$$\Rightarrow \lambda = \frac{10}{9}$$

Substituting $\lambda = \frac{10}{9}$ in equation (3), we obtain

$$\vec{r} \cdot \left(\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k} \right) = 17$$

$$\Rightarrow \vec{r} \cdot \left(38\hat{i} + 68\hat{j} + 3\hat{k} \right) = 153$$

This is the vector equation of the required plane.

Question 11:

Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$

Answer

The equation of the plane through the intersection of the planes, $x + y + z = 1$ and $2x + 3y + 4z = 5$, is

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0 \qquad ...(1)$$

The direction ratios, a_1, b_1, c_1, of this plane are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(4\lambda + 1)$.

The plane in equation (1) is perpendicular to $x - y + z = 0$

Its direction ratios, a_2, b_2, c_2, are 1, −1, and 1.

Since the planes are perpendicular,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow (2\lambda + 1) - (3\lambda + 1) + (4\lambda + 1) = 0$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$$

$$\Rightarrow x - z + 2 = 0$$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$\vec{r} \cdot \left(2\hat{i} + 2\hat{j} - 3\hat{k} \right) = 5$ and $\vec{r} \cdot \left(3\hat{i} - 3\hat{j} + 5\hat{k} \right) = 3$.

Answer

The equations of the given planes are $\vec{r} \cdot \left(2\hat{i} + 2\hat{j} - 3\hat{k} \right) = 5$ and $\vec{r} \cdot \left(3\hat{i} - 3\hat{j} + 5\hat{k} \right) = 3$

It is known that if $\vec{n}_1$ and $\vec{n}_2$ are normal to the planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then the angle between them, Q, is given by,

$$\cos Q = \left| \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \right| \qquad ...(1)$$

Here, $\vec{n}_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{n}_2 = 3\hat{i} - 3\hat{j} + 5\hat{k}$

$$\therefore \vec{n}_1 \cdot \vec{n}_2 = \left(2\hat{i} + 2\hat{j} - 3\hat{k} \right)\left(3\hat{i} - 3\hat{j} + 5\hat{k} \right) = 2.3 + 2.(-3) + (-3).5 = -15$$

$$|\vec{n}_1| = \sqrt{(2)^2 + (2)^2 + (-3)^2} = \sqrt{17}$$

$$|\vec{n}_2| = \sqrt{(3)^2 + (-3)^2 + (5)^2} = \sqrt{43}$$

Substituting the value of $\vec{n} \cdot \vec{n}_2$, $|\vec{n}_1| \text{ and } |\vec{n}_2|$ in equation (1), we obtain

$$\cos Q = \left| \frac{-15}{\sqrt{17} \cdot \sqrt{43}} \right|$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1} = \left(\frac{15}{\sqrt{731}}\right)$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Answer

The direction ratios of normal to the plane, $L_1 : a_1x + b_1y + c_1z = 0$, are a_1, b_1, c_1 and $L_2 : a_1x + b_2y + c_2z = 0$ are a_2, b_2, c_2.

$L_1 \parallel L_2$, if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$L_1 \perp L_2$, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

The angle between L_1 and L_2 is given by,

$$Q = \cos^{-1}\left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}}\right|$$

(a) The equations of the planes are $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

Here. $a_1 = 7$. $b_1 = 5$. $c_1 = 6$

$a_2 = 3,\ b_2 = -1, c_2 = -10$

$a_1a_2 + b_1b_2 + c_1c_2 = 7\times 3 + 5\times(-1) + 6\times(-10) = -44 \neq 0$

Therefore, the given planes are not perpendicular.

$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$

It can be seen that, $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq, \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$Q = \cos^{-1}\left|\frac{7\times 3 + 5\times(-1) + 6\times(-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}}\right|$$

$$= \cos^{-1}\left|\frac{21 - 5 - 60}{\sqrt{110} \times \sqrt{110}}\right|$$

$$= \cos^{-1}\frac{44}{110}$$

$$= \cos^{-1}\frac{2}{5}$$

(b) The equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2,\ b_1 = 1,\ c_1 = 3$ and $a_2 = 1,\ b_2 = -2, c_2 = 0$

$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2\times 1 + 1\times(-2) + 3\times 0 = 0$

Thus, the given planes are perpendicular to each other.

(c) The equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2,\ b_1 - 2,\ c_1 = 4$ and

$a_2 = 3,\ b_2 = -3,\ c_2 = 6 \quad a_1a_2 + b_1b_2 + c_1c_2 = 2\times 3 + (-2)(-3) + 4\times 6 = 6 + 6 + 24 = 36 \neq 0$

Thus, the given planes are not perpendicular to each other.

$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{and} \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given planes are parallel to each other.

(d) The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2,\ b_1 = -1, c_1 = 3$ and $a_2 = 2,\ b_2 = -1,\ c_2 = 3$

$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{and} \frac{c_1}{c_2} = \frac{3}{3} = 1$

$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Thus, the given lines are parallel to each other.

Thus, the given lines are parallel to each other.

(e) The equations of the given planes are $4x+8y+z-8=0$ and $y+z-4=0$

Here, $a_1=4,\ b_1=8,\ c_1=1$ and $a_2=0,\ b_2=1,\ c_2=1$

$a_1a_2+b_1b_2+c_1c_2=4\times0+8\times1+1=9\neq0$

Therefore, the given lines are not perpendicular to each other.

$\frac{a_1}{a_2}=\frac{4}{0},\ \frac{b_1}{b_2}=\frac{8}{1}=8,\ \frac{c_1}{c_2}=\frac{1}{1}=1$

$\therefore \frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq,\frac{c_1}{c_2}$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q=\cos^{-1}\left|\frac{4\times0+8\times1+1\times1}{\sqrt{4^2+8^2+1^2}\times\sqrt{0^2+1^2+1^2}}\right|=\cos^{-1}\left|\frac{9}{9\times\sqrt{2}}\right|=\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)=45°$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

(a) (0, 0, 0) $3x-4y+12z=3$

(b) (3, −2, 1) $2x-y+2z+3=0$

(c) (2, 3, −5) $x+2y-2z=9$

(d) (−6, 0, 0) $2x-3y+6z-2=0$

Answer

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d=\left|\frac{Ax_1+By_1+Cz_1-D}{\sqrt{A^2+B^2+C^2}}\right| \quad ...(1)$$

*********** END **********