Q1 : Verify Rolle's Theorem for the function $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$

Answer :

The given function, $f(x) = x^2 + 2x - 8$, being a polynomial function, is continuous in [- 4, 2] and is differentiable in (- 4, 2).

$f(-4) = (-4)^2 + 2 \times (-4) - 8 = 16 - 8 - 8 = 0$

$f(2) = (2)^2 + 2 \times 2 - 8 = 4 + 4 - 8 = 0$

$\therefore f(-4) = f(2) = 0$

$\Rightarrow$ The value of $f(x)$ at - 4 and 2 coincides.

Rolle's Theorem states that there is a point $c \in$ (- 4, 2) such that $f'(c) = 0$

$f(x) = x^2 + 2x - 8$

$\Rightarrow f'(x) = 2x + 2$

$\therefore f'(c) = 0$

$\Rightarrow 2c + 2 = 0$

$\Rightarrow c = -1$, where $c = -1 \in (-4, 2)$

Hence, Rolle's Theorem is verified for the given function.

Q2 : Examine if Rolle's Theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's Theorem from these examples?

(i) $f(x) = [x]$ for $x \in [5, 9]$

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

Answer :

By Rolle's Theorem, for a function $f : [a, b] \rightarrow \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

(c) $f(a) = f(b)$

then, there exists some $c \in$ (a, b) such that $f'(c) = 0$

Therefore, Rolle's Theorem is not applicable to those functions that do not satisfy any of the three conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in [5, 9].

Also, $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore f(5) \neq f(9)$

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^-} \dfrac{f(n+h) - f(n)}{h} = \lim\limits_{h \to 0^-} \dfrac{[n+h] - [n]}{h} = \lim\limits_{h \to 0^-} \dfrac{n - 1 - n}{h} = \lim\limits_{h \to 0^-} \dfrac{-1}{h} = \infty$

The right hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^-} \dfrac{f(n+h) - f(n)}{h} = \lim\limits_{h \to 0^-} \dfrac{[n+h] - [n]}{h} = \lim\limits_{h \to 0^-} \dfrac{n - n}{h} = \lim\limits_{h \to 0^-} 0 = 0$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [5, 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in $[-2, 2]$.

Also, $f(-2) = [-2] = -2$ and $f(2) = [2] = 2$

$\therefore f(-2) \neq f(2)$

The differentiability of f in $(-2, 2)$ is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$$\lim_{h \to 0^-} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^-} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^-} \frac{n - 1 - n}{h} = \lim_{h \to 0^-} \frac{-1}{h} = \infty$$

The right hand limit of f at $x = n$ is,

$$\lim_{h \to 0^+} \frac{f(n+h) - f(n)}{h} = \lim_{h \to 0^+} \frac{[n+h] - [n]}{h} = \lim_{h \to 0^+} \frac{n - n}{h} = \lim_{h \to 0^+} 0 = 0$$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in $(-2, 2)$.

It is observed that f does not satisfy all the conditions of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, 2]$

It is evident that f, being a polynomial function, is continuous in $[1, 2]$ and is differentiable in $(1, 2)$.

$f(1) = (1)^2 - 1 = 0$

$f(2) = (2)^2 - 1 = 3$

$\therefore f(1) \neq f(2)$

It is observed that f does not satisfy a condition of the hypothesis of Rolle's Theorem.

Hence, Rolle's Theorem is not applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Q3 : If $f : [-5,5] \to \mathbf{R}$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Answer :

It is given that $f : [-5,5] \to \mathbf{R}$ is a differentiable function.

Since every differentiable function is a continuous function, we obtain

(a) f is continuous on $[-5, 5]$.

(b) f is differentiable on $(-5, 5)$.

Therefore, by the Mean Value Theorem, there exists $c \in (-5, 5)$ such that

$$f'(c) = \frac{f(5) - f(-5)}{5 - (-5)}$$

$$\Rightarrow 10 f'(c) = f(5) - f(-5)$$

It is also given that $f'(x)$ does not vanish anywhere.

$\therefore f'(c) \neq 0$

$\Rightarrow 10 f'(c) \neq 0$

$\Rightarrow f(5) - f(-5) \neq 0$

$\Rightarrow f(5) \neq f(-5)$

Hence, proved.

Q4 : Verify Mean Value Theorem, if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$, where $a = 1$ and $b = 4$.

Answer :

The given function is $f(x) = x^2 - 4x - 3$

f, being a polynomial function, is continuous in $[1, 4]$ and is differentiable in $(1, 4)$ whose derivative is $2x - 4$.

$f(1) = 1^2 - 4 \times 1 - 3 = -6, f(4) = 4^2 - 4 \times 4 - 3 = -3$

$\therefore \frac{f(b) - f(a)}{b - a} = \frac{f(4) - f(1)}{4 - 1} = \frac{-3 - (-6)}{3} = \frac{3}{3} = 1$

Mean Value Theorem states that there is a point $c \in (1, 4)$ such that $f'(c) = 1$

$f'(c) = 1$

$\Rightarrow 2c - 4 = 1$

$\Rightarrow c = \frac{5}{2}$, where $c = \frac{5}{2} \in (1, 4)$

Hence, Mean Value Theorem is verified for the given function.

Q5 : Verify Mean Value Theorem, if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$, where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$

Answer :

The given function f is $f(x) = x^3 - 5x^2 - 3x$

f, being a polynomial function, is continuous in [1, 3] and is differentiable in (1, 3) whose derivative is $3x^2$ - 10x - 3.

$f(1) = 1^3 - 5 \times 1^2 - 3 \times 1 = -7, \ f(3) = 3^3 - 5 \times 3^2 - 3 \times 3 = -27$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{-27 - (-7)}{3 - 1} = -10$

Mean Value Theorem states that there exist a point $c \in (1, 3)$ such that $f'(c) = -10$

$f'(c) = -10$

$\Rightarrow 3c^2 - 10c - 3 = 10$

$\Rightarrow 3c^2 - 10c + 7 = 0$

$\Rightarrow 3c^2 - 3c - 7c + 7 = 0$

$\Rightarrow 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow (c - 1)(3c - 7) = 0$

$\Rightarrow c = 1, \dfrac{7}{3}$, where $c = \dfrac{7}{3} \in (1, \ 3)$

Hence, Mean Value Theorem is verified for the given function and $c = \dfrac{7}{3} \in (1, \ 3)$ is the only point for which $f'(c) = 0$

Answer needs Correction? **Click Here**

Q6 : Examine the applicability of Mean Value Theorem for all three functions given in the above exercise 2.

Answer :

Mean Value Theorem states that for a function $f : [a, \ b] \rightarrow \mathbf{R}$, if

(a) f is continuous on [a, b]

(b) f is differentiable on (a, b)

then, there exists some $c \in (a, \ b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Therefore, Mean Value Theorem is not applicable to those functions that do not satisfy any of the two conditions of the hypothesis.

(i) $f(x) = [x]$ for $x \in [5, \ 9]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = 5$ and $x = 9$

$\Rightarrow f(x)$ is not continuous in [5, 9].

The differentiability of f in (5, 9) is checked as follows.

Let n be an integer such that $n \in (5, 9)$.

The left hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^-} \dfrac{f(n + h) - f(n)}{h} = \lim\limits_{h \to 0^-} \dfrac{[n + h] - [n]}{h} = \lim\limits_{h \to 0^-} \dfrac{n - 1 - n}{h} = \lim\limits_{h \to 0^-} \dfrac{-1}{h} = \infty$

The right hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^+} \dfrac{f(n + h) - f(n)}{h} = \lim\limits_{h \to 0^+} \dfrac{[n + h] - [n]}{h} = \lim\limits_{h \to 0^+} \dfrac{n - n}{h} = \lim\limits_{h \to 0^+} 0 = 0$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (5, 9).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [5, \ 9]$.

(ii) $f(x) = [x]$ for $x \in [-2, \ 2]$

It is evident that the given function $f(x)$ is not continuous at every integral point.

In particular, $f(x)$ is not continuous at $x = -2$ and $x = 2$

$\Rightarrow f(x)$ is not continuous in [- 2, 2].

The differentiability of f in (- 2, 2) is checked as follows.

Let n be an integer such that $n \in (-2, 2)$.

The left hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^-} \dfrac{f(n + h) - f(n)}{h} = \lim\limits_{h \to 0^-} \dfrac{[n + h] - [n]}{h} = \lim\limits_{h \to 0^-} \dfrac{n - 1 - n}{h} = \lim\limits_{h \to 0^-} \dfrac{-1}{h} = \infty$

The right hand limit of f at $x = n$ is,

$\lim\limits_{h \to 0^+} \dfrac{f(n + h) - f(n)}{h} = \lim\limits_{h \to 0^+} \dfrac{[n + h] - [n]}{h} = \lim\limits_{h \to 0^+} \dfrac{n - n}{h} = \lim\limits_{h \to 0^+} 0 = 0$

Since the left and right hand limits of f at $x = n$ are not equal, f is not differentiable at $x = n$

$\therefore f$ is not differentiable in (- 2, 2).

It is observed that f does not satisfy all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is not applicable for $f(x) = [x]$ for $x \in [-2, \ 2]$.

(iii) $f(x) = x^2 - 1$ for $x \in [1, \ 2]$

It is evident that f, being a polynomial function, is continuous in [1, 2] and is differentiable in (1, 2).

It is observed that f satisfies all the conditions of the hypothesis of Mean Value Theorem.

Hence, Mean Value Theorem is applicable for $f(x) = x^2 - 1$ for $x \in [1, 2]$.

It can be proved as follows.

$f(1) = 1^2 - 1 = 0, \ f(2) = 2^2 - 1 = 3$

$\therefore \dfrac{f(b) - f(a)}{b - a} = \dfrac{f(2) - f(1)}{2 - 1} = \dfrac{3 - 0}{1} = 3$

$f'(x) = 2x$

$\therefore f'(c) = 3$

$\Rightarrow 2c = 3$

$\Rightarrow c = \dfrac{3}{2} = 1.5, \ \text{where } 1.5 \in [1, 2]$

********** END *********