Q17 : A square piece of tin of side 18 cm is to made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer :

Let the side of the square to be cut off be x cm. Then, the length and the breadth of the box will be $(18 - 2x)$ cm each and the height of the box is x cm.

Therefore, the volume $V(x)$ of the box is given by,

$V(x) = x(18 - 2x)^2$

$$\begin{aligned} \therefore V'(x) &= (18-2x)^2 - 4x(18-2x) \\ &= (18-2x)[18-2x-4x] \\ &= (18-2x)(18-6x) \\ &= 6\times 2(9-x)(3-x) \\ &= 12(9-x)(3-x) \end{aligned}$$

$$\begin{aligned} \text{And, } V''(x) &= 12[-(9-x)-(3-x)] \\ &= -12(9-x+3-x) \\ &= -12(12-2x) \\ &= -24(6-x) \end{aligned}$$

Now, $V'(x) = 0 \Rightarrow x = 9$ or $x = 3$

If $x = 9$, then the length and the breadth will become 0.

$\therefore x \neq 9$.

$\Rightarrow x = 3$.

Now, $V''(3) = -24(6-3) = -72 < 0$

$\therefore$ By second derivative test, $x = 3$ is the point of maxima of V.

Hence, if we remove a square of side 3 cm from each corner of the square tin and make a box from the remaining sheet, then the volume of the box obtained is the largest possible.

Answer needs Correction? Click Here

Q18 : A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is the maximum possible?

Answer :

Let the side of the square to be cut off be x cm. Then, the height of the box is x, the length is $45 - 2x$, and the breadth is $24 - 2x$.

Therefore, the volume $V(x)$ of the box is given by,

$$\begin{aligned} V(x) &= x(45-2x)(24-2x) \\ &= x(1080-90x-48x+4x^2) \\ &= 4x^3-138x^2+1080x \end{aligned}$$

$$\begin{aligned} \therefore V'(x) &= 12x^2-276x+1080 \\ &= 12(x^2-23x+90) \\ &= 12(x-18)(x-5) \end{aligned}$$

$V''(x) = 24x - 276 = 12(2x-23)$

Now, $V'(x) = 0 \Rightarrow x = 18$ and $x = 5$

It is not possible to cut off a square of side 18 cm from each corner of the rectangular sheet. Thus, x cannot be equal to 18.

$\therefore x = 5$

Now, $V''(5) = 12(10-23) = 12(-13) = -156 < 0$

$\therefore$ By second derivative test, $x = 5$ is the point of maxima.

Hence, the side of the square to be cut off to make the volume of the box maximum possible is 5 cm.

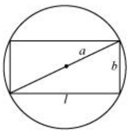
Answer needs Correction? Click Here

Q19 : Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

Answer :

Let a rectangle of length *l* and breadth *b* be inscribed in the given circle of radius *a*.

Then, the diagonal passes through the centre and is of length 2*a* cm.



Now, by applying the Pythagoras theorem, we have:

$$(2a)^2 = l^2 + b^2$$
$$\Rightarrow b^2 = 4a^2 - l^2$$
$$\Rightarrow b = \sqrt{4a^2 - l^2}$$

$\therefore$Area of the rectangle, $A = l\sqrt{4a^2 - l^2}$

$$\therefore \frac{dA}{dl} = \sqrt{4a^2 - l^2} + l\frac{1}{2\sqrt{4a^2 - l^2}}(-2l) = \sqrt{4a^2 - l^2} - \frac{l^2}{\sqrt{4a^2 - l^2}}$$
$$= \frac{4a^2 - 2l^2}{\sqrt{4a^2 - l^2}}$$
$$\frac{d^2A}{dl^2} = \frac{\sqrt{4a^2 - l^2}(-4l) - (4a^2 - 2l^2)\frac{(-2l)}{2\sqrt{4a^2 - l^2}}}{(4a^2 - l^2)}$$
$$= \frac{(4a^2 - l^2)(-4l) + l(4a^2 - 2l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$
$$= \frac{-12a^2 l + 2l^3}{(4a^2 - l^2)^{\frac{3}{2}}} = \frac{-2l(6a^2 - l^2)}{(4a^2 - l^2)^{\frac{3}{2}}}$$

$$\text{Now, } \frac{dA}{dl} = 0 \text{ gives } 4a^2 = 2l^2 \Rightarrow l = \sqrt{2}a$$
$$\Rightarrow b = \sqrt{4a^2 - 2a^2} = \sqrt{2a^2} = \sqrt{2}a$$

$$\text{Now, when } l = \sqrt{2}a,$$
$$\frac{d^2\text{A}}{dl^2} = \frac{-2(\sqrt{2}a)(6a^2 - 2a^2)}{2\sqrt{2}a^3} = \frac{-8\sqrt{2}a^3}{2\sqrt{2}a^3} = -4 < 0$$

$\therefore$ By the second derivative test, when $l = \sqrt{2}a$, then the area of the rectangle is the maximum.

Since $l = b = \sqrt{2}a$, the rectangle is a square.

Hence, it has been proved that of all the rectangles inscribed in the given fixed circle, the square has the maximum area.

Answer needs Correction? **Click Here**

Q20 : Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Answer :

Let *r* and *h* be the radius and height of the cylinder respectively.

Then, the surface area (*S*) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi rh$$
$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$
$$= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let *V* be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$
$$\text{Then, } \frac{dV}{dr} = \frac{S}{2} - 3\pi r^2, \ \frac{d^2V}{dr^2} = -6\pi r$$
$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$$
$$\text{When } r^2 = \frac{S}{6\pi}, \text{ then } \frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0.$$

$\therefore$ By second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

$$\text{Now, when } r^2 = \frac{S}{6\pi}, \text{ then } h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r.$$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.

Answer needs Correction? **Click Here**

Q21 : Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface area?

Answer :

Let *r* and *h* be the radius and height of the cylinder respectively.

Then, volume (*V*) of the cylinder is given by,

$$V = \pi r^2 h = 100 \qquad \text{(given)}$$
$$\therefore h = \frac{100}{\pi r^2}$$

Surface area (*S*) of the cylinder is given by,

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{200}{r}$$
$$\therefore \frac{dS}{dr} = 4\pi r - \frac{200}{r^2}, \quad \frac{d^2S}{dr^2} = 4\pi + \frac{400}{r^3}$$
$$\frac{dS}{dr} = 0 \Rightarrow 4\pi r = \frac{200}{r^2}$$
$$\Rightarrow r^3 = \frac{200}{4\pi} = \frac{50}{\pi}$$
$$\Rightarrow r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$$

Now, it is observed that when $r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} > 0.$

$\therefore$By second derivative test, the surface area is the minimum when the radius of the cylinder is $\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

$$\text{When } r = \left(\frac{50}{\pi}\right)^{\frac{1}{3}}, \; h = \frac{100}{\pi\left(\frac{50}{\pi}\right)^{\frac{2}{3}}} = \frac{2\times 50}{(50)^{\frac{2}{3}}(\pi)^{1-\frac{2}{3}}} = 2\left(\frac{50}{\pi}\right)^{\frac{1}{3}} \text{ cm.}$$

Hence, the required dimensions of the can which has the minimum surface area is given by radius $= \left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm and height = $2\left(\frac{50}{\pi}\right)^{\frac{1}{3}}$ cm.

Q22 : A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?

Answer :

Let a piece of length *l* be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length (28 - *l*) m.

Now, side of square $= \frac{l}{4}$.

Let *r* be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28 - l)$.

The combined areas of the square and the circle (*A*) is given by,

$$A = (\text{side of the square})^2 + \pi r^2$$
$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$
$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$
$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$
$$\frac{d^2A}{dl^2} = \frac{1}{8} + \frac{1}{2\pi} > 0$$
$$\text{Now, } \frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28-l) = 0$$
$$\Rightarrow \frac{\pi l - 4(28-l)}{8\pi} = 0$$
$$\Rightarrow (\pi+4)l - 112 = 0$$
$$\Rightarrow l = \frac{112}{\pi+4}$$

Thus, when $l = \frac{112}{\pi+4}, \frac{d^2A}{dl^2} > 0.$

$\therefore$ By second derivative test, the area (*A*) is the minimum when $l = \frac{112}{\pi+4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi+4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi+4} = \frac{28\pi}{\pi+4}$ cm.

Q23 : Prove that the volume of the largest cone that can be inscribed in a sphere of radius *R* is $\frac{8}{27}$ of the volume of the sphere.

Answer :

Let *r* and *h* be the radius and height of the cone respectively inscribed in a sphere of radius *R*.

Let V be the volume of the cone.

Then, $V = \frac{1}{3}\pi r^2 h$

Height of the cone is given by,

$h = R + AB = R + \sqrt{R^2 - r^2}$ [ABC is a right triangle]

$$\therefore V = \frac{1}{3}\pi r^2\left(R + \sqrt{R^2 - r^2}\right)$$
$$= \frac{1}{3}\pi r^2 R + \frac{1}{3}\pi r^2\sqrt{R^2 - r^2}$$
$$\therefore \frac{dV}{dr} = \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} + \frac{1}{3}\pi r^2 \cdot \frac{(-2r)}{2\sqrt{R^2 - r^2}}$$
$$= \frac{2}{3}\pi rR + \frac{2}{3}\pi r\sqrt{R^2 - r^2} - \frac{1}{3}\pi\frac{r^3}{\sqrt{R^2 - r^2}}$$
$$= \frac{2}{3}\pi rR + \frac{2\pi r\left(R^2 - r^2\right) - \pi r^3}{3\sqrt{R^2 - r^2}}$$
$$= \frac{2}{3}\pi rR + \frac{2\pi rR^2 - 3\pi r^3}{3\sqrt{R^2 - r^2}}$$
$$\frac{d^2V}{dr^2} = \frac{2\pi R}{3} + \frac{3\sqrt{R^2 - r^2}\left(2\pi R^2 - 9\pi r^2\right) - \left(2\pi rR^2 - 3\pi r^3\right)\cdot\frac{(-2r)}{6\sqrt{R^2 - r^2}}}{9\left(R^2 - r^2\right)}$$
$$= \frac{2}{3}\pi R + \frac{9\left(R^2 - r^2\right)\left(2\pi R^2 - 9\pi r^2\right) + 2\pi r^2 R^2 + 3\pi r^4}{27\left(R^2 - r^2\right)^{\frac{3}{2}}}$$
$$\text{Now, } \frac{dV}{dr} = 0 \Rightarrow \frac{2}{3}\pi rR = \frac{3\pi r^3 - 2\pi rR^2}{3\sqrt{R^2 - r^2}}$$
$$\Rightarrow 2R = \frac{3r^2 - 2R^2}{\sqrt{R^2 - r^2}} \Rightarrow 2R\sqrt{R^2 - r^2} = 3r^2 - 2R^2$$
$$\Rightarrow 4R^2\left(R^2 - r^2\right) = \left(3r^2 - 2R^2\right)^2$$
$$\Rightarrow 4R^4 - 4R^2r^2 = 9r^4 + 4R^4 - 12r^2R^2$$
$$\Rightarrow 9r^4 = 8R^2r^2$$
$$\Rightarrow r^2 = \frac{8}{9}R^2$$

When $r^2 = \frac{8}{9}R^2$, then $\frac{d^2V}{dr^2} < 0$.

$\therefore$ By second derivative test, the volume of the cone is the maximum when $r^2 = \frac{8}{9}R^2$.

When $r^2 = \frac{8}{9}R^2$, $h = R + \sqrt{R^2 - \frac{8}{9}R^2} = R + \sqrt{\frac{1}{9}R^2} = R + \frac{R}{3} = \frac{4}{3}R$.

Therefore,

$$= \frac{1}{3}\pi\left(\frac{8}{9}R^2\right)\left(\frac{4}{3}R\right)$$
$$= \frac{8}{27}\left(\frac{4}{3}\pi R^3\right)$$
$$= \frac{8}{27} \times (\text{Volume of the sphere})$$

Hence, the volume of the largest cone that can be inscribed in the sphere is $\frac{8}{27}$ the volume of the sphere.

Answer needs Correction? **Click Here**

Q24 : Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ time the radius of the base.

Answer :

Let r and h be the radius and the height (altitude) of the cone respectively.

Then, the volume (V) of the cone is given as:

Answer needs Correction? **Click Here**

Q25 : Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

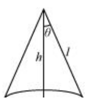
Answer :

Let θ be the semi-vertical angle of the cone.

It is clear that $\theta \in \left[0, \frac{\pi}{2}\right]$.

Let r, h, and l be the radius, height, and the slant height of the cone respectively.

The slant height of the cone is given as constant.



Now, $r = l\sin\theta$ and $h = l\cos\theta$

The volume (V) of the cone is given by,

$$V = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3}\pi\left(l^2\sin^2\theta\right)\left(l\cos\theta\right)$$
$$= \frac{1}{3}\pi l^3\sin^2\theta\cos\theta$$
$$\therefore \frac{dV}{d\theta} = \frac{l^3\pi}{3}\left[\sin^2\theta\left(-\sin\theta\right) + \cos\theta\left(2\sin\theta\cos\theta\right)\right]$$
$$= \frac{l^3\pi}{3}\left[-\sin^3 + 2\sin\theta\cos^2\theta\right]$$
$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[-3\sin^2\theta\cos\theta + 2\cos^3\theta - 4\sin^2\theta\cos\theta\right]$$
$$= \frac{l^3\pi}{3}\left[2\cos^3\theta - 7\sin^2\theta\cos\theta\right]$$

$$\text{Now, } \frac{dV}{d\theta} = 0$$
$$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$$
$$\Rightarrow \tan^2\theta = 2$$
$$\Rightarrow \tan\theta = \sqrt{2}$$
$$\Rightarrow \theta = \tan^{-1}\sqrt{2}$$

Now, when $\theta = \tan^{-1}\sqrt{2}$, then $\tan^2\theta = 2$ or $\sin^2\theta = 2\cos^2\theta$.

Then, we have:

$$\frac{d^2V}{d\theta^2} = \frac{l^3\pi}{3}\left[2\cos^3\theta - 14\cos^3\theta\right] = -4\pi l^3\cos^3\theta < 0 \text{ for } \theta \in \left[0, \frac{\pi}{2}\right]$$

$\therefore$By second derivative test, the volume (V) is the maximum when $\theta = \tan^{-1}\sqrt{2}$.

Hence, for a given slant height, the semi-vertical angle of the cone of the maximum volume is $\tan^{-1}\sqrt{2}$.

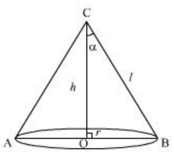
Answer needs Correction? Click Here

Q26 : **Show that semi-vertical angle of right circular cone of given surface area and maximum volume is** $Sin^{-1}\left(\frac{1}{3}\right)$.

Answer :

Let r be the radius, l be the slant height and h be the height of the cone of given surface area, S.

Also, let α be the semi-vertical angle of the cone.



Then $S =$ Ãâ‚¬rl + Ãâ‚¬r^2

$$\Rightarrow l = \frac{S - \pi r^2}{\pi r} \text{....(1)}$$

Let V be the volume of the cone.

$$\text{Then } V = \frac{1}{3}\pi r^2 h$$
$$\Rightarrow V^2 = \frac{1}{9}\pi^2 r^4 h^2$$
$$= \frac{1}{9}\pi^2 r^4 (l^2 - r^2) \quad \left[\text{As } l^2 = r^2 + h^2\right]$$
$$= \frac{1}{9}\pi^2 r^4\left[\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2\right]$$
$$= \frac{1}{9}\pi^2 r^4\left[\frac{(S - \pi r^2)^2 - \pi^2 r^4}{\pi^2 r^2}\right]$$
$$= \frac{1}{9}r^2(S^2 - 2S\pi r^2)$$
$$\Rightarrow V^2 = \frac{1}{9}Sr^2(S - 2\pi r^2)\text{.....(2)}$$

Differentiating (2) with respect to r, we get

$$2V\frac{dV}{dr} = \frac{1}{9}S(2Sr - 8\pi r^3)$$

For maximum or minimum, put $\frac{dV}{dr} = 0$

$$\Rightarrow \frac{1}{9}S(2Sr - 8\pi r^3) = 0$$
$$\Rightarrow 2Sr - 8\pi r^3 = 0 \quad (\text{As } S \neq 0)$$
$$\Rightarrow S = 4\pi r^2 \quad (\text{As } r \neq 0)$$
$$\Rightarrow r^2 = \frac{S}{4\pi}$$

Differentiating again with respect to r, we get

$$2V\frac{d^2V}{dr^2} + 2\left(\frac{dV}{dr}\right)^2 = \frac{1}{9}S(2S - 24\pi r^2)$$
$$\Rightarrow 2V\frac{d^2V}{dr^2} = \frac{1}{9}S\left(2S - 24\pi \times \frac{S}{4\pi}\right) \qquad \left(\text{As } \frac{dV}{dr} = 0 \text{ and } r^2 = \frac{S}{4\pi}\right)$$

$$=\frac{1}{9}S(2S-6S)$$

$$=-\frac{4}{9}S^2<0$$

Thus, V is maximum when S= 4Ãâ‚¬r²

Answer needs Correction? **Click Here**

Q27 : The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is

(A) $\left(2\sqrt{2},4\right)$ (B) $\left(2\sqrt{2},0\right)$

(C) (0, 0) (D) (2, 2)

Answer :

The given curve is $x^2 = 2y$.

For each value of x, the position of the point will be $\left(x,\ \frac{x^2}{2}\right)$.

Let P and A(0, 5) are the given points.
Now distance between the points P and A is given by,

Answer needs Correction? **Click Here**

Q28 : For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(A) 0 (B) 1

(C) 3 (D) $\frac{1}{3}$

Answer :

Let $f(x)=\frac{1-x+x^2}{1+x+x^2}$.

$$\therefore f'(x)=\frac{\left(1+x+x^2\right)\left(-1+2x\right)-\left(1-x+x^2\right)\left(1+2x\right)}{\left(1+x+x^2\right)^2}$$

$$=\frac{-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3}{\left(1+x+x^2\right)^2}$$

$$=\frac{2x^2-2}{\left(1+x+x^2\right)^2}=\frac{2\left(x^2-1\right)}{\left(1+x+x^2\right)^2}$$

$$\therefore f'(x)=0 \;\Rightarrow\; x^2=1 \Rightarrow x=\pm 1$$

$$\text{Now, } f''(x)=\frac{2\left[\left(1+x+x^2\right)^2(2x)-\left(x^2-1\right)(2)\left(1+x+x^2\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$=\frac{4\left(1+x+x^2\right)\left[\left(1+x+x^2\right)x-\left(x^2-1\right)(1+2x)\right]}{\left(1+x+x^2\right)^4}$$

$$=\frac{4\left[x+x^2+x^3-x^2-2x^3+1+2x\right]}{\left(1+x+x^2\right)^3}$$

$$=\frac{4\left(1+3x-x^3\right)}{\left(1+x+x^2\right)^3}$$

$$\text{And, } f''(1)=\frac{4(1+3-1)}{(1+1+1)^3}=\frac{4(3)}{(3)^3}=\frac{4}{9}>0$$

$$\text{Also, } f''(-1)=\frac{4(1-3+1)}{(1-1+1)^3}=4(-1)=-4<0$$

$\therefore$By second derivative test, f is the minimum at $x = 1$ and the minimum value is given by
$f(1)=\frac{1-1+1}{1+1+1}=\frac{1}{3}$.

The correct answer is D.

Answer needs Correction? **Click Here**

Q29 : The maximum value of $\left[x(x-1)+1\right]^{\frac{1}{3}}, 0\le x\le 1$ is

(A) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (B) $\frac{1}{2}$

(C) 1 (D) 0

Answer :

Let $f(x)=\left[x(x-1)+1\right]^{\frac{1}{3}}$.

$$\therefore f'(x)=\frac{2x-1}{3\left[x(x-1)+1\right]^{\frac{2}{3}}}$$

$$\text{Now, } f'(x)=0 \;\Rightarrow\; x=\frac{1}{2}$$

Then, we evaluate the value of f at critical point $x=\frac{1}{2}$ and at the end points of the interval [0, 1] {i.e., at $x = 0$ and $x = 1$}.

$$f(0)=\left[0(0-1)+1\right]^{\frac{1}{3}}=1$$
$$f(1)=\left[1(1-1)+1\right]^{\frac{1}{3}}=1$$
$$f\left(\frac{1}{2}\right)=\left[\frac{1}{2}\left(\frac{-1}{2}\right)+1\right]^{\frac{1}{3}}=\left(\frac{3}{4}\right)^{\frac{1}{3}}$$

Hence, we can conclude that the maximum value of f in the interval [0, 1] is 1.

The correct answer is C.

*********** END **********