CHAPTER 11

GRAVITATION

11.1 HISTORICAL INTRODUCTION

The motion of celestial bodies such as the moon, the earth, the planets etc. has been a subject of great interest for a long time. Famous Indian astronomer and mathematician, Aryabhat, studied these motions in great detail, most likely in the 5th century A.D., and wrote his conclusions in his book *Aryabhatiya*. He established that the earth revolves about its own axis and moves in a circular orbit about the sun, and that the moon moves in a circular orbit about the earth.

About a thousand years after Aryabhat, the brilliant combination of Tycho Brahe (1546-1601) and Johnaase Kepler (1571-1630) studied the planetary motion in great detail. Kepler formulated his important findings in his three laws of planetary motion :

1. *All planets move in elliptical orbits with the sun at a focus.*
2. *The radius vector from the sun to the planet sweeps equal area in equal time.*
3. *The square of the time period of a planet is proportional to the cube of the semimajor axis of the ellipse.*

The year 1665 was very fruitful for Isaac Newton aged 23. He was forced to take rest at his home in Lincolnshire after his college at Cambridge was closed for an indefinite period due to plague. In this year, he performed brilliant theoretical and experimental tasks mainly in the field of mechanics and optics. In this same year he focused his attention on the motion of the moon about the earth.

The moon makes a revolution about the earth in $T = 27{\cdot}3$ days. The distance of the moon from the earth is $R = 3{\cdot}85 \times 10^{5}$ km. The acceleration of the moon is, therefore,

$$a = \omega^{2} R$$

$$= \frac{4\pi^{2} \times (3{\cdot}85 \times 10^{5}\ \text{km})}{(27{\cdot}3\ \text{days})^{2}} = 0{\cdot}0027\ \text{m/s}^{2}.$$

The first question before Newton was, that what is the force that produces this acceleration. The acceleration is towards the centre of the orbit, that is towards the centre of the earth. Hence the force must act towards the centre of the earth. A natural guess was that the earth is attracting the moon. The saying goes that Newton was sitting under an apple tree when an apple fell down from the tree on the earth. This sparked the idea that the earth attracts all bodies towards its centre. The next question was what is the law governing this force.

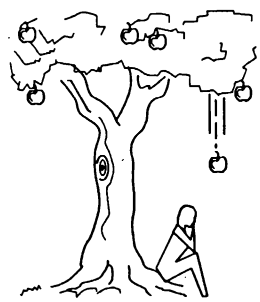

Figure 11.1

Newton had to make several daring assumptions which proved to be turning points in science and philosophy. He declared that the laws of nature are the same for earthly and celestial bodies. The force operating between the earth and an apple and that operating between the earth and the moon, must be governed by the same laws. This statement may look very obvious today but in the era before Newton, there was a general belief in the western countries that the earthly bodies are governed by certain rules and the heavenly bodies are governed by different rules. In particular, this heavenly structure was supposed to be so perfect that there could not be any change in the sky. This distinction was so sharp that when Tycho Brahe saw a new star in the sky, he did not believe his eyes as there could be no change in the sky. So the Newton's declaration was indeed revolutionary.

The acceleration of a body falling near the earth's surface is about $9{\cdot}8\ \mathrm{m/s^2}$. Thus,

$$\frac{a_{apple}}{a_{moon}} = \frac{9{\cdot}8\ \mathrm{m/s^2}}{0{\cdot}0027\ \mathrm{m/s^2}} = 3600.$$

Also,

$$\frac{\text{distance of the moon from the earth}}{\text{distance of the apple from the earth}}$$

$$= \frac{d_{moon}}{d_{apple}} = \frac{3{\cdot}85 \times 10^5\ \mathrm{km}}{6400\ \mathrm{km}}$$

$$= 60.$$

Thus, $$\frac{a_{apple}}{a_{moon}} = \left(\frac{d_{moon}}{d_{apple}}\right)^2.$$

Newton guessed that the acceleration of a body towards the earth is inversely proportional to the square of the distance of the body from the centre of the earth.

Thus, $a \propto \dfrac{1}{r^2}$.

Also, the force is mass times acceleration and so it is proportional to the mass of the body.

Hence,

$$F \propto \frac{m}{r^2}.$$

By the third law of motion, the force on a body due to the earth must be equal to the force on the earth due to the body. Therefore, this force should also be proportional to the mass of the earth. Thus, the force between the earth and a body is

$$F \propto \frac{Mm}{r^2}$$

or, $$F = \frac{GMm}{r^2}. \quad \ldots (11.1)$$

Newton further generalised the law by saying that not only the earth but all material bodies in the universe attract each other according to equation (11.1) with same value of G. The constant G is called *universal constant of gravitation* and its value is found to be $6{\cdot}67 \times 10^{-11}\ \mathrm{N\text{-}m^2/kg^2}$. Equation (11.1) is known as the *universal law of gravitation.*

In this argument, the distance of the apple from the earth is taken to be equal to the radius of the earth. This means we have assumed that earth can be treated as a single particle placed at its centre. This is of course not obvious. Newton had spent several years to prove that indeed this can be done. A spherically symmetric body can be replaced by a point particle of equal mass placed at its centre for the purpose of calculating gravitational force. In the process he discovered the methods of calculus that we have already learnt in Chapter 2. There is evidence that quite a bit of differential calculus was known to the ancient Indian mathematicians but this literature was almost certainly not known to Newton or other scientists of those days.

Example 11.1

Two particles of masses 1·0 kg *and* 2·0 kg *are placed at a separation of* 50 cm. *Assuming that the only forces acting on the particles are their mutual gravitation, find the initial accelerations of the two particles.*

Solution : The force of gravitation exerted by one particle on another is

$$F = \frac{Gm_1m_2}{r^2}$$

$$= \frac{6{\cdot}67 \times 10^{-11}\ \dfrac{\mathrm{N\text{-}m^2}}{\mathrm{kg^2}} \times (1{\cdot}0\ \mathrm{kg}) \times (2{\cdot}0\ \mathrm{kg})}{(0{\cdot}5\ \mathrm{m})^2}$$

$$= 5{\cdot}3 \times 10^{-10}\ \mathrm{N}.$$

The acceleration of 1·0 kg particle is

$$a_1 = \frac{F}{m_1} = \frac{5{\cdot}3 \times 10^{-10}\ \mathrm{N}}{1{\cdot}0\ \mathrm{kg}}$$

$$= 5{\cdot}3 \times 10^{-10}\ \mathrm{m/s^2}.$$

This acceleration is towards the 2·0 kg particle. The acceleration of the 2·0 kg particle is

$$a_2 = \frac{F}{m_2} = \frac{5{\cdot}3 \times 10^{-10}\ \mathrm{N}}{2{\cdot}0\ \mathrm{kg}}$$

$$= 2{\cdot}65 \times 10^{-10}\ \mathrm{m/s^2}.$$

This acceleration is towards the 1·0 kg particle.

11.2 MEASUREMENT OF GRAVITATIONAL CONSTANT G

The gravitational constant G is a small quantity and its measurement needs very sensitive arrangement. The first important successful measurement of this quantity was made by Cavendish in 1736 about 71 years after the law was formulated.

In this method, two small balls of equal mass are attached at the two ends of a light rod to form a dumb-bell. The rod is suspended vertically by a fine quartz wire. Two large spheres of equal mass are placed near the smaller spheres in such a way that all the four spheres are on a horizontal circle. The centre of the circle is at the middle point of the rod (figure 11.2).

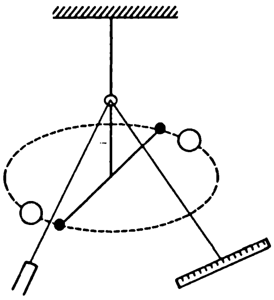

Figure 11.2

Two larger spheres lie on the opposite sides of the smaller balls at equal distance. A small plane mirror is attached to the vertical wire. A light beam, incident on the mirror, falls on a scale after reflection. If the wire rotates by an angle θ, the reflected beam rotates by 2θ and the spot on the scale moves. By measuring this movement of the spot on the scale and the distance between the mirror and the scale, the angle of deviation can be calculated. When the heavy balls are placed close to the small balls, a torque acts on the dumb-bell to rotate it. As the dumb-bell rotates, the suspension wire gets twisted and produces a torque on the dumb-bell in opposite direction. This torque is proportional to the angle rotated. The dumb-bell stays in equilibrium where the two torques have equal magnitude.

Let the mass of a heavy ball = M,

the mass of a small ball = m,

the distance between the centres of a heavy ball and the small ball placed close to it = r,

the deflection of the dumb-bell as it comes to equilibrium = θ,

the torsional constant of the suspension wire = k,

the length of the rod = l and

the distance between the scale and the mirror = D.

The force acting on each of the small balls is

$$F = G\frac{Mm}{r^2}.$$

Here we have used the fact that the gravitational force due to a uniform sphere is same as that due to a single particle of equal mass placed at the centre of the sphere. As the four balls are on the same horizontal circle and the heavy balls are placed close to the smaller balls, this force acts in a horizontal direction perpendicular to the length of the dumb-bell. The torque due to each of these gravitational forces about the suspension wire is $F(l/2)$.

The total gravitational torque on the dumb-bell is, therefore,

$$\Gamma = 2F(l/2) = Fl.$$

The opposing torque produced by the suspension wire is $k\theta$. For rotational equilibrium,

$$Fl = k\theta$$

or, $$\frac{GMml}{r^2} = k\theta$$

$$G = \frac{k\theta r^2}{Mml}. \quad \text{... (i)}$$

In an experiment, the heavy balls are placed close to the smaller balls as shown in the figure and the dumb-bell is allowed to settle down. The light beam is adjusted so that the beam reflected by the plane mirror falls on the scale. Now the heavy balls are shifted in such a way that they are placed on the same horizontal circle at same distance from the smaller balls but on the opposite side. In figure (11.3), the original positions of the heavy balls are shown by A, B and the shifted positions by A', B'.

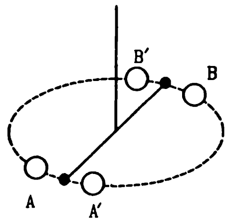

Figure 11.3

As the heavy balls are shifted to the new position, the dumb-bell rotates. If it was settled previously at an angle θ deviated from the mean position, it will now settle at the same angle θ on the other side. Thus, the total deflection of the dumb-bell due to the change in the positions of the heavy balls is 2θ. The reflected light beam deviates by an angle of 4θ.

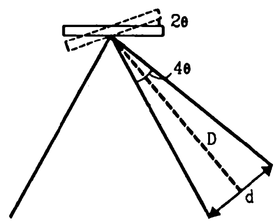

Figure 11.4

If the linear displacement of the light spot is d, we have (figure 11.4)

$$4\theta = \frac{d}{D}$$

or, $$\theta = \frac{d}{4D}.$$

Substituting in (i),

$$G = \frac{kdr^2}{4MmlD}.$$

All the quantities on the right hand side are experimentally known and hence the value of G may be calculated.

11.3 GRAVITATIONAL POTENTIAL ENERGY

The concept of potential energy of a system was introduced in Chapter-8. The potential energy of a system corresponding to a conservative force was defined as

$$U_f - U_i = -\int_i^f \vec{F} . d\vec{r}.$$

The change in potential energy is equal to the negative of the work done by the internal forces. We also calculated the change in gravitational potential energy of the earth-particle system when the particle was raised through a small height over earth's surface. In this case the force mg may be treated as constant and the change in potential energy is

$$U_f - U_i = mgh$$

where the symbols have their usual meanings. We now derive the general expression for the change in gravitational potential energy of a two-particle system.

Let a particle of mass m_1 be kept fixed at a point A (figure 11.5) and another particle of mass m_2 is taken from a point B to a point C. Initially, the distance between the particles is $AB = r_1$ and finally it becomes $AC = r_2$. We have to calculate the change in potential energy of the system of the two particles as the distance changes from r_1 to r_2.

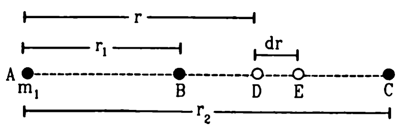

Figure 11.5

Consider a small displacement when the distance between the particles changes from r to $r + dr$. In the figure, this corresponds to the second particle going from D to E.

The force on the second particle is

$$F = \frac{Gm_1 m_2}{r^2} \text{ along } \vec{DA}.$$

The work done by the gravitational force in the displacement is

$$dW = -\frac{Gm_1 m_2}{r^2} dr.$$

The increase in potential energy of the two particle system during this displacement is

$$dU = -dW = \frac{Gm_1 m_2}{r^2} dr.$$

The increase in potential energy as the distance between the particles changes from r_1 to r_2 is

$$U(r_2) - U(r_1) = \int dU$$

$$= \int_{r_1}^{r_2} \frac{Gm_1 m_2}{r^2} dr = Gm_1 m_2 \int_{r_1}^{r_2} \frac{1}{r^2} dr$$

$$= Gm_1 m_2 \left[-\frac{1}{r} \right]_{r_1}^{r_2}$$

$$= Gm_1 m_2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right). \quad \ldots (11.2)$$

We choose the potential energy of the two-particle system to be zero when the distance between them is infinity. This means that we choose $U(\infty) = 0$. By (11.2) the potential energy $U(r)$, when the separation between the particles is r, is

$$U(r) = U(r) - U(\infty)$$

$$= Gm_1 m_2 \left[\frac{1}{\infty} - \frac{1}{r} \right] = -\frac{Gm_1 m_2}{r}.$$

The gravitational potential energy of a two particle system is

$$U(r) = -\frac{Gm_1 m_2}{r} \quad \ldots (11.3)$$

where m_1 and m_2 are the masses of the particles, r is the separation between the particles and the potential energy is chosen to be zero when the separation is infinite.

We have proved this result by assuming that one of the particles is kept at rest and the other is displaced. However, as the potential energy depends only on the separation and not on the location of the particles, equation (11.3) is general.

Equation (11.3) gives the potential energy of a pair of particles. If there are three particles A, B and C, there are three pairs AB, AC and BC. The potential energy of the three-particle system is equal to the sum of the potential energies of the three pairs. For an N-particle system there are $N(N-1)/2$ pairs and the potential energy is calculated for each pair and added to get the total potential energy of the system.

Example 11.2

Find the work done in bringing three particles, each having a mass of 100 g, *from large distances to the vertices of an equilateral triangle of side* 20 cm.

Solution : When the separations are large, the gravitational potential energy is zero. When the particles are brought at the vertices of the triangle *ABC*, three pairs *AB*, *BC* and *CA* are formed. The potential energy of each pair is $-Gm_1 m_2/r$ and hence the total potential energy becomes

$$U = 3 \times \left[-\frac{Gm_1 m_2}{r} \right]$$

$$= 3 \times \left[-\frac{6{\cdot}67 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2 \times (0{\cdot}1\ \text{kg}) \times (0{\cdot}1\ \text{kg})}{0{\cdot}20\ \text{m}} \right]$$

$$= -1{\cdot}0 \times 10^{-11}\ \text{J}.$$

The work done by the gravitational forces is $W = -U = 1{\cdot}0 \times 10^{-11}$ J. If the particles are brought by some external agency without changing the kinetic energy, the work done by the external agency is equal to the change in potential energy $= -1{\cdot}0 \times 10^{-11}$ J.

11.4 GRAVITATIONAL POTENTIAL

Suppose a particle of mass m is taken from a point A to a point B while keeping all other masses fixed. Let U_A and U_B denote the gravitational potential energy when the mass m is at point A and point B respectively.

We define the "change in potential" $V_B - V_A$ between the two points as

$$V_B - V_A = \frac{U_B - U_A}{m}. \qquad \text{... (11.4)}$$

The equation defines only the change in potential. We can choose any point to have zero potential. Such a point is called a *reference point*. If A be the reference point, $V_A = 0$ and

$$V_B = \frac{U_B - U_A}{m}. \qquad \text{... (11.5)}$$

Thus, *gravitational potential at a point is equal to the change in potential energy per unit mass, as the mass is brought from the reference point to the given point.* If the particle is slowly brought without increasing the kinetic energy, the work done by the external agent equals the change in potential energy. Thus, the potential at a point may also be defined as *the work done per unit mass by an external agent in bringing a particle slowly from the reference point to the given point.* Generally the reference point is chosen at infinity so that the potential at infinity is zero.

The SI unit of gravitational potential is J/kg.

11.5 CALCULATION OF GRAVITATIONAL POTENTIAL

(A) Potential due to a Point Mass

Suppose a particle of mass M is kept at a point A (figure 11.6) and we have to calculate the potential at a point P at a distance r away from A. The reference point is at infinity.

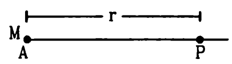

Figure 11.6

From equation (11.5), the potential at the point P is

$$V(r) = \frac{U(r) - U(\infty)}{m}.$$

But $\quad U(r) - U(\infty) = -\frac{GMm}{r}.$

so that

$$V = -\frac{GM}{r}. \qquad \text{... (11.6)}$$

The gravitational potential due to a point mass M at a distance r is $-\frac{GM}{r}$.

(B) Potential due to a Uniform Ring at a Point on its Axis

Let the mass of the ring be M and its radius be a. We have to calculate the gravitational potential at a point P on the axis of the ring (figure 11.7). The centre is at O and $OP = r$.

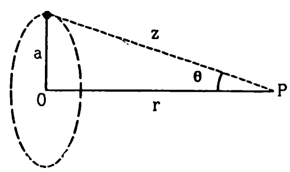

Figure 11.7

Consider any small part of the ring of mass dm. The point P is at a distance $z = \sqrt{a^2 + r^2}$ from dm.

The potential at P due to dm is

$$dV = -\frac{G\,dm}{z} = -\frac{G\,dm}{\sqrt{a^2 + r^2}}.$$

The potential V due to the whole ring is obtained by summing the contributions from all the parts. As the potential is a scalar quantity, we have

$$V = \int dV$$
$$= \int -\frac{G\,dm}{\sqrt{a^2 + r^2}}$$
$$= -\frac{G}{\sqrt{a^2 + r^2}} \int dm$$
$$= -\frac{GM}{\sqrt{a^2 + r^2}} \cdot \quad \dots (11.7)$$

In terms of the distance z between the point P and any point of the ring, the expression for the potential is given by

$$V = -\frac{GM}{z} \cdot \quad \dots (11.8)$$

(C) Potential due to a Uniform Thin Spherical Shell

Let the mass of the given spherical shell be M and the radius a. We have to calculate the potential due to this shell at a point P. The centre of the shell is at O and $OP = r$ (figure 11.8).

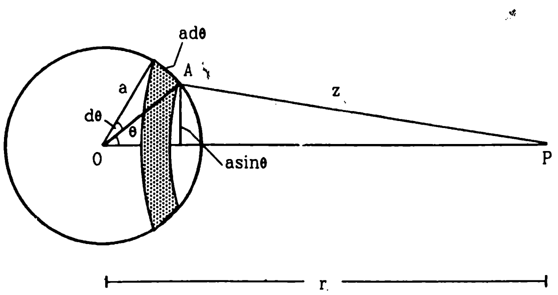

Figure 11.8

Let us draw a radius OA making an angle θ with OP. Let us rotate this radius about OP keeping the angle AOP fixed at value θ. The point A traces a circle on the surface of the shell. Let us now consider another radius at an angle $\theta + d\theta$ and likewise rotate it about OP. Another circle is traced on the surface of the shell. The part of the shell included between these two circles (shown shaded in the figure) may be treated as a ring.

The radius of this ring is $a \sin\theta$ and hence the perimeter is $2\pi a \sin\theta$. The width of the ring is $a\,d\theta$. The area of the ring is

$$(2\pi a \sin\theta)(a d\theta)$$
$$= 2\pi a^2 \sin\theta\, d\theta.$$

The total area of the shell is $4\pi a^2$. As the shell is uniform, the mass of the ring enclosed is

$$dm = \frac{M}{4\pi a^2}(2\pi a^2 \sin\theta\, d\theta)$$
$$= \frac{M}{2} \sin\theta\, d\theta .$$

Let the distance of any point of the ring from P be $AP = z$. From the triangle OAP,

$$z^2 = a^2 + r^2 - 2ar\cos\theta$$

or, $\quad 2z\,dz = 2ar \sin\theta\, d\theta$

or, $\quad \sin\theta\, d\theta = \frac{z\,dz}{ar} \cdot$

Thus, the mass of the ring is

$$dm = \frac{M}{2}\sin\theta\, d\theta = \frac{M}{2ar} z\,dz.$$

As the distance of any point of the ring from P is z, the potential at P due to the ring is

$$dV = -\frac{G\,dm}{z}$$
$$= -\frac{GM}{2ar}dz .$$

As we vary θ from 0 to π, the rings formed on the shell cover up the whole shell. The potential due to the whole shell is obtained by integrating dV within the limits $\theta = 0$ to $\theta = \pi$.

Case I : *P* is outside the shell ($r > a$)

As figure (11.8) shows, when $\theta = 0$, the distance $z = AP = r - a$. When $\theta = \pi$, it is $z = r + a$. Thus, as θ varies from 0 to π, the distance z varies from $r - a$ to $r + a$. Thus,

$$V = \int dV = -\frac{GM}{2ar}\int_{r-a}^{r+a} dz$$
$$= -\frac{GM}{2ar}[z]_{r-a}^{r+a}$$
$$= -\frac{GM}{2ar}[(r + a) - (r - a)]$$
$$= -\frac{GM}{r} \cdot \quad \dots (11.9)$$

To calculate the potential at an external point, a uniform spherical shell may be treated as a point particle of equal mass placed at its centre.

Case II : *P* is inside the shell ($r < a$)

In this case when $\theta = 0$, the distance $z = AP = a - r$ and when $\theta = \pi$ it is $z = a + r$ (figure 11.9). Thus, as θ varies from 0 to π, the distance z varies from $a - r$ to $a + r$. Thus, the potential due to the shell is

$$V = \int dV$$
$$= -\frac{GM}{2ar}[z]_{a-r}^{a+r}$$
$$= -\frac{GM}{2ar}[(a + r) - (a - r)]$$
$$= -\frac{GM}{a} \cdot \quad \dots (11.10)$$

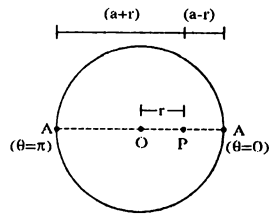

Figure 11.9

This does not depend on r. Thus, *the potential due to a uniform spherical shell is constant throughout the cavity of the shell.*

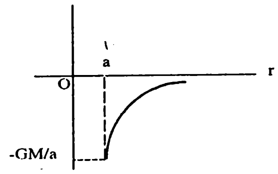

Figure 11.10

Figure (11.10) shows graphically the variation of potential with the distance from the centre of the shell.

Example 11.3

A particle of mass M is placed at the centre of a uniform spherical shell of equal mass and radius a. Find the gravitational potential at a point P at a distance a/2 from the centre.

Solution : The gravitational potential at the point P due to the particle at the centre is

$$V_1 = -\frac{GM}{a/2} = -\frac{2GM}{a}.$$

The potential at P due to the shell is

$$V_2 = -\frac{GM}{a}.$$

The net potential at P is $V_1 + V_2 = -\dfrac{3GM}{a}$.

(D) Potential due to a Uniform Solid Sphere

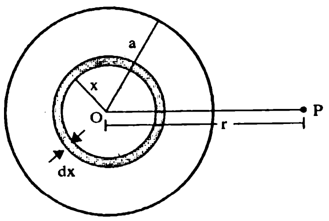

Figure 11.11

The situation is shown in figure (11.11). Let the mass of the sphere be M and its radius a. We have to calculate the gravitational potential at a point P. Let $OP = r$.

Let us draw two spheres of radii x and $x + dx$ concentric with the given sphere. These two spheres enclose a thin spherical shell of volume $4\pi x^2 dx$. The volume of the given sphere is $\frac{4}{3}\pi a^3$. As the sphere is uniform, the mass of the shell is

$$dm = \frac{M}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3M}{a^3} x^2 dx.$$

The potential due to this shell at the point P is

$$dV = -\frac{G\,dm}{r} \text{ if } x < r \text{ and } dV = -\frac{G\,dm}{x} \text{ if } x > r.$$

Case I : Potential at an external point

Suppose the point P is outside the sphere (figure 11.11). The potential at P due to the shell considered is

$$dV = -\frac{G\,dm}{r}.$$

Thus, the potential due to the whole sphere is

$$V = \int dV = -\frac{G}{r}\int dm$$

$$= -\frac{GM}{r}. \qquad \dots (11.11)$$

The gravitational potential due to a uniform sphere at an external point is same as that due to a single particle of equal mass placed at its centre.

Case II : Potential at an internal point

Let us divide the sphere in two parts by imagining a concentric spherical surface passing through P. The inner part has a mass

$$M' = \frac{M}{\frac{4}{3}\pi a^3} \times \frac{4}{3}\pi r^3 = \frac{Mr^3}{a^3}.$$

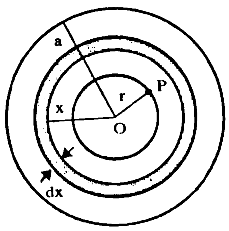

Figure 11.12

The potential at P due to this inner part is by equation (11.11)

$$V_1 = -\frac{GM'}{r}$$

$$= -\frac{GMr^2}{a^3}. \qquad \dots \text{(i)}$$

To get the potential at P due to the outer part of the sphere, we divide this part in concentric shells. The mass of the shell between radii x and $x + dx$ is

$$dm = \frac{M}{\frac{4}{3}\pi a^3} 4\pi x^2 dx = \frac{3Mx^2 dx}{a^3}.$$

The potential at P due to this shell is,

$$\frac{-G\,dm}{x} = -3\frac{GM}{a^3} x dx \cdot$$

The potential due to the outer part is

$$V_2 = \int_r^a -\frac{3GM}{a^3} x\,dx$$

$$= -\frac{3GM}{a^3}\left[\frac{x^2}{2}\right]_r^a$$

$$= \frac{-3GM}{2a^3}(a^2 - r^2). \quad \text{... (ii)}$$

By (i) and (ii), the total potential at P is

$$V = V_1 + V_2$$

$$= -\frac{GMr^2}{a^3} - \frac{3GM}{2a^3}(a^2 - r^2)$$

$$= -\frac{GM}{2a^3}(3a^2 - r^2). \quad \text{... (11.12)}$$

At the centre of the sphere the potential is

$$V = -\frac{3GM}{2a}.$$

11.6 GRAVITATIONAL FIELD

We have been saying all through that a body A exerts a force of gravitation on another body B kept at a distance. This is called *action at a distance* viewpoint. However, this viewpoint creates certain problems when one deals with objects separated by large distances. It is now assumed that a body can not directly interact with another body kept at a distance. The force between two objects is seen to be a two-step process.

In the first step, it is assumed that the body A creates a *gravitational field* in the space around it. The field has its own existence and has energy and momentum. This field has a definite direction at each point of the space and its intensity varies from point to point.

In the second step, it is assumed that when a body B is placed in a gravitational field, this field exerts a force on it. The direction and the intensity of the field is defined in terms of the force it exerts on a body placed in it. We define the *intensity of gravitational field* $\vec{E}$ at a point by the equation

$$\vec{E} = \frac{\vec{F}}{m} \quad \text{... (11.13)}$$

where $\vec{F}$ is the force exerted by the field on a body of mass m placed in the field. Quite often the intensity of gravitational field is abbreviated as *gravitational field*. Its SI unit is N/kg.

Gravitational field adds according to the rules of vector addition. If $\vec{E}_1$ is the field due to a source S_1 and $\vec{E}_2$ is the field at the same point due to another source S_2, the resultant field when both the sources are present is $\vec{E}_1 + \vec{E}_2$.

If a mass m is placed close to the surface of the earth, the force on it is mg. We say that the earth has set up a gravitational field and this field exerts a force on the mass. The intensity of the field is

$$\vec{E} = \frac{\vec{F}}{m} = \frac{m\vec{g}}{m} = \vec{g}.$$

Thus, the intensity of the gravitational field near the surface of the earth is equal to the acceleration due to gravity. It should be clearly understood that the intensity of the gravitational field and the acceleration due to gravity are two separate physical quantities having equal magnitudes and directions.

Example 11.4

A particle of mass 50 g experiences a gravitational force of 2·0 N when placed at a particular point. Find the gravitational field at that point.

Solution : The gravitational field has a magnitude

$$E = \frac{F}{m} = \frac{2{\cdot}0\text{ N}}{(50 \times 10^{-3}\text{ kg})} = 40\text{ N/kg}.$$

This field is along the direction of the force.

11.7 RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

Suppose the gravitational field at a point $\vec{r}$ due to a given mass distribution is $\vec{E}$. By definition (equation 11.13), the force on a particle of mass m when it is at $\vec{r}$ is

$$\vec{F} = m\vec{E}.$$

As the particle is displaced from $\vec{r}$ to $\vec{r} + d\vec{r}$ the work done by the gravitational force on it is

$$dW = \vec{F} \cdot d\vec{r}$$

$$= m\vec{E} \cdot d\vec{r}.$$

The change in potential energy during this displacement is

$$dU = -dW = -m\vec{E} \cdot d\vec{r}.$$

The change in potential is, by equation (11.4),

$$dV = \frac{dU}{m} = -\vec{E}\,.\,d\vec{r}. \qquad \ldots (11.14)$$

Integrating between $\vec{r}_1$ and $\vec{r}_2$

$$V(\vec{r}_2) - V(\vec{r}_1) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{E}\,.\,d\vec{r} \qquad \ldots (11.15)$$

If $\vec{r}_1$ is taken at the reference point, $V(\vec{r}_1) = 0$. The potential $V(\vec{r})$ at any point $\vec{r}$ is, therefore,

$$V(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\,.\,d\vec{r} \qquad \ldots (11.16)$$

where $\vec{r}_0$ denotes the reference point.

If we work in Cartesian coordinates, we can write

$$\vec{E} = \vec{i}\,E_x + \vec{j}\,E_y + \vec{k}\,E_z$$

and $\qquad d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$

so that

$$\vec{E}\,.\,d\vec{r} = E_x\,dx + E_y\,dy + E_z\,dz\,.$$

Equation (11.14) may be written as

$$dV = -E_x\,dx - E_y\,dy - E_z\,dz.$$

If y and z remain constant, $dy = dz = 0$.

Thus, $\qquad E_x = -\frac{\partial V}{\partial x} \qquad \ldots (11.17)$

Similarly, $\qquad E_y = -\frac{\partial V}{\partial y}$ and $E_z = -\frac{\partial V}{\partial z}$.

The symbol $\frac{\partial}{\partial x}$ means partial differentiation with respect to x treating y and z to be constants. Similarly for $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$.

If the field is known, the potential may be obtained by integrating the field according to equation (11.16) and if the potential is known, the field may be obtained by differentiating the potential according to equation (11.17).

Example 11.5

The gravitational field due to a mass distribution is given by $E = K/x^3$ in X-direction. Taking the gravitational potential to be zero at infinity, find its value at a distance x.

Solution : The potential at a distance x is

$$V(x) = -\int_{\infty}^{x} E\,dx = -\int_{\infty}^{x} \frac{K}{x^3}\,dx$$

$$= \left[\frac{K}{2x^2}\right]_{\infty}^{x} = \frac{K}{2x^2}.$$

Example 11.6

The gravitational potential due to a mass distribution is $V = \frac{A}{\sqrt{x^2 + a^2}}$. Find the gravitational field.

Solution : $\qquad V = \frac{A}{\sqrt{x^2 + a^2}} = A(x^2 + a^2)^{-1/2}.$

If the gravitational field is E,

$$E_x = -\frac{\partial V}{\partial x} = -A\left(-\frac{1}{2}\right)(x^2 + a^2)^{-3/2}(2x)$$

$$= \frac{Ax}{(x^2 + a^2)^{3/2}}$$

$$E_y = -\frac{\partial V}{\partial y} = 0 \text{ and } E_z = -\frac{\partial V}{\partial z} = 0.$$

The gravitational field is $\frac{Ax}{(x^2 + a^2)^{3/2}}$ in the x-direction.

11.8 CALCULATION OF GRAVITATIONAL FIELD

(A) Field due to a Point Mass

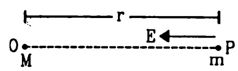

Figure 11.13

Suppose a particle of mass M is placed at a point O (figure 11.13) and a second particle of mass m is placed at a point P. Let $OP = r$. The mass M creates a field $\vec{E}$ at the site of mass m and this field exerts a force

$$\vec{F} = m\vec{E}$$

on the mass m. But the force $\vec{F}$ on the mass m due to the mass M is

$$F = \frac{GMm}{r^2}$$

acting along $\vec{PO}$. Thus, the gravitational field at P is

$$E = \frac{GM}{r^2} \qquad \ldots (11.18)$$

along $\vec{PO}$. If O is taken as the origin, the position vector of mass m is $\vec{r} = \vec{OP}$. Equation (11.18) may be rewritten in vector form as

$$= -\frac{GM}{r^2}\vec{e}_r \qquad \ldots (11.19)$$

where $\vec{e}_r$ is the unit vector along $\vec{r}$.

(B) Field due to a Uniform Circular Ring at a Point on its Axis

Figure (11.14) shows a uniform circular ring of radius a and mass M. Let P be a point on its axis at a distance r from the centre. We have to obtain the gravitational field at P due to the ring. By symmetry the field must be towards the centre that is along $\vec{PO}$.

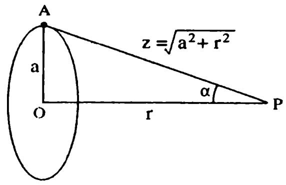

Figure 11.14

Consider any particle of mass dm on the ring, say at point A. The distance of this particle from P is $AP = z = \sqrt{a^2 + r^2}$. The gravitational field at P due to dm is along $\vec{PA}$ and its magnitude is

$$dE = \frac{G\,dm}{z^2}.$$

The component along PO is

$$dE\cos\alpha = \frac{G\,dm}{z^2}\cos\alpha.$$

The net gravitational field at P due to the ring is

$$E = \int \frac{G\,dm}{z^2}\cos\alpha = \frac{G\cos\alpha}{z^2}\int dm = \frac{GM\cos\alpha}{z^2}$$

$$= \frac{GMr}{(a^2 + r^2)^{3/2}}. \qquad \dots (11.20)$$

The field is directed towards the centre of the ring.

(C) Field due to a Uniform Disc at a Point on its Axis

The situation is shown in figure (11.15). Let the mass of the disc be M and its radius be a. Let O be the centre of the disc and P be a point on its axis at a distance r from the centre. We have to find the gravitational field at P due to the disc.

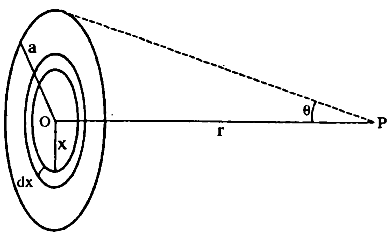

Figure 11.15

Let us draw a circle of radius x with the centre at O. We draw another concentric circle of radius $x + dx$. The part of the disc enclosed between these two circles can be treated as a uniform ring of radius x. The point P is on its axis at a distance r from the centre. The area of this ring is $2\pi x\,dx$. The area of the whole disc is πa^2. As the disc is uniform, the mass of this ring is

$$dm = \frac{M}{\pi a^2}2\pi x\,dx$$

$$= \frac{2M x\,dx}{a^2}.$$

The gravitational field at P due to the ring is, by equation (11.20),

$$dE = \frac{G\left(\frac{2M x\,dx}{a^2}\right)r}{(r^2 + x^2)^{3/2}}$$

$$= \frac{2GMr}{a^2}\frac{x\,dx}{(r^2 + x^2)^{3/2}}.$$

As x varies from 0 to a, the rings cover up the whole disc. The field due to each of these rings is in the same direction PO. Thus, the net field due to the whole disc is along PO and its magnitude is

$$E = \int_0^a \frac{2GMr}{a^2}\frac{x\,dx}{(r^2 + x^2)^{3/2}}$$

$$= \frac{2GMr}{a^2}\int_0^a \frac{x\,dx}{(r^2 + x^2)^{3/2}}. \qquad \dots \text{(i)}$$

Let $r^2 + x^2 = z^2$.

Then $2x\,dx = 2z\,dz$ and

$$\int \frac{x\,dx}{(r^2 + x^2)^{3/2}} = \int \frac{z\,dz}{z^3}$$

$$= \int \frac{1}{z^2}dz = -\frac{1}{z} = -\frac{1}{\sqrt{r^2 + x^2}}.$$

From (i), $$E = \frac{2GMr}{a^2}\left[-\frac{1}{\sqrt{r^2 + x^2}}\right]_0^a$$

$$= \frac{2GMr}{a^2}\left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + a^2}}\right] \qquad \dots (11.21)$$

Equation (11.21) may be expressed in terms of the angle θ subtended by a radius of the disc at P as,

$$E = \frac{2GM}{a^2}(1 - \cos\theta).$$

(D) Field due to a Uniform Thin Spherical Shell

We can use the construction of figure (11.8) to find the gravitational field at a point due to a uniform thin spherical shell. The figure is reproduced here (figure 11.16) with symbols having same meanings. The shaded ring has mass $dm = \frac{M}{2}\sin\theta\,d\theta$. The field at P due to this ring is

$$dE = \frac{Gdm}{z^2}\cos\alpha = \frac{GM}{2}\frac{\sin\theta\,d\theta\cos\alpha}{z^2}. \qquad \dots \text{(i)}$$

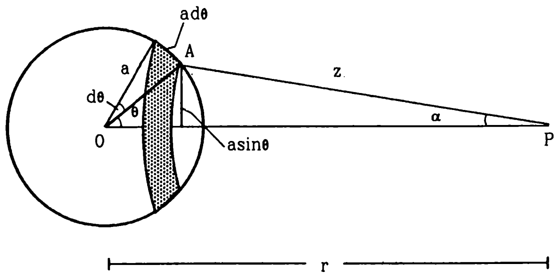

Figure 11.16

From the triangle OAP,

$$z^2 = a^2 + r^2 - 2ar\cos\theta$$

or, $$2z\,dz = 2ar\sin\theta\,d\theta$$

or, $$\sin\theta\,d\theta = \frac{z\,dz}{ar}. \quad \text{... (ii)}$$

Also from the triangle OAP,

$$a^2 = z^2 + r^2 - 2zr\cos\alpha$$

or, $$\cos\alpha = \frac{z^2 + r^2 - a^2}{2zr}. \quad \text{... (iii)}$$

Putting from (ii) and (iii) in (i),

$$dE = \frac{GM}{4ar^2}\left(1 - \frac{a^2 - r^2}{z^2}\right)dz$$

or, $$\int dE = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]$$

Case I : *P* is outside the shell ($r > a$)

In this case z varies from $r - a$ to $r + a$. The field due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]_{r-a}^{r+a} = \frac{GM}{r^2}. \quad \text{... (11.22)}$$

We see that *the shell may be treated as a point particle of the same mass placed at its centre to calculate the gravitational field at an external point.*

Case II : *P* is inside the shell

In this case z varies from $a - r$ to $a + r$ (figure 11.9). The field at P due to the whole shell is

$$E = \frac{GM}{4ar^2}\left[z + \frac{a^2 - r^2}{z}\right]_{a-r}^{a+r} = 0.$$

Hence *the field inside a uniform spherical shell is zero.*

(E) Gravitational Field due to a Uniform Solid Sphere

Case I : Field at an external point

Let the mass of the sphere be M and its radius be a. We have to calculate the gravitational field due to the sphere at a point outside the sphere at a distance r from the centre. Figure (11.17) shows the situation. The centre of the sphere is at O and the field is to be calculated at P.

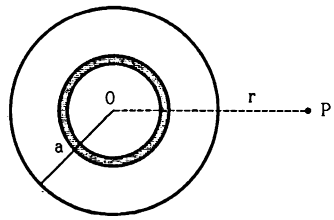

Figure 11.17

Let us divide the sphere into thin spherical shells each centred at O. Let the mass of one such shell be dm. To calculate the gravitational field at P, we can replace the shell by a single particle of mass dm placed at the centre of the shell that is at O. The field at P due to this shell is then

$$dE = \frac{G\,dm}{r^2}$$

towards PO. The field due to the whole sphere may be obtained by summing the fields of all the shells making the solid sphere.

Thus, $$E = \int dE$$

$$= \int \frac{G\,dm}{r^2} = \frac{G}{r^2}\int dm$$

$$= \frac{GM}{r^2}. \quad \text{... (11.23)}$$

Thus, *a uniform sphere may be treated as a single particle of equal mass placed at its centre for calculating the gravitational field at an external point.*

This allows us to treat the earth as a point particle placed at its centre while calculating the force between the earth and an apple.

Case II : Field at an internal point

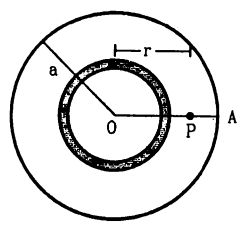

Figure 11.18

Suppose the point P is inside the solid sphere (figure 11.18). In this case $r < a$. The sphere may be divided into thin spherical shells all centered at O. Suppose the mass of such a shell is dm. If the radius of the shell is less than r, the point P is outside the shell and the field due to the shell is

$$dE = \frac{G\,dm}{r^2} \text{ along } PO.$$

If the radius of the shell considered is greater than r, the point P is internal and the field due to such a shell is zero. The total field due to the whole sphere is obtained by summing the fields due to all the shells. As all these fields are along the same direction, the net field is

$$E = \int dE$$

$$= \int \frac{G\,dm}{r^2} = \frac{G}{r^2}\int dm. \qquad \text{... (i)}$$

Only the masses of the shells with radii less than r should be added to get $\int dm$. These shells form a solid sphere of radius r. The volume of this sphere is $\frac{4}{3}\pi r^3$. The volume of the whole sphere is $\frac{4}{3}\pi a^3$. As the given sphere is uniform, the mass of the sphere of radius r is

$$\frac{M}{\frac{4}{3}\pi a^3}\cdot\left(\frac{4}{3}\pi r^3\right) = \frac{Mr^3}{a^3}.$$

Thus, $$\int dm = \frac{Mr^3}{a^3}$$

and by (i) $$E = \frac{G}{r^2}\frac{Mr^3}{a^3}$$

$$= \frac{GM}{a^3}r. \qquad \text{... (11.24)}$$

The gravitational field due to a uniform sphere at an internal point is proportional to the distance of the point from the centre of the sphere. At the centre itself, $r = 0$ and the field is zero. This is also expected from symmetry because any particle at the centre is equally pulled from all sides and the resultant must be zero. At the surface of the sphere, $r = a$ and

$$E = \frac{GM}{a^2}.$$

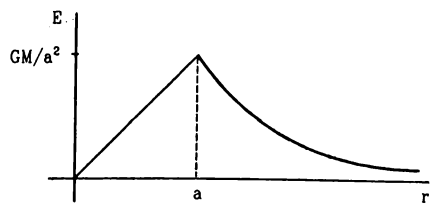

Figure 11.19

The formula (11.23) for the field at an external point also gives $E = \frac{GM}{a^2}$ at the surface of the sphere. The two formulae agree at $r = a$. Figure (11.19) shows graphically the variation of gravitational field due to a solid sphere with the distance from its centre.

Example 11.7

Find the gravitational field due to the moon at its surface. The mass of the moon is 7.36×10^{22} kg and the radius of the moon is 1.74×10^6 m. Assume the moon to be a spherically symmetric body.

Solution : To calculate the gravitational field at an external point, the moon may be replaced by a single particle of equal mass placed at its centre. Then the field at the surface is

$$E = \frac{GM}{a^2}$$

$$= \frac{6.67 \times 10^{-11}\,\text{N-m}^2/\text{kg}^2 \times 7.36 \times 10^{22}\,\text{kg}}{(1.74 \times 10^6\,\text{m})^2}$$

$$= 1.62\ \text{N/kg}.$$

This is about one sixth of the gravitational field due to the earth at its surface.

11.9 VARIATION IN THE VALUE OF *g*

The acceleration due to gravity is given by

$$g = \frac{F}{m}$$

where F is the force exerted by the earth on an object of mass m. This force is affected by a number of factors and hence g also depends on these factors.

(a) Height from the Surface of the Earth

If the object is placed at a distance h above the surface of the earth, the force of gravitation on it due to the earth is

$$F = \frac{GMm}{(R+h)^2}$$

where M is the mass of the earth and R is its radius.

Thus, $$g = \frac{F}{m} = \frac{GM}{(R+h)^2}.$$

We see that the value of g decreases as one goes up. We can write,

$$g = \frac{GM}{R^2\left(1+\frac{h}{R}\right)^2} = \frac{g_0}{\left(1+\frac{h}{R}\right)^2}$$

where $g_0 = \frac{GM}{R^2}$ is the value of g at the surface of the earth. If $h << R$,

$$g = g_0\left(1+\frac{h}{R}\right)^{-2} \approx g_0\left(1-\frac{2h}{R}\right).$$

If one goes a distance h inside the earth such as in mines etc., the value of g again decreases. The force by the earth is, by equation (11.24),

$$F = \frac{GMm}{R^3}(R - h)$$

or, $$g = \frac{F}{m} = \frac{GM}{R^2}\left(\frac{R-h}{R}\right)$$

$$= g_0\left(1 - \frac{h}{R}\right).$$

The value of g is maximum at the surface of the earth and decreases with the increase in height as well as with depth similar to that shown in figure (11.19).

Example 11.8

Calculate the value of acceleration due to gravity at a point (a) 5·0 km *above the earth's surface and (b)* 5·0 km *below the earth's surface. Radius of earth* = 6400 km *and the value of g at the surface of the earth is* 9·80 m/s^2.

Solution :

(a) The value of g at a height h is (for $h << R$)

$$g = g_0\left(1 - \frac{2h}{R}\right)$$

$$= (9\cdot80 \text{ m/s}^2)\left(1 - \frac{2 \times 5\cdot0 \text{ km}}{6400 \text{ km}}\right)$$

$$= 9\cdot78 \text{ m/s}^2.$$

(b) The value at a depth h is

$$g = g_0\left(1 - \frac{h}{R}\right)$$

$$= (9\cdot8 \text{ m/s}^2)\left(1 - \frac{5\cdot0 \text{ km}}{6400 \text{ km}}\right)$$

$$= 9\cdot79 \text{ m/s}^2.$$

(b) Rotation of the Earth

As the earth rotates about its own axis the frame attached to the earth is noninertial. If we wish to use the familiar Newton's laws, we have to include pseudo forces. For an object at rest with respect to the earth, a centrifugal force $m\omega^2 r$ is to be added where m is the mass of the object, ω is the angular velocity of the earth and r is the radius of the circle in which the particle rotates.

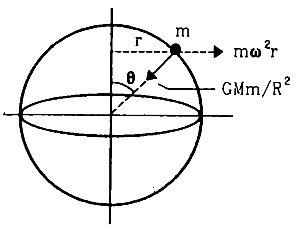

Figure 11.20

If the colatitude of the location of the particle is θ (figure 11.20), $r = R \sin\theta$ where R is the radius of the earth. Acceleration of an object falling near the earth's surface, as measured from the earth frame, is F/m where F is the vector sum of the gravitational force $\frac{GMm}{R^2} = mg$ and the centrifugal force $m\omega^2 r = m\omega^2 R \sin\theta$. The acceleration $F/m = g'$ is the apparent value of the acceleration due to gravity.

At the equator, $\theta = \pi/2$ and the centrifugal force is just opposite to the force of gravity. The resultant of these two is

$$F = mg - m\omega^2 R$$

or, $$g' = g - \omega^2 R.$$

At the poles, $\theta = 0$ and the centrifugal force $m\omega^2 R \sin\theta = 0$. Thus, $F = mg$ and $g' = g$. Thus, the observed value of the acceleration due to gravity is minimum at the equator and is maximum at the poles. This effect had been discussed in the chapter on circular motion.

(c) Non-sphericity of the Earth

All formulae and equations have been derived by assuming that the earth is a uniform solid sphere. The shape of the earth slightly deviates from the perfect sphere. The radius in the equatorial plane is about 21 km larger than the radius along the poles. Due to this the force of gravity is more at the poles and less at the equator. The value of g is accordingly larger at the poles and less at the equator. Note that due to rotation of earth also, the value of g is smaller at the equator than that at the poles.

(d) Non-uniformity of the Earth

The earth is not a uniformly dense object. There are a variety of minerals, metals, water, oil etc., inside the earth. Then at the surface there are mountains, seas, etc. Due to these non-uniformities in the mass distribution, the value of g is locally affected.

"Weighing" the Earth

The force exerted by the earth on a body is called the *weight* of the body. In this sense "weight of the earth" is a meaningless concept. However, the mass of the earth can be determined by noting the acceleration due to gravity near the surface of the earth. We have,

$$g = \frac{GM}{R^2}$$

or, $$M = gR^2/G$$

Putting $g = 9\cdot8 \text{ m/s}^2$, $R = 6400$ km and $G = 6{\cdot}67 \times 10^{-11} \frac{\text{N-m}^2}{\text{kg}^2}$

the mass of the earth comes out to be $5{\cdot}98 \times 10^{24}$ kg.

11.10 PLANETS AND SATELLITES

Planets

Planets move round the sun due to the gravitational attraction of the sun. The path of these planets are elliptical with the sun at a focus. However, the difference in major and minor axes is not large. The largest difference in the planets of our solar system is for Pluto and that is only 3%. The orbits can, therefore, be treated as nearly circular for not too sophisticated calculations. Let us derive certain characteristics of the planetory motion in terms of the radius of the orbit assuming it to be perfectly circular.

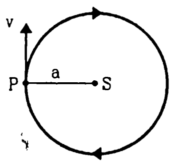

Figure 11.21

Let the mass of the sun be M and that of the planet under study be m. The mass of the sun is many times larger than the mass of the planet. The sun may, therefore, be treated as an inertial frame of reference.

Speed

Let the radius of the orbit be a and the speed of the planet in the orbit be v. By Newton's second law, the force on the planet equals its mass times the acceleration. Thus,

$$\frac{GMm}{a^2} = m\left(\frac{v^2}{a}\right)$$

or, $$v = \sqrt{\frac{GM}{a}} . \quad \dots (11.25)$$

The speed of a planet is inversely proportional to the square root of the radius of its orbit.

Time period

The time taken by a planet in completing one revolution is its time period T. In one revolution it covers a linear distance of $2\pi a$ at speed v. Thus,

$$T = \frac{2\pi a}{v}$$

$$= \frac{2\pi a}{\sqrt{\frac{GM}{a}}} = \frac{2\pi}{\sqrt{GM}} a^{3/2}$$

or, $$T^2 = \frac{4\pi^2}{GM} a^3 . \quad \dots (11.26)$$

Energy

The kinetic energy of the planet is

$$K = \frac{1}{2} mv^2 .$$

Using (11.25),

$$K = \frac{1}{2} m \frac{GM}{a} = \frac{GMm}{2a} .$$

The gravitational potential energy of the sun-planet system is

$$U = -\frac{GMm}{a} .$$

The total mechanical energy of the sun-planet system is

$$E = K + U = \frac{GMm}{2a} - \frac{GMm}{a} = -\frac{GMm}{2a} .$$

The total energy is negative. This is true for any bound system if the potential energy is taken to be zero at infinite separation.

Satellite

Satellites are launched from the earth so as to move round it. A number of rockets are fired from the satellite at proper time to establish the satellite in the desired orbit. Once the satellite is placed in the desired orbit with the correct speed for that orbit, it will continue to move in that orbit under gravitational attraction of the earth. All the equations derived above for planets are also true for satellites with M representing the mass of the earth and m representing the mass of the satellite.

Example 11.9

A satellite is revolving round the earth at a height of 600 km. *Find (a) the speed of the satellite and (b) the time period of the satellite. Radius of the earth* = 6400 km *and mass of the earth* = 6×10^{24} kg.

Solution : The distance of the satellite from the centre of the earth is 6400 km + 600 km = 7000 km.

The speed of the satellite is

$$v = \sqrt{\frac{GM}{a}}$$

$$= \sqrt{\frac{6{\cdot}67 \times 10^{-11}\ \text{N-m}^2/\text{kg}^2 \times 6 \times 10^{24}\ \text{kg}}{7000 \times 10^3\ \text{m}}}$$

$$= 7{\cdot}6 \times 10^3\ \text{m/s} = 7{\cdot}6\ \text{km/s} .$$

The time period is

$$T = \frac{2\pi a}{v}$$

$$= \frac{2\pi \times 7000 \times 10^3 \text{ m}}{7{\cdot}6 \times 10^3 \text{ m/s}} = 5{\cdot}8 \times 10^3 \text{ s}.$$

Geostationary Satellite

The earth rotates about its own axis (the line joining the north pole and the south pole) once in 24 hours. Suppose a satellite is established in an orbit in the plane of the equator. Suppose the height is such that the time period of the satellite is 24 hours and it moves in the same sense as the earth. The satellite will always be overhead a particular place on the equator. As seen from the earth, this satellite will appear to be stationary. Such a satellite is called *a geostationary satellite*. Such satellites are used for telecommunication, weather forecast and other applications.

According to equation (11.26),

$$T^2 = \frac{4\pi^2}{GM} a^3$$

or, $$a = \left(\frac{GMT^2}{4\pi^2} \right)^{1/3}.$$

Putting the values of G, M $(6 \times 10^{24}$ kg) and T (24 hours); the radius of the geostationary orbit comes out to be $a = 4{\cdot}2 \times 10^4$ km. The height above the surface of the earth is about $3{\cdot}6 \times 10^4$ km.

11.11 KEPLER'S LAWS

From the observations of Tycho Brahe, Kepler formulated the laws of planetary motion which we have listed in the first section of this chapter. The first law states that the path of a planet is elliptical with the sun at a focus. Circular path is a special case of an ellipse when the major and minor axes are equal. For a circular path, the planet should have velocity perpendicular to the line joining it with the sun and the magnitude should satisfy equation (11.25), that is $v = \sqrt{\frac{GM}{a}}$. If these conditions are not satisfied, the planet moves in an ellipse.

The second law states that the radius vector from the sun to the planet sweeps out equal area in equal time. For a circular orbit, this is obvious because the speed of the particle remains constant.

The third law of Kepler states that the square of the time period of a planet is proportional to the cube of the semimajor axis. For a circular orbit semimajor axis is same as the radius. We have already proved this law for circular orbits in equation (11.26). As M denotes mass of the sun, $\frac{4\pi^2}{GM}$ is fixed for all planets and $T^2 \propto a^3$.

11.12 WEIGHTLESSNESS IN A SATELLITE

A satellite moves round the earth in a circular orbit under the action of gravity. The acceleration of the satellite is $\frac{GM}{R^2}$ towards the centre of the earth, where M is the mass of the earth and R is the radius of the orbit of the satellite. Consider a body of mass m placed on a surface inside a satellite moving round the earth. The forces on the body are

(a) the gravitational pull of the earth $= \frac{GMm}{R^2}$,

(b) the contact force $\mathcal{N}$ by the surface.

By Newton's law,

$$G\frac{Mm}{R^2} - \mathcal{N} = m\left(\frac{GM}{R^2}\right) \quad \text{or,} \quad \mathcal{N} = 0.$$

Thus, the surface does not exert any force on the body and hence its apparent weight is zero. No support is needed to hold a body in the satellite. All positions shown in figure (11.22) are equally comfortable.

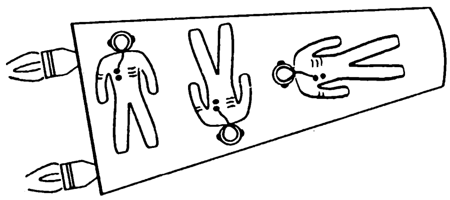

Figure 11.22

One can analyse the situation from the frame of the satellite. Working in the satellite frame we have to add a centrifugal force on all bodies. If the mass of a body is m, the centrifugal force is $m\left(\frac{GM}{R^2}\right)$ away from the centre of the earth. This psuedo force exactly balances the weight of the body which is $\frac{GMm}{R^2}$ towards the centre of the earth. A body needs no support to stay at rest in the satellite and hence all positions are equally comfortable. Water will not fall down from the glass even if it is inverted. It will act like a "gravity free hall". Such a state is called *weightlessness*.

It should be clear that the earth still attracts a body with the same force $\frac{GMm}{R^2}$. The feeling of weightlessness arises because one stays in a rotating frame.

11.13 ESCAPE VELOCITY

When a stone is thrown up, it goes up to a maximum height and then returns. As the particle goes up, the gravitational potential energy increases and the kinetic energy of the particle decreases. The particle will continue to go up till its kinetic energy becomes zero and will return from there.

Let the initial velocity of the particle be u. The kinetic energy of the particle is $K = \frac{1}{2} mu^2$ and the gravitational potential energy of the earth-particle system is $U = -\frac{GMm}{R}$, where M is the mass of the earth, m is the mass of the particle and R is the radius of the earth. When it reaches a height h above the earth's surface, its speed becomes v. The kinetic energy there is $\frac{1}{2} mv^2$ and the gravitational potential energy is $-\frac{GMm}{R+h}$.

By conservation of energy

$$\frac{1}{2} mu^2 - \frac{GMm}{R} = \frac{1}{2} mv^2 - \frac{GMm}{R+h}$$

or, $$\frac{1}{2} mv^2 = \left[\frac{1}{2} mu^2 - \frac{GMm}{R}\right] + \frac{GMm}{R+h}. \quad \ldots \text{(i)}$$

The particle will reach the maximum height when v becomes zero.

If $\frac{1}{2} mu^2 - \frac{GMm}{R} \geq 0$, the right hand side of (i) is greater than zero for all values of h. Thus, $\frac{1}{2} mv^2$ never becomes zero. The particle's velocity never reaches zero and so the particle will continue to go farther and farther away from the earth. Thus, the particle will never return to the earth if

$$\frac{1}{2} mu^2 - \frac{GMm}{R} \geq 0$$

or, $$u \geq \sqrt{\frac{2GM}{R}}. \quad \ldots (11.27)$$

This critical initial velocity is called the *escape velocity*. Putting the values of G, M and R, the escape velocity from the earth comes out to be 11·6 km/s. In this we have neglected the effect of other planets, stars and other objects in space. In fact, even if the initial velocity is somewhat less than the escape velocity, the particle may get attracted by some other celestial object and land up there.

Equation (11.27) is valid for any celestial object. For example, if something is thrown up from the surface of the moon, it will never return to the moon if the initial velocity is greater than $\sqrt{\frac{2GM}{R}}$, where M is the mass of the moon and R is the radius of the moon.

Example 11.10

Calculate the escape velocity from the moon. The mass of the moon $= 7{\cdot}4 \times 10^{22}$ kg *and radius of the moon* = 1740 km.

Solution : The escape velocity is

$$v = \sqrt{\frac{2GM}{R}}$$

$$= \sqrt{\frac{2 \times 6{\cdot}67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2 \times 7{\cdot}4 \times 10^{22} \text{ kg}}{1740 \times 10^3 \text{ m}}}$$

$= 2{\cdot}4$ km/s.

11.14 GRAVITATIONAL BINDING ENERGY

We have seen that if a particle of mass m placed on the earth is given an energy $\frac{1}{2} mu^2 = \frac{GMm}{R}$ or more, it finally escapes from earth. The minimum energy needed to take the particle infinitely away from the earth is called the *binding energy* of the earth-particle system. Thus, the binding energy of the earth-particle system is $\frac{GMm}{R}$.

11.15 BLACK HOLES

Consider a spherical body of mass M and radius R. Suppose, due to some reason the volume goes on decreasing while the mass remains the same. The escape velocity $\sqrt{\frac{2GM}{R}}$ from such a dense material will be very high. Suppose the radius is so small that

$$\sqrt{\frac{2GM}{R}} \geq c$$

where $c = 3 \times 10^8$ m/s is the speed of light. The escape velocity for such an object is equal to or greater than the speed of light. This means, anything starting from the object with a speed less than the speed of light will return to the object (neglecting the effect of other objects in space). According to the theory of relativity it is not possible to achieve a velocity greater than c for any material object. Thus, nothing can escape from such a dense material. Such objects are known as *black holes*. A number of such black holes exist in space. Even light cannot escape from a black hole.

11.16 INERTIAL AND GRAVITATIONAL MASS

Given two objects A and B, how can we determine the ratio of the mass of A to the mass of B. One way is to use Newton's second law of motion. If we apply equal forces F on each of the two objects,

$$F = m_A a_A \quad \text{and also} \quad F = m_B a_B.$$

Thus, $\frac{m_A}{m_B} = \frac{a_B}{a_A}$

or, $m_A = \frac{a_B}{a_A} m_B$. ... (i)

This equation may be used to "define the mass" of an object. Taking the object B to be the standard kilogram ($m_B = 1$ kg), mass of any object may be obtained by measuring their accelerations under equal force and using (i). The mass so defined is called *inertial mass.*

Another way to compare masses of two objects is based on the law of gravitation. The gravitational force exerted by a massive body on an object is proportional to the mass of the object. If F_A and F_B be the forces of attraction on the two objects due to the earth,

$$F_A = \frac{G m_A M}{R^2} \quad \text{and} \quad F_B = \frac{G m_B M}{R^2}.$$

Thus, $\frac{m_A}{m_B} = \frac{F_A}{F_B}$

or, $m_A = \frac{F_A}{F_B} m_B$. ... (ii)

We can use this equation to "define the mass" of an object. If B is a standard unit mass, by measuring the gravitational forces F_A and F_B we can obtain the mass of the object A. The mass so defined is called *gravitational mass.* when we measure the mass using a spring balance, we actually measure the gravitational mass.

Equivalence of Inertial and Gravitational Mass

The two definitions of mass, described above, are quite independent of each other. There is no obvious reason why the two should be identical. However, they happen to be identical. Several sophisticated experiments have been performed to test this equivalence and none of them has supplied any evidence against it. The general theory of relativity is based on the principle of equivalence of inertial and gravitational mass.

11.17 POSSIBLE CHANGES IN THE LAW OF GRAVITATION

There is some indication that the force between two masses is not as described in this chapter. The deviation from the simple law $F = \frac{GMm}{R^2}$ is being taken as an indication of the existence of a fifth interaction besides gravitational, electromagnetic, nuclear and weak. It has been reported (Phys. Rev. Lett. Jan 6, 1986) that the force between two masses may be better represented by

$$F = \frac{G_\infty m_1 m_2}{r^2}\left[1 + \left(1 + \frac{r}{\lambda}\right)\alpha e^{-\frac{r}{\lambda}}\right]$$

with $\alpha \approx -0{\cdot}007$ and $\lambda \approx 200$ m. As α is negative, the second term in the square bracket represents a repulsive force. For $r >> 200$ m,

$$F = \frac{G_\infty m_1 m_2}{r^2}$$

which is the force operative between the earth and other objects. For $r << 200$ m,

$$F = \frac{G_\infty m_1 m_2(1+\alpha)}{r^2} = \frac{G' m_1 m_2}{r^2}$$

where $G' = G_\infty(1+\alpha)$.

This is the force we measure in a Cavendish-experiment. The value of G for small distances is about 1% less than the value of G for large distances.

Worked Out Examples

1. *Three particles A, B and C, each of mass m, are placed in a line with AB = BC = d. Find the gravitational force on a fourth particle P of same mass, placed at a distance d from the particle B on the perpendicular bisector of the line AC.*

Solution :

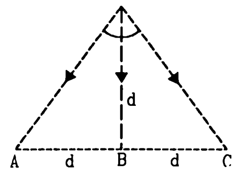

Figure 11-W1

The force at P due to A is

$$F_A = \frac{G m^2}{(AP)^2} = \frac{G m^2}{2 d^2}$$

along PA. The force at P due to C is

$$F_C = \frac{G m^2}{(CP)^2} = \frac{G m^2}{2 d^2}$$

along PC. The force at P due to B is

$$F_B = \frac{G m^2}{d^2} \text{ along } PB.$$

The resultant of F_A, F_B and F_C will be along PB.

Clearly $\angle APB = \angle BPC = 45°$.

Component at F_A along $PB = F_A \cos 45° = \frac{G m^2}{2\sqrt{2} d^2}$.

Component at F_C along $PB = F_C \cos 45° = \dfrac{G m^2}{2\sqrt{2} d^2}$.

Component at F_B along $PB = \dfrac{G m^2}{d^2}$.

Hence, the resultant of the three forces is

$$\frac{G m^2}{d^2}\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} + 1\right) = \frac{G m^2}{d^2}\left(1 + \frac{1}{\sqrt{2}}\right) \text{ along } PB.$$

2. *Find the distance of a point from the earth's centre where the resultant gravitational field due to the earth and the moon is zero. The mass of the earth is* $6\cdot0 \times 10^{24}$ kg *and that of the moon is* $7\cdot4 \times 10^{22}$ kg. *The distance between the earth and the moon is* $4\cdot0 \times 10^5$ km.

Solution : The point must be on the line joining the centres of the earth and the moon and in between them. If the distance of the point from the earth is x, the distance from the moon is $(4\cdot0 \times 10^5 \text{ km} - x)$. The magnitude of the gravitational field due to the earth is

$$E_1 = \frac{GM_e}{x^2} = \frac{G \times 6 \times 10^{24} \text{ kg}}{x^2}$$

and magnitude of the gravitational field due to the moon is

$$E_2 = \frac{GM_m}{(4\cdot0 \times 10^5 \text{ km} - x)^2} = \frac{G \times 7\cdot4 \times 10^{22} \text{ kg}}{(4\cdot0 \times 10^5 \text{ km} - x)^2}.$$

These fields are in opposite directions. For the resultant field to be zero $E_1 = E_2$.

or, $$\frac{6 \times 10^{24} \text{ kg}}{x^2} = \frac{7\cdot4 \times 10^{22} \text{ kg}}{(4\cdot0 \times 10^5 \text{ km} - x)^2}$$

or, $$\frac{x}{4\cdot0 \times 10^5 \text{ km} - x} = \sqrt{\frac{6 \times 10^{24}}{7\cdot4 \times 10^{22}}} = 9$$

or, $$x = 3\cdot6 \times 10^5 \text{ km}.$$

3. *Two particles of equal mass go round a circle of radius* R *under the action of their mutual gravitational attraction. Find the speed of each particle.*

Solution : The particles will always remain diametrically opposite so that the force on each particle will be directed along the radius. Consider the motion of one of the particles. The force on the particle is $F = \dfrac{G m^2}{4 R^2}$. If the speed is v, its acceleration is v^2/R.

Thus, by Newton's law,

$$\frac{G m^2}{4 R^2} = \frac{m v^2}{R}$$

or, $$v = \sqrt{\frac{G m}{4 R}}.$$

4. *Two particles* A *and* B *of masses* 1 kg *and* 2 kg *respectively are kept* 1 m *apart and are released to move under mutual attraction. Find the speed of* A *when that of* B *is* 3·6 cm/hour. *What is the separation between the particles at this instant?*

Solution : The linear momentum of the pair $A + B$ is zero initially. As only mutual attraction is taken into account, which is internal when $A + B$ is taken as the system, the linear momentum will remain zero. The particles move in opposite directions. If the speed of A is v when the speed of B is 3·6 cm/hour $= 10^{-5}$ m/s,

$$(1 \text{ kg})\, v = (2 \text{ kg})\,(10^{-5} \text{ m/s})$$

or, $$v = 2 \times 10^{-5} \text{ m/s}.$$

The potential energy of the pair is $-\dfrac{G m_A m_B}{R}$ with usual symbols. Initial potential energy

$$= -\frac{6\cdot67 \times 10^{-11} \text{ N-m}^2/\text{kg}^2 \times 2 \text{ kg} \times 1 \text{ kg}}{1 \text{ m}}$$

$$= -13\cdot34 \times 10^{-11} \text{ J}.$$

If the separation at the given instant is d, using conservation of energy,

$$-13\cdot34 \times 10^{-11} \text{ J} + 0$$

$$= -\frac{13\cdot34 \times 10^{-11} \text{ J-m}}{d} + \frac{1}{2}(2 \text{ kg})(10^{-5} \text{ m/s})^2$$

$$+ \frac{1}{2}(1 \text{ kg})(2 \times 10^{-5} \text{ m/s})^2$$

Solving this, $d = 0\cdot31$ m.

5. *The gravitational field in a region is given by* $\vec{E} = (10 \text{ N/kg})(\vec{i} + \vec{j})$. *Find the work done by an external agent to slowly shift a particle of mass* 2 kg *from the point* (0,0) *to a point* (5 m, 4 m).

Solution : As the particle is slowly shifted, its kinetic energy remains zero. The total work done on the particle is thus zero. The work done by the external agent should be negative of the work done by the gravitational field. The work done by the field is

$$\int_i^f \vec{F} \cdot d\vec{r}$$

Consider figure (11-W2). Suppose the particle is taken from O to A and then from A to B. The force on the particle is

$$\vec{F} = m\vec{E} = (2 \text{ kg})(10 \text{ N/kg})(\vec{i} + \vec{j}) = (20 \text{ N})(\vec{i} + \vec{j}).$$

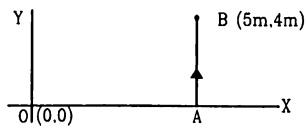

Figure 11-W2

The work done by the field during the displacement OA is

$$W_1 = \int_0^{5m} F_x\,dx$$

$$= \int_0^{5m} (20\text{ N})\,dx = 20\text{ N} \times 5\text{ m} = 100\text{ J}.$$

Similarly, the work done in displacement AB is

$$W_2 = \int_0^{4m} F_y\,dy = \int_0^{4m} (20\text{ N})\,dy$$

$$= (20\text{ N})(4\text{ m}) = 80\text{ J}.$$

Thus, the total work done by the field, as the particle is shifted from O to B, is 180 J.

The work done by the external agent is –180 J.

Note that the work is independent of the path so that we can choose any path convenient to us from O to B.

6. *A uniform solid sphere of mass M and radius a is surrounded symmetrically by a uniform thin spherical shell of equal mass and radius 2 a. Find the gravitational field at a distance (a) $\frac{3}{2}a$ from the centre, (b) $\frac{5}{2}a$ from the centre.*

Solution :

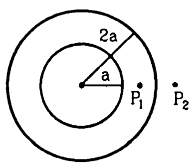

Figure 11-W3

Figure (11-W3) shows the situation. The point P_1 is at a distance $\frac{3}{2}a$ from the centre and P_2 is at a distance $\frac{5}{2}a$ from the centre. As P_1 is inside the cavity of the thin spherical shell, the field here due to the shell is zero. The field due to the solid sphere is

$$E = \frac{GM}{\left(\frac{3}{2}a\right)^2} = \frac{4\,GM}{9\,a^2}.$$

This is also the resultant field. The direction is towards the centre. The point P_2 is outside the sphere as well as the shell. Both may be replaced by single particles of the same mass at the centre. The field due to each of them is

$$E' = \frac{GM}{\left(\frac{5}{2}a\right)^2} = \frac{4\,GM}{25\,a^2}.$$

The resultant field is $E = 2\,E' = \dfrac{8\,GM}{25\,a^2}$ towards the centre.

7. *The density inside a solid sphere of radius a is given by $\rho = \rho_0\, a/r$ where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance 2 a from its centre.*

Solution **:** The field is required at a point outside the sphere. Dividing the sphere in concentric shells, each shell can be replaced by a point particle at its centre having mass equal to the mass of the shell. Thus, the whole sphere can be replaced by a point particle at its centre having mass equal to the mass of the given sphere. If the mass of the sphere is M, the gravitational field at the given point is

$$E = \frac{GM}{(2a)^2} = \frac{GM}{4a^2}. \qquad \text{... (i)}$$

The mass M may be calculated as follows. Consider a concentric shell of radius r and thickness dr. Its volume is

$$dV = (4\pi r^2)\,dr$$

and its mass is

$$dM = \rho dV = \left(\rho_0 \frac{a}{r}\right)(4\pi r^2 dr).$$

$$= 4\pi\rho_0\, ar\, dr.$$

The mass of the whole sphere is

$$M = \int_0^a 4\pi\rho_0\, ar\, dr$$

$$= 2\pi\rho_0 a^3.$$

Thus, by (i) the gravitational field is

$$E = \frac{2\pi G\rho_0 a^3}{4a^2} = \frac{1}{2}\pi G\rho_0 a.$$

8. *A uniform ring of mass m and radius a is placed directly above a uniform sphere of mass M and of equal radius. The centre of the ring is at a distance √3 a from the centre of the sphere. Find the gravitational force exerted by the sphere on the ring.*

Solution **:** The gravitational field at any point on the ring due to the sphere is equal to the field due to a single particle of mass M placed at the centre of the sphere. Thus, the force on the ring due to the sphere is also equal to the force on it by a particle of mass M placed at this point. By Newton's third law it is equal to the force on the particle by the ring. Now the gravitational field due to the ring at a distance $d = \sqrt{3}\,a$ on its axis is

$$E = \frac{Gmd}{(a^2 + d^2)^{3/2}} = \frac{\sqrt{3}\,Gm}{8a^2}.$$

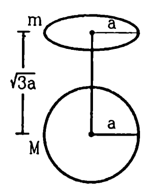

Figure 11-W4

The force on a particle of mass M placed here is

$$F = ME$$
$$= \frac{\sqrt{3}GMm}{8a^2}.$$

This is also the force due to the sphere on the ring.

9. *A particle is fired vertically upward with a speed of* 9·8 km/s *Find the maximum height attained by the particle. Radius of earth* = 6400 km *and g at the surface* = 9·8 m/s^2. *Consider only earth's gravitation.*

Solution : At the surface of the earth, the potential energy of the earth-particle system is $-\frac{GMm}{R}$ with usual symbols. The kinetic energy is $\frac{1}{2}mv_0^2$ where v_0 = 9·8 km/s. At the maximum height the kinetic energy is zero. If the maximum height reached is H, the potential energy of the earth-particle system at this instant is $-\frac{GMm}{R+H}$. Using conservation of energy,

$$-\frac{GMm}{R} + \frac{1}{2}mv_0^2 = -\frac{GMm}{R+H}.$$

Writing $GM = gR^2$ and dividing by m,

$$-gR + \frac{v_0^2}{2} = \frac{-gR^2}{R+H}$$

or, $$\frac{R^2}{R+H} = R - \frac{v_0^2}{2g}$$

or, $$R + H = \frac{R^2}{R - \frac{v_0^2}{2g}}.$$

Putting the values of R, v_0 and g on the right side,

$$R + H = \frac{(6400\text{ km})^2}{6400\text{ km} - \frac{(9{\cdot}8\text{ km/s})^2}{2 \times 9{\cdot}8\text{ m/s}^2}}$$
$$= \frac{(6400\text{ km})^2}{1500\text{ km}} = 27300\text{ km}$$

or, $$H = (27300 - 6400)\text{ km} = 20900\text{ km}.$$

10. *A particle hanging from a spring stretches it by* 1 cm *at earth's surface. How much will the same particle stretch the spring at a place* 800 km *above the earth's surface? Radius of the earth* = 6400 km.

Solution : Suppose the mass of the particle is m and the spring constant of the spring is k. The acceleration due to gravity at earth's surface is $g = \frac{GM}{R^2}$ with usual symbols. The extension in the spring is mg/k.

Hence, $$1\text{ cm} = \frac{GMm}{kR^2}. \qquad \dots \text{(i)}$$

At a height h = 800 km, the extension is given by

$$x = \frac{GMm}{k(R+h)^2}. \qquad \dots \text{(ii)}$$

By (i) and (ii), $$\frac{x}{1\text{ cm}} = \frac{R^2}{(R+h)^2}$$
$$= \frac{(6400\text{ km})^2}{(7200\text{ km})^2} = 0{\cdot}79.$$

Hence $$x = 0{\cdot}79\text{ cm}.$$

11. *A simple pendulum has a time period exactly* 2 s *when used in a laboratory at north pole. What will be the time period if the same pendulum is used in a laboratory at equator? Account for the earth's rotation only. Take* $g = \frac{GM}{R^2}$ = 9·8 m/s^2 *and radius of earth* = 6400 km.

Solution : Consider the pendulum in its mean position at the north pole. As the pole is on the axis of rotation, the bob is in equilibrium. Hence in the mean position, the tension T is balanced by earth's attraction. Thus, $T = \frac{GMm}{R^2} = mg$. The time period t is

$$t = 2\pi\sqrt{\frac{l}{T/m}} = 2\pi\sqrt{\frac{l}{g}}. \qquad \dots \text{(i)}$$

At equator, the lab and the pendulum rotate with the earth at angular velocity $\omega = \frac{2\pi\text{ radian}}{24\text{ hour}}$ in a circle of radius equal to 6400 km. Using Newton's second law,

$$\frac{GMm}{R^2} - T' = m\omega^2 R \quad \text{or,} \quad T' = m(g - \omega^2 R)$$

where T' is the tension in the string.

The time period will be

$$t' = 2\pi\sqrt{\frac{l}{(T'/m)}} = 2\pi\sqrt{\frac{l}{g - \omega^2 R}}. \qquad \dots \text{(ii)}$$

By (i) and (ii),

$$\frac{t'}{t} = \sqrt{\frac{g}{g - \omega^2 R}} = \left(1 - \frac{\omega^2 R}{g}\right)^{-1/2}$$

or, $$t' \approx t\left(1 + \frac{\omega^2 R}{2g}\right).$$

Putting the values, t' = 2·004 seconds.

12. *A satellite is to revolve round the earth in a circle of radius* 8000 km. *With what speed should this satellite be projected into orbit? What will be the time period? Take g at the surface* = 9·8 m/s^2 *and radius of the earth* = 6400 km.

Solution : Suppose, the speed of the satellite is v. The acceleration of the satellite is v^2/r, where r is the radius of the orbit. The force on the satellite is $\frac{GMm}{r^2}$ with usual symbols. Using Newton's second law,

$$\frac{GMm}{r^2} = m\frac{v^2}{r}$$

or, $$v^2 = \frac{GM}{r} = \frac{gR^2}{r} = \frac{(9\cdot8\ \text{m/s}^2)(6400\ \text{km})^2}{(8000\ \text{km})}$$

giving $v = 7\cdot08$ km/s.

The time period is $\dfrac{2\pi r}{v} = \dfrac{2\pi(8000\ \text{km})}{(7\cdot08\ \text{km/s})} \approx 118$ minutes.

13. *Two satellites S_1 and S_2 revolve round a planet in coplanar circular orbits in the same sense. Their periods of revolution are 1 h and 8 h respectively. The radius of the orbit of S_1 is 10^4 km. When S_2 is closest to S_1, find (a) the speed of S_2 relative to S_1 and (b) the angular speed of S_2 as observed by an astronaut in S_1.*

Solution : Let the mass of the planet be M, that of S_1 be m_1 and of S_2 be m_2. Let the radius of the orbit of S_1 be R_1 ($= 10^4$ km) and of S_2 be R_2.

Let v_1 and v_2 be the linear speeds of S_1 and S_2 with respect to the planet. Figure (11-W5) shows the situation.

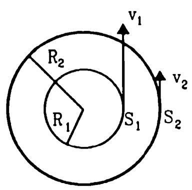

Figure 11-W5

As the square of the time period is proportional to the cube of the radius,

$$\left(\frac{R_2}{R_1}\right)^3 = \left(\frac{T_2}{T_1}\right)^2 = \left(\frac{8\ \text{h}}{1\ \text{h}}\right)^2 = 64$$

or, $$\frac{R_2}{R_1} = 4$$

or, $$R_2 = 4R_1 = 4 \times 10^4\ \text{km}.$$

Now the time period of S_1 is 1 h. So,

$$\frac{2\pi R_1}{v_1} = 1\ \text{h}$$

or, $$v_1 = \frac{2\pi R_1}{1\ \text{h}} = 2\pi \times 10^4\ \text{km/h}.$$

similarly, $$v_2 = \frac{2\pi R_2}{8\ h} = \pi \times 10^4\ \text{km/h}.$$

(a) At the closest separation, they are moving in the same direction. Hence the speed of S_2 with respect to S_1 is $|v_2 - v_1| = \pi \times 10^4$ km/h.

(b) As seen from S_1, the satellite S_2 is at a distance $R_2 - R_1 = 3 \times 10^4$ km at the closest separation. Also it is moving at $\pi \times 10^4$ km/h in a direction perpendicular to the line joining them. Thus, the angular speed of S_2 as observed by S_1 is

$$\omega = \frac{\pi \times 10^4\ \text{km/h}}{3 \times 10^4\ \text{km}} = \frac{\pi}{3}\ \text{rad/h}.$$

□

QUESTIONS FOR SHORT ANSWER

1. Can two particles be in equilibrium under the action of their mutual gravitational force ? Can three particles be ? Can one of the three particles be ?
2. Is there any meaning of "Weight of the earth" ?
3. If heavier bodies are attracted more strongly by the earth, why don't they fall faster than the lighter bodies ?
4. Can you think of two particles which do not exert gravitational force on each other ?
5. The earth revolves round the sun because the sun attracts the earth. The sun also attracts the moon and this force is about twice as large as the attraction of the earth on the moon. Why does the moon not revolve round the sun ? Or does it ?
6. At noon, the sun and the earth pull the objects on the earth's surface in opposite directions. At midnight the sun and the earth pull these objects in same direction. Is the weight of an object, as measured by a spring balance on the earth's surface, more at midnight as compared to its weight at noon ?
7. An apple falls from a tree. An insect in the apple finds that the earth is falling towards it with an acceleration g. Who exerts the force needed to accelerate the earth with this acceleration g ?
8. Suppose the gravitational potential due to a small system is k/r^2 at a distance r from it. What will be the gravitational field ? Can you think of any such system ? What happens if there were negative masses ?
9. The gravitational potential energy of a two-particle system is derived in this chapter as $U = -\dfrac{Gm_1m_2}{r}$. Does it follow from this equation that the potential energy for $r = \infty$ must be zero ? Can we choose the potential energy for $r = \infty$ to be 20 J and still use this formula ? If no, what formula should be used to calculate the gravitational potential energy at separation r ?
10. The weight of an object is more at the poles than at the equator. Is it benificial to purchase goods at equator and sell them at the pole ? Does it matter whether a spring balance is used or an equal-beam balance is used ?
11. The weight of a body at the poles is greater than the weight at the equator. Is it the actual weight or the apparent weight we are talking about ? Does your

answer depend on whether only the earth's rotation is taken into account or the flattening of the earth at the poles is also taken into account?

12. If the radius of the earth decreases by 1% without changing its mass, will the acceleration due to gravity at the surface of the earth increase or decrease? If so, by what per cent?
13. A nut becomes loose and gets detached from a satellite revolving around the earth. Will it land on the earth? If yes, where will it land? If no, how can an astronaut make it land on the earth?
14. Is it necessary for the plane of the orbit of a satellite to pass through the centre of the earth?
15. Consider earth satellites in circular orbits. A geostationary satellite must be at a height of about 36000 km from the earth's surface. Will any satellite moving at this height be a geostationary satellite? Will any satellite moving at this height have a time period of 24 hours?
16. No part of India is situated on the equator. Is it possible to have a geostationary satellite which always remains over New Delhi?
17. As the earth rotates about its axis, a person living in his house at the equator goes in a circular orbit of radius equal to the radius of the earth. Why does he/she not feel weightless as a satellite passenger does?
18. Two satellites going in equatorial plane have almost same radii. As seen from the earth one move from east to west and the other from west to east. Will they have the same time period as seen from the earth? If not, which one will have less time period?
19. A spacecraft consumes more fuel in going from the earth to the moon than it takes for a return trip. Comment on this statement.

OBJECTIVE I

1. The acceleration of moon with respect to earth is $0{\cdot}0027\ \text{m/s}^2$ and the acceleration of an apple falling on earth's surface is about $10\ \text{m/s}^2$. Assume that the radius of the moon is one fourth of the earth's radius. If the moon is stopped for an instant and then released, it will fall towards the earth. The initial acceleration of the moon towards the earth will be

   (a) $10\ \text{m/s}^2$ (b) $0{\cdot}0027\ \text{m/s}^2$ (c) $6{\cdot}4\ \text{m/s}^2$ (d) $5{\cdot}0\ \text{m/s}^2$.
2. The acceleration of the moon just before it strikes the earth in the previous question is

   (a) $10\ \text{m/s}^2$ (b) $0{\cdot}0027\ \text{m/s}^2$ (c) $6{\cdot}4\ \text{m/s}^2$ (d) $5{\cdot}0\ \text{m/s}^2$
3. Suppose, the acceleration due to gravity at the earth's surface is $10\ \text{m/s}^2$ and at the surface of Mars it is $4{\cdot}0\ \text{m/s}^2$. A 60 kg passenger goes from the earth to the Mars in a spaceship moving with a constant velocity. Neglect all other objects in the sky. Which part of figure (11-Q1) best represents the weight (net gravitational force) of the passenger as a function of time.

   (a) A. (b) B. (c) C. (d) D.

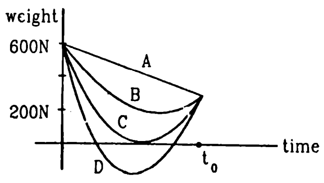

Figure 11-Q1

4. Consider a planet in some solar system which has a mass double the mass of the earth and density equal to the average density of the earth. An object weighing W on the earth will weigh

   (a) W (b) $2W$ (c) $W/2$ (d) $2^{1/3}W$ at the planet.
5. If the acceleration due to gravity at the surface of the earth is g, the work done in slowly lifting a body of mass m from the earth's surface to a height R equal to the radius of the earth is

   (a) $\frac{1}{2}mgR$ (b) $2mgR$ (c) mgR (d) $\frac{1}{4}mgR$.
6. A person brings a mass of 1 kg from infinity to a point A. Initially the mass was at rest but it moves at a speed of 2 m/s as it reaches A. The work done by the person on the mass is -3 J. The potential at A is

   (a) -3 J/kg (b) -2 J/kg (c) -5 J/kg (d) none of these.
7. Let V and E be the gravitational potential and gravitational field at a distance r from the centre of a uniform spherical shell. Consider the following two statements :

   (A) The plot of V against r is discontinuous.
   (B) The plot of E against r is discontinuous.
   (a) Both A and B are correct.
   (b) A is correct but B is wrong.
   (c) B is correct but A is wrong.
   (d) Both A and B are wrong.
8. Let V and E represent the gravitational potential and field at a distance r from the centre of a uniform solid sphere. Consider the two statements:

   (A) the plot of V against r is discontinuous.
   (B) The plot of E against r is discontinuous.
   (a) Both A and B are correct.
   (b) A is correct but B is wrong.
   (c) B is correct but A is wrong.
   (d) Both A and B are wrong.
9. Take the effect of bulging of earth and its rotation in account. Consider the following statements:

   (A) There are points outside the earth where the value of g is equal to its value at the equator.
   (B) There are points outside the earth where the value of g is equal to its value at the poles.

(a) Both A and B are correct.
(b) A is correct but B is wrong.
(c) B is correct but A is wrong.
(d) Both A and B are wrong.

10. The time period of an earth-satellite in circular orbit is independent of
(a) the mass of the satellite (b) radius of the orbit
(c) none of them (d) both of them.

11. The magnitude of gravitational potential energy of the moon-earth system is U with zero potential energy at infinite separation. The kinetic energy of the moon with respect to the earth is K.
(a) $U < K$. (b) $U > K$. (c) $U = K$.

12. Figure (11-Q2) shows the elliptical path of a planet about the sun. The two shaded parts have equal area. If t_1 and t_2 be the time taken by the planet to go from a to b and from c to d respectively,
(a) $t_1 < t_2$ (b) $t_1 = t_2$ (c) $t_1 > t_2$
(d) insufficient information to deduce the relation between t_1 and t_2.

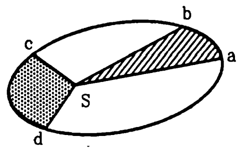

Figure 11-Q2

13. A person sitting in a chair in a satellite feels weightless because
(a) the earth does not attract the objects in a satellite
(b) the normal force by the chair on the person balances the earth's attraction
(c) the normal force is zero
(d) the person in satellite is not accelerated.

14. A body is suspended from a spring balance kept in a satellite. The reading of the balance is W_1 when the satellite goes in an orbit of radius R and is W_2 when it goes in an orbit of radius $2R$.
(a) $W_1 = W_2$. (b) $W_1 < W_2$. (c) $W_1 > W_2$. (d) $W_1 \neq W_2$

15. The kinetic energy needed to project a body of mass m from the earth's surface to infinity is
(a) $\frac{1}{4}mgR$ (b) $\frac{1}{2}mgR$ (c) mgR (d) $2mgR$.

16. A particle is kept at rest at a distance R (earth's radius) above the earth's surface. The minimum speed with which it should be projected so that it does not return is
(a) $\sqrt{\frac{GM}{4R}}$ (b) $\sqrt{\frac{GM}{2R}}$ (c) $\sqrt{\frac{GM}{R}}$ (d) $\sqrt{\frac{2GM}{R}}$.

17. A satellite is orbiting the earth close to its surface. A particle is to be projected from the satellite to just escape from the earth. The escape speed from the earth is v_e. Its speed with respect to the satellite
(a) will be less than v_e
(b) will be more than v_e
(c) will be equal to v_e
(d) will depend on direction of projection.

OBJECTIVE II

1. Let V and E denote the gravitational potential and gravitational field at a point. It is possible to have
(a) $V = 0$ and $E = 0$ (b) $V = 0$ and $E \neq 0$
(c) $V \neq 0$ and $E = 0$ (d) $V \neq 0$ and $E \neq 0$.

2. Inside a uniform spherical shell
(a) the gravitational potential is zero
(b) the gravitational field is zero
(c) the gravitational potential is same everywhere
(d) the gravitational field is same everywhere.

3. A uniform spherical shell gradually shrinks maintaining its shape. The gravitational poential at the centre
(a) increases (b) decreases
(c) remains constant (d) oscillates.

4. Consider a planet moving in an elliptical orbit round the sun. The work done on the planet by the gravitational force of the sun
(a) is zero in any small part of the orbit
(b) is zero in some parts of the orbit
(c) is zero in one complete revolution
(d) is zero in no part of the motion.

5. Two satellites A and B move round the earth in the same orbit. The mass of B is twice the mass of A.
(a) Speeds of A and B are equal.
(b) The potential energy of earth $+ A$ is same as that of earth $+ B$.
(c) The kinetic energy of A and B are equal.
(d) The total energy of earth $+ A$ is same as that of earth $+ B$.

6. Which of the following quantities remain constant in a planetory motion (consider elliptical orbits) as seen from the sun ?
(a) Speed. (b) Angular speed.
(c) Kinetic energy. (d) Angular momentum.

EXERCISES

1. Two spherical balls of mass 10 kg each are placed 10 cm apart. Find the gravitational force of attraction between them.

2. Four particles having masses m, $2m$, $3m$ and $4m$ are placed at the four corners of a square of edge a. Find the gravitational force acting on a particle of mass m placed at the centre.

3. Three equal masses m are placed at the three corners of an equilateral triangle of side a. Find the force exerted by this system on another particle of mass m placed at (a) the mid-point of a side, (b) at the centre of the triangle.
4. Three uniform spheres each having a mass M and radius a are kept in such a way that each touches the other two. Find the magnitude of the gravitational force on any of the spheres due to the other two.
5. Four particles of equal masses M move along a circle of radius R under the action of their mutual gravitational attraction. Find the speed of each particle.
6. Find the acceleration due to gravity of the moon at a point 1000 km above the moon's surface. The mass of the moon is $7{\cdot}4 \times 10^{22}$ kg and its radius is 1740 km.
7. Two small bodies of masses 10 kg and 20 kg are kept a distance 1·0 m apart and released. Assuming that only mutual gravitational forces are acting, find the speeds of the particles when the separation decreases to 0·5 m.
8. A semicircular wire has a length L and mass M. A particle of mass m is placed at the centre of the circle. Find the gravitational attraction on the particle due to the wire.
9. Derive an expression for the gravitational field due to a uniform rod of length L and mass M at a point on its perpendicular bisector at a distance d from the centre.
10. Two concentric spherical shells have masses M_1, M_2 and radii R_1, R_2 $(R_1 < R_2)$. What is the force exerted by this system on a particle of mass m_1 if it is placed at a distance $(R_1 + R_2)/2$ from the centre?
11. A tunnel is dug along a diameter of the earth. Find the force on a particle of mass m placed in the tunnel at a distance x from the centre.
12. A tunnel is dug along a chord of the earth at a perpendicular distance $R/2$ from the earth's centre. The wall of the tunnel may be assumed to be frictionless. Find the force exerted by the wall on a particle of mass m when it is at a distance x from the centre of the tunnel.
13. A solid sphere of mass m and radius r is placed inside a hollow thin spherical shell of mass M and radius R as shown in figure (11-E1). A particle of mass m' is placed on the line joining the two centres at a distance x from the point of contact of the sphere and the shell. Find the magnitude of the resultant gravitational force on this particle due to the sphere and the shell if (a) $r < x < 2r$, (b) $2r < x < 2R$ and (c) $x > 2R$.

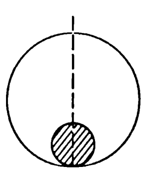

Figure 11-E1

14. A uniform metal sphere of radius a and mass M is surrounded by a thin uniform spherical shell of equal mass and radius $4a$ (figure 11-E2). The centre of the shell falls on the surface of the inner sphere. Find the gravitational field at the points P_1 and P_2 shown in the figure.

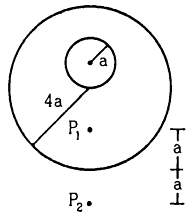

Figure 11-E2

15. A thin spherical shell having uniform density is cut in two parts by a plane and kept separated as shown in figure (11-E3). The point A is the centre of the plane section of the first part and B is the centre of the plane section of the second part. Show that the gravitational field at A due to the first part is equal in magnitude to the gravitational field at B due to the second part.

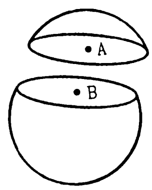

Figure 11-E3

16. Two small bodies of masses 2·00 kg and 4·00 kg are kept at rest at a separation of 2·0 m. Where should a particle of mass 0·10 kg be placed to experience no net gravitational force from these bodies? The particle is placed at this point. What is the gravitational potential energy of the system of three particles with usual reference level?
17. Three particles of mass m each are placed at the three corners of an equilateral triangle of side a. Find the work which should be done on this system to increase the sides of the triangle to $2a$.
18. A particle of mass 100 g is kept on the surface of a uniform sphere of mass 10 kg and radius 10 cm. Find the work to be done against the gravitational force between them to take the particle away from the sphere.
19. The gravitational field in a region is given by $\vec{E} = (5\text{ N/kg})\,\vec{i} + (12\text{ N/kg})\,\vec{j}$. (a) Find the magnitude of the gravitational force acting on a particle of mass 2 kg placed at the origin. (b) Find the potential at the points (12 m, 0) and (0, 5 m) if the potential at the origin is taken to be zero. (c) Find the change in gravitational potential energy if a particle of mass 2 kg is taken from the origin to the point (12 m, 5 m). (d) Find the change in potential energy if the particle is taken from (12 m, 0) to (0, 5 m).

20. The gravitational pontential in a region is given by $V = (20\ \text{N/kg})(x + y)$. (a) Show that the equation is dimensionally correct. (b) Find the gravitational field at the point (x, y). Leave your answer in terms of the unit vectors $\vec{i}, \vec{j}, \vec{k}$. (c) Calculate the magnitude of the gravitational force on a particle of mass 500 g placed at the origin.

21. The gravitational field in a region is given by $E = (2\vec{i} + 3\vec{j})$ N/kg. Show that no work is done by the gravitational field when a particle is moved on the line $3y + 2x = 5$.

[Hint : If a line $y = mx + c$ makes angle θ with the X-axis, $m = \tan\theta$.]

22. Find the height over the earth's surface at which the weight of a body becomes half of its value at the surface.

23. What is the acceleration due to gravity on the top of Mount Everest ? Mount Everest is the highest mountain peak of the world at the height of 8848 m. The value at sea level is $9{\cdot}80\ \text{m/s}^2$.

24. Find the acceleration due to gravity in a mine of depth 640 m if the value at the surface is $9{\cdot}800\ \text{m/s}^2$. The radius of the earth is 6400 km.

25. A body is weighed by a spring balance to be 1·000 kg at the north pole. How much will it weigh at the equator ? Account for the earth's rotation only.

26. A body stretches a spring by a particular length at the earth's surface at equator. At what height above the south pole will it stretch the same spring by the same length ? Assume the earth to be spherical.

27. At what rate should the earth rotate so that the apparent g at the equator becomes zero ? What will be the length of the day in this situation ?

28. A pendulum having a bob of mass m is hanging in a ship sailing along the equator from east to west. When the ship is stationary with respect to water the tension in the string is T_0. (a) Find the speed of the ship due to rotation of the earth about its axis. (b) Find the difference between T_0 and the earth's attraction on the bob. (c) If the ship sails at speed v, what is the tension in the string ? Angular speed of earth's rotation is ω and radius of the earth is R.

29. The time taken by Mars to revolve round the sun is 1·88 years. Find the ratio of average distance between Mars and the sun to that between the earth and the sun.

30. The moon takes about 27·3 days to revolve round the earth in a nearly circular orbit of radius $3{\cdot}84 \times 10^5$ km. Calculate the mass of the earth from these data.

31. A Mars satellite moving in an orbit of radius $9{\cdot}4 \times 10^3$ km takes 27540 s to complete one revolution. Calculate the mass of Mars.

32. A satellite of mass 1000 kg is supposed to orbit the earth at a height of 2000 km above the earth's surface. Find (a) its speed in the orbit, (b) its kinetic energy, (c) the potential energy of the earth-satellite system and (d) its time period. Mass of the earth $= 6 \times 10^{24}$ kg.

33. (a) Find the radius of the circular orbit of a satellite moving with an angular speed equal to the angular speed of earth's rotation. (b) If the satellite is directly above the north pole at some instant, find the time it takes to come over the equatorial plane. Mass of the earth $= 6 \times 10^{24}$ kg.

34. What is the true weight of an object in a geostationary satellite that weighed exactly 10·0 N at the north pole ?

35. The radius of a planet is R_1 and a satellite revolves round it in a circle of radius R_2. The time period of revolution is T. Find the acceleration due to the gravitation of the planet at its surface.

36. Find the minimum colatitude which can directly receive a signal from a geostationary satellite.

37. A particle is fired vertically upward from earth's surface and it goes upto a maximum height of 6400 km. Find the initial speed of the particle.

38. A particle is fired vertically upward with a speed of 15 km/s. With what speed will it move in intersteller space. Assume only earth's gravitational field.

39. A mass of 6×10^{24} kg (equal to the mass of the earth) is to be compressed in a sphere in such a way that the escape velocity from its surface is 3×10^8 m/s. What should be the radius of the sphere ?

□

ANSWERS

OBJECTIVE I

1. (b)	2. (c)	3. (c)	4. (d)	5. (a)	6. (c)
7. (c)	8. (d)	9. (b)	10. (a)	11 (b)	12. (b)
13. (c)	14. (a)	15. (c)	16. (c)	17. (d)	

OBJECTIVE II

1. all 2. (b), (c), (d) 3. (b) 4. (b), (c) 5. (a) 6. (d)

EXERCISES

1. $6{\cdot}67 \times 10^{-7}$ N
2. $\dfrac{4\sqrt{2}\,Gm^2}{a^2}$
3. (a) $\dfrac{4Gm^2}{3a^2}$, (b) zero
4. $\dfrac{\sqrt{3}\,GM^2}{4a^2}$

5. $\sqrt{\frac{GM}{R}\left(\frac{2\sqrt{2}+1}{4}\right)}$
6. $0{\cdot}65\ \mathrm{m/s^2}$
7. $4{\cdot}2 \times 10^{-5}$ m/s and $2{\cdot}1 \times 10^{-5}$ m/s
8. $\frac{2\pi\, GMm}{L^2}$
9. $\frac{2\, Gm}{d\sqrt{L^2 + 4\, d^2}}$
10. $\frac{4GM_1 m}{(R_1 + R_2)^2}$
11. $\frac{GM_e m}{R^3}\, x$
12. $\frac{GM_e m}{2R^2}$
13. (a) $\frac{Gmm'(x - r)}{r^3}$ (b) $\frac{Gmm'}{(x - r)^2}$ (c) $\frac{GMm'}{(x - R)^2} + \frac{Gmm'}{(x - r)^2}$
14. $\frac{GM}{16a^2}$, $\frac{61\, GM}{900a^2}$
16. 0·83 m from the 2.00 kg body towards the other body, $-3{\cdot}06 \times 10^{-10}$ J
17. $\frac{3Gm^2}{2a}$
18. $6{\cdot}67 \times 10^{-10}$ J
19. (a) 26 N (b) − 60 J/kg, − 60 J/kg (c) − 240 J (d) zero
20. (b) $-20(\vec{i} + \vec{j})$ N/kg (c) $10\sqrt{2}$ N
22. $(\sqrt{2} - 1)$ times the radius of the earth
23. $9{\cdot}77\ \mathrm{m/s^2}$
24. $9{\cdot}799\ \mathrm{m/s^2}$
25. 0·997 kg
26. 10 km approx.
27. $1{\cdot}237 \times 10^{-3}$ rad/sec, 1·41 h
28. (a) ωR (b) $m\omega^2 R$ (c) $T_0 + 2\, m\omega v$ approx.
29. 1·52
30. $6{\cdot}02 \times 10^{24}$ kg
31. $6{\cdot}5 \times 10^{23}$ kg
32. (a) 6·90 km/s (b) $2{\cdot}38 \times 10^{10}$ J (c) $-4{\cdot}76 \times 10^{10}$ J with usual reference (d) 2·01 hours
33. (a) 42300 km (b) 6 hours
34. 0·23 N
35. $\frac{4\pi^2 R_2^3}{T^2 R_1^2}$
36. $\sin^{-1}(0{\cdot}15)$
37. 7·9 km/s
38. 10·0 km/s
39. ≈ 9 mm