CHAPTER 10

ROTATIONAL MECHANICS

Consider a pulley fixed at a typical Indian well on which a rope is wound with one end attached to a bucket. When the bucket is released, the pulley starts rotating. As the bucket goes down, the pulley rotates more rapidly till the bucket goes into the water.

Take the pulley as the system. The centre of mass of the pulley is at its geometrical centre which remains at rest. However, the other particles of the pulley move and are accelerated. The pulley is said to be executing rotational motion. Also, the rotational motion is not uniform. Since $\vec{a}_{CM} = 0$, the resultant external force $\vec{F}$ acting on the pulley must be zero. Even then the pulley is not in rotational equilibrium. We shall now study this type of motion.

10.1 ROTATION OF A RIGID BODY ABOUT A GIVEN FIXED LINE

Take a rigid body like a plate or a ball or your tennis racket or anything else present nearby and hold it between your fingers at two points. Now keep these two points fixed and then displace the body (try it with any rigid body at hand).

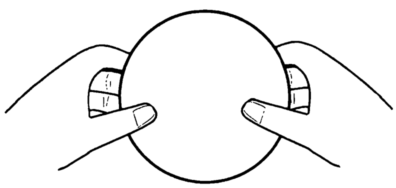


Figure 10.1

Notice the kind of displacement you can produce. In particular, notice that each particle of the rigid body goes in a circle, the centre being on the line joining the two fixed points between your finger tips. Let us call this line the axis of rotation. In fact, the centre of the circular path of a particle is at the foot of the perpendicular from the particle to this axis. Different particles move in different circles, the planes of all these circles are parallel to each other, and the radii depend on the distances of the particles from this axis. The particles on the axis remain stationary, those close to this line move on smaller circles and those far away from this line move in larger circles. However, each particle takes equal time to complete its circle.

Such a displacement of a rigid body in which a given line is held fixed, is called *rotation of the rigid body about the given line.* The line itself is called the *axis of rotation.*

Examples : (1) Consider the door of your almirah. When you open the door, the vertical line passing through the hinges is held fixed and that is the axis of rotation. Each point of the door describes a circle with the centre at the foot of the perpendicular from the particle on the axis. All these circles are horizontal and thus perpendicular to the axis.

(2) Consider the ceiling fan in your room. When it is on, each point on its body goes in a circle. Locate the centres of the circles traced by different particles on the three blades of the fan and the body covering the motor. All these centres lie on a vertical line through the centre of the body. The fan rotates about this vertical line.

(3) Look at the on–off switch on the front panel of your gas stove in the kitchen. To put the gas on, you push the switch a little, and then you rotate it. While rotating, each particle of the switch moves on a circle. Think about the centres of all such circles. They lie on a straight line (generally horizontal, towards the operator). This is the axis of rotation and the switch rotates about this axis.

Sometimes the axis may not pass through the body. Consider a record rotating on the turntable of a record player. Suppose a fly is sitting on the record near the rim. Look at the path of any particle of the fly. It is a circle with the centre on the vertical line through the centre of the record. The fly is "rotating about this vertical line" (can you consider the fly as a rigid body ?). The axis of rotation is lying completely outside the fly.

If each particle of a rigid body moves in a circle, with centres of all the circles on a straight line and with planes of the circles perpendicular to this line, we say that the body is rotating about this line. The straight line itself is called the *axis of rotation.*

10.2 KINEMATICS

Consider a rigid body rotating about a given fixed line. Take this line as the Z-axis. Consider a particle P of the body (figure 10.2). Look at its position P_0 at $t = 0$. Draw a perpendicular P_0Q to the axis of rotation. At time t, the particle moves to P. Let $\angle PQP_0 = \theta$. We say that the particle P has rotated through an angle θ. In fact, all the particles of the body have also rotated through the same angle θ and so we say that the whole rigid body has rotated through an angle θ. The "angular position" of the body at time t is said to be θ. If P has made a complete revolution on its circle, every particle has done so and we say that the body has rotated through an angle of 2π. So the rotation of a rigid body is measured by the rotation of the line QP from its initial position.

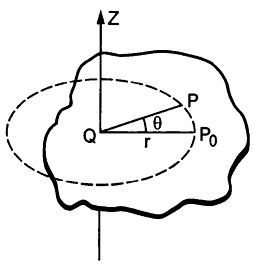


Figure 10.2

Now, suppose the angular position of the body at time t is θ. During a time Δt, it further rotates through $\Delta\theta$, so that its angular position becomes $\theta + \Delta\theta$. The average angular velocity during the time interval Δt is $\omega = \frac{\Delta\theta}{\Delta t}$.

The instantaneous angular velocity at time t is

$$\omega = \frac{d\theta}{dt}.$$

We associate the direction of the axis of rotation with the angular velocity. If the body rotates anticlockwise as seen through the axis, the angular velocity is towards the viewer. If it rotates clockwise, the angular velocity is away from the reader. It turns out that the angular velocity adds like a vector and hence it becomes a vector quantity. The magnitude of angular velocity is called *angular speed* . However, we shall continue to use the word angular velocity if the direction of the axis is clear from the context. The SI unit for angular velocity is radian/sec (rad/s). Quite often the angular velocity is given in revolutions per second (rev/s). The conversion in radian per second may be made using 1 rev = 2 π radian.

If the body rotates through equal angles in equal time intervals (irrespective of the smallness of the intervals), we say that it rotates with uniform angular velocity. In this case $\omega = d\theta/dt =$ constant and thus $\theta = \omega t$. If it is not the case, the body is said to be rotationally "accelerated". The angular acceleration is defined as

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d^2\theta}{dt^2}.$$

If the angular acceleration α is constant, we have

$$\omega = \omega_0 + \alpha t \quad \dots (10.1)$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \quad \dots (10.2)$$

and

$$\omega^2 = \omega_0^2 + 2\,\alpha\theta \quad \dots (10.3)$$

where ω_0 is the angular velocity of the body at $t = 0$.

As an example, think of your ceiling fan. Switch on the fan. The fan rotates about a vertical line (axis). The angle rotated by the fan in the first second is small, that in the second second is larger, that in the third second is still larger and so on. The fan, thus, has an angular acceleration. The angular velocity $\omega = d\theta/dt$ increases with time. Wait for about a couple of minutes. The fan has now attained full speed. The angle rotated in any time interval is now equal to the angle rotated in the successive equal time interval. The fan is rotating uniformly about the vertical axis. Now switch off the fan. The angle rotated in any second is smaller than the angle rotated in the previous second. The angular velocity $d\omega/dt$ decreases as time passes, and finally it becomes zero when the fan stops. The fan has an angular deceleration.

Given the axis of rotation, the body can rotate in two directions. Looking through the axis, it may be clockwise or anticlockwise. One has to define the "positive" rotation. This may be defined according to the convenience of the problem, but once decided, one has to stick to the choice. The angular displacement, angular velocity and the angular acceleration may accordingly be positive or negative.

Notice the similarity between the motion of a particle on a straight line and the rotation of a rigid body about a fixed axis. The position of the particle was decided by a single variable x, which could be positive or negative according to the choice of the positive direction of the X-axis. The rate of change of position gave the velocity and the rate of change of velocity gave the acceleration.

Example 10.1

The motor of an engine is rotating about its axis with an angular velocity of 100 rev/minute. *It comes to rest* *in* 15 s, *after being switched off. Assuming constant angular deceleration, calculate the number of revolutions made by it before coming to rest.*

Solution : The initial angular velocity = 100 rev/minute
$= (10\,\pi/3)$ rad/s.

Final angular velocity = 0.

Time interval = 15 s.

Let the angular acceleration be α. Using the equation $\omega = \omega_0 + \alpha t$, we obtain $\alpha = (-2\pi/9)$ rad/s^2.

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$= \left(\frac{10\pi}{3}\frac{\text{rad}}{\text{s}}\right)(15\text{ s}) - \frac{1}{2}\left(\frac{2\pi}{9}\frac{\text{rad}}{\text{s}^2}\right)(15\text{ s})^2$$

$$= 25\pi \text{ rad} = 12{\cdot}5 \text{ revolutions.}$$

Hence the motor rotates through 12·5 revolutions before coming to rest.

Example 10.2

Starting from rest, a fan takes five seconds to attain the maximum speed of 400 rpm*(revolutions per minute). Assuming constant acceleration, find the time taken by the fan in attaining half the maximum speed.*

Solution : Let the angular acceleration be α. According to the question,

$$400 \text{ rev/min} = 0 + \alpha\, 5\text{ s} \qquad \text{... (i)}$$

Let t be the time taken in attaining the speed of 200 rev/min which is half the maximum.

Then, $$200 \text{ rev/min} = 0 + \alpha t \qquad \text{... (ii)}$$

Dividing (i) by (ii), we get,

$$2 = 5\text{ s}/t \quad \text{or,} \quad t = 2{\cdot}5\text{ s.}$$

Relation between the Linear Motion of a Particle of a Rigid Body and its Rotation

Consider a point P of the rigid body rotating about a fixed axis as shown in figure (10.2). As the body rotates, this point moves on a circle. The radius of this circle is the perpendicular distance of the particle from the axis of rotation. Let it be r. If the body rotates through an angle θ, so does the radius joining the particle with the centre of its circle. The linear distance moved by the particle is $s = r\theta$ along the circle.

The linear speed along the tangent is

$$v = \frac{ds}{dt} = r\cdot\frac{d\theta}{dt} = r\omega \qquad \text{... (10.4)}$$

and the linear acceleration along the tangent, i.e., the tangential acceleration, is

$$a = \frac{dv}{dt} = r\cdot\frac{d\omega}{dt} = r\alpha. \qquad \text{... (10.5)}$$

The relations $v = r\omega$ and $a = r\alpha$ are very useful and their meanings should be clearly understood. For different particles of the rigid body, the radius r of their circles has different values, but ω and α are same for all the particles. Thus, the linear speed and the tangential acceleration of different particles are different. For $r = 0$, i.e., for the particles on the axis, $v = r\omega = 0$ and $a = r\alpha = 0$, consistent with the fact that the particles on the axis do not move at all.

Example 10.3

A bucket is being lowered down into a well through a rope passing over a fixed pulley of radius 10 cm. *Assume that the rope does not slip on the pulley. Find the angular velocity and angular acceleration of the pulley at an instant when the bucket is going down at a speed of* 20 cm/s *and has an acceleration of* 4·0 m/s^2.

Solution : Since the rope does not slip on the pulley, the linear speed v of the rim of the pulley is same as the speed of the bucket.

The angular velocity of the pulley is then

$$\omega = v/r = \frac{20\text{ cm/s}}{10\text{ cm}} = 2\text{ rad/s}$$

and the angular acceleration of the pulley is

$$\alpha = a/r = \frac{4{\cdot}0\text{ m/s}^2}{10\text{ cm}} = 40\text{ rad/s}^2.$$

10.3 ROTATIONAL DYNAMICS

When one switches a fan on, the centre of the fan remains unmoved while the fan rotates with an angular acceleration. As the centre of mass remains at rest, the external forces acting on the fan must add to zero. This means that one can have angular acceleration even if the resultant external force is zero. But then why do we need to switch on the fan in order to start it ? If an angular acceleration may be achieved with zero total external force, why does not a wheel chair start rotating on the floor as soon as one wishes it to do so. Why are we compelled to use our muscles to set it into rotation ? In fact, one cannot have angular acceleration without external forces.

What is then the relation between the force and the angular acceleration ? We find that even if the resultant external force is zero, we may have angular acceleration. We also find that without applying an external force we cannot have an angular acceleration. What is responsible for producing angular acceleration ? The answer is *torque* which we define below.

10.4 TORQUE OF A FORCE ABOUT THE AXIS OF ROTATION

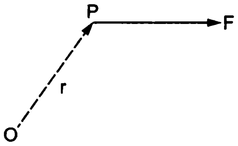


Figure 10.3

Consider a force $\vec{F}$ acting on a particle P. Choose an origin O and let $\vec{r}$ be the position vector of the particle experiencing the force. We define *the torque of the force F about O* as

$$\vec{\Gamma} = \vec{r} \times \vec{F} \qquad \dots (10.6)$$

This is a vector quantity having its direction perpendicular to $\vec{r}$ and $\vec{F}$ according to the rule of cross product. Now consider a rigid body rotating about a given axis of rotation AB (figure 10.4). Let F be a force acting on the particle P of the body. F may not be in the plane ABP. Take the origin O somewhere on the axis of rotation.

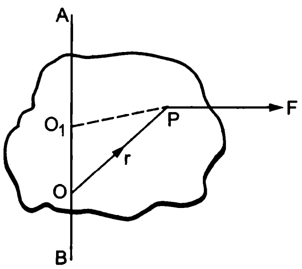


Figure 10.4

The torque of F about O is $\vec{\Gamma} = \vec{r} \times \vec{F}$. Its component along OA is called the torque of $\vec{F}$ *about OA*. To calculate it, we should find the vector $\vec{r} \times \vec{F}$ and then find out the angle θ it makes with OA. The torque about OA is then $|\vec{r} \times \vec{F}| \cos\theta$. The torque of a force about a line is independent of the choice of the origin as long as it is chosen on the line. This can be shown as given below. Let O_1 be any point on the line AB (figure 10.4). The torque of F about O_1 is

$$\vec{O_1P} \times \vec{F} = (\vec{O_1O} + \vec{OP}) \times \vec{F} = \vec{O_1O} \times \vec{F} + \vec{OP} \times \vec{F}.$$

As $\vec{O_1O} \times \vec{F} \perp \vec{O_1O}$, this term will have no component along AB.

Thus, the component of $\vec{O_1P} \times \vec{F}$ is equal to that of $\vec{OP} \times \vec{F}$.

There are some special cases which occur frequently.

Case I

$$\vec{F} \parallel \vec{AB}.$$

$\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel \vec{AB}$, hence $\vec{r} \times \vec{F}$ is perpendicular to $\vec{AB}$. The component of $\vec{r} \times \vec{F}$ along $\vec{AB}$ is, therefore, zero.

Case II

F intersects AB (say at O)

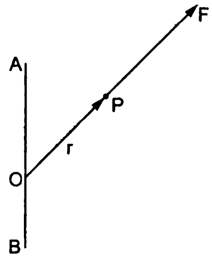


Figure 10.5

Taking the point of intersection as the origin, we see that $\vec{r} (= \vec{OP})$ and $\vec{F}$ are in the same line. The torque about O is $\vec{r} \times \vec{F} = 0$. Hence the component along OA is zero.

Case III

$\vec{F} \perp \vec{AB}$ but $\vec{F}$ and AB do not intersect.

In three dimensions, two lines may be perpendicular without intersecting each other. For example, a vertical line on the surface of a wall of your room and a horizontal line on the opposite wall are mutually perpendicular but they never intersect. Two nonparallel and nonintersecting lines are called *skew* lines.

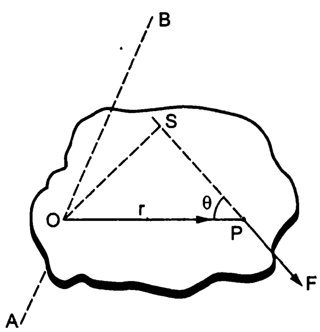


Figure 10.6

Figure (10.6) shows the plane through the particle P that is perpendicular to the axis of rotation AB. Suppose the plane intersects the axis at the point O. The force F is in this plane. Taking the origin at O,

$$\vec{\Gamma} = \vec{r} \times \vec{F} = \vec{OP} \times \vec{F}.$$

Thus, $\Gamma = rF \sin\theta = F.\ (OS)$

where OS is the perpendicular from O to the line of action of the force $\vec{F}$. The line OS is also perpendicular to the axis of rotation. It is thus the length of the common perpendicular to the force and the axis of rotation.

The direction of $\vec{\Gamma} = \vec{OP} \times \vec{F}$ is along the axis AB because $\vec{AB} \perp \vec{OP}$ and $\vec{AB} \perp \vec{F}$. The torque about AB is, therefore, equal to the magnitude of $\vec{\Gamma}$ that is $F.(OS)$.

Thus, the torque of F about AB = magnitude of the force $F \times$ length of the common perpendicular to the force and the axis. The common perpendicular OS is called the *lever arm or moment arm* of this torque.

The torque may try to rotate the body clockwise or anticlockwise about AB. Depending on the convenience of the problem one may be called positive and the other negative. It is conventional to take the torque positive if the body rotates anticlockwise as viewed through the axis.

Case IV

$\vec{F}$ and $\vec{OA}$ are skew but not perpendicular.

Take components of $\vec{F}$ parallel and perpendicular to the axis.

The torque of the parallel part is zero from case I and that of the perpendicular part may be found as in case III.

In most of the applications that we shall see, cases I, II or III will apply.

Example 10.4

Consider a pulley fixed at its centre of mass by a clamp. A light rope is wound over it and the free end is tied to a block. The tension in the rope is T. (a) Write the forces acting on the pulley. How are they related ? (b) Locate the axis of rotation. (c) Find the torque of the forces about the axis of rotation.

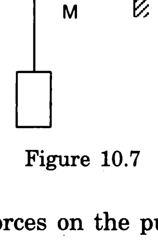


Figure 10.7

Solution : (a) The forces on the pulley are (figure 10.7)

(i) attraction by the earth, Mg vertically downward,

(ii) tension T by the rope, along the rope,

(iii) contact force $\mathcal{N}$ by the support at the centre.

$\mathcal{N} = T + Mg$ (centre of mass of the pulley is at rest, so Newton's 1st law applies).

(b) The axis of rotation is the line through the centre of the pulley and perpendicular to the plane of the pulley.

(c) Let us take the positive direction of the axis towards the reader.

The force Mg passes through the centre of mass and it intersects the axis of rotation. Hence the torque of Mg about the axis is zero (Case II). Similarly, the torque of the contact force $\mathcal{N}$ is also zero.

The tension T is along the tangent of the rim in the vertically downward direction. The tension and the axis of rotation are perpendicular but never intersect. Case III applies. Join the point where the rope leaves the rim to the centre. This line is the common perpendicular to the tension and the axis. Hence the torque is $T.r$ (positive, since it will try to rotate the pulley anticlockwise).

If there are more than one forces $\vec{F}_1, \vec{F}_2, \vec{F}_3, \ldots$ acting on a body, one can define the *total torque* acting on the body about a given line.

To obtain the total torque, we have to get separately the torques of the individual forces and then add them.

$$\vec{\Gamma} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$$

You may be tempted to add the forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3, \ldots$ vectorially and then obtain the torque of resultant force about the axis. But that won't always work. Even if $\vec{F}_1 + \vec{F}_2 + \ldots = 0$, $\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \ldots$ may not. However, if the forces act on the same particle, one can add the forces and then take the torque of the resultant.

10.5 $\Gamma = I\alpha$

We are now in a position to tell how the angular acceleration is produced when the resultant force on the body is zero. It is the total torque that decides the angular acceleration. Although the resultant force on the fan in our example is zero, the total torque is not. Whereas, if one does not apply any force, the torque is also zero and no angular acceleration is produced. For angular acceleration, there must be a torque.

To have linear acceleration of a particle, the total force on the particle should be nonzero. The acceleration of the particle is proportional to the force applied on it. To have angular acceleration about an axis you must have a nonzero torque on the body about the axis of rotation. Do we also have the relation that the angular acceleration is proportional to the total torque on the body ? Let us hope so.

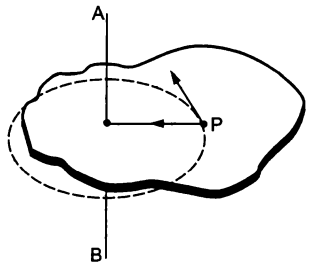


Figure 10.8

Consider a rigid body rotating about a fixed axis *AB* (figure 10.8). Consider a particle *P* of mass *m* rotating in a circle of radius *r*.

The radial acceleration of the particle $= \frac{v^2}{r} = \omega^2 r$.

Thus, the radial force on it $= m\omega^2 r$.

The tangential acceleration of the particle $= \frac{dv}{dt}$.

Thus, the tangential force on it

$$= m\frac{dv}{dt} = mr\frac{d\omega}{dt} = m\,r\,\alpha .$$

The torque of $m\omega^2 r$ about AB is zero as it intersects the axis and that of $mr\alpha$ is $mr^2\alpha$ as the force and the axis are skew and perpendicular. Thus, the torque of the resultant force acting on P is $mr^2\alpha$. Summing over all the particles, the total torque of all the forces acting on all the particles of the body is

$$\Gamma^{total} = \sum_i m_i r_i^2 \alpha = I\alpha \qquad \text{... (i)}$$

where $$I = \sum_i m_i r_i^2 . \qquad \text{... (10.7)}$$

The quantity I is called the *moment of inertia* of the body about the axis of rotation. Note that m_i is the mass of the ith particle and r_i is its perpendicular distance from the axis.

We have $\Gamma^{total} = \sum_i (\vec{r_i} \times \vec{F_i})$ where $\vec{F_i}$ is the resultant force on the ith particle. This resultant force consists of forces by all the other particles as well as other external forces applied on the ith particle. Thus,

$$\Gamma^{total} = \sum_i \vec{r_i} \times \left(\sum_{j \neq i} \vec{F}_{ij} + \vec{F_i}^{ext} \right)$$

where $\vec{F}_{ij}$ is the force on the ith particle by the jth particle and $\vec{F_i}^{ext}$ is the external force applied on the ith particle. When summation is made on both i and j, the first summation contains pairs like $\vec{r_i} \times \vec{F}_{ij} + \vec{r_j} \times \vec{F}_{ji}$. Newton's third law tells us that $\vec{F}_{ij} = -\vec{F}_{ji}$ so that such pairs become $(\vec{r_i} - \vec{r_j}) \times \vec{F}_{ij}$. Also the force $\vec{F}_{ij}$ is along the line joining the particles so that $(\vec{r_i} - \vec{r_j}) \parallel \vec{F}_{ij}$ and the cross product is zero. Thus, it is necessary to consider only the torques of the external forces applied on the rigid body and (i) becomes

$$\Gamma^{ext} = I\alpha \qquad \text{... (10.8)}$$

where the torque and the moment of inertia are both evaluated about the axis of rotation.

Note the similarity between $\Gamma = I\alpha$ and $F = Ma$. Also, note the dissimilarity between the behaviour of M and I. The mass M is a property of the body and does not depend on the choice of the origin or the axes or the kind of motion it undergoes (as long as we are dealing with velocities much less than 3×10^8 m/s). But the moment of inertia $I = \sum_i m_i r_i^2$ depends on the choice of the axis about which it is calculated. The quantity r_i is the perpendicular distance of the ith particle from the "axis". Changing the axis changes r_i and hence I.

Moment of inertia of bodies of simple geometrical shapes may be calculated using the techniques of integration. We shall discuss the calculation for bodies of different shapes in somewhat greater detail in a later section.

Note that $\Gamma = I\alpha$ is not an independent rule of nature. It is derived from the more basic Newton's laws of motion.

Example 10.5

A wheel of radius 10 cm *can rotate freely about its centre as shown in figure (10.9). A string is wrapped over its rim and is pulled by a force of* 5·0 N. *It is found that the torque produces an angular acceleration* 2·0 rad/s² *in the wheel. Calculate the moment of inertia of the wheel.*

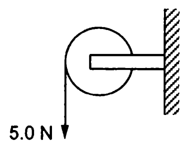


Figure 10.9

Solution : The forces acting on the wheel are (i) W due to gravity, (ii) $\mathcal{N}$ due to the support at the centre and (iii) F due to tension. The torque of W and $\mathcal{N}$ are separately zero and that of F is $F.r$. The net torque is

$$\Gamma = (5{\cdot}0 \text{ N}).(10 \text{ cm}) = 0{\cdot}50 \text{ N-m}.$$

The moment of inertia is

$$I = \frac{\Gamma}{\alpha} = \frac{0{\cdot}50 \text{ N-m}}{2 \text{ rad/s}^2} = 0{\cdot}25 \text{ kg-m}^2 .$$

10.6 BODIES IN EQUILIBRIUM

The centre of mass of a body remains in equilibrium if the total external force acting on the body is zero. This follows from the equation $F = Ma$. Similarly, a body remains in rotational equilibrium if the total external torque acting on the body is zero. This follows from the equation $\Gamma = I\alpha$. Thus, if a body remains at rest in an inertial frame, the total external force acting on the body should be zero in any direction and the total external torque should be zero about any line.

We shall often find situations in which all the forces acting on a body lie in a single plane as shown in figure (10.10).

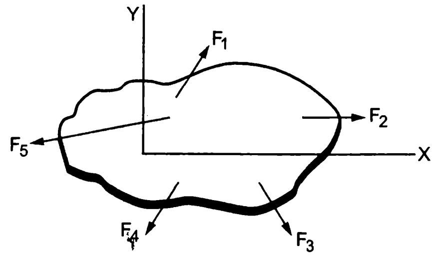


Figure 10.10

Let us take this plane as the X-Y plane. For translational equilibrium

$$\sum F_x = 0 \qquad \dots \text{(i)}$$

and $$\sum F_y = 0. \qquad \dots \text{(ii)}$$

As all the forces are in the X-Y plane, F_z is identically zero for each force and so $\sum F_z = 0$ is automatically satisfied. Now consider rotational equilibrium. The torque of each force about the X-axis is identically zero because either the force intersects the axis or it is parallel to it. Similarly, the torque of each force about the Y-axis is identically zero. In fact, the torque about any line in the X-Y plane is zero.

Thus, the condition of rotational equilibrium is

$$\sum \Gamma_z = 0. \qquad \dots \text{(iii)}$$

While taking torque about the Z-axis, the origin can be chosen at any point in the plane of the forces. That is, the torque can be taken about any line perpendicular to the plane of the forces. In general, the torque is different about different lines but it can be shown that if the resultant force is zero, the total torque about any line perpendicular to the plane of the forces is equal. If it is zero about one such line, it will be zero about all such lines.

If a body is placed on a horizontal surface, the torque of the contact forces about the centre of mass should be zero to maintain the equilibrium. This may happen only if the vertical line through the centre of mass cuts the base surface at a point within the contact area or the area bounded by the contact points. That is why a person leans in the opposite direction when he or she lifts a heavy load in one hand.

The equilibrium of a body is called *stable* if the body tries to regain its equilibrium position after being slightly displaced and released. It is called *unstable* if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in *neutral equilibrium.*

In the case of stable equilibrium, the centre of mass goes higher on being slightly displaced. For unstable equilibrium it goes lower and for neutral equilibrium it stays at the same height.

10.7 BENDING OF A CYCLIST ON A HORIZONTAL TURN

Suppose a cyclist is going at a speed v on a circular horizontal road of radius r which is not banked. Consider the cycle and the rider together as the system. The centre of mass C (figure 10.11a) of the system is going in a circle with the centre at O and radius r.

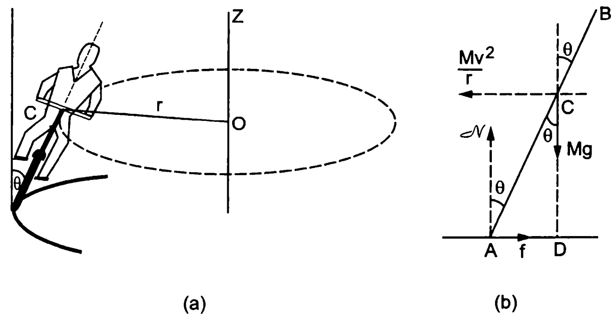


Figure 10.11

Let us choose O as the origin, OC as the X-axis and vertically upward as the Z-axis. This frame is rotating at an angular speed $\omega = v/r$ about the Z-axis. In this frame the system is at rest. Since we are working from a rotating frame of reference, we will have to apply a centrifugal force on each particle. The net centrifugal force on the system will be $M\omega^2 r = Mv^2/r$, where M is the total mass of the system. This force will act through the centre of mass. Since the system is at rest in this frame, no other pseudo force is needed.

Figure (10.11b) shows the forces. The cycle is bent at an angle θ with the vertical. The forces are

(i) weight Mg,

(ii) normal force $\mathcal{N}$,

(iii) friction f and,

(iv) centrifugal force Mv^2/r.

In the frame considered, the system is at rest. Thus, the total external force and the total external torque must be zero. Let us consider the torques of all the forces about the point A. The torques of $\mathcal{N}$ and f about A are zero because these forces pass through A. The torque of Mg about A is $Mg(AD)$ in the clockwise direction and that of $\frac{Mv^2}{r}$ is $\frac{Mv^2}{r}(CD)$ in the anti-clockwise direction. For rotational equilibrium,

$$Mg\,(AD) = \frac{Mv^2}{r}\,(CD)$$

or, $$\frac{AD}{CD} = \frac{v^2}{rg}$$

or, $$\tan\theta = \frac{v^2}{rg}. \qquad \ldots (10.9)$$

Thus the cyclist bends at an angle $\tan^{-1}\left(\frac{v^2}{rg}\right)$ with the vertical.

10.8 ANGULAR MOMENTUM

Angular momentum of a particle about a point O is defined as

$$\vec{l} = \vec{r} \times \vec{p} \qquad \ldots (10.10)$$

where $\vec{p}$ is the linear momentum and $\vec{r}$ is the position vector of the particle from the given point O. The angular momentum of a system particles is the vector sum of the angular momenta of the particles of the system. Thus,

$$\vec{L} = \sum_i \vec{l}_i = \sum (\vec{r}_i \times \vec{p}_i).$$

Suppose a particle P of mass m moves at a velocity $\vec{v}$ (figure 10.12). Its angular momentum about a point O is,

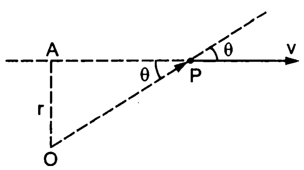


Figure 10.12

$$\vec{l} = \vec{OP} \times (m\vec{v})$$

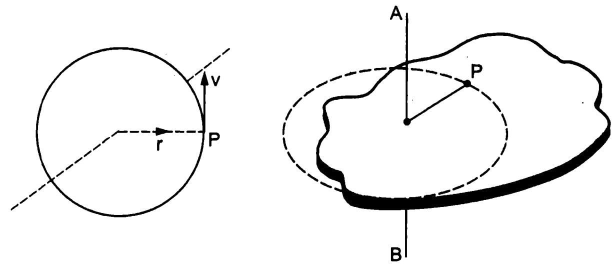
or, $$l = mv\,OP\sin\theta = mvr \qquad \ldots (10.11)$$

where $r = OA = OP\sin\theta$ is the perpendicular distance of the line of motion from O.

As in the case of torque, we define the angular momentum of a particle "about a line" say AB. Take any point O on the line AB and obtain the angular momentum $\vec{r} \times \vec{p}$ of the particle about O. The component of $\vec{r} \times \vec{p}$ along the line AB is called the angular momentum of the particle "about AB". The point O may be chosen anywhere on the line AB.

10.9 $L = I\omega$

Suppose a particle is going in a circle of radius r and at some instant the speed of the particle is v (figure 10.13a). What is the angular momentum of the particle about the axis of the circle ?



(a) (b)

Figure 10.13

As the origin may be chosen anywhere on the axis, we choose it at the centre of the circle. Then $\vec{r}$ is along a radius and $\vec{v}$ is along the tangent so that $\vec{r}$ is perpendicular to $\vec{v}$ and $l = |\vec{r} \times \vec{p}| = mvr$. Also $\vec{r} \times \vec{p}$ is perpendicular to $\vec{r}$ and $\vec{p}$ and hence is along the axis. Thus, the component of $\vec{r} \times \vec{p}$ along the axis is mvr itself.

Next consider a rigid body rotating about an axis AB (figure 10.13b). Let the angular velocity of the body be ω. Consider the ith particle going in a circle of radius r_i with its plane perpendicular to AB. The linear velocity of this particle at this instant is $v_i = r_i\omega$. The angular momentum of this particle about $AB = m_i v_i r_i = m_i r_i^2 \omega$. The angular momentum of the whole body about AB is the sum of these components, i.e.,

$$L = \sum m_i r_i^2\,\omega = I\omega \qquad \ldots (10.12)$$

where I is the moment of inertia of the body about AB.

10.10 CONSERVATION OF ANGULAR MOMENTUM

We have defined the angular momentum of a body as $\vec{L} = \sum (\vec{r}_i \times \vec{p}_i)$. Differentiating with respect to time,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum (\vec{r}_i \times \vec{p}_i)$$

$$= \sum_i \left[\frac{d\vec{r}_i}{dt} \times \vec{p}_i + \vec{r}_i \times \frac{d\vec{p}_i}{dt}\right]$$

$$= \sum_i \left[\vec{v}_i \times m\vec{v}_i + \vec{r}_i \times \vec{F}_i\right]$$

$$= \sum_i (\vec{r}_i \times \vec{F}_i) = \vec{\Gamma}^{\,total} \ldots \text{(i)}$$

where $\vec{F}_i$ is the total force acting on the ith particle. This includes any external force as well as the forces on the ith particle by all the other particles. When summation is taken over all the particles, the internal torques add to zero. Thus, (i) becomes

$$\frac{d\vec{L}}{dt} = \vec{\Gamma}^{ext} \quad \text{... (10.13)}$$

where $\vec{\Gamma}^{ext}$ is the total torque on the system due to all the external forces acting on the system.

For a rigid body rotating about a fixed axis, we can arrive at equation (10.13) in a simpler manner. We have

$$L = I\omega$$

or, $$\frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha$$

or, $$\frac{dL}{dt} = \Gamma^{ext}.$$

Equation (10.13) shows that

If the total external torque on a system is zero, its angular momentum remains constant.

This is known as the *principle of conservation of angular momentum.*

Example 10.6

A wheel is rotating at an angular speed ω about its axis which is kept vertical. An identical wheel initially at rest is gently dropped into the same axle and the two wheels start rotating with a common angular speed. Find this common angular speed.

Solution : Let the moment of inertia of the wheel about the axis be I. Initially the first wheel is rotating at the angular speed ω about the axle and the second wheel is at rest. Take both the wheels together as the system. The total angular momentum of the system before the coupling is $I\omega + 0 = I\omega$. When the second wheel is dropped into the axle, the two wheels slip on each other and exert forces of friction. The forces of friction have torques about the axis of rotation but these are torques of internal forces. No external torque is applied on the two-wheel system and hence the angular momentum of the system remains unchanged. If the common angular speed is ω', the total angular momentum of the two-wheel system is $2I\omega'$ after the coupling. Thus,

$$I\omega = 2I\omega'$$

or, $$\omega' = \omega/2.$$

10.11 ANGULAR IMPULSE

The angular impulse of a torque in a given time interval is defined as

$$J = \int_{t_1}^{t_2} \Gamma \, dt.$$

If Γ be the resultant torque acting on a body

$$\Gamma = \frac{dL}{dt}, \text{ or, } \Gamma dt = dL.$$

Integrating this

$$J = L_2 - L_1 .$$

Thus, the change in angular momentum is equal to the angular impulse of the resultant torque.

10.12 KINETIC ENERGY OF A RIGID BODY ROTATING ABOUT A GIVEN AXIS

Consider a rigid body rotating about a line AB with an angular speed ω. The ith particle is going in a circle of radius r_i with a linear speed $v_i = \omega r_i$. The kinetic energy of this particle is $\frac{1}{2} m_i(\omega r_i)^2$. The kinetic energy of the whole body is

$$\sum \frac{1}{2} m_i \omega^2 r_i^2 = \frac{1}{2} \sum (m_i r_i^2)\omega^2 = \frac{1}{2} I\omega^2.$$

Sometimes it is called rotational kinetic energy. It is not a new kind of kinetic energy as is clear from the derivation. It is the sum of $\frac{1}{2}mv^2$ of all the particles.

Example 10.7

A wheel of moment of inertia I and radius r is free to rotate about its centre as shown in figure (10.14). *A string is wrapped over its rim and a block of mass m is attached to the free end of the string. The system is released from rest. Find the speed of the block as it descends through a height h.*

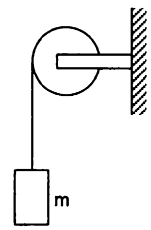


Figure 10.14

Solution : Let the speed of the block be v when it descends through a height h. So is the speed of the string and hence of a particle at the rim of the wheel. The angular velocity of the wheel is v/r and its kinetic energy at this instant is $\frac{1}{2}I(v/r)^2$. Using the principle of conservation of energy, the gravitational potential energy lost by the block must be equal to the kinetic energy gained by the block and the wheel. Thus,

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2}$$

or, $$v = \left[\frac{2\,mgh}{m + I/r^2}\right]^{1/2}.$$

10.13 POWER DELIVERED AND WORK DONE BY A TORQUE

Consider a rigid body rotating about a fixed axis on which a torque acts. The torque produces angular acceleration and the kinetic energy increases. The rate of increase of the kinetic energy equals the rate of doing work on it, i.e., the power delivered by the torque.

$$P = \frac{dW}{dt} = \frac{dK}{dt}$$

$$= \frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = I\omega\frac{d\omega}{dt} = I\alpha\omega = \Gamma\omega.$$

The work done in an infinitesimal angular displacement $d\theta$ is

$$dW = Pdt = \Gamma\omega\, dt = \Gamma\, d\theta.$$

The work done in a finite angular displacement θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} \Gamma\, d\theta. \quad \ldots (10.14)$$

10.14 CALCULATION OF MOMENT OF INERTIA

We have defined the moment of inertia of a system about a given line as

$$I = \sum_i m_i r_i^2$$

where m_i is the mass of the ith particle and r_i is its perpendicular distance from the given line. If the system is considered to be a collection of discrete particles, this definition may directly be used to calculate the moment of inertia.

Example 10.8

Consider a light rod with two heavy mass particles at its ends. Let AB be a line perpendicular to the rod as shown in figure (10.15). *What is the moment of inertia of the system about AB ?*

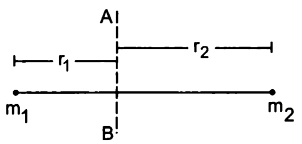


Figure 10.15

Solution : Moment of inertia of the particle on the left is $m_1 r_1^2$.

Moment of inertia of the particle on the right is $m_2 r_2^2$.

Moment of inertia of the rod is negligible as the rod is light.

Thus, the moment of inertia of the system about AB is

$$m_1 r_1^2 + m_2 r_2^2.$$

Example 10.9

Three particles, each of mass m, are situated at the vertices of an equilateral triangle ABC of side L (figure 10.16). *Find the moment of inertia of the system about the line AX perpendicular to AB in the plane of ABC.*

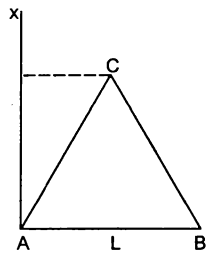


Figure 10.16

Solution : Perpendicular distance of A from $AX = 0$

" " B " " $= L$

" " C " " $= L/2$.

Thus, the moment of inertia of the particle at $A = 0$, of the particle at $B = mL^2$, and of the particle at $C = m(L/2)^2$. The moment of inertia of the three-particle system about AX is

$$0 + mL^2 + m(L/2)^2 = \frac{5\,mL^2}{4}\cdot$$

Note that the particles on the axis do not contribute to the moment of inertia.

Moment of Inertia of Continuous Mass Distributions

If the body is assumed to be continuous, one can use the technique of integration to obtain its moment of inertia about a given line. Consider a small element of the body. The element should be so chosen that the perpendiculars from different points of the element to the given line differ only by infinitesimal amounts. Let its mass be dm and its perpendicular distance from the given line be r. Evaluate the product r^2dm and integrate it over the appropriate limits to cover the whole body. Thus,

$$I = \int r^2 dm$$

under proper limits.

We can call r^2dm the moment of inertia of the small element. Moment of inertia of the body about the given line is the sum of the moments of inertia of its constituent elements about the same line.

(A) Uniform rod about a perpendicular bisector

Consider a uniform rod of mass M and length l (figure 10.17) and suppose the moment of inertia is to be calculated about the bisector AB. Take the origin at the middle point O of the rod. Consider the element of the rod between a distance x and $x + dx$ from the origin. As the rod is uniform,

Mass per unit length of the rod $= M/l$

so that the mass of the element $= (M/l)dx$.

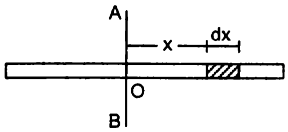


Figure 10.17

The perpendicular distance of the element from the line AB is x. The moment of inertia of this element about AB is

$$dI = \frac{M}{l} dx\, x^2.$$

When $x = -l/2$, the element is at the left end of the rod. As x is changed from $-l/2$ to $l/2$, the elements cover the whole rod.

Thus, the moment of inertia of the entire rod about AB is

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \left[\frac{M}{l}\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{Ml^2}{12}.$$

(B) Moment of inertia of a rectangular plate about a line parallel to an edge and passing through the centre

The situation is shown in figure (10.18). Draw a line parallel to AB at a distance x from it and another at a distance $x + dx$. We can take the strip enclosed between the two lines as the small element.

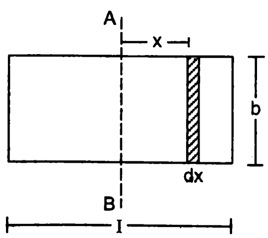


Figure 10.18

It is "small" because the perpendiculars from different points of the strip to AB differ by not more than dx. As the plate is uniform,

its mass per unit area $= \frac{M}{bl}$.

Mass of the strip $= \frac{M}{bl} b\, dx = \frac{M}{l} dx.$

The perpendicular distance of the strip from $AB = x$. The moment of inertia of the strip about $AB = dI = \frac{M}{l} dx\, x^2$. The moment of inertia of the given plate is, therefore,

$$I = \int_{-l/2}^{l/2} \frac{M}{l} x^2\, dx = \frac{Ml^2}{12}.$$

The moment of inertia of the plate about the line parallel to the other edge and passing through the centre may be obtained from the above formula by replacing l by b and thus,

$$I = \frac{Mb^2}{12}.$$

(C) Moment of inertia of a circular ring about its axis (the line perpendicular to the plane of the ring through its centre)

Suppose the radius of the ring is R and its mass is M. As all the elements of the ring are at the same perpendicular distance R from the axis, the moment of inertia of the ring is

$$I = \int r^2 dm = \int R^2 dm = R^2 \int dm = MR^2.$$

(D) Moment of inertia of a uniform circular plate about its axis

Let the mass of the plate be M and its radius R (figure 10.19). The centre is at O and the axis OX is perpendicular to the plane of the plate.

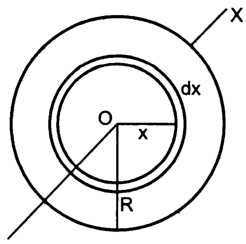


Figure 10.19

Draw two concentric circles of radii x and $x + dx$, both centred at O and consider the area of the plate in between the two circles.

This part of the plate may be considered to be a circular ring of radius x. As the periphery of the ring is $2\pi x$ and its width is dx, the area of this elementary ring is $2\pi x dx$. The area of the plate is πR^2. As the plate is uniform,

its mass per unit area $= \frac{M}{\pi R^2}$.

Mass of the ring $= \frac{M}{\pi R^2} 2\pi x\, dx = \frac{2 M x\, dx}{R^2}$.

Using the result obtained above for a circular ring, the moment of inertia of the elementary ring about OX is

$$dI = \left[\frac{2\,Mxdx}{R^2}\right] x^2.$$

The moment of inertia of the plate about OX is

$$I = \int_0^R \frac{2\,M}{R^2} x^3\, dx = \frac{MR^2}{2}.$$

(E) Moment of inertia of a hollow cylinder about its axis

Suppose the radius of the cylinder is R and its mass is M. As every element of this cylinder is at the same perpendicular distance R from the axis, the moment of inertia of the hollow cylinder about its axis is

$$I = \int r^2 dm = \int r^2 dm = R^2 \int dm = MR^2.$$

(F) Moment of inertia of a uniform solid cylinder about its axis

Let the mass of the cylinder be M and its radius R. Draw two cylindrical surfaces of radii x and $x + dx$ coaxial with the given cylinder. Consider the part of the cylinder in between the two surfaces (figure 10.20). This part of the cylinder may be considered to be a hollow cylinder of radius x. The area of cross-section of the wall of this hollow cylinder is $2\pi\, x\, dx$. If the length of the cylinder is l, the volume of the material of this elementary hollow cylinder is $2\pi\, x\, dx\, l$.

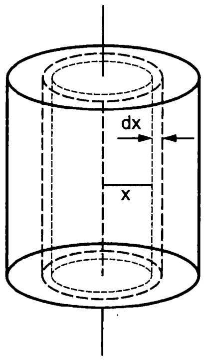


Figure 10.20

The volume of the solid cylinder is $\pi R^2 l$ and it is uniform, hence its mass per unit volume is

$$\rho = \frac{M}{\pi R^2 l}.$$

The mass of the hollow cylinder considered is

$$\frac{M}{\pi R^2 l} 2\pi x\, dx\, l = \frac{2\,M}{R^2}\, xdx.$$

As its radius is x, its moment of inertia about the given axis is

$$dI = \left[\frac{2\,M}{R^2} x\, dx\right] x^2.$$

The moment of inertia of the solid cylinder is, therefore,

$$I = \int_0^R \frac{2\,M}{R^2} x^3 dx = \frac{MR^2}{2}.$$

Note that the formula does not depend on the length of the cylinder.

(G) Moment of inertia of a uniform hollow sphere about a diameter

Let M and R be the mass and the radius of the sphere, O its centre and OX the given axis (figure 10.21). The mass is spread over the surface of the sphere and the inside is hollow.

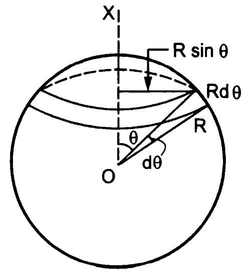


Figure 10.21

Let us consider a radius OA of the sphere at an angle θ with the axis OX and rotate this radius about OX. The point A traces a circle on the sphere. Now change θ to $\theta + d\theta$ and get another circle of somewhat larger radius on the sphere. The part of the sphere between these two circles, shown in the figure, forms a ring of radius $R \sin\theta$. The width of this ring is $Rd\theta$ and its periphery is $2\pi R \sin\theta$. Hence,

the area of the ring $= (2\pi R \sin\theta)\,(Rd\theta)$.

Mass per unit area of the sphere $= \dfrac{M}{4\pi R^2}$.

The mass of the ring

$$= \frac{M}{4\pi R^2}(2\pi R \sin\theta)\,(Rd\theta) = \frac{M}{2} \sin\theta\, d\theta.$$

The moment of inertia of this elemental ring about OX is

$$dI = \left(\frac{M}{2} \sin\theta\, d\theta\right)(R \sin\theta)^2.$$

$$= \frac{M}{2} R^2 \sin^3\theta\, d\theta$$

As θ increases from 0 to π, the elemental rings cover the whole spherical surface. The moment of inertia of the hollow sphere is, therefore,

$$I = \int_0^\pi \frac{M}{2} R^2 \sin^3\theta \, d\theta = \frac{MR^2}{2} \left[\int_0^\pi (1 - \cos^2\theta) \sin\theta \, d\theta \right]$$

$$= \frac{MR^2}{2} \left[\int_{\theta = 0}^{\pi} - (1 - \cos^2\theta) \, d(\cos\theta) \right]$$

$$= \frac{-MR^2}{2} \left[\cos\theta - \frac{\cos^3\theta}{3} \right]_0^\pi = \frac{2}{3} MR^2 .$$

Alternative method

Consider any particle P of the surface, having coordinates (x_i, y_i, z_i) with respect to the centre O as the origin (figure 10.22) and OX as the X-axis. Let PQ be the perpendicular to OX. Then $OQ = x_i$. That is the definition of x-coordinate.

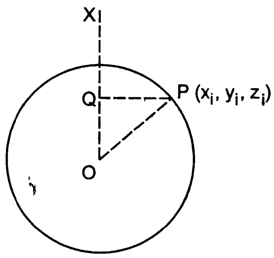


Figure 10.22

Thus, $PQ^2 = OP^2 - OQ^2$

$$= \left(x_i^2 + y_i^2 + z_i^2\right) - x_i^2 = y_i^2 + z_i^2 .$$

The moment of inertia of the particle P about the X-axis

$$= m_i (y_i^2 + z_i^2).$$

The moment of inertia of the hollow sphere about the X-axis is, therefore,

$$I_x = \sum_i m_i \left(y_i^2 + z_i^2\right) \cdot$$

Similarly, the moment of inertia of the hollow sphere about the Y-axis is

$$I_y = \sum_i m_i \left(z_i^2 + x_i^2\right)$$

and about the Z-axis it is

$$I_z = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

Adding these three equations we get

$$I_x + I_y + I_z = \sum_i 2\, m_i \left(x_i^2 + y_i^2 + z_i^2\right)$$

$$= \sum_i 2\, m_i R^2 = 2\, MR^2 .$$

As the mass is uniformly distributed over the entire surface of the sphere, all diameters are equivalent. Hence I_x, I_y and I_z must be equal.

Thus, $$I = \frac{I_x + I_y + I_z}{3} = \frac{2}{3} MR^2 .$$

(H) Moment of inertia of a uniform solid sphere about a diameter

Let M and R be the mass and radius of the given solid sphere. Let O be the centre and OX the given axis. Draw two spheres of radii x and $x + dx$ concentric with the given solid sphere. The thin spherical shell trapped between these spheres may be treated as a hollow sphere of radius x.

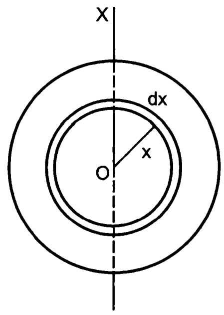


Figure 10.23

The mass per unit volume of the solid sphere

$$= \frac{M}{\frac{4}{3}\pi R^3} = \frac{3\,M}{4\,\pi R^3} .$$

The thin hollow sphere considered above has a surface area $4\,\pi\,x^2$ and thickness dx. Its volume is $4\,\pi\,x^2 dx$ and hence its mass is

$$= \left(\frac{3\,M}{4\,\pi R^3}\right)(4\,\pi\,x^2 dx)$$

$$= \frac{3\,M}{R^3}\,x^2 dx.$$

Its moment of inertia about the diameter OX is, therefore,

$$dI = \frac{2}{3}\left[\frac{3\,M}{R^3}\,x^2 dx\right] x^2 = \frac{2\,M}{R^3}\,x^4 dx.$$

If $x = 0$, the shell is formed at the centre of the solid sphere. As x increases from 0 to R, the shells cover the whole solid sphere.

The moment of inertia of the solid sphere about OX is, therefore,

$$I = \int_0^R \frac{2\,M}{R^3}\,x^4 dx = \frac{2}{5} MR^2 .$$

10.15 TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA

Theorem of Parallel Axes

Suppose we have to obtain the moment of inertia of a body about a given line AB (figure 10.24). Let C

be the centre of mass of the body and let CZ be the line parallel to AB through C. Let I and I_0 be the moments of inertia of the body about AB and CZ respectively. The parallel axes theorem states that

$$I = I_0 + Md^2$$

where d is the perpendicular distance between the parallel lines AB and CZ and m is the mass of the body.

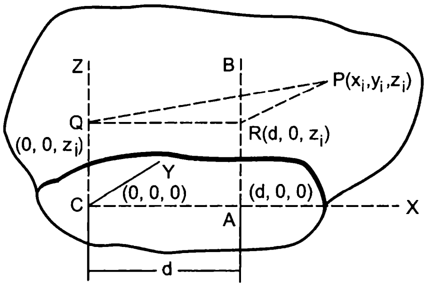


Figure 10.24

Take C to be the origin and CZ the Z-axis. Let CA be the perpendicular from C to AB. Take CA to be the X-axis. As $CA = d$, the coordinates of A are $(d, 0, 0)$.

Let P be an arbitrary particle of the body with the coordinates (x_i, y_i, z_i). Let PQ and PR be the perpendiculars from P to CZ and AB respectively. Note that P may not be in the plane containing CZ and AB. We have $CQ = z_i$. Also $AR = CQ = z_i$. Thus, the point Q has coordinates $(0, 0, z_i)$ and the point R has coordinates$(d, 0, z_i)$.

$$I = \sum_i m_i (PR)^2$$

$$= \sum_i m_i \left[(x_i - d)^2 + (y_i - 0)^2 + (z_i - z_i)^2\right]$$

$$= \sum_i m_i (x_i^2 + y_i^2 + d^2 - 2x_i d)$$

$$= \sum_i m_i (x_i^2 + y_i^2) + \sum_i m_i d^2 - 2d \sum_i m_i x_i \quad \ldots \text{(i)}$$

We have

$$\sum_i m_i x_i = MX_{CM} = 0.$$

The moment of inertia about CZ is,

$$I_0 = \sum_i (PQ)^2$$

$$= \sum_i m_i [(x_i - 0)^2 + (y_i - 0)^2 + (z_i - z_i)^2]$$

$$= \sum_i m_i (x_i^2 + y_i^2)$$

From (i),

$$I = I_0 + \sum m_i d^2 = I_0 + Md^2.$$

Theorem of Perpendicular Axes

This theorem is applicable only to the plane bodies. Let X and Y-axes be chosen in the plane of the body and Z-axis perpendicular to this plane, three axes being mutually perpendicular. Then the theorem states that

$$I_z = I_x + I_y.$$

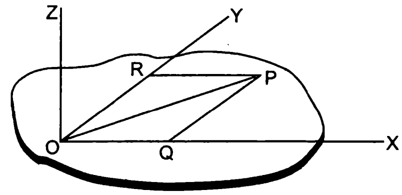


Figure 10.25

Consider an arbitrary particle P of the body (figure 10.25). Let PQ and PR be the perpendiculars from P on the X and the Y-axes respectively. Also PO is the perpendicular from P to the Z-axis. Thus, the moment of inertia of the body about the Z-axis is

$$I_z = \sum_i m_i (PO)^2 = \sum_i m_i (PQ^2 + OQ^2)$$

$$= \sum_i m_i (PQ^2 + PR^2)$$

$$= \sum_i m_i (PQ)^2 + \sum_i m_i (PR)^2$$

$$= I_x + I_y.$$

Example 10.10

Find the moment of inertia of a uniform ring of mass M and radius R about a diameter.

Solution :

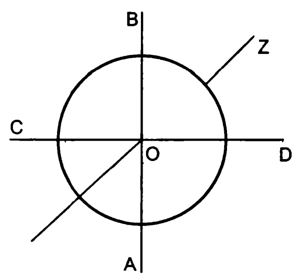


Figure 10.26

Let AB and CD be two mutually perpendicular diameters of the ring. Take them as X and Y-axes and the line perpendicular to the plane of the ring through the centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = MR^2$. As the ring is uniform, all of its diameters are equivalent and so $I_x = I_y$. From perpendicular axes theorem,

$$I_z = I_x + I_y. \text{ Hence } I_x = \frac{I_z}{2} = \frac{MR^2}{2}.$$

Similarly, the moment of inertia of a uniform disc about a diameter is $MR^2/4$.

Example 10.11

Find the moment of inertia of a solid cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.

Solution : The moment of inertia of the cylinder about its axis $= \frac{MR^2}{2}$.

Using parallel axes theorem

$$I = I_0 + MR^2 = \frac{MR^2}{2} + MR^2 = \frac{3}{2}MR^2.$$

Similarly, the moment of inertia of a solid sphere about a tangent is

$$\frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

Radius of Gyration

The radius of gyration k of a body about a given line is defined by the equation

$$I = Mk^2$$

where I is its moment of inertia about the given line and M is its total mass. It is the radius of a ring with the given line as the axis such that if the total mass of the body is distributed on the ring, it will have the same moment of inertia I. For example, the radius of gyration of a uniform disc of radius r about its axis is $r/\sqrt{2}$.

10.16 COMBINED ROTATION AND TRANSLATION

We now consider the motion of a rigid body which is neither pure translational nor pure rotational as seen from a lab. Suppose instead, there is a frame of reference A in which the motion of the rigid body is a pure rotation about a fixed line. If the frame A is also inertial, the motion of the body with respect to A is governed by the equations developed above. The motion of the body in the lab may then be obtained by adding the motion of A with respect to the lab to the motion of the body in A.

If the frame A is noninertial, we do not hope $\Gamma^{ext} = I\alpha$ to hold. In the derivation of this equation we used $F = m\,a$ for each particle and this holds good only if a is measured from an inertial frame. If the frame A has an acceleration $\vec{a}$ in a fixed direction with respect to an inertial frame, we have to apply a pseudo force $-m\vec{a}$ to each particle. These pseudo forces produce a pseudo torque about the axis.

Pleasantly, there exists a very special and very useful case where $\Gamma^{ext} = I\alpha$ does hold even if the angular acceleration α is measured from a noninertial frame A. And that special case is, when the axis of rotation in the frame A passes through the centre of mass.

Take the origin at the centre of mass. The total torque of the pseudo forces is

$$\sum \vec{r_i} \times (-m_i\vec{a}) = -\left(\sum m_i\vec{r_i}\right) \times \vec{a} = -M\left(\frac{\sum m_i\vec{r_i}}{M}\right) \times \vec{a}$$

where $\vec{r_i}$ is the position vector of the ith particle as measured from the centre of mass.

But $\frac{\sum m_i\vec{r_i}}{M}$ is the position vector of the centre of mass and that is zero as the centre of mass is at the origin. Hence the pseudo torque is zero and we get $\Gamma^{ext} = I\alpha$. To make the point more explicit, we write $\Gamma_{cm} = I_{cm}\alpha$, reminding us that the equation is valid in a noninertial frame, only if the axis of rotation passes through the centre of mass and the torques and the moment of inertia are evaluated about the axis through the centre of mass.

So, the working rule for discussing combined rotation and translation is as follows. List the external forces acting on the body. The vector sum divided by the mass of the body gives the acceleration of the centre of mass. Then find the torque of the external forces and the moment of inertia of the body about a line through the centre of mass and perpendicular to the plane of motion of the particles. Note that this line may not be the axis of rotation in the lab frame. Still calculate Γ and I about this line. The angular acceleration α about the centre of mass will be obtained by $\alpha = \Gamma/I$.

$$\left.\begin{array}{ll}\text{Thus} & \vec{a}_{cm} = \vec{F}^{ext}/M \\ \text{and} & \alpha = \Gamma^{ext}_{cm}/I_{cm}\end{array}\right] . \qquad \ldots (10.15)$$

These equations together with the initial conditions completely determine the motion.

10.17 ROLLING

When you go on a bicycle on a straight road what distance on the road is covered during one full pedal ? Suppose a particular spoke of the bicycle is painted black and is vertical at some instant pointing downward. After one full pedal the spoke is again vertical in the similar position. We say that the wheel has made one full rotation. During this period the bicycle has moved through a distance $2\pi R$ in normal cycling on a good, free road. Here R is the radius of the wheel. The wheels are said to be 'rolling' on the road.

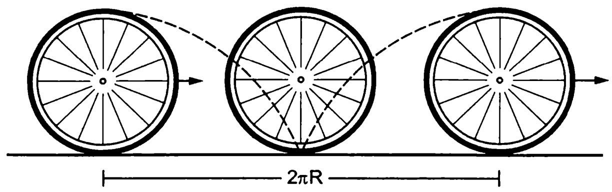


Figure 10.27

Looking from the road frame, the wheel is not making pure rotation about a fixed line. The particles of the wheel do not go on circles. The path of a particle at the rim will be something like that shown in figure (10.27), whereas the centre of the wheel goes in a straight line. But we still say that during one pedal the wheel has made one rotation, i.e., it has rotated through an angle of 2π. By this we mean that the spoke that was vertical (pointing downward from the centre) again became vertical in the similar position. In this period the centre of the wheel has moved through a distance $2\pi R$. In half of this period, the wheel has moved through a distance πR and the spoke makes an angle of π with its original direction. During a short time-interval Δt, the wheel moves through a distance Δx and the spoke rotates by $\Delta\theta$. Thus the wheel rotates and at the same time moves forward. The relation between the displacement of (the centre of) the wheel and the angle rotated by (a spoke of) the wheel is $\Delta x = R\Delta\theta$. Dividing by Δt and taking limits, we get

$$v = R\omega,$$

where v is the linear speed of the centre of mass and ω is the angular velocity of the wheel.

This type of motion of a wheel (or any other object with circular boundary) in which the centre of the wheel moves in a straight line and the wheel rotates in its plane about its centre with $v = R\omega$, is called pure *rolling*.

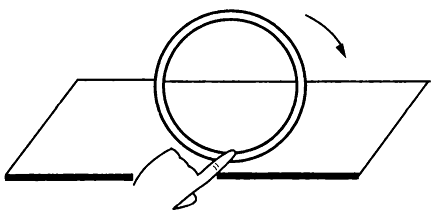


Figure 10.28

Place a ring on a horizontal surface as shown in figure (10.28) and put your finger on the lowest part. Use other hand to rotate the ring a little while the finger is kept on the lowest point. This is approximately a small part of rolling motion. Note the displacements of different particles of the ring. The centre has moved forward a little, say Δx. The topmost point has moved approximately double of this distance. The part in contact with the horizontal surface below the finger has almost been in the same position.

In pure rolling, the velocity of the contact point is zero. The velocity of the centre of mass is $v_{cm} = R\omega$ and that of the topmost point is $v_{top} = 2R\omega = 2v_{cm}$.

Next, consider another type of combination of rotation and translation, in which the wheel moves through a distance greater than $2\pi R$ in one full rotation. Hold the ring of figure (10.28) between three fingers, apply a forward force to move it fast on the table and rotate it slowly through the fingers. Its angular velocity $\omega = d\theta/dt$ is small and $v_{cm} > R\omega$. This is a case of rolling with forward slipping. This type of motion occurs when you apply sudden brakes to the bicycle on a road which is fairly smooth after rain. The cycle stops after a long distance and the wheel rotates only little during this period. If you look at the particles in contact, these will be found rubbing the road in the forward direction. The particles in contact have a velocity in the forward direction. In this case $v_{cm} > R\omega$. An extreme example of this type occurs when the wheel does not rotate at all and translates with linear velocity v. Then $v_{cm} = v$ and $\omega = 0$.

Yet another type of rolling with slipping occurs when the wheel moves a distance shorter than $2\pi R$ while making one rotation. In this case, the velocity $v_{cm} < R\omega$. Hold the ring of figure (10.28) between three fingers, rotate it fast and translate it slowly. It will move a small distance on the table and rotate fast. If you drive a bicycle on a road on which a lot of mud is present, sometimes the wheel rotates fast but moves a little. If you look at the particles in contact, they rub the road in the backward direction. The centre moves less than $2\pi R$ during one full rotation and $v_{cm} < R\omega$.

These situations may be visualised in a different manner which gives another interpretation of rolling. Consider a wheel of radius r with its axis fixed in a second-hand car. The wheel may rotate freely about this axis. Suppose the floor of the second-hand car has a hole in it and the wheel just touches the road through the hole. Suppose the person sitting on the back seat rotates the wheel at a uniform angular velocity ω and the driver drives the car at a uniform velocity v on the road which is parallel to the plane of the wheel as shown in figure (10.29). The two motions are independent. The backseater is rotating the wheel at an angular velocity according to his will and the driver is driving the car at a velocity according to his will.

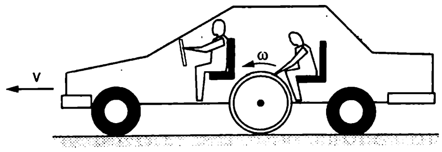


Figure 10.29

Look at the wheel from the road. If the persons inside the car agree to choose v and ω in such a way that $v = \omega r$, the wheel is in pure rolling on the road. Looking from the road, the centre of the wheel is moving forward at a speed v and the wheel is rotating at an angular velocity ω with $v = \omega r$. The velocity of the lowest particle with respect to the road = its velocity with respect to the car + velocity of the car with respect to the road. So,

$$v_{contact, road} = v_{contact, car} + v_{car, road} = -\omega r + v = 0.$$

If the driver drives the car at a higher speed, $v > \omega r$, the wheel rubs the road and we have rolling with forward slipping. In this case

$$v_{contact, road} = v_{contact, car} + v_{car, road} = -\omega r + v > 0$$

Similarly, if $v < \omega r$, we have rolling with backward slipping,

$$v_{contact, road} = -\omega r + v < 0,$$

the particles at contact rub the road backward.

10.18 KINETIC ENERGY OF A BODY IN COMBINED ROTATION AND TRANSLATION

Consider a body in combined translational and rotational motion in the lab frame. Suppose in the frame of the centre of mass, the body is making a pure rotation with an angular velocity ω. The centre of mass itself is moving in the lab frame at a velocity $\vec{v}_0$. The velocity of a particle of mass m_i is $\vec{v}_{i, cm}$ with respect to the centre-of-mass frame and $\vec{v}_i$ with respect to the lab frame. We have,

$$\vec{v}_i = \vec{v}_{i, cm} + \vec{v}_0$$

The kinetic energy of the particle in the lab frame is

$$\frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\vec{v}_{i, cm} + \vec{v}_0) \cdot (\vec{v}_{i, cm} + \vec{v}_0)$$

$$= \frac{1}{2} m_i v_{i, cm}^2 + \frac{1}{2} m_i v_0^2 + \frac{1}{2} m_i (2\vec{v}_{i, cm} \cdot \vec{v}_0).$$

Summing over all the particles, the total kinetic energy of the body in the lab frame is

$$K = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i v_{i, cm}^2 + \frac{1}{2} \sum_i m_i v_0^2 + \left(\sum_i m_i \vec{v}_{i, cm} \right) \cdot \vec{v}_0.$$

Now $\sum_i \frac{1}{2} m_i v_{i, cm}^2$ is the kinetic energy of the body in the centre of mass frame. In this frame, the body is making pure rotation with an angular velocity ω. Thus, this term is equal to $\frac{1}{2} I_{cm} \omega^2$. Also $\frac{\sum m_i \vec{v}_{i, cm}}{M}$ is the velocity of the centre of mass in the centre of mass frame which is obviously zero. Thus,

$$K = \frac{1}{2} I_{cm} \omega^2 + \frac{1}{2} M v_0^2.$$

In the case of pure rolling, $v_0 = R\omega$ so that

$$K = \frac{1}{2}(I_{cm} + MR^2)\omega^2.$$

Using the parallel axes theorem, $I_{cm} + MR^2 = I$, which is the moment of inertia of the wheel about the line through the point of contact and parallel to the axis. Thus, $K = \frac{1}{2} I\omega^2$.

This gives another interpretation of rolling. At any instant a rolling body may be considered to be in pure rotation about an axis through the point of contact. This axis translates forward with a speed v_0.

Example 10.12

A uniform sphere of mass 200 g *rolls without slipping on a plane surface so that its centre moves at a speed of* 2·00 cm/s. *Find its kinetic energy.*

Solution : As the sphere rolls without slipping on the plane surface, its angular speed about the center is $\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$$K = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$$

$$= \frac{1}{2} \cdot \frac{2}{5} Mr^2\omega^2 + \frac{1}{2} Mv_{cm}^2$$

$$= \frac{1}{5} Mv_{cm}^2 + \frac{1}{2} Mv_{cm}^2 = \frac{7}{10} Mv_{cm}^2$$

$$= \frac{7}{10} (0{\cdot}200 \text{ kg}) (0{\cdot}02 \text{ m/s})^2 = 5{\cdot}6 \times 10^{-5} \text{ J}.$$

10.19 ANGULAR MOMENTUM OF A BODY IN COMBINED ROTATION AND TRANSLATION

Consider the situation described in the previous section. Let O be a fixed point in the lab which we take as the origin. Angular momentum of the body about O is

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$= \sum_i m_i (\vec{r}_{i, cm} + \vec{r}_0) \times (\vec{v}_{i, cm} + \vec{v}_0).$$

Here, $\vec{r}_0$ is the position vector of the centre of mass. Thus,

$$\vec{L} = \sum_i m_i (\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + \left(\sum_i m_i \vec{r}_{i,cm}\right) \times \vec{v}_0 + \vec{r}_0 \times \left(\sum_i m_i \vec{v}_{i,cm}\right) + \left(\sum_i m_i\right) \vec{r}_0 \times \vec{v}_0 .$$

Now, $\quad \sum_i m_i \vec{r}_{i,cm} = M \vec{R}_{cm,cm} = 0$

and $\quad \sum_i m_i \vec{v}_{i,cm} = M \vec{V}_{cm,cm} = 0.$

Thus, $\quad \vec{L} = \sum_i m_i (\vec{r}_{i,cm} \times \vec{v}_{i,cm}) + M \vec{r}_0 \times \vec{v}_0$

$$= \vec{L}_{cm} + M \vec{r}_0 \times \vec{v}_0 .$$

The first term $\vec{L}_{cm}$ represents the angular momentum of the body as seen from the centre-of-mass frame. The second term $M \vec{r}_0 \times \vec{v}_0$ equals the angular momentum of the body if it is assumed to be concentrated at the centre of mass translating with the velocity $\vec{v}_0$.

10.20 WHY DOES A ROLLING SPHERE SLOW DOWN ?

When a sphere is rolled on a horizontal table it slows down and eventually stops. Figure (10.30) shows the situation. The forces acting on the sphere are (a) weight mg, (b) friction at the contact and (c) the normal force. As the centre of the sphere decelerates, the friction should be opposite to its velocity, that is towards left in figure (10.30). But this friction will have a clockwise torque that should increase the angular velocity of the sphere. There must be an anticlockwise torque that causes the decrease in the angular velocity.

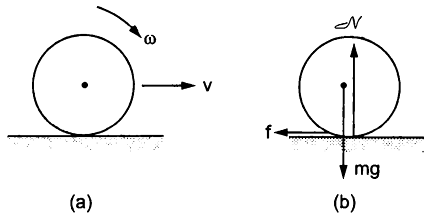


Figure 10.30

In fact, when the sphere rolls on the table, both the sphere and the surface deform near the contact. The contact is not at a single point as we normally assume, rather there is an area of contact. The front part pushes the table a bit more strongly than the back part. As a result, the normal force does not pass through the centre, it is shifted towards the right. This force, then, has an anticlockwise torque. The net torque causes an angular deceleration.

Worked Out Examples

1. *A wheel rotates with a constant acceleration of* 2·0 rad/s^2. *If the wheel starts from rest, how many revolutions will it make in the first* 10 seconds ?

Solution : The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = \frac{1}{2} (2{\cdot}0 \text{ rad/s}^2)(10 \text{ s})^2 = 100 \text{ rad}.$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in 10 s is

$$n = \frac{100}{2\pi} = 16.$$

2. *The wheel of a motor, accelerated uniformly from rest, rotates through* 2·5 radian *during the first second. Find the angle rotated during the next second.*

Solution : As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2} \alpha t^2 = \frac{1}{2} \alpha t^2.$$

Thus, $\quad 2{\cdot}5 \text{ rad} = \frac{1}{2} \alpha (1 \text{ s})^2$

$$\alpha = 5 \text{ rad/s}^2$$

or, $\quad \alpha = 5 \text{ rad/s}^2.$

The angle rotated during the first two seconds is

$$= \frac{1}{2} \times (5 \text{ rad/s}^2)(2 \text{ s})^2 = 10 \text{ rad}.$$

Thus, the angle rotated during the 2nd second is

$$10 \text{ rad} - 2{\cdot}5 \text{ rad} = 7{\cdot}5 \text{ rad}.$$

3. *A wheel having moment of inertia* 2 kg-m^2 *about its axis, rotates at* 50 rpm *about this axis. Find the torque that can stop the wheel in one minute.*

Solution : The initial angular velocity

$$= 50 \text{ rpm} = \frac{5\pi}{3} \text{ rad/s}.$$

Using $\omega = \omega_0 + \alpha t$,

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - \frac{5\pi}{3}}{60} \text{ rad/s}^2 = -\frac{\pi}{36} \text{ rad/s}^2.$$

The torque that can produce this deceleration is

$$\Gamma = I \mid \alpha \mid = (2 \text{ kg-m}^2)\left(\frac{\pi}{36} \text{ rad/s}^2\right) = \frac{\pi}{18} \text{ N-m}.$$

4. *A string is wrapped around the rim of a wheel of moment of inertia* $0{\cdot}20\ \text{kg-m}^2$ *and radius* 20 cm. *The wheel is free to rotate about its axis. Initially, the wheel is at rest. The string is now pulled by a force of* 20 N. *Find the angular velocity of the wheel after* 5·0 seconds.

Solution :

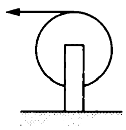


Figure 10-W1

The torque applied to the wheel is

$$\Gamma = F.\,r = (20\ \text{N})\,(0{\cdot}20\ \text{m}) = 4{\cdot}0\ \text{N-m}.$$

The angular acceleration produced is

$$\alpha = \frac{\Gamma}{I} = \frac{4{\cdot}0\ \text{N-m}}{0{\cdot}20\ \text{kg-m}^2} = 20\ \text{rad/s}^2.$$

The angular velocity after 5·0 seconds is

$$\omega = \omega_0 + \alpha\, t = (20\ \text{rad/s}^2)\,(5{\cdot}0\ \text{s}) = 100\ \text{rad/s}.$$

5. *A wheel of radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure (10-W2). A string is wrapped round the wheel and its free end supports a block of mass M which can slide on the plane. Initially, the wheel is rotating at a speed ω in a direction such that the block slides up the plane. How far will the block move before stopping ?*

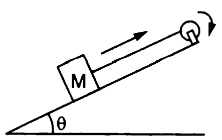


Figure 10-W2

Solution : Suppose the deceleration of the block is a. The linear deceleration of the rim of the wheel is also a. The angular deceleration of the wheel is $\alpha = a/r$. If the tension in the string is T, the equations of motion are as follows:

$$Mg\,\sin\theta - T = Ma$$

and $$Tr = I\alpha = I\,a/r.$$

Eliminating T from these equations,

$$Mg\,\sin\theta - I\frac{a}{r^2} = Ma$$

giving, $$a = \frac{Mg\,r^2\sin\theta}{I + Mr^2}.$$

The initial velocity of the block up the incline is $v = \omega\, r$. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega^2 r^2 (I + Mr^2)}{2\,Mg\,r^2\sin\theta} = \frac{(I + Mr^2)\omega^2}{2\,Mg\,\sin\theta}.$$

6. *The pulley shown in figure (10-W3) has a moment of inertia I about its axis and its radius is R. Find the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.*

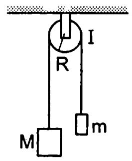


Figure 10-W3

Solution : Suppose the tension in the left string is T_1 and that in the right string is T_2. Suppose the block of mass M goes down with an acceleration a and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, $\alpha = a/R$. The equations of motion for the mass M, the mass m and the pulley are as follows :

$$Mg - T_1 = Ma \qquad \text{... (i)}$$

$$T_2 - mg = ma \qquad \text{... (ii)}$$

$$T_1 R - T_2 R = I\alpha = Ia/R. \qquad \text{... (iii)}$$

Putting T_1 and T_2 from (i) and (ii) into (iii),

$$[M(g - a) - m(g + a)]\,R = I\,\frac{a}{R}$$

which gives $a = \dfrac{(M - m)gR^2}{I + (M + m)\,R^2}$.

7. *Two small kids weighing* 10 kg *and* 15 kg *respectively are trying to balance a seesaw of total length* 5·0 m, *with the fulcrum at the centre. If one of the kids is sitting at an end, where should the other sit ?*

Solution :

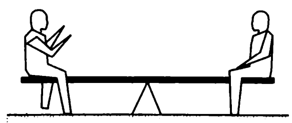


Figure 10-W4

It is clear that the 10 kg kid should sit at the end and the 15 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the kids are in equilibrium, the normal force between a kid and the seesaw equals the weight of that kid. Considering the rotational equilibrium of the seesaw, the torques of the forces acting on it should add to zero. The forces are

(a) (15 kg)g downward by the 15 kg kid,

(b) (10 kg)g downward by the 10 kg kid,
(c) weight of the seesaw and
(d) the normal force by the fulcrum.

Taking torques about the fulcrum,

$$(15\text{ kg})g\,x = (10\text{ kg})g\,(2{\cdot}5\text{ m})$$

or, $$x = 1{\cdot}7\text{ m}.$$

8. *A uniform ladder of mass* 10 kg *leans against a smooth vertical wall making an angle of* 53° *with it. The other end rests on a rough horizontal floor. Find the normal force and the frictional force that the floor exerts on the ladder.*

Solution :

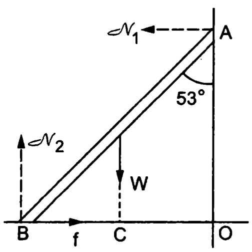


Figure 10-W5

The forces acting on the ladder are shown in figure (10-W5). They are
(a) its weight W,
(b) normal force $\mathcal{N}_1$ by the vertical wall,
(c) normal force $\mathcal{N}_2$ by the floor and
(d) frictional force f by the floor.

Taking horizontal and vertical components

$$\mathcal{N}_1 = f \qquad \text{... (i)}$$

and $$\mathcal{N}_2 = W. \qquad \text{... (ii)}$$

Taking torque about B,

$$\mathcal{N}_1(AO) = W(CB)$$

or, $$\mathcal{N}_1(AB)\cos 53° = W\frac{AB}{2}\sin 53°$$

or, $$\mathcal{N}_1\frac{3}{5} = \frac{W}{2}\frac{4}{5}$$

or, $$\mathcal{N}_1 = \frac{2}{3}W. \qquad \text{... (iii)}$$

The normal force by the floor is

$$\mathcal{N}_2 = W = (10\text{ kg})\,(9{\cdot}8\text{ m/s}^2) = 98\text{ N}.$$

The frictional force is

$$f = \mathcal{N}_1 = \frac{2}{3}W = 65\text{ N}.$$

9. *The ladder shown in figure (10-W6) has negligible mass and rests on a frictionless floor. The crossbar connects the two legs of the ladder at the middle. The angle between the two legs is 60°. The fat person sitting on the ladder has a mass of* 80 kg. *Find the contact force exerted by the floor on each leg and the tension in the crossbar.*

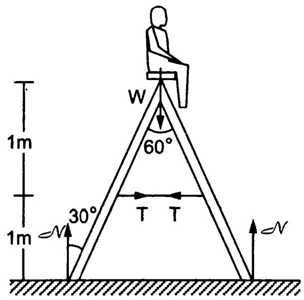


Figure 10-W6

Solution : The forces acting on different parts are shown in figure (10-W6). Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (80 kg)g and the contact force $\mathcal{N} + \mathcal{N} = 2\mathcal{N}$ due to the floor. Thus,

$$2\mathcal{N} = (80\text{ kg})\,g$$

or, $$\mathcal{N} = (40\text{ kg})\,(9{\cdot}8\text{ m/s}^2) = 392\text{ N}.$$

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$$\mathcal{N}\,(2\text{ m})\tan 30° = T(1\text{ m})$$

or, $$T = \mathcal{N}\frac{2}{\sqrt{3}} = (392\text{ N}) \times \frac{2}{\sqrt{3}} \approx 450\text{ N}.$$

10. *Two small balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length d. The structure rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual balls and of the system about the axis of rotation.*

Solution :

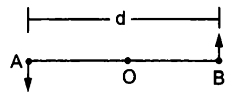


Figure 10-W7

Consider the situation shown in figure (10-W7). The velocity of the ball A with respect to the centre O is $v = \frac{\omega d}{2}$. The angular momentum of the ball with respect to the axis is $L_1 = mvr = m\left(\frac{\omega d}{2}\right)\left(\frac{d}{2}\right) = \frac{1}{4}m\omega d^2$.

The same is the angular momentum L_2 of the second ball. The angular momentum of the system is equal to sum of these two angular momenta i.e., $L = \frac{1}{2}m\omega d^2$.

11. *Two particles of mass m each are attached to a light rod of length d, one at its centre and the other at a free end.*

The rod is fixed at the other end and is rotated in a plane at an angular speed ω. Calculate the angular momentum of the particle at the end with respect to the particle at the centre.

Solution :

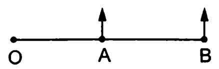


Figure 10-W8

The situation is shown in figure (10-W8). The velocity of the particle A with respect to the fixed end O is $v_A = \omega\left(\frac{d}{2}\right)$ and that of B with respect to O is $v_B = \omega d$. Hence the velocity of B with respect to A is $v_B - v_A = \omega\left(\frac{d}{2}\right)$. The angular momentum of B with respect to A is, therefore,

$$L = mvr = m\omega\left(\frac{d}{2}\right)\frac{d}{2} = \frac{1}{4}m\omega d^2$$

along the direction perpendicular to the plane of rotation.

12. *A particle is projected at time $t = 0$ from a point P with a speed v_0 at an angle of 45° to the horizontal. Find the magnitude and the direction of the angular momentum of the particle about the point P at time $t = v_0/g$.*

Solution : Let us take the origin at P, X-axis along the horizontal and Y-axis along the vertically upward direction as shown in figure (10-W9). For horizontal motion during the time 0 to t,

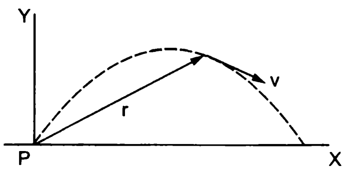


Figure 10-W9

$$v_x = v_0 \cos 45° = v_0/\sqrt{2}$$

and $$x = v_x t = \frac{v_0}{\sqrt{2}} \cdot \frac{v_0}{g} = \frac{v_0^2}{\sqrt{2}g}.$$

For vertical motion,

$$v_y = v_0 \sin 45° - gt = \frac{v_0}{\sqrt{2}} - v_0 = \frac{(1-\sqrt{2})}{\sqrt{2}}v_0$$

and $$y = (v_0 \sin 45°)t - \frac{1}{2}gt^2$$

$$= \frac{v_0^2}{\sqrt{2}g} - \frac{v_0^2}{2g} = \frac{v_0^2}{2g}(\sqrt{2}-1)$$

The angular momentum of the particle at time t about the origin is

$$\vec{L} = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v}$$

$$= m(\vec{i}x + \vec{j}y) \times (\vec{i}v_x + \vec{j}v_y)$$

$$= m(\vec{k}xv_y - \vec{k}yv_x)$$

$$= m\vec{k}\left[\left(\frac{v_0^2}{\sqrt{2}g}\right)\frac{v_0}{\sqrt{2}}(1-\sqrt{2}) - \frac{v_0^2}{2g}(\sqrt{2}-1)\frac{v_0}{\sqrt{2}}\right]$$

$$= -\vec{k}\frac{mv_0^3}{2\sqrt{2}g}.$$

Thus, the angular momentum of the particle is $\frac{mv_0^3}{2\sqrt{2}g}$ in the negative Z-direction, i.e., perpendicular to the plane of motion, going into the plane.

13. *A uniform circular disc of mass* 200 g *and radius* 4·0 cm *is rotated about one of its diameter at an angular speed of* 10 rad/s. *Find the kinetic energy of the disc and its angular momentum about the axis of rotation.*

Solution : The moment of inertia of the circular disc about its diameter is

$$I = \frac{1}{4}Mr^2 = \frac{1}{4}(0{\cdot}200\text{ kg})(0{\cdot}04\text{ m})^2$$

$$= 8{\cdot}0 \times 10^{-5}\text{ kg-m}^2.$$

The kinetic energy is

$$K = \frac{1}{2}I\omega^2 = \frac{1}{2}(8{\cdot}0 \times 10^{-5}\text{ kg-m}^2)(100\text{ rad}^2/\text{s}^2)$$

$$= 4{\cdot}0 \times 10^{-3}\text{ J}$$

and the angular momentum about the axis of rotation is

$$L = I\omega = (8{\cdot}0 \times 10^{-5}\text{ kg-m}^2)(10\text{ rad/s})$$

$$= 8{\cdot}0 \times 10^{-4}\text{ kg-m}^2/\text{s} = 8{\cdot}0 \times 10^{-4}\text{ J-s}.$$

14. *A wheel rotating at an angular speed of* 20 rad/s *is brought to rest by a constant torque in* 4·0 seconds. *If the moment of inertia of the wheel about the axis of rotation is* 0·20 kg-m², *find the work done by the torque in the first two seconds.*

Solution : The angular deceleration of the wheel during the 4·0 seconds may be obtained by the equation

$$\omega = \omega_0 - \alpha t$$

or, $$\alpha = \frac{\omega_0 - \omega}{t} = \frac{20\text{ rad/s}}{4{\cdot}0\text{ s}} = 5{\cdot}0\text{ rad/s}^2.$$

The torque applied to produce this deceleration is

$$\Gamma = I\alpha = (0{\cdot}20\text{ kg-m}^2)(5{\cdot}0\text{ rad/s}^2) = 1{\cdot}0\text{ N-m}.$$

The angle rotated in the first two seconds is

$$\theta = \omega_0 t - \frac{1}{2}\alpha t^2$$

$$= (20\text{ rad/s})(2\text{ s}) - \frac{1}{2}(5{\cdot}0\text{ rad/s}^2)(4{\cdot}0\text{ s}^2)$$

$$= 40\text{ rad} - 10\text{ rad} = 30\text{ rad}.$$

The work done by the torque in the first 2 seconds is, therefore,

$$W = \Gamma\theta = (1{\cdot}0\text{ N-m})(30\text{ rad}) = 30\text{ J}.$$

15. *Two masses M and m are connected by a light string going over a pulley of radius r. The pulley is free to rotate about its axis which is kept horizontal. The moment of inertia of the pulley about the axis is I. The system is released from rest. Find the angular momentum of the system when the mass M has descended through a height h. The string does not slip over the pulley.*

Solution :

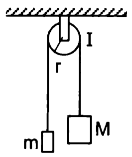


Figure 10-W10

The situation is shown in figure (10-W10). Let the speed of the masses be v at time t. This will also be the speed of a point on the rim of the wheel and hence the angular velocity of the wheel at time t will be v/r. If the height descended by the mass M is h, the loss in the potential energy of the "masses plus the pulley" system is $Mgh - mgh$. The gain in kinetic energy is $\frac{1}{2}Mv^2 + \frac{1}{2}mv^2 + \frac{1}{2}I\left(\frac{v}{r}\right)^2$. As no energy is lost,

$$\frac{1}{2}\left(M + m + \frac{I}{r^2}\right)v^2 = (M - m)gh$$

or,
$$v^2 = \frac{2(M - m)gh}{M + m + \frac{I}{r^2}}.$$

The angular momentum of the mass M is Mvr and that of the mass m is mvr in the same direction. The angular momentum of the pulley is $I\omega = Iv/r$. The total angular momentum is

$$\left[(M + m)r + \frac{I}{r}\right]v = \left[\left(M + m + \frac{I}{r^2}\right)r\right]\sqrt{\frac{2(M - m)gh}{M + m + \frac{I}{r^2}}}$$

$$= \sqrt{2(M - m)\left(M + m + \frac{I}{r^2}\right)r^2 gh}.$$

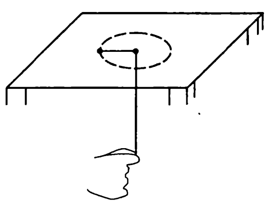


Figure 10-W11

16. *Figure (10-W11) shows a mass m placed on a frictionless horizontal table and attached to a string passing through a small hole in the surface. Initially, the mass moves in a circle of radius r_0 with a speed v_0 and the free end of the string is held by a person. The person pulls on the string slowly to decrease the radius of the circle to r. (a) Find the tension in the string when the mass moves in the circle of radius r. (b) Calculate the change in the kinetic energy of the mass.*

Solution : The torque acting on the mass m about the vertical axis through the hole is zero. The angular momentum about this axis, therefore, remains constant. If the speed of the mass is v when it moves in the cirlce of radius r, we have

$$mv_0 r_0 = mvr$$

or,
$$v = \frac{r_0}{r}v_0. \quad \text{... (i)}$$

(a) The tension $T = \frac{mv^2}{r} = \frac{mr_0^2 v_0^2}{r^3}$.

(b) The change in kinetic energy $= \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$.

By (i) it is

$$= \frac{1}{2}mv_0^2\left[\frac{r_0^2}{r^2} - 1\right].$$

17. *A uniform rod of mass m and length l is kept vertical with the lower end clamped. It is slightly pushed to let it fall down under gravity. Find its angular speed when the rod is passing through its lowest position. Neglect any friction at the clamp. What will be the linear speed of the free end at this instant?*

Solution :

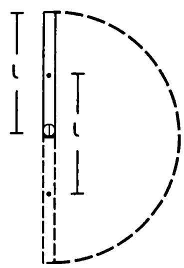


Figure 10-W12

As the rod reaches its lowest position, the centre of mass is lowered by a distance l. Its gravitational potential energy is decreased by mgl. As no energy is lost against friction, this should be equal to the increase in the kinetic energy. As the rotation occurs about the horizontal axis through the clamped end, the moment of inertia is $I = ml^2/3$. Thus,

$$\frac{1}{2}I\omega^2 = mgl$$

$$\frac{1}{2}\left(\frac{ml^2}{3}\right)\omega^2 = mgl$$

or,
$$\omega = \sqrt{\frac{6g}{l}}.$$

The linear speed of the free end is

$$v = l\omega = \sqrt{6gl}.$$

18. *Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.*

Solution :

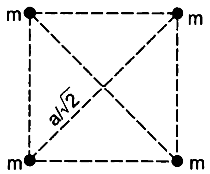


Figure 10-W13

The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore, $4 \times \frac{1}{2}ma^2 = 2ma^2$.

19. *Two identical spheres each of mass* 1·20 kg *and radius* 10·0 cm *are fixed at the ends of a light rod so that the separation between the centres is* 50·0 cm. *Find the moment of inertia of the system about an axis perpendicular to the rod passing through its middle point.*

Solution :

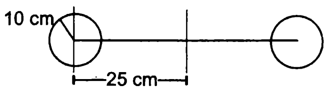


Figure 10-W14

Consider the diameter of one of the spheres parallel to the given axis. The moment of inertia of this sphere about the diameter is

$$I = \frac{2}{5}mR^2 = \frac{2}{5}(1{\cdot}20\text{ kg})(0{\cdot}1\text{ m})^2$$

$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2.$$

Its moment of inertia about the given axis is obtained by using the parallel axes theorem. Thus,

$$I = I_{cm} + md^2$$

$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2 + (1{\cdot}20\text{ kg})(0{\cdot}25\text{ m})^2$$

$$= 4{\cdot}8 \times 10^{-3}\text{ kg-m}^2 + 0{\cdot}075\text{ kg-m}^2$$

$$= 79{\cdot}8 \times 10^{-3}\text{ kg-m}^2.$$

The moment of inertia of the second sphere is also the same so that the moment of inertia of the system is

$$2 \times 79{\cdot}8 \times 10^{-3}\text{ kg-m}^2 \approx 0{\cdot}160\text{ kg-m}^2.$$

20. *Two uniform identical rods each of mass M and length l are joined to form a cross as shown in figure (10-W15). Find the moment of inertia of the cross about a bisector as shown dotted in the figure.*

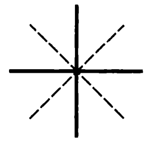


Figure 10-W15

Solution : Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\frac{Ml^2}{12}$ and hence the moment of inertia of the cross is $\frac{Ml^2}{6}\cdot$ The moment of inertia of the cross about the two bisectors are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{Ml^2}{12}\cdot$

21. *A uniform rod of mass M and length a lies on a smooth horizontal plane. A particle of mass m moving at a speed v perpendicular to the length of the rod strikes it at a distance a/4 from the centre and stops after the collision. Find (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.*

Solution :

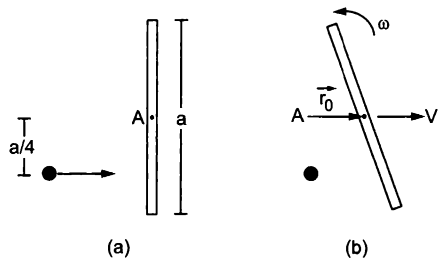


Figure 10-W16

The situation is shown in figure (10-W16a). Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remain constant. Also there is no resultant external torque on the system and so the angular momentum of the system about any line will remain constant.

Suppose the velocity of the centre of the rod is V and the angular velocity about the centre is ω.

(a) The linear momentum before the collision is mv and that after the collision is MV. Thus,

$$mv = MV, \text{ or } V = \frac{m}{M}v.$$

(b) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB. Initially the rod is at rest. The angular momentum of the particle about AB is

$$L = mv(a/4).$$

After the collision, the particle comes to rest. The angular momentum of the rod about A is

$$\vec{L} = \vec{L}_{cm} + M\,\vec{r}_0 \times \vec{V}$$

As $\quad \vec{r}_0 \parallel \vec{V}, \qquad \vec{r}_0 \times \vec{V} = 0.$

Thus, $\qquad \vec{L} = \vec{L}_{cm}.$

Hence the angular momentum of the rod about AB is

$$L = I\omega = \frac{Ma^2}{12}\,\omega.$$

Thus, $\quad \dfrac{mva}{4} = \dfrac{Ma^2}{12}\,\omega \qquad$ or, $\quad \omega = \dfrac{3\,mv}{Ma}.$

22. *A wheel of perimeter* 220 cm *rolls on a level road at a speed of* 9 km/h. *How many revolutions does the wheel make per second ?*

Solution : As the wheel rolls on the road, its angular speed ω about the centre and the linear speed v of the centre are related as $v = \omega r$.

$$\therefore\ \omega = \frac{v}{r} = \frac{9 \text{ km/h}}{220 \text{ cm}/2\pi} = \frac{2\pi \times 9 \times 10^5}{220 \times 3600} \text{ rad/s.}$$

$$= \frac{900}{22 \times 36} \text{ rev/s} = \frac{25}{22} \text{ rev/s.}$$

23. *A cylinder is released from rest from the top of an incline of inclination* θ *and length* l. *If the cylinder rolls without slipping, what will be its speed when it reaches the bottom ?*

Solution : Let the mass of the cylinder be m and its radius r. Suppose the linear speed of the cylinder when it reaches the bottom is v. As the cylinder rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

$$K = \frac{1}{2} I\omega^2 + \frac{1}{2} mv^2$$

$$= \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{4}mv^2 + \frac{1}{2}mv^2 = \frac{3}{4}mv^2.$$

This should be equal to the loss of potential energy $mgl \sin\theta$. Thus, $\quad \dfrac{3}{4}mv^2 = mgl\sin\theta$

or, $$v = \sqrt{\frac{4}{3} gl \sin\theta}\,.$$

24. *A sphere of mass m rolls without slipping on an inclined plane of inclination* θ. *Find the linear acceleration of the sphere and the force of friction acting on it. What should be the minimum coefficient of static friction to support pure rolling ?*

Solution : Suppose the radius of the sphere is r. The forces acting on the sphere are shown in figure (10-W17). They are (a) weight mg, (b) normal force $\mathcal{N}$ and (c) friction f.

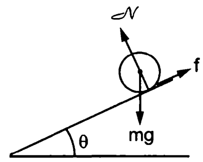


Figure 10-W17

Let the linear acceleration of the sphere down the plane be a. The equation for the linear motion of the centre of mass is

$$mg \sin\theta - f = ma \qquad \text{... (i)}$$

As the sphere rolls without slipping, its angular acceleration about the centre is a/r. The equation of rotational motion about the centre of mass is,

$$fr = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or, $$f = \frac{2}{5}ma. \qquad \text{... (ii)}$$

From (i) and (ii),

$$a = \frac{5}{7} g \sin\theta$$

and $$f = \frac{2}{7} mg \sin\theta.$$

The normal force is equal to $mg\cos\theta$ as there is no acceleration perpendicular to the incline. The maximum friction that can act is, therefore, $\mu\, mg \cos\theta$, where μ is the coefficient of static friction. Thus, for pure rolling

$$\mu\, mg \cos\theta > \frac{2}{7} mg \sin\theta$$

or, $$\mu > \frac{2}{7} \tan\theta.$$

25. *Figure (10-W18) shows two cylinders of radii* r_1 *and* r_2 *having moments of inertia* I_1 *and* I_2 *about their respective axes. Initially, the cylinders rotate about their axes with angular speeds* ω_1 *and* ω_2 *as shown in the figure. The cylinders are moved closer to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Find the angular speeds of the cylinders after the slipping ceases.*

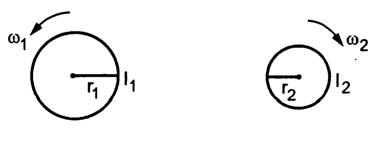


Figure 10-W18

Solution : When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω'_1 *and* ω'_2 be the respective angular speeds, we have

$$\omega'_1 r_1 = \omega'_2 r_2 \qquad \dots \text{(i)}$$

The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fr_1 and that on the second is fr_2. Assuming $\omega_1 r_1 > \omega_2 r_2$, the corresponding angular impluses are $-fr_1 t$ and $fr_2 t$. We, therefore, have

$$-fr_1 t = I_1(\omega'_1 - \omega_1)$$

and $$fr_2 t = I_2(\omega'_2 - \omega_2)$$

or, $$-\frac{I_1}{r_1}(\omega'_1 - \omega_1) = \frac{I_2}{r_2}(\omega'_2 - \omega_2) \qquad \dots \text{(ii)}$$

Solving (i) and (ii),

$$\omega'_1 = \frac{I_1 \omega_1 r_2 + I_2 \omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_2 \text{ and } \omega'_2 = \frac{I_1 \omega_1 r_2 + I_2 \omega_2 r_1}{I_2 r_1^2 + I_1 r_2^2} r_1 .$$

26. *A cylinder of mass m is suspended through two strings wrapped around it as shown in figure (10-W19). Find (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance h.*

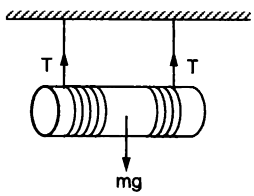


Figure 10-W19

Solution : The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is $\alpha = a/R$, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$mg - 2T = ma \qquad \dots \text{(i)}$$

and for the motion about the centre of mass, it is

$$2\,Tr = \left(\frac{1}{2}mr^2\alpha\right) = \frac{1}{2}mra$$

or, $$2\,T = \frac{1}{2}ma. \qquad \dots \text{(ii)}$$

From (i) and (ii),

$$a = \frac{2}{3}g \text{ and } T = \frac{mg}{6}.$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance h is given by

$$v^2 = 2\left(\frac{2}{3}g\right)h$$

or, $$v = \sqrt{\frac{4gh}{3}}.$$

27. *A force F acts tangentially at the highest point of a sphere of mass m kept on a rough horizontal plane. If the sphere rolls without slipping, find the acceleration of the centre of the sphere.*

Solution :

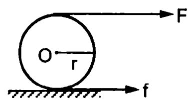


Figure 10-W20

The situation is shown in figure (10-W20). As the force F rotates the sphere, the point of contact has a tendency to slip towards left so that the static friction on the sphere will act towards right. Let r be the radius of the sphere and a be the linear acceleration of the centre of the sphere. The angular acceleration about the centre of the sphere is $\alpha = a/r$, as there is no slipping.

For the linear motion of the centre

$$F + f = ma \qquad \dots \text{(i)}$$

and for the rotational motion about the centre,

$$Fr - fr = I\alpha = \left(\frac{2}{5}mr^2\right)\left(\frac{a}{r}\right)$$

or, $$F - f = \frac{2}{5}ma. \qquad \dots \text{(ii)}$$

From (i) and (ii),

$$2F = \frac{7}{5}ma \text{ or, } a = \frac{10\,F}{7\,m}.$$

28. *A sphere of mass M and radius r shown in figure (10-W21) slips on a rough horizontal plane. At some instant it has translational velocity* v_0 *and rotational velocity about the centre* $\frac{v_0}{2r}$*. Find the translational velocity after the sphere starts pure rolling.*

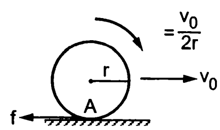


Figure 10-W21

Solution : Velocity of the centre $= v_0$ and the angular velocity about the centre $= \frac{v_0}{2r}$. Thus, $v_0 > \omega_0 r$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is $\mu \mathcal{N} = \mu Mg$ and the sphere will be decelerated by $a_{cm} = f/M$. Hence,

$$v(t) = v_0 - \frac{f}{M} t \cdot \qquad \dots \text{(i)}$$

This friction will also have a torque $\Gamma = fr$ about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f \frac{r}{(2/5) Mr^2} = \frac{5f}{2Mr}$$

and the clockwise angular velocity at time t will be

$$\omega(t) = \omega_0 + \frac{5f}{2Mr} t = \frac{v_0}{2r} + \frac{5f}{2Mr} t$$

Pure rolling starts when $v(t) = r\omega(t)$

i.e., $$v(t) = \frac{v_0}{2} + \frac{5f}{2M} t \qquad \dots \text{(ii)}$$

Eliminating t from (i) and (ii),

$$\frac{5}{2} v(t) + v(t) = \frac{5}{2} v_0 + \frac{v_0}{2}$$

or, $$v(t) = \frac{2}{7} \times 3\, v_0 = \frac{6}{7} v_0 .$$

Thus, the sphere rolls with translational velocity $6v_0/7$ in the forward direction.

Alternative : Let us consider the torque about the initial point of contact A. The force of friction passes through this point and hence its torque is zero. The normal force and the weight balance each other. The net torque about A is zero. Hence the angular momentum about A is conserved.

Initial angular momentum is,

$$L = L_{cm} + Mrv_0 = I_{cm}\, \omega + Mrv_0$$

$$= \left(\frac{2}{5} Mr^2\right)\left(\frac{v_0}{2r}\right) + Mrv_0 = \frac{6}{5} Mrv_0.$$

Suppose the translational velocity of the sphere, after it starts rolling, is v_0. The angular velocity is v/r. The angular momentum about A is,

$$L = L_{cm} + Mrv$$

$$= \left(\frac{2}{5} Mr^2\right)\left(\frac{v}{r}\right) + Mrv = \frac{7}{5} Mrv.$$

Thus, $$\frac{6}{5} Mrv_0 = \frac{7}{5} Mrv$$

or, $$v = \frac{6}{7} v_0 .$$

29. *The sphere shown in figure (10-W22) lies on a rough plane when a particle of mass m travelling at a speed v_0 collides and sticks with it. If the line of motion of the particle is at a distance h above the plane, find (a) the linear speed of the combined system just after the collision, (b) the angular speed of the system about the centre of the sphere just after the collision and (c) the value of h for which the sphere starts pure rolling on the plane. Assume that the mass M of the sphere is large compared to the mass of the particle so that the centre of mass of the combined system is not appreciably shifted from the centre of the sphere.*

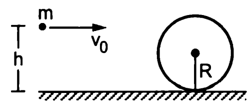


Figure 10-W22

Solution : Take the particle plus the sphere as the system.

(a) Using conservation of linear momentum, the linear speed of the combined system v is given by

$$mv_0 = (M+m)v \quad \text{or,} \quad v = \frac{mv_0}{M+m} \cdot \qquad \dots \text{(i)}$$

(b) Next, we shall use conservation of angular momentum about the centre of mass, which is to be taken at the centre of the sphere ($M >> m$). Angular momentum of the particle before collision is $mv_0(h-R)$. If the system rotates with angular speed ω after collision, the angular momentum of the system becomes

$$\left(\frac{2}{5} MR^2 + mR^2\right)\omega.$$

Hence,

$$mv_0(h-R) = \left(\frac{2}{5} M + m\right) R^2 \omega$$

or, $$\omega = \frac{mv_0(h-R)}{\left(\frac{2}{5} M + m\right) R^2} \cdot$$

(c) The sphere will start rolling just after the collision if

$$v = \omega R, \text{ i.e., } \frac{mv_0}{M+m} = \frac{mv_0(h-R)}{\left(\frac{2}{5} M + m\right) R}$$

giving, $$h = \left(\frac{\frac{7}{5} M + 2m}{M+m}\right) R \approx \frac{7}{5} R.$$

□

QUESTIONS FOR SHORT ANSWER

1. Can an object be in pure translation as well as in pure rotation ?
2. A simple pendulum is a point mass suspended by a light thread from a fixed point. The particle is displaced towards one side and then released. It makes small oscillations. Is the motion of such a simple pendulum a pure rotation ? If yes, where is the axis of rotation ?
3. In a rotating body, $a = \alpha r$ and $v = \omega r$. Thus $\frac{a}{\alpha} = \frac{v}{\omega} \cdot$ Can you use the theorems of ratio and proportion studied in algebra so as to write
$$\frac{a+\alpha}{a-\alpha} = \frac{v+\omega}{v-\omega}$$
4. A ball is whirled in a circle by attaching it to a fixed point with a string. Is there an angular rotation of the ball about its centre ? If yes, is this angular velocity equal to the angular velocity of the ball about the fixed point ?
5. The moon rotates about the earth in such a way that only one hemisphere of the moon faces the earth (figure 10-Q1). Can we ever see the "other face" of the moon from the earth ? Can a person on the moon ever see all the faces of the earth ?

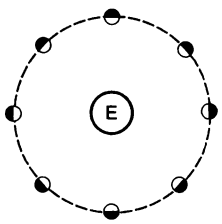


Figure 10-Q1

6. The torque of the weight of any body about any vertical axis is zero. Is it always correct ?
7. The torque of a force $\vec{F}$ about a point is defined as $\vec{\Gamma} = \vec{r} \times \vec{F}$. Suppose $\vec{r}, \vec{F}$ and $\vec{\Gamma}$ are all nonzero. Is $\vec{r} \times \vec{\Gamma} \parallel \vec{F}$ always true ? Is it ever true ?
8. A heavy particle of mass m falls freely near the earth's surface. What is the torque acting on this particle about a point 50 cm east to the line of motion ? Does this torque produce any angular acceleration in the particle ?
9. If several forces act on a particle, the total torque on the particle may be obtained by first finding the resultant force and then taking torque of this resultant. Prove this. Is this result valid for the forces acting on different particles of a body in such a way that their lines of action intersect at a common point ?
10. If the sum of all the forces acting on a body is zero, is it necessarily in equilibrium ? If the sum of all the forces on a particle is zero, is it necessarily in equilibrium ?
11. If the angular momentum of a body is found to be zero about a point, is it necessary that it will also be zero about a different point ?
12. If the resultant torque of all the forces acting on a body is zero about a point, is it necessary that it will be zero about any other point ?
13. A body is in translational equilibrium under the action of coplanar forces. If the torque of these forces is zero about a point, is it necessary that it will also be zero about any other point ?
14. A rectangular brick is kept on a table with a part of its length projecting out. It remains at rest if the length projected is slightly less than half the total length but it falls down if the length projected is slightly more than half the total length. Give reason.
15. When a fat person tries to touch his toes, keeping the legs straight, he generally falls. Explain with reference to figure (10-Q2).

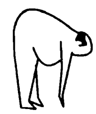


Figure 10-Q2

16. A ladder is resting with one end on a vertical wall and the other end on a horizontal floor. Is it more likely to slip when a man stands near the bottom or near the top ?
17. When a body is weighed on an ordinary balance we demand that the arm should be horizontal if the weights on the two pans are equal. Suppose equal weights are put on the two pans, the arm is kept at an angle with the horizontal and released. Is the torque of the two weights about the middle point (point of support) zero ? Is the total torque zero ? If so, why does the arm rotate and finally become horizontal ?
18. The density of a rod AB continuously increases from A to B. Is it easier to set it in rotation by clamping it at A and applying a perpendicular force at B or by clamping it at B and applying the force at A ?
19. When tall buildings are constructed on earth, the duration of day–night slightly increases. Is it true ?
20. If the ice at the poles melts and flows towards the equator, how will it affect the duration of day–night ?
21. A hollow sphere, a solid sphere, a disc and a ring all having same mass and radius are rolled down on an inclined plane. If no slipping takes place, which one will take the smallest time to cover a given length ?
22. A sphere rolls on a horizontal surface. Is there any point of the sphere which has a vertical velocity ?

OBJECTIVE I

1. Let $\vec{A}$ be a unit vector along the axis of rotation of a purely rotating body and $\vec{B}$ be a unit vector along the velocity of a particle P of the body away from the axis. The value of $\vec{A} \cdot \vec{B}$ is
(a) 1 (b) –1 (c) 0 (d) None of these.

2. A body is uniformly rotating about an axis fixed in an inertial frame of reference. Let $\vec{A}$ be a unit vector along the axis of rotation and $\vec{B}$ be the unit vector along the resultant force on a particle P of the body away from the axis. The value of $\vec{A} \cdot \vec{B}$ is
(a) 1 (b) –1 (c) 0 (d) none of these.

3. A particle moves with a constant velocity parallel to the X-axis. Its angular momentum with respect to the origin
(a) is zero (b) remains constant
(c) goes on increasing (d) goes on decreasing.

4. A body is in pure rotation. The linear speed v of a particle, the distance r of the particle from the axis and the angular velocity ω of the body are related as $\omega = \frac{v}{r}\cdot$ Thus
(a) $\omega \propto \frac{1}{r}$ (b) $\omega \propto r$
(c) $\omega = 0$ (d) ω is independent of r.

5. Figure (10-Q3) shows a small wheel fixed coaxially on a bigger one of double the radius. The system rotates about the common axis. The strings supporting A and B do not slip on the wheels. If x and y be the distances travelled by A and B in the same time interval, then
(a) $x = 2y$ (b) $x = y$ (c) $y = 2x$ (d) none of these.

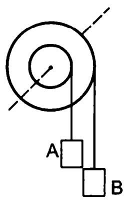


Figure 10-Q3

6. A body is rotating uniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(a) vertical (b) horizontal and skew with the axis
(c) horizontal and intersecting the axis
(d) none of these.

7. A body is rotating nonuniformly about a vertical axis fixed in an inertial frame. The resultant force on a particle of the body not on the axis is
(a) vertical (b) horizontal and skew with the axis
(c) horizontal and intersecting the axis
(d) none of these.

8. Let $\vec{F}$ be a force acting on a particle having position vector $\vec{r}$. Let $\vec{\Gamma}$ be the torque of this force about the origin, then
(a) $\vec{r} \cdot \vec{\Gamma} = 0$ and $\vec{F} \cdot \vec{\Gamma} = 0$ (b) $\vec{r} \cdot \vec{\Gamma} = 0$ but $\vec{F} \cdot \vec{\Gamma} \neq 0$
(c) $\vec{r} \cdot \vec{\Gamma} \neq 0$ but $\vec{F} \cdot \vec{\Gamma} = 0$ (d) $\vec{r} \cdot \vec{\Gamma} \neq 0$ and $\vec{F} \cdot \vec{\Gamma} \neq 0$.

9. One end of a uniform rod of mass m and length l is clamped. The rod lies on a smooth horizontal surface and rotates on it about the clamped end at a uniform angular velocity ω. The force exerted by the clamp on the rod has a horizontal component
(a) $m\omega^2 l$ (b) zero (c) mg (d) $\frac{1}{2} m\omega^2 l$.

10. A uniform rod is kept vertically on a horizontal smooth surface at a point O. If it is rotated slightly and released, it falls down on the horizontal surface. The lower end will remain
(a) at O (b) at a distance less than $l/2$ from O
(c) at a distance $l/2$ from O
(d) at a distance larger than $l/2$ from O.

11. A circular disc A of radius r is made from an iron plate of thickness t and another circular disc B of radius $4r$ is made from an iron plate of thickness $t/4$. The relation between the moments of inertia I_A and I_B is
(a) $I_A > I_B$ (b) $I_A = I_B$ (c) $I_A < I_B$
(d) depends on the actual values of t and r.

12. Equal torques act on the discs A and B of the previous problem, initially both being at rest. At a later instant, the linear speeds of a point on the rim of A and another point on the rim of B are v_A and v_B respectively. We have
(a) $v_A > v_B$ (b) $v_A = v_B$ (c) $v_A < v_B$
(d) the relation depends on the actual magnitude of the torques.

13. A closed cylindrical tube containing some water (not filling the entire tube) lies in a horizontal plane. If the tube is rotated about a perpendicular bisector, the moment of inertia of water about the axis
(a) increases (b) decreases (c) remains constant
(d) increases if the rotation is clockwise and decreases if it is anticlockwise.

14. The moment of inertia of a uniform semicircular wire of mass M and radius r about a line perpendicular to the plane of the wire through the centre is
(a) Mr^2 (b) $\frac{1}{2}Mr^2$ (c) $\frac{1}{4}Mr^2$ (d) $\frac{2}{5}Mr^2$.

15. Let I_1 and I_2 be the moments of inertia of two bodies of identical geometrical shape, the first made of aluminium and the second of iron.
(a) $I_1 < I_2$ (b) $I_1 = I_2$ (c) $I_1 > I_2$
(d) relation between I_1 and I_2 depends on the actual shapes of the bodies.

16. A body having its centre of mass at the origin has three of its particles at $(a,0,0)$, $(0,a,0)$, $(0,0,a)$. The moments of inertia of the body about the X and Y axes are 0·20 kg–m ^{2} each. The moment of inertia about the Z-axis
(a) is 0·20 kg-m ^{2} (b) is 0·40 kg-m ^{2}
(c) is $0{\cdot}20\sqrt{2}$ kg-m ^{2}
(d) cannot be deduced with this information.

17. A cubical block of mass M and edge a slides down a rough inclined plane of inclination θ with a uniform

velocity. The torque of the normal force on the block about its centre has a magnitude
(a) zero (b) Mga (c) $Mga \sin\theta$ (d) $\frac{1}{2} Mga \sin\theta$.

18. A thin circular ring of mass M and radius r is rotating about its axis with an angular speed ω. Two particles having mass m each are now attached at diametrically opposite points. The angular speed of the ring will become
(a) $\frac{\omega M}{M+m}$ (b) $\frac{\omega M}{M+2m}$
(c) $\frac{\omega(M-2m)}{M+2m}$ (d) $\frac{\omega(M+2m)}{M}$.

19. A person sitting firmly over a rotating stool has his arms stretched. If he folds his arms, his angular momentum about the axis of rotation
(a) increases (b) decreases
(c) remains unchanged (d) doubles.

20. The centre of a wheel rolling on a plane surface moves with a speed v_0. A particle on the rim of the wheel at the same level as the centre will be moving at speed
(a) zero (b) v_0 (c) $\sqrt{2}v_0$ (d) $2v_0$.

21. A wheel of radius 20 cm is pushed to move it on a rough horizontal surface. It is found to move through a distance of 60 cm on the road during the time it completes one revolution about the centre. Assume that the linear and the angular accelerations are uniform. The frictional force acting on the wheel by the surface is
(a) along the velocity of the wheel
(b) opposite to the velocity of the wheel
(c) perpendicular to the velocity of the wheel
(d) zero.

22. The angular velocity of the engine (and hence of the wheel) of a scooter is proportional to the petrol input per second. The scooter is moving on a frictionless road with uniform velocity. If the petrol input is increased by 10%, the linear velocity of the scooter is increased by
(a) 50% (b) 10% (c) 20% (d) 0%.

23. A solid sphere, a hollow sphere and a disc, all having same mass and radius, are placed at the top of a smooth incline and released. Least time will be taken in reaching the bottom by
(a) the solid sphere (b) the hollow sphere
(c) the disc (d) all will take same time.

24. A solid sphere, a hollow sphere and a disc, all having same mass and radius, are placed at the top of an incline and released. The friction coefficients between the objects and the incline are same and not sufficient to allow pure rolling. Least time will be taken in reaching the bottom by
(a) the solid sphere (b) the hollow sphere
(c) the disc (d) all will take same time.

25. In the previous question, the smallest kinetic energy at the bottom of the incline will be achieved by
(a) the solid sphere (b) the hollow sphere
(c) the disc (d) all will achieve same kinetic energy.

26. A string of negligible thickness is wrapped several times around a cylinder kept on a rough horizontal surface. A man standing at a distance l from the cylinder holds one end of the string and pulls the cylinder towards him (figure 10-Q4). There is no slipping anywhere. The length of the string passed through the hand of the man while the cylinder reaches his hands is
(a) l (b) $2l$ (c) $3l$ (d) $4l$.

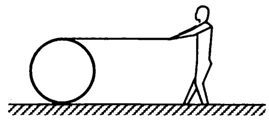


Figure 10-Q4

OBJECTIVE II

1. The axis of rotation of a purely rotating body
(a) must pass through the centre of mass
(b) may pass through the centre of mass
(c) must pass through a particle of the body
(d) may pass through a particle of the body.

2. Consider the following two equations
(A) $L = I\omega$ (B) $\frac{dL}{dt} = \Gamma$
In noninertial frames
(a) both A and B are true (b) A is true but B is false
(c) B is true but A is false (d) both A and B are false.

3. A particle moves on a straight line with a uniform velocity. Its angular momentum
(a) is always zero
(b) is zero about a point on the straight line
(c) is not zero about a point away from the straight line
(d) about any given point remains constant.

4. If there is no external force acting on a nonrigid body, which of the following quantities must remain constant ?
(a) angular momentum (b) linear momentum
(c) kinetic energy (d) moment of inertia.

5. Let I_A and I_B be moments of inertia of a body about two axes A and B respectively. The axis A passes through the centre of mass of the body but B does not.
(a) $I_A < I_B$ (b) If $I_A < I_B$, the axes are parallel
(c) If the axes are parallel, $I_A < I_B$
(d) If the axes are not parallel, $I_A \geq I_B$.

6. A sphere is rotating about a diameter.
(a) The particles on the surface of the sphere do not have any linear acceleration.
(b) The particles on the diameter mentioned above do not have any linear acceleration.
(c) Different particles on the surface have different

angular speeds.
(d) All the particles on the surface have same linear speed.

7. The density of a rod gradually decreases from one end to the other. It is pivoted at an end so that it can move about a vertical axis through the pivot. A horizontal force F is applied on the free end in a direction perpendicular to the rod. The quantities, that do not depend on which end of the rod is pivoted, are
(a) angular acceleration
(b) angular velocity when the rod completes one rotation
(c) angular momentum when the rod completes one rotation
(d) torque of the applied force.

8. Consider a wheel of a bicycle rolling on a level road at a linear speed v_0 (figure 10-Q5).
(a) the speed of the particle A is zero
(b) the speed of B, C and D are all equal to v_0
(c) the speed of C is $2\,v_0$
(d) the speed of B is greater than the speed of O.

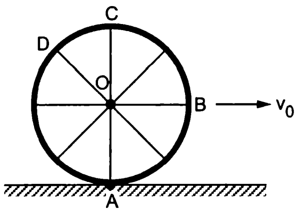


Figure 10-Q5

9. Two uniform solid spheres having unequal masses and unequal radii are released from rest from the same height on a rough incline. If the spheres roll without slipping,
(a) the heavier sphere reaches the bottom first
(b) the bigger sphere reaches the bottom first
(c) the two spheres reach the bottom together
(d) the information given is not sufficient to tell which sphere will reach the bottom first.

10. A hollow sphere and a solid sphere having same mass and same radii are rolled down a rough inclined plane.
(a) The hollow sphere reaches the bottom first.
(b) The solid sphere reaches the bottom with greater speed.
(c) The solid sphere reaches the bottom with greater kinetic energy.
(d) The two spheres will reach the bottom with same linear momentum.

11. A sphere cannot roll on
(a) a smooth horizontal surface
(b) a smooth inclined surface
(c) a rough horizontal surface
(d) a rough inclined surface.

12. In rear-wheel drive cars, the engine rotates the rear wheels and the front wheels rotate only because the car moves. If such a car accelerates on a horizontal road, the friction
(a) on the rear wheels is in the forward direction
(b) on the front wheels is in the backward direction
(c) on the rear wheels has larger magnitude than the friction on the front wheels
(d) on the car is in the backward direction.

13. A sphere can roll on a surface inclined at an angle θ if the friction coefficient is more than $\frac{2}{7} g \tan\theta$. Suppose the friction coefficient is $\frac{1}{7} g \tan\theta$. If a sphere is released from rest on the incline,
(a) it will stay at rest
(b) it will make pure translational motion
(c) it will translate and rotate about the centre
(d) the angular momentum of the sphere about its centre will remain constant.

14. A sphere is rolled on a rough horizontal surface. It gradually slows down and stops. The force of friction tries to
(a) decrease the linear velocity
(b) increase the angular velocity
(c) increase the linear momentum
(d) decrease the angular velocity.

15. Figure (10-Q6) shows a smooth inclined plane fixed in a car accelerating on a horizontal road. The angle of incline θ is related to the acceleration a of the car as $a = g \tan\theta$. If the sphere is set in pure rolling on the incline,
(a) it will continue pure rolling
(b) it will slip down the plane
(c) its linear velocity will increase
(d) its linear velocity will slowly decrease.

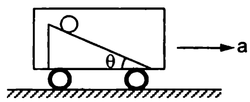


Figure 10-Q6

EXERCISES

1. A wheel is making revolutions about its axis with uniform angular acceleration. Starting from rest, it reaches 100 rev/sec in 4 seconds. Find the angular acceleration. Find the angle rotated during these four seconds.

2. A wheel rotating with uniform angular acceleration covers 50 revolutions in the first five seconds after the start. Find the angular acceleration and the angular velocity at the end of five seconds.

3. A wheel starting from rest is uniformly accelerated at 4 rad/s ^{2} for 10 seconds.It is allowed to rotate uniformly

for the next 10 seconds and is finally brought to rest in the next 10 seconds. Find the total angle rotated by the wheel.

4. A body rotates about a fixed axis with an angular acceleration of one radian/second/second. Through what angle does it rotate during the time in which its angular velocity increases from 5 rad/s to 15 rad/s.

5. Find the angular velocity of a body rotating with an acceleration of 2 rev/s ^{2} as it completes the 5th revolution after the start.

6. A disc of radius 10 cm is rotating about its axis at an angular speed of 20 rad/s. Find the linear speed of
(a) a point on the rim,
(b) the middle point of a radius.

7. A disc rotates about its axis with a constant angular acceleration of 4 rad/s^2. Find the radial and tangential accelerations of a particle at a distance of 1 cm from the axis at the end of the first second after the disc starts rotating.

8. A block hangs from a string wrapped on a disc of radius 20 cm free to rotate about its axis which is fixed in a horizontal position. If the angular speed of the disc is 10 rad/s at some instant, with what speed is the block going down at that instant?

9. Three particles, each of mass 200 g, are kept at the corners of an equilateral triangle of side 10 cm. Find the moment of inertia of the system about an axis
(a) joining two of the particles and
(b) passing through one of the particles and perpendicular to the plane of the particles.

10. Particles of masses 1 g, 2 g, 3 g, ..., 100 g are kept at the marks 1 cm, 2 cm, 3 cm, ..., 100 cm respectively on a metre scale. Find the moment of inertia of the system of particles about a perpendicular bisector of the metre scale.

11. Find the moment of inertia of a pair of spheres, each having a mass m and radius r, kept in contact about the tangent passing through the point of contact.

12. The moment of inertia of a uniform rod of mass 0·50 kg and length 1 m is 0·10 kg-m^2 about a line perpendicular to the rod. Find the distance of this line from the middle point of the rod.

13. Find the radius of gyration of a circular ring of radius r about a line perpendicular to the plane of the ring and passing through one of its particles.

14. The radius of gyration of a uniform disc about a line perpendicular to the disc equals its radius. Find the distance of the line from the centre.

15. Find the moment of inertia of a uniform square plate of mass m and edge a about one of its diagonals.

16. The surface density (mass/area) of a circular disc of radius a depends on the distance from the centre as $\rho(r) = A + Br$. Find its moment of inertia about the line perpendicular to the plane of the disc through its centre.

17. A particle of mass m is projected with a speed u at an angle θ with the horizontal. Find the torque of the weight of the particle about the point of projection when the particle is at the highest point.

18. A simple pendulum of length l is pulled aside to make an angle θ with the vertical. Find the magnitude of the torque of the weight w of the bob about the point of suspension. When is the torque zero?

19. When a force of 6·0 N is exerted at 30° to a wrench at a distance of 8 cm from the nut, it is just able to loosen the nut. What force F would be sufficient to loosen it if it acts perpendicularly to the wrench at 16 cm from the nut?

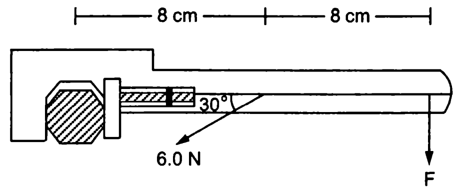


Figure 10-E1

20. Calculate the total torque acting on the body shown in figure (10-E2) about the point O.

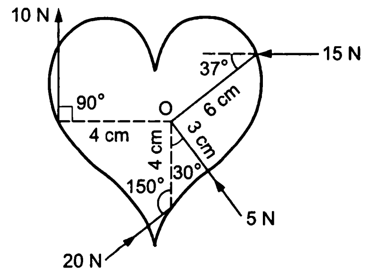


Figure 10-E2

21. A cubical block of mass m and edge a slides down a rough inclined plane of inclination θ with a uniform speed. Find the torque of the normal force acting on the block about its centre.

22. A rod of mass m and length L, lying horizontally, is free to rotate about a vertical axis through its centre. A horizontal force of constant magnitude F acts on the rod at a distance of $L/4$ from the centre. The force is always perpendicular to the rod. Find the angle rotated by the rod during the time t after the motion starts.

23. A square plate of mass 120 g and edge 5·0 cm rotates about one of the edges. If it has a uniform angular acceleration of 0·2 rad/s ^{2}, what torque acts on the plate?

24. Calculate the torque on the square plate of the previous problem if it rotates about a diagonal with the same angular acceleration.

25. A flywheel of moment of inertia 5·0 kg-m^2 is rotated at a speed of 60 rad/s. Because of the friction at the axle, it comes to rest in 5·0 minutes. Find (a) the average torque of the friction, (b) the total work done by the friction and (c) the angular momentum of the wheel 1 minute before it stops rotating.

26. Because of the friction between the water in oceans with the earth's surface, the rotational kinetic energy of the earth is continuously decreasing. If the earth's angular speed decreases by 0·0016 rad/day in 100 years, find the

average torque of the friction on the earth. Radius of the earth is 6400 km and its mass is $6{\cdot}0 \times 10^{24}$ kg.

27. A wheel rotating at a speed of 600 rpm (revolutions per minute) about its axis is brought to rest by applying a constant torque for 10 seconds. Find the angular deceleration and the angular velocity 5 seconds after the application of the torque.

28. A wheel of mass 10 kg and radius 20 cm is rotating at an angular speed of 100 rev/min when the motor is turned off. Neglecting the friction at the axle, calculate the force that must be applied tangentially to the wheel to bring it to rest in 10 revolutions.

29. A cylinder rotating at an angular speed of 50 rev/s is brought in contact with an identical stationary cylinder. Because of the kinetic friction, torques act on the two cylinders, accelerating the stationary one and decelerating the moving one. If the common magnitude of the acceleration and deceleration be one revolution per second square, how long will it take before the two cylinders have equal angular speed ?

30. A body rotating at 20 rad/s is acted upon by a constant torque providing it a deceleration of 2 rad/s^2 . At what time will the body have kinetic energy same as the initial value if the torque continues to act ?

31. A light rod of length 1 m is pivoted at its centre and two masses of 5 kg and 2 kg are hung from the ends as shown in figure (10-E3). Find the initial angular acceleration of the rod assuming that it was horizontal in the beginning.

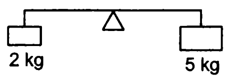


Figure 10-E3

32. Suppose the rod in the previous problem has a mass of 1 kg distributed uniformly over its length.
(a) Find the initial angular acceleration of the rod.
(b) Find the tension in the supports to the blocks of mass 2 kg and 5 kg.

33. Figure (10-E4) shows two blocks of masses m and M connected by a string passing over a pulley. The horizontal table over which the mass m slides is smooth. The pulley has a radius r and moment of inertia I about its axis and it can freely rotate about this axis. Find the acceleration of the mass M assuming that the string does not slip on the pulley.

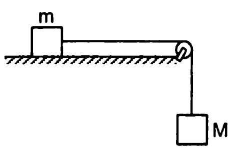


Figure 10-E4

34. A string is wrapped on a wheel of moment of inertia $0{\cdot}20$ kg-m^2 and radius 10 cm and goes through a light pulley to support a block of mass $2{\cdot}0$ kg as shown in figure (10-E5). Find the acceleration of the block.

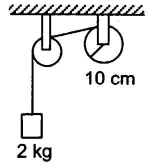


Figure 10-E5

35. Suppose the smaller pulley of the previous problem has its radius $5{\cdot}0$ cm and moment of inertia $0{\cdot}10$ kg-m^2. Find the tension in the part of the string joining the pulleys.

36. The pulleys in figure (10-E6) are identical, each having a radius R and moment of inertia I. Find the acceleration of the block M.

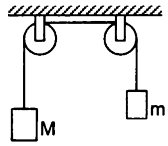


Figure 10-E6

37. The descending pulley shown in figure (10-E7) has a radius 20 cm and moment of inertia $0{\cdot}20$ kg-m^2. The fixed pulley is light and the horizontal plane frictionless. Find the acceleration of the block if its mass is $1{\cdot}0$ kg.

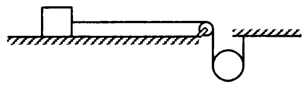


Figure 10-E7

38. The pulley shown in figure (10-E8) has a radius 10 cm and moment of inertia $0{\cdot}5$ kg-m^2 about its axis. Assuming the inclined planes to be frictionless, calculate the acceleration of the $4{\cdot}0$ kg block.

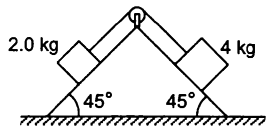


Figure 10-E8

39. Solve the previous problem if the friction coefficient between the $2{\cdot}0$ kg block and the plane below it is $0{\cdot}5$ and the plane below the $4{\cdot}0$ kg block is frictionless.

40. A uniform metre stick of mass 200 g is suspended from the ceiling through two vertical strings of equal lengths fixed at the ends. A small object of mass 20 g is placed on the stick at a distance of 70 cm from the left end. Find the tensions in the two strings.

41. A uniform ladder of length $10{\cdot}0$ m and mass $16{\cdot}0$ kg is resting against a vertical wall making an angle of 37° with it. The vertical wall is frictionless but the ground is rough. An electrician weighing $60{\cdot}0$ kg climbs up the ladder. If he stays on the ladder at a point $8{\cdot}00$ m from the lower end, what will be the normal force and the force of friction on the ladder by the ground? What should be the minimum coefficient of friction for the elctrician to work safely?

42. Suppose the friction coefficient between the ground and the ladder of the previous problem is 0·540. Find the maximum weight of a mechanic who could go up and do the work from the same position of the ladder.

43. A 6·5 m long ladder rests against a vertical wall reaching a height of 6·0 m. A 60 kg man stands half way up the ladder. (a) Find the torque of the force exerted by the man on the ladder about the upper end of the ladder. (b) Assuming the weight of the ladder to be negligible as compared to the man and assuming the wall to be smooth, find the force exerted by the ground on the ladder.

44. The door of an almirah is 6 ft high, 1·5 ft wide and weighs 8 kg. The door is supported by two hinges situated at a distance of 1 ft from the ends. If the magnitudes of the forces exerted by the hinges on the door are equal, find this magnitude.

45. A uniform rod of length L rests against a smooth roller as shown in figure (10-E9). Find the friction coefficient between the ground and the lower end if the minimum angle that the rod can make with the horizontal is θ.

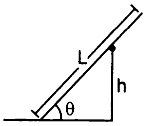


Figure 10-E9

46. A uniform rod of mass 300 g and length 50 cm rotates at a uniform angular speed of 2 rad/s about an axis perpendicular to the rod through an end. Calculate (a) the angular momentum of the rod about the axis of rotation, (b) the speed of the centre of the rod and (c) its kinetic energy.

47. A uniform square plate of mass 2·0 kg and edge 10 cm rotates about one of its diagonals under the action of a constant torque of 0·10 N-m. Calculate the angular momentum and the kinetic energy of the plate at the end of the fifth second after the start.

48. Calculate the ratio of the angular momentum of the earth about its axis due to its spinning motion to that about the sun due to its orbital motion. Radius of the earth = 6400 km and radius of the orbit of the earth about the sun = $1·5 \times 10^8$ km.

49. Two particles of masses m_1 and m_2 are joined by a light rigid rod of length r. The system rotates at an angular speed ω about an axis through the centre of mass of the system and perpendicular to the rod. Show that the angular momentum of the system is $L = \mu r^2 \omega$ where μ is the reduced mass of the system defined as $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

50. A dumb-bell consists of two identical small balls of mass 1/2 kg each connected to the two ends of a 50 cm long light rod. The dumb-bell is rotating about a fixed axis through the centre of the rod and perpendicular to it at an angular speed of 10 rad/s. An impulsive force of average magnitude 5·0 N acts on one of the masses in the direction of its velocity for 0·10 s. Find the new angular velocity of the system.

51. A wheel of moment of inertia 0·500 kg-m^2 and radius 20·0 cm is rotating about its axis at an angular speed of 20·0 rad/s. It picks up a stationary particle of mass 200 g at its edge. Find the new angular speed of the wheel.

52. A diver having a moment of inertia of 6·0 kg-m^2 about an axis through its centre of mass rotates at an angular speed of 2 rad/s about this axis. If he folds his hands and feet to decrease the moment of inertia to 5·0 kg-m^2, what will be the new angular speed?

53. A boy is seated in a revolving chair revolving at an angular speed of 120 revolutions per minute. Two heavy balls form part of the revolving system and the boy can pull the balls closer to himself or may push them apart. If by pulling the balls closer, the boy decreases the moment of inertia of the system from 6 kg-m^2 to 2 kg-m^2, what will be the new angular speed?

54. A boy is standing on a platform which is free to rotate about its axis. The boy holds an open umbrella in his hand. The axis of the umbrella coincides with that of the platform. The moment of inertia of "the platform plus the boy system" is $3·0 \times 10^{-3}$ kg-m^2 and that of the umbrella is $2·0 \times 10^{-3}$ kg-m^2. The boy starts spinning the umbrella about the axis at an angular speed of 2·0 rev/s with respect to himself. Find the angular velocity imparted to the platform.

55. A wheel of moment of inertia 0·10 kg-m^2 is rotating about a shaft at an angular speed of 160 rev/minute. A second wheel is set into rotation at 300 rev/minute and is coupled to the same shaft so that both the wheels finally rotate with a common angular speed of 200 rev/minute. Find the moment of inertia of the second wheel.

56. A kid of mass M stands at the edge of a platform of radius R which can be freely rotated about its axis. The moment of inertia of the platform is I. The system is at rest when a friend throws a ball of mass m and the kid catches it. If the velocity of the ball is v horizontally along the tangent to the edge of the platform when it was caught by the kid, find the angular speed of the platform after the event.

57. Suppose the platform of the previous problem is brought to rest with the ball in the hand of the kid standing on the rim. The kid throws the ball horizontally to his friend in a direction tangential to the rim with a speed v as seen by his friend. Find the angular velocity with which the platform will start rotating.

58. Suppose the platform with the kid in the previous problem is rotating in anticlockwise direction at an angular speed ω. The kid starts walking along the rim with a speed v relative to the platform also in the anticlockwise direction. Find the new angular speed of the platform.

59. A uniform rod of mass m and length l is struck at an end by a force F perpendicular to the rod for a short time interval t. Calculate
(a) the speed of the centre of mass, (b) the angular speed of the rod about the centre of mass, (c) the kinetic energy of the rod and (d) the angular momentum of the rod about the centre of mass after the force has stopped to act. Assume that t is so small that the rod does not appreciably change its direction while the force acts.

60. A uniform rod of length L lies on a smooth horizontal table. A particle moving on the table strikes the rod perpendicularly at an end and stops. Find the distance travelled by the centre of the rod by the time it turns through a right angle. Show that if the mass of the rod is four times that of the particle, the collision is elastic.

61. Suppose the particle of the previous problem has a mass m and a speed v before the collision and it sticks to the rod after the collision. The rod has a mass M. (a) Find the velocity of the centre of mass C of the system constituting "the rod plus the particle". (b) Find the velocity of the particle with respect to C before the collision. (c) Find the velocity of the rod with respect to C before the collision. (d) Find the angular momentum of the particle and of the rod about the centre of mass C before the collision. (e) Find the moment of inertia of the system about the vertical axis through the centre of mass C after the collision. (f) Find the velocity of the centre of mass C and the angular velocity of the system about the centre of mass after the collision.

62. Two small balls A and B, each of mass m, are joined rigidly by a light horizontal rod of length L. The rod is clamped at the centre in such a way that it can rotate freely about a vertical axis through its centre. The system is rotated with an angular speed ω about the axis. A particle P of mass m kept at rest sticks to the ball A as the ball collides with it. Find the new angular speed of the rod.

63. Two small balls A and B, each of mass m, are joined rigidly to the ends of a light rod of lengh L (figure 10-E10). The system translates on a frictionless horizontal surface with a velocity v_0 in a direction perpendicular to the rod. A particle P of mass m kept at rest on the surface sticks to the ball A as the ball collides with it. Find
(a) the linear speeds of the balls A and B after the collision, (b) the velocity of the centre of mass C of the system $A + B + P$ and (c) the angular speed of the system about C after the collision.

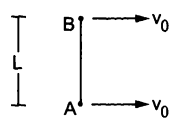


Figure 10-E10

[**Hint :** The light rod will exert a force on the ball B only along its length.]

64. Suppose the rod with the balls A and B of the previous problem is clamped at the centre in such a way that it can rotate freely about a horizontal axis through the clamp. The system is kept at rest in the horizontal position. A particle P of the same mass m is dropped from a height h on the ball B. The particle collides with B and sticks to it. (a) Find the angular momentum and the angular speed of the system just after the collision. (b) What should be the minimum value of h so that the system makes a full rotation after the collision.

65. Two blocks of masses 400 g and 200 g are connected through a light string going over a pulley which is free to rotate about its axis. The pulley has a moment of inertia $1 \cdot 6 \times 10^{-4}$ kg-m^2 and a radius 2·0 cm. Find (a) the kinetic energy of the system as the 400 g block falls through 50 cm, (b) the speed of the blocks at this instant.

66. The pulley shown in figure (10-E11) has a radius of 20 cm and moment of inertia 0·2 kg-m^2. The string going over it is attached at one end to a vertical spring of spring constant 50 N/m fixed from below, and supports a 1 kg mass at the other end. The system is released from rest with the spring at its natural length. Find the speed of the block when it has descended through 10 cm. Take $g = 10$ m/s^2.

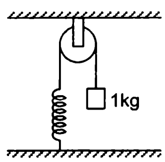


Figure 10-E11

67. A metre stick is held vertically with one end on a rough horizontal floor. It is gently allowed to fall on the floor. Assuming that the end at the floor does not slip, find the angular speed of the rod when it hits the floor.

68. A metre stick weighing 240 g is pivoted at its upper end in such a way that it can freely rotate in a vertical plane through this end (figure 10-E12). A particle of mass 100 g is attached to the upper end of the stick through a light string of length 1 m. Initially, the rod is kept vertical and the string horizontal when the system is released from rest. The particle collides with the lower end of the stick and sticks there. Find the maximum angle through which the stick will rise.

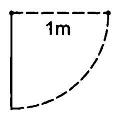


Figure 10-E12

69. A uniform rod pivoted at its upper end hangs vertically. It is displaced through an angle of 60° and then released. Find the magnitude of the force acting on a particle of mass dm at the tip of the rod when the rod makes an angle of 37° with the vertical.

70. A cylinder rolls on a horizontal plane surface. If the speed of the centre is 25 m/s, what is the speed of the highest point ?

71. A sphere of mass m rolls on a plane surface. Find its kinetic energy at an instant when its centre moves with speed v.

72. A string is wrapped over the edge of a uniform disc and the free end is fixed with the ceiling. The disc moves down, unwinding the string. Find the downward acceleration of the disc.

73. A small spherical ball is released from a point at a height h on a rough track shown in figure (10-E13). Assuming that it does not slip anywhere, find its linear speed when it rolls on the horizontal part of the track.

Figure 10-E13

74. A small disc is set rolling with a speed v on the horizontal part of the track of the previous problem from right to left. To what height will it climb up the curved part ?

75. A sphere starts rolling down an incline of inclination θ. Find the speed of its centre when it has covered a distance l.

76. A hollow sphere is released from the top of an inclined plane of inclination θ. (a) What should be the minimum coefficient of friction between the sphere and the plane to prevent sliding ? (b) Find the kinetic energy of the ball as it moves down a length l on the incline if the friction coefficient is half the value calculated in part (a).

77. A solid sphere of mass m is released from rest from the rim of a hemispherical cup so that it rolls along the surface. If the rim of the hemisphere is kept horizontal, find the normal force exerted by the cup on the ball when the ball reaches the bottom of the cup.

78. Figure (10-E14) shows a rough track, a portion of which is in the form of a cylinder of radius R. With what minimum linear speed should a sphere of radius r be set rolling on the horizontal part so that it completely goes round the circle on the cylindrical part.

Figure 10-E14

79. Figure (10-E15) shows a small spherical ball of mass m rolling down the loop track. The ball is released on the linear portion at a vertical height H from the lowest point. The circular part shown has a radius R.
(a) Find the kinetic energy of the ball when it is at a point A where the radius makes an angle θ with the horizontal.
(b) Find the radial and the tangential accelerations of the centre when the ball is at A.
(c) Find the normal force and the frictional force acting on the ball if $H = 60$ cm, $R = 10$ cm, $\theta = 0$ and $m = 70$ g.

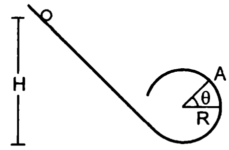


Figure 10-E15

80. A thin spherical shell of radius R lying on a rough horizontal surface is hit sharply and horizontally by a cue. Where should it be hit so that the shell does not slip on the surface ?

81. A uniform wheel of radius R is set into rotation about its axis at an angular speed ω. This rotating wheel is now placed on a rough horizontal surface with its axis horizontal. Because of friction at the contact, the wheel accelerates forward and its rotation decelerates till the wheel starts pure rolling on the surface. Find the linear speed of the wheel after it starts pure rolling.

82. A thin spherical shell lying on a rough horizontal surface is hit by a cue in such a way that the line of action passes through the centre of the shell. As a result, the shell starts moving with a linear speed v without any initial angular velocity. Find the linear speed of the shell after it starts pure rolling on the surface.

83. A hollow sphere of radius R lies on a smooth horizontal surface. It is pulled by a horizontal force acting tangentially from the highest point. Find the distance travelled by the sphere during the time it makes one full rotation.

84. A solid sphere of mass 0·50 kg is kept on a horizontal surface. The coefficient of static friction between the surfaces in contact is 2/7. What maximum force can be applied at the highest point in the horizontal direction so that the sphere does not slip on the surface ?

85. A solid sphere is set into motion on a rough horizontal surface with a linear speed v in the forward direction and an angular speed v/R in the anticlockwise direction as shown in figure (10-E16). Find the linear speed of the sphere (a) when it stops rotating and (b) when slipping finally ceases and pure rolling starts.

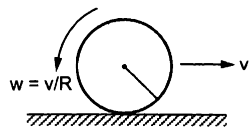


Figure 10-E16

86. A solid sphere rolling on a rough horizontal surface with a linear speed v collides elastically with a fixed, smooth, vertical wall. Find the speed of the sphere after it has started pure rolling in the backward direction.

□

ANSWERS

OBJECTIVE I

1. (c)	2. (c)	3. (b)	4. (d)	5. (c)	6. (c)
7. (b)	8. (a)	9. (d)	10. (c)	11. (c)	12. (a)
13. (a)	14. (a)	15. (a)	16. (d)	17. (d)	18. (b)
19. (c)	20. (c)	21. (a)	22. (d)	23. (d)	24. (d)
25. (b)	26. (b)				

OBJECTIVE II

1. (b), (d)	2. (b)	3. (b), (c), (d)
4. (a), (b)	5. (c)	6. (b)
7. (d)	8. (a), (c), (d)	9. (c)
10. (b)	11. (b)	12. (a), (b), (c)
13. (c)	14. (a), (b)	15. (a)

EXERCISES

1. 25 rev/s ^{2}, 400 π rad
2. 4 rev/s ^{2}, 20 rev/s
3. 800 rad
4. 100 rad
5. 2 $\sqrt{5}$ rev/s
6. 2 m/s, 1 m/s
7. 16 cm/s ^{2}, 4 cm/s ^{2}
8. 2 m/s
9. $1{\cdot}5 \times 10^{-3}$ kg–m ^{2}, $4{\cdot}0 \times 10^{-3}$ kg–m ^{2}
10. 0·43 kg–m ^{2}
11. $\dfrac{14\, mr^2}{5}$
12. 0·34 m
13. $\sqrt{2}\, r$
14. $r / \sqrt{2}$
15. $ma^2/12$
16. $2\pi\left(\dfrac{Aa^4}{4} + \dfrac{Ba^5}{5}\right)$
17. $mu^2 \sin\theta \cos\theta$ perpendicular to the plane of motion
18. $wl \sin\theta$, when the bob is at the lowest point
19. 1·5 N
20. 0·54 N-m
21. $\dfrac{1}{2} mg\, a \sin\theta$
22. $\dfrac{3\, Ft^2}{2\, ml}$
23. $2{\cdot}0 \times 10^{-5}$ N-m
24. $0{\cdot}5 \times 10^{-5}$ N-m
25. (a) 1·0 N-m (b) 9·0 kJ (c) 60 kg-m ^{2}/s
26. $5{\cdot}8 \times 10^{20}$ N-m
27. 1 rev/s ^{2}, 5 rev/s
28. 0·87 N
29. 25 s
30. 20 s
31. 8·4 rad/s ^{2}
32. 8·0 rad/s ^{2}, 27·6 N, 29 N
33. $\dfrac{Mg}{M + m + I/r^2}$
34. 0·89 m/s ^{2}
35. 6·3 N
36. $\dfrac{(M - m)g}{M + m + 2I/r^2}$
37. 10 m/s ^{2}
38. 0·25 m/s ^{2}
39. 0·125 m/s ^{2}
40. 1·04 N in the left string and 1·12 N in the right
41. 745 N, 412 N, 0·553
42. 44·0 kg
43. (a) 740 N-m
    (b) 590 N vertical and 120 N horizontal
44. 43 N
45. $\dfrac{L \cos\theta \sin^2\theta}{2h - L \cos^2\theta \sin\theta}$
46. (a) 0·05 kg-m ^{2}/s (b) 50 cm/s (c) 0·05 J
47. 0·5 kg-m ^{2}/s, 75 J
48. $2{\cdot}66 \times 10^{-7}$
50. 12 rad/s
51. 19·7 rad/s
52. 2·4 rad/s
53. 360 rev/minute
54. 0·8 rev/s
55. 0·04 kg-m ^{2}
56. $\dfrac{mvR}{I + (M + m)R^2}$
57. $\dfrac{mvR}{I + MR^2}$
58. $\omega - \dfrac{Mvr}{I + MR^2}$
59. (a) $\dfrac{Ft}{m}$ (b) $\dfrac{6\, Ft}{ml}$ (c) $\dfrac{2\, F^2 t^2}{m}$ (d) $\dfrac{Flt}{2}$
60. $\pi L/12$
61. (a) $\dfrac{mv}{M + m}$ (b) $\dfrac{Mv}{M + m}$ (c) $-\dfrac{mv}{M + m}$
    (d) $\dfrac{M^2 mvl}{2(M + m)^2}$, $\dfrac{Mm^2 vl}{2(M + m)^2}$ (e) $\dfrac{M(M + 4m)L^2}{12(M + m)}$
    (f) $\dfrac{mv}{M + m}$, $\dfrac{6\, mv}{(M + 4m)L}$
62. $2\,\omega/3$

63. (a) $\frac{v_0}{2}$, v_0 (b) $\frac{2}{3}v_0$ along the initial motion of the rod

(c) $\frac{v_0}{2L}$

64. (a) $\frac{mL\sqrt{gh}}{\sqrt{2}}$, $\frac{\sqrt{8gh}}{3L}$ (b) $\frac{3}{2}L$

65. (a) 0·98 J (b) 1·4 m/s

66. 0·5 m/s

67. 5·4 rad/s

68. 41°

69. $0{\cdot}9\sqrt{2}\,dm\,g$

70. 50 m/s

71. $\frac{7}{10}mv^2$

72. $\frac{2}{3}g$

73. $\sqrt{10\,gh/7}$

74. $\frac{3v^2}{4g}$

75. $\sqrt{\frac{10}{7}g\,l\sin\theta}$

76. (a) $\frac{2}{5}\tan\theta$ (b) $\frac{7}{8}mgl\sin\theta$

77. $17\,mg/7$

78. $\sqrt{\frac{27}{7}g(R-r)}$

79. (a) $mg(H-R-R\sin\theta)$,

(b) $\frac{10}{7}g\left(\frac{H}{R}-1-\sin\theta\right)$, $-\frac{5}{7}g\cos\theta$

(c) 4·9 N, 0·196 N upward

80. $2R/3$ above the centre

81. $\omega R/3$

82. $3\,v/5$

83. $4\pi R/3$

84. 3·3 N

85. (a) $3\,v/5$ (b) $3\,v/7$

86. $3\,v/7$

□