Mean Value Theorems Ex 15.1 Q3(xv)

Here,

$$f(x) = 4^{\sin x} \text{ on } [0, \pi]$$

We know that exponential and $\sin x$ both are continuous and differentiable, so $f(x)$ is continuous is $[0, \pi]$ and differentiable is $(0, \pi)$.

Now,

$$f(0) = 4^{\sin 0} = 4^0 = 1$$

$$f(\pi) = 4^{\sin \pi} = 4^0 = 1$$

$$\Rightarrow \quad f(0) = f(\pi)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (0, \pi)$ such that $f'(c) = 0$.

Now,

$$f(x) = 4^{\sin x}$$

$$f'(x) = 4^{\sin x} \log 4 \times \cos x$$

Now,

$$f'(c) = 0$$

$$4^{\sin c} \times \cos x c \log 4 = 0$$

$$\Rightarrow \quad \cos c = 0$$

$$\Rightarrow \quad c = \frac{\pi}{2} \in (0, \pi)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvi)

Here,

$$f(x) = x^2 - 5x + 4 \text{ on } [1, 4]$$

$f(x)$ is continuous and differentiable as it is a polynomial funciton.

Now,

$$f(1) = (1)^2 - 5(1) + 4 = 0$$

$$f(4) = (4)^2 - 5(4) + 4 = 0$$

$$\Rightarrow \quad f(1) = f(4)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in (1, 4)$ such that $f'(c) = 0$.

Now,

$$f(x) = x^2 - 5x + 4$$

$$f'(x) = 2x - 5$$

So,

$$f'(c) = 0$$

$$\Rightarrow \quad 2c - 5 = 0$$

$$\Rightarrow \quad c = \frac{5}{2} \in (1, 4)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Here,

$$f(x) = \sin^4 x + \cos^4 x \text{ on } \left[0, \frac{\pi}{2}\right]$$

We know that *sine* and *cosine* function are differentiable and continuous.

So, $f(x)$ is continuous is $\left[0, \frac{\pi}{2}\right]$ and it is differentiable is $\left(0, \frac{\pi}{2}\right)$.

Now,

$$f(0) = \sin^4(0) + \cos^4(0) = 1$$

$$f\left(\frac{\pi}{2}\right) = \sin^4\left(\frac{\pi}{2}\right) + \cos^4\left(\frac{\pi}{2}\right) = 1$$

$$\Rightarrow \quad f(0) = f\left(\frac{\pi}{2}\right)$$

So, Rolle's theorem is applicable, so there must exist a point $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

Now,

$$f(x) = \sin^4 x + \cos^4 x$$

$$f'(x) = 4\sin^3 x \cos x - 4\cos^3 x \sin x$$

$$= -2(2\sin x \cos x)(\cos^2 x - \cos^2 x)$$

$$= -2\sin 2x \cos 2x$$

$$f'(x) = -\sin 4x$$

Now,

$$f'(c) = 0$$

$$-\sin 4x = 0$$

$$\sin 4x = 0$$

$$\Rightarrow \quad 4x = 0 \quad \text{or} \quad 4x = \pi$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence, Rolle's theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xvii)

Since trigonometric functions are differentiable and continuous,

the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2(2\cos^2 x - 1)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$\Rightarrow \cos x = \dfrac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180^\circ - \cos^{-1}(0.5931) \quad \left[\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)\right]$

$\Rightarrow x = 32^\circ 32' \text{ or } x = 126^\circ 23'$

Both $32^\circ 32'$ and $126^\circ 23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

Mean Value Theorems Ex 15.1 Q3(xviii)

Since trigonometric functions are differentiable and continuous,
the given function, $f(x) = \sin x - \sin 2x$ is also continuous and differentiable.

Now $f(0) = \sin 0 - \sin 2 \times 0 = 0$

and

$f(\pi) = \sin \pi - \sin 2 \times \pi = 0$

$\Rightarrow f(0) = f(\pi)$

Thus, $f(x)$ satisfies conditions of the Rolle's Theorem on $[0, \pi]$.

Therefore, there exists $c \in [0, \pi]$ such that $f'(c) = 0$

Now $f(x) = \sin x - \sin 2x$

$\Rightarrow f'(x) = \cos x - 2\cos 2x = 0$

$\Rightarrow \cos x = 2\cos 2x$

$\Rightarrow \cos x = 2(2\cos^2 x - 1)$

$\Rightarrow \cos x = 4\cos^2 x - 2$

$\Rightarrow 4\cos^2 x - \cos x - 2 = 0$

$\Rightarrow \cos x = \dfrac{1 \pm \sqrt{33}}{8} = 0.8431 \text{ or } -0.5931$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } \cos^{-1}(-0.5931)$

$\Rightarrow x = \cos^{-1}(0.8431) \text{ or } 180° - \cos^{-1}(0.5931) \quad \left[\because \cos^{-1}(-x) = \pi - \cos^{-1}(x)\right]$

$\Rightarrow x = 32°32' \text{ or } x = 126°23'$

Both $32°32'$ and $126°23' \in [0, \pi]$ such that $f'(c) = 0$.

Hence Rolle's Theorem is verified.

*********** END **********