Dynamic Programming | Set 9 (Binomial Coefficient)

Following are common definition of Binomial Coefficients.
1) A binomial coefficient C(n, k) can be defined as the coefficient of X^k in the expansion of (1 + X)^n.

2) A binomial coefficient C(n, k) also gives the number of ways, disregarding order, that k objects can be chosen from among n objects; more formally, the number of k-element subsets (or k-combinations) of an n-element set.

The Problem
Write a function that takes two parameters n and k and returns the value of Binomial Coefficient C(n, k). For example, your function should return 6 for n = 4 and k = 2, and it should return 10 for n = 5 and k = 2.

1) Optimal Substructure
The value of C(n, k) can recursively calculated using following standard formula for Binomial Cofficients.

```
   C(n, k) = C(n-1, k-1) + C(n-1, k)
   C(n, 0) = C(n, n) = 1
```

2) Overlapping Subproblems
Following is simple recursive implementation that simply follows the recursive structure mentioned above.

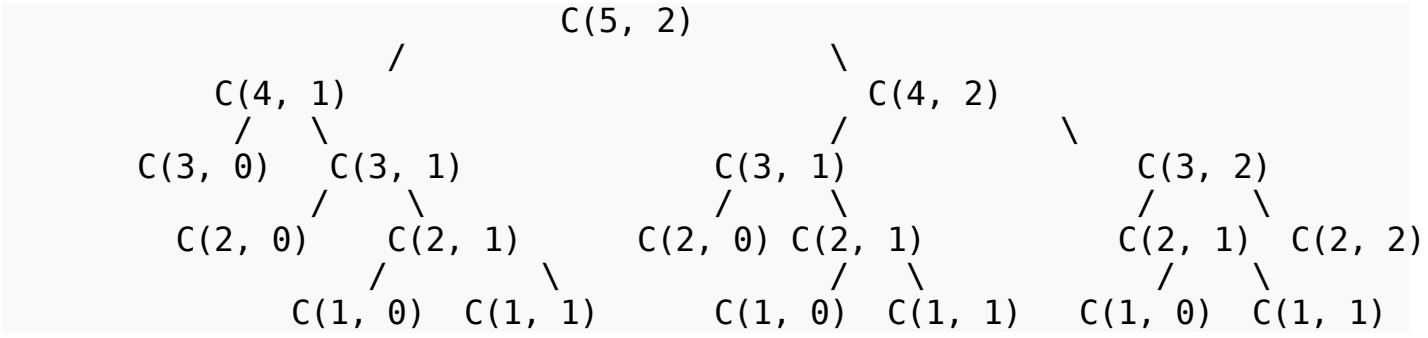
```
// A Naive Recursive Implementation
#include<stdio.h>

// Returns value of Binomial Coefficient C(n, k)
int binomialCoeff(int n, int k)
{
  // Base Cases
  if (k==0 || k==n)
    return 1;

  // Recur
  return  binomialCoeff(n-1, k-1) + binomialCoeff(n-1, k);
}

/* Drier program to test above function*/
int main()
{
    int n = 5, k = 2;
    printf("Value of C(%d, %d) is %d ", n, k, binomialCoeff(n, k));
    return 0;
}
```

It should be noted that the above function computes the same subproblems again and again. See the following recursion tree for n = 5 an k = 2. The function C(3, 1) is called two times. For large values of n, there will be many common subproblems.



Since same suproblems are called again, this problem has Overlapping Subprolems property. So the Binomial Coefficient problem has both properties (see this and this) of a dynamic programming problem. Like other typical Dynamic Programming(DP) problems, recomputations of same subproblems can be avoided by constructing a temporary array C[][] in bottom up manner. Following is Dynamic Programming based implementation.

```
// A Dynamic Programming based solution that uses table C[][] to calculate the
```

```
// Binomial Coefficient
#include<stdio.h>

// Prototype of a utility function that returns minimum of two integers
int min(int a, int b);

// Returns value of Binomial Coefficient C(n, k)
int binomialCoeff(int n, int k)
{
    int C[n+1][k+1];
    int i, j;

    // Caculate value of Binomial Coefficient in bottom up manner
    for (i = 0; i <= n; i++)
    {
        for (j = 0; j <= min(i, k); j++)
        {
            // Base Cases
            if (j == 0 || j == i)
                C[i][j] = 1;

            // Calculate value using previosly stored values
            else
                C[i][j] = C[i-1][j-1] + C[i-1][j];
        }
    }

    return C[n][k];
}

// A utility function to return minimum of two integers
int min(int a, int b)
{
    return (a<b)? a: b;
}

/* Drier program to test above function*/
int main()
{
    int n = 5, k = 2;
    printf ("Value of C(%d, %d) is %d ", n, k, binomialCoeff(n, k) );
    return 0;
}
```

Time Complexity: O(n*k)
Auxiliary Space: O(n*k)

Following is a space optimized version of the above code. The following code only uses O(k). Thanks to AK for suggesting this method.

```
// A space optimized Dynamic Programming Solution
int binomialCoeff(int n, int k)
{
    int* C = (int*)calloc(k+1, sizeof(int));
    int i, j, res;

    C[0] = 1;

    for(i = 1; i <= n; i++)
    {
        for(j = min(i, k); j > 0; j--)
            C[j] = C[j] + C[j-1];
    }

    res = C[k];  // Store the result before freeing memory

    free(C);  // free dynamically allocated memory to avoid memory leak

    return res;
}
```

Time Complexity: O(n*k)
Auxiliary Space: O(k)

References:
http://www.csl.mtu.edu/cs4321/www/Lectures/Lecture%2015%20-%20Dynamic%20Programming%20Binomial%20Coefficients.htm