Definite Integrals Ex 20.4B Q13

Let $I = \int_0^{\pi} x \sin^3 x \, dx$

$= \int_0^{\pi} (\pi - x) \sin^3 (\pi - x) dx \qquad \left[\because \int_0^{a} f(x) dx = \int_0^{a} f(a - x) dx\right]$

$= \int_0^{\pi} \pi \sin^3 x \, dx - \int_0^{\pi} x \sin^3 x \, dx$

$\therefore \; I = \int_0^{\pi} \pi \sin^3 x \, dx - I$

$\Rightarrow 2I = \pi \int_0^{\pi} \sin^3 x \, dx$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{3 \sin x - \sin 3x}{4} dx$

$= \frac{\pi}{4} \int_0^{\pi} (3 \sin x - \sin 3x) dx$

$= \frac{\pi}{4} \left[-3 \cos x + \frac{\cos 3x}{3}\right]_0^{\pi}$

$= \frac{\pi}{4} \left[\left(-3 \cos \pi + \frac{\cos 3\pi}{3}\right) - \left(-3 \cos 0 + \frac{\cos 0}{3}\right)\right]$

$= \frac{\pi}{4} \left[\left(3 - \frac{1}{3}\right) - \left(-3 + \frac{1}{3}\right)\right]$

$= \frac{\pi}{4} \left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right]$

$\frac{\pi}{4} \left[6 - \frac{2}{3}\right]$

$= \frac{\pi}{4} \times \frac{16}{3} \quad = \quad \frac{4\pi}{3}$

$\therefore \; I = \frac{2\pi}{3}$

Definite Integrals Ex 20.4B Q14

We have,

$$I = \int_0^{\pi} x\log\sin x\,dx = \int_0^{\pi}(\pi - x)\log\sin(\pi - x)\,dx$$

$$I = \pi\int_0^{\pi}\log\sin(x)\,dx - \int_0^{\pi} x\log\sin x\,dx$$

$$2I = \pi\int_0^{\pi}\log\sin x\,dx$$

Since $f(x) = f(-x)$, f(x) is an even function.

$$\therefore\ 2I = 2\pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx$$

$$I = \pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx \qquad \ldots(i)$$

$$\Rightarrow I = \pi\int_0^{\frac{\pi}{2}}\log\sin\left(\frac{\pi}{2} - x\right)dx = \pi\int_0^{\frac{\pi}{2}}\log\cos x\,dx \quad \ldots(ii)$$

Now adding (i) & (ii) we get

$$2I = \pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx + \pi\int_0^{\frac{\pi}{2}}\log\cos x\,dx = \pi\int_0^{\frac{\pi}{2}}(\log\sin x + \log\cos x)\,dx = \pi\int_0^{\frac{\pi}{2}}\log\sin x.\cos x\,dx$$

$$\Rightarrow 2I = \pi\int_0^{\frac{\pi}{2}}\log\left(\frac{2\sin x.\cos x}{2}\right)dx = \pi\int_0^{\frac{\pi}{2}}\log\left(\frac{\sin 2x}{2}\right)dx = \pi\int_0^{\frac{\pi}{2}}\log\sin 2x\,dx - \pi\int_0^{\frac{\pi}{2}}\log 2\,dx \qquad \ldots(iii)$$

Now let $I = \int_0^{\frac{\pi}{2}}\log\sin 2x\,dx$

Putting $2x = t$ we get

$$I_1 = \int_0^{\pi}\log\sin t\,\frac{dt}{2} = \frac{1}{2}\int_0^{\pi}\log\sin t.dt = \frac{1}{2}\times 2\pi\int_0^{\frac{\pi}{2}}\log\sin t\,dt = \pi\int_0^{\frac{\pi}{2}}\log\sin x\,dx = I$$

So from (iii) we get

$$2I = I - \pi\frac{\pi}{2}\log 2$$

$$I = -\frac{\pi^2}{2}\log 2$$

Definite Integrals Ex 20.4B Q15

Let $I = \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin x} dx \qquad \left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$I = \int_0^{\pi} \frac{\pi \sin x}{1+\sin x} dx - \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \times \frac{(1-\sin x)}{(1-\sin x)} dx$$

$$2I = \pi \int_0^{\pi} \frac{\sin x - \sin^2 x}{1+\sin^2 x} dx$$

$$2I = \pi \int_0^{\pi} \frac{(\sin x - \sin^2 x)}{\cos^2 x} dx$$

$$2I = \pi \int_0^{\pi} (\tan x.\sec x - \tan^2 x) dx$$

$$2I = \pi \int_0^{\pi} \left[\tan x.\sec x - (\sec^2 x - 1)\right] dx$$

$$2I = \pi \int_0^{\pi} (\sec x.\tan x - \sec^2 x + 1) dx$$

$$2I = \pi \int_0^{\pi} (\sec x.\tan x - \sec^2 x + 1) dx$$

$$2I = \pi\left[\sec x - \tan x + x\right]_0^{\pi}$$

$$2I = \pi\left[(\sec \pi - \tan \pi + \pi) - (\sec 0 - \tan 0 + 0)\right]$$

$$2I = \pi\left[(-1 - 0 + \pi) - (1 - 0 + 0)\right]$$

$$2I = \pi(\pi - 1 - 1)$$

$$I = \frac{\pi}{2}(\pi - 2)$$

$$\therefore \int_0^{\pi} \frac{x \sin x}{1+\sin x} dx = \pi\left(\frac{\pi}{2} - 1\right)$$

Definite Integrals Ex 20.4B Q16

We have

$$I = \int_0^{\pi} \frac{x\,dx}{1+\cos\alpha\sin x} \qquad --(i)$$

$$\therefore \int_0^{a} f(x)\,dx = \int_0^{a} f(a-x)\,dx$$

$$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos\alpha\sin(\pi - x)} = \int_0^{\pi} \frac{(\pi - x)\,dx}{1+\cos\alpha\sin x} \qquad --(ii)$$

Adding (i) & (ii) we get

$$2I = \pi\int_0^{\pi} \frac{\pi}{1+\cos\alpha.\sin x}\,dx$$

Substituting $\sin x = \dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$2I = \pi\int_0^{\pi} \frac{\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}\,2\cos\alpha.\tan\frac{x}{2}}\,dx = \pi\int_0^{\pi} \frac{\sec^2\frac{x}{2}dx}{1-\cos^2\alpha + \left(\cos\alpha.\tan\frac{x}{2}\right)^2}\,dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx = dt$

When $x = 0 \quad t = 0$

$\pi \;\Rightarrow t = \alpha$

$$2I = \int_0^{\alpha} \frac{dt}{\left(1+\cos^2\alpha\right)+\left(\cos\alpha + t\right)^2}\,dx = 2\pi.\frac{1}{\sqrt{1+\cos^2\alpha}}.\left[\tan^{-1}\left(\frac{\cos\alpha+1}{\sqrt{1+\cos^2\alpha}}\right)\right]_0^{\alpha}$$

$$= \frac{2\pi}{\sin\alpha}\left[\frac{\pi}{2} - \tan^{-1}\cot\alpha\right]$$

$$= \frac{2\pi}{\sin\alpha}\left[\cot^{-1}\left(\cot\alpha\right)\right]$$

$$= \frac{2\pi}{\sin\alpha}.\alpha$$

$$\Rightarrow I = \frac{\pi\alpha}{\sin\alpha}$$

*********** END **********