Maxima and Minima 18.5 Q7

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28 - l)$ m.

Now, side of square $= \frac{l}{4}$.

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$.

The combined areas of the square and the circle (A) is given by,

$$A = (\text{side of the square})^2 + \ r^2$$
$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28 - l)\right]^2$$
$$= \frac{l^2}{16} + \frac{1}{4\pi}(28 - l)^2$$
$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28 - l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28 - l)$$
$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$
$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$$
$$\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} = 0$$
$$\Rightarrow (\pi + 4)l - 112 = 0$$
$$\Rightarrow l = \frac{112}{\pi + 4}$$

Thus, when $l = \frac{112}{\pi + 4}, \frac{d^2A}{dl^2} > 0$.

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi + 4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi + 4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi + 4} = \frac{28\pi}{\pi + 4}$ cm.

Maxima and Minima 18.5 Q8

Let the wire of length 20 m be cut into x cm and y cm and bent into a square and equilateral triangle, so that the sum of area of square and triangle is minimum.

Now,

$$x + y = 20 \qquad \text{---(i)}$$
$$x = 4l \text{ and } y = 3a$$

Let s = sum of area of square and triangle

$$s = l^2 + \frac{\sqrt{3}}{4}a^2 \qquad \text{---(ii)}$$

$$\left[\because \text{area of equilateral } \Delta = \frac{\sqrt{3}}{4}(\text{one side})^2\right]$$

We have, $4l+3a=20$

$\Rightarrow 4l=20-3a$

$\Rightarrow l=\dfrac{20-3a}{4}$

From (i), we have,

$s=\left(\dfrac{20-3a}{4}\right)^2+\dfrac{\sqrt{3}}{4}a^2$

$\dfrac{ds}{da}=2\left(\dfrac{20-3a}{4}\right)\left(\dfrac{-3}{4}\right)+2a\times\dfrac{\sqrt{3}}{4}$

To find the maximum or minimum, $\dfrac{ds}{da}=0$

$\Rightarrow 2\left(\dfrac{20-3a}{4}\right)\left(\dfrac{-3}{4}\right)+2a\times\dfrac{\sqrt{3}}{4}=0$

$\Rightarrow -3(20-3a)+4a\sqrt{3}=0$

$\Rightarrow -60+9a+4a\sqrt{3}=0$

$\Rightarrow 9a+4a\sqrt{3}=60$

$\Rightarrow a\left(9+4\sqrt{3}\right)=60$

$\Rightarrow a=\dfrac{60}{9+4\sqrt{3}}$

Differentiating once again, we have,

$\dfrac{d^2s}{da^2}=\dfrac{9+4\sqrt{3}}{8}>0$

Thus, the sum of the areas of the square and triangle is minimum when $a=\dfrac{60}{9+4\sqrt{3}}$

We know that, $l=\dfrac{20-3a}{4}$

$\Rightarrow l=\dfrac{20-3\left(\frac{60}{9+4\sqrt{3}}\right)}{4}$

$\Rightarrow l=\dfrac{180+80\sqrt{3}-180}{4\left(9+4\sqrt{3}\right)}$

$\Rightarrow l=\dfrac{20\sqrt{3}}{9+4\sqrt{3}}$

Maxima and Minima 18.5 Q9

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$2\pi r+4a=k$ (where k is constant)

$\Rightarrow a=\dfrac{k-2\pi r}{4}$

The sum of the areas of the circle and the square (A) is given by,

$A=\pi r^2+a^2=\pi r^2+\dfrac{(k-2\pi r)^2}{16}$

$\therefore \dfrac{dA}{dr}=2\pi r+\dfrac{2(k-2\pi r)(-2\pi)}{16}=2\pi r-\dfrac{\pi(k-2\pi r)}{4}$

Now, $\dfrac{dA}{dr}=0$

$\Rightarrow 2\pi r=\dfrac{\pi(k-2\pi r)}{4}$

$8r=k-2\pi r$

$\Rightarrow (8+2\pi)r = k$

$\Rightarrow r = \dfrac{k}{8+2\pi} = \dfrac{k}{2(4+\pi)}$

Now, $\dfrac{d^2A}{dr^2} = 2\pi + \dfrac{\pi^2}{2} > 0$

$\therefore$ When $r = \dfrac{k}{2(4+\pi)}$, $\dfrac{d^2A}{dr^2} > 0.$

$\therefore$ The sum of the areas is least when $r = \dfrac{k}{2(4+\pi)}$.

When $r = \dfrac{k}{2(4+\pi)}$, $a = \dfrac{k - 2\pi\left[\dfrac{k}{2(4+\pi)}\right]}{4} = \dfrac{k(4+\pi)-\pi k}{4\,4(4+\pi)} = \dfrac{4k}{4\,4(4+\pi)} = \dfrac{k}{4+\pi} = 2r.$

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

*********** END **********