Monotone cubic interpolation

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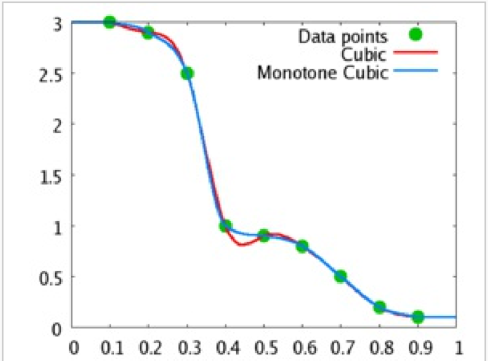
In the mathematical subfield of numerical analysis, **monotone cubic interpolation** is a variant of cubic interpolation that preserves monotonicity of the data set being interpolated.

Monotonicity is preserved by linear interpolation but not guaranteed by cubic interpolation.

Contents [hide]

Monotone cubic Hermite interpolation [edit]



Example showing non-monotone cubic interpolation (in red) and monotone cubic interpolation (in blue) of a monotone data set.

Monotone interpolation can be accomplished using cubic Hermite spline with the tangents m_i modified to ensure the monotonicity of the resulting Hermite spline.

An algorithm is also available for monotone quintic Hermite interpolation.

Interpolant selection [edit]

There are several ways of selecting interpolating tangents for each data point. This section will outline the use of the Fritsch–Carlson method.

Let the data points be (x_k, y_k) for $k = 1, ..., n$

1. Compute the slopes of the secant lines between successive points:

$$\Delta_k = \frac{y_{k+1} - y_k}{x_{k+1} - x_k}$$

for $k = 1, \ldots, n-1$.

2. Initialize the tangents at every data point as the average of the secants,

$$m_k = \frac{\Delta_{k-1} + \Delta_k}{2}$$

for $k = 2, \ldots, n-1$; if Δ_{k-1} and Δ_k have different sign, set $m_k = 0$. These may be updated in further steps. For the endpoints, use one-sided differences:

$$m_1 = \Delta_1 \quad \text{and} \quad m_n = \Delta_{n-1}$$

3. For $k = 1, \ldots, n-1$, if $\Delta_k = 0$ (if two successive $y_k = y_{k+1}$ are equal), then set $m_k = m_{k+1} = 0$, as the spline connecting these points must be flat to preserve monotonicity. Ignore step 4 and 5 for those k.
4. Let $\alpha_k = m_k / \Delta_k$ and $\beta_k = m_{k+1} / \Delta_k$. If α_k or β_{k-1} are computed to be less than zero, then the input data points are not strictly monotone, and (x_k, y_k) is a local extremum. In such cases, piecewise monotone curves can still be generated by choosing $m_k = 0$, although global strict monotonicity is not possible.
5. To prevent overshoot and ensure monotonicity, at least one of the following conditions must be met:

1. the function

$$\phi(\alpha,\beta) = \alpha - \frac{(2\alpha+\beta-3)^2}{3(\alpha+\beta-2)}$$

must have a value greater than or equal to zero;
2. $\alpha + 2\beta - 3 \leq 0$; or
3. $2\alpha + \beta - 3 \leq 0$.

If monotonicity must be strict then $\phi(\alpha,\beta)$ must have a value strictly greater than zero.

One simple way to satisfy this constraint is to restrict the magnitude of vector (α_k, β_k) to a circle of radius 3. That is, if $\alpha_k^2 + \beta_k^2 > 9$, then set $m_k = \tau_k \alpha_k \Delta_k$ and $m_{k+1} = \tau_k \beta_k \Delta_k$ where $\tau_k = \frac{3}{\sqrt{\alpha_k^2 + \beta_k^2}}$.

Alternatively it is sufficient to restrict $\alpha_k \leq 3$ and $\beta_k \leq 3$. To accomplish this if $\alpha_k > 3$, then set $m_k = 3 \times \Delta_k$. Similarly for β.

Note that only one pass of the algorithm is required.

Cubic interpolation [edit]

After the preprocessing, evaluation of the interpolated spline is equivalent to cubic Hermite spline, using the data x_k, y_k, and m_k for $k = 1, ..., n$.

To evaluate at x, find the smallest value larger than x, x_{upper}, and the largest value smaller than x, x_{lower}, among x_k such that $x_{\text{lower}} \leq x \leq x_{\text{upper}}$. Calculate

$h = x_{\text{upper}} - x_{\text{lower}}$ and $t = \frac{x - x_{\text{lower}}}{h}$

then the interpolant is

$$f_{\text{interpolated}}(x) = y_{\text{lower}} h_{00}(t) + h m_{\text{lower}} h_{10}(t) + y_{\text{upper}} h_{01}(t) + h m_{\text{upper}} h_{11}(t)$$

where h_{ii} are the basis functions for the cubic Hermite spline.

Example implementation [edit]

The following JavaScript implementation takes a data set and produces a monotone cubic spline interpolant function:

```
/* Monotone cubic spline interpolation
   Usage example:
 var f = createInterpolant([0, 1, 2, 3, 4], [0, 1, 4, 9, 16]);
 var message = '';
 for (var x = 0; x <= 4; x += 0.5) {
  var xSquared = f(x);
  message += x + ' squared is about ' + xSquared + '\n';
 }
 alert(message);
*/
var createInterpolant = function(xs, ys) {
 var i, length = xs.length;

 // Deal with length issues
 if (length != ys.length) { throw 'Need an equal count of xs and ys.'; }
 if (length === 0) { return function(x) { return 0; }; }
 if (length === 1) {
  // Impl: Precomputing the result prevents problems if ys is mutated later and
allows garbage collection of ys
  // Impl: Unary plus properly converts values to numbers
  var result = +ys[0];
  return function(x) { return result; };
 }

 // Rearrange xs and ys so that xs is sorted
 var indexes = [];
 for (i = 0; i < length; i++) { indexes.push(i); }
 indexes.sort(function(a, b) { return xs[a] < xs[b] ? -1 : 1; });
```

```
 var oldXs = xs, oldYs = ys;
 // Impl: Creating new arrays also prevents problems if the input arrays are mutated
later
 xs = []; ys = [];
 // Impl: Unary plus properly converts values to numbers
 for (i = 0; i < length; i++) { xs.push(+oldXs[indexes[i]]);
ys.push(+oldYs[indexes[i]]); }

 // Get consecutive differences and slopes
 var dys = [], dxs = [], ms = [];
 for (i = 0; i < length - 1; i++) {
  var dx = xs[i + 1] - xs[i], dy = ys[i + 1] - ys[i];
  dxs.push(dx); dys.push(dy); ms.push(dy/dx);
 }

 // Get degree-1 coefficients
 var c1s = [ms[0]];
 for (i = 0; i < dxs.length - 1; i++) {
  var m = ms[i], mNext = ms[i + 1];
  if (m*mNext <= 0) {
   c1s.push(0);
  } else {
   var dx = dxs[i], dxNext = dxs[i + 1], common = dx + dxNext;
   c1s.push(3*common/((common + dxNext)/m + (common + dx)/mNext));
  }
 }
 c1s.push(ms[ms.length - 1]);

 // Get degree-2 and degree-3 coefficients
 var c2s = [], c3s = [];
 for (i = 0; i < c1s.length - 1; i++) {
  var c1 = c1s[i], m = ms[i], invDx = 1/dxs[i], common = c1 + c1s[i + 1] - m - m;
  c2s.push((m - c1 - common)*invDx); c3s.push(common*invDx*invDx);
 }

 // Return interpolant function
 return function(x) {
  // The rightmost point in the dataset should give an exact result
  var i = xs.length - 1;
  if (x == xs[i]) { return ys[i]; }

  // Search for the interval x is in, returning the corresponding y if x is one of
the original xs
  var low = 0, mid, high = c3s.length - 1;
  while (low <= high) {
   mid = Math.floor(0.5*(low + high));
   var xHere = xs[mid];
   if (xHere < x) { low = mid + 1; }
   else if (xHere > x) { high = mid - 1; }
   else { return ys[mid]; }
  }
  i = Math.max(0, high);

  // Interpolate
  var diff = x - xs[i], diffSq = diff*diff;
  return ys[i] + c1s[i]*diff + c2s[i]*diffSq + c3s[i]*diff*diffSq;
 };
};
```

References [edit]

- Fritsch, F. N.; Carlson, R. E. (1980). "Monotone Piecewise Cubic Interpolation". *SIAM Journal on Numerical Analysis* (SIAM) **17** (2): 238–246. doi:10.1137/0717021.
- Dougherty, R.L.; Edelman, A.; Hyman, J.M. (April 1989). "Positivity-, monotonicity-, or convexity-preserving cubic and quintic Hermite interpolation". *Mathematics of Computation* **52** (186): 471–494.

External links [edit]

- GPLv3 licensed C++ implementation: MonotCubicInterpolator.cpp MonotCubicInterpolator.hpp

Categories: Interpolation | Splines