Question 10:

Find which of the operations given above has identity.

Answer

An element $e \in \mathbf{Q}$ will be the identity element for the operation * if

$a * e = a = e * a, \ \forall a \in \mathbf{Q}$.

However, there is no such element $e \in \mathbf{Q}$ with respect to each of the six operations satisfying the above condition.

Thus, none of the six operations has identity.

Question 11:

Let A = $\mathbf{N} \times \mathbf{N}$ and * be the binary operation on A defined by

$(a, b) * (c, d) = (a + c, b + d)$

Show that * is commutative and associative. Find the identity element for * on A, if any.

Answer

A = $\mathbf{N} \times \mathbf{N}$

* is a binary operation on A and is defined by:

$(a, b) * (c, d) = (a + c, b + d)$

Let $(a, b), (c, d) \in A$

Then, $a, b, c, d \in \mathbf{N}$

We have:

$(a, b) * (c, d) = (a + c, b + d)$

$(c, d) * (a, b) = (c + a, d + b) = (a + c, b + d)$

[Addition is commutative in the set of natural numbers]

$\therefore (a, b) * (c, d) = (c, d) * (a, b)$

Therefore, the operation * is commutative.

Now, let $(a, b), (c, d), (e, f) \in A$

Then, $a, b, c, d, e, f \in \mathbf{N}$

We have:

$$((a,b)*(c,d))*(e,f)=(a+c,\ b+d)*(e,f)=(a+c+e,\ b+d+f)$$

$$(a,b)*((c,d)*(e,f))=(a,b)*(c+e,\ d+f)=(a+c+e,\ b+d+f)$$

$$\therefore ((a,b)*(c,d))*(e,f)=(a,b)*((c,d)*(e,f))$$

Therefore, the operation * is associative.

An element $e=(e_1,e_2)\in A$ will be an identity element for the operation * if $a*e=a=e*a \ \forall \ a=(a_1,a_2)\in A$, i.e., $(a_1+e_1,a_2+e_2)=(a_1,a_2)=(e_1+a_1,e_2+a_2)$, which is not true for any element in A.

Therefore, the operation * does not have any identity element.

Question 12:

State whether the following statements are true or false. Justify.

(i) For an arbitrary binary operation * on a set **N**, $a * a = a \ \forall a * \mathbf{N}$.

(ii) If * is a commutative binary operation on **N**, then $a * (b * c) = (c * b) * a$

Answer

(i) Define an operation * on **N** as:

$a * b = a + b \ \forall a, b \in \mathbf{N}$

Then, in particular, for $b = a = 3$, we have:

$3 * 3 = 3 + 3 = 6 \neq 3$

Therefore, statement (i) is false.

(ii) R.H.S. = $(c * b) * a$

$= (b * c) * a$ [* is commutative]

$= a * (b * c)$ [Again, as * is commutative]

= L.H.S.

$\therefore a * (b * c) = (c * b) * a$

Therefore, statement (ii) is true.

Question 13:

Consider a binary operation * on **N** defined as $a * b = a^3 + b^3$. Choose the correct answer.

(A) Is * both associative and commutative?
(B) Is * commutative but not associative?
(C) Is * associative but not commutative?
(D) Is * neither commutative nor associative?

Answer

On **N**, the operation * is defined as $a * b = a^3 + b^3$.

For, a, b, ∈ **N**, we have:

$a * b = a^3 + b^3 = b^3 + a^3 = b * a$ [Addition is commutative in **N**]

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1^3+2^3)*3=9*3=9^3+3^3=729+27=756$$

$$1*(2*3)=1*(2^3+3^3)=1*(8+27)=1\times35=1^3+35^3=1+(35)^3=1+42875=42876$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where 1, 2, 3 ∈ **N**

Therefore, the operation * is not associative.

Hence, the operation * is commutative, but not associative. Thus, the correct answer is B.

Miscellaneous Questions:

Question 1:

Let f: **R** → **R** be defined as $f(x) = 10x + 7$. Find the function g: **R** → **R** such that $g \circ f = f \circ g = 1_{\mathbf{R}}$.

Answer

It is given that f: **R** → **R** is defined as $f(x) = 10x + 7$.

One-one:

Let $f(x) = f(y)$, where x, y ∈**R**.

$\Rightarrow 10x + 7 = 10y + 7$

$\Rightarrow x = y$

∴ f is a one-one function.

Onto:

For y ∈ **R**, let $y = 10x + 7$.

$$\Rightarrow x=\frac{y-7}{10}\in\mathbf{R}$$

Therefore, for any y ∈ **R**, there exists $x=\frac{y-7}{10}\in\mathbf{R}$ such that

$$f(x)=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y.$$

∴ f is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define g: **R** → **R** as $g(y)=\frac{y-7}{10}$.

Now, we have:

$$gof(x)=g(f(x))=g(10x+7)=\frac{(10x+7)-7}{10}=\frac{10x}{10}=10$$

And,

$$fog(y)=f(g(y))=f\left(\frac{y-7}{10}\right)=10\left(\frac{y-7}{10}\right)+7=y-7+7=y$$

$$\therefore gof=\mathrm{I}_{\mathbf{R}} \text{ and } fog=\mathrm{I}_{\mathbf{R}}$$

Hence, the required function g: **R** → **R** is defined as $g(y)=\frac{y-7}{10}$.

Question 2:

Let f: W → W be defined as $f(n) = n - 1$, if is odd and $f(n) = n + 1$, if n is even. Show that f is invertible. Find the inverse of f. Here, W is the set of all whole numbers.

Answer

It is given that:

f: W → W is defined as $f(n)=\begin{cases} n-1, & \text{if } n \text{ is odd} \\ n+1, & \text{if } n \text{ is even} \end{cases}$

One-one:

Let $f(n) = f(m)$.

It can be observed that if n is odd and m is even, then we will have $n - 1 = m + 1$.

$\Rightarrow n - m = 2$

However, this is impossible.

Similarly, the possibility of n being even and m being odd can also be ignored under a similar argument.

∴Both n and m must be either odd or even.

Now, if both n and m are odd, then we have:

$f(n) = f(m) \Rightarrow n - 1 = m - 1 \Rightarrow n = m$

Again, if both n and m are even, then we have:

$f(n) = f(m) \Rightarrow n + 1 = m + 1 \Rightarrow n = m$

$\therefore f$ is one-one.

It is clear that any odd number $2r + 1$ in co-domain **N** is the image of $2r$ in domain **N** and any even number $2r$ in co-domain **N** is the image of $2r + 1$ in domain **N**.

$\therefore f$ is onto.

Hence, f is an invertible function.

Let us define g: W → W as:

$$g(m) = \begin{cases} m+1, & \text{if } m \text{ is even} \\ m-1, & \text{if } m \text{ is odd} \end{cases}$$

Now, when n is odd:

$$gof(n) = g(f(n)) = g(n-1) = n-1+1 = n$$

And, when n is even:

$$gof(n) = g(f(n)) = g(n+1) = n+1-1 = n$$

Similarly, when m is odd:

$$fog(m) = f(g(m)) = f(m-1) = m-1+1 = m$$

When m is even:

$$fog(m) = f(g(m)) = f(m+1) = m+1-1 = m$$

$\therefore$ $\therefore gof = \mathrm{I_W}$ and $fog = \mathrm{I_W}$

Thus, f is invertible and the inverse of f is given by $f^{-1} = g$, which is the same as f.

Hence, the inverse of f is f itself.

Question 3:

If f: **R** → **R** is defined by $f(x) = x^2 - 3x + 2$, find $f(f(x))$.

Answer

It is given that f: **R** → **R** is defined as $f(x) = x^2 - 3x + 2$.

$$\begin{aligned} f(f(x)) &= f(x^2 - 3x + 2) \\ &= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2 \\ &= x^4 + 9x^2 + 4 - 6x^3 - 12x + 4x^2 - 3x^2 + 9x - 6 + 2 \\ &= x^4 - 6x^3 + 10x^2 - 3x \end{aligned}$$

Question 4:

*********** END **********