$\Rightarrow \cos Q = \frac{19}{7\times 3}$

$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$\therefore \left|\vec{b}_1\right| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

$\left|\vec{b}_2\right| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$

$\vec{b}_1 \cdot \vec{b}_2 = \left(\hat{i} - \hat{j} - 2\hat{k}\right)\cdot\left(3\hat{i} - 5\hat{j} - 4\hat{k}\right)$

$= 1\cdot 3 - 1(-5) - 2(-4)$

$= 3 + 5 + 8$

$= 16$

$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$\Rightarrow \cos Q = \frac{16}{\sqrt{6}\cdot 5\sqrt{2}} = \frac{16}{\sqrt{2}\cdot\sqrt{3}\cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$

$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$

$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$

Question 11:

Find the angle between the following pairs of lines:

(i) $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii) $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

i. Answer

ii. Let $\vec{b}_1$ and $\vec{b}_2$ be the vectors parallel to the pair of lines, $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$, respectively.

$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k}$ and $\vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$

$\left|\vec{b}_1\right| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$

$\left|\vec{b}_2\right| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$

$\vec{b}_1 \cdot \vec{b}_2 = \left(2\hat{i} + 5\hat{j} - 3\hat{k}\right)\cdot\left(-\hat{i} + 8\hat{j} + 4\hat{k}\right)$

$= 2(-1) + 5\times 8 + (-3)\cdot 4$

$= -2 + 40 - 12$

$= 26$

The angle, Q, between the given pair of lines is given by the relation,

$\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}\right|$

$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$

$\Rightarrow Q = \cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$

(ii) Let $\vec{b}_1, \vec{b}_2$ be the vectors parallel to the given pair of lines, $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and $\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}$, respectively.

$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$

$$\therefore |\vec{b}_1| = \sqrt{(2)^2+(2)^2+(1)^2} = \sqrt{9} = 3$$

$$|\vec{b}_2| = \sqrt{4^2+1^2+8^2} = \sqrt{81} = 9$$

$$\begin{aligned}\vec{b}_1 \cdot \vec{b}_2 &= \left(2\hat{i}+2\hat{j}+\hat{k}\right)\cdot\left(4\hat{i}+\hat{j}+8\hat{k}\right)\\ &= 2\times4+2\times1+1\times8\\ &= 8+2+8\\ &= 18\end{aligned}$$

If Q is the angle between the given pair of lines, then $\cos Q = \left|\frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}\right|$

$$\Rightarrow \cos Q = \frac{18}{3\times9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

Question 12:

Find the values of p so the line $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$ and $\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles.

Answer

The given equations can be written in the standard form as

$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2}$ and $\frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$

The direction ratios of the lines are -3, $\frac{2p}{7}$, 2 and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore (-3)\cdot\left(\frac{-3p}{7}\right)+\left(\frac{2p}{7}\right)\cdot(1)+2\cdot(-5)=0$$

$$\Rightarrow \frac{9p}{7}+\frac{2p}{7}=10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

The direction ratios of the given lines are 7, −5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2, are perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$

$= 7 - 10 + 3$

$= 0$

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$\vec{r} = \left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and

$\vec{r} = 2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$

Answer

The equations of the given lines are

$$\vec{r} = \left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$$

$$\vec{r} = 2\hat{i}-\hat{j}-\hat{k}+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$\left|\left(\vec{b}_1\times\vec{b}_2\right)\cdot\left(\vec{a}_2-\vec{a}_1\right)\right|$$

$$d = \left|\frac{(\vec{b}_1 \times \vec{b}_2)\cdot(\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|}\right| \qquad ...(1)$$

Comparing the given equations, we obtain

$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$

$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$

$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$

$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$

$\vec{a}_2 - \vec{a}_1 = \left(2\hat{i} - \hat{j} - \hat{k}\right) - \left(\hat{i} + 2\hat{j} + \hat{k}\right) = \hat{i} - 3\hat{j} - 2\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$

$\Rightarrow \left|\vec{b}_1 \times \vec{b}_2\right| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$

Substituting all the values in equation (1), we obtain

$$d = \left|\frac{\left(-3\hat{i} + 3\hat{k}\right)\cdot\left(\hat{i} - 3\hat{j} - 2\hat{k}\right)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-3.1 + 3(-2)}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \left|\frac{-9}{3\sqrt{2}}\right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

*********** END **********