Exercise Miscellaneous : Solutions of Questions on Page Number : **191**

Q1 : $\left(3x^2-9x+5\right)^9$

Answer :

Let $y=\left(3x^2-9x+5\right)^9$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\left(3x^2-9x+5\right)^9\\&=9\left(3x^2-9x+5\right)^8\cdot\frac{d}{dx}\left(3x^2-9x+5\right)\\&=9\left(3x^2-9x+5\right)^8\cdot(6x-9)\\&=9\left(3x^2-9x+5\right)^8\cdot 3(2x-3)\\&=27\left(3x^2-9x+5\right)^8(2x-3)\end{aligned}$$

Answer needs Correction? **Click Here**

Q2 : $\sin^3 x+\cos^6 x$

Answer :

$$\begin{aligned}&\text{Let } y=\sin^3 x+\cos^6 x\\&\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\sin^3 x\right)+\frac{d}{dx}\left(\cos^6 x\right)\\&\quad=3\sin^2 x\cdot\frac{d}{dx}(\sin x)+6\cos^5 x\cdot\frac{d}{dx}(\cos x)\\&\quad=3\sin^2 x\cdot\cos x+6\cos^5 x\cdot(-\sin x)\\&\quad=3\sin x\cos x\left(\sin x-2\cos^4 x\right)\end{aligned}$$

Answer needs Correction? **Click Here**

Q3 : $(5x)^{3\cos 2x}$

Answer :

Let $y=(5x)^{3\cos 2x}$

Taking logarithm on both the sides, we obtain

$\log y=3\cos 2x\log 5x$

Differentiating both sides with respect to *x*, we obtain

$$\begin{aligned}&\frac{1}{y}\frac{dy}{dx}=3\left[\log 5x\cdot\frac{d}{dx}(\cos 2x)+\cos 2x\cdot\frac{d}{dx}(\log 5x)\right]\\&\Rightarrow\frac{dy}{dx}=3y\left[\log 5x(-\sin 2x)\cdot\frac{d}{dx}(2x)+\cos 2x\cdot\frac{1}{5x}\cdot\frac{d}{dx}(5x)\right]\\&\Rightarrow\frac{dy}{dx}=3y\left[-2\sin 2x\log 5x+\frac{\cos 2x}{x}\right]\\&\Rightarrow\frac{dy}{dx}=3y\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]\\&\therefore\frac{dy}{dx}=(5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x}-6\sin 2x\log 5x\right]\end{aligned}$$

Answer needs Correction? **Click Here**

Q4 : $\sin^{-1}\left(x\sqrt{x}\right),\ 0\le x\le 1$

Answer :

Let $y=\sin^{-1}\left(x\sqrt{x}\right)$

Using chain rule, we obtain

$$\begin{aligned}\frac{dy}{dx}&=\frac{d}{dx}\sin^{-1}\left(x\sqrt{x}\right)\\&=\frac{1}{\quad}\times\frac{d}{dx}\left(x\sqrt{x}\right)\end{aligned}$$

$$\frac{1}{\sqrt{1-\left(x\sqrt{x}\right)^2}}\cdot\frac{d}{dx}\left(x\sqrt{x}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\cdot\frac{d}{dx}\left(x^{\frac{3}{2}}\right)$$

$$=\frac{1}{\sqrt{1-x^3}}\times\frac{3}{2}\cdot x^{\frac{1}{2}}$$

$$=\frac{3\sqrt{x}}{2\sqrt{1-x^3}}$$

$$=\frac{3}{2}\sqrt{\frac{x}{1-x^3}}$$

Answer needs Correction? **Click Here**

Q5 : $\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}},\ -2<x<2$

Answer :

Let $y=\dfrac{\cos^{-1}\frac{x}{2}}{\sqrt{2x+7}}$

By quotient rule, we obtain

$$\frac{dy}{dx}=\frac{\sqrt{2x+7}\frac{d}{dx}\left(\cos^{-1}\frac{x}{2}\right)-\left(\cos^{-1}\frac{x}{2}\right)\frac{d}{dx}\left(\sqrt{2x+7}\right)}{\left(\sqrt{2x+7}\right)^2}$$

$$=\frac{\sqrt{2x+7}\left[\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{d}{dx}\left(\frac{x}{2}\right)\right]-\left(\cos^{-1}\frac{x}{2}\right)\frac{1}{2\sqrt{2x+7}}\cdot\frac{d}{dx}(2x+7)}{2x+7}$$

$$=\frac{\sqrt{2x+7}\frac{-1}{\sqrt{4-x^2}}-\left(\cos^{-1}\frac{x}{2}\right)\frac{2}{2\sqrt{2x+7}}}{2x+7}$$

$$=\frac{-\sqrt{2x+7}}{\sqrt{4-x^2}\times(2x+7)}-\frac{\cos^{-1}\frac{x}{2}}{\left(\sqrt{2x+7}\right)(2x+7)}$$

$$=-\left[\frac{1}{\sqrt{4-x^2}\sqrt{2x+7}}+\frac{\cos^{-1}\frac{x}{2}}{(2x+7)^{\frac{3}{2}}}\right]$$

Answer needs Correction? **Click Here**

Q6 : $\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right],0<x<\frac{}{2}$

Answer :

Let $y=\cot^{-1}\left[\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right]$ ...(1)

Then, $\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$

$$=\frac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)}$$

$$=\frac{(1+\sin x)+(1-\sin x)+2\sqrt{(1-\sin x)(1+\sin x)}}{(1+\sin x)-(1-\sin x)}$$

$$=\frac{2+2\sqrt{1-\sin^2 x}}{2\sin x}$$

$$=\frac{1+\cos x}{\sin x}$$

$$=\frac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$=\cot\frac{x}{2}$$

Therefore, equation (1) becomes

$$y=\cot^{-1}\left(\cot\frac{x}{2}\right)$$

$$\Rightarrow y=\frac{x}{2}$$

$$\therefore\frac{dy}{dx}=\frac{1}{2}\frac{d}{dx}(x)$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{2}$$

Answer needs Correction? **Click Here**

Q7 : $(\log x)^{\log x},\ x>1$

Answer :

Let $y = (\log x)^{\log x}$

Taking logarithm on both the sides, we obtain

$\log y = \log x \cdot \log(\log x)$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[\log x \cdot \log(\log x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\log x).\frac{d}{dx}(\log x) + \log x \cdot \frac{d}{dx}\left[\log(\log x)\right]$$
$$\Rightarrow \frac{dy}{dx} = y\left[\log(\log x).\frac{1}{x} + \log x \cdot \frac{1}{\log x}\cdot\frac{d}{dx}(\log x)\right]$$
$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x}\log(\log x) + \frac{1}{x}\right]$$
$$\therefore \frac{dy}{dx} = (\log x)^{\log x}\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

Answer needs Correction? Click Here

Q8 : $\cos(a\cos x + b\sin x)$, for some constant *a* and *b*.

Answer :

Let $y = \cos(a\cos x + b\sin x)$

By using chain rule, we obtain

$$\frac{dy}{dx} = \frac{d}{dx}\cos(a\cos x + b\sin x)$$
$$\Rightarrow \frac{dy}{dx} = -\sin(a\cos x + b\sin x)\cdot\frac{d}{dx}(a\cos x + b\sin x)$$
$$= -\sin(a\cos x + b\sin x)\cdot\left[a(-\sin x) + b\cos x\right]$$
$$= (a\sin x - b\cos x)\cdot\sin(a\cos x + b\sin x)$$

Answer needs Correction? Click Here

Q9 : $(\sin x - \cos x)^{(\sin x - \cos x)}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Answer :

Let $y = (\sin x - \cos x)^{(\sin x - \cos x)}$

Taking logarithm on both the sides, we obtain

$$\log y = \log\left[(\sin x - \cos x)^{(\sin x - \cos x)}\right]$$
$$\Rightarrow \log y = (\sin x - \cos x)\cdot\log(\sin x - \cos x)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[(\sin x - \cos x)\log(\sin x - \cos x)\right]$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot\frac{d}{dx}(\sin x - \cos x) + (\sin x - \cos x)\cdot\frac{d}{dx}\log(\sin x - \cos x)$$
$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log(\sin x - \cos x)\cdot(\cos x + \sin x) + (\sin x - \cos x)\cdot\frac{1}{(\sin x - \cos x)}\cdot\frac{d}{dx}(\sin x - \cos x)$$
$$\Rightarrow \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}\left[(\cos x + \sin x)\cdot\log(\sin x - \cos x) + (\cos x + \sin x)\right]$$
$$\therefore \frac{dy}{dx} = (\sin x - \cos x)^{(\sin x - \cos x)}(\cos x + \sin x)\left[1 + \log(\sin x - \cos x)\right]$$

Answer needs Correction? Click Here

Q10 : $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$

Answer :

Let $y = x^x + x^a + a^x + a^a$

Also, let $x^x = u,\ x^a = v,\ a^x = w,$ and $a^a = s$

$\therefore y = u + v + w + s$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \frac{ds}{dx} \qquad ...(1)$$

$u = x^x$

$\Rightarrow \log u = \log x^x$

$\Rightarrow \log u = x\log x$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{u}\frac{du}{dx} = \log x\cdot\frac{d}{dx}(x) + x\cdot\frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{du}{dx} = u\left[\log x\cdot 1 + x\cdot\frac{1}{x}\right]$$
$$\Rightarrow \frac{du}{dx} = x^x[\log x + 1] = x^x(1 + \log x) \qquad ...(2)$$

$$\Rightarrow \frac{du}{dx} = x^x [\log x + 1] = x^x (1 + \log x) \quad ...(2)$$

$$v = x^a$$
$$\therefore \frac{dv}{dx} = \frac{d}{dx}(x^a)$$
$$\Rightarrow \frac{dv}{dx} = ax^{a-1} \quad ...(3)$$
$$w = a^x$$
$$\Rightarrow \log w = \log a^x$$
$$\Rightarrow \log w = x \log a$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{w} \cdot \frac{dw}{dx} = \log a \cdot \frac{d}{dx}(x)$$
$$\Rightarrow \frac{dw}{dx} = w \log a$$
$$\Rightarrow \frac{dw}{dx} = a^x \log a \quad ...(4)$$

$s = a^a$

Since a is constant, a^a is also a constant.

$$\therefore \frac{ds}{dx} = 0 \quad ...(5)$$

From (1), (2), (3), (4), and (5), we obtain

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a + 0$$
$$= x^x(1+\log x) + ax^{a-1} + a^x \log a$$

Answer needs Correction? **Click Here**

Q11 : $x^{x^2-3} + (x-3)^{x^2}$, for $x > 3$

Answer :

Let $y = x^{x^2-3} + (x-3)^{x^2}$

Also, let $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$\therefore y = u + v$

Differentiating both sides with respect to *x*,we obtain

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad ...(1)$$

$$u = x^{x^2-3}$$
$$\therefore \log u = \log\left(x^{x^2-3}\right)$$
$$\log u = (x^2-3)\log x$$

Differentiating with respect to *x*, we obtain

$$\frac{1}{u} \cdot \frac{du}{dx} = \log x \cdot \frac{d}{dx}(x^2-3) + (x^2-3) \cdot \frac{d}{dx}(\log x)$$
$$\Rightarrow \frac{1}{u}\frac{du}{dx} = \log x \cdot 2x + (x^2-3) \cdot \frac{1}{x}$$
$$\Rightarrow \frac{du}{dx} = x^{x^2-3} \cdot \left[\frac{x^2-3}{x} + 2x \log x\right]$$

Also,

$$v = (x-3)^{x^2}$$
$$\therefore \log v = \log(x-3)^{x^2}$$
$$\Rightarrow \log v = x^2 \log(x-3)$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log(x-3) \cdot \frac{d}{dx}(x^2) + x^2 \cdot \frac{d}{dx}[\log(x-3)]$$
$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \log(x-3) \cdot 2x + x^2 \cdot \frac{1}{x-3} \cdot \frac{d}{dx}(x-3)$$
$$\Rightarrow \frac{dv}{dx} = v\left[2x \log(x-3) + \frac{x^2}{x-3} \cdot 1\right]$$
$$\Rightarrow \frac{dv}{dx} = (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x \log(x-3)\right]$$

Substituting the expressions of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in equation (1), we obtain

$$\frac{dy}{dx} = x^{x^2-3}\left[\frac{x^2-3}{x} + 2x \log x\right] + (x-3)^{x^2}\left[\frac{x^2}{x-3} + 2x \log(x-3)\right]$$

Answer needs Correction? **Click Here**

Q12 : Find $\frac{dy}{dx}$, if $y = 12(1-\cos t), x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$

Answer :

It is given that, $y = 12(1-\cos t), x = 10(t-\sin t)$

$$\therefore \frac{dx}{dt}=\frac{d}{dt}\left[10(t-\sin t)\right]=10\cdot\frac{d}{dt}(t-\sin t)=10(1-\cos t)$$

$$\frac{dy}{dt}=\frac{d}{dt}\left[12(1-\cos t)\right]=12\cdot\frac{d}{dt}(1-\cos t)=12\cdot\left[0-(-\sin t)\right]=12\sin t$$

$$\therefore \frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{12\sin t}{10(1-\cos t)}=\frac{12\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{10\cdot 2\sin^2\frac{t}{2}}=\frac{6}{5}\cot\frac{t}{2}$$

Answer needs Correction? Click Here

Q13 : Find $\frac{dy}{dx}$, if $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2},\ -1\le x\le 1$

Answer :

It is given that, $y=\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}$

$$\therefore \frac{dy}{dx}=\frac{d}{dx}\left[\sin^{-1}x+\sin^{-1}\sqrt{1-x^2}\right]$$

$$\Rightarrow \frac{dy}{dx}=\frac{d}{dx}\left(\sin^{-1}x\right)+\frac{d}{dx}\left(\sin^{-1}\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}\cdot\frac{d}{dx}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{x}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}\left(1-x^2\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}+\frac{1}{2x\sqrt{1-x^2}}(-2x)$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}$$

$$\therefore \frac{dy}{dx}=0$$

Answer needs Correction? Click Here

Q14 : If $x\sqrt{1+y}+y\sqrt{1+x}=0$, for, - 1 < *x*<1, prove that

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

Answer :

It is given that,

$$x\sqrt{1+y}+y\sqrt{1+x}=0$$

$$\Rightarrow x\sqrt{1+y}=-y\sqrt{1+x}$$

Squaring both sides, we obtain

$$x^2(1+y)=y^2(1+x)$$

$$\Rightarrow x^2+x^2y=y^2+xy^2$$

$$\Rightarrow x^2-y^2=xy^2-x^2y$$

$$\Rightarrow x^2-y^2=xy(y-x)$$

$$\Rightarrow (x+y)(x-y)=xy(y-x)$$

$$\therefore x+y=-xy$$

$$\Rightarrow (1+x)y=-x$$

$$\Rightarrow y=\frac{-x}{(1+x)}$$

Differentiating both sides with respect to *x*, we obtain

$$y=\frac{-x}{(1+x)}$$

$$\frac{dy}{dx}=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

Hence, proved.

Answer needs Correction? Click Here

Q15 : If $(x-a)^2+(y-b)^2=c^2$, for some $c>0$, prove that

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$ is a constant independent of *a* and *b*.

Answer :

It is given that, $(x-a)^2+(y-b)^2=c^2$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}\left[(x-a)^2\right]+\frac{d}{dx}\left[(y-b)^2\right]=\frac{d}{dx}\left(c^2\right)$$

$$\Rightarrow 2(x-a)\cdot\frac{d}{dx}(x-a)+2(y-b)\cdot\frac{d}{dx}(y-b)=0$$

$$\Rightarrow 2(x-a)\cdot 1+2(y-b)\cdot\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=\frac{-(x-a)}{y-b} \qquad ...(1)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{-(x-a)}{y-b}\right]$$

$$=-\left[\frac{(y-b)\cdot\frac{d}{dx}(x-a)-(x-a)\cdot\frac{d}{dx}(y-b)}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\frac{dy}{dx}}{(y-b)^2}\right]$$

$$=-\left[\frac{(y-b)-(x-a)\cdot\left\{\frac{-(x-a)}{y-b}\right\}}{(y-b)^2}\right] \qquad [\text{Using } (1)]$$

$$=-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left[1+\frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}=\frac{\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\left[\frac{(y-b)^2+(x-a)^2}{(y-b)^3}\right]}$$

$$=\frac{\left[\frac{c^2}{(y-b)^2}\right]^{\frac{3}{2}}}{-\frac{c^2}{(y-b)^3}}=\frac{\frac{c^3}{(y-b)^3}}{-\frac{c^2}{(y-b)^3}}$$

$=-c$, which is constant and is independent of a and b

Hence, proved.

Answer needs Correction? Click Here

Q16 : If $\cos y = x\cos(a+y)$, with $\cos a \neq \pm 1$, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Answer :

It is given that, $\cos y = x\cos(a+y)$

$$\therefore \frac{d}{dx}[\cos y]=\frac{d}{dx}[x\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y)\cdot\frac{d}{dx}(x)+x\cdot\frac{d}{dx}[\cos(a+y)]$$

$$\Rightarrow -\sin y\frac{dy}{dx}=\cos(a+y)+x\cdot[-\sin(a+y)]\frac{dy}{dx}$$

$$\Rightarrow [x\sin(a+y)-\sin y]\frac{dy}{dx}=\cos(a+y) \qquad ...(1)$$

Since $\cos y = x\cos(a+y)$, $x=\frac{\cos y}{\cos(a+y)}$

Then, equation (1) reduces to

$$\left[\frac{\cos y}{\cos(a+y)}\cdot\sin(a+y)-\sin y\right]\frac{dy}{dx}=\cos(a+y)$$

$$\Rightarrow [\cos y\cdot\sin(a+y)-\sin y\cdot\cos(a+y)]\cdot\frac{dy}{dx}=\cos^2(a+y)$$

$$\Rightarrow \sin(a+y-y)\frac{dy}{dx}=\cos^2(a+b)$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^2(a+b)}{\sin a}$$

Hence, proved.

Answer needs Correction? Click Here

Q17 : If $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$, find $\frac{d^2y}{dx^2}$

Answer :

It is given that, $x=a(\cos t+t\sin t)$ and $y=a(\sin t-t\cos t)$

$$\therefore \frac{dx}{dt}=a\cdot\frac{d}{dt}(\cos t+t\sin t)$$

$$=a\left[-\sin t+\sin t\cdot\frac{d}{dx}(t)+t\cdot\frac{d}{dt}(\sin t)\right]$$

$$=a[-\sin t+\sin t+t\cos t]=at\cos t$$

$$\frac{dy}{dt}=a\cdot\frac{d}{dt}(\sin t-t\cos t)$$

$$=a\left[\cos t-\left\{\cos t\cdot\frac{d}{dt}(t)+t\cdot\frac{d}{dt}(\cos t)\right\}\right]$$

$$=a[\cos t-\{\cos t-t\sin t\}]=at\sin t$$

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{at\sin t}{at\cos t} = \tan t$$

$$\text{Then, } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan t) = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \sec^2 t \cdot \frac{1}{at\cos t} \qquad \left[\frac{dx}{dt} = at\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{at\cos t}\right]$$

$$= \frac{\sec^3 t}{at}, 0 < t < \frac{\pi}{2}$$

Answer needs Correction? Click Here

Q18 : If $f(x) = |x|^3$, show that $f''(x)$ exists for all real *x*, and find it.

Answer :

It is known that, $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Therefore, when $x \geq 0$, $f(x) = |x|^3 = x^3$

In this case, $f'(x) = 3x^2$ and hence, $f''(x) = 6x$

When $x < 0$, $f(x) = |x|^3 = (-x)^3 = -x^3$

In this case, $f'(x) = -3x^2$ and hence, $f''(x) = -6x$

Thus, for $f(x) = |x|^3$, $f''(x)$ exists for all real *x*and is given by,

$$f''(x) = \begin{cases} 6x, & \text{if } x \geq 0 \\ -6x, & \text{if } x < 0 \end{cases}$$

Answer needs Correction? Click Here

Q19 : Using mathematical induction prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers *n*.

Answer :

To prove: $P(n): \frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n

For *n*= 1,

$$P(1): \frac{d}{dx}(x) = 1 = 1 \cdot x^{1-1}$$

∴P(*n*) is true for *n*= 1

Let P(*k*) is true for some positive integer *k*.

That is, $P(k): \frac{d}{dx}(x^k) = kx^{k-1}$

It has to be proved that P(*k*+ 1) is also true.

$$\text{Consider } \frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$$

$$= x^k \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}(x^k) \qquad [\text{By applying product rule}]$$

$$= x^k \cdot 1 + x \cdot k \cdot x^{k-1}$$

$$= x^k + kx^k$$

$$= (k+1) \cdot x^k$$

$$= (k+1) \cdot x^{(k+1)-1}$$

Thus, P(*k* + 1) is true whenever P (*k*) is true.

Therefore, by the principle of mathematical induction, the statement P(*n*) is true for every positive integer *n*.

Hence, proved.

Answer needs Correction? Click Here

Q20 : Using the fact that sin (*A* + *B*) = sin *A*cos *B* + cos *A* sin *B* and the differentiation, obtain the sum formula for cosines.

Answer :

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

Differentiating both sides with respect to *x*, we obtain

$$\frac{d}{dx}[\sin(A+B)] = \frac{d}{dx}(\sin A\cos B) + \frac{d}{dx}(\cos A\sin B)$$

$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \frac{d}{dx}(\sin A) + \sin A \cdot \frac{d}{dx}(\cos B) + \sin B \cdot \frac{d}{dx}(\cos A) + \cos A \cdot \frac{d}{dx}(\sin B)$$

$$\Rightarrow \cos(A+B) \cdot \frac{d}{dx}(A+B) = \cos B \cdot \cos A\frac{dA}{dx} + \sin A(-\sin B)\frac{dB}{dx} + \sin B(-\sin A) \cdot \frac{dA}{dx} + \cos A\cos B\frac{dB}{dx}$$

$$\Rightarrow \cos(A+B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]=(\cos A\cos B-\sin A\sin B)\cdot\left[\frac{dA}{dx}+\frac{dB}{dx}\right]$$
$$\therefore \cos(A+B)=\cos A\cos B-\sin A\sin B$$

Answer needs Correction? **Click Here**

Q21 : Does there exist a function which is continuos everywhere but not differentiable at exactly two points? Justify your answer ?

Answer :

Answer needs Correction? **Click Here**

Q22 : If $y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, prove that $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer :

$$y=\begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$
$$\Rightarrow y=(mc-nb)f(x)-(lc-na)g(x)+(lb-ma)h(x)$$
$$\text{Then, } \frac{dy}{dx}=\frac{d}{dx}\left[(mc-nb)f(x)\right]-\frac{d}{dx}\left[(lc-na)g(x)\right]+\frac{d}{dx}\left[(lb-ma)h(x)\right]$$
$$=(mc-nb)f'(x)-(lc-na)g'(x)+(lb-ma)h'(x)$$
$$=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Thus, $\frac{dy}{dx}=\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$

Answer needs Correction? **Click Here**

Q23 : If $y=e^{a\cos^{-1}x}, -1\le x\le 1$, show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

Answer :

It is given that, $y=e^{a\cos^{-1}x}$

Taking logarithm on both the sides, we obtain

$$\log y=a\cos^{-1}x\log e$$
$$\log y=a\cos^{-1}x$$

Differentiating both sides with respect to x, we obtain

$$\frac{1}{y}\frac{dy}{dx}=a\times\frac{-1}{\sqrt{1-x^2}}$$
$$\Rightarrow \frac{dy}{dx}=\frac{-ay}{\sqrt{1-x^2}}$$

By squaring both the sides, we obtain

$$\left(\frac{dy}{dx}\right)^2=\frac{a^2y^2}{1-x^2}$$
$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

$$(1-x^2)\left(\frac{dy}{dx}\right)^2=a^2y^2$$

Again differentiating both sides with respect to x, we obtain

$$\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)+(1-x^2)\times\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^2\right]=a^2\frac{d}{dx}(y^2)$$
$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x)+(1-x^2)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)^2(-2x)+(1-x^2)\times 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}=a^2.2y.\frac{dy}{dx}$$
$$\Rightarrow -x\frac{dy}{dx}+(1-x^2)\frac{d^2y}{dx^2}=a^2.y \qquad \left[\frac{dy}{dx}\neq 0\right]$$
$$\Rightarrow (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$$

Hence, proved.

*********** END **********