Lecture 4: Implied Volatility and Monte Carlo Method

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Outline

Asset Model and Black-Scholes PDE

In Lecture 3, we first introduced the following asset model

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}. \tag{1}$$

Then we derived the famous *Black* − *Scholes* partial differential equation (PDE).

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{2}$$

Black-Scholes Formulas

- The closed-form formulas for European Call/Put:

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (3)$$
$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \quad (4)$$

where $N(x)$ is the cumulative density function of $\mathbf{N}(0,1)$, and

$$d_1 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t},$$
$$d_2 = \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}.$$

Option Valuation

The key points:

- The drift term μ in the asset model doesn't matter for option valuation.
- Risk can be eliminated by holding a portfolio in which the random parts of two different sub-portfolios cancel each other
- No-arbitrage principle implies that a portfolio from which risk has been eliminated must grow at the risk-free rate.
- A European option's value can be replicated by a (self-financing) portfolio consisting of dynamically trading in stock and risk-free bond.

Practicalities of Trading Options

- So far, we have learned quite a bit about options.
- You might start thinking to get your feet wet and trade some options. But before that you have to answer the following question first.
  - With the same underlying asset, there are many different options with different strikes, different maturities, and different payoff types (Call or Put).
  - How will you decide which one to buy or sell? Intuitively, you should buy cheaper ones and sell more expensive ones.
  - You need a systematic way to decide the relative cheapness/richness among different options.
- Can we directly look at the prices of options, just similarly to what we normally do with stocks?
- Not a good idea! Why?
- The rest of this lecture gives you a new tool that could help you on this.

What is Implied Volatility

- The Black-Scholes formula gives the value of an option as a function of several inputs: S_0, K, T, r, and σ.
- Of these only one is not specified in the contract or readily observable, the volatility σ.
- What we can observe from the market are option prices.
- Since the option price is a monotonic function with respect to σ, given the option price V, there exists a unique σ when substituted into Black-Scholes formula that gives the option price V, which is called *implied volatility*.
- In practice, we calculate the implied volatilities for different strikes and different maturities, and then generate an implied volatility surface using some numerical interpolations.

Why implied volatility

- A convenient quantity to measure the cheapness or dearness of an option.
- We can look at the option premium directly for cheapness/dearness. Not a good idea:
  - Options with different strikes, different maturities, and different underlying assets are essentially different contracts. Direct comparison is meaningless.
- With implied volatilities, there are two ways of judging the cheapness or dearness of options.
- The first is simply by comparing current implied volatility with past levels of implied volatility on the same underlying asset.
- The second is by comparing current implied volatility with the historical volatility of the underlying itself (will be discussed in next lecture).

Why implied volatility

- Implied volatility is relatively more stable than stock levels.
- Complex products need the volatilities implied by the prices of simple options observed from the market.
- Later we will introduce some much more complex options, e.g., Asian options, basket options, etc.
- All of these are OTC products. Not standardized products. Their prices cannot be observed directly from the market. You have to check with various banks for the price.
- Then how do banks get the volatility information to price them?
- They use the implied volatility surface obtained from the market prices of simpler products (European/American call/put options).

Nonlinear implicit Equation

- Suppose that on a certain date we observe an asset price of S_0 and an interest rate of r.
- If we also observe the value V of a call option with expiry time T and strike price K.
- Recall the Black-Scholes formula

$$C(\sigma) = S_0 N(d_1) - Ke^{-rT} N(d_2) = V,$$

where

$$d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

- Since everything else is known, this is an equation for σ.
- But it is an implicit equation. We cannot just rearrange it to isolate σ and thus read off its value.
- We have to do some work to find the value of σ^* that makes $C(\sigma^*) - V = 0$.

Solving nonlinear implicit equations

Nonlinear implicit equation

- Consider the following general problem: given some *continuous* nonlinear monotonic function $f(x)$, find a real number x^* such that

$$f(x^*) = 0.$$

- There are a couple of options open to us.
  - Use guesswork.
  - Use inspired guesswork: use our previous guesses to make new, better ones.

Bisection Method

- If we have two guesses, x_a and x_b, with $f(x_a)f(x_b) < 0$, then we know that $f(x)$ must cross zero somewhere between x_a and x_b.
- We can use a divide-and-conquer approach to find x^*.

Bisection Method

- The bisection method goes as follows:

  Step 1: Find x_a and x_b with $x_a < x_b$ such that $f(x_a)f(x_b) < 0$.

  Step 2: Set $x_{\mathrm{mid}} := \frac{x_a + x_b}{2}$ and evaluate $f(x_{\mathrm{mid}})$.

  Step 3: If $f(x_{\mathrm{mid}}) = 0$ then stop. If $f(x_{\mathrm{mid}})f(x_a) < 0$, then reset $x_b = x_{\mathrm{mid}}$. Otherwise, reset $x_a = x_{\mathrm{mid}}$.

  Step 4: If $x_b - x_a < \epsilon$, then stop and use $\frac{x_a + x_b}{2}$ as the approximation to x^*. Otherwise, return to Step 2.

- Note that we must choose a value $\epsilon > 0$ for our stopping criterion $x_b - x_a < \epsilon$.
- The approximation error is guaranteed to be no more than $\epsilon/2$.

Bisection Method

- There is no foolproof procedure for finding suitable x_a and x_b in Step 1.
- Without specific knowledge of the function $f(x)$, we must resort to trial and error.
- The bisection method halves the length of the interval $[x_a, x_b]$ on each iteration, the error at the k-th iteration is bounded by $\frac{L}{2^{k+1}}$, where L is the length of the original interval, $x_b - x_a$.
- This is referred to as a *linear convergence bound* because the error bound decreases by a linear factor (in this case $\frac{1}{2}$).
- Next, we consider a faster method.

Newton-Raphson method

- Let's look at a faster method: Newton-Raphson method.
- Suppose that we have a current guess x_n. Then, assuming $f(x)$ is differentiable, and writing $\epsilon_n = x^* - x_n$, the Taylor series expansion gives

$$0 = f(x^*) = f(x_n + \epsilon_n) = f(x_n) + \epsilon_n f'(x_n) + \frac{\epsilon^2}{2} f''(x_n) + \cdots .$$

- If we ignore the terms of second order or higher, we get

$$\epsilon_n \approx -\frac{f(x_n)}{f'(x_n)} \Rightarrow x^* \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

- We set this estimate as our next guess:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton-Raphson method

- To see how quickly the error decreases, let's go back and look at the Taylor expansion:

$$\begin{aligned}
\epsilon_{n+1} &= x^* - x_{n+1} = x^* - x_n + \frac{f(x_n)}{f'(x_n)} = \epsilon_n + \frac{f(x_n)}{f'(x_n)} \\
&\approx -\frac{\epsilon_n^2}{2}\frac{f''(x_n)}{f'(x_n)}
\end{aligned}$$

- If we could assume that $\frac{f''(x_n)}{f'(x_n)}$ is bounded by a constant C, then the new error is proportional to the square of the old.
- Second order convergence.
- Note that the result requires the starting value x_0 is sufficiently close to x^*. Otherwise, it may fail to converge.

Examples

Find the reciprocal

- Given a number $a > 0$, find $1/a$ without doing any division!
- Set $f(x) = \frac{1}{ax} - 1$. Then Newton-Raphson gives

$$x_{n+1} = x_n - \frac{\frac{1}{ax_n} - 1}{-\frac{1}{ax_n^2}} = x_n + x_n - ax_n^2 = x_n(2 - ax_n).$$

Find the square root

- Given $a > 0$, find $\sqrt{a}$.
- Set $f(x) = x^2 - a$. Then Newton-Raphson gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{x_n^2 + a}{2x_n}.$$

General Observations

- Newton-Raphson will normally converge very quickly, roughly doubling the number of decimal places of accuracy at each iteration.
- It will fail to produce this kind of convergence if $f'(x^*) = 0$, e.g., $f(x) = x^3$.
- A suitable initial guess has to be supplied, otherwise, it may produce disastrous results.
- It is always a good idea to do a sanity check on the results.
- Bisection can be used to generate a good initial guess.
- Ababu Teklemariam Tiruneh, W. N. Ndlela, and S. J. Nkambule, "**A Two-Point Newton Method Suitable for Nonconvergent Cases and with Super-Quadratic Convergence**", Advances in Numerical Analysis, Volume 2013, Article ID 687382

Implied Volatility

- Recall our problem of solving implied volatility: given S_0, K, r, T, and the call option value V, find σ such that
$$C(\sigma) - V = 0.$$
- Write $f(\sigma) = C(\sigma) - V = S_0 N(d_1(\sigma)) - Ke^{-rT} N(d_2(\sigma)) - V$, where
$$d_1(\sigma) = d_2(\sigma) + \sigma\sqrt{T} = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$
- Then $f'(\sigma) = S_0\sqrt{T} N'(d_1(\sigma)) = \frac{S_0\sqrt{T}}{\sqrt{2\pi}} e^{-d_1(\sigma)^2/2}$.
- Note that $f'(\sigma) > 0$, so $f(\sigma)$ is monotonically increasing with respect to σ.
- $\lim_{\sigma\to 0^+} C(\sigma) = \max(S_0 - Ke^{-rT}, 0)$.
- $\lim_{\sigma\to\infty} C(\sigma) = S_0$.

Implied Volatility

- Thus if $V \in (\max(S_0 - Ke^{-rT}, 0), S_0)$, there will be a unique solution to the equation $f(\sigma) = 0$.
- From the Black-Scholes formula for a European call option,

$$
\begin{aligned}
\frac{\partial C}{\partial \sigma} &= S_0 \sqrt{T} N'(d_1) \\
\frac{\partial^2 C}{\partial \sigma^2} &= -\frac{S_0 \sqrt{T}}{\sqrt{2\pi}} e^{-\frac{1}{2} d_1^2} d_1 \frac{\partial d_1}{\partial \sigma} = \frac{d_1 d_2}{\sigma} \frac{\partial C}{\partial \sigma}
\end{aligned}
$$

- Thus $\partial C / \partial \sigma$ is maximum over $[0, \infty]$ at $\sigma = \hat{\sigma}$, where

$$
\hat{\sigma} = \sqrt{2 \left| \frac{\ln S_0/K + rT}{T} \right|}
$$

Implied Volatility

- For $\sigma < \hat{\sigma}$, $f'(\sigma)$ is increasing, and for $\sigma > \hat{\sigma}$ $f'(\sigma)$ is decreasing.
- Recall that

$$\epsilon_{n+1} = \epsilon_n + \frac{f(x_n)}{f'(\sigma_n)}$$

  Also the Mean Value Theorem tells us that $f(\sigma^*) - f(x_n) = f'(\xi_n)(\sigma^* - x_n)$ for some ξ_n between σ_n and σ^*. Thus

$$\epsilon_{n+1} = \epsilon_n - \epsilon_n \frac{f'(\xi_n)}{f'(\sigma_n)} \Rightarrow \frac{\epsilon_{n+1}}{\epsilon_n} = 1 - \frac{f'(\xi_n)}{f'(\sigma_n)}$$

- We would start our iteration with $\sigma_0 = \hat{\sigma}$, and since $f'(\sigma)$ is decreasing away from $\hat{\sigma}$, this will guarantee that each ϵ_n has the same sign as ϵ_0 and is smaller.

Example

Let's see an python implementation of the Newton's method for implied volatility.

```
import black_scholes as bs
from math import sqrt, log
#====================== parameters =======================
r = 0.03; S = 2; K = 2; T = 3; sigma_true = 0.3;optionType='call'
#=========================================================
C_true = bs.euro_vanilla(S,K,T,r,sigma_true, optionType)
print('the option price is ', C_true)

# the option price is 0.48413599739115154
```

Example

```
#==========================================================
# now use Newton's method to get the implied volatility
#===================== Newton's Method ====================
#starting value
sigmahat = sqrt(2*abs( (log(S/K) + r*T)/T ) )
tol = 1e-8; nmax = 100
sigmadiff = 1
n = 1
sigma = sigmahat
while (sigmadiff >= tol and n < nmax):
    C = bs.euro_vanilla(S,K,T,r,sigma,optionType)
    Cvega = bs.vega(S,K,T,r,sigma)
    increment = (C-C_true)/Cvega
    sigma = sigma - increment
    n = n+1
    sigmadiff = abs(increment)

print('the implied volatility is ', sigma)
# the implied volatility is 0.2999999919061244
```

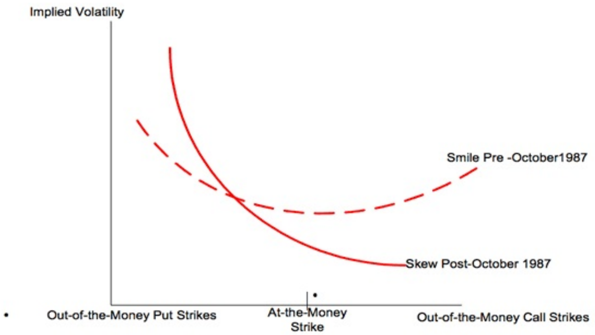
Implied Volatility Surface

- The implied volatilities from a call and a put with the same strike and the same maturity on the same asset should be the same.
- In practice, the implied volatilities from options on the same underlying asset but with different strike/maturity are different.
- These are different contracts after all: their prices/implied volatilities are subject to supply/demand.
- Strictly speaking, the only constraints are the no-arbitrage conditions mentioned in Assignment 1.
- In general, the out-of-money put options have higher implied volatilities.
- If we plot the implied volatilities against the options strikes, we see a skewed curve.
- Closely related to the stock market crash of 1987.

Equity Implied Volatility Smile

- Implied volatility curve:

Chart 1. The S&P 500 Implied Volatility Curve Pre-and Post- 1987



Source: CBOE

Black Monday 19 Oct 1987

What happened? A brief time line of the 1987 Crash:

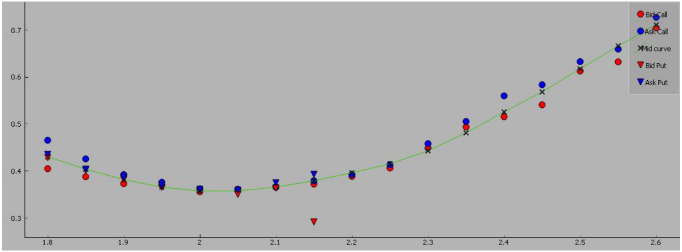
- Wednesday, October 14: The Dow Jones Industrial Average Index (DJIA) dropped 95 points to 2412 (about 4%)
- Thursday, October 15: DJIA down 48 points.
- Friday, Oct 16: DJIA down 108 points to 2246 on record volume
- Monday, Oct. 19: The Hong Kong Hang Seng Index (HSI) down 11%, the London the Financial Times 30 Index fell 10%; DJIA dropped 508 points, to 1,738, down 22.6%, on record volume of 604.3 million shares.
- Tuesday, Oct. 20: No bids, big delays in opening stocks, DJIA down over 100 points initially, then rallied and ended up 103 points at 1,841.

Implied Volatility Smile

- You can google “market crash 1987” to read more about it, e.g. `http://en.wikipedia.org/wiki/Black_Monday_(1987)`
- After the crash of 1987, the implied volatilities with different strikes look very different.
- People start to realize that a giant market indeed could drop by 20% or more in a day or two.
- An investor has to pay *relatively* more for low-strike puts than for high-strike call.

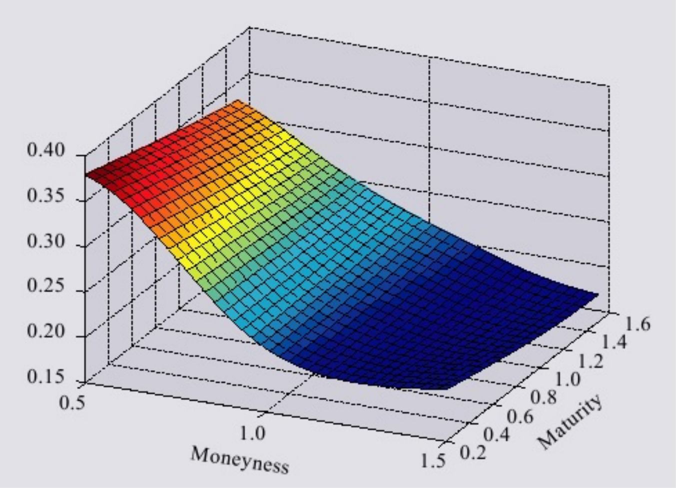
Equity Implied Volatility Smile

- Implied volatility curve from A50 ETF options listed in Shanghai Stock Exchange:



Equity Implied Volatility Smile

- A typical implied volatility surface for stocks. Moneyness refers to the ratio $K/S(0)$.



State-of-the-Art Technique for Implied Volatility

The Newton-Raphson method is both easy and efficient, and used to be the most widely used technique for implied volatility calculation. In practice, however, as the trading volume has been increasing explosively, researchers from industry have been striving to improve it. The current state-of-the-art technique can be found from the following two papers by Peter Jackel. Both papers can be found from Jackel's webpage `http://www.jaeckel.org/`

- Let's Be Rational, Wilmott, pages 40-53, January 2015.
- By Implication, Wilmott, pages 60-66, November 2006.

These papers are for your information only, and they are not part of the course requirements.

Option Valuation With Monte Carlo

In the next two lectures, we discuss the Monte Carlo method for option valuation.

- First, discuss the concept of "Risk neutrality".
- Second, discuss the general idea of the Monte Carlo method
- Third, apply the Monte Carlo method to option valuation.
- Fourth, discuss various techiques to speed up the Monte Carlo method.

Risk Neutrality

- In Lecture 3, we first introduced the following asset model

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}. \tag{5}$$

- The drift term μ in the asset model doesn't matter for option valuation, but can help you decide to take long or short positions.
- Now we introduce a new concept "risk neutrality" which can help us tackle the derivative valuation problem with more mathematical tools.

Forward Price and Risk-free Rate

- Recall the forward price for time t is $F(t) = S_0 e^{rt}$, where r is the risk-free interest rate.
- You are asked to prove it using no arbitrage rule.
- The main idea is that we can replicate the payout of a forward contract by a portfolio of underlying asset and zero-coupon bond.
- Alternatively, from the probability theory perspective, the forward price should be the expected asset price at t.
- Recall that our asset model is:

$$S(t) = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}.$$

- It is easy to show that $\mathbb{E}(S(t)) = S_0 e^{\mu t}$.

Forward Price and Risk-free Rate

In case you are wondering how to show $\mathbb{E}(S(t)) = S_0 e^{\mu t}$

$$
\begin{aligned}
\mathbb{E}(S(t)) &= \mathbb{E}\left(S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}\right) \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} S_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}x} e^{-\frac{x^2}{2}}\,dx \\
&= \frac{S_0 e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-\sigma\sqrt{t})^2}{2}}\,dx \\
&= \frac{S_0 e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx \\
&= S_0 e^{\mu t}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx}_{\text{this is 1}} \\
&= S_0 e^{\mu t}.
\end{aligned}
$$

Forward Price and Risk-free Rate

- Actually we have never mentioned how to specify μ.
- There could be many different methods/techniques to do this:
  - Statistical methods using the historical data
  - Fundamental economic analysis (profits, sales, etc.)
- Different persons using different methods could end up with very different μ, and hence very different forward prices.
- There could be only *one* forward price $S_0 e^{rt}$ implied by the no arbitrage rule.
- Hence, to make $\mathbb{E}(S(t)) = S_0 e^{rt}$, there must be $\mu = r$.
- In plain English, the forward price is the expected asset price if the drift term is the risk free interest rate.

European Options and Risk-free Rate

- We have derived the European call/put formulas using Black-Scholes PDE.
- Alternatively, from the probability theory perspective, the call/put option price should just be the expected payoff at maturity with proper discounting for interest rate.
- Mathematically, it means that the call option value $C(T)$ satisfies

$$
\begin{aligned}
C(T) &= e^{-rT}\mathbb{E}(\max(S(T)-K,0)) \\
&= e^{-rT}\mathbb{E}(\max(S_0 e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}-K,0)) \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\max(S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K,0))e^{-\frac{x^2}{2}}dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}}\int_{-d_2}^{\infty}(S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x}-K)e^{-\frac{x^2}{2}}dx,
\end{aligned}
$$

where $d_2 = \frac{\ln(S_0/K)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$

European Options and Risk-free Rate

- Continues from last slide

$$
\begin{aligned}
C(T) &= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} (S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} - K) e^{-\frac{x^2}{2}}\, dx \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \left(\int_{-d_2}^{\infty} S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\, dx - \int_{-d_2}^{\infty} K e^{-\frac{x^2}{2}}\, dx \right) \\
&= \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} S_0 e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}x} e^{-\frac{x^2}{2}}\, dx - K e^{-rT} N(d_2) \\
&= \frac{S_0 e^{(\mu-r)T}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\, dx - K e^{-rT} N(d_2) \\
&= \frac{S_0 e^{(\mu-r)T}}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{T}}^{\infty} e^{-\frac{y^2}{2}}\, dy - K e^{-rT} N(d_2) \\
&= S_0 e^{(\mu-r)T} N(d_1) - K e^{-rT} N(d_2)
\end{aligned}
$$

where $d_1 = d_2 + \sigma\sqrt{T} = \frac{\ln(S_0/K)+(\mu+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$.

European Options and Risk-free Rate

- Obviously, when $\mu = r$, we have

$$S_0 e^{(\mu - r)T} N(d_1) - Ke^{-rT} N(d_2) = S_0 N(d_1) - Ke^{-rT} N(d_2).$$

- In other words, when the drift term $\mu = r$, the European call value is just the discounted expected payoff at maturity.
- When setting $\mu = r$, we usually say we are in a *risk neutral* world.
- In this world, the expected rate of return (the drift term) of any tradable asset is r, which implies that there is no any arbitrage.
- The value of an option is the discounted expected payoff at maturity.
- This allows us to develop computational methods for valuating options where analytical formulas are not available.

Risk Neutral Asset Model

- It can be mathematically proved that in a risk neutral world, the value of other types options can also be calculated as the discounted expected cash flow.
- From now on, we always assume that the underlying asset follows the following model:

$$S(t) = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma\sqrt{t}Z}. \tag{6}$$

- In the following lectures, we would introduce some numerical techniques to valuate option products.

Monte Carlo: General Idea

- Consider a general random variable X.
- Its exact analytical distribution function is unknown, but we want to calculate the value $\mathbb{E}(X)$.
- The good news is that we are able to generate samples of X.
- Then we can proceed by generating a (large) number of samples (let's say, M) and computing the sample mean a_M, and treat it as an approximation to $\mathbb{E}(X)$. This is essentially what the Monte Carlo method does.
- How do we measure how good our approximation is?
- We can use the sample variance b_M^2 to help us monitor the approximation error.
- For later convenience, let $\mathbb{E}(X) = a$ and $\mathrm{var}(X) = b^2$

Sample Mean and variance

- Given M independent samples $X_1, X_2, \ldots, X_M$ of X, the sample mean a_M and the sample variance b_M^2 are defined by

$$a_M := \frac{1}{M}\sum_{i=1}^{M} X_i \text{ and } b_M^2 := \frac{1}{M-1}\sum_{i=1}^{M}(X_i - a_M)^2.$$

- The Central Limit Theorem tells us that $a_M \sim a + \frac{b}{\sqrt{M}} N(0,1)$.
- If M is large enough, a_M is a good estimate of a.
- Let's find a quantitative way to measure the approximation accuracy.

Sample Mean and variance

- To measure the uncertainty of a random variable Y, we often look at the confidence interval. Suppose that

$$\mathbb{P}(\alpha \leq Y \leq \beta) = 95\%,$$

then we say that $[\alpha, \beta]$ is a 95% *confidence interval* for Y.

- For a stand normal random variable $Y \sim N(0, 1)$, we have

$$\mathbb{P}(-1.96 \leq Y \leq 1.96) = 95\%,$$

so $[-1.96, 1.96]$ is a 95% confidence interval for Y. Verify this yourself!

- This means that if we generate many samples of Y, then 95% of the samples fall in the range $[-1.96, 1.96]$.

Sample Mean and variance

- Now that $\frac{a_M - a}{\left(\frac{b}{\sqrt{M}}\right)} \sim N(0,1)$,

$$\mathbb{P}(-1.96 \leq \frac{a_M - a}{\frac{b}{\sqrt{M}}} \leq 1.96) = 95\%, \tag{7}$$

$$\mathbb{P}\left(a - \frac{1.96b}{\sqrt{M}} < a_M < a + \frac{1.96b}{\sqrt{M}}\right) = 95\% \tag{8}$$

- Re-write it to get the 95% confidence interval for a

$$\mathbb{P}\left(a_M - \frac{1.96b}{\sqrt{M}} < a < a_M + \frac{1.96b}{\sqrt{M}}\right) = 95\% \tag{9}$$

- Replace the unknown b by b_M, approximately

$$\mathbb{P}\left(a_M - \frac{1.96b_M}{\sqrt{M}} < a < a_M + \frac{1.96b_M}{\sqrt{M}}\right) = 95\% \tag{10}$$

Monte Carlo Simulation

Summary

Let's summarize the basic Monte Carlo simulation method for approximating $\mathbb{E}(X)$:

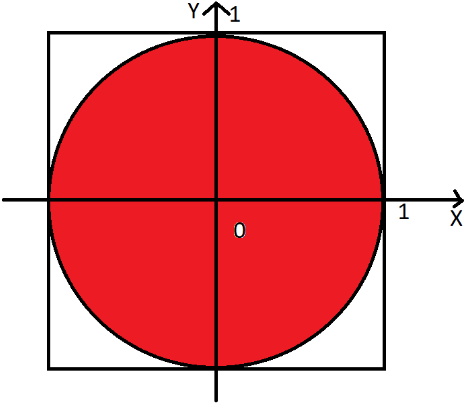
- We compute M independent samples of X and calculate a_M.
- In order to monitor the approximation error, we also compute b_M, which allows us to compute the confidence interval.

Example: Computing π

- Consider a random variable X taking values uniformly in the square $[-1, 1]^2$.
- The probability that $||X|| = \sqrt{X_x^2 + X_y^2} < 1$ is $\frac{\pi}{4}$.

Example: Computing π

A circle with radius 1 is inscribed in a square. If we randomly sample points in the square, the probability of points inside the circle is $\frac{\pi}{4}$.



Example: Computing π

- Now define a random variable Y by

$$Y = \begin{cases} 4 & \text{if } ||X|| < 1, \\ 0 & \text{otherwise} \end{cases}$$

- Then $\mathbb{E}(Y) = 4 \times \mathbb{P}(||X|| < 1) = \pi$.
- If $Y_1, Y_2, \ldots, Y_M$ are M independent samples of Y, the sample mean a_M will be our estimate of π.

Example: Computing π

- A short piece of Python code:

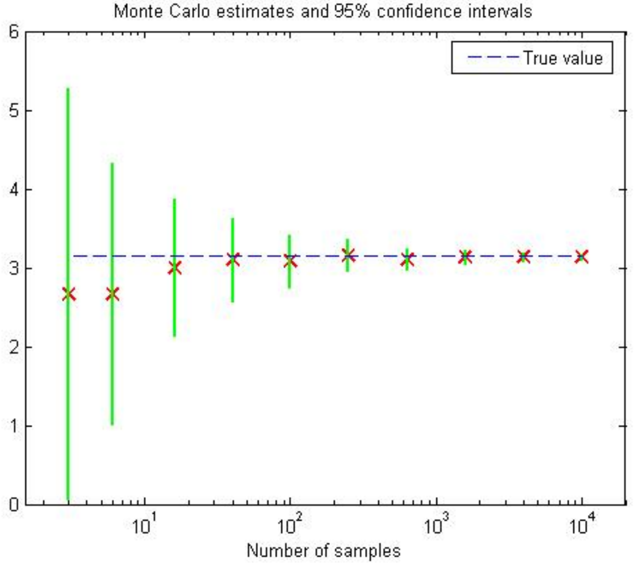
```
import numpy as np

# set the random seed
np.random.seed(1000)

M=int(1e4);
X = np.random.uniform(low=[-1,-1],high=[1.0,1.0],size=[M,2
X2 = np.square(X[:,0]) + np.square(X[:,1])
Y = (abs(X2) < 1).astype(float) * 4
aM = np.mean(Y)
bM = np.std(Y)

print('the mean is ', aM, ' the std is ', bM)
```

Example: Computing π



Monte Carlo option valuation

- Let's discuss how to apply Monte Carlo method to option valuation.
- Consider a European call option. Recall its payoff function at maturity T is $\max(S(T) - K, 0)$.
- In risk-neutral world, the value of the call option:

$$V = e^{-rT}\mathbb{E}\Big(\max\Big(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K, 0\Big)\Big),$$

  where $Z \sim N(0,1)$
- For later convenience, let

$$X = e^{-rT}\max\Big(S(0)e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z} - K, 0\Big),$$

- So our goal is to estimate:

$$V = \mathbb{E}(X)$$

.

Call option valuation

- Let's consider a call option with the following parameters: $S(0) = 10$; $K = 9$, $r = 0.06$; $\sigma = 0.10$; $T = 1.0$; True value is 1.5429.
- To estimate $\mathbb{E}(X)$, let's first draw 10 samples from standard normal distribution, $Z_1, Z_2, \cdots, Z_{10}$.
- Next, let's calculate the corresponding 10 samples of X,

$$X_i = e^{-rT} \max\left(S(0) e^{(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} Z_i} - K, 0 \right), \quad i = 1, \cdots, 10$$

- Finally, we get an estimation of the call option value:

$$\mathbb{E}(X) = \frac{1}{10} \sum_{i=1}^{10} X_i$$

Call option valuation

Monte Carlo estimations with different numbers of samples.