Gosper's algorithm

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In mathematics, **Gosper's algorithm** is a procedure for finding sums of hypergeometric terms that are themselves hypergeometric terms. That is: suppose we have $a(1) + ... + a(n) = S(n) - S(0)$, where $S(n)$ is a hypergeometric term (i.e., $S(n + 1)/S(n)$ is a rational function of n); then necessarily $a(n)$ is itself a hypergeometric term, and given the formula for $a(n)$ Gosper's algorithm finds that for $S(n)$.

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Outline of the algorithm [edit]

Step 1: Find a polynomial p such that, writing $b(n) = a(n)/p(n)$, the ratio $b(n)/b(n - 1)$ has the form $q(n)/r(n)$ where q and r are polynomials and no $q(n)$ has a nontrivial factor with $r(n + j)$ for $j = 0, 1, 2, ...$. (This is always possible, whether or not the series is summable in closed form.)

Step 2: Find a polynomial $ƒ$ such that $S(n) = q(n + 1)/p(n)\ ƒ(n)\ a(n)$. If the series is summable in closed form then clearly a rational function $ƒ$ with this property exists; in fact it must always be a polynomial, and an upper bound on its degree can be found. Determining $ƒ$ (or finding that there is no such $ƒ$) is then a matter of solving a system of linear equations.

Relationship to Wilf–Zeilberger pairs [edit]

Gosper's algorithm can be used to discover Wilf–Zeilberger pairs, where they exist. Suppose that $F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$ where F is known but G is not. Then feed $a(k) := F(n + 1, k) - F(n, k)$ into Gosper's algorithm. (Treat this as a function of k whose coefficients happen to be functions of n rather than numbers; everything in the algorithm works in this setting.) If it successfully finds $S(k)$ with $S(k) - S(k - 1) = a(k)$, then we are done: this is the required G. If not, there is no such G.

Definite versus indefinite summation [edit]

Gosper's algorithm finds (where possible) a hypergeometric closed form for the *indefinite* sum of hypergeometric terms. It can happen that there is no such closed form, but that the sum over *all* n, or some particular set of values of n, has a closed form. This question is only meaningful when the coefficients are themselves functions of some other variable. So, suppose a(n,k) is a hypergeometric term in both n and k: that is, $a(n, k)/a(n - 1,k)$ and $a(n, k)/a(n, k - 1)$ are rational functions of n and k. Then Zeilberger's algorithm and Petkovšek's algorithm may be used to find closed forms for the sum over k of $a(n, k)$.

History [edit]

Bill Gosper discovered this algorithm in the 1970s while working on the Macsyma computer algebra system at SAIL and MIT.

Further reading [edit]

- Marko Petkovšek, Herbert Wilf and Doron Zeilberger, *A = B*, AK Peters 1996, ISBN 1-56881-063-6. Full text online.[1]
- Gosper's 1977 article in PNAS reporting on the algorithm.