CHAPTER 2

PHYSICS AND MATHEMATICS

Mathematics is the language of physics. It becomes easier to describe, understand and apply the physical principles, if one has a good knowledge of mathematics. In the present course we shall constantly be using the techniques of algebra, trigonometry and geometry as well as vector algebra, differential calculus and integral calculus. In this chapter we shall discuss the latter three topics. Errors in measurement and the concept of significant digits are also introduced.

2.1 VECTORS AND SCALARS

Certain physical quantities are completely described by a numerical value alone (with units specified) and are added according to the ordinary rules of algebra. As an example the mass of a system is described by saying that it is 5 kg. If two bodies one having a mass of 5 kg and other having a mass of 2 kg are added together to make a composite system, the total mass of the system becomes 5 kg + 2 kg = 7 kg. Such quantities are called *scalars*.

The complete description of certain physical quantities requires a numerical value (with units specified) as well as a direction in space. Velocity of a particle is an example of this kind. The magnitude of velocity is represented by a number such as 5 m/s and tells us how fast a particle is moving. But the description of velocity becomes complete only when the direction of velocity is also specified. We can represent this velocity by drawing a line parallel to the velocity and putting an arrow showing the direction of velocity. We can decide beforehand a particular length to represent 1 m/s and the length of the line representing a velocity of 5 m/s may be taken as 5 times this unit length. Figure (2.1) shows representations of several velocities in this scheme. The front end (carrying the arrow) is called the head and the rear end is called the tail.

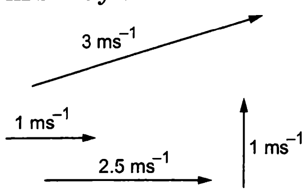

Figure 2.1

Further, if a particle is given two velocities simultaneously its resultant velocity is different from the two velocities and is obtained by using a special rule. Suppose a small ball is moving inside a long tube at a speed 3 m/s and the tube itself is moving in the room at a speed 4 m/s along a direction perpendicular to its length. In which direction and how fast is the ball moving as seen from the room ?

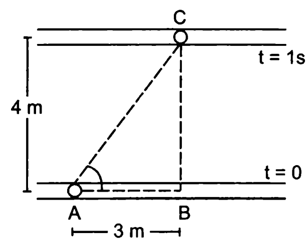

Figure 2.2

Figure (2.2) shows the positions of the tube and the ball at t = 0 and t = 1 s. Simple geometry shows that the ball has moved 5 m in a direction $\theta = 53°$ from the tube. So the resultant velocity of the ball is 5 m/s along this direction. The general rule for finding the resultant of two velocities may be stated as follows.

Draw a line *AB* representing the first velocity with *B* as the head. Draw another line *BC* representing the second velocity with its tail *B* coinciding with the head of the first line. The line *AC* with *A* as the tail and *C* as the head represents the resultant velocity. Figure (2.3) shows the construction.

The resultant is also called the sum of the two velocities. We have added the two velocities *AB* and *BC* and have obtained the sum *AC*. This rule of addition is called the "triangle rule of addition".

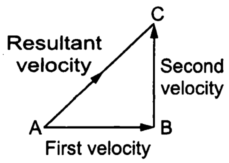

Figure 2.3

The physical quantities which have magnitude and direction and which can be added according to the triangle rule, are called *vector quantities*. Other examples of vector quantities are force, linear momentum, electric field, magnetic field etc.

The vectors are denoted by putting an arrow over the symbols representing them. Thus, we write $\overrightarrow{AB}$, $\overrightarrow{BC}$ etc. Sometimes a vector is represented by a single letter such as $\vec{v}$, $\vec{F}$ etc. Quite often in printed books the vectors are represented by bold face letters like **AB**, **BC**, **v**, **f** etc.

If a physical quantity has magnitude as well as direction but does not add up according to the triangle rule, it will not be called a vector quantity. Electric current in a wire has both magnitude and direction but there is no meaning of triangle rule there. Thus, electric current is not a vector quantity.

2.2 EQUALITY OF VECTORS

Two vectors (representing two values of the same physical quantity) are called equal if their magnitudes and directions are same. Thus, a parallel translation of a vector does not bring about any change in it.

2.3 ADDITION OF VECTORS

The triangle rule of vector addition is already described above. If $\vec{a}$ and $\vec{b}$ are the two vectors to be added, a diagram is drawn in which the tail of $\vec{b}$ coincides with the head of $\vec{a}$. The vector joining the tail of $\vec{a}$ with the head of $\vec{b}$ is the vector sum of $\vec{a}$ and $\vec{b}$. Figure (2.4a) shows the construction. The same rule

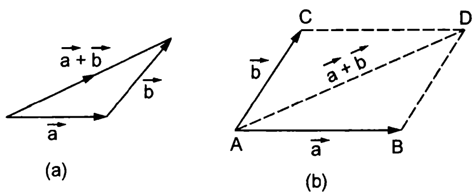

Figure 2.4

may be stated in a slightly different way. We draw the vectors $\vec{a}$ and $\vec{b}$ with both the tails coinciding (figure 2.4b). Taking these two as the adjacent sides we complete the parallelogram. The diagonal through the common tails gives the sum of the two vectors. Thus, in figure, (2.4b) $\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AD}$.

Suppose the magnitude of $\vec{a} = a$ and that of $\vec{b} = b$. What is the magnitude of $\vec{a} + \vec{b}$ and what is its direction ? Suppose the angle between $\vec{a}$ and $\vec{b}$ is θ. It is easy to see from figure (2.5) that

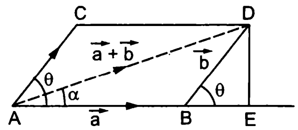

Figure 2.5

$$AD^2 = (AB + BE)^2 + (DE)^2$$
$$= (a + b\cos\theta)^2 + (b\sin\theta)^2$$
$$= a^2 + 2ab\cos\theta + b^2.$$

Thus, the magnitude of $\vec{a} + \vec{b}$ is

$$\sqrt{a^2 + b^2 + 2ab\cos\theta}. \quad \ldots (2.1)$$

Its angle with $\vec{a}$ is α where

$$\tan\alpha = \frac{DE}{AE} = \frac{b\sin\theta}{a + b\cos\theta}. \quad \ldots (2.2)$$

Example 2.1

Two vectors having equal magnitudes A make an angle θ with each other. Find the magnitude and direction of the resultant.

Solution : The magnitude of the resultant will be

$$B = \sqrt{A^2 + A^2 + 2AA\cos\theta}$$
$$= \sqrt{2A^2(1 + \cos\theta)} = \sqrt{4A^2\cos^2\frac{\theta}{2}}$$
$$= 2A\cos\frac{\theta}{2}.$$

The resultant will make an angle α with the first vector where

$$\tan\alpha = \frac{A\sin\theta}{A + A\cos\theta} = \frac{2A\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2A\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

or, $\alpha = \dfrac{\theta}{2}$

Thus, the resultant of two equal vectors bisects the angle between them.

2.4 MULTIPLICATION OF A VECTOR BY A NUMBER

Suppose $\vec{a}$ is a vector of magnitude a and k is a number. We define the vector $\vec{b} = k\vec{a}$ as a vector of magnitude $|ka|$. If k is positive the direction of the vector $\vec{b} = k\vec{a}$ is same as that of $\vec{a}$. If k is negative, the direction of $\vec{b}$ is opposite to $\vec{a}$. In particular, multiplication by (–1) just inverts the direction of the vector. The vectors $\vec{a}$ and $-\vec{a}$ have equal magnitudes but opposite directions.

If $\vec{a}$ is a vector of magnitude a and $\vec{u}$ is a vector of unit magnitude in the direction of $\vec{a}$, we can write $\vec{a} = a\vec{u}$.

2.5 SUBTRACTION OF VECTORS

Let $\vec{a}$ and $\vec{b}$ be two vectors. We define $\vec{a} - \vec{b}$ as the sum of the vector $\vec{a}$ and the vector $(-\vec{b})$. To subtract $\vec{b}$ from $\vec{a}$, invert the direction of $\vec{b}$ and add to $\vec{a}$. Figure (2.6) shows the process.

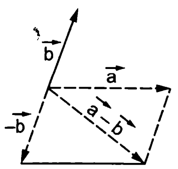

Figure 2.6

Example 2.2

Two vectors of equal magnitude 5 unit *have an angle* 60° *between them. Find the magnitude of* (a) *the sum of the vectors and* (b) *the difference of the vectors.*

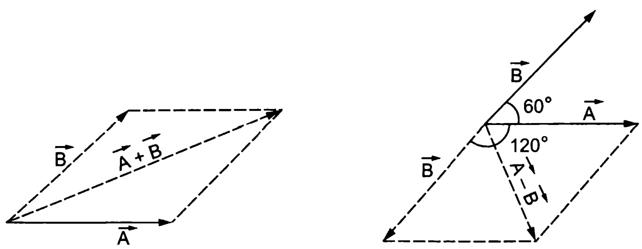

Figure 2.7

Solution : Figure (2.7) shows the construction of the sum $\vec{A} + \vec{B}$ and the difference $\vec{A} - \vec{B}$.

(a) $\vec{A} + \vec{B}$ is the sum of $\vec{A}$ and $\vec{B}$. Both have a magnitude of 5 unit and the angle between them is 60°. Thus, the magnitude of the sum is

$$|\vec{A} + \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 60^\circ}$$
$$= 2 \times 5 \cos 30^\circ = 5\sqrt{3} \text{ unit.}$$

(b) $\vec{A} - \vec{B}$ is the sum of $\vec{A}$ and $(-\vec{B})$. As shown in the figure, the angle between $\vec{A}$ and $(-\vec{B})$ is 120°. The magnitudes of both $\vec{A}$ and $(-\vec{B})$ is 5 unit. So,

$$|\vec{A} - \vec{B}| = \sqrt{5^2 + 5^2 + 2 \times 5 \times 5 \cos 120^\circ}$$
$$= 2 \times 5 \cos 60^\circ = 5 \text{ unit.}$$

2.6 RESOLUTION OF VECTORS

Figure (2.8) shows a vector $\vec{a} = \overrightarrow{OA}$ in the X-Y plane drawn from the origin O. The vector makes an angle α with the X-axis and β with the Y-axis. Draw perpendiculars AB and AC from A to the X and Y axes respectively. The length OB is scalled the projection of $\overrightarrow{OA}$ on X-axis. Similarly OC is the projection of $\overrightarrow{OA}$ on Y-axis. According to the rules of vector addition

$$\vec{a} = \overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{OC}.$$

Thus, we have resolved the vector $\vec{a}$ into two parts, one along OX and the other along OY. The magnitude of the part along OX is $OB = a \cos\alpha$ and the magnitude of the part along OY is $OC = a \cos\beta$. If $\vec{i}$ and $\vec{j}$ denote vectors of unit magnitude along OX and OY respectively, we get

$$\overrightarrow{OB} = a \cos\alpha\, \vec{i} \text{ and } \overrightarrow{OC} = a \cos\beta\, \vec{j}$$

so that $$\vec{a} = a \cos\alpha\, \vec{i} + a \cos\beta\, \vec{j}.$$

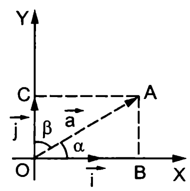

Figure 2.8

If the vector $\vec{a}$ is not in the X-Y plane, it may have nonzero projections along X, Y, Z axes and we can resolve it into three parts i.e., along the X, Y and Z axes. If α, β, γ be the angles made by the vector $\vec{a}$ with the three axes respectively, we get

$$\vec{a} = a \cos\alpha\, \vec{i} + a \cos\beta\, \vec{j} + a \cos\gamma\, \vec{k} \qquad \ldots \text{(2.3)}$$

where $\vec{i}$, $\vec{j}$ and $\vec{k}$ are the unit vectors along X, Y and Z axes respectively. The magnitude ($a \cos\alpha$) is called the component of $\vec{a}$ along X-axis, ($a \cos\beta$) is called the component along Y-axis and ($a \cos\gamma$) is called the component along Z-axis. In general, the component of a vector $\vec{a}$ along a direction making an angle θ with it