



Question 6:

Prove $\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$

Answer

Let $\sin^{-1} \frac{3}{5} = x$. Then, $\sin x = \frac{3}{5} \Rightarrow \cos x = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$.

$\therefore \tan x = \frac{3}{4} \Rightarrow x = \tan^{-1} \frac{3}{4}$

$\therefore \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{3}{4} \quad \dots(1)$

Now, let $\cos^{-1} \frac{12}{13} = y$. Then, $\cos y = \frac{12}{13} \Rightarrow \sin y = \frac{5}{13}$.

$\therefore \tan y = \frac{5}{12} \Rightarrow y = \tan^{-1} \frac{5}{12}$

$\therefore \cos^{-1} \frac{12}{13} = \tan^{-1} \frac{5}{12} \quad \dots(2)$

Let $\sin^{-1} \frac{56}{65} = z$. Then, $\sin z = \frac{56}{65} \Rightarrow \cos z = \frac{33}{65}$.

$\therefore \tan z = \frac{56}{33} \Rightarrow z = \tan^{-1} \frac{56}{33}$

$\therefore \sin^{-1} \frac{56}{65} = \tan^{-1} \frac{56}{33} \quad \dots(3)$

Now, we have:

$$\begin{aligned} \text{L.H.S.} &= \cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{3}{4} \quad \left[\text{Using (1) and (2)} \right] \\ &= \tan^{-1} \frac{\frac{5}{12} + \frac{3}{4}}{1 - \frac{5}{12} \times \frac{3}{4}} \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan^{-1} \frac{20+36}{48-15} \\ &= \tan^{-1} \frac{56}{33} \\ &= \sin^{-1} \frac{56}{65} = \text{R.H.S.} \quad \left[\text{Using (3)} \right] \end{aligned}$$

Question 7:

Prove $\tan^{-1} \frac{63}{16} = \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5}$

Answer

Let $\sin^{-1} \frac{5}{13} = x$. Then, $\sin x = \frac{5}{13} \Rightarrow \cos x = \frac{12}{13}$.

$\therefore \tan x = \frac{5}{12} \Rightarrow x = \tan^{-1} \frac{5}{12}$

$\therefore \sin^{-1} \frac{5}{13} = \tan^{-1} \frac{5}{12} \quad \dots(1)$

Let $\cos^{-1} \frac{3}{5} = y$. Then, $\cos y = \frac{3}{5} \Rightarrow \sin y = \frac{4}{5}$.

$\therefore \tan y = \frac{4}{3} \Rightarrow y = \tan^{-1} \frac{4}{3}$

$\therefore \cos^{-1} \frac{3}{5} = \tan^{-1} \frac{4}{3} \quad \dots(2)$

Using (1) and (2), we have

$$\begin{aligned} \text{R.H.S.} &= \sin^{-1} \frac{5}{13} + \cos^{-1} \frac{3}{5} \\ &= \tan^{-1} \frac{5}{12} + \tan^{-1} \frac{4}{3} \\ &= \tan^{-1} \left(\frac{\frac{5}{12} + \frac{4}{3}}{1 - \frac{5}{12} \times \frac{4}{3}} \right) \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan^{-1} \left(\frac{15+48}{36-20} \right) \end{aligned}$$

$$= \tan^{-1} \frac{63}{16}$$

$$= \text{L.H.S.}$$

Question 8:

Prove $\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} = \frac{\pi}{4}$

Answer

$$\begin{aligned} \text{L.H.S.} &= \tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{8} \\ &= \tan^{-1} \left(\frac{\frac{1}{5} + \frac{1}{7}}{1 - \frac{1}{5} \times \frac{1}{7}} \right) + \tan^{-1} \left(\frac{\frac{1}{3} + \frac{1}{8}}{1 - \frac{1}{3} \times \frac{1}{8}} \right) \quad \left[\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \right] \\ &= \tan^{-1} \left(\frac{7+5}{35-1} \right) + \tan^{-1} \left(\frac{8+3}{24-1} \right) \\ &= \tan^{-1} \frac{12}{34} + \tan^{-1} \frac{11}{23} \\ &= \tan^{-1} \frac{6}{17} + \tan^{-1} \frac{11}{23} \\ &= \tan^{-1} \left(\frac{\frac{6}{17} + \frac{11}{23}}{1 - \frac{6}{17} \times \frac{11}{23}} \right) \\ &= \tan^{-1} \left(\frac{138+187}{391-66} \right) \\ &= \tan^{-1} \left(\frac{325}{325} \right) = \tan^{-1} 1 \\ &= \frac{\pi}{4} = \text{R.H.S.} \end{aligned}$$

Question 9:

Prove $\tan^{-1} \sqrt{x} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right), x \in [0, 1]$

Answer

Let $x = \tan^2 \theta$. Then, $\sqrt{x} = \tan \theta \Rightarrow \theta = \tan^{-1} \sqrt{x}$.

$$\therefore \frac{1-x}{1+x} = \frac{1-\tan^2 \theta}{1+\tan^2 \theta} = \cos 2\theta$$

Now, we have:

$$\text{R.H.S.} = \frac{1}{2} \cos^{-1} \left(\frac{1-x}{1+x} \right) = \frac{1}{2} \cos^{-1} (\cos 2\theta) = \frac{1}{2} \times 2\theta = \theta = \tan^{-1} \sqrt{x} = \text{L.H.S.}$$

Question 10:

Prove $\cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4} \right)$

Answer

$$\begin{aligned} \text{Consider } & \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \\ &= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2} \quad (\text{by rationalizing}) \\ &= \frac{(1+\sin x) + (1-\sin x) + 2\sqrt{(1+\sin x)(1-\sin x)}}{1+\sin x - 1+\sin x} \\ &= \frac{2(1+\sqrt{1-\sin^2 x})}{2\sin x} = \frac{1+\cos x}{\sin x} = \frac{2\cos^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}} \\ &= \cot \frac{x}{2} \end{aligned}$$

$$\therefore \text{L.H.S.} = \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right) = \cot^{-1} \left(\cot \frac{x}{2} \right) = \frac{x}{2} = \text{R.H.S.}$$

Question 11:

Prove $\tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$

[Hint: put $x = \cos 2\theta$]

Answer

Put $x = \cos 2\theta$ so that $\theta = \frac{1}{2} \cos^{-1} x$. Then, we have:

$$\begin{aligned} \text{L.H.S.} &= \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right) \\ &= \tan^{-1} \left(\frac{\sqrt{1+\cos 2\theta} - \sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta} + \sqrt{1-\cos 2\theta}} \right) \end{aligned}$$

$$\begin{aligned}
&= \tan^{-1} \left(\frac{\sqrt{2} \cos^2 \theta - \sqrt{2} \sin^2 \theta}{\sqrt{2} \cos^2 \theta + \sqrt{2} \sin^2 \theta} \right) \\
&= \tan^{-1} \left(\frac{\sqrt{2} \cos \theta - \sqrt{2} \sin \theta}{\sqrt{2} \cos \theta + \sqrt{2} \sin \theta} \right) \\
&= \tan^{-1} \left(\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right) = \tan^{-1} \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) \\
&= \tan^{-1} 1 - \tan^{-1} (\tan \theta) \qquad \left[\tan^{-1} \left(\frac{x-y}{1+xy} \right) = \tan^{-1} x - \tan^{-1} y \right] \\
&= \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x = \text{R.H.S.}
\end{aligned}$$

Question 12:

Prove $\frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3}$

Answer

$$\begin{aligned}
\text{L.H.S.} &= \frac{9\pi}{8} - \frac{9}{4} \sin^{-1} \frac{1}{3} \\
&= \frac{9}{4} \left(\frac{\pi}{2} - \sin^{-1} \frac{1}{3} \right) \\
&= \frac{9}{4} \left(\cos^{-1} \frac{1}{3} \right) \qquad \dots (1) \quad \left[\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right]
\end{aligned}$$

Now, let $\cos^{-1} \frac{1}{3} = x$. Then, $\cos x = \frac{1}{3} \Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{3}\right)^2} = \frac{2\sqrt{2}}{3}$.

$$\therefore x = \sin^{-1} \frac{2\sqrt{2}}{3} \Rightarrow \cos^{-1} \frac{1}{3} = \sin^{-1} \frac{2\sqrt{2}}{3}$$

$$\therefore \text{L.H.S.} = \frac{9}{4} \sin^{-1} \frac{2\sqrt{2}}{3} = \text{R.H.S.}$$

Question 13:

Solve $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$

Answer

$$\begin{aligned}
2 \tan^{-1} (\cos x) &= \tan^{-1} (2 \operatorname{cosec} x) \\
\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) &= \tan^{-1} (2 \operatorname{cosec} x) \qquad \left[2 \tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2} \right] \\
\Rightarrow \frac{2 \cos x}{1 - \cos^2 x} &= 2 \operatorname{cosec} x \\
\Rightarrow \frac{2 \cos x}{\sin^2 x} &= \frac{2}{\sin x} \\
\Rightarrow \cos x &= \sin x \\
\Rightarrow \tan x &= 1 \\
\therefore x &= \frac{\pi}{4}
\end{aligned}$$

Question 14:

Solve $\tan^{-1} \frac{1-x}{1+x} = \frac{1}{2} \tan^{-1} x, (x > 0)$

Answer

$$\begin{aligned}
\tan^{-1} \frac{1-x}{1+x} &= \frac{1}{2} \tan^{-1} x \\
\Rightarrow \tan^{-1} 1 - \tan^{-1} x &= \frac{1}{2} \tan^{-1} x \qquad \left[\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy} \right] \\
\Rightarrow \frac{\pi}{4} &= \frac{3}{2} \tan^{-1} x \\
\Rightarrow \tan^{-1} x &= \frac{\pi}{6} \\
\Rightarrow x &= \tan \frac{\pi}{6} \\
\therefore x &= \frac{1}{\sqrt{3}}
\end{aligned}$$

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