Question 19:

The area bounded by the y-axis, $y = \cos x$ and $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$

A. $2\left(\sqrt{2}-1\right)$

B. $\sqrt{2}-1$

C. $\sqrt{2}+1$

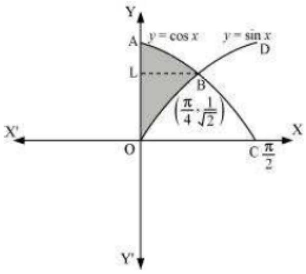
D. $\sqrt{2}$

Answer

The given equations are

$y = \cos x$ ... (1)

And, $y = \sin x$ ... (2)



Required area = Area (ABLA) + area (OBLO)

$$= \int_{\frac{1}{\sqrt{2}}}^{1} x\,dy + \int_{0}^{\frac{1}{\sqrt{2}}} x\,dy$$

$$= \int_{\frac{1}{\sqrt{2}}}^{1} \cos^{-1} y\,dy + \int_{0}^{\frac{1}{\sqrt{2}}} \sin^{-1} x\,dy$$

Integrating by parts, we obtain

$$= \left[y\cos^{-1} y - \sqrt{1-y^2}\right]_{\frac{1}{\sqrt{2}}}^{1} + \left[x\sin^{-1} x + \sqrt{1-x^2}\right]_{0}^{\frac{1}{\sqrt{2}}}$$

$$= \left[\cos^{-1}(1) - \frac{1}{\sqrt{2}}\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}}\right] + \left[\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sqrt{1-\frac{1}{2}} - 1\right]$$

$$= \frac{-\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} - 1$$

$$= \frac{2}{\sqrt{2}} - 1$$

$$= \sqrt{2} - 1 \text{ units}$$

Thus, the correct answer is B.

$$\text{Put } 2x = t \Rightarrow dx = \frac{dt}{2}$$

$$\text{When } x = \frac{3}{2},\ t = 3 \text{ and when } x = \frac{1}{2},\ t = 1$$

$$= \int_{0}^{\frac{1}{2}} 2\sqrt{x}\,dx + \frac{1}{4}\int_{1}^{3} \sqrt{(3)^2 - (t)^2}\,dt$$

$$= 2\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_{0}^{\frac{1}{2}} + \frac{1}{4}\left[\frac{t}{2}\sqrt{9-t^2} + \frac{9}{2}\sin^{-1}\left(\frac{t}{3}\right)\right]_{1}^{3}$$

$$= 2\left[\frac{2}{3}\left(\frac{1}{2}\right)^{\frac{3}{2}}\right] + \frac{1}{4}\left[\left\{\frac{3}{2}\sqrt{9-(3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{3}{3}\right)\right\} - \left\{\frac{1}{2}\sqrt{9-(1)^2} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{2}{3\sqrt{2}} + \frac{1}{4}\left[\left\{0 + \frac{9}{2}\sin^{-1}(1)\right\} - \left\{\frac{1}{2}\sqrt{8} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right\}\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{1}{4}\left[\frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= \frac{\sqrt{2}}{3} + \frac{9\pi}{16} - \frac{\sqrt{2}}{4} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right)$$

$$= \frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}$$

Therefore, the required area is $\left[2\times\left(\frac{9\pi}{16} - \frac{9}{8}\sin^{-1}\left(\frac{1}{3}\right) + \frac{\sqrt{2}}{12}\right)\right] = \frac{9\pi}{8} - \frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right) + \frac{1}{3\sqrt{2}}$ units

*********** END **********