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Prüfer sequence

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In [combinatorial mathematics](#), the **Prüfer sequence** (also **Prüfer code** or **Prüfer numbers**) of a [labeled tree](#) is a unique [sequence](#) associated with the tree. The sequence for a tree on n vertices has length $n - 2$, and can be generated by a simple iterative algorithm. Prüfer sequences were first used by [Heinz Prüfer](#) to prove [Cayley's formula](#) in 1918.^[1]

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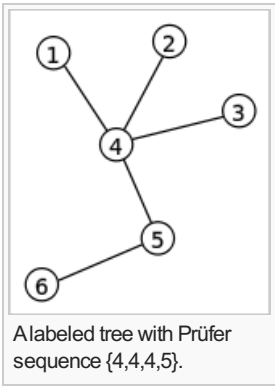
Algorithm to convert a tree into a Prüfer sequence [\[edit\]](#)

One can generate a labeled tree's Prüfer sequence by iteratively removing vertices from the tree until only two vertices remain. Specifically, consider a labeled tree T with vertices $\{1, 2, \dots, n\}$. At step i , remove the leaf with the smallest label and set the i th element of the Prüfer sequence to be the label of this leaf's neighbour.

The Prüfer sequence of a labeled tree is unique and has length $n - 2$.

Example [\[edit\]](#)

Consider the above algorithm run on the tree shown to the right. Initially, vertex 1 is the leaf with the smallest label, so it is removed first and 4 is put in the Prüfer sequence. Vertices 2 and 3 are removed next, so 4 is added twice more. Vertex 4 is now a leaf and has the smallest label, so it is removed and we append 5 to the sequence. We are left with only two vertices, so we stop. The tree's sequence is $\{4, 4, 4, 5\}$.



Algorithm to convert a Prüfer sequence into a tree [\[edit\]](#)

Let $\{a[1], a[2], \dots, a[n]\}$ be a Prüfer sequence:

The tree will have $n+2$ nodes, numbered from 1 to $n+2$. For each node set its degree to the number of times it appears in the sequence plus 1. For instance, in pseudo-code:

```
Convert-Prüfer-to-Tree( $a$ )
1  $n \leftarrow \text{length}[a]$ 
2  $T \leftarrow$  a graph with  $n + 2$  isolated nodes, numbered 1 to  $n + 2$ 
3  $\text{degree} \leftarrow$  an array of integers
4 for each node  $i$  in  $T$ 
5   do  $\text{degree}[i] \leftarrow 1$ 
6 for each value  $i$  in  $a$ 
7   do  $\text{degree}[i] \leftarrow \text{degree}[i] + 1$ 
```

Next, for each number in the sequence $a[i]$, find the first (lowest-numbered) node, j , with degree equal to 1, add the edge $(j, a[i])$ to the tree, and decrement the degrees of j and $a[i]$. In pseudo-code:

```
8 for each value  $i$  in  $a$ 
9   for each node  $j$  in  $T$ 
10    if  $\text{degree}[j] = 1$ 
11    then Insert  $\text{edge}[i, j]$  into  $T$ 
```

```

12         degree[i] ← degree[i] - 1
13         degree[j] ← degree[j] - 1
14         break

```

At the end of this loop two nodes with degree 1 will remain (call them u , v). Lastly, add the edge (u, v) to the tree.^[2]

```

14 u ← v ← 0
15 for each node i in T
16     if degree[i] = 1
17         then if u = 0
18             then u ← i
19             else v ← i
20         break
21 Insert edge[u, v] into T
22 degree[u] ← degree[u] - 1
23 degree[v] ← degree[v] - 1
24 return T

```

Cayley's formula ^[edit]

The Prüfer sequence of a labeled tree on n vertices is a unique sequence of length $n - 2$ on the labels 1 to n — this much is clear. Somewhat less obvious is the fact that for a given sequence S of length $n-2$ on the labels 1 to n , **there is a unique labeled tree whose Prüfer sequence is S .**

The immediate consequence is that Prüfer sequences provide a [bijection](#) between the set of labeled trees on n vertices and the set of sequences of length $n-2$ on the labels 1 to n . The latter set has size n^{n-2} , so the existence of this bijection proves [Cayley's formula](#), i.e. that there are n^{n-2} labeled trees on n vertices.

Other applications^[3] ^[edit]

- Cayley's formula can be strengthened to prove the following claim:



The number of spanning trees in a complete graph K_n with a degree d_i specified for each vertex i is equal to the [multinomial coefficient](#)

$$\binom{n-2}{d_1-1, d_2-1, \dots, d_n-1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)! \cdots (d_n-1)!}.$$


The proof follows by observing that in the Prüfer sequence number i appears exactly $(d_i - 1)$ times.

- Cayley's formula can be generalized: a labeled tree is in fact a [spanning tree](#) of the labeled [complete graph](#). By placing restrictions on the enumerated Prüfer sequences, similar methods can give the number of spanning trees of a complete [bipartite graph](#). If G is the complete bipartite graph with vertices 1 to n_1 in one partition and vertices $n_1 + 1$ to n in the other partition, the number of labeled spanning trees of G is $n_1^{n_2-1} n_2^{n_1-1}$, where $n_2 = n - n_1$.
- Generating uniformly distributed random Prüfer sequences and converting them into the corresponding trees is a straightforward method of generating uniformly distributed random labelled trees.

References ^[edit]

- [↑] Prüfer, H. (1918). "Neuer Beweis eines Satzes über Permutationen". *Arch. Math. Phys.* **27**: 742–744.
- [↑] Jens Gottlieb, Bryant A. Julstrom, Günther R. Raidl, and Franz Rothlauf. (2001). "[Prüfer numbers: A poor representation of spanning trees for evolutionary search](#)"  (PDF). *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO-2001)*: 343–350.
- [↑] Kajimoto, H. (2003). "An Extension of the Prüfer Code and Assembly of Connected Graphs from Their Blocks". *Graphs and Combinatorics* **19**: 231–239. doi:10.1007/s00373-002-0499-3 .

External links ^[edit]

- [Prüfer code](#)  – from [MathWorld](#)

Categories: [Enumerative combinatorics](#) | [Trees \(graph theory\)](#)

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