Mathematical Induction Ex 12.2 Q33

$P(n)$: $\frac{n^{11}}{11}+\frac{n^5}{5}+\frac{n^3}{3}+\frac{62}{165}n$ is a positive integer

For $n = 1$

$$\frac{1}{11}+\frac{1}{5}+\frac{1}{3}+\frac{62}{165}$$
$$=\frac{15+33+55+62}{165}$$
$$=\frac{165}{165}$$

Which is a positive integer

Let $P(n)$ is true for $n = k$, so

$$\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165} \text{ is a positive integer}$$
$$\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165}=\lambda \qquad \text{---(i)}$$

For $n = k+1$

$$\frac{(k+1)^{11}}{11}+\frac{(k+1)^5}{5}+\frac{(k+1)^3}{3}+\frac{62}{165}(k+1)$$
$$=\frac{1}{11}\left[k^{11}+11k^{10}+55k^9+165k^8+330k^7+462k^6+462k^5+330k^4+165k^3+55k^2+11k+1\right]$$
$$+\frac{1}{5}\left[k^5+5k^4+10k^3+10k^2+5k+1\right]+\frac{1}{3}\left[k^3+3k^2+3k+1\right]+\frac{62}{165}[k+1]$$
$$=\left[\frac{k^{11}}{11}+\frac{k^5}{5}+\frac{k^3}{3}+\frac{62k}{165}\right]+k^{10}+5k^9+15k^8+30k^7+42k^6+42k^5+30k^4+15k^3+5k^2+1+\frac{1}{11}$$
$$+k^4+2k^3+2k^2+k+\frac{1}{5}+k^2+k+\frac{1}{3}+\frac{62}{165}$$
$$=\lambda+k^{10}+5k^9+15k^8+30k^7+42k^6+42k^5+31k^4+17k^3+8k^2+2k+1$$
$$= \text{An integer}$$

$\Rightarrow$ $P(n)$ is true for $n = k+1$

$\Rightarrow$ $P(n)$ is true for all $n \in N$ by PMI

Mathematical Induction Ex 12.2 Q34

Let $P(n): \frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right)=\frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)-\cot x$

For $n=1$

$\frac{1}{2}\tan\frac{x}{2}=\frac{1}{2}\cot\left(\frac{x}{2}\right)-\cot x$

$$=\frac{1}{2}\frac{1}{\tan\frac{x}{2}}-\frac{1}{\tan x}$$

$$=\frac{1}{2\tan\frac{x}{2}}-\frac{1}{\left(\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}\right)}$$

$$=\frac{1}{2\tan\frac{x}{2}}-\frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{1-1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

$$=\frac{1}{2}\tan\frac{x}{2}$$

$\Rightarrow$ $P(n)$ is true for $n=1$

Let $P(n)$ is true for $n=k$, so

$$\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x \qquad ---(1)$$

We have to show that,

$$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)=\frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right)-\cot x$$

Now,

$$\left\{\frac{1}{2}\tan\frac{x}{2}+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\ldots+\frac{1}{2^k}\tan\left(\frac{x}{2^k}\right)\right\}+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x+\frac{1}{2^{k+1}}\tan\left(\frac{x}{2^{k+1}}\right)$$

$$=\frac{1}{2^k}\cot\left(\frac{x}{2^k}\right)-\cot x+\frac{1}{2.2^k}\frac{1}{\cot\left(\frac{x}{2^k}\cdot\frac{1}{2}\right)}$$

$$=\frac{1}{2^k}\left[\frac{1}{\tan\left(\frac{x}{2^k}\right)}+\frac{1}{2}.\tan\left\{\left(\frac{x}{2^k}\right).\frac{1}{2}\right\}\right]-\cot x$$

$$=\frac{1}{2^k}\left[\frac{1-\tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)}+\frac{1}{2}\tan\left(\frac{x}{2.2^k}\right)\right]-\cot x$$

$$=\frac{1}{2^k}\left[\frac{1-\tan^2\left(\frac{x}{2^{k+1}}\right)+\tan^2\left(\frac{x}{2^{k+1}}\right)}{2\tan\left(\frac{x}{2^{k+1}}\right)}\right]-\cot x$$

$$=\frac{1}{2^{k+1}}\left[\frac{1}{\tan\left(\frac{x}{2^{k+1}}\right)}\right]-\cot x$$

$$=\frac{1}{2^{k+1}}\cot\left(\frac{x}{2^{k+1}}\right)-\cot x$$

$\Rightarrow$ $P(n)$ is true for $n=k+1$

$\Rightarrow$ $P(n)$ is true for all $n\in N$ by PMI

Mathematical Induction Ex 12.2 Q35

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots\left(1-\frac{1}{n^2}\right)$$

Above can be written as

$$=\left(\frac{2^2-1}{2^2}\right)\left(\frac{3^2-1}{3^2}\right)\left(\frac{4^2-1}{4^2}\right)\ldots\left(\frac{n^2-1}{n^2}\right)$$

$$=\left(\frac{(2+1)(2-1)}{2^2}\right)\left(\frac{(3+1)(3-1)}{3^2}\right)$$

$$\left(\frac{(4+1)(4-1)}{4^2}\right)\ldots\ldots\left(\frac{(n+1)(n-1)}{n^2}\right)$$

$$=\left(\frac{3\cdot 1}{2^2}\right)\left(\frac{4\cdot 2}{3^2}\right)\left(\frac{5\cdot 3}{4^2}\right)\ldots\ldots\left(\frac{(n+1)\cdot(n-1)}{n^2}\right)$$

In the above product, there are two series in numerator

$3\cdot 4\cdot 5\ldots\ldots(n+1)\,and\,1\cdot 2\cdot 3\ldots\ldots(n-1)$

All numbers from 3 to (n-1) are repeated twice

and 1, 2, n are appeared once in numerator

So after cancelling like terms we get

$$=\frac{(n+1)}{2n}$$

*********** END **********