The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of the given rectangle are given as:

$$\overrightarrow{AB}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{BC}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{(-2)^2}=2$$

Now, it is known that the area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $\left|\vec{a}\times\vec{b}\right|$.

Hence, the area of the given rectangle is $\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=2$ square units.

The correct answer is C.

Miscellaneous Solutions

Question 1:

Write down a unit vector in XY-plane, making an angle of 30° with the positive direction of *x*-axis.

Answer

If $\vec{r}$ is a unit vector in the XY-plane, then $\vec{r}=\cos\theta\hat{i}+\sin\theta\hat{j}$.

Here, θ is the angle made by the unit vector with the positive direction of the *x*-axis.

Therefore, for θ = 30°:

$$\vec{r}=\cos 30°\hat{i}+\sin 30°\hat{j}=\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$$

Hence, the required unit vector is $\frac{\sqrt{3}}{2}\hat{i}+\frac{1}{2}\hat{j}$

Question 2:

Find the scalar components and magnitude of the vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

Answer

The vector joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ can be obtained by,

$$\overrightarrow{PQ}=\text{Position vector of Q}-\text{Position vector of P}$$
$$=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$$
$$\left|\overrightarrow{PQ}\right|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Hence, the scalar components and the magnitude of the vector joining the given points are respectively $\{(x_2-x_1),(y_2-y_1),(z_2-z_1)\}$ and $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

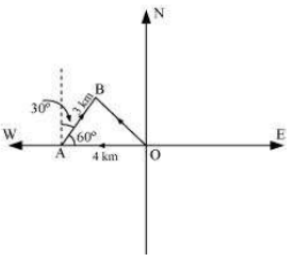
Question 3:

A girl walks 4 km towards west, then she walks 3 km in a direction 30° east of north and stops. Determine the girl's displacement from her initial point of departure.

Answer

Let O and B be the initial and final positions of the girl respectively.

Then, the girl's position can be shown as:



Now, we have:

$\overrightarrow{OA} = -4\hat{i}$

$\overrightarrow{AB} = \hat{i}\left|\overrightarrow{AB}\right|\cos 60° + \hat{j}\left|\overrightarrow{AB}\right|\sin 60°$

$= \hat{i}\,3 \times \frac{1}{2} + \hat{j}\,3 \times \frac{\sqrt{3}}{2}$

$= \frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

By the triangle law of vector addition, we have:

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$

$= \left(-4\hat{i}\right) + \left(\frac{3}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}\right)$

$= \left(-4 + \frac{3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \left(\frac{-8+3}{2}\right)\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

$= \frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$

Hence, the girl's displacement from her initial point of departure is

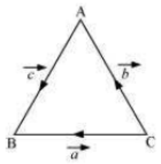
$\frac{-5}{2}\hat{i} + \frac{3\sqrt{3}}{2}\hat{j}$.

Question 4:

If $\vec{a} = \vec{b} + \vec{c}$, then is it true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$? Justify your answer.

Answer

In ΔABC, let $\overrightarrow{CB} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, and $\overrightarrow{AB} = \vec{c}$ (as shown in the following figure).



Now, by the triangle law of vector addition, we have $\vec{a} = \vec{b} + \vec{c}$.

It is clearly known that $|\vec{a}|$, $|\vec{b}|$, and $|\vec{c}|$ represent the sides of ΔABC.

Also, it is known that the sum of the lengths of any two sides of a triangle is greater than the third side.

$\therefore |\vec{a}| < |\vec{b}| + |\vec{c}|$

Hence, it is not true that $|\vec{a}| = |\vec{b}| + |\vec{c}|$.

Question 5:

Find the value of x for which $x\left(\hat{i} + \hat{j} + \hat{k}\right)$ is a unit vector.

Answer

$x\left(\hat{i} + \hat{j} + \hat{k}\right)$ is a unit vector if $\left|x\left(\hat{i} + \hat{j} + \hat{k}\right)\right| = 1$.

Now,

$\left|x\left(\hat{i} + \hat{j} + \hat{k}\right)\right| = 1$

$\Rightarrow \sqrt{x^2 + x^2 + x^2} = 1$

$\Rightarrow \sqrt{3x^2} = 1$

$\Rightarrow \sqrt{3}\,x = 1$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

Hence, the required value of x is $\pm\frac{1}{\sqrt{3}}$.

Question 6:

Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$.

Answer

We have,

$\vec{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$.

Then,

$\vec{c} = \vec{a} + \vec{b} = (2+1)\hat{i} + (3-2)\hat{j} + (-1+1)\hat{k} = 3\hat{i} + \hat{j}$

$$\therefore |\vec{c}| = \sqrt{3^2+1^2} = \sqrt{9+1} = \sqrt{10}$$

$$\therefore \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{\left(3\hat{i}+\hat{j}\right)}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$$\pm 5\cdot\hat{c} = \pm 5\cdot\frac{1}{\sqrt{10}}\left(3\hat{i}+\hat{j}\right) = \pm\frac{3\sqrt{10}\hat{i}}{2} \pm \frac{\sqrt{10}}{2}\hat{j}.$$

Question 7:

If $\vec{a} = \hat{i}+\hat{j}+\hat{k},\ \vec{b} = 2\hat{i}-\hat{j}+3\hat{k}$ and $\vec{c} = \hat{i}-2\hat{j}+\hat{k}$, find a unit vector parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

Answer

We have,

$$\vec{a} = \hat{i}+\hat{j}+\hat{k},\ \vec{b} = 2\hat{i}-\hat{j}+3\hat{k} \text{ and } \vec{c} = \hat{i}-2\hat{j}+\hat{k}$$

$$\begin{aligned} 2\vec{a}-\vec{b}+3\vec{c} &= 2\left(\hat{i}+\hat{j}+\hat{k}\right)-\left(2\hat{i}-\hat{j}+3\hat{k}\right)+3\left(\hat{i}-2\hat{j}+\hat{k}\right) \\ &= 2\hat{i}+2\hat{j}+2\hat{k}-2\hat{i}+\hat{j}-3\hat{k}+3\hat{i}-6\hat{j}+3\hat{k} \\ &= 3\hat{i}-3\hat{j}+2\hat{k} \end{aligned}$$

$$\left|2\vec{a}-\vec{b}+3\vec{c}\right| = \sqrt{3^2+(-3)^2+2^2} = \sqrt{9+9+4} = \sqrt{22}$$

Hence, the unit vector along $2\vec{a}-\vec{b}+3\vec{c}$ is

$$\frac{2\vec{a}-\vec{b}+3\vec{c}}{\left|2\vec{a}-\vec{b}+3\vec{c}\right|} = \frac{3\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{22}} = \frac{3}{\sqrt{22}}\hat{i}-\frac{3}{\sqrt{22}}\hat{j}+\frac{2}{\sqrt{22}}\hat{k}.$$

Question 8:

Show that the points A (1, −2, −8), B (5, 0, −2) and C (11, 3, 7) are collinear, and find the ratio in which B divides AC.

Answer

The given points are A (1, −2, −8), B (5, 0, −2), and C (11, 3, 7).

$$\therefore \overrightarrow{AB} = (5-1)\hat{i}+(0+2)\hat{j}+(-2+8)\hat{k} = 4\hat{i}+2\hat{j}+6\hat{k}$$

$$\overrightarrow{BC} = (11-5)\hat{i}+(3-0)\hat{j}+(7+2)\hat{k} = 6\hat{i}+3\hat{j}+9\hat{k}$$

$$\overrightarrow{AC} = (11-1)\hat{i}+(3+2)\hat{j}+(7+8)\hat{k} = 10\hat{i}+5\hat{j}+15\hat{k}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{4^2+2^2+6^2} = \sqrt{16+4+36} = \sqrt{56} = 2\sqrt{14}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{6^2+3^2+9^2} = \sqrt{36+9+81} = \sqrt{126} = 3\sqrt{14}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{10^2+5^2+15^2} = \sqrt{100+25+225} = \sqrt{350} = 5\sqrt{14}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Thus, the given points A, B, and C are collinear.

Now, let point B divide AC in the ratio $\lambda:1$. Then, we have:

$$\overrightarrow{OB} = \frac{\lambda\overrightarrow{OC}+\overrightarrow{OA}}{(\lambda+1)}$$

$$\Rightarrow 5\hat{i}-2\hat{k} = \frac{\lambda\left(11\hat{i}+3\hat{j}+7\hat{k}\right)+\left(\hat{i}-2\hat{j}-8\hat{k}\right)}{\lambda+1}$$

$$\Rightarrow (\lambda+1)\left(5\hat{i}-2\hat{k}\right) = 11\lambda\hat{i}+3\lambda\hat{j}+7\lambda\hat{k}+\hat{i}-2\hat{j}-8\hat{k}$$

$$\Rightarrow 5(\lambda+1)\hat{i}-2(\lambda+1)\hat{k} = (11\lambda+1)\hat{i}+(3\lambda-2)\hat{j}+(7\lambda-8)\hat{k}$$

On equating the corresponding components, we get:

$$5(\lambda+1) = 11\lambda+1$$

$$\Rightarrow 5\lambda+5 = 11\lambda+1$$

$$\Rightarrow 6\lambda = 4$$

*********** END **********