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Gauss–Legendre algorithm

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The **Gauss–Legendre algorithm** is an [algorithm](#) to compute the digits of π. It is notable for being rapidly convergent, with only 25 iterations producing 45 million correct digits of π. However, the drawback is that it is memory intensive and it is therefore sometimes not used over [Machin-like formulas](#).

The method is based on the individual work of [Carl Friedrich Gauss](#) (1777–1855) and [Adrien-Marie Legendre](#) (1752–1833) combined with modern algorithms for multiplication and [square roots](#). It repeatedly replaces two numbers by their [arithmetic](#) and [geometric mean](#), in order to approximate their [arithmetic-geometric mean](#).

The version presented below is also known as the **Gauss–Euler, Brent–Salamin (or Salamin–Brent) algorithm**;[1] it was independently discovered in 1975 by [Richard Brent](#) and [Eugene Salamin](#). It was used to compute the first 206,158,430,000 decimal digits of π on September 18 to 20, 1999, and the results were checked with [Borwein's algorithm](#).

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Algorithm [\[edit\]](#)

- Initial value setting:

$$a_0 = 1 \qquad b_0 = \frac{1}{\sqrt{2}} \qquad t_0 = \frac{1}{4} \qquad p_0 = 1.$$

- Repeat the following instructions until the difference of *a*_{*n*} and *b*_{*n*} is within the desired accuracy:

$$a_{n+1} = \frac{a_n + b_n}{2},$$

$$b_{n+1} = \sqrt{a_n b_n},$$

$$t_{n+1} = t_n - p_n(a_n - a_{n+1})^2,$$

$$p_{n+1} = 2p_n.$$

- π is then approximated as:

$$\pi \approx \frac{(a_{n+1} + b_{n+1})^2}{4t_{n+1}}.$$

The first three iterations give (approximations given up to and including the first incorrect digit):

3.140...
3.14159264...
3.1415926535897932382...

The algorithm has second-order convergent nature, which essentially means that the number of correct digits doubles with each step of the algorithm.

Mathematical background [\[edit\]](#)

Limits of the arithmetic–geometric mean [\[edit\]](#)

The [arithmetic–geometric mean](#) of two numbers, *a*₀ and *b*₀, is found by calculating the limit of the sequences

$$a_{n+1} = \frac{a_n + b_n}{2},$$

$$b_{n+1} = \sqrt{a_n b_n},$$

which both converge to the same limit.

If $a_0 = 1$ and $b_0 = \cos \varphi$ then the limit is $\frac{\pi}{2K(\sin \varphi)}$ where $K(k)$ is the [complete elliptic integral of the first kind](#)

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

If $c_0 = \sin \varphi$, $c_{i+1} = a_i - a_{i+1}$, then

$$\sum_{i=0}^{\infty} 2^{i-1} c_i^2 = 1 - \frac{E(\sin \varphi)}{K(\sin \varphi)}$$

where $E(k)$ is the [complete elliptic integral of the second kind](#):

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} d\theta.$$

Gauss knew of both of these results.^[2] ^[3] ^[4]

Legendre's identity ^[edit]

For φ and θ such that $\varphi + \theta = \frac{1}{2}\pi$ Legendre proved the identity:

$$K(\sin \varphi)E(\sin \theta) + K(\sin \theta)E(\sin \varphi) - K(\sin \varphi)K(\sin \theta) = \frac{1}{2}\pi.^[2]$$

Gauss–Euler method ^[edit]

The values $\varphi = \theta = \frac{\pi}{4}$ can be substituted into Legendre's identity and the approximations to K, E can be

found by terms in the sequences for the arithmetic geometric mean with $a_0 = 1$ and $b_0 = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.^[5]$

See also ^[edit]

- [Numerical approximations of π](#)

References ^[edit]

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