# Gauss–Legendre algorithm

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The **Gauss–Legendre algorithm** is an algorithm to compute the digits of π. It is notable for being rapidly convergent, with only 25 iterations producing 45 million correct digits of π. However, the drawback is that it is memory intensive and it is therefore sometimes not used over Machin-like formulas.

The method is based on the individual work of Carl Friedrich Gauss (1777–1855) and Adrien-Marie Legendre (1752–1833) combined with modern algorithms for multiplication and square roots. It repeatedly replaces two numbers by their arithmetic and geometric mean, in order to approximate their arithmetic-geometric mean.

The version presented below is also known as the **Gauss–Euler, Brent–Salamin (or Salamin–Brent) algorithm**;[1] it was independently discovered in 1975 by Richard Brent and Eugene Salamin. It was used to compute the first 206,158,430,000 decimal digits of π on September 18 to 20, 1999, and the results were checked with Borwein's algorithm.

**Contents** [hide]



## Algorithm [edit]

1. Initial value setting:

$$a_0 = 1 \qquad b_0 = \frac{1}{\sqrt{2}} \qquad t_0 = \frac{1}{4} \qquad p_0 = 1.$$

2. Repeat the following instructions until the difference of $a_n$ and $b_n$ is within the desired accuracy:

$$\begin{aligned} a_{n+1} &= \frac{a_n + b_n}{2}, \\ b_{n+1} &= \sqrt{a_n b_n}, \\ t_{n+1} &= t_n - p_n(a_n - a_{n+1})^2, \\ p_{n+1} &= 2p_n. \end{aligned}$$

3. π is then approximated as:

$$\pi \approx \frac{(a_{n+1} + b_{n+1})^2}{4t_{n+1}}.$$

The first three iterations give (approximations given up to and including the first incorrect digit):

$$\begin{aligned} &3.140\ldots \\ &3.14159264\ldots \\ &3.14159265358979323382\ldots \end{aligned}$$

The algorithm has second-order convergent nature, which essentially means that the number of correct digits doubles with each step of the algorithm.

## Mathematical background [edit]

### Limits of the arithmetic–geometric mean [edit]

The arithmetic–geometric mean of two numbers, $a_0$ and $b_0$, is found by calculating the limit of the sequences

$$\begin{aligned} a_{n+1} &= \frac{a_n + b_n}{2}, \\ b_{n+1} &= \sqrt{a_n b_n}, \end{aligned}$$

which both converge to the same limit.

If $a_0 = 1$ and $b_0 = \cos\varphi$ then the limit is $\frac{\pi}{2K(\sin\varphi)}$ where $K(k)$ is the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

If $c_0 = \sin\varphi$, $c_{i+1} = a_i - a_{i+1}$, then

$$\sum_{i=0}^{\infty} 2^{i-1} c_i^2 = 1 - \frac{E(\sin\varphi)}{K(\sin\varphi)}$$

where $E(k)$ is the complete elliptic integral of the second kind:

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta.$$

Gauss knew of both of these results.[2] [3] [4]

### Legendre’s identity [edit]

For $\varphi$ and $\theta$ such that $\varphi + \theta = \frac{1}{2}\pi$ Legendre proved the identity:

$$K(\sin\varphi)E(\sin\theta) + K(\sin\theta)E(\sin\varphi) - K(\sin\varphi)K(\sin\theta) = \frac{1}{2}\pi.$$ [2]

### Gauss–Euler method [edit]

The values $\varphi = \theta = \frac{\pi}{4}$ can be substituted into Legendre’s identity and the approximations to K, E can be found by terms in the sequences for the arithmetic geometric mean with $a_0 = 1$ and $b_0 = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$.[5]

## See also [edit]

- Numerical approximations of π

## References [edit]


1. ^ Brent, Richard, *Old and New Algorithms for pi*, Letters to the Editor, Notices of the AMS 60(1), p. 7
2. ^ *a* *b* Brent, Richard (1975), Traub, J F, ed., "Multiple-precision zero-finding methods and the complexity of elementary function evaluation", *Analytic Computational Complexity* (New York: Academic Press): 151–176, retrieved 8 September 2007
3. ^ Salamin, Eugene, *Computation of pi*, Charles Stark Draper Laboratory ISS memo 74–19, 30 January 1974, Cambridge, Massachusetts
4. ^ Salamin, Eugene (1976), "Computation of pi Using Arithmetic–Geometric Mean", *Mathematics of Computation* **30** (135): 565–570, ISSN 0025-5718
5. ^ Adlaj, Semjon, *An eloquent formula for the perimeter of an ellipse*, Notices of the AMS 59(8), p. 1096


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