Question 2:

Check the injectivity and surjectivity of the following functions:

(i) f: **N** → **N** given by $f(x) = x^2$

(ii) f: **Z** → **Z** given by $f(x) = x^2$

(iii) f: **R** → **R** given by $f(x) = x^2$

(iv) f: **N** → **N** given by $f(x) = x^3$

(v) f: **Z** → **Z** given by $f(x) = x^3$

Answer

(i) f: **N** → **N** is given by,

$f(x) = x^2$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^2 = y^2 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in$ **N**. But, there does not exist any x in **N** such that $f(x) = x^2 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

(ii) f: **Z** → **Z** is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in$ **Z**. But, there does not exist any element $x \in$ **Z** such that $f(x) = x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iii) f: **R** → **R** is given by,

$f(x) = x^2$

It is seen that $f(-1) = f(1) = 1$, but $-1 \neq 1$.

$\therefore f$ is not injective.

Now, $-2 \in$ **R**. But, there does not exist any element $x \in$ **R** such that $f(x) = x^2 = -2$.

$\therefore f$ is not surjective.

Hence, function f is neither injective nor surjective.

(iv) f: **N** → **N** given by,

$f(x) = x^3$

It is seen that for $x, y \in$ **N**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in$ **N**. But, there does not exist any element x in domain **N** such that $f(x) = x^3 = 2$.

$\therefore f$ is not surjective

Hence, function f is injective but not surjective.

(v) f: **Z** → **Z** is given by,

$f(x) = x^3$

It is seen that for $x, y \in$ **Z**, $f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$.

$\therefore f$ is injective.

Now, $2 \in$ **Z**. But, there does not exist any element x in domain **Z** such that $f(x) = x^3 = 2$.

$\therefore f$ is not surjective.

Hence, function f is injective but not surjective.

Question 3:

Prove that the Greatest Integer Function f: **R** → **R** given by $f(x) = [x]$, is neither one-once nor onto, where $[x]$ denotes the greatest integer less than or equal to x.

Answer

f: **R** → **R** is given by,

$f(x) = [x]$

It is seen that $f(1.2) = [1.2] = 1$, $f(1.9) = [1.9] = 1$.

$\therefore f(1.2) = f(1.9)$, but $1.2 \neq 1.9$.

$\therefore f$ is not one-one.

Now, consider $0.7 \in$ **R**.

It is known that $f(x) = [x]$ is always an integer. Thus, there does not exist any element $x \in$ **R** such that $f(x) = 0.7$.

$\therefore f$ is not onto.

Hence, the greatest integer function is neither one-one nor onto.

Question 4:

Show that the Modulus Function *f*: **R** → **R** given by $f(x)=|x|$, is neither one-one nor onto, where $|x|$ is *x*, if *x* is positive or 0 and $|x|$ is − *x*, if *x* is negative.

Answer

f: **R** → **R** is given by,

$$f(x)=|x|=\begin{cases}x, \text{ if } x\geq 0\\ -x, \text{ if } x<0\end{cases}$$

It is seen that $f(-1)=|-1|=1, f(1)=|1|=1$.

∴$f(-1) = f(1)$, but $-1 \neq 1$.

∴ *f* is not one-one.

Now, consider $-1 \in$ **R**.

It is known that $f(x) = |x|$ is always non-negative. Thus, there does not exist any element *x* in domain **R** such that $f(x) = |x| = -1$.

∴ *f* is not onto.

Hence, the modulus function is neither one-one nor onto.

Question 5:

Show that the Signum Function *f*: **R** → **R**, given by

$$f(x)=\begin{cases}1, \text{ if } x>0\\ 0, \text{ if } x=0\\ -1, \text{ if } x<0\end{cases}$$

is neither one-one nor onto.

Answer

f: **R** → **R** is given by,

$$f(x)=\begin{cases}1, \text{ if } x>0\\ 0, \text{ if } x=0\\ -1, \text{ if } x<0\end{cases}$$

It is seen that $f(1) = f(2) = 1$, but $1 \neq 2$.

∴*f* is not one-one.

Now, as $f(x)$ takes only 3 values (1, 0, or −1) for the element −2 in co-domain **R**, there does not exist any *x* in domain **R** such that $f(x) = -2$.

∴ *f* is not onto.

Hence, the signum function is neither one-one nor onto.

Question 6:

Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from *A* to *B*. Show that *f* is one-one.

Answer

It is given that $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$.

$f: A \to B$ is defined as $f = \{(1, 4), (2, 5), (3, 6)\}$.

∴ $f(1) = 4, f(2) = 5, f(3) = 6$

It is seen that the images of distinct elements of *A* under *f* are distinct.

Hence, function *f* is one-one.

Question 7:

In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.

(i) *f*: **R** → **R** defined by $f(x) = 3 - 4x$

(ii) *f*: **R** → **R** defined by $f(x) = 1 + x^2$

Answer

(i) *f*: **R** → **R** is defined as $f(x) = 3 - 4x$.

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1)=f(x_2)$.

$$\Rightarrow 3-4x_1=3-4x_2$$
$$\Rightarrow -4x_1=-4x_2$$
$$\Rightarrow x_1=x_2$$

∴ *f* is one-one.

For any real number (y) in **R**, there exists $\frac{3-y}{4}$ in **R** such that

$$f\left(\frac{3-y}{4}\right)=3-4\left(\frac{3-y}{4}\right)=y.$$

∴*f* is onto.

Hence, *f* is bijective.

(ii) *f*: **R** → **R** is defined as

$f(x)=1+x^2$.

Let $x_1, x_2 \in \mathbf{R}$ such that $f(x_1)=f(x_2)$.

$$\Rightarrow 1+x_1^2=1+x_2^2$$

$\Rightarrow x_1^2 = x_2^2$
$\Rightarrow x_1 = \pm x_2$

$\therefore f(x_1) = f(x_2)$ does not imply that $x_1 = x_2$.

For instance,

$f(1) = f(-1) = 2$

$\therefore$ *f* is not one-one.

Consider an element −2 in co-domain **R**.

It is seen that $f(x) = 1 + x^2$ is positive for all $x \in$ **R**.

Thus, there does not exist any *x* in domain **R** such that *f*(*x*) = −2.

$\therefore$ *f* is not onto.

Hence, *f* is neither one-one nor onto.

Question 8:

Let *A* and *B* be sets. Show that *f*: *A* × *B* → *B* × *A* such that (*a*, *b*) = (*b*, *a*) is bijective function.

Answer

f: *A* × *B* → *B* × *A* is defined as *f*(*a*, *b*) = (*b*, *a*).

Let $(a_1, b_1), (a_2, b_2) \in A \times B$ such that $f(a_1, b_1) = f(a_2, b_2)$.

$\Rightarrow (b_1, a_1) = (b_2, a_2)$
$\Rightarrow b_1 = b_2$ and $a_1 = a_2$
$\Rightarrow (a_1, b_1) = (a_2, b_2)$

$\therefore$ *f* is one-one.

Now, let (*b*, *a*) ∈ *B* × *A* be any element.

Then, there exists (*a*, *b*) ∈ *A* × *B* such that *f*(*a*, *b*) = (*b*, *a*). [By definition of *f*]

$\therefore$ *f* is onto.

Hence, *f* is bijective.

Question 9:

Let *f*: **N** → **N** be defined by $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

State whether the function f is bijective. Justify your answer.

Answer

f: **N** → **N** is defined as $f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$ for all $n \in \mathbf{N}$.

It can be observed that:

$f(1) = \frac{1+1}{2} = 1$ and $f(2) = \frac{2}{2} = 1$ [By definition of f]

$\therefore f(1) = f(2)$, where $1 \neq 2$.

$\therefore$ *f* is not one-one.

*********** END **********