Definite Integrals Ex 20.2 Q53

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{3}{2}}}\,dx$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2\cos^2\frac{x}{2}}}{\left(2\sin^2\frac{x}{2}\right)^{\frac{3}{2}}}\,dx \qquad \left[\begin{array}{l} \because\ 1+\cos x = 2\cos^2\frac{x}{2} \\ \quad 1-\cos x = 2\sin^2\frac{x}{2} \end{array}\right]$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{2}\cos\frac{x}{2}}{2\sqrt{2}\sin^3\frac{x}{2}}\,dx$$

$$= \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,\operatorname{cosec}^2\frac{x}{2}\,dx \qquad \left[\begin{array}{l} \because\ \operatorname{cosec}^2\frac{x}{2} = \dfrac{1}{\sin^2\frac{x}{2}} \\ \quad \cot\frac{x}{2} = \dfrac{\cos\frac{x}{2}}{\sin\frac{x}{2}} \end{array}\right]$$

Let $\cot\frac{x}{2} = t$

Differentiating w.r.t. x, we get

$$\frac{-1}{2}\operatorname{cosec}^2\frac{x}{2} = dt$$

Now, $x = \frac{\pi}{3} \Rightarrow t = \sqrt{3}$

$x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cot\frac{x}{2}\,\operatorname{cosec}^2\frac{x}{2}\,dx = -\int_{\sqrt{3}}^{1} t\,dt = -\left[\frac{t^2}{2}\right]_{\sqrt{3}}^{1} = \frac{-1}{2}[1-3]$$

$$= 1$$

$$\therefore \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{\frac{3}{2}}}\,dx = 1$$

Definite Integrals Ex 20.2 Q54

Substitute $x^2 = a^2 \cos 2\theta$

Differentiating w.r.t. x, we get

$$2x\,dx = -2a^2 \sin 2\theta d\theta$$

Now, $x = 0 \Rightarrow \theta = \frac{\pi}{4}$

$x = a \Rightarrow \theta = 0$

$$\therefore \int_0^a x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}dx = \int_{\frac{\pi}{4}}^0 \sqrt{\frac{a^2(1 - \cos 2\theta)}{a^2 - (1 - \cos 2\theta)}}\left(-a^2 \sin 2\theta\right)d\theta$$

$$= -a^2 \int_{\frac{\pi}{4}}^0 \frac{\sin\theta}{\cos\theta} \sin 2\theta d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} 2\sin^2 \theta d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} \left(1 - \cos 2\theta\right)d\theta$$

$$= a^2 \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}$$

$$= a^2 \left[\frac{\pi}{4} - \frac{1}{2}\right]$$

$$\therefore \int_0^a x\sqrt{\frac{a^2 - x^2}{a^2 + x^2}}dx = a^2 \left[\frac{\pi}{4} - \frac{1}{2}\right]$$

Definite Integrals Ex 20.2 Q55

Let $x = a\cos 2\theta$

Differentiating w.r.t. x, we get

$dx = -2a \sin 2\theta$

Now, $x = -a \Rightarrow \theta = \frac{\pi}{2}$

$x = a \Rightarrow \theta = 0$

$$\therefore \int_{-a}^a \sqrt{\frac{a - x}{a + x}}dx = \int_{\frac{\pi}{2}}^0 \sqrt{\frac{a(1 - \cos 2\theta)}{a(1 + \cos 2\theta)}}\left(-2\sin 2\theta\right)d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta}.\sin 2\theta d\theta \qquad \left[\begin{array}{l} \because 1 - \cos 2\theta = 2\sin^2\theta \\ \quad 1 + \cos 2\theta = 2\cos^2\theta \\ -\int_a^b f(x)dx = \int_b^a f(x)dx \end{array}\right]$$

$$= 2a\int_0^{\frac{\pi}{2}} \frac{\sin\theta.2\sin\theta\cos\theta}{\cos\theta}$$

$$= 4a\int_0^{\frac{\pi}{2}} \sin^2\theta d\theta$$

$$= 2a\int_0^{\frac{\pi}{2}} \left(1 - \cos 2\theta\right)d\theta$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2a\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2a\left[\frac{\pi}{2} - 0 - 0 + 0\right] = \pi a$$

$$\therefore \int_{-a}^a \sqrt{\frac{a - x}{a + x}}dx = \pi a$$

Definite Integrals Ex 20.2 Q56

Let $\cos x = t$

Differentiating w.r.t. x, we get

$-\sin x dx = dt$

Now, $x = 0 \Rightarrow t = 1$

$x = \frac{\pi}{2} \Rightarrow t = 0$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{\cos^2 x + 3\cos x + 2}$$

$$= -\int_1^0 \frac{t dt}{t^2 + 3t + 2}$$

$$= \int_0^1 \frac{t dt}{(t+2)(t+1)} \qquad \left[\because -\int_a^b f(x) = \int_b^a f(x)\right]$$

$$= \int_0^1 \left(-\frac{1}{t+1} + \frac{2}{t+2}\right) dt \qquad \left[\text{Applying partial fraction}\right]$$

$$= \left[-\log|1+t| + 2\log|t+2|\right]_0^1$$

$$= -\log 2 + 2\log 3 + 0 - 2\log 2$$

$$= 2\log 3 - 3\log 2$$

$$= \log\frac{9}{8}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\sin x \cos x dx}{\cos^2 x + 3\cos x + 2} = \log\frac{9}{8}$$

*********** END *********