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Matrix chain multiplication

From Wikipedia, the free encyclopedia
(Redirected from Chain matrix multiplication)

Matrix chain multiplication (or Matrix Chain Ordering Problem, MCOP) is an optimization problem that can be solved using dynamic programming. Given a sequence of matrices, the goal is to find the most efficient way to multiply these matrices. The problem is not actually to *perform* the multiplications, but merely to decide the sequence of the matrix multiplications involved.

We have many options because matrix multiplication is associative. In other words, no matter how we parenthesize the product, the result obtained will remain the same. For example, if we had four matrices *A*, *B*, *C*, and *D*, we would have:

$$((AB)C)D = ((A(BC))D) = (AB)(CD) = A((BC)D) = A(B(CD)).$$

However, the order in which we parenthesize the product affects the number of simple arithmetic operations needed to compute the product, or the *efficiency*. For example, suppose A is a 10 × 30 matrix, B is a 30 × 5 matrix, and C is a 5 × 60 matrix. Then,

$$(AB)C = (10\times30\times5) + (10\times5\times60) = 1500 + 3000 = 4500 \text{ operations}$$
$$A(BC) = (30\times5\times60) + (10\times30\times60) = 9000 + 18000 = 27000 \text{ operations.}$$

Clearly the first method is more efficient. With this information, the problem statement can be refined, how do we determine the optimal parenthesization of a product of *n* matrices? We could go through each possible parenthesization (brute force), requiring a run-time that is exponential in the number of matrices, which is very slow and impractical for large *n*. A quicker solution to this problem can be achieved by breaking up the problem into a set of related subproblems. By solving subproblems one time and reusing these solutions, we can drastically reduce the run-time required. This concept is known as dynamic programming.

Contents [hide]

A Dynamic Programming Algorithm [edit]

To begin, let us assume that all we really want to know is the minimum cost, or minimum number of arithmetic operations, needed to multiply out the matrices. If we are only multiplying two matrices, there is only one way to multiply them, so the minimum cost is the cost of doing this. In general, we can find the minimum cost using the following recursive algorithm:

- Take the sequence of matrices and separate it into two subsequences.
- Find the minimum cost of multiplying out each subsequence.
- Add these costs together, and add in the cost of multiplying the two result matrices.
- Do this for each possible position at which the sequence of matrices can be split, and take the minimum over all of them.

For example, if we have four matrices *ABCD*, we compute the cost required to find each of (*A*)(*BCD*), (*AB*)(*CD*), and (*ABC*)(*D*), making recursive calls to find the minimum cost to compute *ABC*, *AB*, *CD*, and *BCD*. We then choose the best one. Better still, this yields not only the minimum cost, but also demonstrates the best way of doing the multiplication: group it the way that yields the lowest total cost, and do the same for each factor.

Unfortunately, if we implement this algorithm we discover that it is just as slow as the naive way of trying all permutations! What went wrong? The answer is that we're doing a lot of redundant work. For example, above we made a recursive call to find the best cost for computing both *ABC* and *AB*. But finding the best cost for computing ABC also requires finding the best cost for *AB*. As the recursion grows deeper, more and more of this type of unnecessary repetition occurs.

One simple solution is called memoization: each time we compute the minimum cost needed to multiply out a specific subsequence, we save it. If we are ever asked to compute it again, we simply give the saved answer, and do not recompute it. Since there are about $n^2/2$ different subsequences, where n is the number of matrices, the space required to do this is reasonable. It can be shown that this simple trick brings the runtime down to O(n^3) from O(2^n), which is more than efficient enough for real applications. This is top-down dynamic programming.

From [1] Pseudocode:

```
// Matrix Ai has dimension p[i-1] x p[i] for i = 1..n
MatrixChainOrder(int p[])
{
    // length[p] = n + 1
    n = p.length - 1;
    // m[i,j] = Minimum number of scalar multiplications (i.e., cost)
    // needed to compute the matrix A[i]A[i+1]...A[j] = A[i..j]
    // cost is zero when multiplying one matrix
    for (i = 1; i <= n; i++)
       m[i,i] = 0;

    for (L=2; L<=n; L++) { // L is chain length
        for (i=1; i<=n-L+1; i++) {
            j = i+L-1;
            m[i,j] = MAXINT;
            for (k=i; k<=j-1; k++) {
                // q = cost/scalar multiplications
                q = m[i,k] + m[k+1,j] + p[i-1]*p[k]*p[j];
                if (q < m[i,j]) {
                    m[i,j] = q;
                    s[i,j]=k;  // s[i,j] = Second auxiliary table that stores k
                               // k      = Index that achieved optimal cost

                }
            }
        }
    }
}
```

- Note : The first index for p is 0 and the first index for m and s is 1

Another solution is to anticipate which costs we will need and precompute them. It works like this:

- For each *k* from 2 to *n*, the number of matrices:
  - Compute the minimum costs of each subsequence of length *k*, using the costs already computed.

The code in java using zero based array indexes along with a convenience method for printing the solved order of operations:

```
public class MatrixOrderOptimization {
    protected int[][]m;
    protected int[][]s;
    public void matrixChainOrder(int[] p) {
        int n = p.length - 1;
        m = new int[n][n];
        s = new int[n][n];

        for (int ii = 1; ii < n; ii++) {
            for (int i = 0; i < n - ii; i++) {
                int j = i + ii;
                m[i][j] = Integer.MAX_VALUE;
                for (int k = i; k < j; k++) {
                    int q = m[i][k] + m[k+1][j] + p[i]*p[k+1]*p[j+1];
                    if (q < m[i][j]) {
                        m[i][j] = q;
                        s[i][j] = k;
                    }
                }
            }
        }
    }
    public void printOptimalParenthesizations() {
        boolean[] inAResult = new boolean[s.length];
        printOptimalParenthesizations(s, 0, s.length - 1, inAResult);
    }

    void printOptimalParenthesizations(int[][]s, int i, int j,  /* for pretty
printing: */ boolean[] inAResult) {
        if (i != j) {
            printOptimalParenthesizations(s, i, s[i][j], inAResult);
            printOptimalParenthesizations(s, s[i][j] + 1, j, inAResult);
            String istr = inAResult[i] ? "_result " : " ";
            String jstr = inAResult[j] ? "_result " : " ";
            System.out.println(" A_" + i + istr + "* A_" + j + jstr);
            inAResult[i] = true;
            inAResult[j] = true;
        }
    }
}
```

At the end of this program, we have the minimum cost for the full sequence. Although, this algorithm requires O(n^3) time, this approach has practical advantages that it requires no recursion, no testing if a value has already been computed, and we can save space by throwing away some of the subresults that are no longer required. This is bottom-up dynamic programming: a second way by which this problem can be solved.

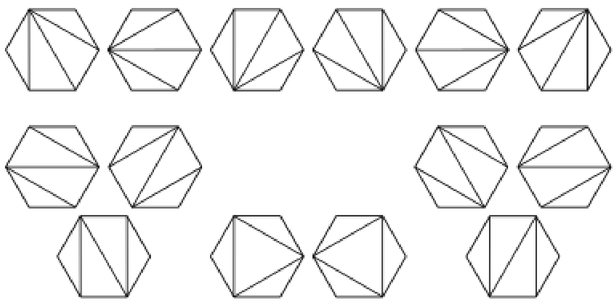
More Efficient Algorithms [edit]

There are algorithms that are more efficient than the $O(n^3)$ dynamic programming algorithm, though they are more complex.

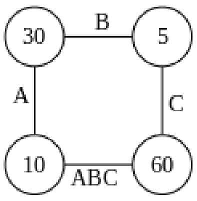
Hu & Shing (1981) [edit]

An algorithm published in 1981 by Hu and Shing achieves $O(n \log n)$ complexity.[2][3][4] They showed how the matrix chain multiplication problem can be transformed (or reduced) into the problem of triangulation of a regular polygon. The polygon is oriented such that there is a horizontal bottom side, called the base, which represents the final result. The other n sides of the polygon, in the clockwise direction, represent the matrices. The vertices on each end of a side are the dimensions of the matrix represented by that side. With n matrices in the multiplication chain there are n−1 binary operations and C_{n-1} ways of placing parenthesizes, where C_{n-1} is the (n−1)-th Catalan number. The algorithm exploits that there are also C_{n-1} possible triangulations of a polygon with n+1 sides.

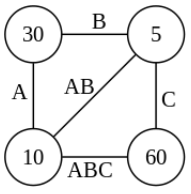
This image illustrates possible triangulations of a regular hexagon. These correspond to the different ways that parentheses can be placed to order the multiplications for a product of 5 matrices.



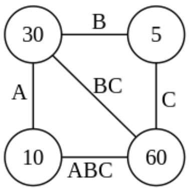
For the example below, there are four sides: A, B, C and the final result ABC. A is a 10×30 matrix, B is a 30×5 matrix, C is a 5×60 matrix, and the final result is a 10×60 matrix. The regular polygon for this example is a 4-gon, i.e. a square:



The matrix product AB is a 10x5 matrix and BC is a 30x60 matrix. The two possible triangulations in this example are:



Polygon representation of (AB)C



Polygon representation of A(BC)

The cost of a single triangle in terms of the number of multiplications needed is the product of its vertices. The total cost of a particular triangulation of the polygon is the sum of the costs of all its triangles:

(*AB*)*C*: (10×30×5) + (10×5×60) = 1500 + 3000 = 4500 multiplications
A(*BC*): (30×5×60) + (10×30×60) = 9000 + 18000 = 27000 multiplications

Hu & Shing developed an algorithm that finds an optimum solution for the minimum cost partition problem in *O*(*n* log *n*) time.

This section requires expansion. *(April 2009)*

Generalizations [edit]

The matrix chain multiplication problem generalizes to solving a more abstract problem: given a linear sequence of objects, an associative binary operation on those objects, and a way to compute the cost of performing that operation on any two given objects (as well as all partial results), compute the minimum cost way to group the objects to apply the operation over the sequence.[5] One somewhat contrived special case of this is string concatenation of a list of strings. In C, for example, the cost of concatenating two strings of length *m* and *n* using *strcat* is O(*m* + *n*), since we need O(*m*) time to find the end of the first string and O(*n*) time to copy the second string onto the end of it. Using this cost function, we can write a dynamic programming algorithm to find the fastest way to concatenate a sequence of strings. However, this optimization is rather useless because we can straightforwardly concatenate the strings in time proportional to the sum of their lengths. A similar problem exists for singly linked lists.

Another generalization is to solve the problem when parallel processors are available. In this case, instead of adding the costs of computing each factor of a matrix product, we take the maximum because we can do them simultaneously. This can drastically affect both the minimum cost and the final optimal grouping; more "balanced" groupings that keep all the processors busy are favored. There are even more sophisticated approaches.[6]

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