Binomial Coefficients: Key Properties and Formulas

Definition and basic properties

- **Binomial coefficients, definition:** For $n = 1, 2, \dots$ and $k = 0, 1, \dots, n$, the binomial coefficient $\binom{n}{k}$ ("n choose k") is defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
  (Here $0! = 1$ by definition, so the above formula makes sense when $k = 0$ or $k = n$.)
  NOTE: Use only the $\binom{n}{k}$ notation. Avoid "C" notations such as C_n^k or C_k^n.

- **Alternate definition:** $\boxed{\binom{n}{k} = \frac{n(n-1)\dots(n-k+1)}{k!}}$
  This version is convenient for hand-calculating binomial coefficients.

- **Symmetry property:** $\binom{n}{k} = \binom{n}{n-k}$

- **Special cases:** $\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n$

- **Pascal's Triangle:** In this triangle each entry (except for the 1's on the outside) is obtained by adding the two entries above it. The rows in this triangle give the binomial coefficients $\binom{n}{0}, \binom{n}{1}, \dots, \binom{n}{n-1}, \binom{n}{n}$ with a fixed "numerator" n:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

- **Binomial Theorem:** $\boxed{(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}}$

Combinatorial interpretations of the binomial coefficients

The binomial coefficient $\binom{n}{k}$ can be interpreted in the following ways. *Try to convince yourself that these interpretations are all equivalent, i.e., that they have the same formula as answer.*

- **Main combinatorial interpretation:** The number of ways to select k (distinct) objects out of n given (distinct) objects.
- **Committee selection:** The number of ways to choose a committee of k people in a group of n people.
- **Subset counting:** The number of k-element subsets of an n-element set.
- **Word/string counting:** The number of binary strings of length n with exactly k 1's.
- **Head/tail counting:** The number of ways to get exactly k heads in n coin tosses;
- **Polynomial coefficients:** The coefficient of $x^k y^{n-k}$ when expanding $(x+y)^n$ and collecting terms.

Binomial Formulas

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n} \quad \text{(Binomial Theorem)}$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \quad \text{(Lower summation)}$$

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1} \quad \text{(Upper summation)}$$

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n} \quad \text{(Parallel summation)}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \quad \text{(Square summation)}$$

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r} \quad \text{(Vandermonde identity)}$$

Hints for proofs

- **Binomial Theorem:** Imagine expanding the right-hand side completely, and collecting terms that involve the same number of x's and the same number of y's. How many terms are there that involve k x's and $n-k$ y's? Write out the first few cases (say, $n = 1, 2, 3, 4$) explicitly, to see the pattern.
- **Lower Summation Formula:** Derive from the Binomial Theorem.
- **Parallel Summation Formula:** Derive from the Upper Summation Formula.
- **Square Summation Formula:** Derive from the Vandermonde Identity.
- **Lower Summation Formula, combinatorial proof:** Count the total number of subsets of an n-element set in two ways.
- **Vandermonde Identity, combinatorial proof:** Pick a committee of r people from a group of n men and m women.
- **Upper Summation Formula, combinatorial proof:** Pick a $(k+1)$-element subset of $\{1, 2, \ldots, n+1\}$, and classify according to the largest element.