Chapter 5 Algebra of Matrices Ex 5.3 Q51

Given,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

To prove $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ we will use the principle of mathematical induction.

Step 1: Put $n = 1$

$$A^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

So,

A^n is true for $n = 1$

Step 2: Let, A^n be true for $n = k$, then

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \qquad \text{---(i)}$$

Step 3: We have to show that $A^{k+1} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}$

So,

$$A^{k+1} = A^k \times A$$

$$= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \{\text{using equation (i) and given}\}$$

$$= \begin{bmatrix} 1+0 & 1+k \\ 0+0 & 0+1 \end{bmatrix}$$

$$A^{k+1} = \begin{bmatrix} 1 & 1+k \\ 0 & 1 \end{bmatrix}$$

This shows that A^n is true for $n = k+1$ whenever it is true for $n = k$

Hence, by the principle of mathematical induction A^n is true for all positive integer.

Chapter 5 Algebra of Matrices Ex 5.3 Q52

Given,

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

To prove $A^n = \begin{bmatrix} a^n & \dfrac{b\left(a^n - 1\right)}{a - 1} \\ 0 & 1 \end{bmatrix}$ we will use the principle of mathematical induction.

Step 1: Put $n = 1$

$$A^1 = \begin{bmatrix} a^1 & \dfrac{b\left(a^1 - 1\right)}{a - 1} \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$$

So,

A^n is true for $n = 1$

Step 2: Let, A^n is true for $n = k$, so,

$$A^k = \begin{bmatrix} a^k & \dfrac{b\left(ak - 1\right)}{a - 1} \\ 0 & 1 \end{bmatrix} \qquad \text{---(i)}$$

Step 3: We have to show that

$$A^{k+1} = \begin{bmatrix} a^{k+1} & \dfrac{b\left(a^k - 1\right)}{a - 1} \\ 0 & 1 \end{bmatrix}$$

Now,

$$A^{k+1} = A^k \times A$$

$$= \begin{bmatrix} a^k & \dfrac{b\left(a^k - 1\right)}{a - 1} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \qquad \{\text{using equation (i) and given}\}$$

$$= \begin{bmatrix} a^{k+1} + 0 & a^k b + \dfrac{b\left(a^k - 1\right)}{a - 1} \\ 0 + 0 & 0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} a^{k+1} & \dfrac{a^{k+1}b - a^k b + a^k b - b}{a - 1} \\ 0 & 1 \end{bmatrix}$$

Chapter 5 Algebra of Matrices Ex 5.3 Q53

Given,

$$A = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

To show that,

$$A^n = \begin{bmatrix} \cos n\theta & i\sin n\theta \\ i\sin n\theta & \cos n\theta \end{bmatrix} \text{ for all } n \in N.$$

Put $n = 1$

$$A^1 = \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

So,

A^n is true for $n = 1$

Let, A^n is true for $n = k$, so

$$A^k = \begin{bmatrix} \cos k\theta & i\sin k\theta \\ i\sin k\theta & \cos k\theta \end{bmatrix} \qquad \text{---(i)}$$

Now, we have to show that,

$$A^{k+1} = \begin{bmatrix} \cos(k+1)\theta & i\sin(k+1)\theta \\ i\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

Now, $A^{k+1} = A^k \times A$

$$= \begin{bmatrix} \cos k\theta & i\sin k\theta \\ i\sin k\theta & \cos k\theta \end{bmatrix} \begin{bmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta + i^2\sin k\theta\sin\theta & i^2\cos k\theta\sin\theta + i\sin k\theta\cos\theta \\ i\sin k\theta\cos\theta + i\cos k\theta\sin\theta & i^2\sin k\theta\sin\theta + \cos\theta\cos k\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos k\theta\cos\theta - \sin k\theta\sin\theta & i(\cos k\theta\sin\theta + \sin k\theta\cos\theta) \\ i(\sin k\theta\cos k\theta\sin\theta) & \cos k\theta\cos\theta - \sin k\theta\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(k+1)\theta & i\sin(k+1)\theta \\ i\sin(k+1)\theta & \cos(k+1)\theta \end{bmatrix}$$

So, A^n is true for $n = k+1$ whenever it is true for $n = k$.

Hence, By principle of mathematical induction A^n is true for all positive integer.

Chapter 5 Algebra of Matrices Ex 5.3 Q54

Given,

$$A = \begin{bmatrix} \cos\alpha + \sin\alpha & \sqrt{2}\sin\alpha \\ -\sqrt{2}\sin\alpha & \cos\alpha - \sin\alpha \end{bmatrix}$$

To prove P(n): $A^n = \begin{bmatrix} \cos n\alpha + \sin n\alpha & \sqrt{2}\sin n\alpha \\ -\sqrt{2}\sin n\alpha & \cos n\alpha - \sin n\alpha \end{bmatrix}$ we use mathematical induction.

Step 1: To show P(1) is true.

A^n is true for $n = 1$

Step 2: Let, $P(k)$ be true, so

$$A^k = \begin{bmatrix} \cos k\alpha + \sin k\alpha & \sqrt{2}\sin k\alpha \\ -\sqrt{2}\sin k\alpha & \cos k\alpha - \sin k\alpha \end{bmatrix} \qquad \text{---(i)}$$

Step 3: Let, $P(k)$ is true.

Now, we have to show that

$$A^{k+1} = \begin{bmatrix} \cos(k+1)\alpha + \sin(k+1)\alpha & \sqrt{2}\sin(k+1)\alpha \\ -\sqrt{2}\sin(k1)\alpha & \cos(k+1)\alpha - \sin(k+1)\alpha \end{bmatrix}$$

Now,

$A^{k+1} = A^k \times A$

$$= \begin{bmatrix} \cos k\alpha + \sin k\alpha & \sqrt{2}\sin k\alpha \\ -\sqrt{2}\sin k\alpha & \cos k\alpha - \sin k\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha + \sin\alpha & \sqrt{2}\sin\alpha \\ -\sqrt{2}\sin\alpha & \cos\alpha - \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} (\cos k\alpha + \sin k\alpha)(\cos\alpha + \sin\alpha) - 2\sin\alpha\sin k\alpha & \begin{matrix} (\cos k\alpha + \sin k\alpha)\sqrt{2}\sin\alpha \\ +\sqrt{2}\sin k\alpha(\cos\alpha - \sin\alpha) \end{matrix} \\ (\cos\alpha + \sin\alpha)\left(-\sqrt{2}\sin k\alpha\right) - \sqrt{2}\sin\alpha(\cos k\alpha - \sin k\alpha) & \begin{matrix} -2\sin k\alpha\sin\alpha + (\cos k\alpha - \sin k\alpha) \\ (\cos\alpha - \sin\alpha) \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} \cos k\alpha\cos\alpha + \sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha \\ +\sin\alpha\sin k\alpha - 2\sin\alpha\sin k\alpha \end{matrix} & \begin{matrix} \sqrt{2}\cos k\alpha\sin\alpha + \sqrt{2}\sin\alpha\sin k\alpha + \\ \sqrt{2}\sin k\alpha\cos\alpha - \sqrt{2}\sin k\alpha\sin\alpha \end{matrix} \\ \begin{matrix} -\sqrt{2}\cos\alpha\sin\alpha - \sqrt{2}\sin\alpha\sin k\alpha - \sqrt{2}\sin\alpha \\ \cos k\alpha + \sqrt{2}\sin\alpha\sin k\alpha \end{matrix} & \begin{matrix} -2\sin k\alpha\sin\alpha + \cos k\alpha\cos\alpha - \cos\alpha \\ \sin k\alpha - \sin\alpha\cos k\alpha\sin\alpha\sin k\alpha \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{matrix} \cos\alpha\cos k\alpha + \sin\alpha\sin k\alpha \\ \sin\alpha\cos k\alpha + \sin k\alpha\cos\alpha \end{matrix} & \sqrt{2}\left(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha\right) \\ -\sqrt{2}\left(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha\right) & \begin{matrix} \cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha - \\ \left(\sin k\alpha\cos\alpha + \sin\alpha\cos k\alpha\right) \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(k+1\right)\alpha + \sin\left(k+1\right)\alpha & \sqrt{2}\sin\left(k+1\right)\alpha \\ -\sqrt{2}\sin\left(k+1\right)\alpha & \cos\left(k+1\right)\alpha - \sin\left(k+1\right)\alpha \end{bmatrix}$$

So, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by principle of mathematical induction $P(n)$ is true for all positive integer.

*********** END **********