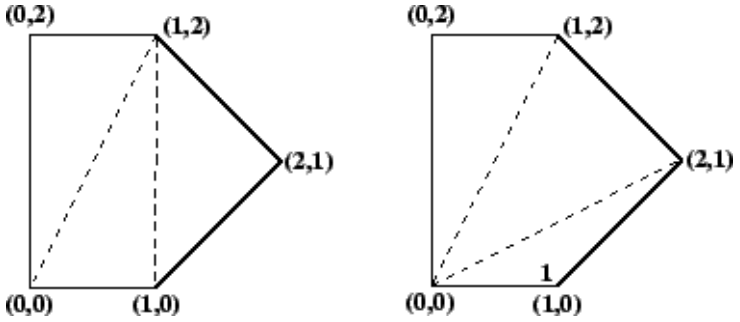
Minimum Cost Polygon Triangulation

A triangulation of a convex polygon is formed by drawing diagonals between non-adjacent vertices (corners) such that the diagonals never intersect. The problem is to find the cost of triangulation with the minimum cost. The cost of a triangulation is sum of the weights of its component triangles. Weight of each triangle is its perimeter (sum of lengths of all sides)

See following example taken from this source.

Two triangulations of the same convex pentagon. The triangulation on the left has a cost of 8 + 2√2 + 2√5 (approximately 15.30), the one on the right has a cost of 4 + 2√2 + 4√5 (approximately 15.77).



This problem has recursive substructure. The idea is to divide the polygon into three parts: a single triangle, the sub-polygon to the left, and the sub-polygon to the right. We try all possible divisions like this and find the one that minimizes the cost of the triangle plus the cost of the triangulation of the two sub-polygons.

```
Let Minimum Cost of triangulation of vertices from i to j be minCost(i, j)
If j <= i + 2 Then
  minCost(i, j) = 0
Else
  minCost(i, j) = Min { minCost(i, k) + minCost(k, j) + cost(i, k, j) }
                  Here k varies from 'i+1' to 'j-1'

Cost of a triangle formed by edges (i, j), (j, k) and (k, j) is
  cost(i, j, k)  = dist(i, j) + dist(j, k) + dist(k, j)
```

Following is C++ implementation of above naive recursive formula.

```
// Recursive implementation for minimum cost convex poly
#include <iostream>
#include <cmath>
#define MAX 1000000.0
using namespace std;
```

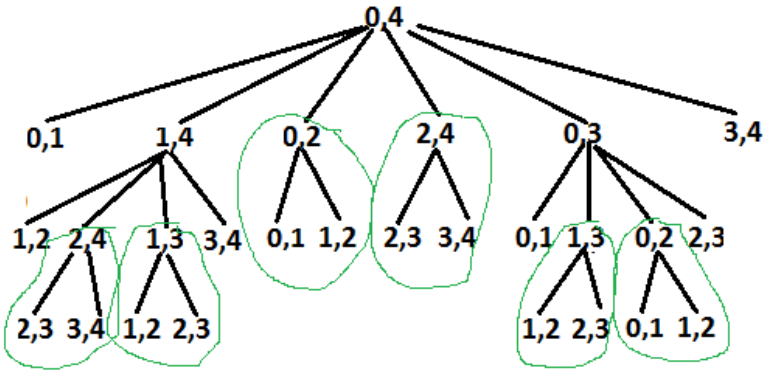
```
// Structure of a point in 2D plane
struct Point
{
    int x, y;
};

// Utility function to find minimum of two double values
double min(double x, double y)
{
    return (x <= y)? x : y;
}

// A utility function to find distance between two point
double dist(Point p1, Point p2)
{
    return sqrt((p1.x - p2.x)*(p1.x - p2.x) +
                (p1.y - p2.y)*(p1.y - p2.y));
}

// A utility function to find cost of a triangle. The co
// as perimeter (sum of lengths of all edges) of the tri
double cost(Point points[], int i, int j, int k)
{
    Point p1 = points[i], p2 = points[j], p3 = points[k]
    return dist(p1, p2) + dist(p2, p3) + dist(p3, p1);
}

// A recursive function to find minimum cost of polygon
// The polygon is represented by points[i..j].
double mTC(Point points[], int i, int j)
{
   // There must be at least three points between i and
   // (including i and j)
   if (j < i+2)
      return 0;

   // Initialize result as infinite
   double res = MAX;

   // Find minimum triangulation by considering all
   for (int k=i+1; k<j; k++)
        res = min(res, (mTC(points, i, k) + mTC(points,
                        cost(points, i, k, j)));
   return  res;
}

// Driver program to test above functions
int main()
{
    Point points[] = {{0, 0}, {1, 0}, {2, 1}, {1, 2}, {0
    int n = sizeof(points)/sizeof(points[0]);
```

```
    cout << mTC(points, 0, n-1);
    return 0;
}
```

Output:

```
15.3006
```

The above problem is similar to Matrix Chain Multiplication. The following is recursion tree for mTC(points[], 0, 4).



Recursion Tree for recursive implementation. Overlapping subproblems are encircled.

It can be easily seen in the above recursion tree that the problem has many overlapping subproblems. Since the problem has both properties: Optimal Substructure and Overlapping Subproblems, it can be efficiently solved using dynamic programming.

Following is C++ implementation of dynamic programming solution.

```
// A Dynamic Programming based program to find minimum c
// polygon triangulation
#include <iostream>
#include <cmath>
#define MAX 1000000.0
using namespace std;

// Structure of a point in 2D plane
struct Point
{
    int x, y;
};

// Utility function to find minimum of two double values
double min(double x, double y)
{
```

```
    return (x <= y)? x : y;
}

// A utility function to find distance between two point
double dist(Point p1, Point p2)
{
    return sqrt((p1.x - p2.x)*(p1.x - p2.x) +
                (p1.y - p2.y)*(p1.y - p2.y));
}

// A utility function to find cost of a triangle. The co
// as perimeter (sum of lengths of all edges) of the tri
double cost(Point points[], int i, int j, int k)
{
    Point p1 = points[i], p2 = points[j], p3 = points[k]
    return dist(p1, p2) + dist(p2, p3) + dist(p3, p1);
}

// A Dynamic programming based function to find minimum
// polygon triangulation.
double mTCDP(Point points[], int n)
{
   // There must be at least 3 points to form a triangle
   if (n < 3)
      return 0;

   // table to store results of subproblems.  table[i][j
   // triangulation of points from i to j.  The entry tal
   // the final result.
   double table[n][n];

   // Fill table using above recursive formula. Note tha
   // is filled in diagonal fashion i.e., from diagonal
   // table[0][n-1] which is the result.
   for (int gap = 0; gap < n; gap++)
   {
      for (int i = 0, j = gap; j < n; i++, j++)
      {
          if (j < i+2)
             table[i][j] = 0.0;
          else
          {
              table[i][j] = MAX;
              for (int k = i+1; k < j; k++)
              {
                double val = table[i][k] + table[k][j] +
                if (table[i][j] > val)
                     table[i][j] = val;
              }
          }
      }
```

```
    }
    return  table[0][n-1];
}

// Driver program to test above functions
int main()
{
    Point points[] = {{0, 0}, {1, 0}, {2, 1}, {1, 2}, {0
    int n = sizeof(points)/sizeof(points[0]);
    cout << mTCDP(points, n);
    return 0;
}
```

Output:

```
15.3006
```

Time complexity of the above dynamic programming solution is $O(n^3)$.

Please note that the above implementations assume that the points of covnvex polygon are given in order (either clockwise or anticlockwise)

Exercise:
Extend the above solution to print triangulation also. For the above example, the optimal triangulation is 0 3 4, 0 1 3, and 1 2 3.

Sources:
http://www.cs.utexas.edu/users/djimenez/utsa/cs3343/lecture12.html
http://www.cs.utoronto.ca/~heap/Courses/270F02/A4/chains/node2.html