Algebra of Matrices Ex 5.3 Q60

Given,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

To prove, $A^n = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$, we will use the principle of mathematical induction.

Step 1: Put $n = 1$

$$A^1 = \begin{bmatrix} 1 & 1 & \frac{1(1+1)}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

So, A^n is true for $n = 1$

Step 2: Let, A^n be true for $n = k$, so,

$$A^k = \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3: We will prove that A^n be true for $n = k+1$

Now,

$$A^{k+1} = A^k \times A$$

$$= \begin{bmatrix} 1 & k & \frac{k(k+1)}{2} \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{\text{using equation (i) and given}\}$$

$$= \begin{bmatrix} 1+0+0 & 1+k+0 & 1+k+\frac{k(k+1)}{2} \\ 0+0+0 & 0+1+0 & 0+1+k \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (k+1) & \frac{(k+1)(k+2)}{2} \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, A^n is true for $n = k+1$ whenever it is true for $n = k$.

So, by principle of mathematical induction A^n is true for all positive integer n.

Algebra of Matrices Ex 5.3 Q61

We will prove P(n): $A^{n+1} = B^n\left[B + (n+1)C\right]$ is true for all natural numbers using mathematical induction.

Given,

$$A = B + C,\ BC = CB,\ C^2 = 0$$
$$A = B + C$$

Squaring both the sides, so

$$A^2 = (B+C)^2$$
$$\Rightarrow\quad A^2 = (B+C)(B+C)$$
$$\Rightarrow\quad A^2 = B \times B + BC + CB + C \times C \qquad \{\text{using distributive property}\}$$
$$\Rightarrow\quad A^2 = B^2 + BC + BC + C^2 \qquad \{\text{using } BC = CB \text{ given}\}$$
$$\Rightarrow\quad A^2 = B^2 + 2BC + 0 \qquad \left\{\text{since, given } C^2 = 0\right\}$$
$$\Rightarrow\quad A^2 = B^2 + 2BC \qquad ---(1)$$
$$A^2 = B(B + 2C)$$

Now, consider

$$P(n):\ A^{n+1} = B^n\left[B + (n+1)C\right]$$

Step 1: Tp prove P(1) is true, put $n = 1$

$$A^{1+1} = B^1\left[B + (1+1)C\right]$$
$$A^2 = B\left[B + 2C\right]$$
$$A^2 = B^2 + 2BC$$

From equation (i), P(1) is true.

Step 2: Suppose P(k) is true.

$$\therefore\ A^{k+1} = B^k\left[B + (k+1)C\right] \qquad ---(2)$$

Step 3: Now, we have to show that P(k+1) is true.

That is we need to prove that,

$$A^{k+2} = B^{k+1}\left[B + (k+2)C\right]$$

Now,

$$A^{k+2} = A^k \times A^2$$
$$= B^{(k-1)}\left[B + kC\right] \times \left[B(B + 2C)\right]$$
$$= B^k\left[B + kC\right] \times \left[B + 2C\right]$$
$$= B^k\left[B \times B + B \times 2C + kC \times B + 2kC^2\right]$$
$$= B^k\left[B^2 + 2BC + kBC + 2k \times 0\right] \qquad \left\{\text{since } BC = CB,\ C^2 = 0\right\}$$
$$= B^k\left[B^2 + BC(2+k)\right]$$
$$= B^k \times B\left[B + (k+2)C\right]$$
$$= B^{k+1}\left[B + (k+2)C\right]$$

So, $\quad P(n)$ is true for $n = k+1$ whenever $P(n)$ is true for $n = k$

Therefore by principle of mathematical induction $P(n)$ is true for all natural number.

*********** END **********