Dynamic Programming | Set 7 (Coin Change)

Given a value N, if we want to make change for N cents, and we have infinite supply of each of S = { S1, S2, .. , Sm} valued coins, how many ways can we make the change? The order of coins doesn't matter.

For example, for N = 4 and S = {1,2,3}, there are four solutions: {1,1,1,1},{1,1,2},{2,2},{1,3}. So output should be 4. For N = 10 and S = {2, 5, 3, 6}, there are five solutions: {2,2,2,2,2}, {2,2,3,3}, {2,2,6}, {2,3,5} and {5,5}. So the output should be 5.

1) Optimal Substructure
To count total number solutions, we can divide all set solutions in two sets.
1) Solutions that do not contain mth coin (or Sm).
2) Solutions that contain at least one Sm.
Let count(S[], m, n) be the function to count the number of solutions, then it can be written as sum of count(S[], m-1, n) and count(S[], m, n-Sm).

Therefore, the problem has optimal substructure property as the problem can be solved using solutions to subproblems.

2) Overlapping Subproblems
Following is a simple recursive implementation of the Coin Change problem. The implementation simply follows the recursive structure mentioned above.

```
#include<stdio.h>

// Returns the count of ways we can sum  S[0...m-1] coin
int count( int S[], int m, int n )
{
    // If n is 0 then there is 1 solution (do not includ
    if (n == 0)
        return 1;

    // If n is less than 0 then no solution exists
    if (n < 0)
        return 0;

    // If there are no coins and n is greater than 0, th
```

```
    if (m <=0 && n >= 1)
        return 0;

    // count is sum of solutions (i) including S[m-1] (i
    return count( S, m - 1, n ) + count( S, m, n-S[m-1]
}

// Driver program to test above function
int main()
{
    int i, j;
    int arr[] = {1, 2, 3};
    int m = sizeof(arr)/sizeof(arr[0]);
    printf("%d ", count(arr, m, 4));
    getchar();
    return 0;
}
```

It should be noted that the above function computes the same subproblems again and again. See the following recursion tree for S = {1, 2, 3} and n = 5. The function C({1}, 3) is called two times. If we draw the complete tree, then we can see that there are many subproblems being called more than once.

```
C() --> count()
                              C({1,2,3}, 5)
                           /                 \
                         /                     \
             C({1,2,3}, 2)                 C({1,2}, 5)
            /     \                        /         \
           /        \                     /           \
C({1,2,3}, -1)  C({1,2}, 2)        C({1,2}, 3)    C({1}, 5)
               /     \             /    \            /     \
             /         \          /      \          /       \
    C({1,2},0)  C({1},2)   C({1,2},1) C({1},3)    C({1}, 4)  C({}, 5)
                   / \       / \        / \        /     \
                  /   \     /   \      /   \      /       \
                .       .  .     .    .     .   C({1}, 3) C({}, 4)
                                                 /  \
                                                /    \
                                               .      .
```

Since same suproblems are called again, this problem has Overlapping Subprolems property. So the Coin Change problem has both properties (see this and this) of a dynamic programming problem. Like other typical Dynamic Programming(DP) problems, recomputations of same subproblems can be avoided by constructing a temporary array table[][] in bottom up manner.

Dynamic Programming Solution

```
#include<stdio.h>

int count( int S[], int m, int n )
{
    int i, j, x, y;

    // We need n+1 rows as the table is consturcted in bo
    // the base case 0 value case (n = 0)
    int table[n+1][m];

    // Fill the enteries for 0 value case (n = 0)
    for (i=0; i<m; i++)
        table[0][i] = 1;

    // Fill rest of the table enteries in bottom up manne
    for (i = 1; i < n+1; i++)
    {
        for (j = 0; j < m; j++)
        {
            // Count of solutions including S[j]
            x = (i-S[j] >= 0)? table[i - S[j]][j]: 0;

            // Count of solutions excluding S[j]
            y = (j >= 1)? table[i][j-1]: 0;

            // total count
            table[i][j] = x + y;
        }
    }
    return table[n][m-1];
}

// Driver program to test above function
int main()
{
    int arr[] = {1, 2, 3};
    int m = sizeof(arr)/sizeof(arr[0]);
    int n = 4;
```

```
    printf(" %d ", count(arr, m, n));
    return 0;
}
```

Time Complexity: O(mn)

Following is a simplified version of method 2. The auxiliary space required here is O(n) only.

```
int count( int S[], int m, int n )
{
    // table[i] will be storing the number of solutions f
    // value i. We need n+1 rows as the table is consturc
    // in bottom up manner using the base case (n = 0)
    int table[n+1];

    // Initialize all table values as 0
    memset(table, 0, sizeof(table));

    // Base case (If given value is 0)
    table[0] = 1;

    // Pick all coins one by one and update the table[] v
    // after the index greater than or equal to the value
    // picked coin
    for(int i=0; i<m; i++)
        for(int j=S[i]; j<=n; j++)
            table[j] += table[j-S[i]];

    return table[n];
}
```