Show that function f: $\mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$ is one-one and onto function.

Answer

It is given that f: $\mathbf{R} \to \{x \in \mathbf{R}: -1 < x < 1\}$ is defined as $f(x) = \frac{x}{1+|x|}$, $x \in \mathbf{R}$.

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$$\Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$$

It can be observed that if x is positive and y is negative, then we have:

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since x is positive and y is negative:

$x > y \Rightarrow x - y > 0$

But, $2xy$ is negative.

Then, $2xy \neq x - y$.

Thus, the case of x being positive and y being negative can be ruled out.

Under a similar argument, x being negative and y being positive can also be ruled out

$\therefore$ x and y have to be either positive or negative.

When x and y are both positive, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$$

When x and y are both negative, we have:

$$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - yx \Rightarrow x = y$$

$\therefore$ f is one-one.

Now, let $y \in \mathbf{R}$ such that $-1 < y < 1$.

If y is negative, then there exists $x = \frac{y}{1+y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y.$$

If y is positive, then there exists $x = \frac{y}{1-y} \in \mathbf{R}$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\left(\frac{y}{1-y}\right)\right|} = \frac{\frac{y}{1-y}}{1+\frac{y}{1-y}} = \frac{y}{1-y+y} = y.$$

$\therefore$ f is onto.

Hence, f is one-one and onto.

Question 5:

Show that the function f: $\mathbf{R} \to \mathbf{R}$ given by $f(x) = x^3$ is injective.

Answer

f: $\mathbf{R} \to \mathbf{R}$ is given as $f(x) = x^3$.

Suppose $f(x) = f(y)$, where $x, y \in \mathbf{R}$.

$\Rightarrow x^3 = y^3$ ... (1)

Now, we need to show that $x = y$.

Suppose $x \neq y$, their cubes will also not be equal.

$\Rightarrow x^3 \neq y^3$

However, this will be a contradiction to (1).

$\therefore$ $x = y$

Hence, f is injective.

Question 6:

Give examples of two functions f: $\mathbf{N} \to \mathbf{Z}$ and g: $\mathbf{Z} \to \mathbf{Z}$ such that $g \circ f$ is injective but g is not injective.

(Hint: Consider $f(x) = x$ and $g(x) = |x|$)

Answer

Define f: $\mathbf{N} \to \mathbf{Z}$ as $f(x) = x$ and g: $\mathbf{Z} \to \mathbf{Z}$ as $g(x) = |x|$.

We first show that g is not injective.

It can be observed that:

$g(-1) = |-1| = 1$

$g(1) = |1| = 1$

$\therefore g(-1) = g(1)$, but $-1 \neq 1$.

$\therefore g$ is not injective.

Now, gof: $\mathbf{N} \to \mathbf{Z}$ is defined as $gof(x) = g(f(x)) = g(x) = |x|$.

Let $x, y \in \mathbf{N}$ such that $gof(x) = gof(y)$.

$\Rightarrow |x| = |y|$

Since x and $y \in \mathbf{N}$, both are positive.

$\therefore |x| = |y| \Rightarrow x = y$

Hence, gof is injective

Question 7:

Given examples of two functions f: $\mathbf{N} \to \mathbf{N}$ and g: $\mathbf{N} \to \mathbf{N}$ such that gof is onto but f is not onto.

(Hint: Consider $f(x) = x + 1$ and $g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$

Answer

Define f: $\mathbf{N} \to \mathbf{N}$ by,

$f(x) = x + 1$

And, g: $\mathbf{N} \to \mathbf{N}$ by,

$$g(x) = \begin{cases} x-1 & \text{if } x > 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We first show that g is not onto.

For this, consider element 1 in co-domain $\mathbf{N}$. It is clear that this element is not an image of any of the elements in domain $\mathbf{N}$.

$\therefore f$ is not onto.

Now, gof: $\mathbf{N} \to \mathbf{N}$ is defined by,

$$\begin{aligned} gof(x) = g(f(x)) = g(x+1) &= (x+1) - 1 \qquad [x \in \mathbf{N} \Rightarrow (x+1) > 1] \\ &= x \end{aligned}$$

Then, it is clear that for $y \in \mathbf{N}$, there exists $x = y \in \mathbf{N}$ such that $gof(x) = y$.

Hence, gof is onto.

Question 8:

Given a non empty set X, consider P(X) which is the set of all subsets of X.

Define the relation R in P(X) as follows:

For subsets A, B in P(X), ARB if and only if $A \subset B$. Is R an equivalence relation on P(X)? Justify you answer:

Answer

Since every set is a subset of itself, ARA for all $A \in$ P(X).

$\therefore$R is reflexive.

Let $ARB \Rightarrow A \subset B$.

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A.

$\therefore$ R is not symmetric.

Further, if ARB and BRC, then $A \subset B$ and $B \subset C$.

$\Rightarrow A \subset C$

$\Rightarrow ARC$

$\therefore$ R is transitive.

Hence, R is not an equivalence relation since it is not symmetric.

Question 9:

Given a non-empty set X, consider the binary operation $*$: P(X) × P(X) → P(X) given by $A * B = A \cap B$ □ A, B in P(X) is the power set of X. Show that X is the identity element for this operation and X is the only invertible element in P(X) with respect to the operation$*$.

Answer

It is given that $*: \mathrm{P}(X) \times \mathrm{P}(X) \to \mathrm{P}(X)$ is defined as $A * B = A \cap B \ \forall\ A, B \in \mathrm{P}(X)$.

We know that $A \cap X = A = X \cap A \ \forall\ A \in \mathrm{P}(X)$.

$\Rightarrow A * X = A = X * A \ \forall\ A \in \mathrm{P}(X)$

Thus, X is the identity element for the given binary operation $*$.

Thus, X is the identity element for the given binary operation *.

Now, an element $A \in P(X)$ is invertible if there exists $B \in P(X)$ such that

$A * B = X = B * A$. (As X is the identity element)

i.e.,

$A \cap B = X = B \cap A$

This case is possible only when $A = X = B$.

Thus, X is the only invertible element in P(X) with respect to the given operation*.

Hence, the given result is proved.

Question 10:

Find the number of all onto functions from the set {1, 2, 3, ... , n) to itself.

Answer

Onto functions from the set {1, 2, 3, ... ,n} to itself is simply a permutation on n symbols 1, 2, ..., n.

Thus, the total number of onto maps from {1, 2, ... , n} to itself is the same as the total number of permutations on n symbols 1, 2, ..., n, which is n.

Question 11:

Let $S = \{a, b, c\}$ and $T = \{1, 2, 3\}$. Find F^{-1} of the following functions F from S to T, if it exists.

(i) $F = \{(a, 3), (b, 2), (c, 1)\}$ (ii) $F = \{(a, 2), (b, 1), (c, 1)\}$

Answer

$S = \{a, b, c\}$, $T = \{1, 2, 3\}$

(i) F: $S \rightarrow T$ is defined as:

F = {(a, 3), (b, 2), (c, 1)}

⇒ F (a) = 3, F (b) = 2, F(c) = 1

Therefore, F^{-1}: $T \rightarrow S$ is given by

$F^{-1} = \{(3, a), (2, b), (1, c)\}$.

(ii) F: $S \rightarrow T$ is defined as:

F = {(a, 2), (b, 1), (c, 1)}

Since F (b) = F (c) = 1, F is not one-one.

Hence, F is not invertible i.e., F^{-1} does not exist.

Question 12:

Consider the binary operations*: **R** ×**R** → and o: **R** × **R** → **R** defined as $a * b = |a - b|$ and a o b = a, □a, b ∈ **R**. Show that * is commutative but not associative, o is associative but not commutative. Further, show that □a, b, c ∈ **R**, a*(b o c) = (a * b) o (a * c). [If it is so, we say that the operation * distributes over the operation o]. Does o distribute over *? Justify your answer.

*********** END **********