Exercise 7.10 : Solutions of Questions on Page Number : **340**

Q1 : $\int_0^1 \frac{x}{x^2+1}dx$

Answer :

$\int_0^1 \frac{x}{x^2+1}dx$

Let $x^2+1=t \Rightarrow 2x\,dx=dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\begin{aligned}\therefore \int_0^1 \frac{x}{x^2+1}dx &= \frac{1}{2}\int_1^2 \frac{dt}{t}\\ &= \frac{1}{2}\left[\log|t|\right]_1^2\\ &= \frac{1}{2}\left[\log 2-\log 1\right]\\ &= \frac{1}{2}\log 2\end{aligned}$$

Answer needs Correction? **Click Here**

Q2 : $\int_0^1 \frac{x}{x^2+1}dx$

Answer :

$\int_0^1 \frac{x}{x^2+1}dx$

Let $x^2+1=t \Rightarrow 2x\,dx=dt$

When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$

$$\begin{aligned}\therefore \int_0^1 \frac{x}{x^2+1}dx &= \frac{1}{2}\int_1^2 \frac{dt}{t}\\ &= \frac{1}{2}\left[\log|t|\right]_1^2\\ &= \frac{1}{2}\left[\log 2-\log 1\right]\\ &= \frac{1}{2}\log 2\end{aligned}$$

Answer needs Correction? **Click Here**

Q3 : $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Answer :

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^4\phi\cos\phi\, d\phi$

Also, let $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi=0$, $t=0$ and when $\phi=\frac{\pi}{2}$, $t=1$

$$\begin{aligned}\therefore I &= \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt\\ &= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt\\ &= \int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt\\ &= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1\\ &= \frac{2}{3}+\frac{2}{11}-\frac{4}{7}\\ &= \frac{154+42-132}{231}\\ &= \frac{64}{231}\end{aligned}$$

Answer needs Correction? **Click Here**

Q4 : $\int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi$

Answer :

Let $I = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\cos^5\phi\, d\phi = \int_0^{\frac{\pi}{2}} \sqrt{\sin\phi}\, \cos^4\phi \cos\phi\, d\phi$

Also, let $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi = 0,\ t = 0$ and when $\phi = \frac{\pi}{2},\ t = 1$

$$\therefore I = \int_0^1 \sqrt{t}\left(1-t^2\right)^2 dt$$
$$= \int_0^1 t^{\frac{1}{2}}\left(1+t^4-2t^2\right)dt$$
$$= \int_0^1\left[t^{\frac{1}{2}}+t^{\frac{9}{2}}-2t^{\frac{5}{2}}\right]dt$$
$$= \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}+\frac{t^{\frac{11}{2}}}{\frac{11}{2}}-\frac{2t^{\frac{7}{2}}}{\frac{7}{2}}\right]_0^1$$
$$= \frac{2}{3}+\frac{2}{11}-\frac{4}{7}$$
$$= \frac{154+42-132}{231}$$
$$= \frac{64}{231}$$

Answer needs Correction? **Click Here**

Q5 : $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Answer :

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\sin 2\theta\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 2\theta\cdot\sec^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\, d\theta$$

Takingθas first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$
$$= 2\left[\theta\tan\theta - \int\tan\theta\, d\theta\right]_0^{\frac{\pi}{4}}$$
$$= 2\left[\theta\tan\theta + \log\left|\cos\theta\right|\right]_0^{\frac{\pi}{4}}$$
$$= 2\left[\frac{\pi}{4}\tan\frac{\pi}{4} + \log\left|\cos\frac{\pi}{4}\right| - \log\left|\cos 0\right|\right]$$
$$= 2\left[\frac{\pi}{4} + \log\left(\frac{1}{\sqrt{2}}\right) - \log 1\right]$$
$$= 2\left[\frac{\pi}{4} - \frac{1}{2}\log 2\right]$$
$$= \frac{\pi}{2} - \log 2$$

Answer needs Correction? **Click Here**

Q6 : $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Answer :

Let $I = \int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Also, let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} \sin^{-1}\left(\sin 2\theta\right)\sec^2\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{4}} 2\theta\cdot\sec^2\theta\, d\theta$$
$$= 2\int_0^{\frac{\pi}{4}} \theta\cdot\sec^2\theta\, d\theta$$

Takingθas first function and $\sec^2\theta$ as second function and integrating by parts, we obtain

$$I = 2\left[\theta\int\sec^2\theta\, d\theta - \int\left\{\left(\frac{d}{dx}\theta\right)\int\sec^2\theta\, d\theta\right\}d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta-\int\tan\theta\,d\theta\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\theta\tan\theta+\log|\cos\theta|\right]_0^{\frac{\pi}{4}}$$

$$=2\left[\frac{\pi}{4}\tan\frac{\pi}{4}+\log\left|\cos\frac{\pi}{4}\right|-\log|\cos 0|\right]$$

$$=2\left[\frac{\pi}{4}+\log\left(\frac{1}{\sqrt{2}}\right)-\log 1\right]$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right]$$

$$=\frac{\pi}{2}-\log 2$$

Answer needs Correction? Click Here

Q7 : $\int_0^2 x\sqrt{x+2}\ \left(\text{Put } x+2=t^2\right)$

Answer :

$\int_0^2 x\sqrt{x+2}dx$

Let $x + 2 = t^2 \Rightarrow dx = 2tdt$

When $x = 0$, $t=\sqrt{2}$ and when $x = 2$, $t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}dx=\int_{\sqrt{2}}^2\left(t^2-2\right)\sqrt{t^2}\,2tdt$$

$$=2\int_{\sqrt{2}}^2\left(t^2-2\right)t^2dt$$

$$=2\int_{\sqrt{2}}^2\left(t^4-2t^2\right)dt$$

$$=2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$=2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$

$$=2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$=2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$=\frac{16\left(2+\sqrt{2}\right)}{15}$$

$$=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Answer needs Correction? Click Here

Q8 : $\int_0^2 x\sqrt{x+2}\ \left(\text{Put } x+2=t^2\right)$

Answer :

$\int_0^2 x\sqrt{x+2}dx$

Let $x + 2 = t^2 \Rightarrow dx = 2tdt$

When $x = 0$, $t=\sqrt{2}$ and when $x = 2$, $t = 2$

$$\therefore \int_0^2 x\sqrt{x+2}dx=\int_{\sqrt{2}}^2\left(t^2-2\right)\sqrt{t^2}\,2tdt$$

$$=2\int_{\sqrt{2}}^2\left(t^2-2\right)t^2dt$$

$$=2\int_{\sqrt{2}}^2\left(t^4-2t^2\right)dt$$

$$=2\left[\frac{t^5}{5}-\frac{2t^3}{3}\right]_{\sqrt{2}}^2$$

$$=2\left[\frac{32}{5}-\frac{16}{3}-\frac{4\sqrt{2}}{5}+\frac{4\sqrt{2}}{3}\right]$$

$$=2\left[\frac{96-80-12\sqrt{2}+20\sqrt{2}}{15}\right]$$

$$=2\left[\frac{16+8\sqrt{2}}{15}\right]$$

$$=\frac{16\left(2+\sqrt{2}\right)}{15}$$

$$=\frac{16\sqrt{2}\left(\sqrt{2}+1\right)}{15}$$

Answer needs Correction? Click Here

Q9 : $\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx$

Answer :

$\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

When $x = 0$, $t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$
$$= -\left[-\frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

Answer needs Correction? **Click Here**

Q10 : $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Answer :

$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$

Let $\cos x = t \Rightarrow -\sin x \, dx = dt$

When $x = 0$, $t = 1$ and when $x = \frac{\pi}{2}, t = 0$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx = -\int_1^0 \frac{dt}{1+t^2}$$
$$= -\left[\tan^{-1} t\right]_1^0$$
$$= -\left[\tan^{-1} 0 - \tan^{-1} 1\right]$$
$$= -\left[-\frac{\pi}{4}\right]$$
$$= \frac{\pi}{4}$$

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