Definite Integrals Ex 20.2 Q18

Let $\sin^2 x = t$

Differentiating w.r.t. x, we get

$$2\sin x \cos x dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{1+t^2}$$

$$= \frac{1}{2}\left[\tan^{-1} t\right]_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\tan\frac{\pi}{4}\right) - \tan^{-1}(\tan 0)\right]$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x}{1+\sin^4 x} dx = \frac{\pi}{8}$$

Definite Integrals Ex 20.2 Q19

$$\text{Putting } \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$\sin x = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$\therefore \qquad I = \int_0^{\frac{\pi}{2}} \frac{1}{a\cos x + b\sin x}dx = \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{a\left(1-\tan^2\frac{x}{2}\right)+2b\tan^2\frac{x}{2}}dx$$

Put $\tan\frac{x}{2} = t$

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

If $x = 0$, $t = 0$ and if $x = \frac{\pi}{2}$, $t = 1$

$$\Rightarrow \qquad I = 2\int_0^1 \frac{dt}{a\left(1-t^2\right)+2bt}$$

$$= 2\int_0^1 \frac{dt}{-at^2+2bt+a}$$

$$= 2\int_0^1 \frac{dt}{-a\left[t^2-\frac{2b}{a}t-1\right]}$$

$$= \frac{2}{a}\int_0^1 \frac{dt}{-\left[\left(t-\frac{b}{a}\right)^2-1-\frac{b^2}{a^2}\right]}$$

$$= \frac{2}{a}\int_0^1 \frac{dt}{\left(\frac{b^2}{a^2}+1\right)-\left(t-\frac{b}{a}\right)^2}$$

$$= \frac{2}{a}\left[\frac{1}{2\sqrt{\frac{b^2+a^2}{a^2}}}\left(\log\left|\frac{\frac{\sqrt{b^2+a^2}}{a^2}+\left(t-\frac{b}{a}\right)}{\frac{\sqrt{b^2+a^2}}{a^2}-\left(t-\frac{b}{a}\right)}\right|\right)\right]_0^1$$

$$= \frac{1}{\sqrt{b^2+a^2}}\log\left(\frac{a+b+\sqrt{a^2+b^2}}{a+b-\sqrt{a^2+b^2}}\right)$$

Definite Integrals Ex 20.2 Q20

We know that $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{5+4\sin\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\dfrac{5\left(1+\tan^2\frac{x}{2}\right)+4\left(2\tan\frac{x}{2}\right)}{1+\tan^2\frac{x}{2}}}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\tan^2\frac{x}{2}}{\left(5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}\right)}\,dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}\,dx$$

Let $\tan\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sec^2\frac{x}{2}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}\,dx$$

$$= \int_0^1 \frac{2dt}{5 + 5t^2 + 8t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{1 + t^2 + \frac{8}{5}t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{1 - \frac{16}{25} + \frac{16}{25} + t^2 + \frac{8}{5}t} \qquad \left[\text{Adding and subtracting } \frac{16}{25}\right]$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(\frac{3}{2}\right)^2 + \left(t + \frac{4}{5}\right)^2}$$

$$= \frac{2}{5}\left[\frac{5}{3}\tan^{-1}\left(t+\frac{4}{5}\right)\times\frac{5}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\left(1+\frac{4}{5}\right)\times\frac{5}{3} - \tan^{-1}\frac{4}{5}\times\frac{5}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}3 - \tan^{-1}\frac{4}{3}\right]_0^1$$

$$= \frac{2}{3}\left[\tan^{-1}\left(\frac{3-\frac{4}{3}}{1+3\times\frac{4}{3}}\right)\right]_0^1 \qquad \left[\because \tan^{-1}A - \tan^{-1}B = \tan^{-1}\left(\frac{A-B}{1+AB}\right)\right]$$

$$= \frac{2}{3}\left[\tan^{-1}\frac{\frac{5}{3}}{5}\right]$$

$$= \frac{2}{3}\tan^{-1}\frac{1}{3}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{1}{5 + 4\sin x}dx = \frac{2}{3}\tan^{-1}\frac{1}{3}$$

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