EXERCISE 9.3

Question 1:

Find the 20^{th} and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Ans:

The given G.P. is $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \ldots$

Here, a = First term = $\frac{5}{2}$

r = Common ratio = $\frac{\frac{5}{4}}{\frac{5}{2}} = \frac{1}{2}$

$$a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{(2)(2)^{19}} = \frac{5}{(2)^{20}}$$

$$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{(2)(2)^{n-1}} = \frac{5}{(2)^n}$$

Question 2:

Find the 12^{th} term of a G.P. whose 8^{th} term is 192 and the common ratio is 2.

Ans:

Common ratio, $r = 2$

Let a be the first term of the G.P.

$a_8 = ar^{8-1} = ar^7$

$ar^7 = 192$

$a(2)^7 = 192$

$a(2)^7 = (2)^6(3)$

$$\Rightarrow a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$$

$$\therefore a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072$$

Question 3:

The 5^{th}, 8^{th} and 11^{th} terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.

Ans:

Let a be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_5 = a\,r^{5-1} = a\,r^4 = p$ … (1)

$a_8 = a\,r^{8-1} = a\,r^7 = q$ … (2)

$a_{11} = a\,r^{11-1} = a\,r^{10} = s$ … (3)

Dividing equation (2) by (1), we obtain

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$$r^3 = \frac{q}{p} \qquad \ldots(4)$$

Dividing equation (3) by (2), we obtain

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$

$$\Rightarrow r^3 = \frac{s}{q} \qquad \ldots(5)$$

Equating the values of r^3 obtained in (4) and (5), we obtain

$$\frac{q}{p} = \frac{s}{q}$$

$$\Rightarrow q^2 = ps$$

Thus, the given result is proved.

Question 4:

The 4^{th} term of a G.P. is square of its second term, and the first term is -3. Determine its 7^{th} term.

Ans:

Let a be the first term and r be the common ratio of the G.P.

$a = -3$

It is known that, $a_n = ar^{n-1}$

$a_4 = ar^3 = (-3)\, r^3$

$a_2 = a\, r^1 = (-3)\, r$

According to the given condition,

$(-3)\, r^3 = [\, (-3)\, r]^2$

$-3r^3 = 9\, r^2$

$r = -3$

$a_7 = a\, r^{7-1} = a\, r^6 = (-3)\, (-3)^6 = -\, (3)^7 = -2187$

Thus, the seventh term of the G.P. is -2187.

Question 5:

Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4, \ldots$ is 128? (b) $\sqrt{3}, 3, 3\sqrt{3}, \ldots$ is 729? (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$ is $\frac{1}{19683}$?

Ans:

(a) The given sequence is $2, 2\sqrt{2}, 4, \ldots$

Here, $a = 2$ and $r = \frac{2\sqrt{2}}{2} = \sqrt{2}$

Let the n^{th} term of the given sequence be 128.

$$a_n = a r^{n-1}$$
$$\Rightarrow (2)\left(\sqrt{2}\right)^{n-1} = 128$$
$$\Rightarrow (2)(2)^{\frac{n-1}{2}} = (2)^7$$
$$\Rightarrow (2)^{\frac{n-1}{2}+1} = (2)^7$$
$$\therefore \frac{n-1}{2} + 1 = 7$$
$$\Rightarrow \frac{n-1}{2} = 6$$
$$\Rightarrow n - 1 = 12$$
$$\Rightarrow n = 13$$

Thus, the 13^{th} term of the given sequence is 128.

(b) The given sequence is $\sqrt{3}, 3, 3\sqrt{3},...$

Here, $a=\sqrt{3} \text{ and } r=\frac{3}{\sqrt{3}}=\sqrt{3}$

Let the n^{th} term of the given sequence be 729.

$$a_n = ar^{n-1}$$
$$\therefore ar^{n-1}=729$$
$$\Rightarrow (\sqrt{3})(\sqrt{3})^{n-1}=729$$
$$\Rightarrow (3)^{\frac{1}{2}}(3)^{\frac{n-1}{2}}=(3)^6$$
$$\Rightarrow (3)^{\frac{1}{2}+\frac{n-1}{2}}=(3)^6$$

$$\therefore \frac{1}{2}+\frac{n-1}{2}=6$$
$$\Rightarrow \frac{1+n-1}{2}=6$$
$$\Rightarrow n=12$$

Thus, the 12^{th} term of the given sequence is 729.

(c) The given sequence is $\frac{1}{3},\frac{1}{9},\frac{1}{27},...$

Here, $a=\frac{1}{3} \text{ and } r=\frac{1}{9}\div\frac{1}{3}=\frac{1}{3}$

Let the n^{th} term of the given sequence be $\frac{1}{19683}$.

$$a_n = ar^{n-1}$$
$$\therefore ar^{n-1}=\frac{1}{19683}$$
$$\Rightarrow \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}=\frac{1}{19683}$$
$$\Rightarrow \left(\frac{1}{3}\right)^{n}=\left(\frac{1}{3}\right)^{9}$$
$$\Rightarrow n=9$$

Thus, the 9^{th} term of the given sequence is $\frac{1}{19683}$.

Question 6:

For what values of x, the numbers $\frac{2}{7}, x, -\frac{7}{2}$ are in G.P?

Ans:

The given numbers are $\frac{-2}{7}, x, \frac{-7}{2}$.

Common ratio = $\frac{x}{\frac{-2}{7}} = \frac{-7x}{2}$

Also, common ratio = $\frac{\frac{-7}{2}}{x} = \frac{-7}{2x}$

$$\therefore \frac{-7x}{2} = \frac{-7}{2x}$$
$$\Rightarrow x^2 = \frac{-2\times 7}{-2\times 7} = 1$$
$$\Rightarrow x = \sqrt{1}$$
$$\Rightarrow x = \pm 1$$

Thus, for $x = \pm 1$, the given numbers will be in G.P.

Question 7:

Find the sum to 20 terms in the geometric progression 0.15, 0.015, 0.0015 …

Ans:

The given G.P. is 0.15, 0.015, 0.00015, ...

Here, $a = 0.15$ and $r = \frac{0.015}{0.15} = 0.1$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$
$$\therefore S_{20} = \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1}$$
$$= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right]$$
$$= \frac{15}{90}\left[1-(0.1)^{20}\right]$$
$$= \frac{1}{6}\left[1-(0.1)^{20}\right]$$

Question 8:

Find the sum to n terms in the geometric progression $\sqrt{7}, \sqrt{21}, 3\sqrt{7}$...

Ans:

The given G.P. is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Here, $a = \sqrt{7}$

$$r = \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{3}$$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}}$$

$$= \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} \qquad \text{(By rationalizing)}$$

$$= \frac{\sqrt{7}\left(1+\sqrt{3}\right)\left[1-\left(\sqrt{3}\right)^n\right]}{1-3}$$

$$= \frac{-\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[1-(3)^{\frac{n}{2}}\right]$$

$$= \frac{\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

Question 9:

Find the sum to n terms in the geometric progression $1, -a, a^2, -a^3 \ldots$ $(\text{if } a \neq -1)$

Ans:

The given G.P. is $1, -a, a^2, -a^3, \ldots\ldots\ldots\ldots$

Here, first term = $a_1 = 1$

Common ratio = $r = -a$

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{1\left[1-(-a)^n\right]}{1-(-a)} = \frac{\left[1-(-a)^n\right]}{1+a}$$

Question 10:

Find the sum to n terms in the geometric progression $x^3, x^5, x^7 \ldots$ $(\text{if } x \neq \pm 1)$

Ans:

The given G.P. is $x^3, x^5, x^7, \ldots$

Here, $a = x^3$ and $r = x^2$

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{x^3\left[1-(x^2)^n\right]}{1-x^2} = \frac{x^3(1-x^{2n})}{1-x^2}$$

Question 11:

Evaluate $\sum_{k=1}^{11}(2+3^k)$

Ans:

$$\sum_{k=1}^{11}(2+3^k) = \sum_{k=1}^{11}(2) + \sum_{k=1}^{11}3^k = 2(11) + \sum_{k=1}^{11}3^k = 22 + \sum_{k=1}^{11}3^k \quad \ldots(1)$$

$$\sum_{k=1}^{11}3^k = 3^1 + 3^2 + 3^3 + \ldots + 3^{11}$$

The terms of this sequence $3, 3^2, 3^3, \ldots$ forms a G.P.

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$\Rightarrow S_{11} = \frac{3\left[(3)^{11}-1\right]}{3-1}$$

$$\Rightarrow S_{11} = \frac{3}{2}(3^{11}-1)$$

$$\therefore \sum_{k=1}^{11}3^k = \frac{3}{2}(3^{11}-1)$$

Substituting this value in equation (1), we obtain

$$\sum_{k=1}^{11}(2+3^k) = 22 + \frac{3}{2}(3^{11}-1)$$

Question 12:

The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Ans:

Let $\frac{a}{r}, a, ar$ be the first three terms of the G.P.

$$\frac{a}{r}+a+ar=\frac{39}{10} \qquad ...(1)$$

$$\left(\frac{a}{r}\right)(a)(ar)=1 \qquad ...(2)$$

From (2), we obtain

$a^3 = 1$

$a = 1$ (Considering real roots only)

Substituting $a = 1$ in equation (1), we obtain

$$\frac{1}{r}+1+r=\frac{39}{10}$$

$$\Rightarrow 1+r+r^2=\frac{39}{10}r$$

$$\Rightarrow 10+10r+10r^2-39r=0$$

$$\Rightarrow 10r^2-29r+10=0$$

$$\Rightarrow 10r^2-25r-4r+10=0$$

$$\Rightarrow 5r(2r-5)-2(2r-5)=0$$

$$\Rightarrow (5r-2)(2r-5)=0$$

$$\Rightarrow r=\frac{2}{5} \text{ or } \frac{5}{2}$$

Thus, the three terms of G.P. are $\frac{5}{2}, 1, \text{and } \frac{2}{5}$.

Question 13:

How many terms of G.P. $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

Ans:

The given G.P. is $3, 3^2, 3^3, \ldots$

Let n terms of this G.P. be required to obtain the sum as 120.

$$S_n=\frac{a(r^n-1)}{r-1}$$

Here, $a = 3$ and $r = 3$

$$\therefore S_n=120=\frac{3(3^n-1)}{3-1}$$

$$\Rightarrow 120=\frac{3(3^n-1)}{2}$$

$$\Rightarrow \frac{120\times 2}{3}=3^n-1$$

$$\Rightarrow 3^n-1=80$$

$$\Rightarrow 3^n=81$$

$$\Rightarrow 3^n=3^4$$

$n = 4$

Thus, four terms of the given G.P. are required to obtain the sum as 120.

Question 14:

The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

Ans:

According to the given condition,

$a + ar + ar^2 = 16$ and $ar^3 + ar^4 + ar^5 = 128$

$a\ (1 + r + r^2) = 16\ ...\ (1)$

$ar^3(1 + r + r^2) = 128\ ...\ (2)$

Dividing equation (2) by (1), we obtain

$$\frac{ar^3\left(1+r+r^2\right)}{a\left(1+r+r^2\right)}=\frac{128}{16}$$
$$\Rightarrow r^3=8$$
$$\therefore r=2$$

Substituting $r = 2$ in (1), we obtain

$a\ (1 + 2 + 4) = 16$

$a\ (7) = 16$

$$\Rightarrow a=\frac{16}{7}$$

$$S_n=\frac{a\left(r^n-1\right)}{r-1}$$

$$\Rightarrow S_n=\frac{16}{7}\frac{\left(2^n-1\right)}{2-1}=\frac{16}{7}\left(2^n-1\right)$$

Question 15:

Given a G.P. with $a = 729$ and 7^{th} term 64, determine S_7.

Ans:

Let r be the common ratio of the G.P.

It is known that, $a_n = a\, r^{n-1}$

$a_7 = ar^{7-1} = (729)r^6$

$64 = 729\, r^6$

$$\Rightarrow r^6 = \frac{64}{729}$$

$$\Rightarrow r^6 = \left(\frac{2}{3}\right)^6$$

$$\Rightarrow r = \frac{2}{3}$$

Also, it is known that, $S_n = \frac{a\left(1-r^n\right)}{1-r}$

$$\therefore S_7 = \frac{729\left[1-\left(\frac{2}{3}\right)^7\right]}{1-\frac{2}{3}}$$

$$= 3 \times 729\left[1-\left(\frac{2}{3}\right)^7\right]$$

$$= (3)^7\left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$

$$= (3)^7-(2)^7$$

$$= 2187-128$$

$$= 2059$$

Question 16:

Find a G.P. for which sum of the first two terms is –4 and the fifth term is 4 times the third term.

Ans:

Let a be the first term and r be the common ratio of the G.P.

According to the given conditions,

$$S_2 = -4 = \frac{a\left(1-r^2\right)}{1-r} \qquad \ldots(1)$$

$a_5 = 4 \times a_3$

$ar^4 = 4ar^2$

$r^2 = 4$

$r = \pm 2$

From (1), we obtain

$$-4 = \frac{a\left[1-(2)^2\right]}{1-2} \text{ for } r = 2$$
$$\Rightarrow -4 = \frac{a(1-4)}{-1}$$
$$\Rightarrow -4 = a(3)$$

$$\Rightarrow a = \frac{-4}{3}$$

$$\text{Also, } -4 = \frac{a\left[1-(-2)^2\right]}{1-(-2)} \text{ for } r = -2$$

$$\Rightarrow -4 = \frac{a(1-4)}{1+2}$$

$$\Rightarrow -4 = \frac{a(-3)}{3}$$

$$\Rightarrow a = 4$$

Thus, the required G.P. is

$\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$ or $4, -8, 16, -32, \ldots$

Question 17:

If the 4^{th}, 10^{th} and 16^{th} terms of a G.P. are x, y and z, respectively. Prove that x, y, z are in G.P.

Ans:

Let a be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_4 = a\, r^3 = x$ ... (1)

$a_{10} = a\, r^9 = y$ ... (2)

$a_{16} = a\, r^{15} = z$ ... (3)

Dividing (2) by (1), we obtain

$$\frac{y}{x} = \frac{ar^9}{ar^3} \Rightarrow \frac{y}{x} = r^6$$

Dividing (3) by (2), we obtain

$$\frac{z}{y} = \frac{ar^{15}}{ar^9} \Rightarrow \frac{z}{y} = r^6$$

$$\frac{y}{x} = \frac{z}{y}$$

Thus, x, y, z are in G. P.

Question 18:

Find the sum to n terms of the sequence, 8, 88, 888, 8888…

Ans:

The given sequence is 8, 88, 888, 8888...

This sequence is not a G.P. However, it can be changed to G.P. by writing the terms as

S_n = 8 + 88 + 888 + 8888 + ... to n terms

$$= \frac{8}{9}[9 + 99 + 999 + 9999 + \text{..........to } n \text{ terms}]$$

$$= \frac{8}{9}\left[(10-1) + (10^2-1) + (10^3-1) + (10^4-1) + \text{........to } n \text{ terms}\right]$$

$$= \frac{8}{9}\left[(10 + 10^2 + \text{......} n \text{ terms}) - (1+1+1+\text{....} n \text{ terms})\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{10-1} - n\right]$$

$$= \frac{8}{9}\left[\frac{10(10^n-1)}{9} - n\right]$$

$$= \frac{80}{81}(10^n-1) - \frac{8}{9}n$$

Question 19:

Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, $\frac{1}{2}$.

Ans:

Required sum = $2 \times 128 + 4 \times 32 + 8 \times 8 + 16 \times 2 + 32 \times \frac{1}{2}$

$= 64\left[4 + 2 + 1 + \frac{1}{2} + \frac{1}{2^2}\right]$

Here, 4, 2, 1, $\frac{1}{2}, \frac{1}{2^2}$ is a G.P.

First term, $a = 4$

Common ratio, $r = \frac{1}{2}$

It is known that, $S_n = \frac{a(1-r^n)}{1-r}$

$$\therefore S_5 = \frac{4\left[1-\left(\frac{1}{2}\right)^5\right]}{1-\frac{1}{2}} = \frac{4\left[1-\frac{1}{32}\right]}{\frac{1}{2}} = 8\left(\frac{32-1}{32}\right) = \frac{31}{4}$$

Required sum = $64\left(\frac{31}{4}\right) = (16)(31) = 496$

Question 20:

Show that the products of the corresponding terms of the sequences $a, ar, ar^2, \ldots ar^{n-1}$ and $A, AR, AR^2, \ldots AR^{n-1}$ form a G.P, and find the common ratio.

Ans:

It has to be proved that the sequence, aA, $arAR$, ar^2AR^2, …$ar^{n-1}AR^{n-1}$, forms a G.P.

$$\frac{\text{Second term}}{\text{First term}} = \frac{arAR}{aA} = rR$$

$$\frac{\text{Third term}}{\text{Second term}} = \frac{ar^2AR^2}{arAR} = rR$$

Thus, the above sequence forms a G.P. and the common ratio is rR.

Question 21:

Find four numbers forming a geometric progression in which third term is greater than the first term by 9, and the second term is greater than the 4th by 18.

Ans:

Let a be the first term and r be the common ratio of the G.P.

$a_1 = a, a_2 = ar, a_3 = ar^2, a_4 = ar^3$

By the given condition,

$a_3 = a_1 + 9$

$ar^2 = a + 9$ ... (1)

$a_2 = a_4 + 18$

$ar = ar^3 + 18$ ... (2)

From (1) and (2), we obtain

$a(r^2 - 1) = 9$ ... (3)

$ar(1 - r^2) = 18$ ... (4)

Dividing (4) by (3), we obtain

$$\frac{ar(1-r^2)}{a(r^2-1)} = \frac{18}{9}$$
$$\Rightarrow -r = 2$$
$$\Rightarrow r = -2$$

Substituting the value of r in (1), we obtain

$4a = a + 9$

$3a = 9$

$a = 3$

Thus, the first four numbers of the G.P. are 3, $3(-2)$, $3(-2)^2$, and $3(-2)^3$ i.e., 3, –6, 12, and –24.

Question 22:

If the p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that $a^{q-r} b^{r-p} c^{p-q} = 1$

Ans:

Let A be the first term and R be the common ratio of the G.P.

According to the given information,

$AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

$a^{q-r} b^{r-p} c^{p-q}$

$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$

$= A^0 \times R^0$

$= 1$

Thus, the given result is proved.

Question 23:

If the first and the n^{th} term of a G.P. are a ad b, respectively, and if P is the product of n terms, prove that $P^2 = (ab)^n$.

Ans:

The first term of the G.P is a and the last term is b.

Therefore, the G.P. is a, ar, ar^2, ar^3, ... ar^{n-1}, where r is the common ratio.

$b = ar^{n-1}$ ... (1)

P = Product of n terms

$= (a)(ar)(ar^2) \dots (ar^{n-1})$

$= (a \times a \times \dots a)(r \times r^2 \times \dots r^{n-1})$

$= a^n r^{1+2+\dots(n-1)}$ ... (2)

Here, 1, 2, ...$(n - 1)$ is an A.P.

$$1 + 2 + \dots\dots\dots + (n-1) = \frac{n-1}{2}\left[2 + (n-1-1)\times 1\right] = \frac{n-1}{2}\left[2+n-2\right] = \frac{n(n-1)}{2}$$

$$P = a^n r^{\frac{n(n-1)}{2}}$$

$$\begin{aligned} \therefore P^2 &= a^{2n} r^{n(n-1)} \\ &= \left[a^2 r^{(n-1)}\right]^n \\ &= \left[a \times ar^{n-1}\right]^n \\ &= (ab)^n \qquad \left[\text{Using } (1)\right] \end{aligned}$$

Thus, the given result is proved.

Question 24:

Show that the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Ans:

Let a be the first term and r be the common ratio of the G.P.

$$\text{Sum of first n terms} = \frac{a(1-r^n)}{(1-r)}$$

Since there are n terms from $(n+1)^{th}$ to $(2n)^{th}$ term,

$$\text{Sum of terms from } (n+1)^{th} \text{ to } (2n)^{th} \text{ term} = \frac{a_{n+1}(1-r^n)}{(1-r)}$$

$a_{n+1} = ar^{n+1-1} = ar^n$

$$\text{Thus, required ratio} = \frac{a(1-r^n)}{(1-r)} \times \frac{(1-r)}{ar^n(1-r^n)} = \frac{1}{r^n}$$

Thus, the ratio of the sum of first n terms of a G.P. to the sum of terms from $(n + 1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Question 25:

If a, b, c and d are in G.P. show that $(a^2+b^2+c^2)(b^2+c^2+d^2) = (ab+bc+cd)^2$.

Ans:

a, b, c, d are in G.P.

Therefore,

$bc = ad$ ... (1)

$b^2 = ac$ ... (2)

$c^2 = bd$ ... (3)

It has to be proved that,

$(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc - cd)^2$

R.H.S.

$= (ab + bc + cd)^2$

$= (ab + ad + cd)^2$ [Using (1)]

$= [ab + d(a + c)]^2$

$= a^2b^2 + 2abd(a + c) + d^2(a + c)^2$

$= a^2b^2 + 2a^2bd + 2acbd + d^2(a^2 + 2ac + c^2)$

$= a^2b^2 + 2a^2c^2 + 2b^2c^2 + d^2a^2 + 2d^2b^2 + d^2c^2$ [Using (1) and (2)]

$= a^2b^2 + a^2c^2 + a^2c^2 + b^2c^2 + b^2c^2 + d^2a^2 + d^2b^2 + d^2b^2 + d^2c^2$

$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \times b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2 \times c^2 + c^2d^2$

[Using (2) and (3) and rearranging terms]

$= a^2(b^2 + c^2 + d^2) + b^2(b^2 + c^2 + d^2) + c^2(b^2 + c^2 + d^2)$

$= (a^2 + b^2 + c^2)(b^2 + c^2 + d^2)$

= L.H.S.

L.H.S. = R.H.S.

$$\left(a^2+b^2+c^2\right)\left(b^2+c^2+d^2\right)=\left(ab+bc+cd\right)^2$$

Question 26:

Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Ans:

Let G_1 and G_2 be two numbers between 3 and 81 such that the series, $3, G_1, G_2, 81$, forms a G.P.

Let a be the first term and r be the common ratio of the G.P.

$81 = (3)(r)^3$

$r^3 = 27$

$r = 3$ (Taking real roots only)

For $r = 3$,

$G_1 = ar = (3)(3) = 9$

$G_2 = ar^2 = (3)(3)^2 = 27$

Thus, the required two numbers are 9 and 27.

Question 27:

Find the value of n so that $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the geometric mean between a and b.

Ans:

G. M. of a and b is $\sqrt{ab}$.

By the given condition, $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$

Squaring both sides, we obtain

$$\frac{\left(a^{n+1}+b^{n+1}\right)^2}{\left(a^n+b^n\right)^2}=ab$$

$$\Rightarrow a^{2n+2}+2a^{n+1}b^{n+1}+b^{2n+2}=(ab)\left(a^{2n}+2a^nb^n+b^{2n}\right)$$

$$\Rightarrow a^{2n+2}+2a^{n+1}b^{n+1}+b^{2n+2}=a^{2n+1}b+2a^{n+1}b^{n+1}+ab^{2n+1}$$

$$\Rightarrow a^{2n+2}+b^{2n+2}=a^{2n+1}b+ab^{2n+1}$$

$$\Rightarrow a^{2n+2}-a^{2n+1}b=ab^{2n+1}-b^{2n+2}$$

$$\Rightarrow a^{2n+1}(a-b)=b^{2n+1}(a-b)$$

$$\Rightarrow \left(\frac{a}{b}\right)^{2n+1}=1=\left(\frac{a}{b}\right)^0$$

$$\Rightarrow 2n+1=0$$

$$\Rightarrow n=\frac{-1}{2}$$

Question 28:

The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Ans:

Let the two numbers be a and b.

G.M. = $\sqrt{ab}$

According to the given condition,

$$a+b=6\sqrt{ab} \qquad ...(1)$$
$$\Rightarrow (a+b)^2=36(ab)$$

Also,

$$(a-b)^2=(a+b)^2-4ab=36ab-4ab=32ab$$
$$\Rightarrow a-b=\sqrt{32}\sqrt{ab}$$
$$=4\sqrt{2}\sqrt{ab} \qquad ...(2)$$

Adding (1) and (2), we obtain

$$2a=\left(6+4\sqrt{2}\right)\sqrt{ab}$$
$$\Rightarrow a=\left(3+2\sqrt{2}\right)\sqrt{ab}$$

Substituting the value of a in (1), we obtain

$$b=6\sqrt{ab}-\left(3+2\sqrt{2}\right)\sqrt{ab}$$
$$\Rightarrow b=\left(3-2\sqrt{2}\right)\sqrt{ab}$$
$$\frac{a}{b}=\frac{\left(3+2\sqrt{2}\right)\sqrt{ab}}{\left(3-2\sqrt{2}\right)\sqrt{ab}}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

Thus, the required ratio is $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Question 29:

If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Ans:

It is given that A and G are A.M. and G.M. between two positive numbers. Let these two positive numbers be a and b.

$$\therefore AM = A = \frac{a+b}{2} \qquad ...(1)$$
$$GM = G = \sqrt{ab} \qquad ...(2)$$

From (1) and (2), we obtain

$a + b = 2A$ ... (3)

$ab = G^2$ ... (4)

Substituting the value of a and b from (3) and (4) in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$(a-b)^2 = 4A^2 - 4G^2 = 4(A^2 - G^2)$

$(a-b)^2 = 4(A+G)(A-G)$

$$(a-b) = 2\sqrt{(A+G)(A-G)} \qquad ...(5)$$

From (3) and (5), we obtain

$$2a = 2A + 2\sqrt{(A+G)(A-G)}$$
$$\Rightarrow a = A + \sqrt{(A+G)(A-G)}$$

Substituting the value of a in (3), we obtain

$$b = 2A - A - \sqrt{(A+G)(A-G)} = A - \sqrt{(A+G)(A-G)}$$

Thus, the two numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Question 30:

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and nth hour?

Ans:

It is given that the number of bacteria doubles every hour. Therefore, the number of bacteria after every hour will form a G.P.

Here, $a = 30$ and $r = 2$

$a_3 = ar^2 = (30)(2)^2 = 120$

Therefore, the number of bacteria at the end of 2nd hour will be 120.

$a_5 = ar^4 = (30)(2)^4 = 480$

The number of bacteria at the end of 4th hour will be 480.

$a_{n+1} = ar^n = (30)\, 2^n$

Thus, number of bacteria at the end of nth hour will be $30(2)^n$.

Question 31:

What will Rs 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Ans:

The amount deposited in the bank is Rs 500.

At the end of first year, amount = $\text{Rs}\,500\left(1+\frac{1}{10}\right)$ = Rs 500 (1.1)

At the end of 2nd year, amount = Rs 500 (1.1) (1.1)

At the end of 3rd year, amount = Rs 500 (1.1) (1.1) (1.1) and so on

Amount at the end of 10 years = Rs 500 (1.1) (1.1) … (10 times)

= Rs $500(1.1)^{10}$

Question 32:

If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Ans:

Let the root of the quadratic equation be a and b.

According to the given condition,

$$\text{A.M.} = \frac{a+b}{2} = 8 \Rightarrow a+b = 16 \qquad ...(1)$$
$$\text{G.M.} = \sqrt{ab} = 5 \Rightarrow ab = 25 \qquad ...(2)$$

The quadratic equation is given by,

$x^2 - x$ (Sum of roots) + (Product of roots) = 0

$x^2 - x(a+b) + (ab) = 0$

$x^2 - 16x + 25 = 0$ [Using (1) and (2)]

Thus, the required quadratic equation is $x^2 - 16x + 25 = 0$

*********** END **********