Therefore, * is a binary operation.

(iii) On **R**, * is defined by $a * b = ab^2$.

It is seen that for each $a, b \in$ **R**, there is a unique element ab^2 in **R**.

This means that * carries each pair (a, b) to a unique element $a * b = ab^2$ in **R**.

Therefore, * is a binary operation.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = |a - b|$.

It is seen that for each $a, b \in \mathbf{Z}^+$, there is a unique element $|a - b|$ in $\mathbf{Z}^+$.

This means that * carries each pair (a, b) to a unique element $a * b =$
$|a - b|$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a$.

* carries each pair (a, b) to a unique element $a * b = a$ in $\mathbf{Z}^+$.

Therefore, * is a binary operation.

Question 2:

For each binary operation * defined below, determine whether * is commutative or associative.

(i) On **Z**, define $a * b = a - b$

(ii) On **Q**, define $a * b = ab + 1$

(iii) On **Q**, define $a * b = \frac{ab}{2}$

(iv) On $\mathbf{Z}^+$, define $a * b = 2^{ab}$

(v) On $\mathbf{Z}^+$, define $a * b = a^b$

(vi) On $\mathbf{R} - \{-1\}$, define $a * b = \frac{a}{b+1}$

Answer

(i) On **Z**, * is defined by $a * b = a - b$.

It can be observed that $1 * 2 = 1 - 2 = 1$ and $2 * 1 = 2 - 1 = 1$.

$\therefore 1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{Z}$

Hence, the operation * is not commutative.

Also we have:

$(1 * 2) * 3 = (1 - 2) * 3 = -1 * 3 = -1 - 3 = -4$

$1 * (2 * 3) = 1 * (2 - 3) = 1 * -1 = 1 - (-1) = 2$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Z}$

Hence, the operation * is not associative.

(ii) On **Q**, * is defined by $a * b = ab + 1$.

It is known that:

$ab = ba$ □ $a, b \in \mathbf{Q}$

$\Rightarrow ab + 1 = ba + 1$ □ $a, b \in \mathbf{Q}$

$\Rightarrow a * b = a * b$ □ $a, b \in \mathbf{Q}$

Therefore, the operation * is commutative.

It can be observed that:

$(1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10$

$1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Q}$

Therefore, the operation * is not associative.

(iii) On **Q**, * is defined by $a * b = \frac{ab}{2}$.

It is known that:

$ab = ba$ □ $a, b \in \mathbf{Q}$

$\Rightarrow \frac{ab}{2} = \frac{ba}{2}$ □ $a, b \in \mathbf{Q}$

$\Rightarrow a * b = b * a$ □ $a, b \in \mathbf{Q}$

Therefore, the operation * is commutative.

For all $a, b, c \in \mathbf{Q}$, we have:

$$(a*b)*c = \left(\frac{ab}{2}\right)*c = \frac{\left(\frac{ab}{2}\right)c}{2} = \frac{abc}{4}$$

$$a*(b*c) = a*\left(\frac{bc}{2}\right) = \frac{a\left(\frac{bc}{2}\right)}{2} = \frac{abc}{4}$$

$\therefore (a*b)*c = a*(b*c)$

Therefore, the operation * is associative.

(iv) On $\mathbf{Z}^+$, * is defined by $a * b = 2^{ab}$.

It is known that:

$ab = ba$ □ $a, b \in \mathbf{Z}^+$

$\Rightarrow 2^{ab} = 2^{ba}$ □ $a, b \in \mathbf{Z}^+$

$\Rightarrow a * b = b * a$ □ $a, b \in \mathbf{Z}^+$

Therefore, the operation * is commutative.

It can be observed that:

$$(1*2)*3 = 2^{(1\times 2)}*3 = 4*3 = 2^{4\times 3} = 2^{12}$$

$$1*(2*3) = 1*2^{2\times 3} = 1*2^6 = 1*64 = 2^{64}$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{Z}^+$

Therefore, the operation * is not associative.

(v) On $\mathbf{Z}^+$, * is defined by $a * b = a^b$.

It can be observed that:

$1*2 = 1^2 = 1$ and $2*1 = 2^1 = 2$

$\therefore 1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{Z}^+$

Therefore, the operation * is not commutative.

It can also be observed that:

$$(2*3)*4 = 2^3*4 = 8*4 = 8^4 = \left(2^3\right)^4 = 2^{12}$$

$$2*(3*4) = 2*3^4 = 2*81 = 2^{81}$$

$\therefore (2 * 3) * 4 \neq 2 * (3 * 4)$; where $2, 3, 4 \in \mathbf{Z}^+$

Therefore, the operation * is not associative.

(vi) On **R**, * − {−1} is defined by $a*b = \dfrac{a}{b+1}$.

It can be observed that $1*2 = \dfrac{1}{2+1} = \dfrac{1}{3}$ and $2*1 = \dfrac{2}{1+1} = \dfrac{2}{2} = 1$.

$\therefore 1 * 2 \neq 2 * 1$; where $1, 2 \in \mathbf{R} - \{-1\}$

Therefore, the operation * is not commutative.

It can also be observed that:

$$(1*2)*3 = \frac{1}{3}*3 = \frac{\frac{1}{3}}{3+1} = \frac{1}{12}$$

$$1*(2*3) = 1*\frac{2}{3+1} = 1*\frac{2}{4} = 1*\frac{1}{2} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$\therefore (1 * 2) * 3 \neq 1 * (2 * 3)$; where $1, 2, 3 \in \mathbf{R} - \{-1\}$

Therefore, the operation * is not associative.

Question 3:

Consider the binary operation ∨ on the set {1, 2, 3, 4, 5} defined by $a \vee b = \min\{a, b\}$. Write the operation table of the operation∨.

Answer

The binary operation ∨ on the set {1, 2, 3, 4, 5} is defined as $a \vee b = \min\{a, b\}$ □ $a, b \in \{1, 2, 3, 4, 5\}$.

Thus, the operation table for the given operation ∨ can be given as:

∨	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

Question 4:

Consider a binary operation * on the set {1, 2, 3, 4, 5} given by the following multiplication table.

(i) Compute (2 * 3) * 4 and 2 * (3 * 4)

(ii) Is * commutative?

(iii) Compute (2 * 3) * (4 * 5).

(Hint: use the following table)

*	1	2	3	4	5
1	1	1	1	1	1

2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

Answer

(i) (2 * 3) * 4 = 1 * 4 = 1

2 * (3 * 4) = 2 * 1 = 1

(ii) For every *a, b* ∈{1, 2, 3, 4, 5}, we have *a* * *b* = *b* * *a*. Therefore, the operation * is commutative.

(iii) (2 * 3) = 1 and (4 * 5) = 1

∴(2 * 3) * (4 * 5) = 1 * 1 = 1

Question 5:

Let*′ be the binary operation on the set {1, 2, 3, 4, 5} defined by *a* *′ *b* = H.C.F. of *a* and *b*. Is the operation *′ same as the operation * defined in Exercise 4 above? Justify your answer.

Answer

The binary operation *′ on the set {1, 2, 3 4, 5} is defined as *a* *′ *b* = H.C.F of *a* and *b*.

The operation table for the operation *′ can be given as:

*′	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	1
3	1	1	3	1	1
4	1	2	1	4	1
5	1	1	1	1	5

We observe that the operation tables for the operations * and *′ are the same.

Thus, the operation *′ is same as the operation*.

Question 6:

Let * be the binary operation on **N** given by *a* * *b* = L.C.M. of *a* and *b*. Find

(i) 5 * 7, 20 * 16 (ii) Is * commutative?

(iii) Is * associative? (iv) Find the identity of * in **N**

*********** END **********