Hence, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

Question 4:

Show that

$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

Answer

$$\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)$$

$$=\left(\vec{a}-\vec{b}\right)\times\vec{a}+\left(\vec{a}-\vec{b}\right)\times\vec{b} \qquad \left[\text{By distributivity of vector product over addition}\right]$$

$$=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b} \qquad \left[\text{Again, by distributivity of vector product over addition}\right]$$

$$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$$

$$=2\vec{a}\times\vec{b}$$

Question 5:

Find λ and μ if $\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$.

Answer

$$\left(2\hat{i}+6\hat{j}+27\hat{k}\right)\times\left(\hat{i}+\lambda\hat{j}+\mu\hat{k}\right)=\vec{0}$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = 0\hat{i}+0\hat{j}+0\hat{k}$$

$$\Rightarrow \hat{i}\left(6\mu-27\lambda\right)-\hat{j}\left(2\mu-27\right)+\hat{k}\left(2\lambda-6\right)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have:

$$6\mu-27\lambda=0$$

$$2\mu-27=0$$

$$2\lambda-6=0$$

Now,

$$2\lambda-6=0 \Rightarrow \lambda=3$$

$$2\mu-27=0 \Rightarrow \mu=\frac{27}{2}$$

Hence, $\lambda=3$ and $\mu=\frac{27}{2}$.

Question 6:

Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Answer

$\vec{a}\cdot\vec{b}=0$

Then,

(i) Either $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$, or $\vec{a}\perp\vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

$\vec{a}\times\vec{b}=0$

(ii) Either $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$, or $\vec{a}\parallel\vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Hence, $\left|\vec{a}\right|=0$ or $\left|\vec{b}\right|=0$.

Question 7:

Let the vectors $\vec{a}, \vec{b}, \vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show that $=\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Answer

We have,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$

$(b+\vec{c}) = (b_1+c_1)i + (b_2+c_2)j + (b_3+c_3)k$

Now, $\vec{a}\times(\vec{b}+\vec{c}) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \end{vmatrix}$

$= \hat{i}[a_2(b_3+c_3) - a_3(b_2+c_2)] - \hat{j}[a_1(b_3+c_3) - a_3(b_1+c_1)] + \hat{k}[a_1(b_2+c_2) - a_2(b_1+c_1)]$

$= \hat{i}[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2] + \hat{j}[-a_1b_3 - a_1c_3 + a_3b_1 + a_3c_1] + \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1]$ ...(1)

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$= \hat{i}[a_2b_3 - a_3b_2] + \hat{j}[b_1a_3 - a_1b_3] + \hat{k}[a_1b_2 - a_2b_1]$ (2)

$\vec{a}\times\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$= \hat{i}[a_2c_3 - a_3c_2] + \hat{j}[a_3c_1 - a_1c_3] + \hat{k}[a_1c_2 - a_2c_1]$ (3)

On adding (2) and (3), we get:

$(\vec{a}\times\vec{b}) + (\vec{a}\times\vec{c}) = \hat{i}[a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2] + \hat{j}[b_1a_3 + a_3c_1 - a_1b_3 - a_1c_3]$

$+ \hat{k}[a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1]$ (4)

Now, from (1) and (4), we have:

$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$

Hence, the given result is proved.

Question 8:

If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example.

Answer

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a} = 2\hat{i} + 3\hat{j} + 4\hat{k},\ \vec{b} = 4\hat{i} + 6\hat{j} + 8\hat{k}$.

Then,

$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{vmatrix} = \hat{i}(24-24) - \hat{j}(16-16) + \hat{k}(12-12) = 0\hat{i} + 0\hat{j} + 0\hat{k} = \vec{0}$

It can now be observed that:

$|\vec{a}| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$

$\therefore \vec{a} \neq \vec{0}$

$|\vec{b}| = \sqrt{4^2+6^2+8^2} = \sqrt{116}$

$\therefore \vec{b} \neq \vec{0}$

Hence, the converse of the given statement need not be true.

Question 9:

Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Answer

The vertices of triangle ABC are given as A (1, 1, 2), B (2, 3, 5), and C (1, 5, 5).

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{BC} = (1-2)\hat{i} + (5-3)\hat{j} + (5-5)\hat{k} = -\hat{i} + 2\hat{j}$

Area of ΔABC $= \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$

$\overrightarrow{AB}\times\overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} = \hat{i}(-6) - \hat{j}(3) + \hat{k}(2+2) = -6\hat{i} - 3\hat{j} + 4\hat{k}$

$\therefore \left|\overrightarrow{AB}\times\overrightarrow{BC}\right| = \sqrt{(-6)^2 + (-3)^2 + 4^2} = \sqrt{36+9+16} = \sqrt{61}$

Hence, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

Question 10:

Find the area of the parallelogram whose adjacent sides are determined by the vector $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Answer

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$.

Adjacent sides are given as:

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}\left(-1+21\right)-\hat{j}\left(1-6\right)+\hat{k}\left(-7+2\right)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$\left|\vec{a}\times\vec{b}\right|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Hence, the area of the given parallelogram is $15\sqrt{2}$ square units.

Question 11:

Let the vectors $\vec{a}$ and $\vec{b}$ be such that $\left|\vec{a}\right|=3$ and $\left|\vec{b}\right|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Answer

It is given that $\left|\vec{a}\right|=3$ and $\left|\vec{b}\right|=\frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b}=\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$.

Now, $\vec{a}\times\vec{b}$ is a unit vector if $\left|\vec{a}\times\vec{b}\right|=1$.

$$\left|\vec{a}\times\vec{b}\right|=1$$

$$\Rightarrow \left|\left|\vec{a}\right|\left|\vec{b}\right|\sin\theta\,\hat{n}\right|=1$$

$$\Rightarrow \left|\vec{a}\right|\left|\vec{b}\right|\left|\sin\theta\right|=1$$

$$\Rightarrow 3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Hence, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

The correct answer is B.

Question 12:

Area of a rectangle having vertices A, B, C, and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Answer

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overrightarrow{OA}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OB}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OC}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k},\ \overrightarrow{OD}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$

*********** END **********