



EXERCISE.13.2

Question-1

Find the derivative of $x^2 - 2$ at $x = 10$.

Ans.

Let $f(x) = x^2 - 2$. Accordingly,

$$\begin{aligned}f'(10) &= \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h} \\&= \lim_{h \rightarrow 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h} \\&= \lim_{h \rightarrow 0} \frac{20h + h^2}{h} \\&= \lim_{h \rightarrow 0} (20 + h) = (20 + 0) = 20\end{aligned}$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question-2

Find the derivative of $99x$ at $x = 100$.

Ans.

Let $f(x) = 99x$. Accordingly,

$$\begin{aligned}f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\&= \lim_{h \rightarrow 0} \frac{99(100+h) - 99(100)}{h} \\&= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h} \\&= \lim_{h \rightarrow 0} \frac{99h}{h} \\&= \lim_{h \rightarrow 0} (99) = 99\end{aligned}$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question-3

Find the derivative of x at $x = 1$.

Ans.

Let $f(x) = x$. Accordingly,

$$\begin{aligned}f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\&= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h} \\&= \lim_{h \rightarrow 0} \frac{h}{h} \\&= \lim_{h \rightarrow 0} (1) \\&= 1\end{aligned}$$

Thus, the derivative of x at $x = 1$ is 1.

Question-4

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Ans.

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \\ &= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\ &= 0 + 3x^2 + 0 = 3x^2 \end{aligned}$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h} \\ &= \lim_{h \rightarrow 0} (2x + h - 3) \\ &= (2x + 0 - 3) \\ &= 2x - 3 \end{aligned}$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2 (x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2 (x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2 (x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-h - 2x}{x^2 (x+h)^2} \right] \\
 &= \frac{0 - 2x}{x^2 (x+0)^2} = \frac{-2}{x^3}
 \end{aligned}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right] \\
 &= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}
 \end{aligned}$$

For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100f'(0)$

Ans.

The given function is

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx}(x) + \frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0 \\ &= x^{99} + x^{98} + \dots + x + 1 \end{aligned}$$

$$\therefore f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At $x = 0$,

$$f'(0) = 1$$

At $x = 1$,

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = [1 + 1 + \dots + 1 + 1]_{100 \text{ terms}} = 1 \times 100 = 100$$

$$\text{Thus, } f'(1) = 100 \times f'(0)$$

Question-6

Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$ for some fixed real number a .

Ans.

$$\text{Let } f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n) \\ &= \frac{d}{dx}(x^n) + a \frac{d}{dx}(x^{n-1}) + a^2 \frac{d}{dx}(x^{n-2}) + \dots + a^{n-1} \frac{d}{dx}(x) + a^n \frac{d}{dx}(1) \end{aligned}$$

On using theorem $\frac{d}{dx} x^n = nx^{n-1}$, we obtain

$$\begin{aligned} f'(x) &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n(0) \\ &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} \end{aligned}$$

Question-7

For some constants a and b , find the derivative of

$$(i) (x - a)(x - b) \quad (ii) (ax^2 + b)^2 \quad (iii) \frac{x - a}{x - b}$$

Ans.

$$(i) \text{ Let } f(x) = (x - a)(x - b)$$

$$\Rightarrow f(x) = x^2 - (a + b)x + ab$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx}(x^2 - (a + b)x + ab) \\ &= \frac{d}{dx}(x^2) - (a + b)\frac{d}{dx}(x) + \frac{d}{dx}(ab) \end{aligned}$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a + b) + 0 = 2x - a - b$$

$$(ii) \quad \text{Let } f(x) = (ax^2 + b)^2$$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2 \frac{d}{dx}(x^4) + 2ab \frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$\begin{aligned} f'(x) &= a^2(4x^3) + 2ab(2x) + b^2(0) \\ &= 4a^2x^3 + 4abx \\ &= 4ax(ax^2 + b) \end{aligned}$$

$$(iii) \text{ Let } f(x) = \frac{(x - a)}{(x - b)}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x - a}{x - b}\right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x - b)\frac{d}{dx}(x - a) - (x - a)\frac{d}{dx}(x - b)}{(x - b)^2} \\ &= \frac{(x - b)(1) - (x - a)(1)}{(x - b)^2} \\ &= \frac{x - b - x + a}{(x - b)^2} \\ &= \frac{a - b}{(x - b)^2} \end{aligned}$$

Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a .

Ans.

$$\text{Let } f(x) = \frac{x^n - a^n}{x - a}$$

$$\Rightarrow f'(x) = \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)$$

By quotient rule,

$$f'(x) = \frac{(x-a) \frac{d}{dx}(x^n - a^n) - (x^n - a^n) \frac{d}{dx}(x-a)}{(x-a)^2}$$

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question-9

Find the derivative of

(i) $2x - \frac{3}{4}$ (ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$ (iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$ (vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Ans.

(i) Let $f(x) = 2x - \frac{3}{4}$

$$\begin{aligned}f'(x) &= \frac{d}{dx} \left(2x - \frac{3}{4} \right) \\&= 2 \frac{d}{dx}(x) - \frac{d}{dx} \left(\frac{3}{4} \right) \\&= 2 - 0 \\&= 2\end{aligned}$$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned}f'(x) &= (5x^3 + 3x - 1) \frac{d}{dx}(x - 1) + (x - 1) \frac{d}{dx}(5x^3 + 3x - 1) \\&= (5x^3 + 3x - 1)(1) + (x - 1)(5 \cdot 3x^2 + 3 - 0) \\&= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3) \\&= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\&= 20x^3 - 15x^2 + 6x - 4\end{aligned}$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$\begin{aligned}f'(x) &= x^{-3} \frac{d}{dx}(5 + 3x) + (5 + 3x) \frac{d}{dx}(x^{-3}) \\&= x^{-3}(0 + 3) + (5 + 3x)(-3x^{-3-1}) \\&= x^{-3}(3) + (5 + 3x)(-3x^{-4}) \\&= 3x^{-3} - 15x^{-4} - 9x^{-3} \\&= -6x^{-3} - 15x^{-4} \\&= -3x^{-3} \left(2 + \frac{5}{x} \right) \\&= \frac{-3x^{-3}}{x} (2x + 5) \\&= \frac{-3}{x^4} (5 + 2x)\end{aligned}$$

(iv) Let $f(x) = x^5(3 - 6x^{-9})$

By Leibnitz product rule,

$$\begin{aligned}f'(x) &= x^5 \frac{d}{dx}(3 - 6x^{-9}) + (3 - 6x^{-9}) \frac{d}{dx}(x^5) \\&= x^5 \{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4) \\&= x^5(54x^{-10}) + 15x^4 - 30x^{-5} \\&= 54x^{-5} + 15x^4 - 30x^{-5} \\&= 24x^{-5} + 15x^4 \\&= 15x^4 + \frac{24}{x^5}\end{aligned}$$

(v) Let $f(x) = x^{-4} (3 - 4x^{-5})$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^{-4} \frac{d}{dx} (3 - 4x^{-5}) + (3 - 4x^{-5}) \frac{d}{dx} (x^{-4}) \\ &= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1} \\ &= x^{-4} (20x^{-6}) + (3 - 4x^{-5})(-4x^{-5}) \\ &= 20x^{-10} - 12x^{-5} + 16x^{-10} \\ &= 36x^{-10} - 12x^{-5} \\ &= -\frac{12}{x^5} + \frac{36}{x^{10}} \end{aligned}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} \right) - \frac{d}{dx} \left(\frac{x^2}{3x-1} \right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \left[\frac{(x+1) \frac{d}{dx} (2) - 2 \frac{d}{dx} (x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx} (x^2) - x^2 \frac{d}{dx} (3x-1)}{(3x-1)^2} \right] \\ &= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2} \end{aligned}$$

Question-10

Find the derivative of $\cos x$ from first principle.

Ans.

Let $f(x) = \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
 &= -\cos x \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\
 &= -\cos x(0) - \sin x(1) \quad \left[\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 &= -\sin x \\
 \therefore f'(x) &= -\sin x
 \end{aligned}$$

Question-11

Find the derivative of the following functions:

(i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5 \sec x + 4 \cos x$

(iv) $\operatorname{cosec} x$ (v) $3 \cot x + 5 \operatorname{cosec} x$

(vi) $5 \sin x - 6 \cos x + 7$ (vii) $2 \tan x - 7 \sec x$

Ans.

(i) Let $f(x) = \sin x \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h) \cos(x+h) - \sin x \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [2 \sin(x+h) \cos(x+h) - 2 \sin x \cos x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2 \cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos \frac{4x+2h}{2} \sin \frac{2h}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\cos(2x+h) \sin h] \\
 &= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos(2x+0) \cdot 1 \\
 &= \cos 2x
 \end{aligned}$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
 &= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x} \\
 &= \sec x \tan x
 \end{aligned}$$

(iii) Let $f(x) = 5 \sec x + 4 \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5 \sec(x+h) + 4 \cos(x+h) - [5 \sec x + 4 \cos x]}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos x \cos h - \sin x \sin h - \cos x]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [-\cos x(1-\cos h) - \sin x \sin h] \\
&= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] + 4 \left[-\cos x \lim_{h \rightarrow 0} \frac{(1-\cos h)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \right] \\
&= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \cdot \frac{\frac{h}{2}}{\frac{h}{2}}}{\cos(x+h)} \right] + 4 [(-\cos x) \cdot (0) - (\sin x) \cdot 1] \\
&= \frac{5}{\cos x} \cdot \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \cdot \lim_{h \rightarrow 0} \frac{\frac{h}{2}}{\frac{h}{2}} \right] - 4 \sin x \\
&= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4 \sin x \\
&= 5 \sec x \tan x - 4 \sin x \\
&= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)} \right] + 4 [(-\cos x) \cdot (0) - (\sin x) \cdot 1] \\
&= \frac{5}{\cos x} \cdot \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right] - 4 \sin x \\
&= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4 \sin x \\
&= 5 \sec x \tan x - 4 \sin x
\end{aligned}$$

(iv) Let $f(x) = \operatorname{cosec} x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
f'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) \sin x} \right]
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h) \sin x} \right] \\
&\quad - \cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x+h) \sin x}{\sin(x+h) \sin x} \\
&= \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h) \sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
&= -\operatorname{cosec} x \cot x
\end{aligned}$$

(v) Let $f(x) = 3 \cot x + 5 \operatorname{cosec} x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{3 \cot(x+h) + 5 \operatorname{cosec}(x+h) - 3 \cot x - 5 \operatorname{cosec} x}{h} \\
&= 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + 5 \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \quad \dots (1)
\end{aligned}$$

$$\begin{aligned}
\text{Now, } \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h) \sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin x \sin(x+h)} \right] \\
&= - \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\sin x \cdot \sin(x+h)} \right) \\
&= -1 \cdot \frac{1}{\sin x \cdot \sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \dots(2)
\end{aligned}$$

$$\begin{aligned}
&\lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h) \sin x} \right]
\end{aligned}$$

$$\begin{aligned}
& -\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
& = \lim_{h \rightarrow 0} \frac{\sin(x+h) \sin x}{\sin(x+h) \sin x} \\
& = \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h) \sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
& = \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
& = -\operatorname{cosec} x \cot x \quad \dots (3)
\end{aligned}$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5\{\sin(x+h) - \sin x\} - 6\{\cos(x+h) - \cos x\}] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2\cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= 5 \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
&= 5 \left[\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \left[(-\cos x) \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \right] \\
&= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
&= 5 \cos x + 6 \sin x
\end{aligned}$$

(vii) Let $f(x) = 2 \tan x - 7 \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2\{\tan(x+h) - \tan x\} - 7\{\sec(x+h) - \sec x\}] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right]
\end{aligned}$$

$$\begin{aligned}
&= 2 \lim_{h \rightarrow 0} \left[\left(\frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \left[\left(\frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left(\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
&= 2 \cdot 1 \cdot \frac{1}{\cos x \cos x} - 7 \cdot 1 \left(\frac{\sin x}{\cos x \cos x} \right) \\
&= 2 \sec^2 x - 7 \sec x \tan x
\end{aligned}$$

***** END *****