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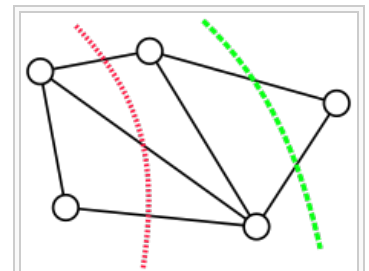
Karger's algorithm

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In [computer science](#) and [graph theory](#), **Karger's algorithm** is a [randomized algorithm](#) to compute a [minimum cut](#) of a connected [graph](#). It was invented by [David Karger](#) and first published in 1993.^[1]

The idea of the algorithm is based on the concept of [contraction of an edge](#) (u, v) in an undirected graph $G = (V, E)$. Informally speaking, the contraction of an edge merges the nodes u and v into one, reducing the total number of nodes of the graph by one. All other edges connecting either u or v are "reattached" to the merged node, effectively producing a [multigraph](#).

Karger's basic algorithm iteratively contracts randomly chosen edges until only two nodes remain; those nodes represent a [cut](#) in the original graph. By iterating this basic algorithm a sufficient number of times, a minimum cut can be found with high probability.



A graph with two cuts. The dotted line in red is a cut with three crossing edges. The dashed line in green is a min-cut of this graph, crossing only two edges.

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The global minimum cut problem [\[edit\]](#)

Main article: [Minimum cut](#)

A *cut* (S, T) in an undirected graph $G = (V, E)$ is a partition of the vertices V into two non-empty, disjoint sets $S \cup T = V$. The *cutset* of a cut consists of the edges $\{uv \in E: u \in S, v \in T\}$ between the two parts. The *size* (or *weight*) of a cut in an unweighted graph is the cardinality of the cutset, i.e., the number of edges between the two parts,

$$w(S, T) = |\{uv \in E: u \in S, v \in T\}|.$$

There are $2^{|V|}$ ways of choosing for each vertex whether it belongs to S or to T , but two of these choices make S or T empty and do not give rise to cuts. Among the remaining choices, swapping the roles of S and T does not change the cut, so each cut is counted twice; therefore, there are $2^{|V|-1} - 1$ distinct cuts. The *minimum cut problem* is to find a cut of smallest size among these cuts.

For weighted graphs with positive edge weights $w: E \rightarrow \mathbf{R}^+$ the weight of the cut is the sum of the weights of edges between vertices in each part

$$w(S, T) = \sum_{uv \in E: u \in S, v \in T} w(uv),$$

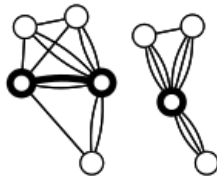
which agrees with the unweighted definition for $w = 1$.

A cut is sometimes called a "global cut" to distinguish it from an "*s*-*t* cut" for a given pair of vertices, which has the additional requirement that $s \in S$ and $t \in T$. Every global cut is an *s*-*t* cut for some $s, t \in V$. Thus, the minimum cut problem can be solved in [polynomial time](#) by iterating over all choices of $s, t \in V$ and solving the resulting minimum *s*-*t* cut problem using the [max-flow min-cut theorem](#) and a polynomial time algorithm for [maximum flow](#), such as the [push-relabel algorithm](#), though this approach is not optimal. There is a deterministic algorithm for the global minimum cut problem with running time $O(mn + n^2 \log n)$.^[2]

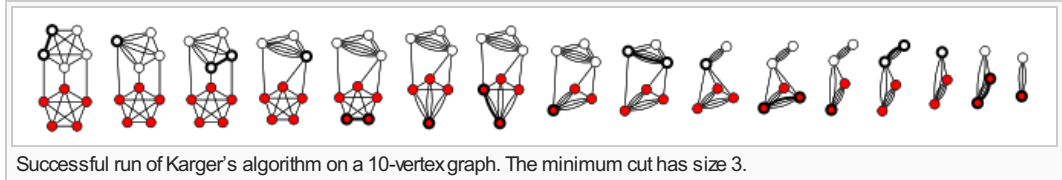
Contraction algorithm [\[edit\]](#)

The fundamental operation of Karger's algorithm is a form of [edge contraction](#). The result of contracting the edge

$e = \{u, v\}$ is new node uv . Every edge $\{w, u\}$ or $\{w, v\}$ for $w \notin \{u, v\}$ to the endpoints of the contracted edge is replaced by an edge $\{w, uv\}$ to the new node. Finally, the contracted nodes u and v with all their incident edges are removed. In particular, the resulting graph contains no self-loops. The result of contracting edge e is denoted G/e .



The contraction algorithm repeatedly contracts random edges in the graph, until only two nodes remain, at which point there is only a single cut.



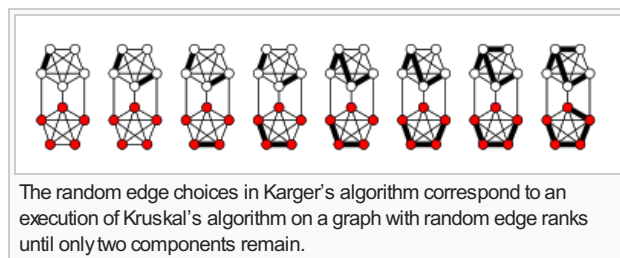
```

procedure contract( $G = (V, E)$ ) :
  while  $|V| > 2$ 
    choose  $e \in E$  uniformly at random
     $G \leftarrow G/e$ 
  return the only cut in  $G$ 

```

When the graph is represented using [adjacency lists](#) or an [adjacency matrix](#), a single edge contraction operation can be implemented with a linear number of updates to the data structure, for a total running time of $O(|V|^2)$.

Alternatively, the procedure can be viewed as an execution of [Kruskal's algorithm](#) for constructing the [minimum spanning tree](#) in a graph where the edges have weights $w(e_i) = \pi(i)$ according to a random permutation π . Removing the heaviest edge of this tree results in two components that describe a cut. In this way, the contraction procedure can be implemented like [Kruskal's algorithm](#) in time $O(|E| \log |E|)$.



The best known implementations use $O(|E|)$ time and space, or $O(|E| \log |E|)$ time and $O(|V|)$ space, respectively.^[1]

Success probability of the contraction algorithm [\[edit\]](#)

In a graph $G = (V, E)$ with $n = |V|$ vertices, the contraction algorithm returns a minimum cut with polynomially small probability $\binom{n}{2}^{-1}$. Every graph has $2^{n-1} - 1$ cuts,^[3] among which at most $\binom{n}{2}$ can be minimum cuts.

Therefore, the success probability for this algorithm is much better than the probability for picking a cut at random, which is at most $\binom{n}{2} / (2^{n-1} - 1)$

For instance, the [cycle graph](#) on n vertices has exactly $\binom{n}{2}$ minimum cuts, given by every choice of 2 edges. The contraction procedure finds each of these with equal probability.

To establish the bound on the success probability in general, let C denote the edges of a specific minimum cut of size k . The contraction algorithm returns C if none of the random edges belongs to the cutset of C . In particular, the first edge contraction avoids C , which happens with probability $1 - k/|E|$. The minimum degree of G is at least k (otherwise a minimum degree vertex would induce a smaller cut), so $|E| \geq nk/2$. Thus, the probability that the contraction algorithm picks an edge from C is

$$\frac{k}{|E|} \leq \frac{k}{nk/2} = \frac{2}{n}.$$

The probability p_n that the contraction algorithm on an n -vertex graph avoids C satisfies the recurrence

$p_n \geq \left(1 - \frac{2}{n}\right)p_{n-1}$, with $p_2 = 1$, which can be expanded as

$$p_n \geq \prod_{i=0}^{n-3} \left(1 - \frac{2}{n-i}\right) = \prod_{i=0}^{n-3} \frac{n-i-2}{n-i} = \frac{n-2}{n} \cdot \frac{n-3}{n-1} \cdot \frac{n-4}{n-2} \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \binom{n}{2}^{-1}.$$

Repeating the contraction algorithm [\[edit\]](#)

By repeating the contraction algorithm $T = \binom{n}{2} \ln n$ times with

independent random choices and returning the smallest cut, the probability of not finding a minimum cut is

$$\left[1 - \binom{n}{2}^{-1}\right]^T \leq \frac{1}{e^{\ln n}} = \frac{1}{n}.$$

The total running time for T repetitions for a graph with n vertices and m edges is $O(Tm) = O(n^2 m \log n)$.

Karger–Stein algorithm [\[edit\]](#)

An extension of Karger's algorithm due to [David Karger](#) and [Clifford Stein](#) achieves an order of magnitude improvement.^[4]

The basic idea is to perform the contraction procedure until the graph reaches t vertices.



10 repetitions of the contraction procedure. The 5th repetition finds the minimum cut of size 3. □

```

procedure contract( $G = (V, E), t$ ):
  while  $|V| > t$ 
    choose  $e \in E$  uniformly at random
     $G \leftarrow G/e$ 
  return  $G$ 

```

The probability $p_{n,t}$ that this contraction procedure avoids a specific cut C in an n -vertex graph is

$$p_{n,t} \geq \prod_{i=0}^{n-t-1} \left(1 - \frac{2}{n-i}\right) = \binom{t}{2} / \binom{n}{2}.$$

This expression is $\Omega(t^2/n^2)$ becomes less than $\frac{1}{2}$ around $t = \lceil 1 + n/\sqrt{2} \rceil$. In particular, the probability that an edge from C is contracted grows towards the end. This motivates the idea of switching to a slower algorithm after a certain number of contraction steps.

```

procedure fastmincut( $G = (V, E)$ ):
  if  $|V| < 6$ :
    return mincut( $V$ )
  else:
     $t \leftarrow \lceil 1 + |V|/\sqrt{2} \rceil$ 
     $G_1 \leftarrow \text{contract}(G, t)$ 
     $G_2 \leftarrow \text{contract}(G, t)$ 
    return min {fastmincut( $G_1$ ), fastmincut( $G_2$ )}

```

Analysis [\[edit\]](#)

The probability $P(n)$ the algorithm finds a specific cutset C is given by the recurrence relation

$$P(n) = 1 - \left(1 - \frac{1}{2}P\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right)\right)^2$$

with solution $P(n) = O\left(\frac{1}{\log n}\right)$. The running time of fastmincut satisfies

$$T(n) = 2T\left(\left\lceil 1 + \frac{n}{\sqrt{2}} \right\rceil\right) + O(n^2)$$

with solution $T(n) = O(n^2 \log n)$. To achieve error probability $O(1/n)$, the algorithm can be repeated $O(\log n / P(n))$ times, for an overall running time of $T(n) \cdot \frac{\log n}{P(n)} = O(n^2 \log^3 n)$. This is an order of magnitude improvement over Karger's original algorithm.

Finding all min-cuts [\[edit\]](#)

Theorem: With high probability we can find all min cuts in the running time of $O(n^2 \ln^3 n)$.

Proof: Since we know that $P(n) = O\left(\frac{1}{\ln n}\right)$, therefore after running this algorithm $O(\ln^2 n)$ times The probability of missing a specific min-cut is

$$\Pr[\text{miss a specific min-cut}] = (1 - P(n))^{O(\ln^2 n)} \leq \left(1 - \frac{c}{\ln n}\right)^{\frac{3 \ln^2 n}{c}} \leq e^{-3 \ln n} = \frac{1}{n^3}$$

And there are at most $\binom{n}{2}$ min-cuts, hence the probability of missing any min-cut is

$$\Pr[\text{miss any min-cut}] \leq \binom{n}{2} \cdot \frac{1}{n^3} = O\left(\frac{1}{n}\right).$$

The probability of failures is considerably small when n is large enough. ■

Improvement bound [\[edit\]](#)

To determine a min-cut, one has to touch every edge in the graph at least once, which is $O(n^2)$ time in a [dense graph](#). The Karger–Stein's min-cut algorithm takes the running time of $O(n^2 \ln^{O(1)} n)$, which is very close to that.

References [\[edit\]](#)

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