Hermite interpolation

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In numerical analysis, **Hermite interpolation**, named after Charles Hermite, is a method of interpolating data points as a polynomial function. The generated **Hermite polynomial** is closely related to the Newton polynomial, in that both are derived from the calculation of divided differences.

Unlike Newton interpolation, Hermite interpolation matches an unknown function both in observed value, and the observed value of its first m derivatives. This means that $n(m+1)$ values

$$\begin{matrix} (x_0, y_0), & (x_1, y_1), & \dots, & (x_{n-1}, y_{n-1}), \\ (x_0, y_0'), & (x_1, y_1'), & \dots, & (x_{n-1}, y_{n-1}'), \\ \vdots & \vdots & & \vdots \\ (x_0, y_0^{(m)}), & (x_1, y_1^{(m)}), & \dots, & (x_{n-1}, y_{n-1}^{(m)}) \end{matrix}$$

must be known, rather than just the first n values required for Newton interpolation. The resulting polynomial may have degree at most $n(m+1)-1$, whereas the Newton polynomial has maximum degree $n-1$. (In the general case, there is no need for m to be a fixed value; that is, some points may have more known derivatives than others. In this case the resulting polynomial may have degree $N-1$, with N the number of data points.)

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Simple case [edit]

When using divided differences to calculate the Hermite polynomial of a function f, the first step is to copy each point m times. (Here we will consider the simplest case $m = 1$ for all points.) Therefore, given $n+1$ data points $x_0, x_1, x_2, \dots, x_n$, and values $f(x_0), f(x_1), \dots, f(x_n)$ and $f'(x_0), f'(x_1), \dots, f'(x_n)$ for a function f that we want to interpolate, we create a new dataset

$$z_0, z_1, \dots, z_{2n+1}$$

such that

$$z_{2i} = z_{2i+1} = x_i.$$

Now, we create a divided differences table for the points $z_0, z_1, \dots, z_{2n+1}$. However, for some divided differences,

$$z_i = z_{i+1} \implies f[z_i, z_{i+1}] = \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} = \frac{0}{0}$$

which is undefined! In this case, we replace the divided difference by $f'(z_i)$. All others are calculated normally.

General case [edit]

In the general case, suppose a given point x_i has k derivatives. Then the dataset $z_0, z_1, \dots, z_N$ contains k identical copies of x_i. When creating the table, divided differences of $j = 2, 3, \dots, k$ identical values will be calculated as

$$\frac{f^{(j)}(x_i)}{j!}.$$

For example,

$$f[x_i, x_i, x_i] = \frac{f''(x_i)}{2}$$

$$f[x_i, x_i, x_i, x_i] = \frac{f^{(3)}(x_i)}{6}$$

etc.

Example [edit]

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Consider the function $f(x) = x^8 + 1$. Evaluating the function and its first two derivatives at $x \in \{-1, 0, 1\}$, we obtain the following data:

x	$f(x)$	$f'(x)$	$f''(x)$
−1	2	−8	56
0	1	0	0
1	2	8	56

Since we have two derivatives to work with, we construct the set $\{z_i\} = \{-1, -1, -1, 0, 0, 0, 1, 1, 1\}$. Our divided difference table is then:

$$
\begin{array}{lllllllll}
z_0 = -1 & f[z_0] = 2 & & & & & & & \\
 & & \frac{f'(z_0)}{1} = -8 & & & & & & \\
z_1 = -1 & f[z_1] = 2 & & \frac{f''(z_1)}{2} = 28 & & & & & \\
 & & \frac{f'(z_1)}{1} = -8 & & f[z_3, z_2, z_1, z_0] = -21 & & & & \\
z_2 = -1 & f[z_2] = 2 & & f[z_3, z_2, z_1] = 7 & & 15 & & & \\
 & & f[z_3, z_2] = -1 & & f[z_4, z_3, z_2, z_1] = -6 & & -10 & & \\
z_3 = 0 & f[z_3] = 1 & & f[z_4, z_3, z_2] = 1 & & 5 & & 4 & \\
 & & \frac{f'(z_3)}{1} = 0 & & f[z_5, z_4, z_3, z_2] = -1 & & -2 & & -1 \\
z_4 = 0 & f[z_4] = 1 & & \frac{f''(z_4)}{2} = 0 & & 1 & & 2 & 1 \\
 & & \frac{f'(z_4)}{1} = 0 & & f[z_6, z_5, z_4, z_3] = 1 & & 2 & & 1 \\
z_5 = 0 & f[z_5] = 1 & & f[z_6, z_5, z_4] = 1 & & 5 & & 4 & \\
 & & f[z_6, z_5] = 1 & & f[z_7, z_6, z_5, z_4] = 6 & & 10 & & \\
z_6 = 1 & f[z_6] = 2 & & f[z_7, z_6, z_5] = 7 & & 15 & & & \\
 & & \frac{f'(z_7)}{1} = 8 & & f[z_8, z_7, z_6, z_5] = 21 & & & & \\
z_7 = 1 & f[z_7] = 2 & & \frac{f''(z_7)}{2} = 28 & & & & & \\
 & & \frac{f'(z_8)}{1} = 8 & & & & & & \\
z_8 = 1 & f[z_8] = 2 & & & & & & &
\end{array}
$$

and the generated polynomial is

$$
\begin{aligned}
P(x) &= 2 - 8(x+1) + 28(x+1)^2 - 21(x+1)^3 + 15x(x+1)^3 - 10x^2(x+1)^3 \\
&\quad + 4x^3(x+1)^3 - 1x^3(x+1)^3(x-1) + x^3(x+1)^3(x-1)^2 \\
&= 2 - 8 + 28 - 21 - 8x + 56x - 63x + 15x + 28x^2 - 63x^2 + 45x^2 - 10x^2 - 21x^3 \\
&\quad + 45x^3 - 30x^3 + 4x^3 + x^3 + x^3 + 15x^4 - 30x^4 + 12x^4 + 2x^4 + x^4 \\
&\quad - 10x^5 + 12x^5 - 2x^5 + 4x^5 - 2x^5 - 2x^5 - x^6 + x^6 - x^7 + x^7 + x^8 \\
&= x^8 + 1.
\end{aligned}
$$

by taking the coefficients from the diagonal of the divided difference table, and multiplying the kth coefficient by $\prod_{i=0}^{k-1}(x - z_i)$, as we would when generating a Newton polynomial.

Error [edit]

Call the calculated polynomial H and original function f. Evaluating a point $x \in [x_0, x_n]$, the error function is

$$f(x) - H(x) = \frac{f^{(K)}(c)}{K!} \prod_i (x - x_i)^{k_i}$$

where c is an unknown within the range $[x_0, x_N]$, K is the total number of data-points, and k_i is the number of derivatives known at each x_i plus one.

See also [edit]

- Cubic Hermite spline
- Newton series, also known as finite differences
- Neville's schema
- Polynomial interpolation
- Lagrange form of the interpolation polynomial
- Bernstein form of the interpolation polynomial
- Chinese remainder theorem - Applications

References [edit]

- Burden, Richard L.; Faires, J. Douglas (2004). *Numerical Analysis*. Belmont: Brooks/Cole.
- Spitzbart, A. (January 1960), "A Generalization of Hermite's Interpolation Formula", *American Mathematical Monthly* **67** (1): 42–46, JSTOR 2308924

External links [edit]

- Hermites Interpolating Polynomial at Mathworld

Categories: Interpolation | Finite differences | Factorial and binomial topics