NUMERICAL COMPUTATION OF INCOMPLETE ELLIPTIC INTEGRALS OF A GENERAL FORM

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(Received: 17 December, 1992; revised 23 November, 1993)

Abstract. We present an algorithm to compute the incomplete elliptic integral of a general form. The algorithm efficiently evaluates some linear combinations of incomplete elliptic integrals of all kinds to a high precision. Some numerical examples are given as illustrations. This enables us to numerically calculate the values and the partial derivatives of incomplete elliptic integrals of all kinds, which are essential when dealing with many problems in celestial mechanics, including the analytic solution of the torque-free rotational motion of a rigid body around its barycenter.

Key words: Incomplete elliptic integrals, numerical computation.

1. Introduction

The Legendre incomplete elliptic integrals of the first kind F, of the second kind E, and of the third kind Π are functions of two or three variables, defined in the form of integrals as

$$F(\varphi, k) = \int_0^\varphi \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta \,, \tag{1}$$

$$E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \,, \tag{2}$$

$$\Pi(\varphi, \alpha^2, k) = \int_0^\varphi \frac{1}{(1 - \alpha^2 \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \, d\theta \,, \tag{3}$$

where the variables φ, k, α^2 are called the argument, the modulus, and the parameter, respectively: see the formulas 110.02 through 110.04 in the handbook by Byrd and Friedman (1964), hereafter referred to as BF. The complete elliptic integrals are defined by the special case of the above three, when $\varphi = \pi/2$, as

$$K(k) \equiv F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta \,, \tag{4}$$

$$E(k) \equiv E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta \,, \tag{5}$$

Celestial Mechanics and Dynamical Astronomy **59:** 237–251, 1994.

$$\Pi(\alpha^2, k) \equiv \Pi\left(\frac{\pi}{2}, \alpha^2, k\right) = \int_0^{\pi/2} \frac{1}{(1-\alpha^2 \sin^2\theta)\sqrt{1-k^2 \sin^2\theta}}\, d\theta \tag{6}$$

(see the formulas 110.06 through 110.08 of BF). Unfortunately, the same notations, E and Π, are used both for the incomplete and complete elliptic integrals. Thus, it is better to write the variables to specify them. The symbol n is sometimes used in place of α^2. Note that the definition of Π, especially the signature of α^2 or n, is given differently by many authors: see Wolfram (1991) for details. Here we follow the notations in BF.

These integrals and their by-products, Jacobi's elliptic functions, appear in many aspects of celestial mechanics; in describing an intermediate orbit of an artificial satellite (Garfinkel, 1958; Garfinkel and Aksnes, 1970); in constructing general planetary theories (Richardson, 1982; Williams *et al.*, 1987; Brumberg, 1993); in dealing with perturbed elliptic oscillators (Deprit and Elipe, 1991); in effectively computing a highly eccentric Keplerian motion (Brumberg, 1992); and in expressing the rigorous solution of the torque-free rotational motion of a rigid body (Whittaker, 1937; Jupp, 1974; Kinoshita, 1992). Even in the case where only the elliptic functions may be needed, F is necessary to evaluate the inverse elliptic functions such as

$$\mathrm{sn}^{-1}(x, k) = F(\sin^{-1} x, k)\,. \tag{7}$$

See also formulas 130.xx of BF.

There are some papers which present an algorithm to evaluate such routines (BF; Burlisch, 1969a; Carlson, 1979) or the routines themselves (Burlisch, 1965, 1969b; Press *et al.*, 1986; Wolfram, 1991). Among them, Burlisch (1969a, 1969b) found an efficient way to compute incomplete and complete elliptic integrals of three kinds based on the Bartky's transformation, a variant of the well-known Landen transformation. Burlisch gave three routines:

(1) $el2$, the routine to compute an arbitrary linear combinations of F and E in the case $|\varphi| \leq \pi/2$ and $k^2 \leq 1$ defined as

$$el2(x, k_c, a, b) \equiv \int_0^{\tan^{-1} x} \frac{a \cos^2\theta + b \sin^2\theta}{\sqrt{\cos^2\theta + k_c^2 \sin^2\theta}}\, d\theta \tag{8}$$

where x and k_c correspond to $\tan\varphi$ and $\sqrt{1-k^2}$, respectively;

(2) $el3$, the routine to compute Π in the case $|\varphi| \leq \pi/2$ and $k^2 \leq 1$ defined as

$$el3(x, k_c, p) \equiv \int_0^{\tan^{-1} x} \frac{1}{(\cos^2\theta + p \sin^2\theta)\sqrt{\cos^2\theta + k_c^2 \sin^2\theta}}\, d\theta \tag{9}$$

where p corresponds to $1 - \alpha^2$; and

(3) *cel*, the routine to compute two forms of arbitrary linear combination of the complete elliptic integrals of three kinds in the case $k^2 \le 1$ defined as

$$cel(k_c, p, a, b) \equiv \int_0^{\pi/2} \frac{a \cos^2\theta + b \sin^2\theta}{(\cos^2\theta + p \sin^2\theta)\sqrt{\cos^2\theta + k_c^2 \sin^2\theta}}\, d\theta\,. \tag{10}$$

It is easy to see that these cover most expressions we need, since

$$\lambda F(\varphi, k) + \mu E(\varphi, k) = el2\left(\tan\varphi,\ \sqrt{1-k^2},\ \lambda+\mu, \lambda + \mu(1-k^2)\right), \tag{11}$$

$$\Pi(\varphi, \alpha^2, k) = el3\left(\tan\varphi,\ \sqrt{1-k^2},\ 1-\alpha^2\right), \tag{12}$$

$$\lambda K(k) + \mu E(k) = cel\left(\sqrt{1-k^2},\ \lambda+\mu,\ \lambda+\mu(1-k^2)\right), \tag{13}$$

$$\lambda K(k) + \mu \Pi(\alpha^2, k) = cel\left(\sqrt{1-k^2},\ 1-\alpha^2,\ \lambda+\mu,\ \lambda(1-\alpha^2)+\mu\right) \tag{14}$$

in the case $|\varphi| \le \pi/2$ and $k^2 \le 1$ where λ and μ are arbitrary constants. These are regarded as the state-of-the-art routines, so that the FORTRAN77 functions for $el2$ and *cel* are given in Press *et al.* (1986), which are the translation of Burlisch's original ALGOL procedures (Burlisch, 1965). And the ALGOL procedure for $el3$ is available from Burlisch (1969b). Unfortunately, Burlisch did not present the incomplete version of *cel*, named *el*, which covers all of $el2$, $el3$, *cel* and is defined as

$$el(x, k_c, p, a, b) \equiv \int_0^{\tan^{-1}x} \frac{a \cos^2\theta + b \sin^2\theta}{(\cos^2\theta + p \sin^2\theta)\sqrt{\cos^2\theta + k_c^2 \sin^2\theta}}\, d\theta\,, \tag{15}$$

although he mentioned it in the phrase that "... *we could define a general elliptic integral of the third kind ... since it would include elliptic integrals of all three kinds as special cases*" (Burlisch, 1969a, p. 279), as

$$F(\varphi, k) = el\left(\tan\varphi,\ \sqrt{1-k^2}, p, 1, p\right),\quad p:\ \text{arbitrary} \tag{16}$$

$$E(\varphi, k) = el\left(\tan\varphi,\ \sqrt{1-k^2}, 1, 1, 1-k^2\right), \tag{17}$$

$$\Pi(\varphi, \alpha^2, k) = el\left(\tan\varphi,\ \sqrt{1-k^2},\ 1-\alpha^2, 1, 1\right), \tag{18}$$

in the case $|\varphi| \le \pi/2$ and $k^2 \le 1$. We would like to stress that this general form is useful because it enables us to evaluate integrals of the type

$$\frac{1}{k^2}\left[F(\varphi,k)-E(\varphi,k)\right]=el\Big(\tan\,\varphi,\ \sqrt{1-k^2},1,0,1\Big)\ , \tag{19}$$

$$\frac{1}{\alpha^2}\left[\Pi(\varphi,\alpha^2,k)-F(\varphi,k)\right]=el\Big(\tan\,\varphi,\ \sqrt{1-k^2},\ 1-\alpha^2,0,1\Big) \tag{20}$$

numerically, without any loss of precision. These types of integrals are essential in computing the partial derivatives of some elliptic integrals and elliptic functions.

Although el is more useful than the combination of $el2$ and $el3$, it has two small defects. One is that the argument expression of el assumes that k_c^2 is non-negative. This excludes the case when k_c is a pure imaginary term, while the integral remains real and finite as long as $1+k_c^2\ \tan^2\ \varphi>0$. Burlisch presented a recipe in such a case only for the routine $el2$ (Burlisch, 1965). The other defect is that el does not cover all possible regions of φ, such as the case $|\varphi|>\pi/2$, although this is minor.

We introduce here an extension of el, named G, as

$$G(\varphi,n_c,m_c,a,b)\equiv\int\limits_0^{\varphi}\frac{a\ \cos^2\ \theta+b\ \sin^2\ \theta}{(\cos^2\ \theta+n_c\ \sin^2\ \theta)\ \sqrt{\cos^2\ \theta+m_c\ \sin^2\ \theta}}\,\mathrm{d}\theta \tag{21}$$

where

$$n_c\equiv 1-\alpha^2=p\ ,\quad m_c\equiv 1-k^2=k_c^2\ . \tag{22}$$

The reason why we choose n_c and m_c instead of α^2 or k as the input arguments is just the same as that given in Burlisch (1969a); to avoid a loss of precision. Clearly G is an extension of el with respect to φ and k_c, since

$$el(x,k_c,p,a,b)=G(\tan^{-1}\ x,p,k_c^2,a,b) \tag{23}$$

and we can compute two forms of linear combination of incomplete elliptic integrals of three kinds by G only as

$$\lambda F(\varphi,k)+\mu E(\varphi,k)=G(\varphi,1,1-k^2,\lambda+\mu,\lambda+\mu(1-k^2))\ , \tag{24}$$

$$\lambda F(\varphi,k)+\mu\Pi(\varphi,\alpha^2,k)=G(\varphi,1-\alpha^2,1-k^2,\lambda+\mu,\lambda(1-\alpha^2)+\mu)\ . \tag{25}$$

We remark that the complete version of real-valued G is already realized by cel as

$$G\left(\frac{\pi}{2},n_c,m_c,a,b\right)=cel(\sqrt{m_c},n_c,a,b) \tag{26}$$

since the negative m_c leads to a complex value of G.

In this note, we will describe the algorithm for computing G following the formulation of Burlisch, based on Bartky's transformation. We will also present some numerical examples for the convenience of readers.

2. Algorithm

Our algorithm is mostly based on an extension of Burlisch's formulation. Namely, it consists of two parts; the part of parameters transformation, reducing the computation of G to that of el, which will be described in the first two subsections below; and the part of computation of el itself, which will be stated in the remaining subsections.

2.1. Check of Input Parameter Range

Before entering the body of the algorithm to compute G numerically, let us examine the parameter domain giving real-valued G. Here we confine ourselves to dealing with real-valued parameters; φ, n_c, m_c, a, and b. Then the effective range of φ and m_c, namely the range when G is real and finite, becomes the following;

$$|\varphi| \geq \frac{\pi}{2} \text{ and } 0 \leq m_c < +\infty \,, \tag{27}$$

or

$$|\varphi| < \frac{\pi}{2} \text{ and } -\cot^2\varphi < m_c < +\infty \,. \tag{28}$$

If we adopt the principal value of G when $\cos^2\theta + n_c \sin^2\theta$ has a root, that for n_c becomes

$$-\infty < n_c < +\infty \text{ and } n_c \neq -\cot^2\varphi \,. \tag{29}$$

Finally, the regions for a and b are

$$-\infty < a < +\infty \text{ and } -\infty < b < +\infty \,. \tag{30}$$

Then the first step of the procedure is to check the input parameters whether they satisfy the above conditions and to return an error flag when the parameters are out of range. We note that, in the actual implementation, ∞ should be replaced by the maximum number allowed by the computer, say around $1.8 \times 10^{+308}$ in IEEE 754 Standard, which is adopted by many FPUs such as Motorola 68881/68882.

2.2. Reduction with Respect to Input Parameters

As the second step, we transform the range of input φ and m_c into the region $0 \leq \varphi < \pi/2$ and $0 \leq m_c$. First, we apply the following reduction formulas with respect to φ;

(1) If φ is negative, we transform φ into the positive range by

$$G(\varphi, n_c, m_c, a, b) = -G(-\varphi, n_c, m_c, a, b) \,. \tag{31}$$

(2) If $\varphi > \pi$, we transform φ into the range $0 \leq \varphi < \pi$ by using cel as

$$G(\varphi, n_c, m_c, a, b) = 2\left[\frac{\varphi}{\pi}\right] cel(k_c, n_c, a, b) + \\ +G\left(\varphi - \left[\frac{\varphi}{\pi}\right]\pi, n_c, m_c, a, b\right) \,, \tag{32}$$

where [ ] is the Gaussian operator realized by the `int` function of `FORTRAN`. In the arguments of cel, we wrote

$$k_c = \sqrt{m_c} \tag{33}$$

since it is assured that m_c will be non-negative in this case by the check described in the previous subsection.

(3) If $\pi/2 < \varphi \leq \pi$, we transform φ into the range $0 \leq \varphi < \pi/2$ by

$$G(\varphi, n_c, m_c, a, b) = cel(k_c, n_c, a, b) - G(\pi - \varphi, n_c, m_c, a, b) \ . \tag{34}$$

Since the routine to compute general incomplete elliptic integral cel is called frequently, it is wise to design it so as to return the last computed value when the input parameters k_c, n_c, a and b are the same as those of the last call.

Second, we compute

$$x = \tan \varphi \tag{35}$$

for later use. Note that the above reduction procedure assures that $0 \leq x < +\infty$.

Third, we make a reduction with respect to m_c.

(1) If m_c is non-negative, then it is easy to see that G is reduced to el as

$$G(\varphi, n_c, m_c, a, b) = el(x, k_c, n_c, a, b) \ . \tag{36}$$

(2) If m_c is negative, then we transform m_c into the positive range by

$$G(\varphi, n_c, m_c, a, b) = el(\tilde{x}, \tilde{k}_c, \tilde{p}, \tilde{a}, \tilde{b}) \tag{37}$$

where

$$\tilde{x} = x\sqrt{\frac{1 - m_c}{1 + m_c x^2}} \ , \quad \tilde{k}_c = \sqrt{\frac{-m_c}{1 - m_c}} \ , \quad \tilde{p} = \frac{n_c - m_c}{1 - m_c} \ ,$$

$$\tilde{a} = \frac{a}{\sqrt{1 - m_c}} \ , \quad \tilde{b} = \frac{b - m_c a}{\sqrt{1 - m_c}^3} \ . \tag{38}$$

This reduction formula is derived from the expression

$$G(\varphi, n_c, m_c, a, b) = \frac{a - b}{1 - n_c} F(\varphi, \sqrt{1 - m_c}) + \\ + \frac{b - n_c a}{1 - n_c} \Pi(\varphi, 1 - n_c, \sqrt{1 - m_c}) \tag{39}$$

and the reduction formulas of F and Π

$$F(\varphi, k) = \tilde{k} F(\tilde{\varphi}, \tilde{k}) \ , \quad \Pi(\varphi, \alpha^2, k) = \tilde{k} \Pi(\tilde{\varphi}, \tilde{\alpha}^2, \tilde{k}) \tag{40}$$

where

$$\tilde{\varphi} = \sin^{-1}(k \ \sin \varphi) \ , \quad \tilde{\alpha}^2 = \frac{\alpha^2}{k^2} \ , \quad \tilde{k} = \frac{1}{k} \tag{41}$$

which are merely the translation of the formula 162.02 of BF.

2.3. Taylor Expansion

Now let us confine ourselves to *el*, defined by

$$el(x, k_c, p, a, b) \equiv \int_0^{\tan^{-1} x} \frac{a \cos^2\theta + b \sin^2\theta}{(\cos^2\theta + p \sin^2\theta)\sqrt{\cos^2\theta + k_c^2 \sin^2\theta}}\, d\theta$$

$$\equiv \int_0^{x} \frac{a + b\xi^2}{(1 + p\xi^2)\sqrt{(1+\xi^2)(1+k_c^2\xi^2)}}\, d\xi \qquad (42)$$

As implicitly written in the ALGOL code in Burilsch (1969b), the Bartky transformation, which will be described in the next subsection, loses its efficiency due to a serious cancellation in the case where the Taylor expansion of the integrand is appropriate; namely when the magnitudes of x, p and k_c are sufficiently small, say all of $px^2, k_c^2 x^2, p$ and k_c^2 are less than 0.1. Unfortunately, the Taylor expansion part of the algorithm for $el3$ is not adequate for the purpose to extend to the general case needed for *el*. We thus give here a different, but naive approach.

Write η in place of ξ^2 and use m_c in place of k_c^2 again. Then, with use of the Taylor expansion formula

$$\frac{1}{\sqrt{1+\eta}} = \sum_{j=0}^{\infty} (-1)^j \frac{(2j-1)!!}{(2j)!!}\, \eta^j , \qquad (43)$$

we can expand the following denominators in the integrand of *el* as

$$\frac{1}{\sqrt{1+m_c\eta}} = \sum_{j=0}^{\infty} X_j \eta^j , \quad \frac{1}{(1+p\eta)\sqrt{1+m_c\eta}} = \sum_{j=0}^{\infty} Y_j \eta^j , \qquad (44)$$

where the coefficients X_j and Y_j are obtained by the recursive formulas

$$X_j = -\frac{2j-1}{2j}\, m_c X_{j-1} , \quad Y_j = X_j - pY_{j-1} , \qquad (45)$$

and their starting values

$$X_0 = Y_0 = 1 . \qquad (46)$$

Then *el* is approximated by

$$el \approx \sum_{j=0}^{J} (aY_j I_j + bY_j I_{j+1}) \qquad (47)$$

where I_j is defined as

$$I_j \equiv \int_0^{x} \frac{\xi^{2j}}{\sqrt{1+\xi^2}}\, d\xi \qquad (48)$$

and is calculated by the recursive formula

$$I_j = \frac{x^{2j-1}}{2j}\sqrt{1+x^2} - \frac{2j-1}{2j} I_{j-1} \tag{49}$$

with the starting value

$$I_0 = \ln\left(x + \sqrt{1+x^2}\right) \tag{50}$$

since $x \geq 0$ is already assured. If the magnitudes of x, p, and k_c^2 are assured to be less than 0.1, it is sufficient that the number of terms taken in the summation J is as many as the number of digits required, 14 in the case of double precision computation, for example. We remark that it is usual to evaluate el for various values of x, a, and b while k_c and p are fixed. Thus, in the actual implementation, it would be wise to save Y_j values and to update them only when the input k_c and p are different from those in the last call.

2.4. Bartky's Transformation

When the Taylor series expansion of the integrand is not adequate, el is evaluated by a modification of the algorithm for computing $el3$ which was developed by Burlisch based on an extension of the Bartky transformation (Burlisch, 1969a, 1969b). Since our algorithm to compute el is a modification of $el3$, we only mention the amendments here.

(1) There are no modifications in the main part of the iteration, the typical form of which is

$$\begin{aligned}
&n \leftarrow 2n\,,\quad e \leftarrow st\,,\quad t \leftarrow t+s\,,\quad s \leftarrow 2\sqrt{e}\,,\quad y \leftarrow y - e/y\,,\\
&j \leftarrow \begin{cases} m & \text{if } y > 0 \\ m+1 & \text{otherwise} \end{cases}\,,\quad m \leftarrow m+j\,,\quad f \leftarrow c\,,\quad g \leftarrow e/q\,,\\
&c \leftarrow c + d/q\,,\quad q \leftarrow q+g\,,\quad d \leftarrow 2(d+fg)\,,\quad w \leftarrow w^2\,u/v\,,\\
&u \leftarrow u + e/u\,,\quad v \leftarrow v + e/v\,.
\end{aligned} \tag{51}$$

where the part for w, u, and v changes depending on the parameter range of p and m_c. We note that n, j and m should be taken as integer variables.

(2) As for the starting values of these variables in the iteration, those of c and d should be modified as; if $\min(m_c, 1) < 2p$ then

$$c = a\,,\quad d = b/q \tag{52}$$

else

$$c = \frac{a-b}{1-p}\,,\quad d = cq - \frac{(b-aq)(1-m_c)}{q(1-p)^2}\,. \tag{53}$$

Here we again used m_c in place of k_c^2.

(3) There is no need to change the criterion of convergence, which is described as

$$|1 - s/t| \leq \varepsilon \tag{54}$$

for the required relative accuracy ε, say a typical value of 10^{-14} for double precision arithmetic.

(4) The approximation formula of $el3$

$$el3 \approx \frac{1}{n}(A + B) \tag{55}$$

where A and B are written in terms of the variables in the above iteration as

$$A = \frac{ct + d}{t(t + q)}\left[\tan^{-1}\left(\frac{y}{t}\right) + j\pi\right], \quad B = \frac{1}{2}\sqrt{\left|\frac{p - 1}{p(p - m_c)}\right|}\ \ln w , \tag{56}$$

should be modified into the form for el as

$$el \approx \frac{1}{n}(A + zB) \tag{57}$$

where

$$z = \frac{ap - b}{p - 1} . \tag{58}$$

We remark that the form of B varies depending on the range of parameters p and m_c in a complicated manner; however, the above modification factor z does not change.

It is easy to see that the above formulas include Burlisch's two sets of formulas for $el3$ and cel as its special cases; namely the former is obtained by substituting 1 for a and b, while the latter is found by dropping the iterations on y, j, m, r, q, w, u, and v. Note that he used a different notation from ours, i.e.:

$$\mu_i \to t , \ \nu_i \to s , \ \mu_i\nu_i \to e , \ x_i \to y , \ M_i \to m ,$$

$$p_i \to q , \ c_i \to c , \ d_i \to d , \ \varepsilon_i \to u , \ \eta_i \to v , \ \zeta_i \to w , \tag{59}$$

for $el3$ (1969b, p. 306) and

$$\mu_i \to t , \ \nu_i \to s , \ \mu_i\nu_i \to e , \ p_i \to q , \ a - i \to c , \ b_i \to d , \tag{60}$$

for cel (1969b, p. 308). Our notation follows that used in his ALGOL procedure for $el3$ (1969b, pp. 309–311).

3. Numerical Examples

3.1. Implementation

The algorithm to compute G described in the previous section was first implemented on our SONY NEWS workstation as a set of few procedures written in EFL, a dialect of FORTRAN developed at AT&T Bell Laboratories. Then we translated them into FORTRAN66 since EFL is a preprocessor language to FORTRAN66. After applying small modifications to enhance the portability, we tested the double precision FORTRAN66 version on a few workstations, including SONY NEWS/NWS821, SUN Sparc 2, DEC Station 3100 and HP 9000/720: and we found that it worked. Since the algorithm is in principle free from a fixed level of precision, one may obtain an implementation in a different precision by changing a few constants, such as J, the maximum number of terms in the Taylor expansion, or ε, the relative tolerance to terminate the iteration of the Bartky transformation. In the following, we will show some numerical examples obtained on our SONY workstation. The results for other workstations were found to be practically the same, namely with differences of order of 10^{-16} or less.

3.2. Check by values

To test our algorithm and implementation of G, we first made a comparison by value with the FORTRAN77 version of $el2$ in Press *et al.* (1986) and with the EFL translation of the original ALGOL version of $el3$ in Burlisch (1969b). The set of the input parameters used in the comparison was taken from the table of Bulirsch (1969b, p. 318). This is to check el, the main body of G, by seeing its two limiting cases

$$el2(x, k_c, 1, 1) = el(x, k_c, p, 1, p) , \tag{61}$$

$$el3(x, k_c, p) = el(x, k_c, p, 1, 1) \tag{62}$$

for various combinations of x, k_c, and p, although these cover only the core range of input parameters for G such as $|\varphi| < \pi/2$ and $m_c \geq 0$. The results are shown in Table I. This assures us that el can be used in place of the pair of $el2$ and $el3$.

3.3. Check by transformation

Next, we made an internal check of the part of parameters reduction of G, especially those of the argument φ, by comparison with the double argument formula of Π as

$$2\Pi(\mathrm{am}(u, k), \alpha^2, k) = \Pi(\mathrm{am}(2u, k), \alpha^2, k) + |\alpha^2| \frac{Z}{\sqrt{|C|}} , \tag{63}$$

where

$$Z = \begin{cases} \tan^{-1}\left(\sqrt{|C|}T\right) & \text{if } C > 0 , \\ \tanh^{-1}\left(\sqrt{|C|}T\right) & \text{otherwise} , \end{cases} \tag{64}$$

TABLE I

Comparison by value with $el2$ and $el3$: Burlisch's set of input parameters.

x	k_c	p	$el2 - el$	$el3 - el$
1.3	0.11	4.21	0.00e+00	0.00e+00
1.3	0.11	0.82	0.00e+00	0.00e+00
1.3	0.92	0.71	0.00e+00	0.00e+00
1.3	0.92	0.23	0.00e+00	0.00e+00
1.3	0.12	–0.11	0.00e+00	–8.15e–17
1.3	0.12	–2.11	0.00e+00	0.00e+00
1.3	0.4	0.1600001	0.00e+00	0.00e+00
1.3	1.0e–10	0.82	–1.07e–16	0.00e+00
1.3e–10	1.0e–10	1.00e–10	–1.42e–20	–7.10e–21
1.6	1.9	9.81	1.23e–16	4.01e–17
1.6	1.9	1.22	1.23e–16	0.00e+00
1.6	1.9	0.87	1.23e–16	0.00e+00
1.6	1.9	0.21	1.23e–16	1.08e–16
1.6	1.9	–0.21	1.23e–16	–1.80e–16
1.6	1.9	–4.3	1.23e–16	0.00e+00
1.6	10.1	–1.0e–5	4.24e–17	–7.96e–17
1.6	1.5	2.24999	–1.18e–16	–6.53e–17
1.6	1.0e+10	1.2	4.14e–25	0.00e+00
–1.6	1.0e+10	1.2	–4.14e–25	0.00e+00
1.0	0.31	0.099	5.94e–17	–2.12e–16

Note: The colums $el2 - el$ and $el - 3el$ show $el2(x, k_c, 1, 1)$ $-el(x, k_c, p, 1, p)$ and $el3(x, k_c, p) - el(x, k_c, p, 1, 1)$, respectively.

$$C = \alpha^2(1 - \alpha^2)(\alpha^2 - k^2) \,, \tag{65}$$

$$T = \frac{\mathrm{sn}^2(u, k)\ \mathrm{sn}(2u, k)}{1 - \alpha^2\ \mathrm{sn}^2(2u, k) + \alpha^2\ \mathrm{sn}^2(u, k)\ \mathrm{cn}(2u, k)\ \mathrm{dn}(2u, k)} \,. \tag{66}$$

This formula is obtained by rewriting the special case of the addition formulas, 116.xx of BF. The deviation from the formula as a function of u is shown in

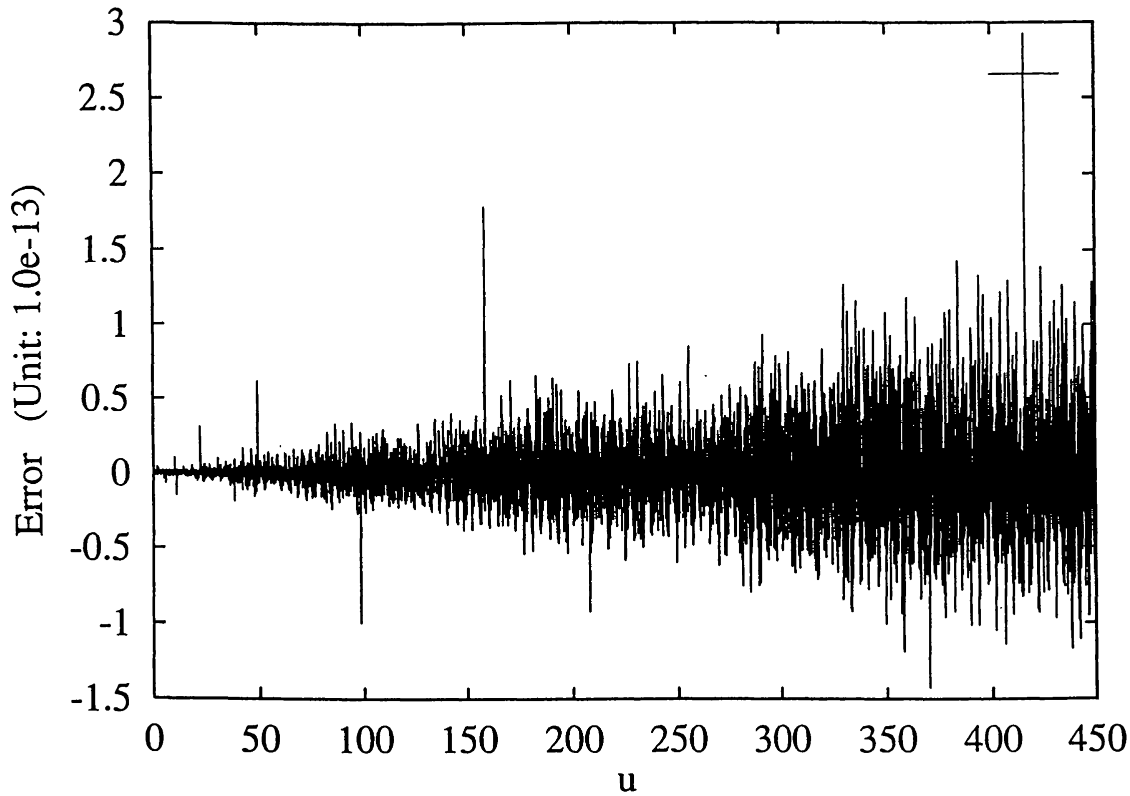

Fig. 1. Check by double argument transformation. The deviation from the double argument formula of $\Pi(\mathrm{am}(u,k),\alpha^2,k)$ is shown as a function of u in the range [0, 450] for the case $\alpha^2=-3$ and $k=\sqrt{3}\tan 25° \approx 0.807\,668\,555\ldots$.

Figure 1 for the case $n_c = 4$ and $m_c = 1-3\tan^2 25° \approx 0.347\,671\,503\ldots$, namely in the case $C \approx 40.172\,058\,04\ldots > 0$. The range of u in the figure is $[0,\ 450]$, which covers roughly 56 revolutions of $\varphi = \mathrm{am}(u,k)$ since $2K(k)/\pi \approx 1.279\,934\,185\,8\ldots$ in this case. The figure shows that the reduction of G works fairly well. In fact, the error increases in proportion to the magnitude of φ; around a few parts in 10^{16} times $|\varphi|$. This tendency to increase in proportion to the magnitude of the argument is a kind of destiny for periodic functions in general.

3.4. Effectiveness of General Form

Finally, we see how G works when the combination of $el2$ and $el3$ fails; the case to evaluate is

$$\frac{1}{k^2}\left[\Pi(\varphi,k^2,k)-F(\varphi,k)\right]$$

$$=\frac{1}{k^2}\left[el3\Big(\tan\varphi,\sqrt{1-k^2},1-k^2\Big)-el2\Big(\tan\varphi,\sqrt{1-k^2},1,1\Big)\right], \tag{67}$$

$$=G(\varphi,1-k^2,1-k^2,0,1)\,, \tag{68}$$

TABLE II

Comparison for small modulus.

k^2	Rigorous	G-Rigorous	$(el3 - el2)$-Rigorous
1.00e–01	0.149 697 534 165 855 0	2.78e–17	7.49e–16
1.00e–02	0.143 370 011 159 605 9	1.39e–16	–1.11e–15
1.00e–03	0.142 765 898 686 430 5	–5.55e–17	1.72e–14
1.00e–04	0.142 705 760 650 496 6	5.55e–17	1.78e–13
1.00e–05	0.142 699 749 566 443 9	1.39e–16	5.93e–12
1.00e–06	0.142 699 148 485 221 5	0.00e+00	–2.10e–10
1.00e–07	0.142 699 088 377 371 1	–5.55e–17	8.34e–10
1.00e–08	0.142 699 082 366 588 9	2.78e–17	3.51e–09
1.00e–09	0.142 699 081 765 510 6	5.55e–17	–6.99e–09
1.00e–10	0.142 699 081 705 402 8	0.00e+00	1.04e–07
1.00e–11	0.142 699 081 699 392 0	5.55e–17	–2.12e–06
1.00e–12	0.142 699 081 698 790 9	8.33e–17	–3.54e–05
1.00e–13	0.142 699 081 698 730 7	5.55e–17	–5.91e–04
1.00e–14	0.142 699 081 698 724 8	–8.33e–17	–9.47e–03
1.00e–15	0.142 699 081 698 724 2	0.00e+00	–3.17e–02

Note: The quantities Rigorous, G, and $el3 - el2$ show

$$\text{Rigorous} = \frac{1}{1-k^2}\left[el2(1,\ \sqrt{1-k^2},\ 1,0) - \frac{1}{2\sqrt{1-k^2/2}}\right],$$

$$G = G(\pi/4, 1-k^2, 1-k^2, 0, 1)\ ,$$

$$el3 - el2 = \frac{1}{k^2}\left[el3(1,\ \sqrt{1-k^2},\ 1-k^2) - el2(1,\ \sqrt{1-k^2},\ 1,1)\right],$$

respectively.

$$= \frac{1}{1-k^2}\left[el2\Big(\tan\varphi,\ \sqrt{1-k^2},\ 1,0\Big) - \frac{\sin\varphi\,\cos\varphi}{\sqrt{1-k^2\sin^2\varphi}}\right]. \tag{69}$$

As is seen from the above forms, at least the last expression is expected to be effective in the limit as k^2 goes to zero. For the case $\varphi = \pi/4$, the deviation from the formula as a function of k^2 is shown in Table II.

4. Conclusion

We have given an algorithm for computing the incomplete elliptic integral of a general form which effectively covers the incomplete elliptic integrals of three kinds and some linear combinations of them. A double precision implementation of this algorithm in FORTRAN gives their numerical values accurate to at least 14 digits. We have developed this algorithm to investigate numerically the torque-free spin motion of a rigid body. Fortunately, in the case of the torque-free motion, there is a proportional relation between α^2 and k^2 as

$$k^2 = -\alpha^2 \tan^2 J_0 \tag{70}$$

where J_0 is a real constant. Then the expressions for the partial derivatives of Π which we give here by means of G are effective. Thus we believe that the function G is sufficient for this purpose. However, to deal with the first-order partial derivatives of Π for all possible cases of α^2 and k, we need one more generalization of G, such as

$$H(\varphi, n_c, l_c, m_c, a, b)$$

$$\equiv \int_0^{\varphi} \frac{a \cos^2 \theta + b \sin^2 \theta}{(\cos^2 \theta + n_c \sin^2 \theta)(\cos^2 \theta + l_c \sin^2 \theta) \sqrt{\cos^2 \theta + m_c \sin^2 \theta}} \, \mathrm{d}\theta \tag{71}$$

because this function includes the partial derivatives of Π as its special cases as

$$\frac{\partial \Pi(\varphi, \alpha^2, k)}{\partial k^2} = H(\varphi, 1 - \alpha^2, 1 - k^2, 1 - k^2, 0, 1) , \tag{72}$$

$$\frac{\partial \Pi(\varphi, \alpha^2, k)}{\partial \alpha^2} = H(\varphi, 1 - \alpha^2, 1 - \alpha^2, 1 - k^2, 0, 1) . \tag{73}$$

To find such a generalization will be a challenging targeet for future work.

In case readers would like to try to use our codes, they are invited to contact us.

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