually.

Interest in first year $= \dfrac{5}{100} \times \text{Rs } 10000 = \text{Rs } 500$

Amount in 15th year = Rs $10000 + \underbrace{500 + 500 + \ldots + 500}_{14 \text{ times}}$

= Rs 10000 + 14 × Rs 500

= Rs 10000 + Rs 7000

= Rs 17000

Amount after 20 years = $\text{Rs } 10000 + \underbrace{500 + 500 + \ldots + 500}_{20 \text{ times}}$

= Rs 10000 + 20 × Rs 500

= Rs 10000 + Rs 10000

= Rs 20000

Question 31:

A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Ans:

Cost of machine = Rs 15625

Machine depreciates by 20% every year.

Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

Value at the end of 5 years = $\underbrace{15625 \times \frac{4}{5} \times \frac{4}{5} \times \dots \times \frac{4}{5}}_{5 \text{ times}}$ = 5 × 1024 = 5120

Thus, the value of the machine at the end of 5 years is Rs 5120.

Question 32:

150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Ans:

Let x be the number of days in which 150 workers finish the work.

According to the given information,

$150x$ = 150 + 146 + 142 + ... $(x + 8)$ terms

The series 150 + 146 + 142 + ... $(x + 8)$ terms is an A.P. with first term 146, common difference –4 and number of terms as $(x + 8)$

$$\Rightarrow 150x = \frac{(x+8)}{2}\left[2(150)+(x+8-1)(-4)\right]$$
$$\Rightarrow 150x = (x+8)\left[150+(x+7)(-2)\right]$$
$$\Rightarrow 150x = (x+8)(150-2x-14)$$
$$\Rightarrow 150x = (x+8)(136-2x)$$
$$\Rightarrow 75x = (x+8)(68-x)$$
$$\Rightarrow 75x = 68x - x^2 + 544 - 8x$$
$$\Rightarrow x^2 + 75x - 60x - 544 = 0$$
$$\Rightarrow x^2 + 15x - 544 = 0$$
$$\Rightarrow x^2 + 32x - 17x - 544 = 0$$
$$\Rightarrow x(x+32) - 17(x+32) = 0$$
$$\Rightarrow (x-17)(x+32) = 0$$
$$\Rightarrow x = 17 \text{ or } x = -32$$

However, x cannot be negative.

$x = 17$

Therefore, originally, the number of days in which the work was completed is 17.

Thus, required number of days = (17 + 8) = 25

*********** END *********