Question 15:

Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

Answer

The given lines are $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$, is given by,

$$d=\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}} \qquad ...(1)$$

Comparing the given equations, we obtain

$x_1=-1,\ y_1=-1,\ z_1=-1$

$a_1=7,\ b_1=-6,\ c_1=1$

$x_2=3,\ y_2=5,\ z_2=7$

$a_2=1,\ b_2=-2,\ c_2=1$

$$\text{Then,} \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$=4(-6+2)-6(7-1)+8(-14+6)$$

$$=-16-36-64$$

$$=-116$$

$$\Rightarrow \sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}=\sqrt{(-6+2)^2+(1+7)^2+(-14+6)^2}$$

$$=\sqrt{16+36+64}$$

$$=\sqrt{116}$$

$$=2\sqrt{29}$$

Substituting all the values in equation (1), we obtain

$$d=\frac{-116}{2\sqrt{29}}=\frac{-58}{\sqrt{29}}=\frac{-2\times 29}{\sqrt{29}}=-2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is $2\sqrt{29}$ units.

Question 16:

Find the shortest distance between the lines whose vector equations are

$\vec{r}=(\hat{i}+2\hat{j}+3\hat{k})+\lambda(\hat{i}-3\hat{j}+2\hat{k})$

and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

Answer

The given lines are $\vec{r}=\hat{i}+2\hat{j}+3\hat{k}+\lambda(\hat{i}-3\hat{j}+2\hat{k})$ and $\vec{r}=4\hat{i}+5\hat{j}+6\hat{k}+\mu(2\hat{i}+3\hat{j}+\hat{k})$

It is known that the shortest distance between the lines, $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, is given by,

$$d=\left|\frac{(\vec{b}_1\times\vec{b}_2)\cdot(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right| \qquad ...(1)$$

Comparing the given equations with $\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ and $\vec{r}=\vec{a}_2+\mu\vec{b}_2$, we obtain

$\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}$

$\vec{b}_1=\hat{i}-3\hat{j}+2\hat{k}$

$\vec{a}_2=4\hat{i}+5\hat{j}+6\hat{k}$

$\vec{b}_2=2\hat{i}+3\hat{j}+\hat{k}$

$\vec{a}_2-\vec{a}_1=(4\hat{i}+5\hat{j}+6\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=3\hat{i}+3\hat{j}+3\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) = \left(-9\hat{i} + 3\hat{j} + 9\hat{k}\right) \cdot \left(3\hat{i} + 3\hat{j} + 3\hat{k}\right)$$
$$= -9 \times 3 + 3 \times 3 + 9 \times 3$$
$$= 9$$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{9}{3\sqrt{19}}\right| = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$ and

$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$

Answer

The given lines are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$

$$\Rightarrow \vec{r} = \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) + t\left(-\hat{i} + \hat{j} - 2\hat{k}\right) \quad ...(1)$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

$$\Rightarrow \vec{r} = \left(\hat{i} - \hat{j} + \hat{k}\right) + s\left(\hat{i} + 2\hat{j} - 2\hat{k}\right) \quad ...(2)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \left|\frac{\left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_2\right)}{\left|\vec{b}_1 \times \vec{b}_2\right|}\right| \quad ...(3)$$

For the given equations,

$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$

$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$

$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$

$$\vec{a}_2 - \vec{a}_1 = \left(\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} - 2\hat{j} + 3\hat{k}\right) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore \left(\vec{b}_1 \times \vec{b}_2\right) \cdot \left(\vec{a}_2 - \vec{a}_1\right) = \left(2\hat{i} - 4\hat{j} - 3\hat{k}\right) \cdot \left(\hat{j} - 4\hat{k}\right) = -4 + 12 = 8$$

Substituting all the values in equation (3), we obtain

$$d = \left|\frac{8}{\sqrt{29}}\right| = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise 11.3

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a)z = 2 (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d)5*y* + 8 = 0

Answer

(a) The equation of the plane is *z* = 2 or 0*x* + 0*y* + *z* = 2 ... (1)

The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$$

Dividing both sides of equation (1) by 1, we obtain

$0.x + 0.y + 1.z = 2$

This is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) *x* + *y* + *z* = 1 ... (1)

The direction ratios of normal are 1, 1, and 1.

$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \qquad ...(2)$$

This equation is of the form *lx* + *my* + *nz* = *d*, where *l*, *m*, *n* are the direction cosines of normal to the plane and *d* is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \text{ and } \frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) 2*x* + 3*y* − *z* = 5 ... (1)

The direction ratios of normal are 2, 3, and −1.

$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

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