(i) f: {1, 2, 3, 4} → {10}defined as:

f = {(1, 10), (2, 10), (3, 10), (4, 10)}

From the given definition of f, we can see that f is a many one function as: $f(1) = f(2) = f(3) = f(4) = 10$

∴f is not one-one.

Hence, function f does not have an inverse.

(ii) g: {5, 6, 7, 8} → {1, 2, 3, 4} defined as:

g = {(5, 4), (6, 3), (7, 4), (8, 2)}

From the given definition of g, it is seen that g is a many one function as: $g(5) = g(7) = 4$.

∴g is not one-one,

Hence, function g does not have an inverse.

(iii) h: {2, 3, 4, 5} → {7, 9, 11, 13} defined as:

h = {(2, 7), (3, 9), (4, 11), (5, 13)}

It is seen that all distinct elements of the set {2, 3, 4, 5} have distinct images under h.

∴Function h is one-one.

Also, h is onto since for every element y of the set {7, 9, 11, 13}, there exists an element x in the set {2, 3, 4, 5}such that $h(x) = y$.

Thus, h is a one-one and onto function. Hence, h has an inverse.

Question 6:

Show that f: [−1, 1] → **R**, given by $f(x)=\frac{x}{(x+2)}$ is one-one. Find the inverse of the function f: [−1, 1] → Range f.

(Hint: For y ∈Range f, y = $f(x)=\frac{x}{x+2}$, for some x in [−1, 1], i.e., $x=\frac{2y}{(1-y)}$)

Answer

f: [−1, 1] → R is given as $f(x)=\frac{x}{(x+2)}$.

Let $f(x) = f(y)$.

$$\Rightarrow \frac{x}{x+2}=\frac{y}{y+2}$$
$$\Rightarrow xy+2x=xy+2y$$
$$\Rightarrow 2x=2y$$
$$\Rightarrow x=y$$

∴ f is a one-one function.

It is clear that f: [−1, 1] → Range f is onto.

∴ f: [−1, 1] → Range f is one-one and onto and therefore, the inverse of the function: f: [−1, 1] → Range f exists.

Let g: Range f → [−1, 1] be the inverse of f.

Let y be an arbitrary element of range f.

Since f: [−1, 1] → Range f is onto, we have:

$$y=f(x) \text{ for same } x\in[-1,\ 1]$$
$$\Rightarrow y=\frac{x}{x+2}$$
$$\Rightarrow xy+2y=x$$
$$\Rightarrow x(1-y)=2y$$
$$\Rightarrow x=\frac{2y}{1-y},\ y\neq 1$$

Now, let us define g: Range f → [−1, 1] as

$$g(y)=\frac{2y}{1-y},\ y\neq 1.$$

$$\text{Now, } (gof)(x)=g(f(x))=g\left(\frac{x}{x+2}\right)=\frac{2\left(\frac{x}{x+2}\right)}{1-\frac{x}{x+2}}=\frac{2x}{x+2-x}=\frac{2x}{2}=x$$

$$(fog)(y)=f(g(y))=f\left(\frac{2y}{1-y}\right)=\frac{\frac{2y}{1-y}}{\frac{2y}{1-y}+2}=\frac{2y}{2y+2-2y}=\frac{2y}{2}=y$$

$\therefore gof = I_{[-1, 1]}$ and $fog = I_{\text{Range} f}$

$\therefore f^{-1} = g$

$\Rightarrow f^{-1}(y) = \dfrac{2y}{1-y}, y \neq 1$

Question 7:

Consider f: **R** → **R** given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

Answer

f: **R** → **R** is given by,

$f(x) = 4x + 3$

One-one:

Let $f(x) = f(y)$.

$\Rightarrow 4x+3 = 4y+3$

$\Rightarrow 4x = 4y$

$\Rightarrow x = y$

$\therefore f$ is a one-one function.

Onto:

For $y \in$ **R**, let $y = 4x + 3$.

$\Rightarrow x = \dfrac{y-3}{4} \in \mathbf{R}$

Therefore, for any $y \in$ **R**, there exists $x = \dfrac{y-3}{4} \in \mathbf{R}$ such that

$f(x) = f\left(\dfrac{y-3}{4}\right) = 4\left(\dfrac{y-3}{4}\right) + 3 = y.$

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: **R**→ **R** by $g(x) = \dfrac{y-3}{4}$.

$$\text{Now}, (gof)(x) = g(f(x)) = g(4x+3) = \frac{(4x+3)-3}{4} = x$$

$$(fog)(y) = f(g(y)) = f\left(\frac{y-3}{4}\right) = 4\left(\frac{y-3}{4}\right) + 3 = y - 3 + 3 = y$$

$\therefore gof = fog = I_{\mathbf{R}}$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \dfrac{y-3}{4}.$

Question 8:

Consider f: $\mathbf{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of given f by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbf{R}_+$ is the set of all non-negative real numbers.

Answer

f: $\mathbf{R}_+ \to [4, \infty)$ is given as $f(x) = x^2 + 4$.

One-one:

Let $f(x) = f(y)$.

$\Rightarrow x^2 + 4 = y^2 + 4$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y \qquad [\text{as } x = y \in \mathbf{R}_+]$

$\therefore f$ is a one-one function.

Onto:

For $y \in [4, \infty)$, let $y = x^2 + 4$.

$\Rightarrow x^2 = y - 4 \geq 0 \qquad [\text{as } y \geq 4]$

$\Rightarrow x = \sqrt{y-4} \geq 0$

Therefore, for any $y \in$ **R**, there exists $x = \sqrt{y-4} \in \mathbf{R}$ such that

$f(x) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = y - 4 + 4 = y$.

$\therefore f$ is onto.

Thus, f is one-one and onto and therefore, f^{-1} exists.

Let us define g: $[4, \infty) \to \mathbf{R}_+$ by,

$g(y) = \sqrt{y-4}$

$$\text{Now}, gof(x) = g(f(x)) = g(x^2+4) = \sqrt{(x^2+4)-4} = \sqrt{x^2} = x$$

$$\text{And}, fog(y) = f(g(y)) = f\left(\sqrt{y-4}\right) = \left(\sqrt{y-4}\right)^2 + 4 = (y-4) + 4 = y$$

$\therefore gof = fog = I_{\mathbf{R}_+}$

Hence, f is invertible and the inverse of f is given by

$f^{-1}(y) = g(y) = \sqrt{y-4}.$

Question 21:

Find the values of $\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)$ is equal to

(A) π **(B)** $-\frac{\pi}{2}$ **(C)** 0 **(D)** $2\sqrt{3}$

Answer

Let $\tan^{-1}\sqrt{3}=x$. Then, $\tan x=\sqrt{3}=\tan\frac{\pi}{3}$ where $\frac{\pi}{3}\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

We know that the range of the principal value branch of $\tan^{-1}$ is $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$\therefore \tan^{-1}\sqrt{3}=\frac{\pi}{3}$

Let $\cot^{-1}\left(-\sqrt{3}\right)=y$.

Then, $\cot y=-\sqrt{3}=-\cot\left(\frac{\pi}{6}\right)=\cot\left(\pi-\frac{\pi}{6}\right)=\cot\frac{5\pi}{6}$ where $\frac{5\pi}{6}\in(0,\ \pi)$.

The range of the principal value branch of $\cot^{-1}$ is $(0,\ \pi)$.

$\therefore \cot^{-1}\left(-\sqrt{3}\right)=\frac{5\pi}{6}$

$\therefore \tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)=\frac{\pi}{3}-\frac{5\pi}{6}=\frac{2\pi-5\pi}{6}=\frac{-3\pi}{6}=-\frac{\pi}{2}$

The correct answer is B.

Question 9:

Consider f: $\mathbf{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with

$f^{-1}(y)=\left(\frac{\left(\sqrt{y+6}\right)-1}{3}\right)$.

Answer

f: $\mathbf{R}_+ \to [-5, \infty)$ is given as $f(x) = 9x^2 + 6x - 5$.

Let y be an arbitrary element of $[-5, \infty)$.

Let $y = 9x^2 + 6x - 5$.

$\Rightarrow y=(3x+1)^2-1-5=(3x+1)^2-6$

$\Rightarrow (3x+1)^2=y+6$

$\Rightarrow 3x+1=\sqrt{y+6} \qquad [\text{as } y\ge -5 \Rightarrow y+6>0]$

$\Rightarrow x=\frac{\sqrt{y+6}-1}{3}$

*********** END **********