Multigrid method

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(Redirected from Multigrid methods)

Multigrid (MG) methods in numerical analysis are a group of algorithms for solving differential equations using a hierarchy of discretizations. They are an example of a class of techniques called multiresolution methods, very useful in (but not limited to) problems exhibiting multiple scales of behavior. For example, many basic relaxation methods exhibit different rates of convergence for short- and long-wavelength components, suggesting these different scales be treated differently, as in a Fourier analysis approach to multigrid.[1] MG methods can be used as solvers as well as preconditioners.

The main idea of multigrid is to accelerate the convergence of a basic iterative method by *global* correction from time to time, accomplished by solving a coarse problem. This principle is similar to interpolation between coarser and finer grids. The typical application for multigrid is in the numerical solution of elliptic partial differential equations in two or more dimensions.[2]

Multigrid methods can be applied in combination with any of the common discretization techniques. For example, the finite element method may be recast as a multigrid method.[3] In these cases, multigrid methods are among the fastest solution techniques known today. In contrast to other methods, multigrid methods are general in that they can treat arbitrary regions and boundary conditions. They do not depend on the separability of the equations or other special properties of the equation. They have also been widely used for more-complicated non-symmetric and nonlinear systems of equations, like the Lamé equations of elasticity or the Navier-Stokes equations.[4]

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Algorithm [edit]

There are many variations of multigrid algorithms, but the common features are that a hierarchy of discretizations (grids) is considered. The important steps are:[5][6]

- **Smoothing** – reducing high frequency errors, for example using a few iterations of the Gauss–Seidel method.
- **Restriction** – downsampling the residual error to a coarser grid.
- **Interpolation** or **prolongation** – interpolating a correction computed on a coarser grid into a finer grid.

Computational cost [edit]

This approach has the advantage over other methods that it often scales linearly with the number of discrete nodes used. In other words, it can solve these problems to a given accuracy in a number of operations that is proportional to the number of unknowns.

Assume that one has a differential equation which can be solved approximately (with a given accuracy) on a grid i with a given grid point density N_i. Assume furthermore that a solution on any grid N_i may be obtained with a given effort $W_i = \rho K N_i$ from a solution on a coarser grid $i+1$. Here, $\rho = N_{i+1}/N_i < 1$ is the ratio of grid points on "neighboring" grids and is assumed to be constant throughout the grid hierarchy, and K is some constant modeling the effort of computing the result for one grid point.

The following recurrence relation is then obtained for the effort of obtaining the solution on grid k:

$$W_k = W_{k+1} + \rho K N_k$$

And in particular, we find for the finest grid N_1 that

$$W_1 = W_2 + \rho K N_1$$

Combining these two expressions (and using $N_k = \rho^{k-1} N_1$) gives

$$W_1 = K N_1 \sum_{p=0}^{n} \rho^p$$

Using the geometric series, we then find (for finite n)

$$W_1 < K N_1 \frac{1}{1-\rho}$$

that is, a solution may be obtained in $O(N)$ time.

Multigrid preconditioning [edit]

A multigrid method with an intentionally reduced tolerance can be used as an efficient preconditioner for an external iterative solver. The solution may still be obtained in $O(N)$ time as well as in the case where the multigrid method is used as a solver. Multigrid preconditioning is used in practice even for linear systems. Its main advantage versus a purely multigrid solver is particularly clear for nonlinear problems, e.g., eigenvalue problems.

Generalized multigrid methods [edit]

Multigrid methods can be generalized in many different ways. They can be applied naturally in a time-stepping solution of parabolic partial differential equations, or they can be applied directly to time-dependent partial differential equations.[7] Research on multilevel techniques for hyperbolic partial differential equations is underway.[8] Multigrid methods can also be applied to integral equations, or for problems in statistical physics.[9]

Other extensions of multigrid methods include techniques where no partial differential equation nor geometrical problem background is used to construct the multilevel hierarchy.[10] Such **algebraic multigrid methods** (AMG) construct their hierarchy of operators directly from the system matrix, and the levels of the hierarchy are simply subsets of unknowns without any geometric interpretation. Thus, AMG methods become true black-box solvers for sparse matrices. However, AMG is regarded as advantageous mainly where geometric multigrid is too difficult to apply.[11]

Another set of multiresolution methods is based upon wavelets. These wavelet methods can be combined with multigrid methods.[12][13] For example, one use of wavelets is to reformulate the finite element approach in terms of a multilevel method.[14]

Adaptive multigrid exhibits adaptive mesh refinement, that is, it adjusts the grid as the computation proceeds, in a manner dependent upon the computation itself.[15] The idea is to increase resolution of the grid only in regions of the solution where it is needed.

Multigrid in time methods [edit]

Multigrid methods have also been adopted for the solution of initial value problems.[16] Of particular interest here are parallel-in-time multigrid methods:[17] in contrast to classical Runge-Kutta or linear multistep methods, they can offer concurrency in temporal direction. The well known Parareal parallel-in-time integration method can also be reformulated as a two-level multigrid in time.

Notes [edit]

1. ^ Roman Wienands; Wolfgang Joppich (2005). *Practical Fourier analysis for multigrid methods*. CRC Press. p. 17. ISBN 1-58488-492-4.
2. ^ U. Trottenberg; C. W. Oosterlee; A. Schüller (2001). *Multigrid*. Academic Press. ISBN 0-12-701070-X.
3. ^ Yu Zhu; Andreas C. Cangellaris (2006). *Multigrid finite element methods for electromagnetic field modeling*. Wiley. p. 132 *ff*. ISBN 0-471-74110-8.
4. ^ Shah, Tasneem Mohammad (1989). *Analysis of the multigrid method* (Thesis). Oxford University. Retrieved 8 January 2013.
5. ^ M. T. Heath (2002). "Section 11.5.7 Multigrid Methods". *Scientific Computing: An Introductory Survey*. McGraw-Hill Higher Education. p. 478 *ff*. ISBN 0-07-112229-X.
6. ^ P. Wesseling (1992). *An Introduction to Multigrid Methods*. Wiley. ISBN 0-471-93083-0.
7. ^ F. Hülsemann; M. Kowarschik; M. Mohr; U. Rüde (2006). "Parallel geometric multigrid". In Are Magnus Bruaset, Aslak Tveito. *Numerical solution of partial differential equations on parallel computers*. Birkhäuser. p. 165.

ISBN 3-540-29076-1.
8. ^ For example, J. Blažek (2001). *Computational fluid dynamics: principles and applications*. Elsevier. p. 305. ISBN 0-08-043009-0. and Achi Brandt and Rima Gandlin (2003). "Multigrid for Atmospheric Data Assimilation: Analysis". In Thomas Y. Hou, Eitan Tadmor. *Hyperbolic problems: theory, numerics, applications: proceedings of the Ninth International Conference on Hyperbolic Problems of 2002*. Springer. p. 369. ISBN 3-540-44333-9.
9. ^ Achi Brandt (2002). "Multiscale scientific computation: review". In Timothy J. Barth, Tony Chan, Robert Haimes. *Multiscale and multiresolution methods: theory and applications*. Springer. p. 53. ISBN 3-540-42420-2.
10. ^ Yair Shapira (2003). "Algebraic multigrid". *Matrix-based multigrid: theory and applications*. Springer. p. 66. ISBN 1-4020-7485-9.
11. ^ U. Trottenberg; C. W. Oosterlee; A. Schüller. op. cit.. p. 417. ISBN 0-12-701070-X.
12. ^ Björn Engquist; Olof Runborg (2002). "Wavelet-based numerical homogenization with applications". In Timothy J. Barth, Tony Chan, Robert Haimes. *Multiscale and Multiresolution Methods*. Vol. 20 of Lecture notes in computational science and engineering. Springer. p. 140 *ff*. ISBN 3-540-42420-2.
13. ^ U. Trottenberg; C. W. Oosterlee; A. Schüller. op. cit.. ISBN 0-12-701070-X.
14. ^ Albert Cohen (2003). *Numerical Analysis of Wavelet Methods*. Elsevier. p. 44. ISBN 0-444-51124-5.
15. ^ U. Trottenberg; C. W. Oosterlee; A. Schüller. "Chapter 9: Adaptive Multigrid". op. cit.. p. 356. ISBN 0-12-701070-X.
16. ^ Hackbusch, Wolfgang (1985). "Parabolic multi-grid methods". *Computing Methods in Applied Sciences and Engineering, VI* (North-Holland Publishing Co): 189–197. Retrieved August 2015.
17. ^ Hortom, Graham (1992). "The time-parallel multigrid method". *Communications in Applied Numerical Methods* (Wiley) **8** (9): 585–595. doi:10.1002/cnm.1630080906. Retrieved August 2015.

References [edit]

- G. P. Astrachancev (1971), An iterative method of solving elliptic net problems. USSR Comp. Math. Math. Phys. 11, 171–182.
- N. S. Bakhvalov (1966), On the convergence of a relaxation method with natural constraints on the elliptic operator. USSR Comp. Math. Math. Phys. 6, 101–13.
- Achi Brandt (April 1977), "Multi-Level Adaptive Solutions to Boundary-Value Problems", *Mathematics of Computation*, **31**: 333–90.
- William L. Briggs, Van Emden Henson, and Steve F. McCormick (2000), *A Multigrid Tutorial* (2nd ed.), Philadelphia: Society for Industrial and Applied Mathematics, ISBN 0-89871-462-1.
- R. P. Fedorenko (1961), A relaxation method for solving elliptic difference equations. USSR Comput. Math. Math. Phys. 1, p. 1092.
- R. P. Fedorenko (1964), The speed of convergence of one iterative process. USSR Comput. Math. Math. Phys. 4, p. 227.
- Press, W. H.; Teukolsky, S. A.; Vetterling, W. T.; Flannery, B. P. (2007). "Section 20.6. Multigrid Methods for Boundary Value Problems". *Numerical Recipes: The Art of Scientific Computing* (3rd ed.). New York: Cambridge University Press. ISBN 978-0-521-88068-8.

External links [edit]

- Repository for multigrid, multilevel, multiscale, aggregation, defect correction, and domain decomposition methods
- Multigrid tutorial
- Algebraic multigrid tutorial[*dead link*]
- Links to AMG presentations[*dead link*]

v · t · e	**Numerical partial differential equations by method** [hide]	
Finite difference	Parabolic	Forward-time central-space (FTCS) · Crank–Nicolson
	Hyperbolic	Lax–Friedrichs · Lax–Wendroff · MacCormack · Upwind · Method of characteristics
	Others	Alternating direction-implicit (ADI) · Finite-difference time-domain (FDTD)
Finite volume	Godunov · High-resolution · Monotonic upstream-centered (MUSCL) · Advection upstream-splitting (AUSM) · Riemann solver	
Finite element	hp-FEM · Extended (XFEM) · Discontinuous Galerkin (DG) · Spectral element (SEM) · Mortar	
Meshless/Meshfree	Smoothed-particle hydrodynamics (SPH) · Material point method (MPM)	
Domain decomposition	Schur complement · Fictitious domain · Schwarz alternating (additive · abstract additive) · Neumann–Dirichlet · Neumann–Neumann · Poincaré–Steklov operator · Balancing (BDD) · Balancing by constraints (BDDC) · Tearing and interconnect (FETI) · FETI-DP	
Others	Spectral · Pseudospectral (DVR) · Method of lines · **Multigrid** · Collocation · Level set · Boundary element · Immersed boundary · Analytic element · Particle-in-cell · Isogeometric analysis	

Categories: Numerical analysis | Partial differential equations | Wavelets