Binary Operations Ex 3.4 Q5

We have,

$$a * b = \frac{ab}{2} \text{ for all } a, b \in Q_0$$

(i)

Commutativity: Let $a, b \in Q_0$, then

$$\Rightarrow \quad a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$$

$$\Rightarrow \quad a * b = b * a$$

Hence, '*' is commutative on Q_0.

Associativity: Let $a, b, c \in Q_0$, then

$$\Rightarrow \quad (a * b) * c = \frac{ab}{2} * c = \frac{abc}{4} \qquad ---(i)$$

$$\text{and,} \quad a * (b * c) = a * \frac{bc}{2} = \frac{abc}{4} \qquad ---(ii)$$

From (i) & (ii)

$$(a * b) * c = a * (b * c)$$

$\Rightarrow$ $*$ is associative on Q_0.

(ii)

Let $e \in Q_0$ be the identity element with respect to $*$.

By identity property, we have,

$$a * e = e * a = a \text{ for all } a \in Q_0$$

$$\Rightarrow \quad \frac{ae}{2} = a \qquad \Rightarrow e = 2$$

Thus, the required identity element is 2.

(iii)

Let $b \in Q_0$ be the inverse of $a \in Q_0$ with respect to $*$, then,

$$a * b = b * a = e \text{ for all } a \in Q_0$$

$$\Rightarrow \quad \frac{ab}{2} = e \qquad \Rightarrow \frac{ab}{2} = 2$$

$$\Rightarrow b = \frac{4}{a}$$

Thus, $b = \frac{4}{a}$ is the inverse of a with respect to $*$.

Binary Operations Ex 3.4 Q6

We have,

$$a * b = a + b - ab \text{ for all } a, b \in R - \{+1\}$$

(i)

Commutative: Let $a, b \in R - \{+1\}$, then,

$\Rightarrow \quad a * b = a + b - ab = b + a - ba = b * a$

$\Rightarrow \quad a * b = b * a$

So, '$*$' is commutative on $R - \{+1\}$.

Associativity: Let $a, b, c \in R - \{+1\}$, then

$$\begin{aligned}(a * b) * c &= (a + b - ab) * c \\ &= a + b - ab + c - ac - bc + abc \\ &= a + b + c - ab - ac - bc + abc \qquad ---(i)\end{aligned}$$

and, $\quad a * (b * c) = a * (b + c - bc)$

$$= a + b + c - bc - ab - ac + abc \qquad ---(ii)$$

From (i) & (ii)

$$(a * b) * c = a * (b * c)$$

So, '$*$' is associative on $R - \{+1\}$.

(ii)

Let $e \in R - \{+1\}$ be the identity element with respect to $*$, then

$$a * e = e * a = a \text{ for all } a \in R - \{+1\}$$

$\Rightarrow \quad a + e - ae = a$

$\Rightarrow \quad e(1 - a) = 0$

$\Rightarrow \quad e = 0 \qquad [\because a \neq 1 \Rightarrow 1 - a \neq 0]$

$\therefore \quad e = 0$ will be the identity element with respect to $*$.

(iii)

Let $b \in R - \{1\}$ be the inverse element of $a \in R - \{1\}$, then

$$a * b = b * a = e$$

$\Rightarrow \quad a + b - ab = 0 \qquad [\because e = 0]$

$\Rightarrow \quad b(1 - a) = -a$

$\Rightarrow \quad b = \dfrac{-a}{1 - a} \neq 1 \qquad \left[\begin{array}{l}\because \text{if } \dfrac{-a}{1-a} = 1 \\ \Rightarrow -a = 1 - a \quad \Rightarrow 1 = 0 \\ \quad \text{Not possible}\end{array}\right]$

$\therefore \quad b = \dfrac{-a}{1 - a}$ is the inverse of $a \in R - \{1\}$ with respect to $*$.

Binary Operations Ex 3.4 Q7

We have,

$$(a,b)*(c,d)=(ac,bd) \text{ for all } (a,b),(c,d)\in A$$

(i)

Let $(a,b),(c,d)\in A$, then

$$\begin{aligned}(a,b)*(c,d)&=(ac,bd)\\&=(ca,db) && [\because ac=ca \text{ and } bd=db]\\&=(c,d)*(a,b)\end{aligned}$$

$\Rightarrow \quad (a,b)*(c,d)=(c,d)*(a,b)$

So, '$*$' is commutative on A

Associativity: Let $(a,b),(c,d),(e,f)\in A$, then

$$\begin{aligned}\Rightarrow \quad ((a,b)*(c,d))*(e,f)&=(ac,bd)*(e,f)\\&=(ace,bdf) && ---(i)\end{aligned}$$

$$\begin{aligned}\text{and,} \quad (a,b)*((c,d)*(e,f))&=(a,b)*(ce,df)\\&=(ace,bdf) && ---(ii)\end{aligned}$$

From (i) & (ii)

$\Rightarrow \quad ((a,b)*(c,d))*(e,f)=(a,b)*((c,d)*(e,f))$

So, '$*$' is associative on A.

(ii)

Let $(x,y)\in A$ be the identity element with respect to $*$.

$$(a,b)*(x,y)=(x,y)*(a,b)=(a,b) \text{ for all } (a,b)\in A$$

$\Rightarrow \quad (ax,by)=(a,b)$

$\Rightarrow \quad ax=a$ and $by=b$

$\Rightarrow \quad x=1$, and $y=1$

$\therefore \quad (1,1)$ will be the identity element

(iii)

Let $(c,d)\in A$ be the inverse of $(a,b)\in A$, then

$$(a,b)*(c,d)=(c,d)*(a,b)=e$$

$\Rightarrow \quad (ac,bd)=(1,1) \qquad [\because e=(1,1)]$

$\Rightarrow \quad ac=1$ and $bd=1$

$\Rightarrow \quad c=\dfrac{1}{a}$ and $d=\dfrac{1}{b}$

$\therefore \quad \left(\dfrac{1}{a},\dfrac{1}{b}\right)$ will be the inverse of (a,b) with respect to $*$.

Binary Operations Ex 3.4 Q8

The binary operation * on **N** is defined as:

$a * b$ = H.C.F. of a and b

It is known that:

H.C.F. of a and b = H.C.F. of b and a, $a, b \in$ **N**.

Therefore, $a * b = b * a$

Thus, the operation * is commutative.

For $a, b, c \in$ **N**, we have:

$(a * b) * c$ = (H.C.F. of a and b) $* c$ = H.C.F. of a, b, and c

$a * (b * c) = a *$(H.C.F. of b and c) = H.C.F. of a, b, and c

Therefore, $(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

Now, an element $e \in$ **N** will be the identity for the operation * if $a * e = a = e * a$, $\forall\ a \in$ **N**.

But this relation is not true for any $a \in$ **N**.

Thus, the operation * does not have any identity in **N**.

*********** END **********