Partial differential equation

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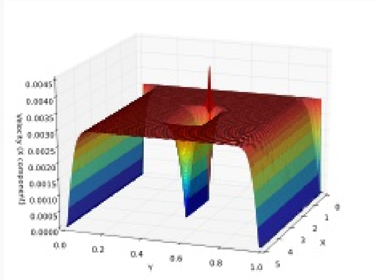
In mathematics, a **partial differential equation** (**PDE**) is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations (ODEs), which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

Differential equations



Navier–Stokes differential equations used to simulate airflow around an obstruction.

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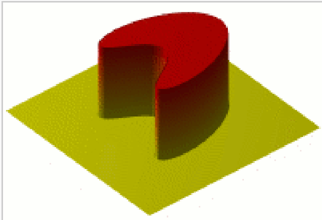

A visualisation of a solution to the heat equation on a three dimensional plane

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Introduction [edit]

Partial differential equations (PDEs) are equations that involve rates of change with respect to continuous variables. The position of a rigid body is specified by six numbers, but the configuration of a fluid is given by the continuous distribution of several parameters, such as the temperature, pressure, and so forth. The dynamics for the rigid body take place in a finite-dimensional configuration space; the dynamics for the fluid occur in an infinite-dimensional configuration space. This distinction usually makes PDEs much harder to solve than ordinary differential equations (ODEs), but here again there will be simple solutions for linear problems. Classic domains where PDEs are used include acoustics, fluid flow, electrodynamics, and heat transfer.

A partial differential equation (PDE) for the function $u(x_1, \cdots, x_n)$ is an equation of the form

$$f\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \ldots\right) = 0.$$

If *f* is a linear function of *u* and its derivatives, then the PDE is called linear. Common examples of linear PDEs include the heat equation, the wave equation, Laplace's equation, Helmholtz equation, Klein–Gordon equation, and Poisson's equation.

A relatively simple PDE is

$$\frac{\partial u}{\partial x}(x, y) = 0.$$

This relation implies that the function *u*(*x*,*y*) is independent of *x*. However, the equation gives no information on the function's dependence on the variable *y*. Hence the general solution of this equation is

$$u(x, y) = f(y),$$

where *f* is an arbitrary function of *y*. The analogous ordinary differential equation is

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x) = 0,$$

which has the solution

$$u(x) = c,$$

where *c* is any constant value. These two examples illustrate that general solutions of ordinary differential equations (ODEs) involve arbitrary constants, but solutions of PDEs involve arbitrary functions. A solution of a PDE is generally not unique; additional conditions must generally be specified on the boundary of the region where the solution is defined. For instance, in the simple example above, the function *f(y)* can be determined if *u* is specified on the line *x* = 0.

Existence and uniqueness [edit]

Although the issue of existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer with the Picard–Lindelöf theorem, that is far from the case for partial differential equations. The Cauchy–Kowalevski theorem states that the Cauchy problem for any partial differential equation whose coefficients are analytic in the unknown function and its derivatives, has a locally unique analytic solution. Although this result might appear to settle the existence and uniqueness of solutions, there are examples of linear partial differential equations whose coefficients have derivatives of all orders (which are nevertheless not analytic) but which have no solutions at all: see Lewy (1957). Even if the solution of a partial differential equation exists and is unique, it may nevertheless have undesirable properties. The mathematical study of these questions is usually in the more powerful context of weak solutions.

An example of pathological behavior is the sequence (depending upon *n*) of Cauchy problems for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with boundary conditions

$$u(x,0) = 0,$$
$$\frac{\partial u}{\partial y}(x,0) = \frac{\sin(nx)}{n},$$

where n is an integer. The derivative of u with respect to y approaches 0 uniformly in x as n increases, but the solution is

$$u(x,y) = \frac{\sinh(ny)\sin(nx)}{n^2}.$$

This solution approaches infinity if nx is not an integer multiple of π for any non-zero value of y. The Cauchy problem for the Laplace equation is called *ill-posed* or *not well posed*, since the solution does not depend continuously upon the data of the problem. Such ill-posed problems are not usually satisfactory for physical applications.

Notation [edit]

In PDEs, it is common to denote partial derivatives using subscripts. That is:

$$u_x = \frac{\partial u}{\partial x}$$
$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$
$$u_{xy} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Especially in physics, del or Nabla (∇) is often used to denote spatial derivatives, and $\dot{u}\,\ddot{u}$ for time derivatives. For example, the wave equation (described below) can be written as

$$\ddot{u} = c^2\nabla^2 u$$

or

$$\ddot{u} = c^2\Delta u$$

where Δ is the Laplace operator.

Examples [edit]

Heat equation in one space dimension [edit]

See also: Heat equation

The equation for conduction of heat in one dimension for a homogeneous body has

$$u_t = \alpha u_{xx}$$

where $u(t,x)$ is temperature, and α is a positive constant that describes the rate of diffusion. The Cauchy problem for this equation consists in specifying $u(0, x)= f(x)$, where $f(x)$ is an arbitrary function.

General solutions of the heat equation can be found by the method of separation of variables. Some examples appear in the heat equation article. They are examples of Fourier series for periodic f and Fourier transforms for non-periodic f. Using the Fourier transform, a general solution of the heat equation has the form

$$u(t,x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\xi)e^{-\alpha\xi^2 t}e^{i\xi x}d\xi,$$

where F is an arbitrary function. To satisfy the initial condition, F is given by the Fourier transform of f, that is

$$F(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}\,dx.$$

If f represents a very small but intense source of heat, then the preceding integral can be approximated by the delta distribution, multiplied by the strength of the source. For a source whose strength is normalized to 1, the result is

$$F(\xi) = \frac{1}{\sqrt{2\pi}},$$

and the resulting solution of the heat equation is

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\xi^2 t} e^{i\xi x} d\xi.$$

This is a Gaussian integral. It may be evaluated to obtain

$$u(t,x) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

This result corresponds to the normal probability density for *x* with mean 0 and variance 2α*t*. The heat equation and similar diffusion equations are useful tools to study random phenomena.

Wave equation in one spatial dimension [edit]

The wave equation is an equation for an unknown function *u*(*k*, *x*) of the form

$$u_{kk} = m^2 u_{xx}.$$

Here *u* might describe the displacement of a stretched string from equilibrium, or the difference in air pressure in a tube, or the magnitude of an electromagnetic field in a tube, and *m* is a number that corresponds to the velocity of the wave. The Cauchy problem for this equation consists in prescribing the initial displacement and velocity of a string or other medium:

$$u(0,x) = f(x),$$
$$u_k(0,x) = g(x),$$

where *f* and *g* are arbitrary given functions. The solution of this problem is given by d'Alembert's formula:

$$u(k,x) = \tfrac{1}{2}\left[f(x-mk) + f(x+mk)\right] + \frac{1}{2m}\int_{x-mk}^{x+mk} g(y)\,dy.$$

This formula implies that the solution at (*k*,*x*) depends only upon the data on the segment of the initial line that is cut out by the characteristic curves

$$x - mk = \text{constant}, \quad x + mk = \text{constant},$$

that are drawn backwards from that point. These curves correspond to signals that propagate with velocity *m* forward and backward. Conversely, the influence of the data at any given point on the initial line propagates with the finite velocity *m*: there is no effect outside a triangle through that point whose sides are characteristic curves. This behavior is very different from the solution for the heat equation, where the effect of a point source appears (with small amplitude) instantaneously at every point in space. The solution given above is also valid if *k* < 0, and the explicit formula shows that the solution depends smoothly upon the data: both the forward and backward Cauchy problems for the wave equation are well-posed.

Generalised heat-like equation in one space dimension [edit]

Where heat-like equation means equations of the form:

$$\frac{\partial u}{\partial t} = \hat{H}u + f(x,t)u + g(x,t)$$

where $\hat{H}$ is a Sturm–Liouville operator subject to the boundary conditions:

$$u(x,0) = h(x).$$

Then:

If:

$$\hat{H}X_n = \lambda_n X_n$$
$$X_n(a) = X_n(b) = 0$$
$$\dot{a}_n(t) - \lambda_n a_n(t) - \sum_m (X_n f(x,t), X_m) a_m(t) = (g(x,t), X_n)$$
$$a_n(0) = \frac{(h(x), X_n)}{(X_n, X_n)}$$
$$u(x,t) = \sum_n a_n(t) X_n(x)$$

where

$$(f,g) = \int_a^b f(x)g(x)w(x)\,dx.$$

Spherical waves [edit]

Spherical waves are waves whose amplitude depends only upon the radial distance *r* from a central point source. For such waves, the three-dimensional wave equation takes the form

$$u_{tt} = c^2 \left[u_{rr} + \frac{2}{r}u_r\right].$$

This is equivalent to

$$(ru)_{tt} = c^2 \left[(ru)_{rr}\right],$$

and hence the quantity *ru* satisfies the one-dimensional wave equation. Therefore a general solution for spherical waves has the form

$$u(t,r) = \frac{1}{r}\left[F(r-ct) + G(r+ct)\right],$$

where *F* and *G* are completely arbitrary functions. Radiation from an antenna corresponds to the case where *G* is identically zero. Thus the wave form transmitted from an antenna has no distortion in time: the only distorting factor is 1/*r*. This feature of undistorted propagation of waves is not present if there are two spatial dimensions.

Laplace equation in two dimensions [edit]

The Laplace equation for an unknown function of two variables φ has the form

$$\varphi_{xx} + \varphi_{yy} = 0.$$

Solutions of Laplace's equation are called harmonic functions.

Connection with holomorphic functions [edit]

Solutions of the Laplace equation in two dimensions are intimately connected with analytic functions of a complex variable (a.k.a. holomorphic functions): the real and imaginary parts of any analytic function are **conjugate harmonic** functions: they both satisfy the Laplace equation, and their gradients are orthogonal. If *f=u+iv*, then the Cauchy–Riemann equations state that

$$u_x = v_y, \quad v_x = -u_y,$$

and it follows that

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

Conversely, given any harmonic function in two dimensions, it is the real part of an analytic function, at least locally. Details are given in Laplace equation.

A typical boundary value problem [edit]

A typical problem for Laplace's equation is to find a solution that satisfies arbitrary values on the boundary of a domain. For example, we may seek a harmonic function that takes on the values *u*(θ) on a circle of radius one. The solution was given by Poisson:

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\theta')}u(\theta')d\theta'.$$

Petrovsky (1967, p. 248) shows how this formula can be obtained by summing a Fourier series for φ. If $r < 1$, the derivatives of φ may be computed by differentiating under the integral sign, and one can verify that φ is analytic, even if *u* is continuous but not necessarily differentiable. This behavior is typical for solutions of elliptic partial differential equations: the solutions may be much more smooth than the boundary data. This is in contrast to solutions of the wave equation, and more general hyperbolic partial differential equations, which typically have no more derivatives than the data.

Euler–Tricomi equation [edit]

The Euler–Tricomi equation is used in the investigation of transonic flow.

$$u_{xx} = xu_{yy}.$$

Advection equation [edit]

The advection equation describes the transport of a conserved scalar ψ in a velocity field **u** = (*u*, *v*, *w*). It is:

$$\psi_t + (u\psi)_x + (v\psi)_y + (w\psi)_z = 0.$$

If the velocity field is solenoidal (that is, ∇·**u** = 0), then the equation may be simplified to

$$\psi_t + u\psi_x + v\psi_y + w\psi_z = 0.$$

In the one-dimensional case where *u* is not constant and is equal to ψ, the equation is referred to as Burgers' equation.

Ginzburg–Landau equation [edit]

The Ginzburg–Landau equation is used in modelling superconductivity. It is

$$iu_t + pu_{xx} + q|u|^2u = i\gamma u$$

where $p,q \in \mathbf{C}$ and $\gamma \in \mathbf{R}$ are constants and *i* is the imaginary unit.

The Dym equation [edit]

The Dym equation is named for Harry Dym and occurs in the study of solitons. It is

$$u_t = u^3 u_{xxx}.$$

Initial-boundary value problems [edit]

Main article: Boundary value problem

Many problems of mathematical physics are formulated as initial-boundary value problems.

Vibrating string [edit]

If the string is stretched between two points where x=0 and x=L and *u* denotes the amplitude of the displacement of the string, then *u* satisfies the one-dimensional wave equation in the region where $0 < x < L$ and *t* is unlimited. Since the string is tied down at the ends, *u* must also satisfy the boundary conditions

$$u(t,0) = 0, \quad u(t,L) = 0,$$

as well as the initial conditions

$$u(0,x) = f(x), \quad u_t(0,x) = g(x).$$

The method of separation of variables for the wave equation

$$u_{tt} = c^2 u_{xx},$$

leads to solutions of the form

$$u(t,x) = T(t)X(x),$$

where

$$T'' + k^2c^2T = 0, \quad X'' + k^2X = 0,$$

where the constant *k* must be determined. The boundary conditions then imply that *X* is a multiple of sin *kx*, and *k* must have the form

$$k = \frac{n\pi}{L},$$

where *n* is an integer. Each term in the sum corresponds to a mode of vibration of the string. The mode with n = 1 is called the fundamental mode, and the frequencies of the other modes are all multiples of this frequency. They form the overtone series of the string, and they are the basis for musical acoustics. The initial conditions may then be satisfied by representing *f* and *g* as infinite sums of these modes. Wind instruments typically correspond to vibrations of an air column with one end open and one end closed. The corresponding boundary conditions are

$$X(0) = 0, \quad X'(L) = 0.$$

The method of separation of variables can also be applied in this case, and it leads to a series of odd overtones.

The general problem of this type is solved in Sturm–Liouville theory.

Vibrating membrane [edit]

If a membrane is stretched over a curve *C* that forms the boundary of a domain *D* in the plane, its vibrations are governed by the wave equation

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy},$$

if t>0 and (x,y) is in *D*. The boundary condition is $u(t,x,y) = 0$ if (x,y) is on *C*. The method of separation of variables leads to the form

$$u(t,x,y) = T(t)v(x,y),$$

which in turn must satisfy

$$\frac{1}{c^2}T'' + k^2T = 0,$$
$$v_{xx} + v_{yy} + k^2v = 0.$$

The latter equation is called the Helmholtz Equation. The constant *k* must be determined to allow a non-trivial *v* to satisfy the boundary condition on *C*. Such values of k^2 are called the eigenvalues of the Laplacian in *D*, and the associated solutions are the eigenfunctions of the Laplacian in *D*. The Sturm–Liouville theory may be extended to this elliptic eigenvalue problem (Jost, 2002).

Other examples [edit]

The Schrödinger equation is a PDE at the heart of non-relativistic quantum mechanics. In the WKB approximation it is the Hamilton–Jacobi equation.

Except for the Dym equation and the Ginzburg–Landau equation, the above equations are **linear** in the sense that they can be written in the form *Au* = *f* for a given linear operator *A* and a given function *f*. Other important non-linear equations include the Navier–Stokes equations describing the flow of fluids, and Einstein's field equations of general relativity.

Also see the list of non-linear partial differential equations.

Classification [edit]

Some linear, second-order partial differential equations can be classified as parabolic, hyperbolic and elliptic. Others such as the Euler–Tricomi equation have different types in different regions. The classification provides a guide to appropriate initial and boundary conditions, and to smoothness of the solutions.

Equations of first order [edit]

Main article: First-order partial differential equation

Equations of second order [edit]

Assuming $u_{xy} = u_{yx}$, the general second-order PDE in two independent variables has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots \text{(lower order terms)} = 0,$$

where the coefficients *A*, *B*, *C* etc. may depend upon *x* and *y*. If $A^2 + B^2 + C^2 > 0$ over a region of the xy plane, the PDE is second-order in that region. This form is analogous to the equation for a conic section:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

More precisely, replacing ∂_x by *X*, and likewise for other variables (formally this is done by a Fourier transform), converts a constant-coefficient PDE into a polynomial of the same degree, with the top degree (a homogeneous polynomial, here a quadratic form) being most significant for the classification.

Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - 4AC$, the same can be done for a second-order PDE at a given point. However, the discriminant in a PDE is given by $B^2 - AC$, due to the convention of the *xy* term being 2*B* rather than *B*; formally, the discriminant (of the associated quadratic form) is $(2B)^2 - 4AC = 4(B^2 - AC)$, with the factor of 4 dropped for simplicity.

1. $B^2 - AC < 0$: solutions of elliptic PDEs are as smooth as the coefficients allow, within the interior of the region where the equation and solutions are defined. For example, solutions of Laplace's equation are analytic within the domain where they are defined, but solutions may assume boundary values that are not smooth. The motion of a fluid at subsonic speeds can be approximated with elliptic PDEs, and the Euler–Tricomi equation is elliptic where $x < 0$.
2. $B^2 - AC = 0$: equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases. The Euler–Tricomi equation has parabolic type on the line where $x = 0$.
3. $B^2 - AC > 0$: hyperbolic equations retain any discontinuities of functions or derivatives in the initial data. An example is the wave equation. The motion of a fluid at supersonic speeds can be approximated with hyperbolic PDEs, and the Euler–Tricomi equation is hyperbolic where $x > 0$.

If there are *n* independent variables x_1, x_2 , ..., x_n, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower-order terms} = 0.$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix $a_{i,j}$..

1. Elliptic: The eigenvalues are all positive or all negative.
2. Parabolic : The eigenvalues are all positive or all negative, save one that is zero.
3. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.
4. Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultrahyperbolic equations (Courant and Hilbert, 1962).

Systems of first-order equations and characteristic surfaces [edit]

The classification of partial differential equations can be extended to systems of first-order equations, where the unknown u is now a vector with m components, and the coefficient matrices A_ν are m by m matrices for $\nu = 1, \ldots, n$. The partial differential equation takes the form

$$Lu = \sum_{\nu=1}^{n} A_\nu \frac{\partial u}{\partial x_\nu} + B = 0,$$

where the coefficient matrices A_ν and the vector B may depend upon x and u. If a hypersurface S is given in the implicit form

$$\varphi(x_1, x_2, \ldots, x_n) = 0,$$

where φ has a non-zero gradient, then S is a **characteristic surface** for the operator L at a given point if the characteristic form vanishes:

$$Q\left(\frac{\partial\varphi}{\partial x_1}, \ldots, \frac{\partial\varphi}{\partial x_n}\right) = \det\left[\sum_{\nu=1}^{n} A_\nu \frac{\partial\varphi}{\partial x_\nu}\right] = 0.$$

The geometric interpretation of this condition is as follows: if data for u are prescribed on the surface S, then it may be possible to determine the normal derivative of u on S from the differential equation. If the data on S and the differential equation determine the normal derivative of u on S, then S is non-characteristic. If the data on S and the differential equation *do not* determine the normal derivative of u on S, then the surface is **characteristic**, and the differential equation restricts the data on S: the differential equation is *internal* to S.

1. A first-order system $Lu=0$ is *elliptic* if no surface is characteristic for L: the values of u on S and the differential equation always determine the normal derivative of u on S.
2. A first-order system is *hyperbolic* at a point if there is a **space-like** surface S with normal ξ at that point. This means that, given any non-trivial vector η orthogonal to ξ, and a scalar multiplier λ, the equation $Q(\lambda\xi + \eta) = 0$ has m real roots $\lambda_1, \lambda_2, \ldots, \lambda_m$. The system is **strictly hyperbolic** if these roots are always distinct. The geometrical interpretation of this condition is as follows: the characteristic form $Q(\zeta) = 0$ defines a cone (the normal cone) with homogeneous coordinates ζ. In the hyperbolic case, this cone has m sheets, and the axis $\zeta = \lambda\,\xi$ runs inside these sheets: it does not intersect any of them. But when displaced from the origin by η, this axis intersects every sheet. In the elliptic case, the normal cone has no real sheets.

Equations of mixed type [edit]

If a PDE has coefficients that are not constant, it is possible that it will not belong to any of these categories but rather be of **mixed type**. A simple but important example is the Euler–Tricomi equation

$$u_{xx} = xu_{yy},$$

which is called **elliptic-hyperbolic** because it is elliptic in the region $x < 0$, hyperbolic in the region $x > 0$, and degenerate parabolic on the line $x = 0$.

Infinite-order PDEs in quantum mechanics [edit]

In the phase space formulation of quantum mechanics, one may consider the quantum Hamilton's equations for trajectories of quantum particles. These equations are infinite-order PDEs. However, in the semiclassical expansion, one has a finite system of ODEs at any fixed order of $\hbar$. The evolution equation of the Wigner function is also an infinite-order PDE. The quantum trajectories are quantum characteristics, with the use of which one could calculate the evolution of the Wigner function.

Analytical methods to solve PDEs [edit]

Separation of variables [edit]

Main article: Separable partial differential equation

Linear PDEs can be reduced to systems of ordinary differential equations by the important technique of separation of variables. This technique rests on a characteristic of solutions to differential equations: if one can find any solution that solves the equation and satisfies the boundary conditions, then it is *the* solution (this also applies to ODEs). We assume as an ansatz that the dependence of a solution on the parameters space and time can be written as a product of terms that each depend on a single parameter, and then see if this can be made to solve the problem.[1]

In the method of separation of variables, one reduces a PDE to a PDE in fewer variables, which is an ordinary differential equation if in one variable – these are in turn easier to solve.

This is possible for simple PDEs, which are called separable partial differential equations, and the domain is generally a rectangle (a product of intervals). Separable PDEs correspond to diagonal matrices – thinking of "the value for fixed *x*" as a coordinate, each coordinate can be understood separately.

This generalizes to the method of characteristics, and is also used in integral transforms.

Method of characteristics [edit]

Main article: Method of characteristics

In special cases, one can find characteristic curves on which the equation reduces to an ODE – changing coordinates in the domain to straighten these curves allows separation of variables, and is called the method of characteristics.

More generally, one may find characteristic surfaces.

Integral transform [edit]

An integral transform may transform the PDE to a simpler one, in particular a separable PDE. This corresponds to diagonalizing an operator.

An important example of this is Fourier analysis, which diagonalizes the heat equation using the eigenbasis of sinusoidal waves.

If the domain is finite or periodic, an infinite sum of solutions such as a Fourier series is appropriate, but an integral of solutions such as a Fourier integral is generally required for infinite domains. The solution for a point source for the heat equation given above is an example for use of a Fourier integral.

Change of variables [edit]

Often a PDE can be reduced to a simpler form with a known solution by a suitable change of variables. For example the Black–Scholes PDE

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

is reducible to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

by the change of variables (for complete details see Solution of the Black Scholes Equation at the Wayback Machine (archived April 11, 2008))

$$\begin{aligned}
&V(S,t) = Kv(x,\tau)\\
&x = \ln\left(\tfrac{S}{K}\right)\\
&\tau = \tfrac{1}{2}\sigma^2(T-t)\\
&v(x,\tau) = \exp(-\alpha x - \beta\tau)u(x,\tau).
\end{aligned}$$

Fundamental solution [edit]

Main article: Fundamental solution

Inhomogeneous equations can often be solved (for constant coefficient PDEs, always be solved) by finding the fundamental solution (the solution for a point source), then taking the convolution with the boundary conditions to get the solution.

This is analogous in signal processing to understanding a filter by its impulse response.

Superposition principle [edit]

Because any superposition of solutions of a linear, homogeneous PDE is again a solution, the particular solutions may then be combined to obtain more general solutions. if u1 and u2 are solutions of a homogeneous linear pde in same region R, then u= c1u1+c2u2 with any constants c1 and c2 is also a solution of that pde in that same region....

Methods for non-linear equations [edit]

See also the list of nonlinear partial differential equations.

There are no generally applicable methods to solve non-linear PDEs. Still, existence and uniqueness results (such as the Cauchy–Kowalevski theorem) are often possible, as are proofs of important qualitative and quantitative properties of solutions (getting these results is a major part of analysis). Computational solution to the nonlinear PDEs, the split-step method, exist for specific equations like nonlinear Schrödinger equation.

Nevertheless, some techniques can be used for several types of equations. The h-principle is the most powerful method to solve underdetermined equations. The Riquier–Janet theory is an effective method for obtaining information about many analytic overdetermined systems.

The method of characteristics (similarity transformation method) can be used in some very special cases to solve partial differential equations.

In some cases, a PDE can be solved via perturbation analysis in which the solution is considered to be a correction to an equation with a known solution. Alternatives are numerical analysis techniques from simple finite difference schemes to the more mature multigrid and finite element methods. Many interesting problems in science and engineering are solved in this way using computers, sometimes high performance supercomputers.

Lie group method [edit]

From 1870 Sophus Lie's work put the theory of differential equations on a more satisfactory foundation. He showed that the integration theories of the older mathematicians can, by the introduction of what are now called Lie groups, be referred to a common source; and that ordinary differential equations which admit the same infinitesimal transformations present comparable difficulties of integration. He also emphasized the subject of transformations of contact.

A general approach to solve PDE's uses the symmetry property of differential equations, the continuous infinitesimal transformations of solutions to solutions (Lie theory). Continuous group theory, Lie algebras and differential geometry are used to understand the structure of linear and nonlinear partial differential equations for generating integrable equations, to find its Lax pairs, recursion operators, Bäcklund transform and finally finding exact analytic solutions to the PDE.

Symmetry methods have been recognized to study differential equations arising in mathematics, physics, engineering, and many other disciplines.

Semianalytical methods [edit]

The adomian decomposition method, the Lyapunov artificial small parameter method, and He's homotopy perturbation method are all special cases of the more general homotopy analysis method. These are series expansion methods, and except for the Lyapunov method, are independent of small physical parameters as compared to the well known perturbation theory, thus giving these methods greater flexibility and solution generality.

Numerical methods to solve PDEs [edit]

The three most widely used numerical methods to solve PDEs are the finite element method (FEM), finite volume methods (FVM) and finite difference methods (FDM). The FEM has a prominent position among these methods and especially its exceptionally efficient higher-order version hp-FEM. Other versions of FEM include the generalized finite element method (GFEM), extended finite element method (XFEM), spectral finite element method (SFEM), meshfree finite element method, discontinuous Galerkin finite element method (DGFEM), Element-Free Galerkin Method (EFGM), Interpolating Element-Free Galerkin Method (IEFGM), etc.

Finite element method [edit]

Main article: Finite element method

The finite element method (FEM) (its practical application often known as finite element analysis (FEA)) is a numerical technique for finding approximate solutions of partial differential equations (PDE) as well as of integral equations. The solution approach is based either on eliminating the differential equation completely (steady state problems), or rendering the PDE into an approximating system of ordinary differential equations, which are then numerically integrated using standard techniques such as Euler's method, Runge–Kutta, etc.

Finite difference method [edit]

Main article: Finite difference method

Finite-difference methods are numerical methods for approximating the solutions to differential equations using finite difference equations to approximate derivatives.

Finite volume method [edit]

Main article: Finite volume method

Similar to the finite difference method or finite element method, values are calculated at discrete places on a meshed geometry. "Finite volume" refers to the small volume surrounding each node point on a mesh. In the finite volume method, surface integrals in a partial differential equation that contain a divergence term are converted to volume integrals, using the Divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative.

See also [edit]

- Dirichlet boundary condition
- Jet bundle
- Laplace transform applied to differential equations
- List of dynamical systems and differential equations topics
- Matrix differential equation
- Neumann boundary condition
- Numerical partial differential equations
- Partial differential algebraic equation
- Recurrence relation
- Robin boundary condition
- Stochastic processes and boundary value problems

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