Comp 7405
Lecture 1: Financial Markets and Probability Review

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Outline

What is Risk

- What comes to mind when you hear the word finance? Money? Investments?
- Arguably risk is the key concept in modern finance. Every transaction can be viewed as the buying or selling of risk.
- Riskless asset:an asset which has a precisely determined future value.
  - The fundamental example is that of a government bond.
  - Buy a government bond for say a \$100 today and know that we will receive say \$5 a year, (called a coupon payment), until a pre-determined date, when we receive our \$100 back.
  - Is it truly riskless?
- Risky asset: an assent which is not riskless. That is, it is an asset of uncertain future value.
- Risk can be regarded as a synonym for uncertainty.
- For example: holding some US dollars, buying an apartment. These are all risky assets.
- The concept of risk is inherent in all investment decisions not just those in the financial markets.

Financial Assets

In this course, the term *Asset* is used to describe any financial object whose value is known at present but is liable to change in the future. Typical examples:

- Stocks: represent part ownership in a company. Owners of stocks may receive dividend payments or a share in the proceeds if the company is sold.
- Indices: are baskets of stocks, may be more liquid and less volatile than individual stocks.
- Bonds: are promises to pay the notional amount (e.g. $100) at some future date, possibly together with fixed periodic interest payments (coupons).
- Commodities: such as gold, oil, or electricity.
- Currencies: for example, the value of HKD in USD.

Risk Diversification and Hedging

- Almost all above financial assets are risky assets (except riskless bonds)
- Two ways to reduce/eliminate risks:
  - diversification: reduces risk by allocating investments among various financial instruments, industries, and other categories.
  - hedging: manipulate an inverstor's risk profile by buying/selling other products.
- Financial Derivatives are such financial instruements: they can be used to increase or reduce risks.

Financial Derivatives

- A derivative is an instrument whose value depends on, or is derived from, the value of another more basic asset. Examples: futures, forwards, swaps, options.
- The more basic asset (or the underlying asset) can be stocks, currencies, interest rates, commodities, debt instruments, electricity, insurance payouts, the weather, etc.
- Many financial transactions have embedded derivatives.
- Call/Put options are the classic and popular examples of financial derivatives. We focus on options in this course.

What are options

An option has a holder (buyer) and a writer (seller).

Call option

- A *call* option gives the *holder* the right (but not the obligation) to purchase from the *writer* a prescribed asset for a prescribed price at a prescribed time in the future.
- Exercise price/strike price: the prescribed purchase price, denoted by K.
- Expiry(maturity) date: the prescribed time, denoted by T.

Put Option

A *put* option gives the *holder* the right (but not the obligation) to sell to the *writer* a prescribed asset for a prescribed price at a prescribed time in the future.

Option types based on exercise style

- *European* options can be exercised only on expiration date; Here "exercise" means "finish the buying (for call) or the selling (for put).
- *American* options can be exercised at any time up to the expiration date.
- The terms *American* and *European* do not refer to the location of the option.
- European options are easier to analyze than American options.
- Do we have *Asian* options? Yes. We will talk about it later.

European call option

Example

Suppose you hold a European call option on HSBC's stock struck at \$95, expiring in three month's time. After three months have elapsed, there could be two possible scenarios:

(a) If HSBC's stock price is higher \$95 at that time, say \$100, you would exercise the option by obtaining one HSBC share from the writer and paying him/her \$95; you then sell the share to the market at \$100 right away, and make a profit $100 - 95 = 5$ dollars.

(b) If the stock price is equal to or below \$95, you would not exercise the option and do nothing, which means the option contract would expire worthless. You make a profit 0 dollars.

European Call Options

Value of Call options

- As the holder, you could either make a profit in case (a) or make no gain/no loss in case (b).
- The writer, on the other hand, cannot gain any money on the expiry date, and may lose unlimited amount in case (a).
- To compensate the risk the writer bears, when the contract is agreed, you as the holder would be expected to pay the writer an amount of money, known as the *value* or *premium* of the option.
- Mathematically, the value of a European call option at the expiry date is

$$C_T^T = \max(S(T) - K, 0). \tag{1}$$

European Put Option

Example

Suppose you hold a European put option on HSBC's stock struck at \$95, expiring in three month's time. After three months:

(a) If HSBC's stock price is higher \$95, you would not exercise the option and do nothing, which means the option contract would expire worthless. You make a profit 0 dollars.

(b) If the stock price is equal to or below \$95, say \$90, you would exercise the option by buying one share from the market with \$90 and selling it to the writer at \$95; you make a profit $95 - 90 = 5$ dollars.

European Put Option

Value of Put options

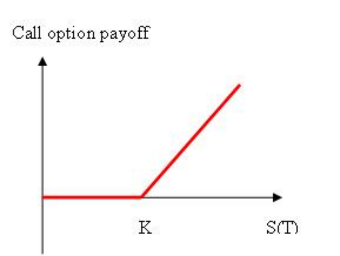
- As the holder, you either make a profit in case (b) or no gain/no loss in case (a).
- The writer cannot gain any money on the expiry date, and may lose a maximum amount \$95 in case (b).
- To compensate the risk, when the contract is agreed, you as the holder would be expected to pay the writer the *value* or *premium* of the option.
- Mathematically, the value of a European put option at the expiry date is

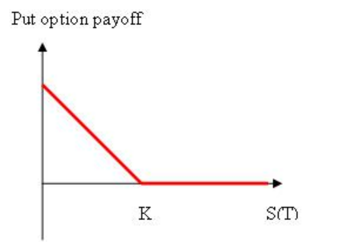
$$P_T^T = \max(K - S(T), 0). \tag{2}$$

Call/Put Option Payoff Diagram for Holder

Hockey Sticks

- At T, the call option payoff is $\max(S(T) - K, 0)$.
- At T, the put option payoff is $\max(K - S(T), 0)$.
- The payoff diagrams can be plotted:





American Options

- Recall that American options can be exercised at any time up to the expiration date while European options can only be exercised at expiration date.
- Intuitively, the buyer of American call/put options has the early exercise privilege, she/he should pay more.
- American call/put options should never be worth less than the corresponding European call/put options with the same strike/maturity.
- The valuation of American options needs more sophisticated numerical techniques.
- We will discuss it later.

Option Terminology

Let $S(t)$ be the current asset price, K be the strike price.

- Intrinsic value: the maximum of zero and the payoff value of the option would have if it were exercised immediately.

$$\text{Intrinsic value} = \begin{cases} \max(S(t) - K, 0), & \text{for a call option,} \\ \max(K - S(t), 0), & \text{for a put option.} \end{cases}$$

- In-the-money option: the intrinsic value > 0

$$\begin{cases} \text{In-the-money call option}: & S(t) > K, \\ \text{In-the-money put option}: & S(t) < K. \end{cases}$$

- At the money option: $S(t) = K$;
- Out-of-the-money option:

$$\begin{cases} \text{Out-of-the-money call option}: & S(t) < K, \\ \text{Out-of-the-money put option}: & S(t) > K. \end{cases}$$

Option Terminology

- The value of an option is determined by
  - the current asset price $S(t)$,
  - the strike price K,
  - the time to maturity $\tau = T - t$,
  - the option type (Call or put, American or European), and
  - the dynamics of the underlying security (e.g., how volatile the security price is)
- Out-of-the-money options do not have intrinsic value, but they have time value.
- Time value is determined by time to maturity of the option and the dynamics of the underlying security.
- Generically, we can decompose the value of each option into two components:

$$\text{option value} = \text{intrinsic value} + \text{time value}$$

Financial Markets

Where are options traded?

Exchange Markets

- Hong Kong Stock Exchange (HKEx), Chicago Board Options Exchange (CBOE), Shanghai Stock Exchange, etc.
- Standardized contracts that have been defined by the exchange.

Over-the-counter (OTC) Markets

- Traders working for banks, fund managers and corporate treasurers contact each other directly.
- Non-standard features that are tailored to the particular needs of the parties involved.

Option Market Making

- Most exchanges use market makers to facilitate options trading.
- A designated market maker is required to provide bid and ask quotes (i.e., provide liquidity), even in distressed market conditions
- Risk: adverse selection, traders with private information take advantage of market makers
- Benefit: bid-ask spread, and rebates
- There can be dozens/hundreds of options written on a single asset. So when the asset price moves, quotes on these options must be updated simultaneously.
- Market makers nowadays all have automated systems to update their quotes, and manage their risks in real time.

Market Participants

Who are trading options

For simplicity, let's assume the size of the option contract is always 100 shares.

Hedgers

use derivatives to reduce risk that they face from potential future movements in market variables.

Example

- An investor owns 100 shares of HSBC stocks whose price is $90 per share.
- He worries about a possible share price decline in the next three months and wants protection.
- He buys one 3-month put option contract on HSBC with strike $85, which gives him the right to sell 100 HSBC shares at $85.

Market Participants

Who are trading options

Example

- If the quoted premium for the option on one share is \$1.5, then the contract would cost $1.5 \times 100 = \$150$.
- With the protection from the put option, in three months, the investor's investment would always be worth equal or more than $100 \times 85 = 8500$ dollars.

Speculators

use derivatives to bet on future direction of a market variable. That is, he uses market instruments to increase his risks.

Example

- Suppose it is January 2014 and a speculator considers that HSBC is likely to increase in value over the next three months. The stock price is currently \$80.

Market Participants

Who are trading options

Example

- A 3-month call option with a \$82 strike price is selling for \$2 (one contract is worth $2 \times 100 =$\$200).
- He is willing to invest \$400 to buy two call option contracts. Another alternative is to buy 5 HSBC shares with \$400.
- Suppose that the speculator's hunch is correct and the HSBC share price rises to \$86 by April.
- The first alternative of buying the call options yields a net profit of $200 \times (86 - 82) - 400 = 400$.
- The second alternative of buying stocks yields a net profit of $5 \times (86 - 80) = 30$. The option strategy is more profitable, because option gives him more leverage.

Market Participants

Who are trading options

Arbitrageurs

exploit market inefficiencies and take offsetting positions in two or more instruments to lock in a profit without risks. A lot of hedge funds belong to this category. We will see some examples in the coming lectures.

Market Participants

Let's look at the roles of some typical market participants.

- Bank: a mixture of speculator and arbitrageur. Every time it sells an option, it is speculating that the premium received will outweigh the risks taken on. Its better access to the markets also allows it to sell products to companies at a margin above what it can buy them for. This is essentially a form of arbitrage.
- Private investors: mainly speculators. They buy risky products such as shares in order to increase their returns.
- Companies: in general, are hedgers. They are exposed to the market because they need to buy and sell commodities, or exchange currency. To reduce these risks, they use derivatives.

Market Participants

A news article published in Bloomberg on Mar 06, 2018.

Cathay Pacific hit by massive $6.45b fuel hedging loss in 2017

The airline deep dives into the red after posting its fourth straight year of deficits.

Bloomberg reports that Cathay Pacific is struggling to take off as it reports its fourth consecutive year of deficits amidst massive fuel-hedging loses worth $6.45b in 2017 with analysts expecting a turnaround by 2019.

Even though the airline has reduced the proportion of fuel it hedges, the effects still weigh on the company's financials as the futures contracts were locked in years ago.

Forward Contract

- A forward contract is an OTC (over-the-counter) agreement between two parties to exchange an asset
  - for an agreed-upon price (the forward price)
  - at a given time in the future (the the expiry date)
- It costs nothing for anyone to enter into a forward contract, i.e. the price (or value) of the forward contract at the initial time is zero.
- The party who agrees to buy the underlying asset is taking a long position; the other party who agrees to sell the underlying asset is taking a short position.

Forward Contract

- Example: on January 29, 2016, Party A signs a forward contract with Party B to buy 100 HSBC shares at $53 per share on July 29, 2016.
  - Party A takes a long position, and Party B takes a short position.
  - On January 29, 2016, sign a contract, shake hands. No money changes hands.
  - On July 29, 2016 (the expiry date), Party A pays $5300 to Party B, and receives 100 HSBC shares from Party B in return.
  - Currently (January 29), the spot price of HSBC share is $52. Six month later (July 29), the share price can be anything (unknown yet).
  - 53 is the forward price.

Profit and Loss (P&L) with Forwards

- By signing a forward contract, one can lock in a price *beforehand* for buying/selling an asset.
- In the end, whether one gains or loses from the contract depends on the spot price at expiry.

  Scenario 1: Suppose that the share price of HSBC \$54 on July 29. Then Party A uses \$5300 ($= \$53 \times 100$) to buy 100 shares rather than using \$5400. The profit-and-loss (P/L) for Party A is $5400 - 5300 = \$100$.

  Scenario 2: Suppose that the share price of HSBC is \$50 on July 29. Then Party A is obligated to use \$5300 to buy the shares rather than using \$5000. The P/L for Party A is $5000 - 5300 = -\$300$.

- Credit risk: there is a small possibility that either side can default on the contract (i.e., one party walks away from its liability). That's why forward contracts are mainly between big institutions.

Futures Contract

Futures contracts are similar to forwards, but

- Buyer and seller negotiate *indirectly* through the exchange.
- Default risk is mitigated through margin account and is borne by the exchange clearinghouse.
- Long/Short positions can be easily reversed at any time before expiry.
- Standardization: quantity, Time, delivery (cash or physical settlement).
- Value is marked to market daily. The delivery price is quoted at the exchange continually. At the end of the date, the contract holder is paid based on the profit and loss (or P&L) with his/her position.

- For example, yesterday Party A bought a futures contract with delivery price $53 for 100 shares. Today the delivery price for the same contract changes to $54, Party A will get paid $100 (the P&L for Party A).
- Easier to go long/short with futures than with stocks. Stocks have strict short selling rules.
- Underlying assets for futures contracts:
  - equity indices: Heng Sheng Index, CSI 300 Index, etc.
  - commodities: oil, gold, soybeans, sugar, etc.
  - Foreign currencies: RMB, HKD, AUD, etc.
  - Interest rates.

Hang Seng Index Futures

Contract Specifications

Hang Seng Index Futures

Hang SengIndex Futures	(updated1 Jun 2002)
UnderlyingIndex	HangSeng Index (Compiled, computed and disseminated by HSI Services Ltd)
ContractMultiplier	HK$50per index point
ContractMonths	Spotmonth, the next calendar month, and the next two calendar quarterly months(March, June, September, and December)
ContractedPrice	Theprice in whole index points at which a Hang Seng Index futures contractis registered by the Clearing House
ContractedValue	Contractedprice multiplied by contract multiplier
MinimumFluctuation	Oneindex point
MaximumFluctuation	Nil
PositionLimits	Position delta for Mini-Hang Seng Index Futures, Hang Seng Index Futures and Hang Seng Index Options combined of 10,000 long or short in all Contract Months combined provided the position delta for Mini-Hang Seng Index Futures shall not at any time exceed 2,000 long or short in all Contract Months combined. For this purpose, the position delta of one Mini-Hang Seng Index Futures Contract will have a value of 0.2
Large Open Positions	500 contracts, in any one contract month, per Exchange Participant for the Exchange Participant's own behalf; and 500 contracts, in any one contract month, per client.
Pre-Market Opening Period	HK 8:45 a.m. - 9:15 a.m. (first trading session) HK 12:45 p.m. - 1:00 p.m. (second trading session)
Trading Hours	HK 9:15 a.m. - 12:00 p.m. (first trading session) HK 1:00 p.m. - 4:15 p.m. (second trading session)
Trading Method	Electronic order matching through the automated trading system
Last Trading Day	The business day immediately preceding the last business day of the contract month
Final Settlement Day	The first business day after the last trading day
Settlement Method	Cash settlement
Final Settlement Price	The final settlement price for Hang Seng Index futures contracts shall be a number, rounded down to the nearest whole number, determined by the Clearing House and shall be the average of quotations of the Hang Seng Index taken at five-minute intervals during the last trading day.

Summary

- Financial Derivatives are mainly for transferring risks. From this respective, similar to insurance.
- Principal difference between derivative products and insurance: derivative products are hedgeable.

Random Variables

- Discrete random variable X:
  - a finite set of numbers $\{x_1, x_2, \dots, x_m\}$ contains the possible values X can take.
  - a set of probabilities $\{p_1, p_2, \dots, p_m\}$.
  - the probability that $X = x_i$ is p_i, that is, $\mathbb{P}(X = x_i) = p_i$.
  - $p_i \geq 0$ and $\sum_{i=1}^{m} p_i = 1$.
- The *mean*, or *expected value*, of a discrete random variable X is defined by

$$\mathbb{E}(X) := \sum_{i=1}^{m} x_i p_i$$

- The expected value of the random variable $f(X)$ is $\mathbb{E}(f(X)) = \sum_{i=1}^{m} f(x_i) p_i$
- Examples: the output from tossing a coin; the output from rolling a dice. What are their expected values?

Continuous Random Variables

- Continuous random variable Y:
  - may take any value in $\mathbb{R}$.
  - a real-valued *density function* f.
  - the probability that $a \leq Y \leq b$ is $\mathbb{P}(a \leq Y \leq b) = \int_a^b f(y)dy$
  - $f(y) \geq 0$ for all y; $\int_{-\infty}^{+\infty} f(y)dy = 1$.
- The *mean*, or *expected value*, of a continuous random variable Y, denoted by $\mathbb{E}(Y)$, is defined by

$$\mathbb{E}(Y) := \int_{-\infty}^{+\infty} yf(y)dy$$

- Example: a random variable Y with a *uniform distribution* over (a, b) has density function

$$f(y) = \begin{cases} (b-a)^{-1} & \text{, for } a < y < b, \\ 0 & \text{otherwise} \end{cases}$$

  The probability that Y falls in the interval $[y_1, y_2]$ is $(y_2 - y_1)/(b - a)$. What is the mean of Y?

Mean, Independence, and Variance

- For any random variables X and Y:
  - $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$
  - $\mathbb{E}(\alpha X) = \alpha \mathbb{E}(X), \text{ for } \alpha \in \mathbb{R}$
- The *mean* of the random variable $h(Y)$ is given by

$$\mathbb{E}(h(Y)) := \int_{-\infty}^{+\infty} h(y) f(y) dy$$

- Independence: two random variables X and Y are *independent* if for any real-valued functions g and h,

$$\mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\mathbb{E}[h(Y)]$$

  In particular, this means that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ if X and Y are independent.

Mean, Independence, and Variance (continued)

- Independent and identically distributed (i.i.d): when random variables $X_1, X_2, X_3, \ldots$ are i.i.d, it means that
  - all X_i have the same distributions: in the discrete case, they have the same possible values and the same probabilities, and in the continuous case they have the same density function.
  - any two X_i and X_j are independent: the value taken by X_i does not depend on any other X_j.
- Variance:
  - two equivalent definitions (Prove they are equivalent!):

$$\begin{cases} \mathrm{var}(X) := \mathbb{E}((X - \mathbb{E}(X))^2) \\ \mathrm{var}(X) := \mathbb{E}(X^2) - (\mathbb{E}(X))^2. \end{cases}$$

  - $\mathrm{var}(\alpha X) = \alpha^2 \mathrm{var}(X)$, $\mathrm{var}(\alpha + X) = \mathrm{var}(X)$, for $\alpha \in \mathbb{R}$.
- X and Y are independent, $\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)$.
- Standard deviation: $\mathrm{std}(X) := \sqrt{\mathrm{var}(X)}$.

Normal Random Variables

- $\mathbf{N}(\alpha, \sigma^2)$: a normal random variable with mean α and variance σ^2, and the density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\alpha)^2}{2\sigma^2}}$$

- $\mathbf{N}(0, 1)$ is called standard normal variable
- Cumulative distribution function: a continuous random variable X with density function $f(x)$, the cumulative distribution function $F(x)$ is defined by

$$F(x) := \mathbb{P}(X \leq x) = \int_{-\infty}^{x} f(s) ds$$

- $N(x)$: cumulative distribution function of a standard normal random variable,

$$N(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

Central Limit Theorem

- Suppose we have a set of i.i.d. random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2, and let

$$S_n := \sum_{i=1}^{n} X_i$$

- The central limit theorem: for large n, S_n behaves like an $\mathbf{N}(n\mu, n\sigma^2)$ random variable. More precisely, $(S_n - n\mu)/(\sigma\sqrt{n})$ is approximately $\mathbf{N}(0,1)$ in the sense that for any x we have

$$\mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \to N(x), \text{ as } n \to \infty$$

Computer Simulation

Motivation

- Natural and social phenomena are mostly random (or too complex to be modeled deterministically)
- Try to describe them with random variables
- Use computer to generate possible outcomes based on the distribution
- Make observations and experiment with the possible outcomes.
- Widely used in various areas (physics, engineering, biology, computer games, and of course finance).

Question

How to generate samples from a specific random distribution, for example, a normal distribution?

Random Number Generation

"True" random numbers vs. Pseudo-random numbers

- True random numbers: http://www.random.org/ provides truly random numbers generated from atmospheric noise.
- True random numbers are impossible on a computer: computers are *deterministic*.
- Pseudo-random numbers: produced by a deterministic algorithm and yet random enough in the sense of satisfying appropriate statistical properties.
- Pseudo-random numbers can be reproduced given the same initial state. A good thing for us in finance!
- Most programming languages have embedded support for pseudo-random number generation. (Matlab, R, C++ (since C++11), Python, Java, etc.).

Question

How to test whether the generated samples are good? Are they sufficiently random for our use?

Statistical tests

Given M samples $\{\xi_i\}_{i=1}^{M}$ from a pseudo-random number generator.

Sample mean and variance

$$\begin{cases} \text{sample mean} & \mu_M := \frac{1}{M}\sum_{i=1}^{M} \xi_i, \\ \text{sample variance} & \sigma_M^2 := \frac{1}{M-1}\sum_{i=1}^{M} (\xi - \mu_M)^2. \end{cases}$$

Kernel density estimation

- Divide the x-axis into subintervals $I_i = [x_i; x_{i+1})$
- Generate M samples. If N_i samples fall in I_i, then $P[X \in I_i] \approx \frac{N_i}{M}$.
- In theory, $P[X \in I_i] = \int_{I_i} f(x)dx \approx \Delta x_i f(x_i^*)$ where $\Delta x_i = x_{i+1} - x_i$ and $x_i^* = (x_i + x_{i+1})/2$.
- $f(x_i^*) \approx \frac{N_i}{M\Delta x_i}$. We can check the plot $\frac{N_i}{M\Delta x_i}$ against x_i.