Functions Ex 2.1 Q5(vi)

$f : Z \to Z$ given by $f(x) = x^2 + x$

Injective: let $x, y \in Z$ such that

$f(x) = f(y)$

$\Rightarrow \quad x^2 + x = y^2 + y$

$\Rightarrow \quad x^2 - y^2 + x - y = 0$

$\Rightarrow \quad (x - y)(x + y + 1) = 0$

$\Rightarrow \quad$ either $x - y = 0$ or $x + y + 1 = 0$

Case I: if $x - y = 0$

$\Rightarrow \quad x = y$

$\therefore \quad f$ is injective

Case II if $x + y + 1 = 0$

$\Rightarrow \quad x + y = -1$

$\Rightarrow \quad x \neq y$

$\therefore \quad f$ is not one to one

Thus, in general, f is not one-one

Surjective:

Since $1 \in Z$ (co-domain)

Now, we wish to find if there is any pre-image in domain Z.

let $x \in Z$ such that $f(x) = 1$

$\Rightarrow \quad x^2 + x = 1 \quad \Rightarrow \quad x^2 + x - 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+4}}{2} \notin Z.$$

So, f is not onto.

Functions Ex 2.1 Q5(vii)

$f : Z \to Z$ given by $f(x) = x - 5$

Injective: let $x, y \in Z$ such that

$f(x) = f(y)$

$\Rightarrow$ $x - 5 = y - 5$

$\Rightarrow$ $x = y$

$\therefore$ f is one-one.

Surjective: let $y \in Z$ be an arbitrary element

then $f(x) = y$

$\Rightarrow$ $x - 5 = y$

$\Rightarrow$ $x = y + 5 \quad \in Z\,(\text{domain})$

Thus, for each element in co-domain Z there exists an element in domain Z such that $f(x) = y$

$\therefore$ f in onto.

Since, f in one-one and onto,

$\therefore$ f in bijective.

Functions Ex 2.1 Q5(viii)

$f : R \to R$ given by $f(x) = \sin x$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $\sin x = \sin y$

$\Rightarrow$ $x = n\pi + (-1)^n y$

$\Rightarrow$ $x \neq y$

$\therefore$ f is not one-one.

Surjective: let $y \in R$ be arbitrary such that

$f(x) = y$

$\Rightarrow$ $\sin x = y$

$\Rightarrow$ $x = \sin^{-1} y$

Now, for $y > 1$ $x \notin R\,(\text{domain})$

$\therefore$ f is not onto.

Functions Ex 2.1 Q5(ix)

$f : R \to R$ difined by $f(x) = x^3+1$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow$ $x^3 + 1 = y^3 + 1$

$\Rightarrow$ $x^3 = y^3$

$\Rightarrow$ $x = y$

$\therefore$ f is one-one.

Surjective:

let $y \in R$, then

$f(x) = y$

$\Rightarrow$ $x^3 + 1 = y$ $\Rightarrow$ $x^3 + 1 - y = 0$

We know that degree 3 equation has atleast one real root.

$\therefore$ let $x = \alpha$ be the real root.

$\therefore$ $\alpha^3 + 1 = y$

$\Rightarrow$ $f(\alpha) = y$

Thus, for each $y \in R$, there exist $\alpha \in R$ such that $f(\alpha) = y$

$\therefore$ f is onto.

Since f is one-one and onto, f is bijective.

Functions Ex 2.1 Q5(x)

$f : R \to R$ defined by $f(x) = x^3 - x$

Injective: let $x, y \in R$ such that

$f(x) = f(y)$

$\Rightarrow \quad x^3 - x = y^3 - y$

$\Rightarrow \quad x^3 - y^3 - (x - y) = 0$

$\Rightarrow \quad (x - y)\left(x^2 + xy + y^2 - 1\right) = 0$

$\because \quad x^2 + xy + y^2 \geq 0 \Rightarrow x^2 + xy + y^2 - 1 \geq -1$

$\therefore \quad x^2 + xy + y^2 - 1 \neq 0$

$\Rightarrow \quad x - y = 0 \quad \Rightarrow \quad x = y$

$\therefore \quad f$ is one-one.

Surjective:

let $y \in R$, then

$f(x) = y$

$\Rightarrow \quad x^3 - x - y = 0$

We know that a degree 3 equation has atleast one real solution.

let $x = \alpha$ be that real solution

$\therefore \quad \alpha^3 - \alpha = y$

$\Rightarrow \quad f(\alpha) = y$

$\therefore$ For each $y \in R$, there exist $x = \alpha \in R$

such that $f(\alpha) = y$

$\therefore \quad f$ is onto.

Functions Ex 2.1 Q5(xi)

$f : R \to R$ defined by $f(x) = sin^2 x + cos^2 x$.

Injective: since $f(x) = sin^2 x + cos^2 x = 1$

$\Rightarrow \quad f(x) = 1$ which is a constant function we know that a constant function in neither injective nor surjective

$\therefore \quad f$ is not one-one and not onto.

Functions Ex 2.1 Q5(xii)

$f : Q - [3] \to Q$ defined by $f(x) = \dfrac{2x+3}{x-3}$

Injective: let $x, y \in Q - [3]$ such that

$f(x) = f(y)$

$\Rightarrow \quad \dfrac{2x+3}{x-3} = \dfrac{2y+3}{y-3}$

$\Rightarrow \quad 2xy - 6x + 3y - 9 = 2xy + 3x - 6y - 9$

$\Rightarrow \quad -6x + 3y - 3x + 6y = 0$

$\Rightarrow \quad -9(x - y) = 0$

$\Rightarrow \quad x = y$

$\Rightarrow \quad f$ is one-one.

Surjective:

let $y \in Q$ be arbitrary. then

$f(x) = y$

$\Rightarrow \quad \dfrac{2x+3}{x-3} = y$

$\Rightarrow \quad 2x + 3 = xy - 3y$

$\Rightarrow \quad x(2 - y) = -3(y + 1)$

$\therefore \quad x = \dfrac{-3(y+1)}{2-y} \notin Q - [3]$ for $y = 2$

$\therefore \quad f$ is not onto

Functions Ex 2.1 Q5(xiii)

$f : Q \to Q$ defined by $f(x) = x^3 + 1$

Injective: let $x, y \in Q$ such that

$$f(x) = f(y)$$

$\Rightarrow$ $x^3 + 1 = y^3 + 1$

$\Rightarrow$ $(x^3 - y^3) = 0$

$\Rightarrow$ $(x - y)(x^2 + xy + y^2) = 0$

but $x^2 + xy + y^2 \geq 0$

$\therefore$ $x - y = 0$

$\Rightarrow$ $x = y$

$\therefore$ f is injective.

Surjective: let $y \in Q$ be arbitrary, then

$$f(x) = y$$

$\Rightarrow$ $x^3 + 1 - y = 0$

we know that a degree 3 equation has alteast one real solution.

let $x = \alpha$ be that solution

$\therefore$ $\alpha^3 + 1 = y$

$\therefore$ $f(\alpha) = y$

$\therefore$ f is onto.

Functions Ex 2.1 Q5(xiv)

$f : R \to R$ defined by $f(x) = 5x^3 + 4$

Injective: let $x, y \in R$ such that

$$f(x) = f(y)$$

$\Rightarrow$ $5x^3 + 4 = 5y^3 + 4$

$\Rightarrow$ $5(x^3 - y^3) = 0$

$\Rightarrow$ $5(x - y)(x^2 + xy + y^2) = 0$

but $5(x^2 + xy + y^2) \geq 0$

$\Rightarrow$ $x - y = 0$ $\Rightarrow$ $x = y$

$\therefore$ f is one-one

Surjective: let $y \in R$ be arbitrary, then

$$f(x) = y$$

$\Rightarrow$ $5x^3 + 4 = y$

$\Rightarrow$ $5x^3 + 4 - y = 0$

we know that a degree 3 equation has alteast one real solution.

let $x = \alpha$ be that real solution

$\therefore$ $5\alpha^3 + 4 = y$

$\therefore$ $f(\alpha) = y$

$\therefore$ For each $y \in Q$, there $\alpha \in R$ such that $f(\alpha) = y$

$\therefore$ f is onto

Since f in one-one and onto

$\therefore$ f in bijective.

Functions Ex 2.1 Q5(xv)

$f : R \to R$ defined by $f(x) = 3 - 4x$

Injective: let $x, y \in R$ such that

$$f(x) = f(y)$$

$\Rightarrow \quad 3 - 4x = 3 - 4y$

$\Rightarrow \quad -4(x - y) = 0$

$\Rightarrow \quad x = y$

$\therefore \quad f$ is one-one.

Surjective: let $y \in R$ be arbitrary, such that

$$f(x) = y$$

$\Rightarrow \quad 3 - 4x = y$

$\Rightarrow \quad x = \dfrac{3 - y}{4} \in R$

Thus for each $y \in R$, there exist $x \in R$ such that

$$f(x) = y$$

$\therefore \quad f$ is onto.

Hence, f is one-one and onto and therefore bijective.

*********** END **********