Stable marriage problem

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In mathematics, economics, and computer science, the **stable marriage problem** (also **stable matching problem** or **SMP**) is the problem of finding a stable matching between two equally sized sets of elements given an ordering of preferences for each element. A matching is a mapping from the elements of one set to the elements of the other set. A matching is stable whenever it is *not* the case that both:

a. some given element *A* of the first matched set prefers some given element *B* of the second matched set over the element to which *A* is already matched, and
b. *B* also prefers *A* over the element to which *B* is already matched

In other words, a matching is stable when there does not exist any match (*A*, *B*) by which both *A* and *B* are individually better off than they would be with the element to which they are currently matched.

The stable marriage problem has been stated as follows:

> Given *n* men and *n* women, where each person has ranked all members of the opposite sex in order of preference, marry the men and women together such that there are no two people of opposite sex who would both rather have each other than their current partners. When there are no such pairs of people, the set of marriages is deemed stable.

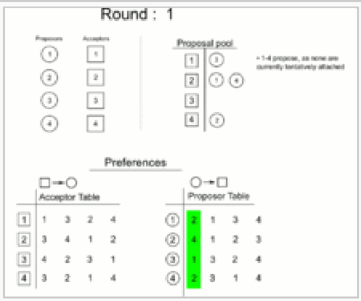
Note that the requirement that the marriages be heterosexual distinguishes this problem from the stable roommates problem.

Algorithms for finding solutions to the stable marriage problem have applications in a variety of real-world situations, perhaps the best known of these being in the assignment of graduating medical students to their first hospital appointments.[1] In 2012, the Nobel Prize in Economics was awarded to Lloyd S. Shapley and Alvin E. Roth "for the theory of stable allocations and the practice of market design."[2]

Contents [hide]

Solution [edit]

In 1962, David Gale and Lloyd Shapley proved that, for any equal number of men and women, it is always possible to solve the SMP and make all marriages stable. They presented an algorithm to do so.[3][4]



Animation showing an example of the Gale–Shapley algorithm

The **Gale–Shapley algorithm** involves a number of "rounds" (or "iterations"). In the first round, first *a*) each unengaged man proposes to the woman he prefers most, and then *b*) each woman replies "maybe" to her suitor she most prefers and "no" to all other suitors. She is then provisionally "engaged" to the suitor she most prefers so far, and that suitor is likewise provisionally engaged to her. In each subsequent round, first *a*) each unengaged man proposes to the most-preferred woman to whom he has not yet proposed (regardless of whether the woman is already engaged), and then *b*) each woman replies "maybe" to her suitor she most prefers (whether her existing provisional partner or someone else) and rejects the rest (again, perhaps including her current provisional partner). The provisional nature of engagements preserves the right of an already-engaged woman to "trade up" (and, in the process, to "jilt" her until-then partner). This process is repeated until everyone is engaged.

The runtime complexity of this algorithm is $O(n^2)$ where n is number of men or women.[5]

This algorithm guarantees that:

Everyone gets married

At the end, there cannot be a man and a woman both unengaged, as he must have proposed to her at some point (since a man will eventually propose to everyone, if necessary) and, being proposed to, she would necessarily be engaged (to someone) thereafter.

The marriages are stable

Let Alice be a woman and Bob be a man who are both engaged, but not to each other. Upon completion of the algorithm, it is not possible for both Alice and Bob to prefer each other over their current partners. If Bob prefers Alice to his current partner, he must have proposed to Alice before he proposed to his current partner. If Alice accepted his proposal, yet is not married to him at the end, she must have dumped him for someone she likes more, and therefore doesn't like Bob more than her current partner. If Alice rejected his proposal, she was already with someone she liked more than Bob.

Algorithm [edit]

```
function stableMatching {
    Initialize all m ∈ M and w ∈ W to free
    while ∃ free man m who still has a woman w to propose to {
       w = highest ranked woman to whom m has not yet proposed
       if w is free
         (m, w) become engaged
       else some pair (m', w) already exists
         if w prefers m to m'
           (m, w) become engaged
           m' becomes free
         else
           (m', w) remain engaged
    }
}
```

Optimality of the solution [edit]

While the solution is stable, it is not necessarily optimal from all individuals' points of view. The traditional form of the algorithm is optimal for the initiator of the proposals and the stable, suitor-optimal solution may or may not be optimal for the reviewer of the proposals. An example is as follows:

There are three suitors (A,B,C) and three reviewers (X,Y,Z) which have preferences of:

A: YXZ   B: ZYX   C: XZY   X: BAC   Y: CBA   Z: ACB

There are 3 stable solutions to this matching arrangement:

suitors get their first choice and reviewers their third (AY, BZ, CX)
all participants get their second choice (AX, BY, CZ)
reviewers get their first choice and suitors their third (AZ, BX, CY)

All three are stable because instability requires both participants to be happier with an alternative match. Giving one group their first choices ensures that the matches are stable because they would be unhappy with any other proposed match. Giving everyone their second choice ensures that any other match would be disliked by one of the parties. The algorithm converges in a single round on the suitor-optimal solution because each reviewer receives exactly one proposal, and therefore selects that proposal as its best choice, ensuring that each suitor has an accepted offer, ending the match. This asymmetry of optimality is driven by the fact that the suitors have the entire set to choose from, but reviewers choose between a limited subset of the suitors at any one time.

Similar problems [edit]

The **assignment problem** seeks to find a matching in a weighted bipartite graph that has maximum weight. Maximum weighted matchings do not have to be stable, but in some applications a maximum weighted matching is better than a stable one.

The **stable roommates problem** is similar to the stable marriage problem, but differs in that all participants belong to a single pool (instead of being divided into equal numbers of "men" and "women").

The **hospitals/residents problem** — also known as the **college admissions problem** — differs from the stable marriage problem in that the "women" can accept "proposals" from more than one "man" (e.g., a hospital can take multiple residents, or a college can take an incoming class of more than one student). Algorithms to solve the hospitals/residents problem can be *hospital-oriented* (female-optimal) or *resident-oriented* (male-optimal).

The **hospitals/residents problem with couples** allows the set of residents to include couples who must be assigned together, either to the same hospital or to a specific pair of hospitals chosen by the couple (e.g., a married couple want to ensure that they will stay together and not be stuck in programs that are far away from each other). The addition of couples to the hospitals/residents problem renders the problem NP-complete.[6]

The matching with contracts problem is a generalization of matching problem, in which participants can be matched with different terms of contracts.[7] An important special case of contracts is matching with flexible wages.[8]

See also [edit]

- Assignment problem a similar problem where edge weights are commutative
- Stable roommates problem a similar problem, but with one set of size n and n-1 preferences
- Nash equilibrium
- Hungarian algorithm an algorithm to solve weighted bipartite matching problem
- Matching (graph theory) generalized matching problem in graphs
- Rainbow matching for edge colored graphs

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2. ^ "The Prize in Economic Sciences 2012". Nobelprize.org. Retrieved 2013-09-09.
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6. ^ Gusfield, D.; Irving, R. W. (1989). *The Stable Marriage Problem: Structure and Algorithms*. MIT Press. p. 54. ISBN 0-262-07118-5.
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External links [edit]

- Interactive Flash Demonstration of SMP
- http://kuznets.fas.harvard.edu/~aroth/alroth.html#NRMP
- http://www.dcs.gla.ac.uk/research/algorithms/stable/EGSapplet/EGS.html
- Gale–Shapley JavaScript Demonstration
- SMP Lecture Notes

v· t· e	**Topics in game theory**	[hide]

Definitions	Normal-form game · Extensive-form game · Graphical game · Cooperative game · Succinct game · Information set · Hierarchy of beliefs · Preference
Equilibrium concepts	Nash equilibrium · Subgame perfection · Mertens-stable equilibrium · Bayesian-Nash · Perfect Bayesian · Trembling hand · Proper equilibrium · Epsilon-equilibrium · Correlated equilibrium · Sequential equilibrium · Quasi-perfect equilibrium · Evolutionarily stable strategy · Risk dominance · Core · Shapley value · Pareto efficiency · Quantal response equilibrium · Self-confirming equilibrium · Strong Nash equilibrium · Markov perfect equilibrium
Strategies	Dominant strategies · Pure strategy · Mixed strategy · Tit for tat · Grim trigger · Collusion · Backward induction · Forward induction · Markov strategy
Classes of games	Symmetric game · Perfect information · Simultaneous game · Sequential game · Repeated game · Signaling game · Cheap talk · Zero-sum game · Mechanism design · Bargaining problem · Stochastic game · *n*-player game · Large Poisson game · Nontransitive game · Global games · Strictly determined game
Games	Prisoner's dilemma · Traveler's dilemma · Coordination game · Chicken · Centipede game · Volunteer's dilemma · Dollar auction · Battle of the sexes · Stag hunt · Matching pennies · Ultimatum game · Rock-paper-scissors · Pirate game · Dictator game · Public goods game · Blotto games · War of attrition · El Farol Bar problem · Fair division · Fair cake-cutting · Cournot game · Deadlock · Diner's dilemma · Guess 2/3 of the average · Kuhn poker · Nash bargaining game · Screening game · Prisoners and hats puzzle · Trust game · Princess and monster game · Monty Hall problem · Rendezvous problem
Theorems	Minimax theorem · Nash's theorem · Purification theorem · Folk theorem · Revelation principle · Arrow's impossibility theorem
Key figures	Kenneth Arrow · Robert Aumann · Kenneth Binmore · Samuel Bowles · Melvin Dresher · Merrill M. Flood · Drew Fudenberg · Donald B. Gillies · John Harsanyi · Leonid Hurwicz · David K. Levine · Daniel Kahneman · David M. Kreps · Harold W. Kuhn · Eric Maskin · Jean-François Mertens · Paul Milgrom · Oskar Morgenstern · Hervé Moulin · Roger Myerson · John Nash · John von Neumann · Ariel Rubinstein · Thomas Schelling · Reinhard Selten · Herbert Simon · Lloyd Shapley · John Maynard Smith · Jean Tirole · Albert W. Tucker · Amos Tversky · William Vickrey · Robert B. Wilson · Peyton Young
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