Definite Integrals Ex 20.2 Q21

We have,

$$\int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx$$

$$\text{Let} \quad \sin x = K\left(\sin x + \cos x\right) + L\frac{d}{dx}\left(\sin x + \cos x\right)$$
$$= K\left(\sin x + \cos x\right) + L\left(\cos x - \sin x\right)$$
$$= \sin x\left(K - L\right) + \cos x\left(K + L\right)$$

Equating similar terms

$$K - L = 1$$
$$K + L = 0$$

$$\Rightarrow K = \frac{1}{2} \text{ and } L = -\frac{1}{2}$$

$$\therefore \int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int_0^{\pi} dx + \left(\frac{-1}{2}\right)\int_0^{\pi} \frac{\cos x - \sin x}{\sin x + \cos x} dx$$
$$= \frac{1}{2}\left[x\right]_0^{\pi} - \frac{1}{2}\left(\log\left|\sin x + \cos x\right|\right)_0^{\pi} = \frac{\pi}{2} - \frac{1}{2}\left(0\right) = \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \frac{\sin x}{\sin x + \cos x} dx = \frac{\pi}{2}$$

Definite Integrals Ex 20.2 Q22

We know,

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$\therefore \frac{1}{3+2\sin x+\cos x}$$

$$= \frac{1}{3+2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \frac{\left(1+\tan^2\frac{x}{2}\right)}{3\left(1+\tan^2\frac{x}{2}\right)+4\tan\frac{x}{2}+\left(1-\tan^2\frac{x}{2}\right)}$$

$$= \frac{\sec^2\frac{x}{2}dx}{3+3\tan^2\frac{x}{2}+4\tan\frac{x}{2}+1-\tan^2\frac{x}{2}}$$

$$\therefore \int_0^{\pi}\frac{1}{3+2\sin x+\cos x}dx = \int_0^{\pi}\frac{\sec^2\frac{x}{2}dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$$

Let $\tan\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

Now,

$x = 0 \Rightarrow t = 0$

$x = \pi \Rightarrow t = \infty$

$$\therefore \int_0^{\pi} \frac{\sec^2 \frac{x}{2} dx}{2 \tan^2 \frac{x}{2} + 4 \tan \frac{x}{2} + 4}$$

$$= \int_0^{\infty} \frac{dt}{t^2 + 2t + 2}$$

$$= \int_0^{\infty} \frac{dt}{(t+1)^2 + 1}$$

$$= \left[\tan^{-1}(t+1)\right]_0^{\infty}$$

$$= \tan^{-1}(\infty) - \tan^{-1}(0+1)$$

$$= \tan^{-1}(\infty) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan \frac{\pi}{2}\right) - \tan^{-1}\left(\tan \frac{\pi}{4}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{2\pi - \pi}{4}$$

$$= \frac{\pi}{4}$$

$$\therefore \int_0^{\pi} \frac{1}{3 + 2 \sin x + \cos x} dx = \frac{\pi}{4}$$

Definite Integrals Ex 20.2 Q23

We have,

$$\int_0^1 1 \cdot \tan^{-1} x \, dx = \tan^{-1} x \int_0^1 dx - \int_0^1 \left(\int dx.\right) \frac{d}{dx}\left(\tan^{-1} x\right) dx$$

$$= \left[x \tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx$$

$$= \left[x \tan^{-1} x - \frac{1}{2} \log\left(1 + x^2\right)\right]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}(\log 2 - 0)$$

$$= \frac{\pi}{4} - \frac{1}{2} \log 2$$

$$\therefore \int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \log 2$$

*********** END **********