Polynomial long division

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In algebra, **polynomial long division** is an algorithm for dividing a polynomial by another polynomial of the same or lower degree, a generalised version of the familiar arithmetic technique called long division. It can be done easily by hand, because it separates an otherwise complex division problem into smaller ones. Sometimes using a shorthand version called synthetic division is faster, with less writing and fewer calculations.

Polynomial long division is an algorithm that implements the Euclidean division of polynomials, which starting from two polynomials *A* (the *dividend*) and *B* (the *divisor*) produces, if *B* is not zero, a *quotient* *Q* and a *remainder* *R* such that

$A = BQ + R,$

and either $R = 0$ or the degree of *R* is lower than the degree of *B*. These conditions define uniquely *Q* and *R*, which means that *Q* and *R* do not depend on the method used to compute them.

Contents [hide]

Example [edit]

Find the quotient and the remainder of the division of $x^3 - 2x^2 - 4$, the *dividend*, by $x - 3$, the *divisor*.

The dividend is first rewritten like this:

$$x^3 - 2x^2 + 0x - 4.$$

The quotient and remainder can then be determined as follows:

1. Divide the first term of the dividend by the highest term of the divisor (meaning the one with the highest power of *x*, which in this case is *x*). Place the result above the bar ($x^3 \div x = x^2$).
$$\begin{array}{r} x^2 \phantom{{}-2x^2+0x-4} \\ x-3\,\overline{)\,x^3-2x^2+0x-4} \end{array}$$
2. Multiply the divisor by the result just obtained (the first term of the eventual quotient). Write the result under the first two terms of the dividend ($x^2 \cdot (x - 3) = x^3 - 3x^2$).
$$\begin{array}{r} x^2 \phantom{{}-2x^2+0x-4} \\ x-3\,\overline{)\,x^3-2x^2+0x-4} \\ x^3-3x^2 \phantom{{}+0x-4} \end{array}$$
3. Subtract the product just obtained from the appropriate terms of the original dividend (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ($(x^3 - 2x^2) - (x^3 - 3x^2) = -2x^2 + 3x^2 = x^2$). Then, "bring down" the next term from the dividend.
$$\begin{array}{r} x^2 \phantom{{}-2x^2+0x-4} \\ x-3\,\overline{)\,x^3-2x^2+0x-4} \\ \underline{x^3-3x^2} \phantom{{}+0x-4} \\ +x^2+0x \phantom{{}-4} \end{array}$$
4. Repeat the previous three steps, except this time use the two terms that have just been written as the dividend.

$$\begin{array}{r} x^2 + x \phantom{{}-4} \\ x-3\overline{)x^3 - 2x^2 + 0x - 4} \\ \underline{x^3 - 3x^2} \phantom{{}+0x-4} \\ +x^2 + 0x \phantom{{}-4} \\ \underline{+x^2 - 3x} \phantom{{}-4} \\ +3x - 4 \end{array}$$

5. Repeat step 4. This time, there is nothing to "pull down".

$$\begin{array}{r} x^2 + x + 3 \\ x-3\overline{)x^3 - 2x^2 + 0x - 4} \\ \underline{x^3 - 3x^2} \phantom{{}+0x-4} \\ +x^2 + 0x \phantom{{}-4} \\ \underline{+x^2 - 3x} \phantom{{}-4} \\ +3x - 4 \\ \underline{+3x - 9} \\ +5 \end{array}$$

The polynomial above the bar is the quotient *q*(*x*), and the number left over (5) is the remainder *r*(*x*).

$$x^3 - 2x^2 - 4 = (x-3)\,\underbrace{(x^2 + x + 3)}_{q(x)} + \underbrace{5}_{r(x)}$$

The long division algorithm for arithmetic is very similar to the above algorithm, in which the variable *x* is replaced by the specific number 10.

Pseudo-code [edit]

The algorithm can be represented in pseudo-code as follows, where +, -, and × represent polynomial arithmetic, and / represents simple division of two terms:

```
function n / d:
  require d ≠ 0
  (q, r) ← (0, n)              # At each step n = d × q + r
  while r ≠ 0 AND degree(r) ≥ degree(d):
     t ← lead(r)/lead(d)       # Divide the leading terms
     (q, r) ← (q + t, r - (t * d))
  return (q, r)
```

Note that this works equally well when degree(n) < degree(d); in that case the result is just the trivial (0, n).

This algorithm describes exactly the above paper and pencil method: d is written on the left of the ")"; q is written, term after term, above the horizontal line, the last term being the value of t; the region under the horizontal line is used to compute and write down the successive values of r.

Euclidean division [edit]

Main article: Euclidean division of polynomials

For every pair of polynomials (*A*, *B*) such that $B \neq 0$, polynomial division provides a *quotient* *Q* and a *remainder* *R* such that

$$A = BQ + R,$$

and either *R*=0 or degree(*R*) < degree(*B*). Moreover (*Q*, *R*) is the unique pair of polynomials having this property.

The process of getting the uniquely defined polynomials *Q* and *R* from *A* and *B* is called **Euclidean division** (sometimes **division transformation**). Polynomial long division is thus an algorithm for Euclidean division.[1]

Applications [edit]

Factoring polynomials [edit]

Sometimes one or more roots of a polynomial are known, perhaps having been found using the rational root theorem. If one root *r* of a polynomial *P*(*x*) of degree *n* is known then polynomial long division can be used to factor *P*(*x*) into the form (*x* - *r*)(*Q*(*x*)) where *Q*(*x*) is a polynomial of degree *n*–1. *Q*(*x*) is simply the quotient obtained from the division process; since r is known to be a root of $P(x)$, it is known that the remainder must be zero.

Likewise, if more than one root is known, a linear factor $(x - r)$ in one of them (r) can be divided out to obtain $Q(x)$, and then a linear term in another root, s, can be divided out of $Q(x)$, etc. Alternatively, they can all be divided out at once: for example the linear factors $x - r$ and $x - s$ can be multiplied together to obtain the quadratic factor $x^2 - (r + s)x + rs$, which can then be divided into the original polynomial $P(x)$ to obtain a quotient of degree $n - 2$.

In this way, sometimes all the roots of a polynomial of degree greater than four can be obtained, even though that is not always possible. For example, if the rational root theorem can be used to obtain a single (rational) root of a quintic polynomial, it can be factored out to obtain a quartic (fourth degree) quotient; the explicit formula for the roots of a quartic polynomial can then be used to find the other four roots of the quintic.

Finding tangents to polynomial functions [edit]

Polynomial long division can be used to find the equation of the line that is tangent to the graph of the function defined by the polynomial $P(x)$ at a particular point $x = r$.[2] If $R(x)$ is the remainder of the division of $P(x)$ by $(x - r)^2$, then the equation of the tangent line at $x = r$ to the graph of the function $y = P(x)$ is $y = R(x)$, regardless of whether or not r is a root of the polynomial.

Example

Find the equation of the line that is tangent to the following curve at $x = 1$:

$$y = x^3 - 12x^2 - 42.$$

Begin by dividing the polynomial by $(x-1)^2 = x^2 - 2x + 1$:

$$\begin{array}{r}
x \quad - 10 \\
x^2 - 2x + 1 \overline{)\, x^3 - 12x^2 + 0x - 42} \\
\underline{x^3 - 2x^2 + x } \\
-10x^2 - x - 42 \\
\underline{-10x^2 + 20x - 10} \\
-21x - 32
\end{array}$$

The tangent line is $y = -21x - 32.$

See also [edit]

- Polynomial remainder theorem
- Synthetic division, a more concise method of performing polynomial long division
- Ruffini's rule
- Euclidean domain
- Gröbner basis
- Greatest common divisor of two polynomials

Notes [edit]

1. ^ S. Barnard (2008). *Higher Algebra*. READ BOOKS. p. 24. ISBN 1-4437-3086-6.
2. ^ Strickland-Constable, Charles, "A simple method for finding tangents to polynomial graphs", *Mathematical Gazette* 89, November 2005: 466-467.

Roe,Spencer and Taylor (2014) http://leicesteripsc.com/index.php?title=Group_3#References