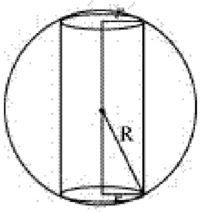
Maxima and Minima 18.5 Q17

A sphere of fixed radius (R) is given.

Let r and h be the radius and the height of the cylinder respectively.



From the given figure, we have $h = 2\sqrt{R^2 - r^2}$.

The volume (V) of the cylinder is given by,

$$V = \pi r^2 h = 2\pi r^2 \sqrt{R^2 - r^2}$$

$$\therefore \frac{dV}{dr} = 4\pi r\sqrt{R^2 - r^2} + \frac{2\pi r^2(-2r)}{2\sqrt{R^2 - r^2}}$$

$$= 4\pi r\sqrt{R^2 - r^2} - \frac{2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi r\left(R^2 - r^2\right) - 2\pi r^3}{\sqrt{R^2 - r^2}}$$

$$= \frac{4\pi rR^2 - 6\pi r^3}{\sqrt{R^2 - r^2}}$$

Now, $\frac{dV}{dr} = 0 \Rightarrow 4\pi rR^2 - 6\pi r^3 = 0$

$$\Rightarrow r^2 = \frac{2R^2}{3}$$

$$\text{Now, } \frac{d^2V}{dr^2} = \frac{\sqrt{R^2 - r^2}\left(4\pi R^2 - 18\pi r^2\right) - \left(4\pi r R^2 - 6\pi r^3\right)\frac{(-2r)}{2\sqrt{R^2 - r^2}}}{\left(R^2 - r^2\right)}$$

$$= \frac{\left(R^2 - r^2\right)\left(4\pi R^2 - 18\pi r^2\right) + r\left(4\pi r R^2 - 6\pi r^3\right)}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

$$= \frac{4\pi R^4 - 22\pi r^2 R^2 + 12\pi r^4 + 4\pi r^2 R^2}{\left(R^2 - r^2\right)^{\frac{3}{2}}}$$

Now, it can be observed that at $r^2 = \frac{2R^2}{3}, \frac{d^2V}{dr^2} < 0$.

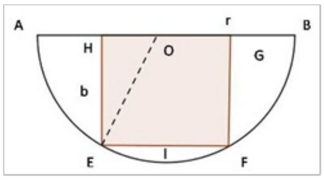
$\therefore$The volume is the maximum when $r^2 = \frac{2R^2}{3}$.

When $r^2 = \frac{2R^2}{3}$, the height of the cylinder is $2\sqrt{R^2 - \frac{2R^2}{3}} = 2\sqrt{\frac{R^2}{3}} = \frac{2R}{\sqrt{3}}$.

Hence, the volume of the cylinder is the maximum when the height of the cylinder is $\frac{2R}{\sqrt{3}}$.

Maxima and Minima 18.5 Q18

Let $EFGH$ be a rectangle inscribed in a semi-circle with radius r.



Let l and b are the length and width of rectangle.

In ΔOHE

$$HE^2 = OE^2 - OH^2$$

$$\Rightarrow \quad HE = b = \sqrt{r^2 - \left(\frac{r}{2}\right)^2} \qquad \text{---(i)}$$

Let $\quad S$ = Area of rectangle

$$= lb = l \times \sqrt{r^2 - \left(\frac{l}{2}\right)^2}$$

$$\therefore \quad S = \frac{1}{2} l \sqrt{4r^2 - l^2}$$

$$\therefore \quad \frac{ds}{dl} = \frac{1}{2}\left[\sqrt{4r^2 - l^2} - \frac{l^2}{\sqrt{4r^2 - l^2}}\right]$$

$$= \frac{1}{2}\left[\frac{4r^2 - l^2 - l^2}{\sqrt{4r^2 - l^2}}\right]$$

$$= \frac{2r^2 - l^2}{\sqrt{4r^2 - l^2}}$$

For maxima and minima,

$$\frac{ds}{dl = 0}$$

$$\Rightarrow \quad \frac{2r^2 - l^2}{\sqrt{4r^2 - l^2}} = 0$$

$$\Rightarrow \quad l = \pm\sqrt{2}r$$

Also,

$$\frac{d^2s}{dl^2} = 0 \text{ at } l = \sqrt{2}r$$

So, the dimension of the rectangle

$$l = \sqrt{2}r, \; b = \sqrt{r^2 - \left(\frac{l}{2}\right)^2} = \frac{r}{\sqrt{2}}$$

Area of rectangle $= lb = \sqrt{2}r \times \frac{r}{\sqrt{2}}$

$= r^2.$

*********** END **********