



WIKIPEDIA  
The Free Encyclopedia

Main page  
Contents  
Featured content  
Current events  
Random article  
Donate to Wikipedia  
Wikipedia store

Interaction

Help  
About Wikipedia  
Community portal  
Recent changes  
Contact page

Tools

What links here  
Related changes  
Upload file  
Special pages  
Permanent link  
Page information  
Wikidata item  
Cite this page

Print/export

Create a book  
Download as PDF  
Printable version

Languages

العربية  
Català  
Deutsch  
Italiano  
Magyar  
Polski  
Русский  
Српски / srpski  
中文

Edit links

Create account Log in

Article Talk

Read

Edit

View history

Search

# Spline interpolation

From Wikipedia, the free encyclopedia

*For a broader coverage related to this topic, see [Spline \(mathematics\)](#).*

In the [mathematical](#) field of [numerical analysis](#), **spline interpolation** is a form of [interpolation](#) where the interpolant is a special type of [piecewise polynomial](#) called a [spline](#). Spline interpolation is often preferred over [polynomial interpolation](#) because the [interpolation error](#) can be made small even when using low degree polynomials for the spline. Spline interpolation avoids the problem of [Runge's phenomenon](#), in which oscillation can occur between points when interpolating using high degree polynomials.

## Contents

- Introduction
- Algorithm to find the interpolating cubic spline
- Example
- See also
- External links

## Introduction

Originally, *spline* was a term for [elastic rulers](#) that were bent to pass through a number of predefined points ("knots"). These were used to make [technical drawings](#) for [shipbuilding](#) and construction by hand, as illustrated by Figure 1.

The approach to mathematically model the shape of such elastic rulers fixed by  $n + 1$  knots

$\{(x_i, y_i) : i = 0, 1, \dots, n\}$  is to interpolate between all the pairs of knots  $(x_{i-1}, y_{i-1})$  and  $(x_i, y_i)$  with polynomials  $y = q_i(x), i = 1, 2, \dots, n$ .

The [curvature](#) of a curve  $y = f(x)$  is given by:

$$\kappa = \frac{y''}{(1 + y'^2)^{3/2}}$$

As the spline will take a shape that minimizes the bending (under the constraint of passing through all knots) both  $y'$  and  $y''$  will be continuous everywhere and at the knots. To achieve this one must have that

$$\begin{cases} q'_i(x_i) = q'_{i+1}(x_i) \\ q''_i(x_i) = q''_{i+1}(x_i) \end{cases} \quad 1 \leq i \leq n - 1$$

This can only be achieved if polynomials of degree 3 or higher are used. The classical approach is to use polynomials of degree 3 — the case of [cubic splines](#).

## Algorithm to find the interpolating cubic spline

A third order polynomial  $q(x)$  for which

$$\begin{aligned} q(x_1) &= y_1 \\ q(x_2) &= y_2 \\ q'(x_1) &= k_1 \\ q'(x_2) &= k_2 \end{aligned}$$

can be written in the symmetrical form

$$q = (1 - t)y_1 + ty_2 + t(1 - t)(a(1 - t) + bt) \tag{1}$$

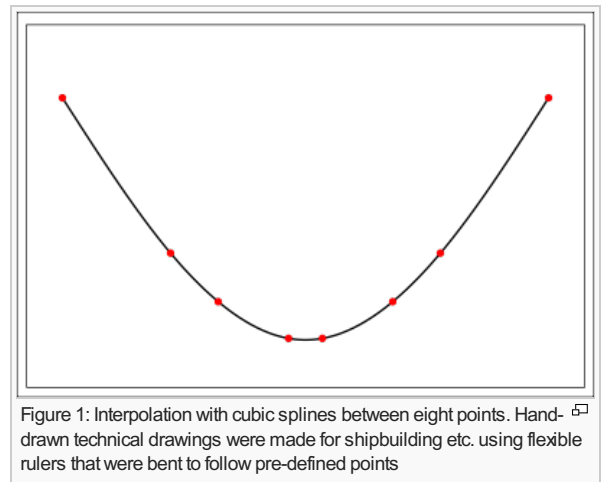
where

$$t = \frac{x - x_1}{x_2 - x_1}, \tag{2}$$

$$a = k_1(x_2 - x_1) - (y_2 - y_1), \tag{3}$$

$$b = -k_2(x_2 - x_1) + (y_2 - y_1). \tag{4}$$

As



$$q' = \frac{dq}{dx} = \frac{dq}{dt} \frac{dt}{dx} = \frac{dq}{dt} \frac{1}{x_2 - x_1}$$

one gets that:

$$q' = \frac{y_2 - y_1}{x_2 - x_1} + (1 - 2t) \frac{a(1 - t) + bt}{x_2 - x_1} + t(1 - t) \frac{b - a}{x_2 - x_1}, \quad (5)$$

$$q'' = 2 \frac{b - 2a + (a - b)3t}{(x_2 - x_1)^2}. \quad (6)$$

Setting  $x = x_1$  and  $x = x_2$  respectively in equations (5) and (6) one gets from (2) that indeed first derivatives  $q'(x_1) = k_1$  and  $q'(x_2) = k_2$  and also second derivatives

$$q''(x_1) = 2 \frac{b - 2a}{(x_2 - x_1)^2} \quad (7)$$

$$q''(x_2) = 2 \frac{a - 2b}{(x_2 - x_1)^2} \quad (8)$$

If now  $(x_i, y_i)$ ,  $i = 0, 1, \dots, n$  are  $n + 1$  points and

$$q_i = (1 - t)y_{i-1} + ty_i + t(1 - t)(a_i(1 - t) + b_it) \quad (9)$$

where  $i = 1, 2, \dots, n$  and  $t = \frac{x - x_{i-1}}{x_i - x_{i-1}}$  are  $n$  third degree polynomials interpolating  $y$  in the interval  $x_{i-1} \leq x \leq x_i$  for  $i = 1, \dots, n$  such that  $q'_i(x_i) = q'_{i+1}(x_i)$  for  $i = 1, \dots, n-1$  then the  $n$  polynomials together define a differentiable function in the interval  $x_0 \leq x \leq x_n$  and

$$a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1}) \quad (10)$$

$$b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1}) \quad (11)$$

for  $i = 1, \dots, n$  where

$$k_0 = q'_1(x_0) \quad (12)$$

$$k_i = q'_i(x_i) = q'_{i+1}(x_i) \quad i = 1, \dots, n-1 \quad (13)$$

$$k_n = q'_n(x_n) \quad (14)$$

If the sequence  $k_0, k_1, \dots, k_n$  is such that in addition  $q''_i(x_i) = q''_{i+1}(x_i)$  holds for  $i = 1, \dots, n-1$ . Then the resulting function will even have a continuous second derivative.

From (7), (8), (10) and (11) follows that this is the case if and only if

$$\frac{k_{i-1}}{x_i - x_{i-1}} + \left( \frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i} \right) 2k_i + \frac{k_{i+1}}{x_{i+1} - x_i} = 3 \left( \frac{y_i - y_{i-1}}{(x_i - x_{i-1})^2} + \frac{y_{i+1} - y_i}{(x_{i+1} - x_i)^2} \right) \quad (15)$$

for  $i = 1, \dots, n-1$ . The relations (15) are  $n - 1$  linear equations for the  $n + 1$  values  $k_0, k_1, \dots, k_n$ .

For the elastic rulers being the model for the spline interpolation one has that to the left of the left-most "knot" and to the right of the right-most "knot" the ruler can move freely and will therefore take the form of a straight line with  $q'' = 0$ . As  $q''$  should be a continuous function of  $x$  one gets that for "Natural Splines" one in addition to the  $n - 1$  linear equations (15) should have that

$$q''_1(x_0) = 2 \frac{3(y_1 - y_0) - (k_1 + 2k_0)(x_1 - x_0)}{(x_1 - x_0)^2} = 0,$$

$$q''_n(x_n) = -2 \frac{3(y_n - y_{n-1}) - (2k_n + k_{n-1})(x_n - x_{n-1})}{(x_n - x_{n-1})^2} = 0,$$

i.e. that

$$\frac{2}{x_1 - x_0} k_0 + \frac{1}{x_1 - x_0} k_1 = 3 \frac{y_1 - y_0}{(x_1 - x_0)^2}, \quad (16)$$

$$\frac{1}{x_n - x_{n-1}} k_{n-1} + \frac{2}{x_n - x_{n-1}} k_n = 3 \frac{y_n - y_{n-1}}{(x_n - x_{n-1})^2}. \quad (17)$$

Eventually, (15) together with (16) and (17) constitute  $n + 1$  linear equations that uniquely define the  $n + 1$  parameters  $k_0, k_1, \dots, k_n$ .

There exists other end conditions: "Clamped spline", that specifies the slope at the ends of the spline, and the popular "not-a-knot spline", that requires that the third derivative is also continuous at the  $x_1$  and  $x_{N-1}$  points. For the "not-a-knot" spline, the additional equations will read:

$$q'''_1(x_1) = q'''_2(x_1) \Rightarrow \frac{1}{\Delta x_1^2} k_0 + \left( \frac{1}{\Delta x_1^2} - \frac{1}{\Delta x_2^2} \right) k_1 - \frac{1}{\Delta x_2^2} k_2 = 2 \left( \frac{\Delta y_1}{\Delta x_1^3} - \frac{\Delta y_2}{\Delta x_2^3} \right)$$

$$q'''_{n-1}(x_{n-1}) = q'''_n(x_{n-1}) \Rightarrow \frac{1}{\Delta x_{n-1}^2} k_{n-2} + \left( \frac{1}{\Delta x_{n-1}^2} - \frac{1}{\Delta x_n^2} \right) k_{n-1} - \frac{1}{\Delta x_n^2} k_n = 2 \left( \frac{\Delta y_{n-1}}{\Delta x_{n-1}^3} - \frac{\Delta y_n}{\Delta x_n^3} \right)$$

where  $\Delta x_i = x_i - x_{i-1}$ ,  $\Delta y_i = y_i - y_{i-1}$

## Example [\[edit\]](#)

In case of three points the values for  $k_0$ ,  $k_1$ ,  $k_2$  are found by solving the [tridiagonal linear equation system](#)

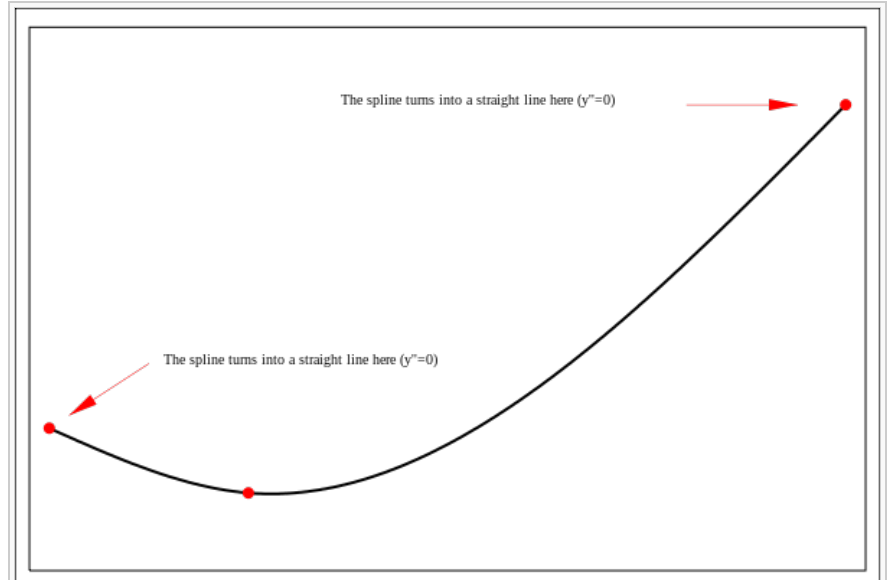


Figure 2: Interpolation with cubic "natural" splines between three points.

$$\begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

with

$$\begin{aligned} a_{11} &= \frac{2}{x_1 - x_0} \\ a_{12} &= \frac{1}{x_1 - x_0} \\ a_{21} &= \frac{1}{x_1 - x_0} \\ a_{22} &= 2 \left( \frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} \right) \\ a_{23} &= \frac{1}{x_2 - x_1} \\ a_{32} &= \frac{1}{x_2 - x_1} \\ a_{33} &= \frac{2}{x_2 - x_1} \\ b_1 &= 3 \frac{y_1 - y_0}{(x_1 - x_0)^2} \\ b_2 &= 3 \left( \frac{y_1 - y_0}{(x_1 - x_0)^2} + \frac{y_2 - y_1}{(x_2 - x_1)^2} \right) \\ b_3 &= 3 \frac{y_2 - y_1}{(x_2 - x_1)^2} \end{aligned}$$

For the three points

$$(-1, 0.5), (0, 0), (3, 3)$$

one gets that

$$k_0 = -0.6875, k_1 = -0.1250, k_2 = 1.5625$$

and from (10) and (11) that

$$\begin{aligned} a_1 &= k_0(x_1 - x_0) - (y_1 - y_0) = -0.1875 \\ b_1 &= -k_1(x_1 - x_0) + (y_1 - y_0) = -0.3750 \\ a_2 &= k_1(x_2 - x_1) - (y_2 - y_1) = -3.3750 \\ b_2 &= -k_2(x_2 - x_1) + (y_2 - y_1) = -1.6875 \end{aligned}$$

In Figure 2 the spline function consisting of the two cubic polynomials  $q_1(x)$  and  $q_2(x)$  given by (9) is displayed

## See also [\[edit\]](#)

- Cubic Hermite spline
- Centripetal Catmull–Rom spline

- [Discrete spline interpolation](#)
- [Monotone cubic interpolation](#)
- [NURBS](#)
- [Multivariate interpolation](#)
- [Polynomial interpolation](#)
- [Smoothing spline](#)
- [Spline wavelet](#)

## External links [\[edit\]](#)

- Hazewinkel, Michiel, ed. (2001), "Spline interpolation" [↗](#), *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- [Dynamic cubic splines with JSXGraph](#) [↗](#)
- [Lectures on the theory and practice of spline interpolation](#) [↗](#)
- [Paper which explains step by step how cubic spline interpolation is done, but only for equidistant knots.](#) [↗](#)
- [Online Cubic Spline Interpolation Utility](#) [↗](#)
- [Numerical Recipes in C](#) [↗](#), [Go to Chapter 3 Section 3-3.](#)
- [A note on cubic splines](#) [↗](#)

Categories: [Splines](#) | [Interpolation](#)

This page was last modified on 20 June 2015, at 04:35.

Text is available under the [Creative Commons Attribution-ShareAlike License](#); additional terms may apply. By using this site, you agree to the [Terms of Use](#) and [Privacy Policy](#). Wikipedia® is a registered trademark of the [Wikimedia Foundation, Inc.](#), a non-profit organization.

[Privacy policy](#) [About Wikipedia](#) [Disclaimers](#) [Contact Wikipedia](#) [Developers](#) [Mobile view](#)

