WIKIPEDIA
The Free Encyclopedia

Main page
Contents
Featured content
Current events
Random article
Donate to Wikipedia
Wikipedia store

Interaction
Help
About Wikipedia
Community portal
Recent changes
Contact page

Tools
What links here
Related changes
Upload file
Special pages
Permanent link
Page information
Wikidata item
Cite this page

Print/export
Create a book
Download as PDF
Printable version

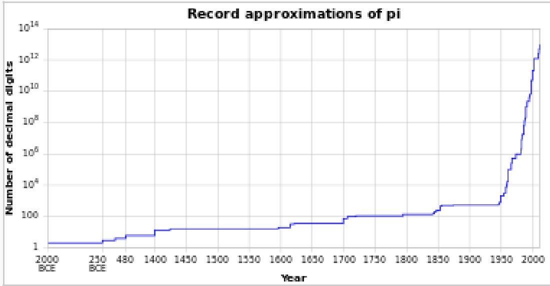
Languages
Italiano
Slovenščina
தமிழ்
中文
Edit links

Approximations of π

From Wikipedia, the free encyclopedia
(Redirected from Computing π)

This page is about the history of approximations; see also chronology of computation of π for a tabular summary. See also the history of π for other aspects of the evolution of our knowledge about mathematical properties of π.

Approximations for the mathematical constant pi (π) in the history of mathematics reached an accuracy within 0.04% of the true value before the beginning of the Common Era (Archimedes). In Chinese mathematics, this was improved to approximations correct to what corresponds to about seven decimal digits by the 5th century.



Graph showing the historical evolution of the record precision of numerical approximations to pi, measured in decimal places (depicted on a logarithmic scale; time before 1400 is not shown to scale).

Further progress was made only from the 15th century (Jamshīd al-Kāshī), and early modern mathematicians reached an accuracy of 35 digits by the 18th century (Ludolph van Ceulen), and 126 digits by the 19th century (Jurij Vega), surpassing the accuracy required for any conceivable application outside of pure mathematics.

The record of manual approximation of π is held by William Shanks, who calculated 527 digits correctly in the years preceding 1873. Since the mid 20th century, approximation of π has been the task of electronic digital computers; the current record (as of May 2015) is at 13.3 trillion digits, calculated in October 2014.[1]

Part of a series of articles on the
mathematical constant π

3.14159265358979...

Uses
Area of disk · Circumference · Use in other formulae

Properties
Irrationality · Transcendence

Value
Less than 22/7 · **Approximations** · Memorization

People
Archimedes · Liu Hui · Zu Chongzhi · Aryabhata · Madhava · Ludolph van Ceulen · Seki Takakazu · Takebe Katahiro · William Jones · John Machin · William Shanks · John Wrench · Chudnovsky brothers · Yasumasa Kanada

History
Chronology · Book

In culture
Legislation · Holiday

Related topics
Squaring the circle · Basel problem · Feynman point · Other topics related to π

v · t · e

Contents

Early history [edit]

The best known approximations to π dating to before the Common Era were accurate to two decimal places; this was improved upon in Chinese mathematics in particular by the mid first millennium, to an accuracy of seven decimal places. After this, no further progress was made until the late medieval period.

Some Egyptologists[2] have claimed that the ancient Egyptians used an approximation of π as $22/7$ from as early as the Old Kingdom.[3] This claim has met with skepticism.[4][5]

Babylonian mathematics usually approximated π to 3, sufficient for the architectural projects of the time (notably also reflected in the description of Solomon's Temple in the Hebrew Bible).[6] The Babylonians were aware that this was an approximation, and one Old Babylonian mathematical tablet excavated near Susa in 1936 (dated to between the 19th and 17th centuries BCE) gives a better approximation of π as $25/8=3.125$, about 0.5 percent below the exact value.[7]

At about the same time, the Egyptian Rhind Mathematical Papyrus (dated to the Second Intermediate Period, c. 1600 BCE, although stated to be a copy of an older, Middle Kingdom text) implies an approximation of π as $256/81 \approx 3.16$ (accurate to 0.6 percent) by calculating the area of a circle by approximating the circle by an octagon.[4][8]

Astronomical calculations in the *Shatapatha Brahmana* (c. 4th century BCE) use a fractional approximation of $339/108 \approx 3.139$ (accuracy $9 \cdot 10^{-4}$).[9]

In the 3rd century BCE, Archimedes proved the sharp inequalities $223/71 < \pi < 22/7$, by means of regular 96-gons (accuracies of $2 \cdot 10^{-4}$ and $4 \cdot 10^{-4}$, respectively).

In the 2nd century CE, Ptolemy, used the value $377/120$, the first known approximation accurate to three decimal places (accuracy $2 \cdot 10^{-5}$).[10]

The Chinese mathematician Liu Hui in 263 CE computed π to between 3.141 024 and 3.142 708 by inscribing an 96-gon and 192-gon; the average of these two values is 3.141864 (accuracy $9 \cdot 10^{-5}$). He also suggested that 3.14 was a good enough approximation for practical purposes. He has also frequently been credited with a later and more accurate result $\pi \approx 3927/1250 = 3.1416$ (accuracy $2 \cdot 10^{-6}$), although some scholars instead believe that this is due to the later (5th-century) Chinese mathematician Zu Chongzhi.[11] Zu Chongzhi is known to have computed π between 3.1415926 and 3.1415927, which was correct to seven decimal places. He gave two other approximations of π: $\pi \approx 22/7$ and $\pi \approx 355/113$. The latter fraction is the best possible rational approximation of π using fewer than five decimal digits in the numerator and denominator. Zu Chongzhi's result surpasses the accuracy reached in Hellenistic mathematics, and would remain without improvement for close to a millennium.

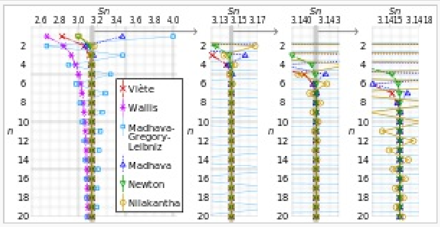
In Gupta-era India (6th century), mathematician Aryabhata in his astronomical treatise Āryabhaṭīya calculated the value of π to five significant figures ($\pi \approx 62832/20000 = 3.1416$).[12] using it to calculate an approximation of the earth's circumference.[13] Aryabhata stated that his result "approximately" (*āsanna* "approaching") gave the circumference of a circle. His 15th-century commentator Nilakantha Somayaji (Kerala school of astronomy and mathematics) has argued that the word means not only that this is an approximation, but that the value is incommensurable (irrational).[14]

Middle ages [edit]

By the 5th century CE, π was known to about seven digits in Chinese mathematics, and to about five in Indian mathematics. Further progress was not made for nearly a millennium, until the 14th century, when Indian mathematician and astronomer Madhava of Sangamagrama, founder of the Kerala school of astronomy and mathematics, discovered the infinite series for π, now known as the Madhava–Leibniz series,[15][16] and gave two methods for computing the value of π. One of these methods is to obtain a rapidly converging series by transforming the original infinite series of π. By doing so, he obtained the infinite series

$$\pi = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-3)^{-k}}{2k+1} = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-\frac{1}{3})^k}{2k+1} = \sqrt{12}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots\right)$$

and used the first 21 terms to compute an approximation of π correct to 11 decimal places as 3.141 592 653 59.



Comparison of the convergence of two Madhava series (the one with √12 in dark blue) and several historical infinite series for π. S_n is the approximation after taking n terms. Each subsequent subplot magnifies the shaded area horizontally by 10 times. (click for detail)

The other method he used was to add a remainder term to the original series of π. He used the remainder term

$$\frac{n^2+1}{4n^3+5n}$$

in the infinite series expansion of $\pi/4$ to improve the approximation of π to 13 decimal places of accuracy when $n = 75$.

Jamshīd al-Kāshī (Kāshānī), a Persian astronomer and mathematician, correctly computed 2π to 9 sexagesimal digits in 1424.[17] This figure is equivalent to 17 decimal digits as

$$2\pi \approx 6.28318530717958648,$$

which equates to

$$\pi \approx 3.14159265358979324.$$

He achieved this level of accuracy by calculating the perimeter of a regular polygon with 3×2^{28} sides.[18]

16th to 19th centuries [edit]

In the second half of the 16th century the French mathematician François Viète discovered an infinite product which converged on Pi known as Viète's formula.

The German/Dutch mathematician Ludolph van Ceulen (*circa* 1600) computed the first 35 decimal places of π with a 2^{62}-gon. He was so proud of this accomplishment that he had them inscribed on his tombstone.

In *Cyclometricus* (1621), Willebrord Snellius demonstrated that the perimeter of the inscribed polygon converges on the circumference twice as fast as does the perimeter of the corresponding circumscribed polygon. This was proved by Christiaan Huygens in 1654. Snellius was able to obtain 7 digits of Pi from a 96-sided polygon.[19]

The Slovene mathematician Jurij Vega in 1789 calculated the first 140 decimal places for π of which the first 126 were correct [20] and held the world record for 52 years until 1841, when William Rutherford calculated 208 decimal places of which the first 152 were correct. Vega improved John Machin's formula from 1706 and his method is still mentioned today.

The magnitude of such precision (152 decimal places) can be put into context by the fact that the circumference of the largest known thing, the observable universe, can be calculated from its diameter (93 billion light-years) to a precision of less than one Planck length (at 1.6162×10^{-35} meters, the shortest unit of length that has real meaning) using π expressed to just 62 decimal places.

The English amateur mathematician William Shanks, a man of independent means, spent over 20 years calculating π to 707 decimal places. This was accomplished in 1873, although only the first 527 were correct. He would calculate new digits all morning and would then spend all afternoon checking his morning's work. This was the longest expansion of π until the advent of the electronic digital computer three-quarters of a century later.

20th century [edit]

In 1910, the Indian mathematician Srinivasa Ramanujan found several rapidly converging infinite series of π, including

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

which computes a further eight decimal places of π with each term in the series. His series are now the basis for the fastest algorithms currently used to calculate π. See also Ramanujan–Sato series.

From the mid-20th century onwards, all calculations of π have been done with the help of calculators or computers.

In 1944, D. F. Ferguson, with the aid of a mechanical desk calculator, found that William Shanks had made a mistake in the 528th decimal place, and that all succeeding digits were incorrect.

In the early years of the computer, an expansion of π to 100 000 decimal places[21]:78 was computed by Maryland mathematician Daniel Shanks (no relation to the above-mentioned William Shanks) and his team at the United States Naval Research Laboratory in Washington, D.C. In 1961, Shanks and his team used two different power series for calculating the digits of π. For one it was known that any error would produce a value slightly high, and for the other, it was known that any error would produce a value slightly low. And hence, as long as the two series produced the same digits, there was a very high confidence that they were correct. The first 100,265 digits of π were published in 1962.[21]:80–99 The authors outlined what would be needed to calculate π to 1 million decimal places and concluded that the task was beyond that day's technology, but would be possible in five to seven years.[21]:78

In 1989, the Chudnovsky brothers correctly computed π to over 1 billion decimal places on the supercomputer IBM 3090 using the following variation of Ramanujan's infinite series of π:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}.$$

In 1999, Yasumasa Kanada and his team at the University of Tokyo correctly computed π to over 200 billion decimal places on the supercomputer HITACHI SR8000/MPP (128 nodes) using another variation of Ramanujan's infinite series of π. In October 2005 they claimed to have calculated it to 1.24 trillion places.[22]

21st century – current claimed world record [edit]

In November 2002, Yasumada Kanada and a team of 9 others used the Hitachi SR8000 to calculate π to roughly 1.24 trillion digits in around 600 hours.

In August 2009, a Japanese supercomputer called the T2K Open Supercomputer more than doubled the previous record by calculating π to roughly 2.6 trillion digits in approximately 73 hours and 36 minutes.

In December 2009, Fabrice Bellard used a home computer to compute 2.7 trillion decimal digits of π. Calculations were performed in base 2 (binary), then the result was converted to base 10 (decimal). The calculation, conversion, and verification steps took a total of 131 days.[23]

In August 2010, Shigeru Kondo used Alexander Yee's y-cruncher to calculate 5 trillion digits of π. This was the world record for any type of calculation, but significantly it was performed on a home computer built by Kondo.[24] The calculation was done between 4 May and 3 August, with the primary and secondary verifications taking 64 and 66 hours respectively.[25]

In October 2011, Shigeru Kondo broke his own record by computing ten trillion (10^{13}) and fifty digits using the same method but with better hardware.[26][27]

In December 2013 Kondo broke his own record for a third time when he computed 12.1 trillion digits of π.[28]

In October 2014 someone going by the pseudonym "houkouonchi" used y-cruncher to calculate 13.3 trillion digits of π. [29]

Practical approximations [edit]

Depending on the purpose of a calculation, π can be approximated by using fractions for ease of calculation. The most notable such approximations are 22⁄7 (accuracy $2 \cdot 10^{-4}$) and 355⁄113 (accuracy $8 \cdot 10^{-8}$).

Of some notability are legal or historical texts purportedly "defining π" to have some rational value, notably the "Indiana Pi Bill" of 1897, which stated "the ratio of the diameter and circumference is as five-fourths to four" (which would imply "$\pi = 3.2$") and a passage in the Hebrew Bible which seems to imply that "π equals three".

Imputed biblical value [edit]

See also: Molten Sea

It is sometimes claimed that the Hebrew Bible implies that "π equals three", based on a passage in 1 Kings 7:23 and 2 Chronicles 4:2 giving measurements for the round basin located in front of the Temple in Jerusalem as having a diameter of 10 cubits and a circumference of 30 cubits.

The issue is discussed in the Talmud and in Rabbinic literature.[30] Among the many explanations and comments are these:

- Rabbi Nehemiah explained this in his *Mishnat ha-Middot* (the earliest known Hebrew text on geometry, ca. 150 CE) by saying that the diameter was measured from the *outside* rim while the circumference was measured along the *inner* rim. This interpretation implies a brim about 0.225 cubit (or, assuming an 18-inch "cubit", some 4 inches), or one and a third "handbreadths," thick (cf. 1 Kings 7:24 and 2 Chronicles 4:3).
- Maimonides states (ca. 1168 CE) that π can only be known approximately, so the value 3 was given as accurate enough for religious purposes. This is taken by some[31] as the earliest assertion that π is irrational.
- Another rabbinical explanation[*by whom?*][*year needed*] invokes gematria: In 1 Kings 7:23 the word translated 'measuring line' appears in the Hebrew text spelled QWH קוה, but elsewhere the word is most usually spelled QW קו. The ratio of the numerical values of these Hebrew spellings is 111⁄106. If the putative value of 3 is multiplied by this ratio, one obtains 333⁄106 = 3.141509433... – within 1/10,000th of the true value of π.

There is still some debate on this passage in biblical scholarship.[*not in citation given*][32][33] Many reconstructions of the basin show a wider brim (or flared lip) extending outward from the bowl itself by several inches to match the description given in 1 Kings 7:26[34] In the succeeding verses, the rim is described as "a handbreadth thick; and the brim thereof was wrought like the brim of a cup, like the flower of a lily: it received and held three thousand baths" 2 Chronicles 4:5, which suggests a shape that can be encompassed with a string shorter than the total length of the brim, e.g., a Lilium flower or a Teacup.

The Indiana bill [edit]

The "Indiana Pi Bill" of 1897, which was nearly passed by the Indiana General Assembly in the U.S., has been claimed to imply a number of different values for π, although the closest it comes to explicitly asserting one is the wording "the ratio of the diameter and circumference is as five-fourths to four", which would make $\pi = 16/5 = 3.2$, a discrepancy of nearly 2 percent. A mathematics professor who happened to be present the day the bill was brought up for consideration in the Senate, after it had passed in the House, helped to stop the passage of the bill on its second reading, after which the assembly thoroughly ridiculed it before tabling it indefinitely.

Development of efficient formulae [edit]

Main article: List of formulae involving π

Polygon approximation to a circle [edit]

Archimedes, in his *Measurement of a Circle*, created the first algorithm for the calculation of π based on the idea that the perimeter of any (convex) polygon inscribed in a circle is less than the circumference of the circle, which, in turn, is less than the perimeter of any circumscribed polygon. He started with inscribed and circumscribed regular hexagons, whose perimeters are readily determined. He then shows how to calculate the perimeters of regular polygons of twice as many sides that are inscribed and circumscribed about the same circle. This is a recursive procedure which would be described today as follows: Let p_k and P_k denote the perimeters of regular polygons of k sides that are inscribed and circumscribed about the same circle, respectively. Then,

$$P_{2n} = \frac{2 p_n P_n}{p_n + P_n}, \qquad p_{2n} = \sqrt{p_n P_{2n}}.$$

Archimedes uses this to successively compute $P_{12}, p_{12}, P_{24}, p_{24}, P_{48}, p_{48}, P_{96}$ and p_{96}.[35] Using these last values he obtains

$$3\frac{10}{71} < \pi < 3\frac{1}{7}.$$

It is not known why Archimedes stopped at a 96-sided polygon; it only takes patience to extend the computations. Heron reports in his *Metrica* (about 60 CE) that Archimedes continued the computation in a now lost book, but then attributes an incorrect value to him.[36]

Archimedes uses no trigonometry in this computation and the difficulty in applying the method lies in obtaining good approximations for the square roots that are involved. Trigonometry, in the form of a table of chord lengths in a circle, was probably used by Claudius Ptolemy of Alexandria to obtain the value of π given in the *Almagest* (circa 150 CE).[37]

Advances in the approximation of π (when the methods are known) were made by increasing the number of sides of the polygons used in the computation. A trigonometric improvement by Willebrord Snell (1621) obtains better bounds from a pair of bounds gotten from the polygon method. Thus, more accurate results were obtained from polygons with fewer sides.[38] Viète's formula, published by François Viète in 1593, was derived by Viète using a closely related polygonal method, but with areas rather than perimeters of polygons whose numbers of sides are powers of two.[39]

The last major attempt to compute π by this method was carried out by Grienberger in 1630 who calculated 39 decimal places of π using Snell's refinement.[38]

Machin-like formulae [edit]

In 1961 the first expansion of π to 100,000 decimal places was computed by Maryland mathematician Dr. Daniel Shanks and his team at the United States Naval Research Laboratory (N.R.L.).

Daniel Shanks and his team used two different power series for calculating the digits of π. For one it was known that any error would produce a value slightly high, and for the other, it was known that any error would produce a value slightly low. And hence, as long as the two series produced the same digits, there was a very high confidence that they were correct. The first 100,000 digits of π were published by the Naval Research Laboratory.

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

together with the Taylor series expansion of the function arctan(*x*). This formula is most easily verified using polar coordinates of complex numbers, producing::$(5+i)^4 \cdot (239-i) = 2^2 \cdot 13^4 (1+i).$ Formulae of this kind are known as *Machin-like formulae*. (Note also that $\{x,y\}$ = {239, 13^2} is a solution to the Pell equation x^2-$2y^2$ = -1.) The first one million digits of π and 1/π are available from Project Gutenberg (see external links below). The record as of December 2002 by Yasumasa Kanada of Tokyo University stood at 1,241,100,000,000 digits, which were computed in September 2002 on a 64-node Hitachi supercomputer with 1 terabyte of main memory, which carries out 2 trillion operations per second, nearly twice as many as the computer used for the previous record (206 billion digits). The following Machin-like formulae were used for this:

$$\frac{\pi}{4} = 12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}$$

K. Takano (1982).

$$\frac{\pi}{4} = 44\arctan\frac{1}{57} + 7\arctan\frac{1}{239} - 12\arctan\frac{1}{682} + 24\arctan\frac{1}{12943}$$

F. C. W. Störmer (1896).

These approximations have so many digits that they are no longer of any practical use, except for testing new supercomputers. (Normality of π will always depend on the infinite string of digits on the end, not on any finite computation.)

Formulae of this kind are known as Machin-like formulae.

Other classical formulae [edit]

Other formulae that have been used to compute estimates of π include:

Liu Hui (see also Viète's formula):

$$\pi \approx 768\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+1}}}}}}}}}$$
$$\approx 3.141590463236763.$$

Madhava:

$$\pi = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-3)^{-k}}{2k+1} = \sqrt{12}\sum_{k=0}^{\infty}\frac{(-\frac{1}{3})^k}{2k+1} = \sqrt{12}\left(\frac{1}{1\cdot 3^0} - \frac{1}{3\cdot 3^1} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3} + \cdots\right)$$

Euler:

$$\pi = 20\arctan\frac{1}{7} + 8\arctan\frac{3}{79}$$

Newton:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty}\frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty}\frac{2^k k!^2}{(2k+1)!} = 1 + \frac{1}{3}\left(1 + \frac{2}{5}\left(1 + \frac{3}{7}(1+\cdots)\right)\right)$$

where (2k+1)!! denotes the product of the odd integers up to 2k+1.

Ramanujan:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

David Chudnovsky and Gregory Chudnovsky:

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

Ramanujan's work is the basis for the Chudnovsky algorithm, the fastest algorithms used, as of the turn of the millennium, to calculate π.

Modern algorithms [edit]

Extremely long decimal expansions of π are typically computed with iterative formulae like the Gauss–Legendre algorithm and Borwein's algorithm. The latter, found in 1985 by Jonathan and Peter Borwein, converges extremely fast:

For $y_0 = \sqrt{2} - 1,\ a_0 = 6 - 4\sqrt{2}$ and

$$y_{k+1} = (1 - f(y_k))/(1 + f(y_k))\ ,\ a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2)$$

where $f(y) = (1 - y^4)^{1/4}$, the sequence $1/a_k$ converges quartically to π, giving about 100 digits in three steps and over a trillion digits after 20 steps. However, it is known that using an algorithm such as the chudnovsky algorithm, that converges linearly is faster than these iterative formulae.

The first one million digits of π and $1/\pi$ are available from Project Gutenberg (see external links below). A former calculation record (December 2002) by Yasumasa Kanada of Tokyo University stood at 1.24 trillion digits, which were computed in September 2002 on a 64-node Hitachi supercomputer with 1 terabyte of main memory, which carries out 2 trillion operations per second, nearly twice as many as the computer used for the previous record (206 billion digits). The following Machin-like formulæ were used for this:

$$\frac{\pi}{4} = 12 \arctan\frac{1}{49} + 32 \arctan\frac{1}{57} - 5 \arctan\frac{1}{239} + 12 \arctan\frac{1}{110443}$$

K. Takano (1982).

$$\frac{\pi}{4} = 44 \arctan\frac{1}{57} + 7 \arctan\frac{1}{239} - 12 \arctan\frac{1}{682} + 24 \arctan\frac{1}{12943}$$ (F. C. W. Störmer (1896)).

These approximations have so many digits that they are no longer of any practical use, except for testing new supercomputers.[40] Properties like the potential normality of π will always depend on the infinite string of digits on the end, not on any finite computation.

Miscellaneous approximations [edit]

Historically, base 60 was used for calculations. In this base, π can be approximated to eight (decimal) significant figures with the number $3{:}8{:}29{:}44_{60}$, which is

$$3 + \frac{8}{60} + \frac{29}{60^2} + \frac{44}{60^3} = 3.14159\ 259^+$$

(The next sexagesimal digit is 0, causing truncation here to yield a relatively good approximation.)

In addition, the following expressions can be used to estimate π:

- accurate to three digits:

  $$\sqrt{2} + \sqrt{3} = 3.146^+$$

  Karl Popper conjectured that Plato knew this expression, that he believed it to be exactly π, and that this is responsible for some of Plato's confidence in the omnicompetence of mathematical geometry—and Plato's repeated discussion of special right triangles that are either isosceles or halves of equilateral triangles.

  $$\sqrt{15} - \sqrt{3} + 1 = 3.140^+$$

- accurate to four digits:

  $$\sqrt[3]{31} = 3.1413^+$$ [41]

- accurate to four digits (or five significant figures):

  $$\sqrt{7 + \sqrt{6 + \sqrt{5}}} = 3.1416^+$$ [42]

- an approximation by Ramanujan, accurate to 4 digits (or five significant figures):

  $$\frac{9}{5} + \sqrt{\frac{9}{5}} = 3.1416^+$$

- accurate to five digits:

  $$\frac{7^7}{4^9} = 3.14156^+$$

- accurate to seven digits:

  $$\frac{355}{113} = 3.14159\ 29^+$$

- accurate to nine digits:

  $$\sqrt[4]{3^4 + 2^4 + \frac{1}{2 + (\frac{2}{3})^2}} = \sqrt[4]{\frac{2143}{22}} = 3.14159\ 2652^+$$

  This is from Ramanujan, who claimed the Goddess of Namagiri appeared to him in a dream and told him the true value of π.[43]

- accurate to ten digits:

$$\frac{63}{25} \times \frac{17+15\sqrt{5}}{7+15\sqrt{5}} = 3.14159\ 26538^+$$

- accurate to ten digits (or eleven significant figures):

$$\sqrt[193]{\frac{10^{100}}{11222.11122}} = 3.14159\ 26536^+$$

  This curious approximation follows the observation that the 193rd power of 1/π yields the sequence 1122211125... Replacing 5 by 2 completes the symmetry without reducing the correct digits of π, while inserting a central decimal point remarkably fixes the accompanying magnitude at 10^{100}.[44]

- accurate to 18 digits:

$$\frac{80\sqrt{15}(5^4+53\sqrt{89})^{\frac{3}{2}}}{3308(5^4+53\sqrt{89})-3\sqrt{89}}$$ [45]

  This is based on the fundamental discriminant $d = 3(89) = 267$ which has class number $h(-d) = 2$ explaining the algebraic numbers of degree 2. Note that the core radical $5^4+53\sqrt{89}$ is 5^3 more than the fundamental unit $U_{89}=500+53\sqrt{89}$ which gives the smallest solution $\{x, y\}$ = {500, 53} to the Pell equation x^2-89y^2 = -1.

- accurate to 30 decimal places:

$$\frac{\ln(640320^3+744)}{\sqrt{163}} = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279^+$$

  Derived from the closeness of Ramanujan constant to the integer 640320^3+744. This does not admit obvious generalizations in the integers, because there are only finitely many Heegner numbers and negative discriminants *d* with class number *h*(−*d*) = 1, and d = 163 is the largest one in absolute value.

- accurate to 52 decimal places:

$$\frac{\ln\left(5280^3(236674+30303\sqrt{61})^3+744\right)}{\sqrt{427}}$$

  Like the one above, a consequence of the j-invariant. Among negative discriminants with class number 2, this *d* the largest in absolute value.

- accurate to 161 decimal places:

$$\frac{\ln\left((2u)^6+24\right)}{\sqrt{3502}}$$

  where *u* is a product of four simple quartic units,

$$u = (a+\sqrt{a^2-1})^2(b+\sqrt{b^2-1})^2(c+\sqrt{c^2-1})(d+\sqrt{d^2-1})$$

  and,

$$a = \tfrac{1}{2}(23+4\sqrt{34})$$
$$b = \tfrac{1}{2}(19\sqrt{2}+7\sqrt{17})$$
$$c = (429+304\sqrt{2})$$
$$d = \tfrac{1}{2}(627+442\sqrt{2})$$

  Based on one found by Daniel Shanks. Similar to the previous two, but this time is a quotient of a modular form, namely the Dedekind eta function, and where the argument involves $\tau = \sqrt{-3502}$. The discriminant *d* = 3502 has *h*(−*d*) = 16.

- The continued fraction representation of π can be used to generate successive best rational approximations. These approximations are the best possible rational approximations of π relative to the size of their denominators. Here is a list of the first thirteen of these:[46][47]

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}, \frac{5419351}{1725033}$$

  Of all of these, $\frac{355}{113}$ is the only fraction in this sequence that gives more exact digits of π (i.e. 7) than the number of digits needed to approximate it (i.e. 6). The accuracy can be improved by using other fractions with larger numerators and denominators, but, for most such fractions, more digits are required in the approximation than correct significant figures achieved in the result.[48]

Summing a circle's area [edit]

Pi can be obtained from a circle if its radius and area are known using the relationship:

$$A = \pi r^2.$$

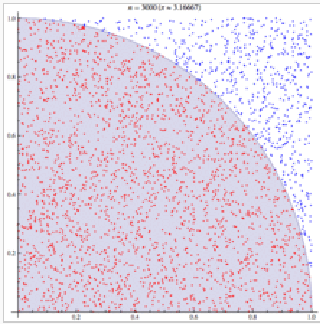
If a circle with radius r is drawn with its center at the point (0, 0), any point whose distance from the origin is less than r will fall inside the circle. The Pythagorean theorem gives the distance from any point (x, y) to the center:

$$d = \sqrt{x^2+y^2}.$$

Mathematical "graph paper" is formed by imagining a 1×1 square centered around each cell (x, y), where x and y are integers between −r and r. Squares whose center resides inside or exactly on the border of the circle can then be counted by testing whether, for each cell (x, y),

$$\sqrt{x^2+y^2} \leq r.$$

The total number of cells satisfying that condition thus approximates the area of the circle, which then can be used to calculate an approximation of π. Closer approximations can be produced by using larger values of r.



Numerical approximation of π: as points are randomly scattered inside the unit square, some fall within the unit circle. The fraction of points inside the circle approaches $\pi/4$ as points are added.

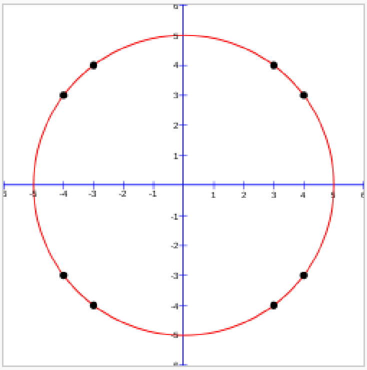
Mathematically, this formula can be written:

$$\pi = \lim_{r\to\infty} \frac{1}{r^2} \sum_{x=-r}^{r} \sum_{y=-r}^{r} \begin{cases} 1 & \text{if } \sqrt{x^2+y^2} \leq r \\ 0 & \text{if } \sqrt{x^2+y^2} > r. \end{cases}$$

In other words, begin by choosing a value for r. Consider all cells (x, y) in which both x and y are integers between −r and r. Starting at 0, add 1 for each cell whose distance to the origin (0,0) is less than or equal to r. When finished, divide the sum, representing the area of a circle of radius r, by r^2 to find the approximation of π. For example, if r is 5, then the cells considered are:

(−5,5)	(−4,5)	(−3,5)	(−2,5)	(−1,5)	(0,5)	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)
(−5,4)	(−4,4)	(−3,4)	(−2,4)	(−1,4)	(0,4)	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)
(−5,3)	(−4,3)	(−3,3)	(−2,3)	(−1,3)	(0,3)	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)
(−5,2)	(−4,2)	(−3,2)	(−2,2)	(−1,2)	(0,2)	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)
(−5,1)	(−4,1)	(−3,1)	(−2,1)	(−1,1)	(0,1)	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)
(−5,0)	(−4,0)	(−3,0)	(−2,0)	(−1,0)	(0,0)	(1,0)	(2,0)	(3,0)	(4,0)	(5,0)
(−5,−1)	(−4,−1)	(−3,−1)	(−2,−1)	(−1,−1)	(0,−1)	(1,−1)	(2,−1)	(3,−1)	(4,−1)	(5,−1)
(−5,−2)	(−4,−2)	(−3,−2)	(−2,−2)	(−1,−2)	(0,−2)	(1,−2)	(2,−2)	(3,−2)	(4,−2)	(5,−2)
(−5,−3)	(−4,−3)	(−3,−3)	(−2,−3)	(−1,−3)	(0,−3)	(1,−3)	(2,−3)	(3,−3)	(4,−3)	(5,−3)
(−5,−4)	(−4,−4)	(−3,−4)	(−2,−4)	(−1,−4)	(0,−4)	(1,−4)	(2,−4)	(3,−4)	(4,−4)	(5,−4)
(−5,−5)	(−4,−5)	(−3,−5)	(−2,−5)	(−1,−5)	(0,−5)	(1,−5)	(2,−5)	(3,−5)	(4,−5)	(5,−5)

The 12 cells (0, ±5), (±5, 0), (±3, ±4), (±4, ±3) are *exactly on* the circle, and 69 cells are *completely inside*, so the approximate area is 81, and π is calculated to be approximately 3.24 because 81 / 5^2 = 3.24. Results for some values of r are shown in the table below:



This circle as it would be drawn on a Cartesian coordinate graph. The cells (±3, ±4) and (±4, ±3) are labeled.

r	area	approximation of π
2	13	3.25
3	29	3.22222
4	49	3.0625
5	81	3.24
10	317	3.17
20	1257	3.1425
100	31417	3.1417
1000	3141549	3.141549

For related results see The circle problem: number of points (x,y) in square lattice with x^2 + y^2 <= n.

Similarly, the more complex approximations of π given below involve repeated calculations of some sort, yielding closer and closer approximations with increasing numbers of calculations.

Continued fractions [edit]

Besides its simple continued fraction representation [3; 7, 15, 1, 292, 1, 1, ...], which displays no discernible pattern, π has many generalized continued fraction representations generated by a simple rule, including these two.

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6 + \cfrac{5^2}{6 + \ddots}}}$$

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{3 + \cfrac{2^2}{5 + \cfrac{3^2}{7 + \ddots}}}}$$

(Other representations are available at The Wolfram Functions Site.)

Trigonometry [edit]

Gregory–Leibniz series [edit]

The Gregory–Leibniz series

$$\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 4\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots\right) = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ddots}}}}$$

is the power series for arctan(x) specialized to x = 1. It converges too slowly to be of practical interest. However, the power series converges much faster for smaller values of x, which leads to formulae where π arises as the sum of small angles with rational tangents, known as Machin-like formulae.

Arctangent [edit]

Further information: Double factorial

Knowing that $4\arctan(1) = \pi$ the formula can be simplified to get:

$$\pi = 2\left(1 + \frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \frac{1\cdot 2\cdot 3\cdot 4}{3\cdot 5\cdot 7\cdot 9} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{3\cdot 5\cdot 7\cdot 9\cdot 11} + \cdots\right)$$
$$= 2\sum_{n=0}^{\infty} \frac{n!}{(2n+1)!!} = \sum_{n=0}^{\infty} \frac{2^{n+1} n!^2}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{\binom{2n}{n}(2n+1)}$$
$$= 2 + \frac{2}{3} + \frac{4}{15} + \frac{4}{35} + \frac{16}{315} + \frac{16}{693} + \frac{32}{3003} + \frac{32}{6435} + \frac{256}{109395} + \frac{256}{230945} + \cdots$$

with a convergence such that each additional 10 terms yields at least three more digits.

Arcsine [edit]

Observing an equilateral triangle and noting that

$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

yields

$$\pi = 6\sin^{-1}\left(\frac{1}{2}\right) = 6\left(\frac{1}{2} + \frac{1}{2\cdot 3\cdot 2^3} + \frac{1\cdot 3}{2\cdot 4\cdot 5\cdot 2^5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7\cdot 2^7} + \cdots\right)$$
$$= \frac{3}{16^0\cdot 1} + \frac{6}{16^1\cdot 3} + \frac{18}{16^2\cdot 5} + \frac{60}{16^3\cdot 7} + \cdots = \sum_{n=0}^{\infty} \frac{3\cdot\binom{2n}{n}}{16^n(2n+1)}$$
$$= 3 + \frac{1}{8} + \frac{9}{640} + \frac{15}{7168} + \frac{35}{98304} + \frac{189}{2883584} + \frac{693}{54525952} + \frac{429}{167772160} + \cdots$$

with a convergence such that each additional five terms yields at least three more digits.

The Salamin–Brent algorithm [edit]

The **Gauss–Legendre algorithm** or **Salamin–Brent algorithm** was discovered independently by Richard Brent and Eugene Salamin in 1975. This can compute π to N digits in time proportional to $N\,\log(N)\,\log(\log(N))$, much faster than the trigonometric formulae.

Digit extraction methods [edit]

The Bailey–Borwein–Plouffe formula (BBP) for calculating π was discovered in 1995 by Simon Plouffe. Using base 16 math, the formula can compute any particular digit of π—returning the hexadecimal value of the digit—without having to compute the intervening digits (digit extraction). [49]

$$\pi = \sum_{n=0}^{\infty}\left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right)\left(\frac{1}{16}\right)^n$$

In 1996, Simon Plouffe derived an algorithm to extract the nth decimal digit of π (using base 10 math to extract a base 10 digit), and which can do so with an improved speed of O($n^3\log(n)^3$) time. The algorithm requires virtually no memory for the storage of an array or matrix so the one-millionth digit of π can be computed using a pocket calculator.[50] However, it would be quite tedious, and impractical to do so.

$$\pi + 3 = \sum_{n=1}^{\infty} \frac{n 2^n n!^2}{(2n)!}$$

The calculation speed of Plouffe's formula was improved to O(n^2) by Fabrice Bellard, who derived an alternative formula (albeit only in base 2 math) for computing π.[51]

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right)$$

Efficient methods [edit]

Many other expressions for π were developed and published by Indian mathematician Srinivasa Ramanujan. He worked with mathematician Godfrey Harold Hardy in England for a number of years.

Extremely long decimal expansions of π are typically computed with the Gauss–Legendre algorithm and Borwein's algorithm; the Salamin–Brent algorithm which was invented in 1976 has also been used.

In 1997, David H. Bailey, Peter Borwein and Simon Plouffe published a paper (Bailey, 1997) on a new formula for π as an infinite series:

$$\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right).$$

This formula permits one to fairly readily compute the kth binary or hexadecimal digit of π, without having to compute the preceding $k-1$ digits. Bailey's website contains the derivation as well as implementations in various programming languages. The PiHex project computed 64-bits around the quadrillionth bit of π (which turns out to be 0).

Fabrice Bellard further improved on BBP with his formula[2]:

$$\pi = \frac{1}{2^6}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{10n}}\left(-\frac{2^5}{4n+1}-\frac{1}{4n+3}+\frac{2^8}{10n+1}-\frac{2^6}{10n+3}-\frac{2^2}{10n+5}-\frac{2^2}{10n+7}+\frac{1}{10n+9}\right)$$

Other formulae that have been used to compute estimates of π include:

$$\frac{\pi}{2} = \sum_{k=0}^{\infty}\frac{k!}{(2k+1)!!} = \sum_{k=0}^{\infty}\frac{2^k k!^2}{(2k+1)!} = 1+\frac{1}{3}\left(1+\frac{2}{5}\left(1+\frac{3}{7}(1+\cdots)\right)\right)$$

Newton.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}$$

Srinivasa Ramanujan.

This converges extraordinarily rapidly. Ramanujan's work is the basis for the fastest algorithms used, as of the turn of the millennium, to calculate π.

$$\frac{1}{\pi} = 12\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!(13591409+545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}$$

David Chudnovsky and Gregory Chudnovsky.

Projects [edit]

Pi Hex [edit]

Pi Hex was a project to compute three specific binary digits of π using a distributed network of several hundred computers. In 2000, after two years, the project finished computing the five trillionth ($5*10^{12}$), the forty trillionth, and the quadrillionth (10^{15}) bits. All three of them turned out to be 0.

Software for calculating π [edit]

Over the years, several programs have been written for calculating π to many digits on personal computers.

General purpose [edit]

Most computer algebra systems can calculate π and other common mathematical constants to any desired precision.

Functions for calculating π are also included in many general libraries for arbitrary-precision arithmetic, for instance CLN and MPFR.

Special purpose [edit]

Programs designed for calculating π may have better performance than general-purpose mathematical software. They typically implement checkpointing and efficient disk swapping to facilitate extremely long-running and memory-expensive computations.

- **y-cruncher** by Alexander Yee[52] is the program which Shigeru Kondo used to compute the current world record number of digits. y-cruncher can also be used to calculate other constants and holds world records for several of them.

- **PiFast** by Xavier Gourdon was the fastest program for Microsoft Windows in 2003. According to its author, it can compute one million digits in 3.5 seconds on a 2.4 GHz Pentium 4.[53] PiFast can also compute other irrational numbers like e and $\sqrt{2}$. It can also work at lesser efficiency with very little memory (down to a few tens of megabytes to compute well over a billion (10^9) digits). This tool is a popular benchmark in the overclocking community. PiFast 4.4 is available from Stu's Pi page. PiFast 4.3 is available from Gourdon's page.
- **QuickPi** by Steve Pagliarulo for Windows is faster than PiFast for runs of under 400 million digits. Version 4.5 is available on Stu's Pi Page below. Like PiFast, QuickPi can also compute other irrational numbers like e, $\sqrt{2}$, and $\sqrt{3}$. The software may be obtained from the Pi-Hacks Yahoo! forum, or from Stu's Pi page.
- **Super PI** by Kanada Laboratory[54] in the University of Tokyo is the program for Microsoft Windows for runs from 16,000 to 33,550,000 digits. It can compute one million digits in 40 minutes, two million digits in 90 minutes and four million digits in 220 minutes on a Pentium 90 MHz. Super PI version 1.1 is available from Super PI 1.1 page.
- **apfloat** provides a Pi Calculator Applet for computing π in a browser. It can compute a million digits of π in a few seconds on a normal PC. Different radixes and algorithms can be used. In theory it can compute more than 10^{15} digits of π.

Notes [edit]

1. ^ http://www.numberworld.org/y-cruncher/
2. ^ Petrie, W.M.F. *Wisdom of the Egyptians* (1940)
3. ^ Based on the Great Pyramid of Giza, supposedly built so that the circle whose radius is equal to the height of the pyramid has a circumference equal to the perimeter of the base (it is 1760 cubits around and 280 cubits in height). Verner, Miroslav. *The Pyramids: The Mystery, Culture, and Science of Egypt's Great Monuments.* Grove Press. 2001 (1997). ISBN 0-8021-3935-3
4. ^ *a b* Rossi, *Corinna Architecture and Mathematics in Ancient Egypt,* Cambridge University Press. 2007. ISBN 978-0-521-69053-9.
5. ^ Legon, J. A. R. *On Pyramid Dimensions and Proportions* (1991) Discussions in Egyptology (20) 25-34 [1]
6. ^ See #Imputed biblical value. There has been concern over the apparent biblical statement of $\pi \approx 3$ from the early times of rabbinical Judaism, addressed by Rabbi Nehemiah in the 2nd century. Petr Beckmann, *A History of Pi*, St. Martin's (1971).[*page needed*]
7. ^ David Gilman Romano, *Athletics and Mathematics in Archaic Corinth: The Origins of the Greek Stadion*, American Philosophical Society, 1993, p. 78. "A group of mathematical clay tablets from the Old Babylonian Period, excavated at Susa in 1936, and published by E.M. Bruins in 1950, provide the information that the Babylonian approximation of π was 3 1/8 or 3.125." E. M. Bruins, *Quelques textes mathématiques de la Mission de Suse*, 1950. E. M. Bruins and M. Rutten, *Textes mathématiques de Suse*, Mémoires de la Mission archéologique en Iran vol. XXXIV (1961). See also Beckmann, Petr (1971), *A History of Pi*, New York: St. Martin's Press, pp. 12, 21–22 "in 1936, a tablet was excavated some 200 miles from Babylon. [...] The mentioned tablet, whose translation was partially published only in 1950, [...] states that the ratio of the perimeter of a regular hexagon to the circumference of the circumscribed circle equals a number which in modern notation is given by $57/60+36/(60)^2$ [i.e. $\pi = 3/0.96 = 25/8$]". Jason Dyer , On the Ancient Babylonian Value for Pi, 3 December 2008.
8. ^ Katz, Victor J. (ed.), Imhausen, Annette et al. *The Mathematics of Egypt, Mesopotamia, China, India, and Islam: A Sourcebook,* Princeton University Press. 2007. ISBN 978-0-691-11485-9
9. ^ Chaitanya, Krishna. A profile of Indian culture. Indian Book Company (1975). P. 133.
10. ^ http://uzweb.uz.ac.zw/science/maths/zimaths/pi.htm
11. ^ Lam, Lay Yong; Ang, Tian Se (1986), "Circle measurements in ancient China", *Historia Mathematica* **13** (4): 325–340, doi:10.1016/0315-0860(86)90055-8, MR 875525. Reprinted in Berggren, J. L.; Borwein, Jonathan M.; Borwein, Peter, eds. (2004), *Pi: A Source Book*, Springer, pp. 20–35, ISBN 9780387205717. See in particular pp. 333–334 (pp. 28–29 of the reprint).
12. ^ Āryabhaṭīya (gaṇitapāda 10):

    > chaturadhikam śatamaṣṭaguṇam dvāśaṣṭistathā sahasrāṇām ayutadvayaviṣkambhasyāsanno vṛttapariṇahaḥ.
    > "Add four to one hundred, multiply by eight and then add sixty-two thousand. The result is approximately the circumference of a circle of diameter twenty thousand. By this rule the relation of the circumference to diameter is given."

    In other words, (4 + 100) × 8 + 62000 is the circumference of a circle with diameter 20000. This provides a value of $\pi \approx 62832/20000 = 3.1416$, Jacobs, Harold R. (2003). *Geometry: Seeing, Doing, Understanding (Third Edition)*. New York: W.H. Freeman and Company. p. 70.
13. ^ "Aryabhata the Elder". *University of St Andrews, School of Mathematics and Statistics*. Retrieved 20 July 2011.
14. ^ S. Balachandra Rao (1998). *Indian Mathematics and Astronomy: Some Landmarks*. Bangalore: Jnana Deep Publications. ISBN 81-7371-205-0.
15. ^ George E. Andrews, Richard Askey, Ranjan Roy (1999). *Special Functions*. Cambridge University Press. p. 58. ISBN 0-521-78988-5.
16. ^ Gupta, R. C. (1992). "On the remainder term in the Madhava–Leibniz's series". *Ganita Bharati* **14** (1-4): 68–71.
17. ^ *Al-Kashi*, author: Adolf P. Youschkevitch, chief editor: Boris A. Rosenfeld, p. 256
18. ^ Azarian, Mohammad K. (2010), "al-Risāla al-muhīṭīyya: A Summary", *Missouri Journal of Mathematical Sciences* **22** (2): 64–85
19. ^ https://docs.google.com/viewer?a=v&q=cache:NTSdP7wNFA8J:www.ijpam.eu/contents/2003-7-2/4/4.pdf+&hl=en&gl=uk&pid=bl&srcid=ADGEESjTAd8gQzDqGaN7fo99joDgBNmLm4PCsT69_WR13A0nR6yT0T-RZFFSpqN-djir-w4lBOV2Juacul9apQNCW2KMOxf0csRinFDa-1DOSRpxTk83Cg4i8qAxvylfWoLRM04qjE8&sig=AHIEtbSw0AiTGzbt4uLFRLrbOXiXNknJCg
20. ^ Sandifer, Dr. Edward. Why 140 Digits of Pi Matter, page 17. NOTE: The last paragraph on that page says "the 127th digit... should be an 8." It is actually a 6.
21. ^ *a b c* Shanks, D.; Wrench, Jr., J. W. (1962). "Calculation of π to 100,000 decimals". *Mathematics of Computation* (American Mathematical Society) **16** (77): 76–99. doi:10.2307/2003813. JSTOR 2003813..
22. ^ Announcement at the Kanada lab web site.
23. ^ Bellard.org
24. ^ McCormick Grad Sets New Pi Record

25. ^ 5 Trillion Digits of Pi - New World Record
26. ^ http://www.newscientist.com/blogs/shortsharpscience/2011/10/pi-10-trillion.html
27. ^ Alexander J. Yee; Shigeru Kondo (22 October 2011). "Round 2... 10 Trillion Digits of Pi".
28. ^ Alexander J. Yee; Shigeru Kondo (28 December 2013). "12.1 Trillion Digits of Pi".
29. ^ "y-cruncher - A Multi-Threaded Pi Program". *numberworld.org*. Retrieved 28 August 2015.
30. ^ Tsaban, Boaz; Garber, David (February 1998). "On the rabbinical approximation of π" (PDF). *Historia Mathematica* **25** (1): 75–84. doi:10.1006/hmat.1997.2185. ISSN 0315-0860. Retrieved 14 July 2009.
31. ^ Wilbur Richard Knorr, The Ancient Tradition of Geometric Problems, New York: Dover Publications, 1993.
32. ^ Aleff, H. Peter. "Ancient Creation Stories told by the Numbers: Solomon's Pi". recoveredscience.com. Archived from the original on 14 October 2007. Retrieved 30 October 2007.
33. ^ O'Connor, J J; E F Robertson (August 2001). "A history of Pi". Archived from the original on 30 October 2007. Retrieved 30 October 2007.
34. ^ Math Forum – Ask Dr. Math
35. ^ Eves 1992, p. 131
36. ^ Beckmann 1971, p. 66
37. ^ Eves 1992, p. 118
38. ^ *a* *b* Eves 1992, p. 119
39. ^ Beckmann 1971, pp. 94–95
40. ^ What Can You Do With a Supercomputer?
41. ^ Gardner, Martin (1995). "New Mathematical Diversions". Mathematical Association of America. p. 92.
42. ^ A nested radical approximation for π
43. ^ "Lost notebook page 16" ,Ramanujan
44. ^ Hoffman, D.W. *College Mathematics Journal*, **40** (2009) 399
45. ^ CetinHakimoglu–Brown
46. ^ "Sloane's A002485 : Numerators of convergents to Pi", *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
47. ^ "Sloane's A002486 : Denominators of convergents to Pi", *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
48. ^ Fractional Approximations of Pi accurate up to 37 digits
49. ^ MathWorld: BBP Formula Wolfram.com
50. ^ Simon Plouffe, *On the computation of the n'th decimal digit of various transcendental numbers*, November 1996; Revised December 2009
51. ^ Bellard's Website: Bellard.org
52. ^ y-cruncher
53. ^ "PiFast timings"
54. ^ Takahashi, Daisuke & Kanada, Yasumasa (10 August 2010). "Kanada Laboratory home page". University of Tokyo. Retrieved 1 May 2011.

References [edit]

- Bailey, David H., Borwein, Peter B., and Plouffe, Simon (April 1997). "On the Rapid Computation of Various Polylogarithmic Constants" (PDF). *Mathematics of Computation* **66** (218): 903–913. doi:10.1090/S0025-5718-97-00856-9.
- Beckmann, Petr (1971), *A History of π (pi)*, St. Martin's Press, ISBN 978-0-88029-418-8, MR 0449960
- Eves, Howard (1992), *An Introduction to the History of Mathematics* (6th ed.), Saunders College Publishing, ISBN 0-03-029558-0
- Joseph, George G. (2000). *The Crest of the Peacock: Non-European Roots of Mathematics* (New ed., London : Penguin ed.). London: Penguin. ISBN 0-14-027778-1.
- Jackson, K and Stamp, J. (2002). *Pyramid: Beyond Imagination. Inside the Great Pyramid of Giza*. London: BBC.

Categories: Approximations | History of mathematics | Pi | Real transcendental numbers