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# Borwein's algorithm

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In **mathematics**, **Borwein's algorithm** is an **algorithm** devised by **Jonathan** and **Peter Borwein** to calculate the value of  $1/\pi$ . They devised several other algorithms. They published a book: Jonathon M. Borwein, Peter B. Borwein, *Pi and the AGM - A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987. Many of their results are available in: Jorg Arndt, Christoph Haenel, *Pi Unleashed*, Springer, Berlin, 2001, ISBN 3-540-66572-2.

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## Jonathan Borwein and Peter Borwein's Version (1993) [\[edit\]](#)

Start out by setting<sup>[*citation needed*]</sup>

$$\begin{aligned} A &= 63365028312971999585426220 \\ &\quad + 28337702140800842046825600\sqrt{5} \\ &\quad + 384\sqrt{5}(10891728551171178200467436212395209160385656017 \\ &\quad \quad + 4870929086578810225077338534541688721351255040\sqrt{5})^{1/2} \\ B &= 7849910453496627210289749000 \\ &\quad + 3510586678260932028965606400\sqrt{5} \\ &\quad + 2515968\sqrt{3110}(6260208323789001636993322654444020882161 \\ &\quad \quad + 2799650273060444296577206890718825190235\sqrt{5})^{1/2} \\ C &= -214772995063512240 \\ &\quad - 96049403338648032\sqrt{5} \\ &\quad - 1296\sqrt{5}(10985234579463550323713318473 \\ &\quad \quad + 4912746253692362754607395912\sqrt{5})^{1/2} \end{aligned}$$

Then

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{n=0}^{\infty} \frac{(6n)!}{(3n)!(n!)^3} \frac{A + nB}{C^{3n}}$$

Each additional term of the series yields approximately 50 digits. This is an example of a **Ramanujan–Sato series**.

## Cubic convergence (1991) [\[edit\]](#)

Start out by setting

$$\begin{aligned} a_0 &= \frac{1}{3} \\ s_0 &= \frac{\sqrt{3}-1}{2} \end{aligned}$$

Then iterate

$$\begin{aligned}r_{k+1} &= \frac{3}{1 + 2(1 - s_k^3)^{1/3}} \\s_{k+1} &= \frac{r_{k+1} - 1}{2} \\a_{k+1} &= r_{k+1}^2 a_k - 3^k (r_{k+1}^2 - 1)\end{aligned}$$

Then  $a_k$  converges cubically to  $1/\pi$ ; that is, each iteration approximately triples the number of correct digits.

## Another formula for $\pi$ (1989) [\[edit\]](#)

Start out by setting<sup>[\[citation needed\]](#)</sup>

$$\begin{aligned}A &= 212175710912\sqrt{61} + 1657145277365 \\B &= 13773980892672\sqrt{61} + 107578229802750 \\C &= (5280(236674 + 30303\sqrt{61}))^3\end{aligned}$$

Then

$$1/\pi = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (A + nB)}{(n!)^3 (3n)! C^{n+1/2}}$$

Each additional term of the partial sum yields approximately 31 digits.

## Quartic algorithm (1985) [\[edit\]](#)

Start out by setting<sup>[\[1\]](#)</sup>

$$\begin{aligned}a_0 &= 6 - 4\sqrt{2} \\y_0 &= \sqrt{2} - 1\end{aligned}$$

Then iterate

$$\begin{aligned}y_{k+1} &= \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}} \\a_{k+1} &= a_k (1 + y_{k+1})^4 - 2^{2k+3} y_{k+1} (1 + y_{k+1} + y_{k+1}^2)\end{aligned}$$

Then  $a_k$  converges quartically against  $1/\pi$ ; that is, each iteration approximately quadruples the number of correct digits.

## Quadratic convergence (1984) [\[edit\]](#)

Start out by setting<sup>[\[2\]](#) [\[3\]](#)</sup>

$$\begin{aligned}a_0 &= \sqrt{2} \\b_0 &= 0 \\p_0 &= 2 + \sqrt{2}\end{aligned}$$

Then iterate

$$\begin{aligned}a_{n+1} &= \frac{\sqrt{a_n} + 1/\sqrt{a_n}}{2} \\b_{n+1} &= \frac{(1 + b_n)\sqrt{a_n}}{a_n + b_n} \\p_{n+1} &= \frac{(1 + a_{n+1})p_n b_{n+1}}{1 + b_{n+1}}\end{aligned}$$

Then  $p_k$  converges quadratically to  $\pi$ ; that is, each iteration approximately doubles the number of correct digits. The algorithm is *not* self-correcting; each iteration must be performed with the desired number of correct digits of  $\pi$ .

## Quintic convergence [\[edit\]](#)

Start out by setting

$$a_0 = \frac{1}{2}$$

$$s_0 = 5(\sqrt{5} - 2)$$

Then iterate

$$x_{n+1} = \frac{5}{s_n} - 1$$

$$y_{n+1} = (x_{n+1} - 1)^2 + 7$$

$$z_{n+1} = \left( \frac{1}{2} x_{n+1} \left( y_{n+1} + \sqrt{y_{n+1}^2 - 4x_{n+1}^3} \right) \right)^{1/5}$$

$$a_{n+1} = s_n^2 a_n - 5^n \left( \frac{s_n^2 - 5}{2} + \sqrt{s_n(s_n^2 - 2s_n + 5)} \right)$$

$$s_{n+1} = \frac{25}{(z_{n+1} + x_{n+1}/z_{n+1} + 1)^2 s_n}$$

Then  $a_k$  converges quintically to  $1/\pi$  (that is, each iteration approximately quintuples the number of correct digits), and the following condition holds:

$$0 < a_n - \frac{1}{\pi} < 16 \cdot 5^n \cdot e^{-5^n} \pi$$

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## Nonic convergence [\[edit\]](#)

Start out by setting

$$a_0 = \frac{1}{3}$$

$$r_0 = \frac{\sqrt{3} - 1}{2}$$

$$s_0 = (1 - r_0^3)^{1/3}$$

Then iterate

$$t_{n+1} = 1 + 2r_n$$

$$u_{n+1} = (9r_n(1 + r_n + r_n^2))^{1/3}$$

$$v_{n+1} = t_{n+1}^2 + t_{n+1}u_{n+1} + u_{n+1}^2$$

$$w_{n+1} = \frac{27(1 + s_n + s_n^2)}{v_{n+1}}$$

$$a_{n+1} = w_{n+1}a_n + 3^{2n-1}(1 - w_{n+1})$$

$$s_{n+1} = \frac{(1 - r_n)^3}{(t_{n+1} + 2u_{n+1})v_{n+1}}$$

$$r_{n+1} = (1 - s_{n+1}^3)^{1/3}$$

Then  $a_k$  converges nonically to  $1/\pi$ ; that is, each iteration approximately multiplies the number of correct digits by nine.

## See also [\[edit\]](#)

- [Gauss–Legendre algorithm](#) – another algorithm to calculate  $\pi$
- [Bailey–Borwein–Plouffe formula](#)

## References [\[edit\]](#)

- ↑ Mak, Ronald (2003). *The Java Programmers Guide to Numerical Computation*. Pearson Educational. p. 353. ISBN 0-13-046041-9.
- ↑ Amdt, Jörg; Haenel, Christoph (1998). *π Unleashed*. Springer-Verlag. p. 236. ISBN 3-540-66572-2.
- ↑  **Template:Pi Unleashed**
- ↑ http://www.cecm.sfu.ca/organics/papers/garvan/paper/html/node12.html

- Pi Formulas**  from Wolfram MathWorld

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