When is *n* a member of a Pythagorean Triple?

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Abstract

Pythagoras' Theorem is perhaps the best known result in the whole of mathematics and yet many things remain unknown (or perhaps just "unstudied") about the consequences of this 'simple' Theorem. In this article we investigate which numbers can be part of triples such as {3, 4, 5} and {5, 12, 13} - right angled triangles with integer sides.

Perhaps the most interesting results concern instances where a number is an element of a triple where none of the members have a common factor. In this case any odd number and any multiple of 4 is the shortest side of some such triangle while an odd number is the middle side *iff* it can be written as xy (x and y co-prime) where $y < x < (1 + \sqrt{2})y$ (x and y are "close"). Similarly, an even number is the middle side *iff* it can be written as $2xy$ where x and y are close, co-prime and of opposite parity. On the other hand a number can be the hypotenuse of an integer right-angled triangle *iff* all of its prime factors are of the form $4k + 1$.

Notation

A triple of positive integers {x, y, z} is said to be a *Pythagorean Triple* (PT) if it satisfies the Pythagorean equation: $x^2 + y^2 = z^2$. A PT will often be written as the triple {s, m, h} ({short, middle, hypotenuse}) so that it is easy to identify which side we are talking about.

If x, y, z have no common factor (i.e. if any pair is co-prime) then this triple is said to be a *Primitive Pythagorean Triple* (PPT), otherwise we have a *Non-primitive Pythagorean Triple* (NPPT).

It is a relatively simple matter to show that all PPTs can be generated by co-prime integers a and b ((a, b) = 1) with opposite parity by letting:

$$x = a^2 - b^2, y = 2ab, z = a^2 + b^2$$

This is often known as Newton's method for generating PPTs and a proof that it generates all such triples can be found in Hardy and Wright's book (Theorem 225 of [1, p. 190]). It follows from Newton's method that all NPPTs can be generated by multiplying these equations by another factor d.

Note: We may use x and y instead of a and b in the remainder of this article, although we hope that the context will make it clear whether these refer to sides of a triangle or the numbers which generate a PT.

1 A Previous Result

In a previous paper [2], we showed that necessary and sufficient conditions for a number to be the shortest side of a PT are as given in Theorem 1.1. For completeness we state this Theorem here, though a proof can be found in that article.

1.1 Theorem

If s = 1, 2 or 4 there is no PT with s as its shortest side. Otherwise if s odd or if 4 | s there is a PPT with s as its shortest side, otherwise there is an NPPT (with common factor d = 2) having s as its shortest side.

2 The Middle Side

Determining which numbers can be the middle side of a PPT is not as clear-cut as which numbers can be the shortest side. To obtain a useful result, we consider even and odd numbers separately though finally it becomes apparent that both cases are reasonably similar.

2.1 Theorem

Let m be odd. Then m is the middle side of a PPT if and only if m = xy where:

$$(*) \qquad y < x < (1 + \sqrt{2})y, \qquad x, y \text{ co-prime}$$

Proof: If m is the middle side of the PPT $\{s, m, h\}$, we can write: $m^2 = (h - s)(h + s)$.

But $h - s$ and $h + s$ must be co-prime since the triple is primitive. From this, they must both be squares and so we write:

$h + s = x^2$, $h - s = y^2$, where x and y are clearly co-prime and odd.

These equations easily yield $m = xy$, $2s = x^2 - y^2$ and so for m to be the middle side we have:

$$x^2 - y^2 < 2xy \Rightarrow (x - y)^2 < 2y^2 \Rightarrow x < (1 + \sqrt{2})y$$

Including the obvious fact that $y < x$ we have the desired inequality.

Conversely, we consider $m = xy$, where x and y are odd and co-prime, satisfying the inequality (*). Now, $x < (1 + \sqrt{2})y \Rightarrow x^2 - y^2 < 2xy = 2m$ so that m is the middle side of the triple:

$$\left\{ xy, \frac{x^2 - y^2}{2}, \frac{x^2 + y^2}{2} \right\}$$

It is a trivial matter to check that this is a PPT since *x* and *y* are both odd and have no common factor. This proves the Theorem.

2.2 Theorem

For m even, m is the middle side of a PPT iff m = 2xy where x and y are co-prime, of opposite parity and satisfy:

$$(\dagger) \qquad y < x < (1 + \sqrt{2})y$$

Proof: If *m* is the even side of a PPT, it can be written as $2xy$ for some co-prime *x* and *y* of opposite parity, where the whole triple can then be written as $\{x^2 - y^2, 2xy, x^2 + y^2\}$. If in addition *m* is to be the middle side of the PPT, we must have:

$$x^2 - y^2 < 2xy \Rightarrow x < (1 + \sqrt{2})y \text{ as before.}$$

From this (†) becomes evident.

Conversely, if $m = 2xy$ satisfying (†) for appropriate *x* and *y*, then $x^2 - y^2 < 2xy$. Since the triple $\{x^2 - y^2, 2xy, x^2 + y^2\}$ is a PPT it follows that *m* is the middle side of this PPT.

3 The Hypotenuse

3.1 Lemma

If p = 4k + 1, a prime, then p is the hypotenuse of a PPT.

Proof: Theorem 366 of Hardy and Wright [1, p. 299] gives the following condition for a number to be the sum of two squares:

"A number n is the sum of two squares iff all prime factors of the form $4k + 3$ have an even exponent in the prime factorisation of n."

From this, it follows that $p = 4k + 1 = x^2 + y^2$, say. *x* and *y* are clearly of opposite parity since *p* is odd and so we only need to show that $(x, y) = 1$.

If instead $(x, y) = m$, say then we would have $m^2 \mid p$, which can only happen if $m = 1$. Thus *p* is the hypotenuse in the PPT $\{x^2 - y^2, 2xy, p\}$.

3.2 Lemma

If h, k are the co-prime hypotenuses of PPTs, then hk is the hypotenuse of another PPT.

Proof: Let $h = a^2 + b^2$, $k = c^2 + d^2$

with $(a, b) = (c, d) = 1$ and take wlog *a*, *c* even, *b*, *d* odd.

We can write:

$$hk = (ac \pm bd)^2 + (ad \ \mathbf{m} \ bc)^2$$

The pairs are clearly of opposite parity and so it remains to check that they are also co-prime.

Suppose $p \mid (ac \pm bd)$ and $p \mid (ad \ \mathbf{m} \ bc)$

Multiplying the first of these by c, the second by d and adding/subtracting we have that $p \mid a(c^2 + d^2)$.

In a similar manner we obtain: $p \mid b(c^2 + d^2)$, $c(a^2 + b^2)$ and $d(a^2 + b^2)$

However, since $(a, b) = (c, d) = 1$, we must have that $p \mid h, k \Rightarrow p = 1$ and so:

$$(ac \pm bd, ad \ \mathbf{m} \ bc) = 1, \text{ as desired.}$$

Note: If $ad < bc$ (or $ac < bd$) then simply multiply the 'negative' bracket by - 1, and we have a pair of numbers that generate a PPT with hk as hypotenuse.

3.3 Lemma

If p is a prime of the form $4k + 1$ then p^n is the hypotenuse of a PPT for all integers $n \geq 1$.

Proof: We prove this result by induction on n. By Lemma 3.1 we know that p is the hypotenuse of some PPT and so it is expressible as the sum of two squares:

$p = a^2 + b^2$, with $(a, b) = 1$ and a, b of opposite parity (wlog take a even, b odd).

Thus the result is true for $n = 1$. Assume it is also true up to n so that $p^n = c^2 + d^2$, where c and d satisfy the usual conditions (and c is even, d odd).

Clearly: $p^{n+1} = (ac \pm bd)^2 + (ad \ \mathbf{m} \ bc)^2$

By the same reasoning as before, we have that:

$q \mid a(c^2 + d^2)$, $b(c^2 + d^2)$, $c(a^2 + b^2)$ and $d(a^2 + b^2)$

where q is a common prime factor within the pairs. From the first two of these results, q is either a factor of a and b or of $c^2 + d^2$. The first of these isn't possible since $(a, b) = 1$ and so $q \mid p^n$. Similarly, $q \mid p$ and so either $q = 1$ (in which case there is no problem) or $q = p$.

If both possible pairs have p as a common factor then $p \mid ac$, ad, bc and bd so that $(a, b) \neq 1$ or $(c, d) \neq 1$. Thus at least one of the pairs $(ac \pm bd, ad \ \mathbf{m} \ bc)$ is co-prime and since the members of that pair are trivially of opposite parity we have that p^{n+1} is also the hypotenuse of a PPT as required. Thus the Lemma is true by induction.

3.4 Theorem

n is the hypotenuse of a PPT iff all of its prime factors are of the form $4k + 1$.

Proof: If *n* is composed entirely of primes of the form $4k + 1$ then it is also the hypotenuse of a PPT as a Corollary of the last two Lemmas.

Conversely, if *n* is the hypotenuse of a PPT then it can be expressed as the sum of two co-prime squares (of opposite parity). Theorem 13 of Hardy and Wright [1, p. 13] states that:

"If a and b have no common factor, then any odd prime divisor of $a^2 + b^2$ is of the form $4n + 1$."

Also, since *a* and *b* are of opposite parity $a^2 + b^2$ is odd.

From this Theorem we also have the following trivial Corollary concerning NPPTs.

3.5 Corollary

n is the hypotenuse of an NPPT iff it has a prime factor of the form $4n + 1$

4 References

[1] Hardy, G H and Wright, E M. "An Introduction to the Theory of Numbers", Oxford University Press, 1960 (Fourth edition)

[2] Vella, A D and Vella, D J R "More interesting properties of Pythagorean Triples", *Math. Gaz.*.**85** (July 2001) pp. 275-278

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