Binary Operations Ex 3.4 Q1

Given,

$$a * b = a + b - 4 \text{ for all } a, b \in Z$$

(i)

Commutative: Let $a, b \in Z$, then

$\Rightarrow \quad a * b = a + b - 4 = b + a - 4 = b * a$

$\Rightarrow \quad a * b = b * a$

So, '$*$' is commutative on Z.

Associativity: Let $a, b, c \in Z$, then

$$(a * b) * c = (a + b - 4) * c = a + b - 4 + c - 4$$
$$= a + b + c - 8 \qquad ---(i)$$

and, $\quad a * (b * c) = a * (b + c - 4) = a + b + c - 8 \qquad ---(ii)$

From (i) & (ii)

$$(a * b) * c = a * (b * c)$$

So, '$*$' is associative on Z.

(ii)

Let $e \in Z$ be the identity element with respect to $*$.

By identity property, we have

$$a * e = e * a = a \quad \text{for all } a \in Z$$

$\Rightarrow \quad a + e - 4 = a$

$\Rightarrow \quad e = 4$

So, $e = 4$ will be the identity element with respect to $*$

(iii)

Let $b \in Z$ be the inverse element of $a \in Z$

Then, $\quad a * b = b * a = e$

$\Rightarrow \quad a + b - 4 = e$

$\Rightarrow \quad a + b - 4 = 4 \qquad [\because e = 4]$

$\Rightarrow \quad b = 8 - a$

Thus, $\quad b = 8 - a$ will be the inverse element of $a \in Z$.

Binary Operations Ex 3.4 Q2

We have,

$$a * b = \frac{3ab}{5} \text{ for all } a, b \in Q_0$$

(i)

Commutative: Let $a, b \in Q_0$, then

$$a * b = \frac{3ab}{5} = \frac{3ba}{5} = b * a$$

$$\Rightarrow \qquad a * b = b * a$$

So, '$*$' is commutative on Q_0

Associativity: Let $a, b, c \in Q_0$, then

$$(a * b) * c = \frac{3ab}{5} * c$$
$$= \frac{9abc}{25} \qquad ---(i)$$

and, $$a * (b * c) = a * \frac{3bc}{5}$$
$$= \frac{9abc}{25} \qquad ---(ii)$$

From (i) & (ii)

$$(a * b) * c = a * (b * c)$$

So, '$*$' is associative on Q_0

(ii)

Let $e \in Q_0$ be the identity element with respect to $*$, then

$$a * e = e * a = a \text{ for all } a \in Q_0$$

$$\Rightarrow \qquad \frac{3ae}{5} = a$$

$$\Rightarrow \qquad e = \frac{5}{3}$$

will be the identity element with respect to $*$.

(iii)

Let $b \in Q_0$ be the inverse element of $a \in Q_0$, then

$$a * b = b * a = e$$

$$\Rightarrow \qquad \frac{3}{5}ab = e$$

$$\Rightarrow \qquad \frac{3}{5}ab = \frac{5}{3} \qquad \left[\because e = \frac{5}{3}\right]$$

$$\Rightarrow \qquad b = \frac{25}{9a}$$

$\therefore \qquad b = \frac{25}{9a}$ is the inverse of $a \in Q_0$

$\therefore \quad b = \frac{}{9a}$ is the inverse of $a \in Q_0$.

Binary Operations Ex 3.4 Q3

We have,

$$a * b = a + b + ab \text{ for all } a, b \in Q - \{-1\}$$

(i)

Commutativity: Let $a, b \in Q - \{-1\}$

$\Rightarrow \quad a * b = a + b + ab = b + a + ba = b * a$

$\Rightarrow \quad a * b = b * a$

$\Rightarrow \quad$ '$*$' is commutative on $Q - \{-1\}$

Associativity: Let $a, b, c \in Q - \{-1\}$, then

$\Rightarrow \quad (a * b) * c = (a + b + ab) * c$

$\qquad = a + b + ab + c + ac + bc + abc \qquad ---(i)$

and, $\quad a * (b * c) = a * (b + c + bc)$

$\qquad = a + b + c + bc + ab + ac + abc \qquad ---(ii)$

From (i) & (ii)

$$(a * b) * c = a * (b * c)$$

$\Rightarrow \quad *$ is associative on $Q - \{-1\}$

(ii)

Let e be identity element with respect to $*$.

By identity property,

$$a * e = a = e * a \text{ for all } a \in Q - \{-1\}$$

$\Rightarrow \quad a + e + ae = a$

$\Rightarrow \quad e(1 + a) = 0 \Rightarrow e = 0 \qquad [\because 1 + a \neq 0 \text{ as } a \neq -1]$

$\therefore \quad e = 0$ is the identity element with respect to $*$

(iii)

Let b be the inverse of $a \in Q - \{-1\}$

Then, $\quad a * b = b * a = e \qquad [e \text{ is the identity element}]$

$\Rightarrow \quad a + b + ab = e$

$\Rightarrow \quad a + b + ab = 0$

$\Rightarrow \quad b(1 + a) = -a$

$\Rightarrow \quad b = \frac{-a}{1 + a} \qquad \left[\begin{array}{l}\because \frac{-a}{1+a} \neq -1, \text{ because if } \frac{-a}{1+a} = -1 \\ \Rightarrow a = 1 + a \Rightarrow 1 = 0 \text{ Not possible}\end{array}\right]$

$\therefore \quad b = \frac{-a}{1 + a}$ is the inverse of a with respect to $*$

Binary Operations Ex 3.4 Q4

We have,

$$(a,b) \odot (c,d) = (ac, bc+d) \text{ for all } (a,b), (c,d) \in R_0 \times R$$

(i)

Commutativity: Let $(a,b), (c,d) \in R_0 \times R$, then

$$\Rightarrow \quad (a,b) \odot (c,d) = (ac, bc+d) \qquad ---(i)$$

$$\text{and,} \quad (c,d) \odot (a,b) = (ca, da+b) \qquad ---(ii)$$

From (i) & (ii)

$$(a,b) \odot (c,d) \neq (c,d) \odot (a,b)$$

$\Rightarrow$ '$\odot$' is not commutative on $R_0 \times R$.

Associativity: Let $(a,b), (c,d), (e,f) \in R_0 \times R$, then

$$\Rightarrow \quad ((a,b) \odot (c,d)) \odot (e,f) = (ac, bc+d) \odot (e,f)$$
$$= (ace, bce, de+f) \qquad ---(i)$$

$$\text{and,} \quad (a,b) \odot (c,d \odot (e,f)) = (a,b) \odot (ce, de+f)$$
$$= (ace, bce+de+f) \qquad ---(ii)$$

$$\Rightarrow \quad ((a,b) \odot (c,d)) \odot (e,f) = (a,b) \odot ((c,d) \odot (e,f))$$

$\Rightarrow$ '$\odot$' is associative on $R_0 \times R$.

(ii)

Let $(x,y) \in R_0 \times R$ be the identity element with respect to $\odot$, then

$$(a,b) \odot (x,y) = (x,y) \odot (a,b) = (a,b) \text{ for all } (a,b) \in R_0 \times R$$

$$\Rightarrow \quad (ax, bx+y) = (a,b)$$
$$\Rightarrow \quad ax = a \text{ and } bx + y = b$$
$$\Rightarrow \quad x = 1, \text{ and } y = 0$$

$\therefore$ $(1,0)$ will be the identity element with respect to $\odot$.

(iii)

Let $(c,d) \in R_0 \times R$ be the inverse of $(a,b) \in R_0 \times R$, then

$$(a,b) \odot (c,d) = (c,d) \odot (a,b) = e$$

$$\Rightarrow \quad (ac, bc+d) = (1,0) \qquad [\because e = (1,0)]$$
$$\Rightarrow \quad ac = 1 \text{ and } bc + d = 0$$
$$\Rightarrow \quad c = \frac{1}{a} \text{ and } d = -\frac{b}{a}$$

$\therefore$ $\left(\frac{1}{a}, -\frac{b}{a}\right)$ will be the inverse of (a,b).

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