



Exercise 10.3

Question 1:

Find the angle between two vectors \vec{a} and \vec{b} with magnitudes $\sqrt{3}$ and 2, respectively having $\vec{a} \cdot \vec{b} = \sqrt{6}$.

Answer

It is given that,

$$|\vec{a}| = \sqrt{3}, |\vec{b}| = 2 \text{ and } \vec{a} \cdot \vec{b} = \sqrt{6}$$

Now, we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6} = \sqrt{3} \times 2 \times \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\sqrt{6}}{\sqrt{3} \times 2}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Hence, the angle between the given vectors \vec{a} and \vec{b} is $\frac{\pi}{4}$.

Question 2:

Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$

Answer

The given vectors are $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + \hat{k}$.

$$|\vec{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1+4+9} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\begin{aligned} \text{Now, } \vec{a} \cdot \vec{b} &= (\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (3\hat{i} - 2\hat{j} + \hat{k}) \\ &= 1.3 + (-2)(-2) + 3.1 \\ &= 3 + 4 + 3 \\ &= 10 \end{aligned}$$

Also, we know that $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

$$\therefore 10 = \sqrt{14}\sqrt{14}\cos\theta$$

$$\Rightarrow \cos\theta = \frac{10}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

Question 3:

Find the projection of the vector $\hat{i} - \hat{j}$ on the vector $\hat{i} + \hat{j}$.

Answer

Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.

Now, projection of vector \vec{a} on \vec{b} is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1 + (-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$

Hence, the projection of vector \vec{a} on \vec{b} is 0.

Question 4:

Find the projection of the vector $\hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $7\hat{i} - \hat{j} + 8\hat{k}$.

Answer

Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$.

Now, projection of vector \vec{a} on \vec{b} is given by,

$$\frac{1}{|\vec{b}|}(\vec{a} \cdot \vec{b}) = \frac{1}{\sqrt{7^2 + (-1)^2 + 8^2}} \{1(7) + 3(-1) + 7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$

Question 5:

Show that each of the given three vectors is a unit vector:

$$\frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}), \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}), \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k})$$

Also, show that they are mutually perpendicular to each other.

Answer

$$\text{Let } \vec{a} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 6\hat{k}) = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k},$$

$$\vec{b} = \frac{1}{7}(3\hat{i} - 6\hat{j} + 2\hat{k}) = \frac{3}{7}\hat{i} - \frac{6}{7}\hat{j} + \frac{2}{7}\hat{k},$$

$$\vec{c} = \frac{1}{7}(6\hat{i} + 2\hat{j} - 3\hat{k}) = \frac{6}{7}\hat{i} + \frac{2}{7}\hat{j} - \frac{3}{7}\hat{k}.$$

$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49} + \frac{9}{49} + \frac{36}{49}} = 1$$

$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2 + \left(-\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49} + \frac{36}{49} + \frac{4}{49}} = 1$$

$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49} + \frac{4}{49} + \frac{9}{49}} = 1$$

Thus, each of the given three vectors is a unit vector.

$$\vec{a} \cdot \vec{b} = \frac{2}{7} \times \frac{3}{7} + \frac{3}{7} \times \left(-\frac{6}{7}\right) + \frac{6}{7} \times \frac{2}{7} = \frac{6}{49} - \frac{18}{49} + \frac{12}{49} = 0$$

$$\vec{b} \cdot \vec{c} = \frac{3}{7} \times \frac{6}{7} + \left(-\frac{6}{7}\right) \times \frac{2}{7} + \frac{2}{7} \times \left(-\frac{3}{7}\right) = \frac{18}{49} - \frac{12}{49} - \frac{6}{49} = 0$$

$$\vec{c} \cdot \vec{a} = \frac{6}{7} \times \frac{2}{7} + \frac{2}{7} \times \frac{3}{7} + \left(-\frac{3}{7}\right) \times \frac{6}{7} = \frac{12}{49} + \frac{6}{49} - \frac{18}{49} = 0$$

Hence, the given three vectors are mutually perpendicular to each other.

Question 6:

Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$ and $|\vec{a}| = 8|\vec{b}|$.

Answer

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 8$$

$$\Rightarrow \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - \vec{b} \cdot \vec{b} = 8$$

$$\Rightarrow |\vec{a}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow (8|\vec{b}|)^2 - |\vec{b}|^2 = 8 \quad [|\vec{a}| = 8|\vec{b}|]$$

$$\Rightarrow 64|\vec{b}|^2 - |\vec{b}|^2 = 8$$

$$\Rightarrow 63|\vec{b}|^2 = 8$$

$$\Rightarrow |\vec{b}|^2 = \frac{8}{63}$$

$$\Rightarrow |\vec{b}| = \sqrt{\frac{8}{63}} \quad [\text{Magnitude of a vector is non-negative}]$$

$$\Rightarrow |\vec{b}| = \frac{2\sqrt{2}}{3\sqrt{7}}$$

$$|\vec{a}| = 8|\vec{b}| = \frac{8 \times 2\sqrt{2}}{3\sqrt{7}} = \frac{16\sqrt{2}}{3\sqrt{7}}$$

Question 7:

Evaluate the product $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$.

Answer

$$(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$$

$$= 3\vec{a} \cdot 2\vec{a} + 3\vec{a} \cdot 7\vec{b} - 5\vec{b} \cdot 2\vec{a} - 5\vec{b} \cdot 7\vec{b}$$

$$= 6\vec{a} \cdot \vec{a} + 21\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35\vec{b} \cdot \vec{b}$$

$$= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$$

Question 8:

Find the magnitude of two vectors \vec{a} and \vec{b} , having the same magnitude and such that

$$\frac{1}{2}$$

the angle between them is 60° and their scalar product is $\frac{1}{2}$.

Answer

Let θ be the angle between the vectors \vec{a} and \vec{b} .

$$\text{It is given that } |\vec{a}| = |\vec{b}|, \vec{a} \cdot \vec{b} = \frac{1}{2}, \text{ and } \theta = 60^\circ. \quad \dots(1)$$

It is given that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

We know that

$$\therefore \frac{1}{2} = |\vec{a}| |\vec{a}| \cos 60^\circ \quad [\text{Using (1)}]$$

$$\Rightarrow \frac{1}{2} = |\vec{a}|^2 \times \frac{1}{2}$$

$$\Rightarrow |\vec{a}|^2 = 1$$

$$\Rightarrow |\vec{a}| = |\vec{b}| = 1$$

Question 9:

Find $|\vec{x}|$, if for a unit vector \vec{a} , $(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$.

Answer

$$(\vec{x} - \vec{a}) \cdot (\vec{x} + \vec{a}) = 12$$

$$\Rightarrow \vec{x} \cdot \vec{x} + \vec{x} \cdot \vec{a} - \vec{a} \cdot \vec{x} - \vec{a} \cdot \vec{a} = 12$$

$$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 12$$

$$\Rightarrow |\vec{x}|^2 - 1 = 12 \quad [|\vec{a}| = 1 \text{ as } \vec{a} \text{ is a unit vector}]$$

$$\Rightarrow |\vec{x}|^2 = 13$$

$$\therefore |\vec{x}| = \sqrt{13}$$

Question 10:

If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that $\vec{a} + \lambda\vec{b}$ is perpendicular to \vec{c} , then find the value of λ .

Answer

The given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$, and $\vec{c} = 3\hat{i} + \hat{j}$.

Now,

$$\vec{a} + \lambda\vec{b} = (2\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-\hat{i} + 2\hat{j} + \hat{k}) = (2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}$$

If $(\vec{a} + \lambda\vec{b})$ is perpendicular to \vec{c} , then

$$(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = 0.$$

$$\Rightarrow [(2 - \lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (3 + \lambda)\hat{k}] \cdot (3\hat{i} + \hat{j}) = 0$$

$$\Rightarrow (2 - \lambda)3 + (2 + 2\lambda)1 + (3 + \lambda)0 = 0$$

$$\Rightarrow 6 - 3\lambda + 2 + 2\lambda = 0$$

$$\Rightarrow -\lambda + 8 = 0$$

$$\Rightarrow \lambda = 8$$

Hence, the required value of λ is 8.

Question 11:

Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors \vec{a} and \vec{b} .

Answer

$$(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}) \cdot (|\vec{a}|\vec{b} - |\vec{b}|\vec{a})$$

$$= |\vec{a}|^2 \vec{b} \cdot \vec{b} - |\vec{a}|\vec{b} \cdot \vec{a} + |\vec{b}|\vec{a} \cdot \vec{b} - |\vec{b}|^2 \vec{a} \cdot \vec{a}$$

$$= |\vec{a}|^2 |\vec{b}|^2 - |\vec{b}|^2 |\vec{a}|^2$$

$$= 0$$

Hence, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

Question 12:

If $\vec{a} \cdot \vec{a} = 0$ and $\vec{a} \cdot \vec{b} = 0$, then what can be concluded about the vector \vec{b} ?

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