CHAPTER 15

WAVE MOTION AND WAVES ON A STRING

15.1 WAVE MOTION

When a particle moves through space, it carries kinetic energy with itself. Wherever the particle goes, the energy goes with it. The energy is associated with the particle and is transported from one region of the space to the other together with the particle just like we ride a car and are taken from Lucknow to Varanasi with the car.

There is another way to transport energy from one part of space to the other without any bulk motion of material together with it. Sound is transmitted in air in this manner. When you say "Hello" to your friend, no material particle is ejected from your lips and falls on your friend's ear. You create some disturbance in the part of the air close to your lips. Energy is transferred to these air particles either by pushing them ahead or pulling them back. The density of the air in this part temporarily increases or decreases. These disturbed particles exert force on the next layer of air, transferring the disturbance to that layer. In this way, the disturbance proceeds in air and finally the air near the ear of the listener gets disturbed.

The disturbance produced in the air near the speaker travels in air, the air itself does not move. The air that is near the speaker at the time of uttering a word remains all the time near the speaker even when the message reaches the listener. This type of motion of energy is called a *wave motion.*

To give another example of propagation of energy without bulk motion of matter, suppose many persons are standing in a queue to buy cinema tickets from the ticket counter. It is not yet time, the counter is closed and the persons are getting annoyed. The last person in the queue is somewhat unruly, he leans forward pushing the man in front of him and then stands straight. The second last person, getting the jerk from behind, is forced to lean forward and push the man in front. This second last person manages to stand straight again but the third last person temporarily loses balance and leans forward. The jerk thus travels down the queue and finally the person at the front of the queue feels it. With the jerk, travels the energy down the queue from one end to another though the last person and the first person are still in their previous positions.

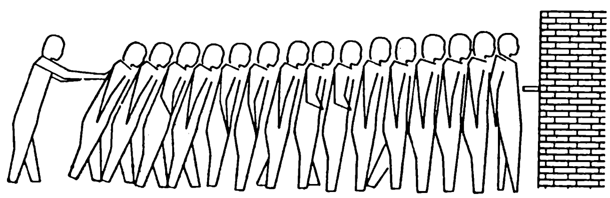

Figure 15.1

The world is full of examples of wave motion. When raindrops hit the surface of calm water, circular waves can be seen travelling on the surface. Any particle of water is only locally displaced for a short time but the disturbance spreads and the particles farther and farther get disturbed when the wave reaches them. Another common example of wave motion is the wave associated with light. One speciality about this wave is that it does not require any material medium for its propagation. The waves requiring a medium are called *mechanical waves* and those which do not require a medium are called *nonmechanical waves.*

In the present chapter, we shall study the waves on a stretched string, a mechanical wave in one dimension.

15.2 WAVE PULSE ON A STRING

Let us consider a long string with one end fixed to a wall and the other held by a person. The person pulls on the string keeping it tight. Suppose the person snaps his hand a little up and down producing a bump in the string near his hand (Figure 15.2). The operation takes a very small time say one tenth of a second after which the person stands still holding the string tight in his hand. What happens as time passes ?

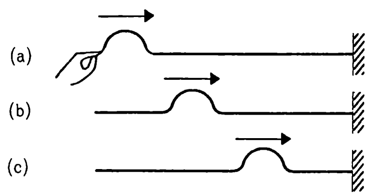

Figure 15.2

Experiments show that if the vertical displacement given is small, the disturbance travels down the string with constant speed. Figure (15.2) also shows the status of the string at successive instants. As time passes, the "bump" travels on the string towards right. For an elastic and homogeneous string, the bump moves with constant speed to cover equal distances in equal time. Also, the shape of the bump is not altered as it moves, provided the bump is small. Notice that no part of the string moves from left to right. The person is holding the left end tight and the string cannot slip from his hand. The part of the string, where the bump is present at an instant, is in up-down motion. As time passes, this part again regains its normal position. The person does some work on the part close to his hand giving some energy to that part. This disturbed part exerts elastic force on the part to the right and transfers the energy, the bump thus moves on to the right. In this way, different parts of the string are successively disturbed, transmitting the energy from left to right.

When a disturbance is localised only to a small part of space at a time, we say that a *wave pulse* is passing through that part of the space. This happens when the source producing the disturbance (hand in this case) is active only for a short time. If the source is active for some extended time repeating its motion several times, we get a *wave train* or a *wave packet*. For example, if the person in figure (15.2) decides to vibrate his hand up and down 10 times and then stop, a wave train consisting of 10 loops will proceed on the string.

Equation of a Travelling Wave

Suppose, in the example of figure (15.2), the man starts snapping his hand at $t = 0$ and finishes his job at $t = \Delta t$. The vertical displacement y of the left end of the string is a function of time. It is zero for $t < 0$, has non-zero value for $0 < t < \Delta t$ and is again zero for $t > \Delta t$. Let us represent this function by $f(t)$. Take the left end of the string as the origin and take the X-axis along the string towards right. The function $f(t)$ represents the displacement y of the particle at $x = 0$ as a function of time

$$y(x = 0, t) = f(t).$$

The disturbance travels on the string towards right with a constant speed v. Thus, the displacement, produced at the left end at time t, reaches the point x at time $t + x/v$. Similarly, the displacement of the particle at point x at time t was originated at the left end at the time $t - x/v$. But the displacement of the left end at time $t - x/v$ is $f(t - x/v)$. Hence,

$$y(x, t) = y(x = 0, t - x/v)$$
$$= f(t - x/v).$$

The displacement of the particle at x at time t i.e., $y(x, t)$ is generally abbreviated as y and the wave equation is written as

$$y = f(t - x/v). \quad \dots (15.1)$$

Equation (15.1) represents a wave travelling in the positive x-direction with a constant speed v. Such a wave is called a *travelling wave* or a *progressive wave*. The function f is arbitrary and depends on how the source moves. The time t and the position x must appear in the wave equation in the combination $t - x/v$ only. For example,

$$y = A \sin\frac{(t - x/v)}{T}, \quad y = A\, e^{-\frac{(t - x/v)}{T}}$$

etc. are valid wave equations. They represent waves travelling in positive x-direction with constant speed. The equation $y = A \sin \frac{(x^2 - v^2 t^2)}{L^2}$ does not represent a wave travelling in x-direction with a constant speed.

If a wave travels in negative x-direction with speed v, its general equation may be written as

$$y = f(t + x/v). \quad \dots (15.2)$$

The wave travelling in positive x-direction (equation 15.1) can also be written as

$$y = f\left(\frac{vt - x}{v}\right)$$

or, $$y = g(x - vt), \quad \dots (15.3)$$

where g is some other function having the following meaning. If we put $t = 0$ in equation (15.3), we get the displacement of various particles at $t = 0$ i.e.,

$$y(x,\ t = 0) = g(x).$$

Thus, $g(x)$ represents the shape of the string at $t = 0$. If the displacement of the different particles at $t = 0$ is represented by the function $g(x)$, the displacement of the particle at x at time t will be $y = g(x - vt)$. Similarly, if the wave is travelling along the negative x-direction and the displacement of different particles at $t = 0$ is $g(x)$, the displacement of the particle at x at time t will be

$$y = g(x + vt). \qquad \ldots \ (15.4)$$

Thus, the function f in equation (15.1) and (15.2) represents the displacement of the point $x = 0$ as time passes and g in (15.3) and (15.4) represents the displacement at $t = 0$ of different particles.

Example 15.1

A wave is propagating on a long stretched string along its length taken as the positive x-axis. The wave equation is given as

$$y = y_0 e^{-\left(\frac{t}{T} - \frac{x}{\lambda}\right)^2}$$

where $y_0 = 4$ mm, $T = 1{\cdot}0$ s *and* $\lambda = 4$ cm. *(a) Find the velocity of the wave. (b) Find the function f(t) giving the displacement of the particle at x = 0. (c) Find the function g(x) giving the shape of the string at t = 0. (d) Plot the shape g(x) of the string at t = 0. (e) Plot the shape of the string at t = 5 s.*

Solution : (a) The wave equation may be written as

$$y = y_0 e^{-\frac{1}{T^2}\left(t - \frac{x}{\lambda/T}\right)^2}.$$

Comparing with the general equation $y = f(t - x/v)$, we see that

$$v = \frac{\lambda}{T} = \frac{4 \text{ cm}}{1{\cdot}0 \text{ s}} = 4 \text{ cm/s}.$$

(b) putting $x = 0$ in the given equation,

$$f(t) = y_0 e^{-(t/T)^2}. \qquad \ldots \ \text{(i)}$$

(c) putting $t = 0$ in the given equation

$$g(x) = y_0 e^{-(x/\lambda)^2}. \qquad \ldots \ \text{(ii)}$$

(d)

fig-15.3 (a)

(e)

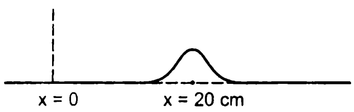

fig-15.3 (b)

15.3 SINE WAVE TRAVELLING ON A STRING

What happens if the person holding the string in figure (15.2) keeps waving his hand up and down continuously. He keeps doing work on the string and the energy is continuously supplied to the string. Any part of the string continues to vibrate up and down once the first disturbance has reached it. It receives energy from the left, transmits it to the right and the process continues till the person is not tired. The nature of vibration of any particle is similar to that of the left end, the only difference being that the motion is repeated after a time delay of x/v.

A very important special case arises when the person vibrates the left end $x = 0$ in a simple harmonic motion. The equation of motion of this end may then be written as

$$f(t) = A \sin \omega t, \qquad \ldots \ (15.5)$$

where A represents the amplitude and ω the angular frequency. The time period of oscillation is $T = 2\pi/\omega$ and the frequency of oscillation is $\nu = 1/T = \omega/2\pi$. The wave produced by such a vibrating source is called a *sine wave* or *sinusoidal wave*.

Since the displacement of the particle at $x = 0$ is given by (15.5), the displacement of the particle at x at time t will be

$$y = f(t - x/v)$$

or,

$$y = A \sin \omega(t - x/v). \qquad \ldots \ (15.6)$$

This follows from the fact that the wave moves along the string with a constant speed v and the displacement of the particle at x at time t was originated at $x = 0$ at time $t - x/v$.

The velocity of the particle at x at time t is given by

$$\frac{\partial y}{\partial t} = A\,\omega \cos \omega(t - x/v). \qquad \ldots \ (15.7)$$

The symbol $\frac{\partial}{\partial t}$ is used in place of $\frac{d}{dt}$ to indicate that while differentiating with respect to t, we should treat x as constant. It is the same particle whose displacement should be considered as a function of time.

This velocity is totally different from the wave velocity v. The wave moves on the string at a constant velocity v along the x-axis, but the particle moves up and down with velocity $\frac{\partial y}{\partial t}$ which changes with x and t according to (15.7).

Figure (15.4) shows the shape of the string as time passes. Each particle of the string vibrates in simple harmonic motion with the same amplitude A and frequency ν. The phases of the vibrations are, however, different. When a particle P (figure 15.4) reaches its extreme position in upward direction, the particle Q little to its right, is still coming up and the particle R little to its left, has already crossed that phase and is going down. The phase difference is larger if the particles are separated farther.

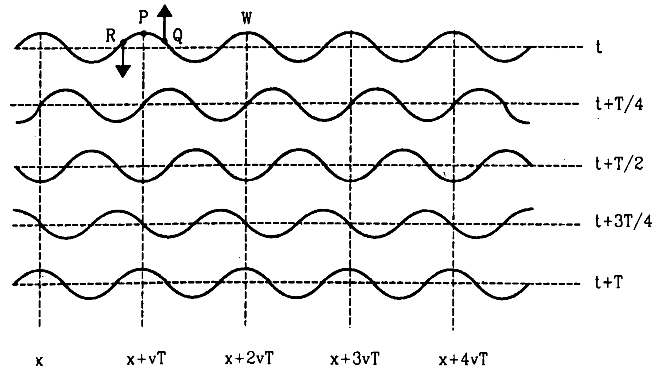

Figure 15.4

Each particle copies the motion of another particle at its left with a time delay of x/v, where x is the separation between the two particles. For the particles P and W, shown in figure (15.4), the separation is $\Delta x = vT$ and the particle W copies the motion of P after a time delay of $\Delta x/v = T$. But the motion of any particle at any instant is identical in all respects to its motion a time period T later. So, a delay of one time period is equivalent to no delay and hence, the particles P and W vibrate in the same phase. They reach their extreme positions together, they cross their mean positions together, their displacements are identical and their velocities are identical at any instant. Same is true for any pair of particles separated by a distance vT. This separation is called the *wavelength* of the wave and is denoted by the Greek letter λ. Thus, $\lambda = vT$.

The above relation can easily be derived mathematically. Suppose, the particles at x and $x + L$ vibrate in the same phase. By equation (15.6) and (15.7),

$$A \sin\left[\omega\left(t - \frac{x}{v}\right)\right] = A \sin\left[\omega\left(t - \frac{x+L}{v}\right)\right]$$

and $$A\,\omega \cos\left[\omega\left(t - \frac{x}{v}\right)\right] = A\,\omega \cos\left[\omega\left(t - \frac{x+L}{v}\right)\right].$$

This gives

$$\omega\left(t - \frac{x}{v}\right) = \omega\left(t - \frac{x+L}{v}\right) + 2\,n\,\pi,$$

where n is an integer.

or, $$0 = -\frac{\omega L}{v} + 2\,n\,\pi$$

or, $$L = \frac{v}{\omega} 2\,n\,\pi.$$

The minimum separation between the particles vibrating in same phase is obtained by putting $n = 1$ in the above equation. Thus, the wavelength is

$$\lambda = \frac{v}{\omega} 2\pi = vT. \qquad \dots (15.8)$$

Also, $$v = \lambda/T = \nu\lambda, \qquad \dots (15.9)$$

where $\nu = 1/T$ is the frequency of the wave.

This represents an important relation between the three characteristic parameters of a sine wave namely, the wave velocity, the frequency and the wavelength.

The quantity $2\pi/\lambda$ is called the *wave number* and is generally denoted by the letter k.

Thus, $$k = \frac{2\,\pi}{\lambda} = \frac{2\pi\nu}{v} = \frac{\omega}{v}.$$

The segment, where the disturbance is positive, is called a *crest* of the wave and the segment, where the disturbance is negative, is called a *trough*. The separation between consecutive crests or between consecutive troughs is equal to the wavelength.

Alternative Forms of Wave Equation

We have written the wave equation of a wave travelling in x-direction as

$$y = A \sin \omega(t - x/v).$$

This can also be written in several other forms such as,

$$y = A \sin(\omega t - kx) \qquad \dots (15.10)$$

$$= A \sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right) \qquad \dots (15.11)$$

$$= A \sin k(vt - x). \qquad \dots (15.12)$$

Also, it should be noted that we have made our particular choice of $t = 0$ in writing equation (15.5) from which the wave equation is deduced. The origin of time is chosen at an instant when the left end $x = 0$ is crossing its mean position $y = 0$ and is going up. For a general choice of the origin of time, we will have to add a phase constant so that the equation will be

$$y = A \sin[\omega(t - x/v) + \phi]. \qquad \dots (15.13)$$

The constant ϕ will be $\pi/2$ if we choose $t = 0$ at an instant when the left end reaches its extreme position $y = A$. The equation will then be

$$y = A \cos \omega(t - x/v). \qquad \dots (15.14)$$

If $t = 0$ is taken at the instant when the left end is crossing the mean position from upward to downward direction, ϕ will be π and the equation will be

$$y = A \sin \omega\left(\frac{x}{v} - t\right)$$

or, $$y = A \sin(kx - \omega t). \qquad \dots (15.15)$$

Example 15.2

Consider the wave $y = (5\text{ mm}) \sin[(1\text{ cm}^{-1})x - (60\text{ s}^{-1})t]$. Find (a) the amplitude (b) the wave number, (c) the *wavelength, (d) the frequency, (e) the time period and (f) the wave velocity.*

Solution : Comparing the given equation with equation (15.15), we find

(a) amplitude $A = 5 \text{ mm}$

(b) wave number $k = 1 \text{ cm}^{-1}$

(c) wavelength $\lambda = \frac{2\pi}{k} = 2\pi \text{ cm}$

(d) frequency $\nu = \frac{\omega}{2\pi} = \frac{60}{2\pi} \text{ Hz}$

$= \frac{30}{\pi} \text{ Hz}$

(e) time period $T = \frac{1}{\nu} = \frac{\pi}{30} \text{ s}$

(f) wave velocity $v = \nu \lambda = 60 \text{ cm/s}$.

15.4 VELOCITY OF A WAVE ON A STRING

The velocity of a wave travelling on a string depends on the elastic and the inertia properties of the string. When a part of the string gets disturbed, it exerts an extra force on the neighbouring part because of the elastic property. The neighbouring part responds to this force and the response depends on the inertia property. The elastic force in the string is measured by its tension F and the inertia by its mass per unit length. We have used the symbol F for tension and not T in order to avoid confusion with the time period.

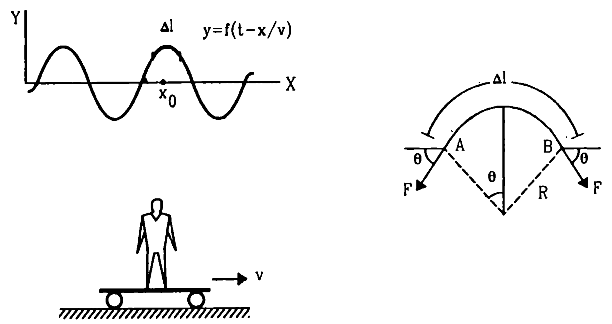

Figure 15.5

Suppose a wave $y = f\left(t - \frac{x}{v}\right)$ is travelling on the string in the positive x-direction with a speed v. Let us choose an observer who is riding on a car that moves along the x-direction with the same velocity v (figure 15.5). Looking from this frame, the pattern of the string is at rest but the entire string is moving towards the negative x-direction with a speed v. If a crest is opposite to the observer at any instant, it will always remain opposite to him with the same shape while the string will pass through this crest in opposite direction like a snake.

Consider a small element AB of the string of length Δl at the highest point of a crest. Any small curve may be approximated by a circular arc. Suppose the small element Δl forms an arc of radius R. The particles of the string in this element go in this circle with a speed v as the string slides through this part. The general situation is shown in figure (15.5a) and the expanded view of the part near Δl is shown in figure (15.5b).

We assume that the displacements are small so that the tension in the string does not appreciably change because of the disturbance. The element AB is pulled by the parts of the string to its right and to its left. Resultant force on this element is in the downward direction as shown in figure (15.5b) and its magnitude is

$$F_r = F\sin\theta + F\sin\theta = 2F\sin\theta.$$

As Δl is taken small, θ will be small and

$$\sin\theta \approx \frac{\Delta l/2}{R}$$

so that the resultant force on Δl is

$$F_r = 2F\left(\frac{\Delta l/2}{R}\right) = F\Delta l/R.$$

If μ be the mass per unit length of the string, the element AB has a mass $\Delta m = \Delta l\, \mu$. Its downward acceleration is

$$a = \frac{F_r}{\Delta m} = \frac{F\Delta l/R}{\mu\, \Delta l} = \frac{F}{\mu R}.$$

But the element is moving in a circle of radius R with a constant speed v. Its acceleration is, therefore, $a = \frac{v^2}{R}$. The above equation becomes

$$\frac{v^2}{R} = \frac{F}{\mu R}$$

or, $$v = \sqrt{F/\mu}. \quad \dots (15.16)$$

The velocity of the wave on a string thus depends only on the tension F and the linear mass density μ. We have used the approximation that the tension F remains almost unchanged as the part of the string vibrates up and down. This approximation is valid only for small amplitudes because as the string vibrates, the lengths of its parts change during the course of vibration and hence, the tension changes.

Example 15.3

Figure (15.6) shows a string of linear mass density 1·0 g/cm *on which a wave pulse is travelling. Find the*

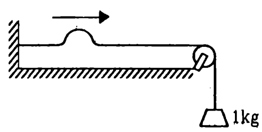

Figure 15.6

time taken by the pulse in travelling through a distance of 50 cm *on the string. Take* $g = 10\ m/s^2$.

Solution : The tension in the string is $F = mg = 10$ N. The mass per unit length is $\mu = 1{\cdot}0$ g/cm $= 0{\cdot}1$ kg/m. The wave velocity is, therefore, $v = \sqrt{F/\mu} = \sqrt{\dfrac{10\ \mathrm{N}}{0{\cdot}1\ \mathrm{kg/m}}} = 10$ m/s. The time taken by the pulse in travelling through 50 cm is, therefore, 0·05 s.

15.5 POWER TRANSMITTED ALONG THE STRING BY A SINE WAVE

When a travelling wave is established on a string, energy is transmitted along the direction of propagation of the wave. Consider again a sine wave travelling along a stretched string in x-direction. The equation for the displacement in y-direction is

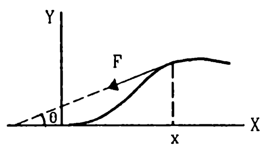

Figure 15.7

$$y = A \sin \omega(t - x/v). \qquad \text{... (i)}$$

Figure (15.7) shows a portion of the string at a time t to the right of position x. The string on the left of the point x exerts a force F on this part. The direction of this force is along the tangent to the string at position x. The component of the force along the Y-axis is

$$F_y = -F \sin\theta \approx -F \tan\theta = -F \frac{\partial y}{\partial x}.$$

The power delivered by the force F to the string on the right of position x is, therefore,

$$P = \left(-F \frac{\partial y}{\partial x}\right) \frac{\partial y}{\partial t}.$$

By (i), it is

$$-F\left[\left(-\frac{\omega}{v}\right) A \cos \omega(t - x/v)\right] [\omega A \cos \omega(t - x/v)]$$

$$= \frac{\omega^2 A^2 F}{v} \cos^2 \omega\,(t - x/v).$$

This is the rate at which energy is being transmitted from left to right across the point at x. The $\cos^2$ term oscillates between 0 and 1 during a cycle and its average value is 1/2. The average power transmitted across any point is, therefore,

$$P_{av} = \frac{1}{2}\frac{\omega^2 A^2 F}{v} = 2\pi^2 \mu\, v\, A^2 \nu^2. \qquad \text{... (15.17)}$$

The power transmitted along the string is proportional to the square of the amplitude and square of the frequency of the wave.

Example 15.4

The average power transmitted through a given point on a string supporting a sine wave is 0·20 W *when the amplitude of the wave is* 2·0 mm. *What power will be transmitted through this point if the amplitude is increased to* 3·0 mm.

Solution : Other things remaining the same, the power transmitted is proportional to the square of the amplitude.

Thus,

$$\frac{P_2}{P_1} = \frac{A_2^2}{A_1^2}$$

or, $$\frac{P_2}{0{\cdot}20\ \mathrm{W}} = \frac{9}{4} = 2{\cdot}25$$

or, $$P_2 = 2{\cdot}25 \times 0{\cdot}20\mathrm{W} = 0{\cdot}45\ \mathrm{W}.$$

15.6 INTERFERENCE AND THE PRINCIPLE OF SUPERPOSITION

So far we have considered a single wave passing on a string. Suppose two persons are holding the string at the two ends and snap their hands to start a wave pulse each. One pulse starts from the left end and travels on the string towards right, the other starts at the right end and travels towards left. The pulses travel at same speed although their shapes depend on how the persons snap their hands. Figure (15.8) shows the shape of the string as time passes.

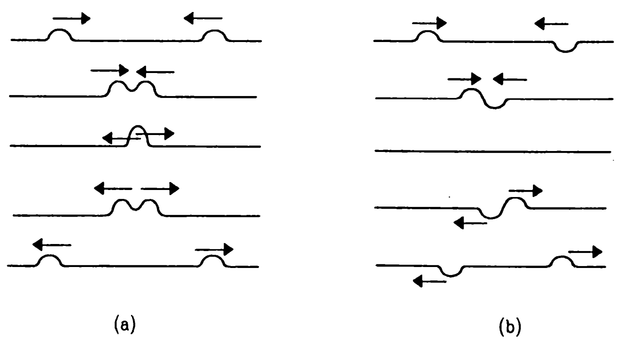

Figure 15.8

The pulses travel towards each other, overlap and recede from each other. The remarkable thing is that the shapes of the pulses, as they emerge after the overlap, are identical to their original shapes. Each pulse has passed the overlap region so smoothly as if the other pulse was not at all there. After the encounter, each pulse looks just as it looked before and each pulse travels just as it did before. The waves can pass through each other freely without being modified.

This is a unique property of the waves. The particles cannot pass through each other, they collide and their course of motion changes. How do we determine the shape of the string at the time when the pulses actually overlap? The mechanism to know the resultant displacement of a particle which is acted upon by two or more waves simultaneously is very simple. The displacement of the particle is equal to the sum of the displacements the waves would have individually produced. If the first wave alone is travelling, let us say it displaces the particle by 0·2 cm upward and if the second wave alone is travelling, suppose the displacement of this same particle is 0·4 cm upward at that instant. The displacement of the particle at that instant will be 0·6 cm upward if both the waves pass through that particle simultaneously. The displacement of the particles, if the first wave alone were travelling, may be written as

$$y_1 = f_1(t - x/v)$$

and the displacement if the second wave alone were travelling may be written as

$$y_2 = f_2(t + x/v).$$

If both the waves are travelling on the string, the displacement of its different particles will be given by

$$y = y_1 + y_2 = f_1(t - x/v) + f_2(t + x/v).$$

The two individual displacements may be in opposite directions. The magnitude of the resulting displacement may be smaller than the magnitudes of the individual displacements.

If two wave pulses, approaching each other, are identical in shape except that one is inverted with respect to the other, at some instant the displacement of all the particles will be zero. However, the velocities of the particles will not be zero as the waves will emerge in the two directions shortly. Such a situation is shown in figure (15.8b). We see that there is an instant when the string is straight every where. But soon the wave pulses emerge which move away from each other

Suppose one person snaps the end up and down whereas the other person snaps his end sideways. The displacements produced are at right angles to each other as indicated in figure (15.9). When the two waves overlap, the resultant displacement of any particle is the vector sum of the two individual displacements.

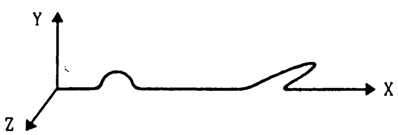

Figure 15.9

The above observations about the overlap of the waves may be summarised in the following statement which is known as the *principle of superposition.*

When two or more waves simultaneously pass through a point, the disturbance at the point is given by the sum of the disturbances each wave would produce in absence of the other wave(s).

In general, the principle of superposition is valid for small disturbances only. If the string is stretched too far, the individual displacements do not add to give the resultant displacement. Such waves are called *nonlinear waves.* In this course, we shall only be talking about linear waves which obey the superposition principle.

When two or more waves pass through the same region simultaneously we say that the waves interfere or the *interference of waves* takes place. The principle of superposition says that the phenomenon of wave interference is remarkably simple. Each wave makes its own contribution to the disturbance no matter what the other waves are doing.

15.7 INTERFERENCE OF WAVES GOING IN SAME DIRECTION

Suppose two identical sources send sinusoidal waves of same angular frequency ω in positive x-direction. Also, the wave velocity and hence, the wave number k is same for the two waves. One source may be started a little later than the other or the two sources may be situated at different points. The two waves arriving at a point then differ in phase. Let the amplitudes of the two waves be A_1 and A_2 and the two waves differ in phase by an angle δ. Their equations may be written as

$$y_1 = A_1 \sin(kx - \omega t)$$

and $$y_2 = A_2 \sin(kx - \omega t + \delta).$$

According to the principle of superposition, the resultant wave is represented by

$$\begin{aligned} y &= y_1 + y_2 = A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t + \delta) \\ &= A_1 \sin(kx - \omega t) + A_2 \sin(kx - \omega t) \cos\delta \\ &\qquad + A_2 \cos(kx - \omega t)\sin\delta \\ &= \sin(kx - \omega t)\,(A_1 + A_2\cos\delta) + \cos(kx - \omega t)\,(A_2\sin\delta). \end{aligned}$$

We can evaluate it using the method described in Chapter-12 to combine two simple harmonic motions.

If we write

$$A_1 + A_2 \cos\delta = A \cos\varepsilon \quad \text{... (i)}$$

and $$A_2 \sin\delta = A \sin\varepsilon, \quad \text{... (ii)}$$

we get

$$\begin{aligned} y &= A\,[\sin(kx - \omega t) \cos\varepsilon + \cos(kx - \omega t) \sin\varepsilon] \\ &= A \sin(kx - \omega t + \varepsilon). \end{aligned}$$

Thus, the resultant is indeed a sine wave of amplitude A with a phase difference ε with the first wave. By (i) and (ii),

$$A^2 = A^2 \cos^2\varepsilon + A^2 \sin^2\varepsilon$$
$$= (A_1 + A_2 \cos\delta)^2 + (A_2 \sin\delta)^2$$
$$= A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta$$

or, $$A = \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos\delta} . \quad \dots (15.18)$$

Also, $$\tan\varepsilon = \frac{A\sin\varepsilon}{A\cos\varepsilon} = \frac{A_2 \sin\delta}{A_1 + A_2 \cos\delta} . \quad \dots (15.19)$$

As discussed in Chapter-12, these relations may be remembered by using a geometrical model. We draw a vector of length A_1 to represent $y_1 = A_1 \sin(kx - \omega t)$ and another vector of length A_2 at an angle δ with the first one to represent $y_2 = A_2 \sin(kx - \omega t + \delta)$. The resultant of the two vectors then represents the resultant wave $y = A \sin(kx - \omega t + \varepsilon)$. Figure (15.10) shows the construction.

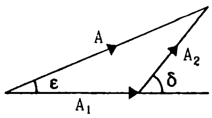

Figure 15.10

Constructive and Destructive Interference

We see from equation (15.18) that the resultant amplitude A is maximum when $\cos\delta = +1$, or $\delta = 2n\pi$ and is minimum when $\cos\delta = -1$, or $\delta = (2n+1)\pi$, where n is an integer. In the first case, the amplitude is $A_1 + A_2$ and in the second case, it is $|A_1 - A_2|$. The two cases are called *constructive* and *destructive* interferences respectively. The conditions may be written as,

constructive interference : $\delta = 2n\pi$
destructive interference : $\delta = (2n+1)\pi$ ... (15.20)

Example 15.5

Two waves are simultaneously passing through a string. The equations of the waves are given by

$$y_1 = A_1 \sin k(x - vt)$$

and $$y_2 = A_2 \sin k(x - vt + x_0),$$

where the wave number $k = 6{\cdot}28 \text{ cm}^{-1}$ *and* $x_0 = 1{\cdot}50$ cm. *The amplitudes are* $A_1 = 5{\cdot}0$ mm *and* $A_2 = 4{\cdot}0$ mm. *Find the phase difference between the waves and the amplitude of the resulting wave.*

Solution : The phase of the first wave is $k(x - vt)$ and of the second is $k(x - vt + x_0)$.

The phase difference is, therefore,

$$\delta = k x_0 = (6{\cdot}28 \text{ cm}^{-1})(1{\cdot}50 \text{ cm}) = 2\pi \times 1{\cdot}5 = 3\pi.$$

The waves satisfy the condition of destructive interference. The amplitude of the resulting wave is given by

$$|A_1 - A_2| = 5{\cdot}0 \text{ mm} - 4{\cdot}0 \text{ mm} = 1{\cdot}0 \text{ mm}.$$

15.8 REFLECTION AND TRANSMISSION OF WAVES

In figure (15.2), a wave pulse was generated at the left end which travelled on the string towards right. When the pulse reaches a particular element, the forces on the element from the left part of the string and from the right part act in such a way that the element is disturbed according to the shape of the pulse.

The situation is different when the pulse reaches the right end which is clamped at the wall. The element at the right end exerts a force on the clamp and the clamp exerts equal and opposite force on the element. The element at the right end is thus acted upon by the force from the string left to it and by the force from the clamp. As this end remains fixed, the two forces are opposite to each other. The force from the left part of the string transmits the forward wave pulse and hence, the force exerted by the clamp sends a return pulse on the string whose shape is similar to the original pulse but is inverted. The original pulse tries to pull the element at the fixed end up and the return pulse sent by the clamp tries to pull it down. The resultant displacement is zero. Thus, the wave is reflected from the fixed end and the reflected wave is inverted with respect to the original wave. The shape of the string at any time, while the pulse is being reflected, can be found by adding an inverted image pulse to the incident pulse (figure 15.11).

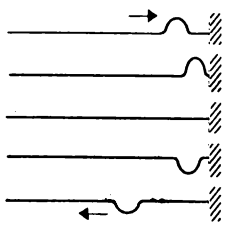

Figure 15.11

Let us now suppose that the right end of the string is attached to a light frictionless ring which can freely move on a vertical rod. A wave pulse is sent on the string from left (Figure 15.12). When the wave reaches the right end, the element at this end is acted on by the force from the left to go up. However, there is no corresponding restoring force from the right as the rod does not exert a vertical force on the ring. As a result, the right end is displaced in upward direction more than the height of the pulse i.e., it overshoots the normal maximum displacement. The lack of restoring force from right can be equivalently described in the following way. An extra force acts from right which sends a wave from right to left with its shape identical to the original one. The element at the end is acted upon by both the incident and the reflected wave and the displacements add. Thus, a wave is reflected by the free end without inversion.

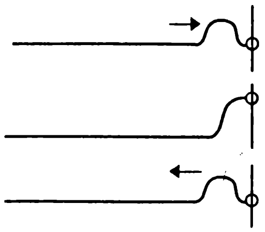

Figure 15.12

Quite often, the end point is neither completely fixed nor completly free to move. As an example, consider a light string attached to a heavier string as shown in figure (15.13). If a wave pulse is produced on the light string moving towards the junction, a part of the wave is reflected and a part is transmitted on the heavier string. The reflected wave is inverted with respect to the original one (figure 15.13a).

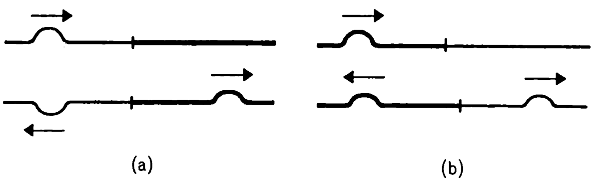

Figure 15.13

On the other hand, if the wave is produced on the heavier string, which moves towards the junction, a part will be reflected and a part transmitted, no inversion of wave shape will take place (figure 15.13b).

The rule about the inversion at reflection may be stated in terms of the wave velocity. The wave velocity is smaller for the heavier string ($v = \sqrt{F/\mu}$) and larger for the lighter string. The above observation may be stated as follows.

If a wave enters a region where the wave velocity is smaller, the reflected wave is inverted. If it enters a region where the wave velocity is larger, the reflected wave is not inverted. The transmitted wave is never inverted.

15.9 STANDING WAVES

Suppose two sine waves of equal amplitude and frequency propagate on a long string in opposite directions. The equations of the two waves are given by

$$y_1 = A\sin(\omega t - kx)$$

and $$y_2 = A\sin(\omega t + kx + \delta).$$

These waves interfere to produce what we call *standing waves*. To understand these waves, let us discuss the special case when $\delta = 0$.

The resultant displacements of the particles of the string are given by the principle of superposition as

$$\begin{aligned} y &= y_1 + y_2 \\ &= A[\sin(\omega t - kx) + \sin(\omega t + kx)] \\ &= 2A\sin\omega t\cos kx \end{aligned}$$

or, $$y = (2A\cos kx)\sin\omega t. \qquad \dots (15.21)$$

This equation can be interpreted as follows. Each particle of the string vibrates in a simple harmonic motion with an amplitude $|2A\cos kx|$. The amplitudes are not equal for all the particles. In particular, there are points where the amplitude $|2A\cos kx| = 0$. This will be the case when

$$\cos kx = 0$$

or, $$kx = \left(n + \frac{1}{2}\right)\pi$$

or, $$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2},$$

where n is an integer.

For these particles, $\cos kx = 0$ and by equation (15.21) the displacement y is zero all the time. Although these points are not physically clamped, they remain fixed as the two waves pass them simultaneously. Such points are known as *nodes*.

For the points where $|\cos kx| = 1$, the amplitude is maximum. Such points are known as *antinodes*.

We also see from equation (15.21) that at a time when $\sin\omega t = 1$, all the particles for which $\cos kx$ is positive reach their positive maximum displacement. At this particular instant, all the particles for which $\cos kx$ is negative, reach their negative maximum displacement. At a time when $\sin\omega t = 0$, all the particles cross their mean positions. Figure (15.14a) shows the change in the shape of the string as time passes. Figure (15.14b) shows the external appearance of the vibrating string. This type of wave is called a *standing wave* or a *stationary wave*. The particles at nodes do not move at all and the particles at the antinodes move with maximum amplitude.

It is clear that the separation between consecutive nodes or consecutive antinodes is $\lambda/2$. As the particles at the nodes do not move at all, energy cannot be transmitted across them. The main differences between a standing wave and a travelling wave are summarised below.

1. In a travelling wave, the disturbance produced in a region propagates with a definite velocity but in a standing wave, it is confined to the region where it is produced.

2. In a travelling wave, the motion of all the particles are similar in nature. In a standing wave, different particles move with different amplitudes.

3. In a standing wave, the particles at nodes always remain in rest. In travelling waves, there is no particle which always remains in rest.

4. In a standing wave, all the particles cross their mean positions together. In a travelling wave, there is no instant when all the particles are at the mean positions together.

5. In a standing wave, all the particles between two successive nodes reach their extreme positions together, thus moving in phase. In a travelling wave, the phases of nearby particles are always different.

6. In a travelling wave, energy is transmitted from one region of space to other but in a standing wave, the energy of one region is always confined in that region.

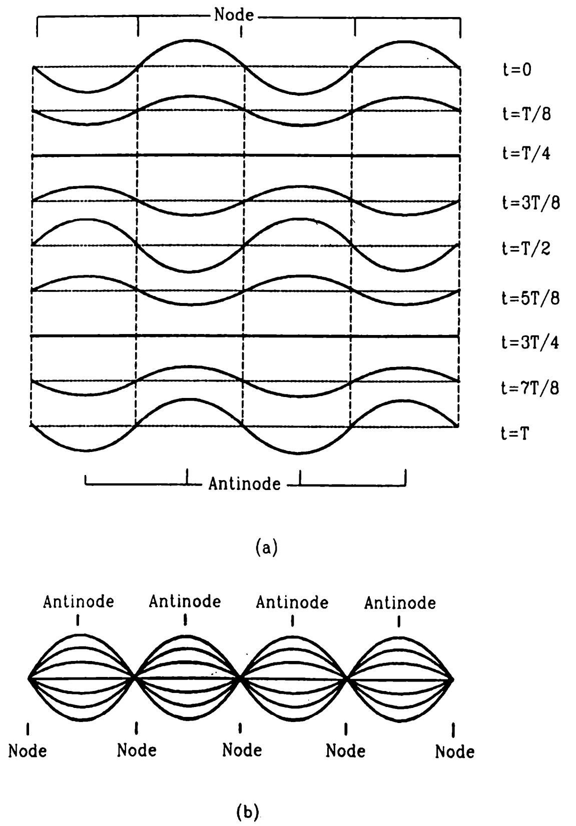

Figure 15.14

Example 15.6

Two travelling waves of equal amplitudes and equal frequencies move in opposite directions along a string. They interfere to produce a standing wave having the equation

$$y = A \cos kx \sin \omega t$$

in which $A = 1{\cdot}0$ mm, $k = 1{\cdot}57\ \text{cm}^{-1}$ and $\omega = 78{\cdot}5\ \text{s}^{-1}$. (a) Find the velocity of the component travelling waves. (b) Find the node closest to the origin in the region $x > 0$. (c) Find the antinode closest to the origin in the region $x > 0$. (d) Find the amplitude of the particle at $x = 2{\cdot}33$ cm.

Solution : (a) The standing wave is formed by the superposition of the waves

$$y_1 = \frac{A}{2}\sin(\omega t - kx) \quad \text{and}$$

$$y_2 = \frac{A}{2}\sin(\omega t + kx).$$

The wave velocity (magnitude) of either of the waves is

$$v = \frac{\omega}{k} = \frac{78{\cdot}5\ \text{s}^{-1}}{1{\cdot}57\ \text{cm}^{-1}} = 50\ \text{cm/s}.$$

(b) For a node, $\cos kx = 0$.

The smallest positive x satisfying this relation is given by

$$kx = \frac{\pi}{2}$$

or, $$x = \frac{\pi}{2k} = \frac{3{\cdot}14}{2 \times 1{\cdot}57\ \text{cm}^{-1}} = 1\ \text{cm}.$$

(c) For an antinode, $|\cos kx| = 1$.

The smallest positive x satisfying this relation is given by

$$kx = \pi$$

or, $$x = \frac{\pi}{k} = 2\ \text{cm}.$$

(d) The amplitude of vibration of the particle at x is given by $|A \cos kx|$. For the given point,

$$kx = (1{\cdot}57\ \text{cm}^{-1})(2{\cdot}33\ \text{cm}) = \frac{7}{6}\pi = \pi + \frac{\pi}{6}.$$

Thus, the amplitude will be

$$(1{\cdot}0\ \text{mm})\ |\cos(\pi + \pi/6)| = \frac{\sqrt{3}}{2}\ \text{mm} = 0{\cdot}86\ \text{mm}.$$

15.10 STANDING WAVES ON A STRING FIXED AT BOTH ENDS (QUALITATIVE DISCUSSION)

Consider a string of length L fixed at one end to a wall and the other end tied to a tuning fork which vibrates longitudinally with a small amplitude (figure 15.15). The fork produces sine waves of amplitude A which travel on the string towards the fixed end and