



(v) Which elements of \mathbf{N} are invertible for the operation $*$?

Answer

The binary operation $*$ on \mathbf{N} is defined as $a * b = \text{L.C.M. of } a \text{ and } b$.

(i) $5 * 7 = \text{L.C.M. of } 5 \text{ and } 7 = 35$

$20 * 16 = \text{L.C.M of } 20 \text{ and } 16 = 80$

(ii) It is known that:

$\text{L.C.M of } a \text{ and } b = \text{L.C.M of } b \text{ and } a \quad a, b \in \mathbf{N}$.

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

(iii) For $a, b, c \in \mathbf{N}$, we have:

$(a * b) * c = (\text{L.C.M of } a \text{ and } b) * c = \text{LCM of } a, b, \text{ and } c$

$a * (b * c) = a * (\text{LCM of } b \text{ and } c) = \text{L.C.M of } a, b, \text{ and } c$

$$\therefore (a * b) * c = a * (b * c)$$

Thus, the operation $*$ is associative.

(iv) It is known that:

$\text{L.C.M. of } a \text{ and } 1 = a = \text{L.C.M. } 1 \text{ and } a \quad a \in \mathbf{N}$

$$\Rightarrow a * 1 = a = 1 * a \quad a \in \mathbf{N}$$

Thus, 1 is the identity of $*$ in \mathbf{N} .

(v) An element a in \mathbf{N} is invertible with respect to the operation $*$ if there exists an element b in \mathbf{N} , such that $a * b = e = b * a$.

Here, $e = 1$

This means that:

$\text{L.C.M of } a \text{ and } b = 1 = \text{L.C.M of } b \text{ and } a$

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of \mathbf{N} with respect to the operation $*$.

Question 7:

Is $*$ defined on the set $\{1, 2, 3, 4, 5\}$ by $a * b = \text{L.C.M. of } a \text{ and } b$ a binary operation?

Justify your answer.

Answer

The operation $*$ on the set $A = \{1, 2, 3, 4, 5\}$ is defined as

$a * b = \text{L.C.M. of } a \text{ and } b$.

Then, the operation table for the given operation $*$ can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

It can be observed from the obtained table that:

$$3 * 2 = 2 * 3 = 6 \notin A, 5 * 2 = 2 * 5 = 10 \notin A, 3 * 4 = 4 * 3 = 12 \notin A$$

$$3 * 5 = 5 * 3 = 15 \notin A, 4 * 5 = 5 * 4 = 20 \notin A$$

Hence, the given operation $*$ is not a binary operation.

Question 8:

Let $*$ be the binary operation on \mathbf{N} defined by $a * b = \text{H.C.F. of } a \text{ and } b$. Is $*$

commutative? Is $*$ associative? Does there exist identity for this binary operation on \mathbf{N} ?

Answer

The binary operation $*$ on \mathbf{N} is defined as:

$a * b = \text{H.C.F. of } a \text{ and } b$

It is known that:

$\text{H.C.F. of } a \text{ and } b = \text{H.C.F. of } b \text{ and } a \quad a, b \in \mathbf{N}$.

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

For $a, b, c \in \mathbf{N}$, we have:

$$(a * b) * c = (\text{H.C.F. of } a \text{ and } b) * c = \text{H.C.F. of } a, b, \text{ and } c$$

$a * (b * c) = a * (\text{H.C.F. of } b \text{ and } c) = \text{H.C.F. of } a, b, \text{ and } c$

$$\therefore (a * b) * c = a * (b * c)$$

Thus, the operation $*$ is associative.

Now, an element $e \in \mathbf{N}$ will be the identity for the operation $*$ if $a * e = a = e * a \quad \forall a \in \mathbf{N}$.

But this relation is not true for any $a \in \mathbf{N}$.

Thus, the operation $*$ does not have any identity in \mathbf{N} .

Question 9:

Let $*$ be a binary operation on the set \mathbf{Q} of rational numbers as follows:

$$(i) a * b = a - b \quad (ii) a * b = a^2 + b^2$$

$$(iii) a * b = a + ab \quad (iv) a * b = (a - b)^2$$

$$a * b = \frac{ab}{4} \quad (v) \quad (vi) a * b = ab^2$$

Find which of the binary operations are commutative and which are associative.

Answer

(i) On \mathbf{Q} , the operation $*$ is defined as $a * b = a - b$.

It can be observed that:

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6} \quad \text{and} \quad \frac{1}{3} * \frac{1}{2} = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}; \text{ where } \frac{1}{2}, \frac{1}{3} \in \mathbf{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that:

$$\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} = \left(\frac{1}{2} - \frac{1}{3}\right) * \frac{1}{4} = \frac{1}{6} * \frac{1}{4} = \frac{1}{6} - \frac{1}{4} = \frac{2-3}{12} = -\frac{1}{12}$$

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) = \frac{1}{2} * \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} * \frac{1}{12} = \frac{1}{2} - \frac{1}{12} = \frac{6-1}{12} = \frac{5}{12}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right); \text{ where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

(ii) On \mathbf{Q} , the operation $*$ is defined as $a * b = a^2 + b^2$.

For $a, b \in \mathbf{Q}$, we have:

$$a * b = a^2 + b^2 = b^2 + a^2 = b * a$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

It can be observed that:

$$(1 * 2) * 3 = (1^2 + 2^2) * 3 = (1 + 4) * 3 = 5 * 3 = 5^2 + 3^2 = 34$$

$$1 * (2 * 3) = 1 * (2^2 + 3^2) = 1 * (4 + 9) = 1 * 13 = 1^2 + 13^2 = 169$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3); \text{ where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

(iii) On \mathbf{Q} , the operation $*$ is defined as $a * b = a + ab$.

It can be observed that:

$$1 * 2 = 1 + 1 \times 2 = 1 + 2 = 3$$

$$2 * 1 = 2 + 2 \times 1 = 2 + 2 = 4$$

$$\therefore 1 * 2 \neq 2 * 1; \text{ where } 1, 2 \in \mathbf{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that:

$$(1 * 2) * 3 = (1 + 1 \times 2) * 3 = 3 * 3 = 3 + 3 \times 3 = 3 + 9 = 12$$

$$1 * (2 * 3) = 1 * (2 + 2 \times 3) = 1 * 8 = 1 + 1 \times 8 = 9$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3); \text{ where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

(iv) On \mathbf{Q} , the operation $*$ is defined by $a * b = (a - b)^2$.

For $a, b \in \mathbf{Q}$, we have:

$$a * b = (a - b)^2$$

$$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

It can be observed that:

$$(1 * 2) * 3 = (1 - 2)^2 * 3 = (-1)^2 * 3 = 1 * 3 = (1 - 3)^2 = (-2)^2 = 4$$

$$1 * (2 * 3) = 1 * (2 - 3)^2 = 1 * (-1)^2 = 1 * 1 = (1 - 1)^2 = 0$$

$$\therefore (1 * 2) * 3 \neq 1 * (2 * 3); \text{ where } 1, 2, 3 \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

(v) On \mathbf{Q} , the operation $*$ is defined as $a * b = \frac{ab}{4}$.

For $a, b \in \mathbf{Q}$, we have:

$$a * b = \frac{ab}{4} = \frac{ba}{4} = b * a$$

$$\therefore a * b = b * a$$

Thus, the operation $*$ is commutative.

For $a, b, c \in \mathbf{Q}$, we have:

$$(a * b) * c = \frac{ab}{4} * c = \frac{\frac{ab}{4} \cdot c}{4} = \frac{abc}{16}$$

$$a * (b * c) = a * \frac{bc}{4} = \frac{a \cdot \frac{bc}{4}}{4} = \frac{abc}{16}$$

$$\therefore (a * b) * c = a * (b * c)$$

Thus, the operation $*$ is associative.

(vi) On \mathbf{Q} , the operation $*$ is defined as $a * b = ab^2$

It can be observed that:

$$\frac{1}{2} * \frac{1}{3} = \frac{1}{2} \cdot \left(\frac{1}{3}\right)^2 = \frac{1}{2} \cdot \frac{1}{9} = \frac{1}{18}$$

$$\frac{1}{3} * \frac{1}{2} = \frac{1}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$\therefore \frac{1}{2} * \frac{1}{3} \neq \frac{1}{3} * \frac{1}{2}; \text{ where } \frac{1}{2}, \frac{1}{3} \in \mathbf{Q}$$

Thus, the operation $*$ is not commutative.

It can also be observed that:

$$\left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} = \left[\frac{1}{2} \cdot \left(\frac{1}{3}\right)^2\right] * \frac{1}{4} = \frac{1}{18} * \frac{1}{4} = \frac{1}{18} \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{18 \times 16}$$

$$\frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right) = \frac{1}{2} * \left[\frac{1}{3} \cdot \left(\frac{1}{4}\right)^2\right] = \frac{1}{2} * \frac{1}{48} = \frac{1}{2} \cdot \left(\frac{1}{48}\right)^2 = \frac{1}{2 \times (48)^2}$$

$$\therefore \left(\frac{1}{2} * \frac{1}{3}\right) * \frac{1}{4} \neq \frac{1}{2} * \left(\frac{1}{3} * \frac{1}{4}\right); \text{ where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \in \mathbf{Q}$$

Thus, the operation $*$ is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

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