∴ f is onto, thereby range $f = [-5, \infty)$.

Let us define $g: [-5, \infty) \to \mathbf{R}_+$ as $g(y) = \dfrac{\sqrt{y+6}-1}{3}$.

We now have:

$$(gof)(x) = g(f(x)) = g(9x^2+6x-5)$$
$$= g((3x+1)^2-6)$$
$$= \frac{\sqrt{(3x+1)^2-6+6}-1}{3}$$
$$= \frac{3x+1-1}{3} = x$$

$$\text{And,} (fog)(y) = f(g(y)) = f\left(\frac{\sqrt{y+6}-1}{3}\right)$$
$$= \left[3\left(\frac{\sqrt{y+6}-1}{3}\right)+1\right]^2 - 6$$
$$= \left(\sqrt{y+6}\right)^2 - 6 = y+6-6 = y$$

∴ $gof = I_{\mathbf{R}_+}$ and $fo\,g = I_{[-5,\infty)}$

Hence, f is invertible and the inverse of f is given by

$$f^{-1}(y) = g(y) = \frac{\sqrt{y+6}-1}{3}.$$

Question 10:

Let $f: X \to Y$ be an invertible function. Show that f has unique inverse.

(Hint: suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$,

$fog_1(y) = I_Y(y) = fog_2(y)$. Use one-one ness of f).

Answer

Let $f: X \to Y$ be an invertible function.

Also, suppose f has two inverses (say g_1 and g_2).

Then, for all $y \in Y$, we have:

$$fog_1(y) = I_Y(y) = fog_2(y)$$
$$\Rightarrow f(g_1(y)) = f(g_2(y))$$
$$\Rightarrow g_1(y) = g_2(y) \qquad [f \text{ is invertible} \Rightarrow f \text{ is one-one}]$$
$$\Rightarrow g_1 = g_2 \qquad [g \text{ is one-one}]$$

Hence, f has a unique inverse.

Question 11:

Consider $f: \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$ and $f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

Answer

Function $f: \{1, 2, 3\} \to \{a, b, c\}$ is given by,

$f(1) = a$, $f(2) = b$, and $f(3) = c$

If we define $g: \{a, b, c\} \to \{1, 2, 3\}$ as $g(a) = 1$, $g(b) = 2$, $g(c) = 3$, then we have:

$$(fog)(a) = f(g(a)) = f(1) = a$$
$$(fo\,g)(b) = f(g(b)) = f(2) = b$$
$$(fog)(c) = f(g(c)) = f(3) = c$$

And,

$$(gof)(1) = g(f(1)) = g(a) = 1$$
$$(gof)(2) = g(f(2)) = g(b) = 2$$
$$(gof)(3) = g(f(3)) = g(c) = 3$$

∴ $gof = I_X$ and $fog = I_Y$, where $X = \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

Thus, the inverse of f exists and $f^{-1} = g$.

∴ $f^{-1}: \{a, b, c\} \to \{1, 2, 3\}$ is given by,

$f^{-1}(a) = 1$, $f^{-1}(b) = 2$, $f^{-1}(c) = 3$

Let us now find the inverse of f^{-1} i.e., find the inverse of g.

If we define h: {1, 2, 3} → {a, b, c} as

$h(1) = a$, $h(2) = b$, $h(3) = c$, then we have:

$$(g\circ h)(1) = g(h(1)) = g(a) = 1$$
$$(g\circ h)(2) = g(h(2)) = g(b) = 2$$
$$(g\circ h)(3) = g(h(3)) = g(c) = 3$$

And,

$$(h\circ g)(a) = h(g(a)) = h(1) = a$$
$$(h\circ g)(b) = h(g(b)) = h(2) = b$$
$$(h\circ g)(c) = h(g(c)) = h(3) = c$$

$\therefore$ $goh = \mathrm{I}_X$ and $hog = \mathrm{I}_Y$, where X = {1, 2, 3} and Y = {a, b, c}.

Thus, the inverse of g exists and $g^{-1} = h \Rightarrow (f^{-1})^{-1} = h$.

It can be noted that $h = f$.

Hence, $(f^{-1})^{-1} = f$.

Question 12:

Let f: $X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

Answer

Let f: $X \to Y$ be an invertible function.

Then, there exists a function g: $Y \to X$ such that $gof = \mathrm{I}_X$ and $fog = \mathrm{I}_Y$.

Here, $f^{-1} = g$.

Now, $gof = \mathrm{I}_X$ and $fog = \mathrm{I}_Y$

$\Rightarrow f^{-1}of = \mathrm{I}_X$ and $fof^{-1} = \mathrm{I}_Y$

Hence, f^{-1}: $Y \to X$ is invertible and f is the inverse of f^{-1}

i.e., $(f^{-1})^{-1} = f$.

Question 13:

If f: $\mathbf{R} \to \mathbf{R}$ be given by $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$, then $fof(x)$ is

(A) $\frac{1}{x^3}$ (B) x^3 (C) x (D) $(3 - x^3)$

Answer

f: $\mathbf{R} \to \mathbf{R}$ is given as $f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$.

$$f(x) = \left(3 - x^3\right)^{\frac{1}{3}}$$

$$\therefore fof(x) = f(f(x)) = f\left(\left(3 - x^3\right)^{\frac{1}{3}}\right) = \left[3 - \left(\left(3 - x^3\right)^{\frac{1}{3}}\right)^3\right]^{\frac{1}{3}}$$

$$= \left[3 - \left(3 - x^3\right)\right]^{\frac{1}{3}} = \left(x^3\right)^{\frac{1}{3}} = x$$

$$\therefore fof(x) = x$$

The correct answer is C.

Question 14:

Let $f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is map g: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ given by

(A) $g(y) = \frac{3y}{3-4y}$ (B) $g(y) = \frac{4y}{4-3y}$

(C) $g(y) = \frac{4y}{3-4y}$ (D) $g(y) = \frac{3y}{4-3y}$

Answer

It is given that $f: \mathbf{R} - \left\{-\frac{4}{3}\right\} \to \mathbf{R}$ is defined as $f(x) = \frac{4x}{3x+4}$.

Let y be an arbitrary element of Range f.

Then, there exists $x \in \mathbf{R} - \left\{-\frac{4}{3}\right\}$ such that $y = f(x)$.

$$\Rightarrow y = \frac{4x}{3x+4}$$
$$\Rightarrow 3xy + 4y = 4x$$
$$\Rightarrow x(4 - 3y) = 4y$$
$$\Rightarrow x = \frac{4y}{4-3y}$$

Let us define g: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$ as $g(y) = \frac{4y}{4-3y}$.

Now, $(g\circ f)(x) = g(f(x)) = g\left(\frac{4x}{3x+4}\right)$

$$4\left(\frac{4x}{\quad}\right)$$

$$= \frac{4\left(\frac{4x}{3x+4}\right)}{4-3\left(\frac{4x}{3x+4}\right)} = \frac{16x}{12x+16-12x} = \frac{16x}{16} = x$$

$$\text{And}, (f \circ g)(y) = f(g(y)) = f\left(\frac{4y}{4-3y}\right)$$

$$= \frac{4\left(\frac{4y}{4-3y}\right)}{3\left(\frac{4y}{4-3y}\right)+4} = \frac{16y}{12y+16-12y} = \frac{16y}{16} = y$$

$\therefore$ $gof = I_{\mathbf{R}-\left\{-\frac{4}{3}\right\}}$ and $fog = I_{\text{Range } f}$

Thus, g is the inverse of f i.e., $f^{-1} = g$.

Hence, the inverse of f is the map g: Range $f \to \mathbf{R} - \left\{-\frac{4}{3}\right\}$, which is given by

$$g(y) = \frac{4y}{4-3y}.$$

The correct answer is B.

Exercise 1.4

Question 1:

Determine whether or not each of the definition of given below gives a binary operation.

In the event that * is not a binary operation, give justification for this.

(i) On $\mathbf{Z}^+$, define * by $a * b = a - b$

(ii) On $\mathbf{Z}^+$, define * by $a * b = ab$

(iii) On $\mathbf{R}$, define * by $a * b = ab^2$

(iv) On $\mathbf{Z}^+$, define * by $a * b = |a - b|$

(v) On $\mathbf{Z}^+$, define * by $a * b = a$

Answer

(i) On $\mathbf{Z}^+$, * is defined by $a * b = a - b$.

It is not a binary operation as the image of (1, 2) under * is $1 * 2 = 1 - 2$

$= -1 \notin \mathbf{Z}^+$.

(ii) On $\mathbf{Z}^+$, * is defined by $a * b = ab$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element ab in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b = ab$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(iii) On $\mathbf{R}$, * is defined by $a * b = ab^2$.

It is seen that for each $a, b \in \mathbf{R}$, there is a unique element ab^2 in $\mathbf{R}$.

This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in $\mathbf{R}$.

*********** END **********