Indefinite Integrals Ex 19.29 Q11

Consider the integral $I=\int (x-3)\sqrt{x^2+3x-18}dx$

Let us express $x-3=\lambda\frac{d}{dx}\left[x^2+3x-18\right]+\mu$

$\Rightarrow x-3=\lambda[2x+3]+\mu$

$\Rightarrow x-3=2\lambda x+3\lambda+\mu$

Comparing the coefficients, we have,

$2\lambda=1$ and $3\lambda+\mu=-3$

$\Rightarrow \lambda=\frac{1}{2}$ and $3\times\frac{1}{2}+\mu=-3$

$\Rightarrow \lambda=\frac{1}{2}$ and $\mu=-3-\frac{3}{2}$

$\Rightarrow \lambda=\frac{1}{2}$ and $\mu=-\frac{9}{2}$

Then

$x-3=\lambda[2x+3]+\mu$

Now the integral $I=\int (x-3)\sqrt{x^2+3x-18}dx$

$=\int\left[\frac{1}{2}[2x+3]-\frac{9}{2}\right]\sqrt{x^2+3x-18}dx$

$I=\frac{1}{2}\int[2x+3]\sqrt{x^2+3x-18}dx-\frac{9}{2}\int\sqrt{x^2+3x-18}dx$

$\Rightarrow I=I_1+I_2$

where, $I_1=\frac{1}{2}\int[2x+3]\sqrt{x^2+3x-18}dx$ and

$I_2=-\frac{9}{2}\int\sqrt{x^2+3x-18}dx$

Let us consider the integral, I_1 :

$I_1=\frac{1}{2}\int[2x+3]\sqrt{x^2+3x-18}dx$

Substituting, $x^2+3x-18=t$

$\Rightarrow (2x+3)dx=dt$

Thus,

$I_1=\frac{1}{2}\int\sqrt{t}dt$

$=\frac{1}{2}\times\frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C$

$$= \frac{1}{2} \times \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{1}{2} \times \frac{2}{3} \times t^{\frac{3}{2}} + C$$

$$= \frac{1}{3} \times t^{\frac{3}{2}} + C$$

$$= \frac{1}{3} \times \left(x^2 + 3x - 18\right)^{\frac{3}{2}} + C$$

Now consider the integral

$$I_2 = -\frac{9}{2}\int \sqrt{x^2 + 3x - 18}\, dx$$

$$= -\frac{9}{2}\int \sqrt{x^2 + 2 \times \frac{3}{2}x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 18}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \frac{9}{4} - 18}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{4} + 18\right)}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9 + 72}{4}\right)}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{81}{4}\right)}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

$$= -\frac{9}{2}\int \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\, dx$$

We know that $\int \sqrt{x^2 - a^2}dx = \frac{1}{2}x\sqrt{x^2 - a^2} - \frac{1}{2}a^2 \log \left|x + \sqrt{x^2 - a^2}\right| + C$

$$\therefore I_2 = -\frac{9}{2}\left\{\frac{1}{2}\left(x + \frac{3}{2}\right)\sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2} - \frac{1}{2}\left(\frac{9}{2}\right)^2 \log \left|\left(x + \frac{3}{2}\right) + \sqrt{\left(x + \frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right|\right\} + C$$

$$= -\frac{9}{4}\left\{\left(\frac{2x + 3}{2}\right)\sqrt{x^2 + 3x - 18} - \left(\frac{729}{4}\right)\log \left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x - 18}\right|\right\} + C$$

$$= -\frac{9}{8}(2x + 3)\sqrt{x^2 + 3x - 18} + \frac{729}{16}\log \left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x - 18}\right| + C$$

Thus, $I = = \frac{1}{3}\left(x^2 + 3x - 18\right)^{\frac{3}{2}} - \frac{9}{8}(2x + 3)\sqrt{x^2 + 3x - 18} + \frac{729}{16}\log \left|\left(x + \frac{3}{2}\right) + \sqrt{x^2 + 3x - 18}\right| + C$

Indefinite Integrals Ex 19.29 Q12

$$\int (x+3)\sqrt{3-4x-x^2}dx$$

$$Let\ \ x+3 = A\frac{d}{dx}(3-4x-x^2)+B$$

$$x+3 = A(-4-2x)+B$$

$$x+3 = -2Ax+B-4A$$

$$-2A = 1, B-4A = 3$$

$$A = -\frac{1}{2},$$

$$B = 4\times\left(-\frac{1}{2}\right)+3 = 1$$

$$\int (x+3)\sqrt{3-4x-x^2}dx$$

$$x+3 = -\frac{1}{2}(-4-2x)+1$$

$$\int\left[-\frac{1}{2}(-4-2x)+1\right]\sqrt{3-4x-x^2}dx$$

$$= -\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}dx + \int\sqrt{3-4x-x^2}dx$$

$$= I_1 + I_2 .....(i)$$

$$I_1 = -\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}dx$$

$$Let\ \ z = 3-4x-x^2$$

$$dz = -4-2x$$

$$I_1 = -\frac{1}{2}\int\sqrt{z}dz$$

$$= -\frac{1}{2}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]$$

$$= -\frac{1}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]$$

$$= -\left[\frac{(3-4x-x^2)^{\frac{3}{2}}}{3}\right]$$

$$I_2 = \int\sqrt{3-4x-x^2}dx$$

$$= \int\sqrt{3-(x^2+4x+4)+4}dx$$

$$= \int \sqrt{7 - (x^2 + 4x + 4)}dx$$

$$= \int \sqrt{\left(\sqrt{7}\right)^2 - (x+2)^2}dx$$

$$= \frac{(x+2)\sqrt{\left(\sqrt{7}\right)^2 - (x+2)^2}}{2} + \frac{1}{2}\left(\sqrt{7}\right)^2 \tan^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

$$= \frac{(x+2)\sqrt{3 - 4x - x^2}}{2} + \frac{7}{2}\tan^{-1}\left(\frac{x+2}{\sqrt{7}}\right)$$

From(i),

$$= I_1 + I_2$$

$$= -\frac{1}{3}\left(3 - 4x - x^2\right)^{\frac{3}{2}} + \frac{1}{2}(x+2)\sqrt{3 - 4x - x^2} + \frac{7}{2}\tan^{-1}\left(\frac{x+2}{\sqrt{7}}\right) + C$$

********** END **********