Multiplicative inverse

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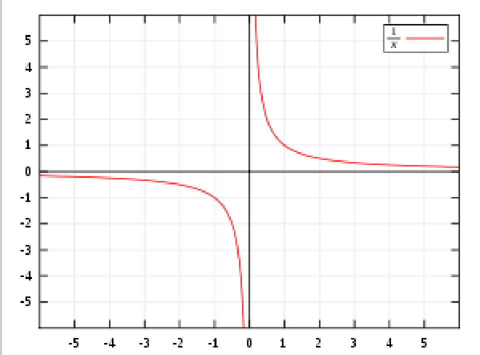
In mathematics, a **multiplicative inverse** or **reciprocal** for a number *x*, denoted by 1/*x* or x^{-1}, is a number which when multiplied by *x* yields the multiplicative identity, 1. The multiplicative inverse of a fraction *a*/*b* is *b*/*a*. For the multiplicative inverse of a real number, divide 1 by the number. For example, the reciprocal of 5 is one fifth (1/5 or 0.2), and the reciprocal of 0.25 is 1 divided by 0.25, or 4. The **reciprocal function**, the function *f*(*x*) that maps *x* to 1/*x*, is one of the simplest examples of a function which is its own inverse (an involution).



The reciprocal function: $y = 1/x$. For every *x* except 0, *y* represents its multiplicative inverse. The graph forms a rectangular hyperbola.

The term *reciprocal* was in common use at least as far back as the third edition of *Encyclopædia Britannica* (1797) to describe two numbers whose product is 1; geometrical quantities in inverse proportion are described as *reciprocall* in a 1570 translation of Euclid's *Elements*.[1]

In the phrase *multiplicative inverse*, the qualifier *multiplicative* is often omitted and then tacitly understood (in contrast to the additive inverse). Multiplicative inverses can be defined over many mathematical domains as well as numbers. In these cases it can happen that $ab \neq ba$; then "inverse" typically implies that an element is both a left and right inverse.

The notation f^{-1} is sometimes also used for the inverse function of the function *f*, which is not in general equal to the multiplicative inverse. For example, the multiplicative inverse $1/(\sin x) = (\sin x)^{-1}$ is different from the inverse sin of *x*, denoted $\sin^{-1} x$ or arcsin *x*. Only for linear maps are they strongly related (see below). The terminology difference *reciprocal* versus *inverse* is not sufficient to make this distinction, since many authors prefer the opposite naming convention, probably for historical reasons (for example in French, the inverse function is preferably called application réciproque).

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Examples and counterexamples [edit]

In the real numbers, zero does not have a reciprocal because no real number multiplied by 0 produces 1 (the product of any number with zero is zero). With the exception of zero, reciprocals of every real number are real, reciprocals of every rational number are rational, and reciprocals of every complex number are complex. The property that every element other than zero has a multiplicative inverse is part of the definition of a field, of which these are all examples. On the other hand, no integer other than 1 and -1 has an integer reciprocal, and so the integers are not a field.

In modular arithmetic, the modular multiplicative inverse of *a* is also defined: it is the number *x* such that $ax \equiv 1 \pmod{n}$. This multiplicative inverse exists if and only if *a* and *n* are coprime. For example, the inverse of 3 modulo 11 is 4 because $4 \cdot 3 \equiv 1 \pmod{11}$. The extended Euclidean algorithm may be used to compute it.

The sedenions are an algebra in which every nonzero element has a multiplicative inverse, but which

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nonetheless has divisors of zero, i.e. nonzero elements *x*, *y* such that *xy* = 0.

A square matrix has an inverse if and only if its determinant has an inverse in the coefficient ring. The linear map that has the matrix A^{-1} with respect to some base is then the reciprocal function of the map having *A* as matrix in the same base. Thus, the two distinct notions of the inverse of a function are strongly related in this case, while they must be carefully distinguished in the general case (as noted above).

The trigonometric functions are related by the reciprocal identity: the cotangent is the reciprocal of the tangent; the secant is the reciprocal of the cosine; the cosecant is the reciprocal of the sine.

A ring in which every nonzero element has a multiplicative inverse is a division ring; likewise an algebra in which this holds is a division algebra.

Complex numbers [edit]

As mentioned above, the reciprocal of every nonzero complex number $z = a + bi$ is complex. It can be found by multiplying both top and bottom of 1/*z* by its complex conjugate $\bar{z} = a - bi$ and using the property that $z\bar{z} = \|z\|^2$, the absolute value of *z* squared, which is the real number $a^2 + b^2$:

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{\|z\|^2} = \frac{a - bi}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i.$$

In particular, if ||z||=1 (*z* has unit magnitude), then $1/z = \bar{z}$. Consequently, the imaginary units, ±*i*, have additive inverse equal to multiplicative inverse, and are the only complex numbers with this property. For example, additive and multiplicative inverses of *i* are −(*i*) = −*i* and 1/*i* = −*i*, respectively.

For a complex number in polar form $z = r(\cos\varphi + i\sin\varphi)$, the reciprocal simply takes the reciprocal of the magnitude and the negative of the angle:

$$\frac{1}{z} = \frac{1}{r}\left(\cos(-\varphi) + i\sin(-\varphi)\right).$$

Calculus [edit]

In real calculus, the derivative of $1/x = x^{-1}$ is given by the power rule with the power −1:

$$\frac{d}{dx}x^{-1} = (-1)x^{(-1)-1} = -x^{-2} = -\frac{1}{x^2}.$$

The power rule for integrals (Cavalieri's quadrature formula) cannot be used to compute the integral of 1/*x*, because doing so would result in division by 0:

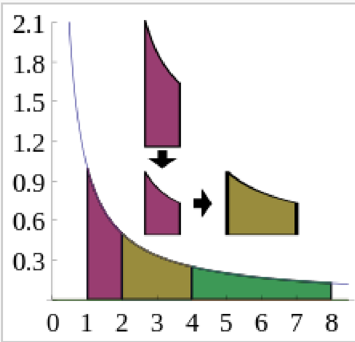
$$\int \frac{1}{x}\,dx = \frac{x^0}{0} + C$$

Instead the integral is given by:

$$\int_1^a \frac{1}{x}\,dx = \ln a,$$
$$\int \frac{1}{x}\,dx = \ln x + C.$$

where ln is the natural logarithm. To show this, note that $\frac{d}{dx}e^x = e^x$, so if $y = e^x$ and $x = \ln y$, we have:[2]

$$\frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = dx \quad \Rightarrow \quad \int \frac{1}{y}\,dy = \int 1\,dx \quad \Rightarrow \quad \int \frac{1}{y}\,dy = x + C = \ln y + C.$$



Geometric intuition for the integral of 1/*x*. The three integrals from 1 to 2, from 2 to 4, and from 4 to 8 are all equal. Each region is the previous region scaled vertically down by 50%, then horizontally by 200%. Extending this, the integral from 1 to 2^k is *k* times the integral from 1 to 2, just as ln 2^k = *k* ln 2.

Algorithms [edit]

The reciprocal may be computed by hand with the use of long division.

Computing the reciprocal is important in many division algorithms, since the quotient *a*/*b* can be computed by first computing 1/*b* and then multiplying it by *a*. Noting that $f(x) = 1/x - b$ has a zero at *x* = 1/*b*, Newton's method can find that zero, starting with a guess x_0 and iterating using the rule:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1/x_n - b}{-1/x_n^2} = 2x_n - bx_n^2 = x_n(2 - bx_n).$$

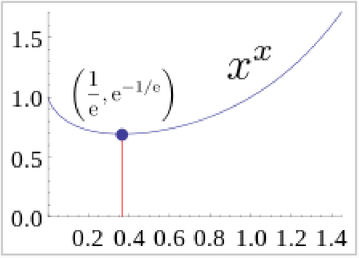
This continues until the desired precision is reached. For example, suppose we wish to compute 1/17 ≈ 0.0588 with 3 digits of precision. Taking x_0 = 0.1, the following sequence is produced:

x_1 = 0.1(2 - 17 × 0.1) = 0.03
x_2 = 0.03(2 - 17 × 0.03) = 0.0447
x_3 = 0.0447(2 - 17 × 0.0447) ≈ 0.0554
x_4 = 0.0554(2 - 17 × 0.0554) ≈ 0.0586
x_5 = 0.0586(2 - 17 × 0.0586) ≈ 0.0588

A typical initial guess can be found by rounding *b* to a nearby power of 2, then using bit shifts to compute its reciprocal.

In constructive mathematics, for a real number *x* to have a reciprocal, it is not sufficient that *x* ≠ 0. There must instead be given a *rational* number *r* such that 0 < *r* < |*x*|. In terms of the approximation algorithm described above, this is needed to prove that the change in *y* will eventually become arbitrarily small.

This iteration can also be generalised to a wider sort of inverses, e.g. matrix inverses.



Graph of f(*x*) = x^x showing the minimum at (1/e, $e^{-1/e}$).

Reciprocals of irrational numbers [edit]

Every number excluding zero has a reciprocal, and reciprocals of certain irrational numbers can have important special properties. Examples include the reciprocal of e (≈ 0.367879)and the golden ratio's reciprocal (≈ 0.618034). The first reciprocal is special because no other positive number can produce a lower number when put to the power of itself; $f(1/e)$ is the global minimum of $f(x) = x^x$. The second number is the only positive number that is equal to its reciprocal plus one:$\phi = 1/\phi + 1$. Its additive inverse is the only negative number that is equal to its reciprocal minus one:$-\phi = -1/\phi - 1$.

The function $f(n) = n + \sqrt{(n^2+1)}, n \in N, n > 0$ gives an infinite number of irrational numbers that differ with their reciprocal by an integer. For example, $f(2)$ is the irrational $2 + \sqrt{5}$. Its reciprocal $1/(2+\sqrt{5})$ is $-2 + \sqrt{5}$, exactly 4 less. Such irrational numbers share a curious property: they have the same fractional part as their reciprocal.

Further remarks [edit]

If the multiplication is associative, an element *x* with a multiplicative inverse cannot be a zero divisor (meaning for some *y*, *xy* = 0 with neither *x* nor *y* equal to zero). To see this, it is sufficient to multiply the equation *xy* = 0 by the inverse of *x* (on the left), and then simplify using associativity. In the absence of associativity, the sedenions provide a counterexample.

The converse does not hold: an element which is not a zero divisor is not guaranteed to have a multiplicative inverse. Within **Z**, all integers except −1, 0, 1 provide examples; they are not zero divisors nor do they have inverses in **Z**. If the ring or algebra is finite, however, then all elements *a* which are not zero divisors do have a (left and right) inverse. For, first observe that the map *f*(*x*) = *ax* must be injective: *f*(*x*) = *f*(*y*) implies *x* = *y*:

$$\begin{aligned} ax = ay \quad & \Rightarrow \quad ax - ay = 0 \\ & \Rightarrow \quad a(x-y) = 0 \\ & \Rightarrow \quad x - y = 0 \\ & \Rightarrow \quad x = y. \end{aligned}$$

Distinct elements map to distinct elements, so the image consists of the same finite number of elements, and the map is necessarily surjective. Specifically, *ƒ* (namely multiplication by *a*) must map some element *x* to 1, *ax* = 1, so that *x* is an inverse for *a*.

Applications [edit]

The expansion of the reciprocal 1/*q* in any base can also act [3] as a source of pseudo-random numbers, if *q* is a "suitable" safe prime, a prime of the form 2*p* + 1 where *p* is also a prime. A sequence of pseudo-random numbers of length *q* − 1 will be produced by the expansion.

See also [edit]

- Division (mathematics)
- Fraction (mathematics)
- Group (mathematics)
- Ring (mathematics)
- Division algebra
- Exponential decay
- Unit fractions – reciprocals of integers
- Hyperbola
- Repeating decimal
- List of sums of reciprocals

Notes [edit]

1. ^ " In equall Parallelipipedons the bases are reciprokall to their altitudes". *OED* "Reciprocal" §3a. Sir Henry Billingsley translation of Elements XI, 34.
2. ^ Anthony, Dr. "Proof that INT(1/x)dx = lnx". *Ask Dr. Math*. Drexel University. Retrieved 22 March 2013.
3. ^ Mitchell, Douglas W., "A nonlinear random number generator with known, long cycle length," *Cryptologia* 17, January 1993, 55-62.

References [edit]

- Maximally Periodic Reciprocals, Matthews R.A.J. *Bulletin of the Institute of Mathematics and its Applications* vol 28 pp 147–148 1992

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