Definite Integrals Ex 20.4B Q9

Let $x = \tan\theta$

$\Rightarrow \quad dx = \sec^2\theta d\theta$

If $\quad x = 0, \ \theta = 0$

If $\quad x = 1, \ \theta = \frac{\pi}{4}$

$\therefore \quad \int_0^1 \frac{\log(1+x)}{1+x^2} dx$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log(1+\tan\theta) d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left\{1+\tan\left(\frac{\pi}{4}-\theta\right)\right\} d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\} d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$\Rightarrow \quad I = \int_0^{\frac{\pi}{4}} \left(\log 2 - \log(1+\tan\theta)\right) d\theta$$

$$\Rightarrow \quad 2I = \int_0^{\frac{\pi}{4}} \log 2 \times d\theta = \frac{\pi}{4}\log 2$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2$$

Definite Integrals Ex 20.4B Q10

$$I = \int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx$$

Let,

$$\frac{x}{(1+x)\left(1+x^2\right)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$$

$$\Rightarrow x = A\left(1+x^2\right) + (Bx+C)(1+x)$$

Equating coeffcients, we get

$$A + B = 0 \Rightarrow A = -B$$
$$B + C = 1 \Rightarrow -2A = 1$$
$$A + C = 0 \Rightarrow A = -C$$

$$\therefore\ A = -\frac{1}{2},\ B = \frac{1}{2},\ C = \frac{1}{2}$$

So,

$$I = \int_0^{\infty} \left(\frac{-\frac{1}{2}}{1+x} + \frac{1}{2}\frac{x+1}{x^2+1} \right) dx$$

$$= \int_0^{\infty} -\frac{1}{2}\frac{dx}{1+x} + \frac{1}{2}\int_0^{\infty} \frac{x}{x^2+1} dx + \frac{1}{2}\int_0^{\infty} \frac{dx}{1+x^2}$$

$$= \left[-\frac{1}{2}\log|1+x| + \frac{1}{4}\log\left|x^2+1\right| + \frac{1}{2}\tan^{-1} x \right]_0^{\infty}$$

$$= 0 + 0 + \frac{\pi}{4} + 0 - 0 - 0$$

$$= \frac{\pi}{4}$$

$$\because \int_0^{\infty} \frac{x}{(1+x)\left(1+x^2\right)} dx = \frac{\pi}{4}$$

Definite Integrals Ex 20.4B Q11

We have,

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx$$

$$I = \int_0^{\pi} \frac{x\left(\frac{\sin x}{\cos x}\right)}{\left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)} dx$$

$$I = \int_0^{\pi} x \sin^2 x\, dx \qquad \ldots(i)$$

$$I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x) dx \qquad \left[\because \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx\right]$$

$$I = \int_0^{\pi} (\pi - x)\sin^2 x\, dx \qquad --(ii)$$

Add (i) and (ii), we get

$$2I = \int_0^{\pi} (\pi)\sin^2 x\, dx = \pi\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{\pi}{2}[\pi - 0 - 0 + 0] = \frac{\pi^2}{2}$$

$$\therefore \int_0^{\pi} \frac{x \tan x}{\sec x \operatorname{cosec} x} dx = \frac{\pi^2}{4}$$

Definite Integrals Ex 20.4B Q12

Let $I=\int_0^{\pi} x\sin x.\cos^4 x\,dx$ $\quad$ --(i)

So,

$$I=\int_0^{\pi}(\pi-x)\sin(\pi-x).\cos^4(\pi-x)\,dx \qquad \left[\because \int_0^{a} f(x)dx=\int_0^{a} f(a-x)\,dx\right]$$

$$=\int_0^{\pi}(\pi-x)\sin x.\cos^4 x\,dx$$

$$=\int_0^{\pi}\pi\sin x.\cos^4 x\,dx-\int_0^{\pi}x\sin x.\cos^4 x\,dx$$

So from equation (i)

$$I=\int_0^{\pi}\pi\sin x.\cos^4 x\,dx-I$$

$$2I=\pi\int_0^{\pi}\sin x.\cos^4 x\,dx$$

Let $t=\cos x\,dx$

$dt=-\sin\ x\,dx$

As,

$x=0 \quad t=1$

$x=\pi \quad t=-1$

Hence

$$2I=\pi\int_{-1}^{+1}t^4\,dt=\pi\left[\frac{t^5}{5}\right]_{-1}^{1}=\pi\left[\frac{1}{5}+\frac{1}{5}\right]$$

$$I=\frac{\pi}{5}$$

*********** END **********