Indefinite Integrals Ex 19.31 Q9

$$\text{Let} \quad I = \int \frac{(x-1)^2}{x^4+x^2+1}\,dx$$

$$= \int \frac{x^2-2x+1}{x^4+x^2+1}\,dx$$

Dividing numerator and denominator by x^2

$$\therefore \quad I = \int \frac{1-\frac{2}{x}+\frac{1}{x^2}}{x^2+1+\frac{1}{x^2}}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x-\frac{1}{x}\right)^2+3}\,dx - \int \frac{2x}{x^4+x^2+1}\,dx$$

$$\text{Let} \left(x-\frac{1}{x}\right) = t$$

$$\Rightarrow \quad \left(1+\frac{1}{x^2}\right)dx = dt \qquad [\text{For Ist part}]$$

$$\text{Let} \quad x^2 = z$$

$$\Rightarrow \quad 2x\,dx = dz \qquad [\text{For IInd part}]$$

$$\therefore \quad I = \int \frac{dt}{t^2+3} - \int \frac{dz}{z^2+z+1}$$

$$= \int \frac{dt}{t^3+3} - \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$\Rightarrow \quad I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c$$

Hence,

$$I = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{3}x}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$$

Indefinite Integrals Ex 19.31 Q10

Let $\quad I = \int \frac{1}{x^4 + 3x^2 + 1} dx$

Dividing numerator and denominator by x^2

$$\therefore \qquad I = \int \frac{\frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{x^2 + 3 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 5} dx - \frac{1}{2} \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 + 1} dx$$

Let $\left(x - \frac{1}{x}\right) = t$

$$\Rightarrow \qquad \left(1 + \frac{1}{x^2}\right) dx = dt$$

and $\quad x + \frac{1}{x} = z$

$$\Rightarrow \qquad \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$\therefore \qquad I = \frac{1}{2} \int \frac{dt}{t^2 + 5} - \frac{1}{2} \int \frac{dz}{z^2 + 1}$$

$$= \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) - \frac{1}{2} \tan^{-1}(z) + c$$

Hence,

$$I = \frac{1}{2\sqrt{5}} \tan^{-1}\left(\frac{x^2 - 1}{\sqrt{5}x}\right) - \frac{1}{2} \tan^{-1}\left(\frac{x^2 + 1}{x}\right) + c$$

Indefinite Integrals Ex 19.31 Q11

Consider the integral

$$I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

Divide both the numerator and the denominator by $\cos^4 x$, we have,

$$I = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}{\cos^4 x}} dx$$

$$= \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{\sec^2 x \times \sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{\left(\tan^2 x + 1\right) \times \sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

Substituting tanx=t; $\sec^2 x dx = dt$

Thus,

$$I = \int \frac{\left(1+t^2\right) dt}{t^4 + t^2 + 1}$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2} + 1\right)}$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2} - 2 + 2 + 1\right)}$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 3}$$

$$= \int \frac{\left(1+\frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 3}$$

Substituting $z = t - \frac{1}{t}$; $dz = \left(1+\frac{1}{t^2}\right) dt$

$$I = \int \frac{dz}{z^2 + 3}$$

$$\Rightarrow I = \int \frac{dz}{z^2 + \left(\sqrt{3}\right)^2}$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{z}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{3}}\right) + C$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{3}}\right) + C$$

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