Exercise Miscellaneous : Solutions of Questions on Page Number : **242**

Q1 : Using differentials, find the approximate value of each of the following.

(a) $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ **(b)** $(33)^{-\frac{1}{5}}$

Answer :

(a) Consider $y = x^{\frac{1}{4}}$. Let $x = \frac{16}{81}$ and $\Delta x = \frac{1}{81}$.

Then, $\Delta y = (x+\Delta x)^{\frac{1}{4}} - x^{\frac{1}{4}}$

$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \left(\frac{16}{81}\right)^{\frac{1}{4}}$$
$$= \left(\frac{17}{81}\right)^{\frac{1}{4}} - \frac{2}{3}$$

$$\therefore \left(\frac{17}{81}\right)^{\frac{1}{4}} = \frac{2}{3} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)\Delta x = \frac{1}{4(x)^{\frac{3}{4}}}(\Delta x) \qquad \left(\text{as } y = x^{\frac{1}{4}}\right)$$
$$= \frac{1}{4\left(\frac{16}{81}\right)^{\frac{3}{4}}}\left(\frac{1}{81}\right) = \frac{27}{4\times 8}\times\frac{1}{81} = \frac{1}{32\times 3} = \frac{1}{96} = 0.010$$

Hence, the approximate value of $\left(\frac{17}{81}\right)^{\frac{1}{4}}$ is $\frac{2}{3} + 0.010$ = 0.667 + 0.010

= 0.677.

(b) Consider $y = x^{-\frac{1}{5}}$. Let x = 32 and Δx = 1.

Then, $\Delta y = (x+\Delta x)^{-\frac{1}{5}} - x^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - (32)^{-\frac{1}{5}} = (33)^{-\frac{1}{5}} - \frac{1}{2}$

$$\therefore (33)^{-\frac{1}{5}} = \frac{1}{2} + \Delta y$$

Now, dy is approximately equal to Δy and is given by,

$$dy = \left(\frac{dy}{dx}\right)(\Delta x) = \frac{-1}{5(x)^{\frac{6}{5}}}(\Delta x) \qquad \left(\text{as } y = x^{-\frac{1}{5}}\right)$$
$$= -\frac{1}{5(2)^{6}}(1) = -\frac{1}{320} = -0.003$$

Hence, the approximate value of $(33)^{-\frac{1}{5}}$ is $\frac{1}{2} + (-0.003)$

= 0.5 - 0.003 = 0.497.

Answer needs Correction? **Click Here**

Q2 : Show that the function given by $f(x) = \frac{\log x}{x}$ has maximum at x = e.

Answer :

The given function is $f(x) = \frac{\log x}{x}$.

$$f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} = \frac{1 - \log x}{x^2}$$

Now, $f'(x) = 0$

$\Rightarrow$ 1 - log x = 0

$$\Rightarrow \log x = 1$$
$$\Rightarrow \log x = \log e$$
$$\Rightarrow x = e$$

$$\text{Now, } f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)(2x)}{x^4}$$
$$= \frac{-x - 2x(1-\log x)}{x^4}$$

$$= \frac{-3+2\log x}{x^3}$$

Now, $f''(e) = \frac{-3+2\log e}{e^3} = \frac{-3+2}{e^3} = \frac{-1}{e^3} < 0$

Therefore, by second derivative test, f is the maximum at $x = e$.

Answer needs Correction? Click Here

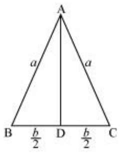
Q3 : The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Answer :

Let ΔABC be isosceles where BC is the base of fixed length b.

Let the length of the two equal sides of ΔABC be a.

Draw AD$\perp$BC.



Now, in ΔADC, by applying the Pythagoras theorem, we have:

$$\text{AD} = \sqrt{a^2 - \frac{b^2}{4}}$$

$\therefore$ Area of triangle $(A) = \frac{1}{2} b \sqrt{a^2 - \frac{b^2}{4}}$

The rate of change of the area with respect to time (t) is given by,

$$\frac{dA}{dt} = \frac{1}{2} b \cdot \frac{2a}{2\sqrt{a^2 - \frac{b^2}{4}}} \frac{da}{dt} = \frac{ab}{\sqrt{4a^2 - b^2}} \frac{da}{dt}$$

It is given that the two equal sides of the triangle are decreasing at the rate of 3 cm per second.

$$\therefore \frac{da}{dt} = -3 \text{ cm/s}$$

$$\therefore \frac{dA}{dt} = \frac{-3ab}{\sqrt{4a^2 - b^2}}$$

Then, when $a = b$, we have:

$$\frac{dA}{dt} = \frac{-3b^2}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3b^2}} = -\sqrt{3}b$$

Hence, if the two equal sides are equal to the base, then the area of the triangle is decreasing at the rate of $\sqrt{3}\,b\ \text{cm}^2/\text{s}$.

Answer needs Correction? Click Here

Q4 : Find the equation of the normal to curve $y^2 = 4x$ at the point (1, 2).

Answer :

The equation of the given curve is $y^2 = 4x$.

Differentiating with respect to x, we have:

$$2y \frac{dy}{dx} = 4$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$$

$$\therefore \left.\frac{dy}{dx}\right]_{(1,2)} = \frac{2}{2} = 1$$

Now, the slope of the normal at point (1, 2) is $\frac{-1}{\left.\frac{dy}{dx}\right]_{(1,2)}} = \frac{-1}{1} = -1.$

$\therefore$Equation of the normal at (1, 2) is y - 2 = - 1(x - 1).

$\Rightarrow$ y - 2 = - x + 1

$\Rightarrow$ x + y - 3 = 0

Answer needs Correction? Click Here

Q5 : Show that the normal at any point θ to the curve

$x = a\cos\theta + a\theta\sin\theta,\ y = a\sin\theta - a\theta\cos\theta$ is at a constant distance from the origin.

Answer :

We have $x = a \cos \theta + a \theta \sin \theta$.

$$\therefore \frac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$$

$$y = a\sin\theta - a\theta\cos\theta$$

$$\therefore \frac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

$\therefore$ Slope of the normal at any point θ is $-\frac{1}{\tan\theta}$.

The equation of the normal at a given point (x, y) is given by,

$$y - a\sin\theta + a\theta\cos\theta = \frac{-1}{\tan\theta}\left(x - a\cos\theta - a\theta\sin\theta\right)$$
$$\Rightarrow y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta = -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$
$$\Rightarrow x\cos\theta + y\sin\theta - a\left(\sin^2\theta + \cos^2\theta\right) = 0$$
$$\Rightarrow x\cos\theta + y\sin\theta - a = 0$$

Now, the perpendicular distance of the normal from the origin is

$$\frac{|-a|}{\sqrt{\cos^2\theta + \sin^2\theta}} = \frac{|-a|}{\sqrt{1}} = |-a|, \text{which is independent of } \theta.$$

Hence, the perpendicular distance of the normal from the origin is constant.

Answer needs Correction? **Click Here**

Q6 : Find the intervals in which the function *f* given by

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$

is (i) increasing (ii) decreasing

Answer :

$$f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}$$
$$\therefore f'(x) = \frac{(2+\cos x)(4\cos x - 2 - \cos x + x\sin x) - (4\sin x - 2x - x\cos x)(-\sin x)}{(2+\cos x)^2}$$
$$= \frac{(2+\cos x)(3\cos x - 2 + x\sin x) + \sin x(4\sin x - 2x - x\cos x)}{(2+\cos x)^2}$$
$$= \frac{6\cos x - 4 + 2x\sin x + 3\cos^2 x - 2\cos x + x\sin x\cos x + 4\sin^2 x - 2x\sin x - x\sin x\cos x}{(2+\cos x)^2}$$
$$= \frac{4\cos x - 4 + 3\cos^2 x + 4\sin^2 x}{(2+\cos x)^2}$$
$$= \frac{4\cos x - 4 + 3\cos^2 x + 4 - 4\cos^2 x}{(2+\cos x)^2}$$
$$= \frac{4\cos x - \cos^2 x}{(2+\cos x)^2} = \frac{\cos x(4 - \cos x)}{(2+\cos x)^2}$$

Now, $f'(x) = 0$

$\Rightarrow$ cos x = 0 or cos x = 4

But, cos $x \neq 4$

$\therefore$ cos x = 0

$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

Now, $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ divides (0, 2π) into three disjoint intervals i.e.,

$\left(0, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$.

In intervals $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$, $f'(x) > 0$.

Thus, $f(x)$ is increasing for $0 < x < \frac{x}{2}$ and $\frac{3\pi}{2} < x < 2\pi$.

In the interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $f'(x) < 0$.

Thus, $f(x)$ is decreasing for $\frac{\pi}{2} < x < \frac{3\pi}{2}$.

Answer needs Correction? **Click Here**

Q7 : Find the intervals in which the function *f* given by $f(x) = x^3 + \frac{1}{x^3}, x \neq 0$ is

(i) increasing (ii) decreasing

Answer :

$$f(x) = x^3 + \frac{1}{x^3}$$
$$\therefore f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3x^6 - 3}{x^4}$$

Then, $f'(x) = 0 \Rightarrow 3x^6 - 3 = 0 \Rightarrow x^6 = 1 \Rightarrow x = \pm 1$

Now, the points x = 1 and x = - 1 divide the real line into three disjoint intervals i.e., $(-\infty, -1), (-1, 1)$, and $(1, \infty)$.

In intervals $(-\infty, -1)$ and $(1, \infty)$ i.e., when x < - 1 and x > 1, $f'(x) > 0$.

Thus, when x < - 1 and x > 1, f is increasing.

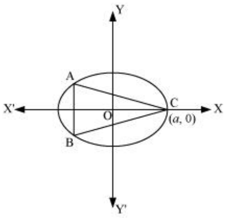
In interval (- 1, 1) i.e., when - 1 < x < 1, $f'(x) < 0$.

Thus, when - 1 < x < 1, f is decreasing.

Answer needs Correction? **Click Here**

Q8 : Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis.

Answer :



The given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let the major axis be along the x - axis.

Let ABC be the triangle inscribed in the ellipse where vertex C is at (a, 0).

Since the ellipse is symmetrical with respect to the x - axis and y - axis, we can assume the coordinates of A to be (- x_1, y_1) and the coordinates of B to be (- x_1, - y_1).

Now, we have $y_1=\pm\frac{b}{a}\sqrt{a^2-x_1^2}$.

$\therefore$Coordinates of A are $\left(-x_1,\ \frac{b}{a}\sqrt{a^2-x_1^2}\right)$ and the coordinates of B are $\left(x_1,\ -\frac{b}{a}\sqrt{a^2-x_1^2}\right)$.

As the point (x_1, y_1) lies on the ellipse, the area of triangle ABC (A)is given by,

$$A=\frac{1}{2}\left|a\left(\frac{2b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)+(-x_1)\left(-\frac{b}{a}\sqrt{a^2-x_1^2}\right)\right|$$

$$\Rightarrow A=b\sqrt{a^2-x_1^2}+x_1\frac{b}{a}\sqrt{a^2-x_1^2} \qquad \ldots(1)$$

$$\therefore \frac{dA}{dx_1}=\frac{-2x_1b}{2\sqrt{a^2-x_1^2}}+\frac{b}{a}\sqrt{a^2-x_1^2}-\frac{2bx_1^2}{a2\sqrt{a^2-x_1^2}}$$

$$=\frac{b}{a\sqrt{a^2-x_1^2}}\left[-x_1a+\left(a^2-x_1^2\right)-x_1^2\right]$$

$$=\frac{b\left(-2x_1^2-x_1a+a^2\right)}{a\sqrt{a^2-x_1^2}}$$

$$\text{Now, } \frac{dA}{dx_1}=0$$

$$\Rightarrow -2x_1^2-x_1a+a^2=0$$

$$\Rightarrow x_1=\frac{a\pm\sqrt{a^2-4(-2)\left(a^2\right)}}{2(-2)}$$

$$=\frac{a\pm\sqrt{9a^2}}{-4}$$

$$=\frac{a\pm 3a}{-4}$$

$$\Rightarrow x_1=-a,\ \frac{a}{2}$$

But, $x_1$cannot be equal to a.

$$\therefore x_1=\frac{a}{2}\Rightarrow y_1=\frac{b}{a}\sqrt{a^2-\frac{a^2}{4}}=\frac{ba}{2a}\sqrt{3}=\frac{\sqrt{3}b}{2}$$

$$\text{Now, } \frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{\sqrt{a^2-x_1^2}\,(-4x_1-a)-\left(-2x_1^2-x_1a+a^2\right)\frac{(-2x_1)}{2\sqrt{a^2-x_1^2}}}{a^2-x_1^2}\right\}$$

$$=\frac{b}{a}\left\{\frac{\left(a^2-x_1^2\right)(-4x_1-a)+x_1\left(-2x_1^2-x_1a+a^2\right)}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

$$=\frac{b}{a}\left\{\frac{2x^3-3a^2x-a^3}{\left(a^2-x_1^2\right)^{\frac{3}{2}}}\right\}$$

Also, when $x_1=\frac{a}{2}$, then

$$\frac{d^2A}{dx_1^2}=\frac{b}{a}\left\{\frac{2\frac{a^3}{8}-3\frac{a^3}{2}-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}=\frac{b}{a}\left\{\frac{\frac{a^3}{4}-\frac{3}{2}a^3-a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}$$

$$=-\frac{b}{a}\left\{\frac{\frac{9}{4}a^3}{\left(\frac{3a^2}{4}\right)^{\frac{3}{2}}}\right\}<0$$

Thus, the area is the maximum when $x_1=\frac{a}{2}$.

$\therefore$ Maximum area of the triangle is given by,

$$A = b\sqrt{a^2 - \frac{a^2}{4}} + \left(\frac{a}{2}\right)\frac{b}{a}\sqrt{a^2 - \frac{a^2}{4}}$$

$$= ab\frac{\sqrt{3}}{2} + \left(\frac{a}{2}\right)\frac{b}{a} \times \frac{a\sqrt{3}}{2}$$

$$= \frac{ab\sqrt{3}}{2} + \frac{ab\sqrt{3}}{4} = \frac{3\sqrt{3}}{4}ab$$

Answer needs Correction? **Click Here**

Q9 : A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq meters for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?

Answer :

Let l, b, and h represent the length, breadth, and height of the tank respectively.

Then, we have height (h) = 2 m

Volume of the tank = 8m^3

Volume of the tank = $l \times b \times h$

$\therefore 8 = l \times b \times 2$

$$\Rightarrow lb = 4 \Rightarrow b = \frac{4}{l}$$

Now, area of the base = lb = 4

Area of the 4 walls (A) = $2h(l + b)$

$$\therefore A = 4\left(l + \frac{4}{l}\right)$$

$$\Rightarrow \frac{dA}{dl} = 4\left(1 - \frac{4}{l^2}\right)$$

$$\text{Now, } \frac{dA}{dl} = 0$$

$$\Rightarrow 1 - \frac{4}{l^2} = 0$$

$$\Rightarrow l^2 = 4$$

$$\Rightarrow l = \pm 2$$

However, the length cannot be negative.

Therefore, we have l = 4.

$$\therefore b = \frac{4}{l} = \frac{4}{2} = 2$$

$$\text{Now, } \frac{d^2A}{dl^2} = \frac{32}{l^3}$$

$$\text{When } l = 2,\ \frac{d^2A}{dl^2} = \frac{32}{8} = 4 > 0.$$

Thus, by second derivative test, the area is the minimum when l = 2.

We have $l = b = h = 2$.

$\therefore$Cost of building the base = Rs 70 ×(lb) = Rs 70 (4) = Rs 280

Cost of building the walls = Rs $2h(l + b)$ ×45 = Rs 90 (2) (2 + 2)

= Rs 8 (90) = Rs 720

Required total cost = Rs (280 + 720) = Rs 1000

Hence, the total cost of the tank will be Rs 1000.

Answer needs Correction? **Click Here**

Q10 : The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Answer :

Let r be the radius of the circle and a be the side of the square.

Then, we have:

$$2\pi r + 4a = k \text{ (where } k \text{ is constant)}$$

$$\Rightarrow a = \frac{k - 2\pi r}{4}$$

The sum of the areas of the circle and the square (A) is given by,

$$A = \pi r^2 + a^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}$$

$$\therefore \frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}$$

$$\text{Now, } \frac{dA}{dr} = 0$$

$$\Rightarrow 2\pi r = \frac{\pi(k - 2\pi r)}{4}$$

$$8r = k - 2\pi r$$

$$\Rightarrow (8 + 2\pi)r = k$$

$$\Rightarrow r = \frac{k}{8 + 2\pi} = \frac{k}{2(4 + \pi)}$$

$$\text{Now, } \frac{d^2A}{dr^2} = 2\pi + \frac{\pi^2}{2} > 0$$

$$\therefore \text{When } r = \frac{k}{2(4 + \pi)},\ \frac{d^2A}{dr^2} > 0.$$

$\therefore$ The sum of the areas is least when $r = \dfrac{k}{2(4+\pi)}$.

When $r = \dfrac{k}{2(4+\pi)}$, $a = \dfrac{k - 2\pi\left[\dfrac{k}{2(4+\pi)}\right]}{4} = \dfrac{k(4+\pi)-\pi k}{4(4+\pi)} = \dfrac{4k}{4(4+\pi)} = \dfrac{k}{4+\pi} = 2r$.

Hence, it has been proved that the sum of their areas is least when the side of the square is double the radius of the circle.

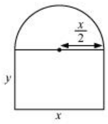
Answer needs Correction? **Click Here**

Q11 : A window is in the form of rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Answer :

Let x and y be the length and breadth of the rectangular window.

Radius of the semicircular opening $= \dfrac{x}{2}$



It is given that the perimeter of the window is 10 m.

$$\therefore\ x + 2y + \frac{\pi x}{2} = 10$$
$$\Rightarrow x\left(1+\frac{\pi}{2}\right) + 2y = 10$$
$$\Rightarrow 2y = 10 - x\left(1+\frac{\pi}{2}\right)$$
$$\Rightarrow y = 5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)$$

$\therefore$Area of the window (A) is given by,

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2$$
$$= x\left[5 - x\left(\frac{1}{2}+\frac{\pi}{4}\right)\right] + \frac{\pi}{8}x^2$$
$$= 5x - x^2\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{8}x^2$$
$$\therefore\ \frac{dA}{dx} = 5 - 2x\left(\frac{1}{2}+\frac{\pi}{4}\right) + \frac{\pi}{4}x$$
$$= 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x$$
$$\therefore\ \frac{d^2A}{dx^2} = -\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4} = -1 - \frac{\pi}{4}$$

Now, $\dfrac{dA}{dx} = 0$

$$\Rightarrow 5 - x\left(1+\frac{\pi}{2}\right) + \frac{\pi}{4}x = 0$$
$$\Rightarrow 5 - x - \frac{\pi}{4}x = 0$$
$$\Rightarrow x\left(1+\frac{\pi}{4}\right) = 5$$
$$\Rightarrow x = \frac{5}{\left(1+\frac{\pi}{4}\right)} = \frac{20}{\pi+4}$$

Thus, when $x = \dfrac{20}{\pi+4}$ then $\dfrac{d^2A}{dx^2} < 0$.

Therefore, by second derivative test, the area is the maximum when length $x = \dfrac{20}{\pi+4}$ m.

Now,

$$y = 5 - \frac{20}{\pi+4}\left(\frac{2+\pi}{4}\right) = 5 - \frac{5(2+\pi)}{\pi+4} = \frac{10}{\pi+4}\ \text{m}$$

Hence, the required dimensions of the window to admit maximum light is given by length $= \dfrac{20}{\pi+4}$ m and breadth $= \dfrac{10}{\pi+4}$ m.

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