Catalan numbers are a sequence of natural numbers that occurs in many interesting counting problems like following.

1) Count the number of expressions containing n pairs of parentheses which are correctly matched. For n = 3, possible expressions are ((())), ()(()), ()()(), (())(), (()()).

2) Count the number of possible Binary Search Trees with n keys (See this)

3) Count the number of full binary trees (A rooted binary tree is full if every vertex has either two children or no children) with n+1 leaves.

See this for more applications.

The first few Catalan numbers for n = 0, 1, 2, 3, … are **1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, …**

Recursive Solution

Catalan numbers satisfy the following recursive formula.

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0;$$

Following is C++ implementation of above recursive formula.

```
#include<iostream>
using namespace std;

// A recursive function to find nth catalan number
unsigned long int catalan(unsigned int n)
{
    // Base case
    if (n <= 1) return 1;

    // catalan(n) is sum of catalan(i)*catalan(n-i-1)
    unsigned long int res = 0;
    for (int i=0; i<n; i++)
        res += catalan(i)*catalan(n-i-1);

    return res;
}

// Driver program to test above function
int main()
{
    for (int i=0; i<10; i++)
        cout << catalan(i) << " ";
    return 0;
}
```

Output :

```
1 1 2 5 14 42 132 429 1430 4862
```

Time complexity of above implementation is equivalent to nth catalan number.

$$T(n) = \sum_{i=0}^{n-1} T(i) * T(n-i) \quad \text{for } n \geq 0;$$

The value of nth catalan number is exponential that makes the time complexity exponential.

Dynamic Programming Solution

We can observe that the above recursive implementation does a lot of repeated work (we can the same by drawing recursion tree). Since there are overlapping subproblems, we can use dynamic programming for this. Following is a Dynamic programming based implementation in C++.

```
#include<iostream>
using namespace std;

// A dynamic programming based function to find nth
// Catalan number
unsigned long int catalanDP(unsigned int n)
{
    // Table to store results of subproblems
    unsigned long int catalan[n+1];

    // Initialize first two values in table
    catalan[0] = catalan[1] = 1;

    // Fill entries in catalan[] using recursive formula
    for (int i=2; i<=n; i++)
    {
        catalan[i] = 0;
        for (int j=0; j<i; j++)
            catalan[i] += catalan[j] * catalan[i-j-1];
    }

    // Return last entry
    return catalan[n];
}

// Driver program to test above function
int main()
{
    for (int i = 0; i < 10; i++)
        cout << catalanDP(i) << " ";
    return 0;
}
```

Output:

```
1 1 2 5 14 42 132 429 1430 4862
```

Time Complexity: Time complexity of above implementation is $O(n^2)$

Using Binomial Coefficient

We can also use the below formula to find nth catalan number in O(n) time.

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

We have discussed a O(n) approach to find binomial coefficient nCr.

```
#include<iostream>
using namespace std;

// Returns value of Binomial Coefficient C(n, k)
unsigned long int binomialCoeff(unsigned int n, unsigned int k)
{
    unsigned long int res = 1;

    // Since C(n, k) = C(n, n-k)
    if (k > n - k)
        k = n - k;

    // Calculate value of [n*(n-1)*---*(n-k+1)] / [k*(k-1)*---*1]
    for (int i = 0; i < k; ++i)
    {
        res *= (n - i);
        res /= (i + 1);
    }
```

```
    return res;
}

// A Binomial coefficient based function to find nth catalan
// number in O(n) time
unsigned long int catalan(unsigned int n)
{
    // Calculate value of 2nCn
    unsigned long int c = binomialCoeff(2*n, n);

    // return 2nCn/(n+1)
    return c/(n+1);
}

// Driver program to test above functions
int main()
{
    for (int i = 0; i < 10; i++)
        cout << catalan(i) << " ";
    return 0;
}
```

Output:

```
1 1 2 5 14 42 132 429 1430 4862
```

Time Complexity: Time complexity of above implementation is O(n).

We can also use below formula to find nth catalan number in O(n) time.

$$C_n = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k} \qquad \text{for } n \ge 0$$

References:

http://en.wikipedia.org/wiki/Catalan_number