Bicubic interpolation

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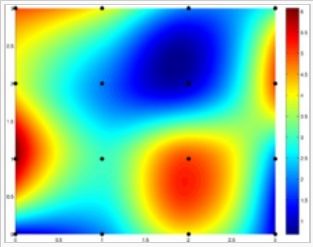
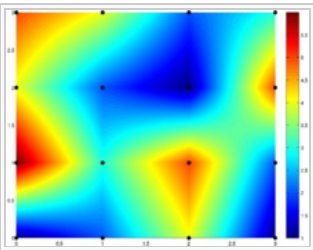
In mathematics, **bicubic interpolation** is an extension of cubic interpolation for interpolating data points on a two dimensional regular grid. The interpolated surface is smoother than corresponding surfaces obtained by bilinear interpolation or nearest-neighbor interpolation. Bicubic interpolation can be accomplished using either Lagrange polynomials, cubic splines, or cubic convolution algorithm.

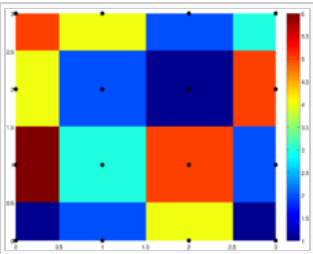
In image processing, bicubic interpolation is often chosen over bilinear interpolation or nearest neighbor in image resampling, when speed is not an issue. In contrast to bilinear interpolation, which only takes 4 pixels (2×2) into account, bicubic interpolation considers 16 pixels (4×4). Images resampled with bicubic interpolation are smoother and have fewer interpolation artifacts.



Bicubic interpolation on the square $[0,3] \times [0,3]$ consisting of 9 unit squares patched together. Bicubic interpolation as per MATLAB's implementation. Colour indicates function value. The black dots are the locations of the prescribed data being interpolated. Note how the color samples are not radially symmetric.



Bilinear interpolation on the same dataset as above. Derivatives of the surface are not continuous over the square boundaries.



Nearest-neighbor interpolation on the same dataset as above. Note that the information content in all these three examples is equivalent.

Suppose the function values f and the derivatives f_x, f_y and f_{xy} are known at the four corners $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ of the unit square. The interpolated surface can then be written

$$p(x,y) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij} x^i y^j.$$

The interpolation problem consists of determining the 16 coefficients a_{ij}. Matching $p(x,y)$ with the function values yields four equations,

1. $f(0,0) = p(0,0) = a_{00}$
2. $f(1,0) = p(1,0) = a_{00} + a_{10} + a_{20} + a_{30}$
3. $f(0,1) = p(0,1) = a_{00} + a_{01} + a_{02} + a_{03}$
4. $f(1,1) = p(1,1) = \sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}$

Likewise, eight equations for the derivatives in the x-direction and the y-direction

1. $f_x(0,0) = p_x(0,0) = a_{10}$
2. $f_x(1,0) = p_x(1,0) = a_{10} + 2a_{20} + 3a_{30}$
3. $f_x(0,1) = p_x(0,1) = a_{10} + a_{11} + a_{12} + a_{13}$
4. $f_x(1,1) = p_x(1,1) = \sum_{i=1}^{3}\sum_{j=0}^{3} a_{ij} i$
5. $f_y(0,0) = p_y(0,0) = a_{01}$
6. $f_y(1,0) = p_y(1,0) = a_{01} + a_{11} + a_{21} + a_{31}$
7. $f_y(0,1) = p_y(0,1) = a_{01} + 2a_{02} + 3a_{03}$
8. $f_y(1,1) = p_y(1,1) = \sum_{i=0}^{3}\sum_{j=1}^{3} a_{ij} j$

And four equations for the cross derivative xy.

1. $f_{xy}(0,0) = p_{xy}(0,0) = a_{11}$
2. $f_{xy}(1,0) = p_{xy}(1,0) = a_{11} + 2a_{21} + 3a_{31}$
3. $f_{xy}(0,1) = p_{xy}(0,1) = a_{11} + 2a_{12} + 3a_{13}$
4. $f_{xy}(1,1) = p_{xy}(1,1) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} ij$

where the expressions above have used the following identities,

$$p_x(x,y) = \sum_{i=1}^{3}\sum_{j=0}^{3} a_{ij} i x^{i-1} y^j$$
$$p_y(x,y) = \sum_{i=0}^{3}\sum_{j=1}^{3} a_{ij} x^i j y^{j-1}$$
$$p_{xy}(x,y) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij} i x^{i-1} j y^{j-1}.$$

This procedure yields a surface $p(x,y)$ on the unit square $[0,1] \times [0,1]$ which is continuous and with continuous derivatives. Bicubic interpolation on an arbitrarily sized regular grid can then be accomplished by patching together such bicubic surfaces, ensuring that the derivatives match on the boundaries.

Grouping the unknown parameters a_{ij} in a vector,

$$\alpha = \begin{bmatrix} a_{00} & a_{10} & a_{20} & a_{30} & a_{01} & a_{11} & a_{21} & a_{31} & a_{02} & a_{12} & a_{22} & a_{32} & a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix}^T$$

and letting

$$x = \begin{bmatrix} f(0,0) & f(1,0) & f(0,1) & f(1,1) & f_x(0,0) & f_x(1,0) & f_x(0,1) & f_x(1,1) & f_y(0,0) & f_y(1,0) & f_y(0,1) & f_y(1,1) & f_{xy}(0,0) & f_{xy}(1,0) & f_{xy}(0,1) & f_{xy}(1,1) \end{bmatrix}^T$$

,

the above system of equations can be reformulated into a matrix for the linear equation $A\alpha = x$.

Inverting the matrix gives the more useful linear equation $A^{-1}x = \alpha$ which allows α to be calculated quickly and easily, where:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 & 1 & 1 & 0 & 0 \\ -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 0 \\ 9 & -9 & -9 & 9 & 6 & 3 & -6 & -3 & 6 & -6 & 3 & -3 & 4 & 2 & 2 & 1 \\ -6 & 6 & 6 & -6 & -3 & -3 & 3 & 3 & -4 & 4 & -2 & 2 & -2 & -2 & -1 & -1 \\ 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ -6 & 6 & 6 & -6 & -4 & -2 & 4 & 2 & -3 & 3 & -3 & 3 & -2 & -1 & -2 & -1 \\ 4 & -4 & -4 & 4 & 2 & 2 & -2 & -2 & 2 & -2 & 2 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

There can be another concise matrix form for 16 coefficients

$$\begin{bmatrix} f(0,0) & f(0,1) & f_y(0,0) & f_y(0,1) \\ f(1,0) & f(1,1) & f_y(1,0) & f_y(1,1) \\ f_x(0,0) & f_x(0,1) & f_{xy}(0,0) & f_{xy}(0,1) \\ f_x(1,0) & f_x(1,1) & f_{xy}(1,0) & f_{xy}(1,1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \end{bmatrix},$$ or

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} f(0,0) & f(0,1) & f_y(0,0) & f_y(0,1) \\ f(1,0) & f(1,1) & f_y(1,0) & f_y(1,1) \\ f_x(0,0) & f_x(0,1) & f_{xy}(0,0) & f_{xy}(0,1) \\ f_x(1,0) & f_x(1,1) & f_{xy}(1,0) & f_{xy}(1,1) \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

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Finding derivatives from function values [edit]

If the derivatives are unknown, they are typically approximated from the function values at points neighbouring the corners of the unit square, e.g. using finite differences.

To find either of the single derivatives, f_x or f_y, using that method, find the slope between the two *surrounding* points in the appropriate axis. For example, to calculate f_x for one of the points, find $f(x, y)$ for the points to the left and right of the target point and calculate their slope, and similarly for f_y.

To find the cross derivative, f_{xy}, take the derivative in both axes, one at a time. For example, one can first use the f_x procedure to find the x derivatives of the points above and below the target point, then use the f_y procedure on those values (rather than, as usual, the values of f for those points) to obtain the value of $f_{xy}(x, y)$ for the target point. (Or one can do it in the opposite direction, first calculating f_y and then f_x off of those. The two give equivalent results.)

At the edges of the dataset, when one is missing some of the surrounding points, the missing points can be approximated by a number of methods. A simple and common method is to assume that the slope from the existing point to the target point continues without further change, and using this to calculate a hypothetical value for the missing point.

Bicubic convolution algorithm [edit]

Bicubic spline interpolation requires the solution of the linear system described above for each grid cell. An interpolator with similar properties can be obtained by applying a convolution with the following kernel in both dimensions:

$$W(x) = \begin{cases} (a+2)|x|^3 - (a+3)|x|^2 + 1 & \text{for } |x| \le 1 \\ a|x|^3 - 5a|x|^2 + 8a|x| - 4a & \text{for } 1 < |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$

where a is usually set to -0.5 or -0.75. Note that $W(0) = 1$ and $W(n) = 0$ for all nonzero integers n.

This approach was proposed by Keys who showed that $a = -0.5$ (which corresponds to cubic Hermite spline) produces third-order convergence with respect to the original function.[1]

If we use the matrix notation for the common case $a = -0.5$, we can express the equation in a more friendly manner:

$$p(t) = \tfrac{1}{2}\begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 2 & -5 & 4 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

for t between 0 and 1 for one dimension. for two dimensions first applied once in x and again in y:

$$b_{-1} = p(t_x, a_{(-1,-1)}, a_{(0,-1)}, a_{(1,-1)}, a_{(2,-1)})$$
$$b_0 = p(t_x, a_{(-1,0)}, a_{(0,0)}, a_{(1,0)}, a_{(2,0)})$$
$$b_1 = p(t_x, a_{(-1,1)}, a_{(0,1)}, a_{(1,1)}, a_{(2,1)})$$

$$b_2 = p(t_x, a_{(-1,2)}, a_{(0,2)}, a_{(1,2)}, a_{(2,2)})$$
$$p(x, y) = p(t_y, b_{-1}, b_0, b_1, b_2)$$

Use in computer graphics [edit]

The bicubic algorithm is frequently used for scaling images and video for display (see bitmap resampling). It preserves fine detail better than the common bilinear algorithm.

However, due to the negative lobes on the kernel, it causes overshoot (haloing). This can cause clipping, and is an artifact (see also ringing artifacts), but it increases acutance (apparent sharpness), and can be desirable.

Bicubic interpolation causes overshoot, which increases acutance.

See also [edit]

- Spatial anti-aliasing
- Bézier surface
- Bilinear interpolation
- Cubic Hermite spline, the one-dimensional analogue of bicubic spline
- Lanczos resampling
- Natural neighbor interpolation
- Sinc filter
- Spline interpolation
- Tricubic interpolation

Mathematics portal

Computer graphics portal

References [edit]

1. ^ R. Keys, (1981). "Cubic convolution interpolation for digital image processing". *IEEE Transactions on Acoustics, Speech, and Signal Processing* **29** (6): 1153–1160. doi:10.1109/TASSP.1981.1163711.

External links [edit]

- Application of interpolation to elevation samples
- Interpolation theory
- Explanation and Java/C++ implementation of (bi)cubic interpolation
- Excel Worksheet Function for Bicubic Lagrange Interpolation

Categories: Image processing | Multivariate interpolation