EXERCISE 4.1

Question 1:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+3+3^2+\ldots+3^{n-1}=\frac{(3^n-1)}{2}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): 1 + 3 + 3^2 + \ldots + 3^{n-1} = \frac{(3^n-1)}{2}$$

For $n = 1$, we have

$$P(1): 1 = \frac{(3^1-1)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$1+3+3^2+\ldots+3^{k-1} = \frac{(3^k-1)}{2} \qquad \ldots(i)$$

We shall now prove that P(k + 1) is true.

Consider

$$1 + 3 + 3^2 + \ldots + 3^{k-1} + 3^{(k+1)-1}$$

$$= (1 + 3 + 3^2 + \ldots + 3^{k-1}) + 3^k$$

$$= \frac{(3^k - 1)}{2} + 3^k \qquad [\text{Using (i)}]$$

$$= \frac{(3^k - 1) + 2.3^k}{2}$$

$$= \frac{(1+2)3^k - 1}{2}$$

$$= \frac{3.3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 2:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Ans :

Let the given statement be P(n), i.e.,

$$\text{P}(n)\text{: } 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

For $n = 1$, we have

$$\text{P}(1)\text{: } 1^3 = 1 = \left(\frac{1(1+1)}{2} \right)^2 = \left(\frac{1.2}{2} \right)^2 = 1^2 = 1, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left(\frac{k(k+1)}{2} \right)^2 \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k + 1)^3$$

$$= (1^3 + 2^3 + 3^3 + \ldots + k^3) + (k + 1)^3$$

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \qquad [\text{Using (i)}]$$

$$=\frac{k^2(k+1)^2}{4}+(k+1)^3$$

$$=\frac{k^2(k+1)^2+4(k+1)^3}{4}$$

$$=\frac{(k+1)^2\{k^2+4(k+1)\}}{4}$$

$$=\frac{(k+1)^2\{k^2+4k+4\}}{4}$$

$$=\frac{(k+1)^2(k+2)^2}{4}$$

$$=\frac{(k+1)^2(k+1+1)^2}{4}$$

$$=\left(\frac{(k+1)(k+1+1)}{2}\right)^2$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 3:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+\frac{1}{(1+2)}+\frac{1}{(1+2+3)}+\ldots+\frac{1}{(1+2+3+\ldots n)}=\frac{2n}{(n+1)}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n):\ 1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots n}=\frac{2n}{n+1}$$

For n = 1, we have

$$P(1):\ 1=\frac{2.1}{1+1}=\frac{2}{2}=1 \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$1+\frac{1}{1+2}+\ldots+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}=\frac{2k}{k+1} \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots+k}+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+\ldots+\frac{1}{1+2+3+\ldots k}\right)+\frac{1}{1+2+3+\ldots+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+\ldots+k+(k+1)} \qquad [\text{Using (i)}]$$

$$= \frac{2k}{k+1} + \frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)} \qquad \left[1+2+3+\ldots+n = \frac{n(n+1)}{2}\right]$$

$$= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)}$$

$$= \frac{2}{(k+1)}\left(k + \frac{1}{k+2}\right)$$

$$= \frac{2}{k+1}\left(\frac{k(k+2)+1}{k+2}\right)$$

$$= \frac{2}{(k+1)}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{2\cdot(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{2(k+1)}{(k+2)}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 4:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): 1.2.3 + 2.3.4 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

For n = 1, we have

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$$

We shall now prove that P(k + 1) is true.

Consider

$$1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$$

$$= \{1.2.3 + 2.3.4 + \ldots + k(k+1)(k+2)\} + (k+1)(k+2)(k+3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \qquad [\text{Using (i)}]$$

$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$

$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$

$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 5:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$1.3+2.3^2+3.3^3+\ldots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): 1.3+2.3^2+3.3^3+\ldots+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

For n = 1, we have

$$P(1): 1.3=3=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.3+2.3^2+3.3^3+\ldots+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$1.3 + 2.3^2 + 3.3^3 + \ldots + k3^k + (k + 1)\ 3^{k+1}$

$= (1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k) + (k + 1)\ 3^{k+1}$

$=\frac{(2k-1)3^{k+1}+3}{4}+(k+1)3^{k+1}$ [Using (i)]

$=\frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$

$=\frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$

$=\frac{3^{k+1}\{6k+3\}+3}{4}$

$=\frac{3^{k+1}.3\{2k+1\}+3}{4}$

$=\frac{3^{(k+1)+1}\{2k+1\}+3}{4}$

$=\frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 6:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$

Ans :

Let the given statement be P(n), i.e.,

P(n): $1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$

For n = 1, we have

P(1): $1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.2+2.3+3.4+\ldots\ldots+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \quad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$1.2 + 2.3 + 3.4 + \ldots + k.(k + 1) + (k + 1).(k + 2)$

$= [1.2 + 2.3 + 3.4 + \ldots + k.(k + 1)] + (k + 1).(k + 2)$

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \text{[Using (i)]}$$

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 7:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+3.5+5.7+\ldots+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n):\ 1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

For n = 1, we have

$$P(1):1.3=3=\frac{1(4.1^2+6.1-1)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.3+3.5+5.7+......+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \quad ...\text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$(1.3 + 3.5 + 5.7 + \ldots + (2k - 1)(2k + 1) + \{2(k + 1) - 1\}\{2(k + 1) + 1\}$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+2-1)(2k+2+1) \qquad [\text{Using (i)}]$$
$$=\frac{k(4k^2+6k-1)}{3}+(2k+1)(2k+3)$$
$$=\frac{k(4k^2+6k-1)}{3}+(4k^2+8k+3)$$
$$=\frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$$
$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$
$$=\frac{4k^3+18k^2+23k+9}{3}$$
$$=\frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$
$$=\frac{k(4k^2+14k+9)+1(4k^2+14k+9)}{3}$$
$$=\frac{(k+1)(4k^2+14k+9)}{3}$$
$$=\frac{(k+1)\{4k^2+8k+4+6k+6-1\}}{3}$$
$$=\frac{(k+1)\{4(k^2+2k+1)+6(k+1)-1\}}{3}$$
$$=\frac{(k+1)\{4(k+1)^2+6(k+1)-1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 8:

Prove the following by using the principle of mathematical induction for all $n \in N$: $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\ 2^{n+1} + 2$

Ans :

Let the given statement be P(n), i.e.,

P(n): $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\, 2^{n+1} + 2$

For $n = 1$, we have

P(1): $1.2 = 2 = (1 - 1)\, 2^{1+1} + 2 = 0 + 2 = 2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k = (k - 1)\, 2^{k+1} + 2 \ldots$ (i)

We shall now prove that P(k + 1) is true.

Consider

$$\begin{aligned}
&\left\{1.2+2.2^2+3.2^3+\ldots+k.2^k\right\}+(k+1)\cdot 2^{k+1}\\
&=(k-1)2^{k+1}+2+(k+1)2^{k+1}\\
&=2^{k+1}\left\{(k-1)+(k+1)\right\}+2\\
&=2^{k+1}.2k+2\\
&=k.2^{(k+1)+1}+2\\
&=\left\{(k+1)-1\right\}2^{(k+1)+1}+2
\end{aligned}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 9:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}=1-\frac{1}{2^n}$$

For $n = 1$, we have

$$P(1): \frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^k}=1-\frac{1}{2^k} \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} \qquad [\text{Using (i)}]$$

$$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$$

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$

$$=1-\frac{1}{2^{k+1}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 10:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): \frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

For n = 1, we have

$$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \quad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+\ldots\ldots+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)} \quad \text{[Using (i)]}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$=\frac{(k+1)}{6k+10}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 11:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Ans :

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$

For n = 1, we have

$P(1):\frac{1}{1\cdot2\cdot3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot4}{4\cdot2\cdot3}=\frac{1}{1\cdot2\cdot3}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+...+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \quad ...(i)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+.....+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \quad \text{[Using (i)]}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+6k+9)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+2k+1)+4(k^2+2k+1)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 12:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$a+ar+ar^2+...+ar^{n-1}=\frac{a(r^n-1)}{r-1}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n): a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

For n = 1, we have

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad ... \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{a + ar + ar^2 + ... + ar^{k-1}\right\} + ar^{(k+1)-1}$$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k \quad \left[\text{Using (i)}\right]$$

Let the given statement be P(n), i.e.,

$$P(n): a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

For n = 1, we have

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad ... \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{a + ar + ar^2 + ... + ar^{k-1}\right\} + ar^{(k+1)-1}$$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k \quad \left[\text{Using (i)}\right]$$

Question 13:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)...\left(1+\frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

For $n = 1$, we have

$$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) \qquad [\text{Using}(1)]$$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

$$=(k+1)^2+2(k+1)+1$$

$$=\{(k+1)+1\}^2$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 14:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{n}\right)=(n+1)$$

For $n = 1$, we have

$$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{k}\right)=(k+1) \qquad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right) \qquad [\text{Using } (1)]$$

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

$$=(k+1)+1$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 15:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Ans :

Let the given statement be P(n), i.e.,

$$P(n)=1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

For $n=1$, we have

$$P(1)=1^2=1=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=1^2+3^2+5^2+...+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \qquad ...(1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{1^2+3^2+5^2+...+(2k-1)^2\right\}+\left\{2(k+1)-1\right\}^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+2-1)^2 \qquad [\text{Using } (1)]$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)\{k(2k-1)+3(2k+1)\}}{3}$$

$$=\frac{(2k+1)\{2k^2-k+6k+3\}}{3}$$

$$=\frac{(2k+1)\{2k^2+5k+3\}}{3}$$

$$=\frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$$

$$=\frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3}$$

$$=\frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

********** END *********