(v) Which elements of **N** are invertible for the operation *?

Answer

The binary operation * on **N** is defined as $a * b$ = L.C.M. of a and b.

(i) 5 * 7 = L.C.M. of 5 and 7 = 35

20 * 16 = L.C.M of 20 and 16 = 80

(ii) It is known that:

L.C.M of a and b = L.C.M of b and a □ a, b ∈ **N**.

∴$a * b = b * a$

Thus, the operation * is commutative.

(iii) For a, b, c ∈ **N**, we have:

$(a * b) * c$ = (L.C.M of a and b) * c = LCM of a, b, and c

$a * (b * c) = a$ * (LCM of b and c) = L.C.M of a, b, and c

∴$(a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(iv) It is known that:

L.C.M. of a and 1 = a = L.C.M. 1 and a □ a ∈ **N**

⇒ $a * 1 = a = 1 * a$ □ a ∈ **N**

Thus, 1 is the identity of * in **N**.

(v) An element a in **N** is invertible with respect to the operation * if there exists an element b in **N**, such that $a * b = e = b * a$.

Here, $e = 1$

This means that:

L.C.M of a and b = 1 = L.C.M of b and a

This case is possible only when a and b are equal to 1.

Thus, 1 is the only invertible element of **N** with respect to the operation *.

Question 7:

Is * defined on the set {1, 2, 3, 4, 5} by $a * b$ = L.C.M. of a and b a binary operation? Justify your answer.

Answer

The operation * on the set A = {1, 2, 3, 4, 5} is defined as

$a * b$ = L.C.M. of a and b.

Then, the operation table for the given operation * can be given as:

*	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

It can be observed from the obtained table that:

3 * 2 = 2 * 3 = 6 ∉ A, 5 * 2 = 2 * 5 = 10 ∉ A, 3 * 4 = 4 * 3 = 12 ∉ A

3 * 5 = 5 * 3 = 15 ∉ A, 4 * 5 = 5 * 4 = 20 ∉ A

Hence, the given operation * is not a binary operation.

Question 8:

Let * be the binary operation on **N** defined by $a * b$ = H.C.F. of a and b. Is * commutative? Is * associative? Does there exist identity for this binary operation on **N**?

Answer

The binary operation * on **N** is defined as:

$a * b$ = H.C.F. of a and b

It is known that:

H.C.F. of a and b = H.C.F. of b and a □ a, b ∈ **N**.

∴$a * b = b * a$

Thus, the operation * is commutative.

For a, b, c ∈ **N**, we have:

$(a * b)* c$ = (H.C.F. of a and b) * c = H.C.F. of a, b, and c

$a * (b * c) = a * $(H.C.F. of b and c) = H.C.F. of a, b, and c

$\therefore (a * b) * c = a * (b * c)$

Thus, the operation * is associative.

Now, an element $e \in \mathbf{N}$ will be the identity for the operation * if $a * e = a = e * a$ $\forall a \in \mathbf{N}$.

But this relation is not true for any $a \in \mathbf{N}$.

Thus, the operation * does not have any identity in **N**.

Question 9:

Let * be a binary operation on the set **Q** of rational numbers as follows:

(i) $a * b = a - b$ (ii) $a * b = a^2 + b^2$

(iii) $a * b = a + ab$ (iv) $a * b = (a - b)^2$

(v) $a*b=\dfrac{ab}{4}$ (vi) $a * b = ab^2$

Find which of the binary operations are commutative and which are associative.

Answer

(i) On **Q**, the operation * is defined as $a * b = a - b$.

It can be observed that:

$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{3-2}{6}=\frac{1}{6}$ and $\frac{1}{3}*\frac{1}{2}=\frac{1}{3}-\frac{1}{2}=\frac{2-3}{6}=\frac{-1}{6}$

$\therefore \frac{1}{2}*\frac{1}{3}\neq\frac{1}{3}*\frac{1}{2}$; where $\frac{1}{2}, \frac{1}{3}\in\mathbf{Q}$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left(\frac{1}{2}-\frac{1}{3}\right)*\frac{1}{4}=\frac{1}{6}*\frac{1}{4}=\frac{1}{6}-\frac{1}{4}=\frac{2-3}{12}=\frac{-1}{12}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{1}{2}*\frac{1}{12}=\frac{1}{2}-\frac{1}{12}=\frac{6-1}{12}=\frac{5}{12}$$

$$\therefore \left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right) \text{; where } \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\in\mathbf{Q}$$

Thus, the operation * is not associative.

(ii) On **Q**, the operation * is defined as $a * b = a^2 + b^2$.

For $a, b \in \mathbf{Q}$, we have:

$a*b=a^2+b^2=b^2+a^2=b*a$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1^2+2^2)*3=(1+4)*4=5*4=5^2+4^2=41$$

$$1*(2*3)=1*(2^2+3^2)=1*(4+9)=1*13=1^2+13^2=169$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in\mathbf{Q}$$

Thus, ,the operation * is not associative.

(iii) On **Q**, the operation * is defined as $a * b = a + ab$.

It can be observed that:

$$1*2=1+1\times2=1+2=3$$

$$2*1=2+2\times1=2+2=4$$

$$\therefore 1*2\neq 2*1 \text{ ; where } 1, 2 \in\mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3=(1+1\times2)*3=3*3=3+3\times3=3+9=12$$

$$1*(2*3)=1*(2+2\times3)=1*8=1+1\times8=9$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in\mathbf{Q}$$

Thus, the operation * is not associative.

(iv) On **Q**, the operation * is defined by $a * b = (a - b)^2$.

For $a, b \in \mathbf{Q}$, we have:

$a * b = (a - b)^2$

$b * a = (b - a)^2 = [-(a - b)]^2 = (a - b)^2$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

It can be observed that:

$$(1*2)*3=(1-2)^2*3=(-1)^2*3=1*3=(1-3)^2=(-2)^2=4$$

$$1*(2*3)=1*(2-3)^2=1*(-1)^2=1*1=(1-1)^2=0$$

$$\therefore (1*2)*3\neq 1*(2*3) \text{ ; where } 1, 2, 3 \in\mathbf{Q}$$

Thus, the operation * is not associative.

(v) On **Q**, the operation * is defined as $a*b=\frac{ab}{4}$.

For a, $b \in$ **Q**, we have:

$$a*b=\frac{ab}{4}=\frac{ba}{4}=b*a$$

$\therefore a * b = b * a$

Thus, the operation * is commutative.

For a, b, $c \in$ **Q**, we have:

$$(a*b)*c=\frac{ab}{4}*c=\frac{\frac{ab}{4}\cdot c}{4}=\frac{abc}{16}$$

$$a*(b*c)=a*\frac{bc}{4}=\frac{a\cdot\frac{bc}{4}}{4}=\frac{abc}{16}$$

$\therefore (a * b) * c = a * (b * c)$

Thus, the operation * is associative.

(vi) On **Q**, the operation * is defined as $a * b = ab^2$

It can be observed that:

$$\frac{1}{2}*\frac{1}{3}=\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2=\frac{1}{2}\cdot\frac{1}{9}=\frac{1}{18}$$

$$\frac{1}{3}*\frac{1}{2}=\frac{1}{3}\cdot\left(\frac{1}{2}\right)^2=\frac{1}{3}\cdot\frac{1}{4}=\frac{1}{12}$$

$$\therefore \frac{1}{2}*\frac{1}{3}\neq\frac{1}{3}*\frac{1}{2}\text{ ; where }\frac{1}{2},\frac{1}{3}\in\mathbf{Q}$$

Thus, the operation * is not commutative.

It can also be observed that:

$$\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}=\left[\frac{1}{2}\cdot\left(\frac{1}{3}\right)^2\right]*\frac{1}{4}=\frac{1}{18}*\frac{1}{4}=\frac{1}{18}\cdot\left(\frac{1}{4}\right)^2=\frac{1}{18\times16}$$

$$\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)=\frac{1}{2}*\left[\frac{1}{3}\cdot\left(\frac{1}{4}\right)^2\right]=\frac{1}{2}*\frac{1}{48}=\frac{1}{2}\cdot\left(\frac{1}{48}\right)^2=\frac{1}{2\times(48)^2}$$

$$\therefore\left(\frac{1}{2}*\frac{1}{3}\right)*\frac{1}{4}\neq\frac{1}{2}*\left(\frac{1}{3}*\frac{1}{4}\right)\text{ ; where }\frac{1}{2},\ \frac{1}{3},\ \frac{1}{4}\in\mathbf{Q}$$

Thus, the operation * is not associative.

Hence, the operations defined in (ii), (iv), (v) are commutative and the operation defined in (v) is associative.

*********** END **********