Comp 7405
Lecture 2: Option valuation preliminaries

J. R. Zhang

Department of Computer Science
The University of Hong Kong

Outline

Review of Lecture 1

- Financial Assets:Stocks, Bonds, Commodities, Currencies, etc.
- Financial Derivatives:futures, forwards, swaps, options, etc.
- Options: call/put options; European/American options;
- Call option payoff function at maturity:

$$C_T^T = \max(S(T) - K, 0)$$

- Put option payoff function at maturity:

$$P_T^T = \max(K - S(T), 0)$$

- Intrinsic value, In-the-money options, at-the-money options, and out-of-the-money options.
- Financial Derivative vs. Insurance: both for transferring risks, but derivatives are hedgeable.
- Forward/Futures.

Futures Contract

Futures contracts are similar to forwards, but

- Buyer and seller negotiate *indirectly* through the exchange.
- Default risk is mitigated through margin account and is borne by the exchange clearinghouse.
- Long/Short positions can be easily reversed at any time before expiry.
- Standardization: quantity, Time, delivery (cash or physical settlement).
- Value is marked to market daily. The delivery price is quoted at the exchange continually. At the end of the date, the contract holder is paid based on the profit and loss (or P&L) with his/her position.

- For example, yesterday Party A bought a futures contract with delivery price \$53 for 100 shares. Today the delivery price for the same contract changes to \$54, Party A will get paid \$100 (the Profit-and-Loss for Party A).
- Let's assume Party B sold the same futures contract yesterday. Then his mark-to-market Profit-and-Loss is -\$100. He would have to pay the exchange \$100.
- With this kind of daily settlement of mark-to-market PnL, default risk is significantly reduced.
- Easier to go long/short with futures than with stocks. Stocks have strict short selling rules.
- Underlying assets for futures contracts:
  - equity indices: Heng Sheng Index, CSI 300 Index, etc.
  - commodities: oil, gold, soybeans, sugar, etc.
  - Foreign currencies: RMB, HKD, AUD, etc.
  - Interest rates.

Time Value of Money

- Cashflows (payments) taking place at different times cannot be directly compared.
- If I have \$100 today, I can put it in an interest-bearing account, and the account will (typically) have more than \$100 in one-year time.
- For example, if the interest rate from the account is 1% for 1-year deposit, I can receive \$101 in a year.
- Mathematically, suppose an interest-bearing account pays an annual interest rate of r compounding m periods per year.
- It means the account gives m interest payments per year, and each time the interest rate is $\frac{r}{m}$.
- If we deposit some money D_0 in this account, then in t years, the money in this account is worth

$$D(t) = D_0 \times \left(1 + \frac{r}{m}\right)^{mt} \tag{1}$$

Present Value

- If the compounding frequency $m \longrightarrow \infty$, the interest rate r is *continuously compounded*, and

$$D(t) = D_0 e^{rt}, \text{since } \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^m = e^r \tag{2}$$

- To receive money P from this account in t years, today you need to deposit

$$P_0 = Pe^{-rt}. \tag{3}$$

  P_0 is the *present value* of the payment P at time t.
- Transforming from P to Pe^{-rt} is called *discounting for interest*.
- To compare the payments happening at different times, we need to look at their *present values*.

Continuously Compounded Interest Rate

- In this course, we always work with continuously compounded interest rates.
- If $D(t)$ grows at a continuously compounded interest rate r, then $D(t) = e^{rt} D(0)$.

$$\frac{dD(t)}{dt} = re^{rt} D(0) = rD(t) \Longrightarrow dD(t) = rD(t)dt. \tag{4}$$

- From (4), over a small time interval δt, we have

$$D(t + \delta t) - D(t) \approx rD(t)\delta t \Longrightarrow D(t + \delta t) \approx D(t)(1 + r\delta t).$$

Zero-coupon Bond

Definition

- A zero-coupon bond is a contract that promises to pay a specified amount at some point in the future.
- "Zero-coupon" means the bond does not pay periodic coupons. You only receive the face value/principal amount of the bond at maturity.
- The price of a zero-coupon bond is normally less than the face value.
- This is how an investor makes money from holding the zero-coupon bond.
- For example, if one unit of a zero-coupon bond paying out \$1 in a year's time costs \$0.80 today, then with \$100 I can buy 125 ($100/0.80 = 125$) units of the bond today. In one year I will receive \$125.

Zero-coupon Bond

Implied Interest Rate

- Putting money in an interest-bearing account (with no possibility of early withdrawal) is equivalent to investing it in zero-coupon bonds.
- Given the present value of a zero-coupon bond, we can calculate the implied continuously compounded interest rate r.
- Let B_0^T denote the current price of a zero-coupon bond mature at time T, and B_T^T be the principal amount paid at maturity T.
- From Equations (2) and (3), we have

$$B_T^T = B_0^T e^{rT} \Longrightarrow r = \frac{1}{T} \ln \left(\frac{B_T^T}{B_0^T} \right) \tag{5}$$

Interest Rate

Risk-free Interest Rate

- In option valuation theory, there is an important term "risk-free interest rate".
- It is essentially means the interest rate from a risk-free zero coupon bond. Here "risk-free" means the there is no risk of financial loss from holding the bond.
- In practice, the interest rate is actually a function of time, so it is an interest rate curve, not a single number.
- Also it can be quite complicated to determine the risk free interest rate, especially after the financial crisis.
- Nevertheless, in this course, we assume that the interest rate can always be observed from the market.

Interest rate

In this course, we make the following assumptions about the risk-free interest rate:

- the fixed interest rate r prevails whenever cash is lent or borrowed.
- the fixed interest rate r applies whatever amount of cash is involved,
- the fixed interest rate r is always positive.

With these assumptions, if somebody were to make you the offer of

(a) $\$100e^{-rT}$ immediately (at time 0), or

(b) $\$100$ at time T.

then you would regard both offers as being of equal value. Similarly, a deal that is guaranteed to produce exactly $\$100$ at time T is worth exactly $\$100e^{-rT}$ at time 0.

Short selling

Portfolio

We use the term *portfolio* to describe a combination of

(i) assets,

(ii) options, and

(iii) cash (invested in a bank or in a bond).

We assume that it is possible to hold negative amounts of each. Specifically,

- How to hold negative amount of cash? Easy! borrow money from a bank!
- How to hold negative amount of options? Easy! write an option!
- How to hold negative amount of assets (say, stocks)? Sounds a bit ridiculous! In practice, be implemented through *short selling*.

Short selling

Definition

Short selling means you sell something you don't really own. To short sell an asset, you first borrow it from somebody who owns it, and later buy it back and return it.

Let $S(t)$ denote the value of an asset at time t. Let's see the process:

- At t_1, we borrowed a share from somebody (usually from your broker), sold it to the market, and gained an amount $S(t_1)$ at time $t = t_1$ from the short sale,
- At t_2, we paid out an amount $S(t_2)$ to buy back the share, and returned it to the owner.

Then the overall profit/loss at time $t = t_2$ from the short selling is

$$PnL = e^{r(t_2 - t_1)} S(t_1) - S(t_2). \tag{6}$$

Short selling

- To give you some intuition about short selling, let's assume the interest rate $r = 0$, then the PnL (profit/loss) becomes

$$PnL = S(t_1) - S(t_2) = -1 \times (S(t_2) - S(t_1)). \tag{7}$$

  Just look like the PnL from holding -1 share of the stock.
- In this course, we assume that this is always possible, at no cost, and that the short seller is free to choose when to buy back and return the item.

No arbitrage principle

Definition

- *No arbitrage* is one of the key principles on which option valuation theory rests. There are different ways to describe it, but the basic idea is the same.
- Essentially, it means that one cannot *consistently* make money for nothing. *No free lunch*!

An example

Let's consider an example in foreign exchange.

- Suppose 1 pound (GBP) is worth 1.6 U.S. dollars and 1 U.S. dollar is worth 100 Japanese yen.
- How much yen is 1 pound worth? It has to be worth exactly 160 yen. But why?

No arbitrage principle: An Example

- If 1 pound is worth more than 160 yen, suppose we start with 1 pound. We take the following actions:
  - Sell this one pound for yen (> 160 yen),
  - Then sell these yen for dollars (> 1.6 dollars),
  - Finally sell the dollars for pounds (> 1 pound).
  - We end up with more pounds than what we started with.
  - We keep on doing the same thing as long as we can.
  - This process is called *taking advantage of an arbitrage opportunity*.
  - This process will cause the arbitrage opportunity disappear due to supply and demand.
  - Specifically, buying yen will drive the exchange rate pound/yen down (i.e. 1 pound worth less yen), buying dollars will drive the yen/dollar rate down, and so on.
- If 1 pound is worth less than 160 yen, we take the opposite actions.

No arbitrage principle

- Due to the above actions from market participants, the arbitrage opportunity will be short-lived.
- In real market, arbitrage opportunities can exist but will generally be very small and disappear quickly.
- In the mathematical finance theory, it is therefore convenient to assume that there is no arbitrage.
- From another perspective, our job is to find the fair price in an arbitrage-free market. If the observed price from the market is not in agreement, then we know there is an arbitrage opportunity to be exploited.

Call-Put Parity

- A beautiful relationship between the value C_0^T of a European call and the value P_0^T of a European put, with the same strike price K and expiry date T. Let r be the risk-free rate and $S(0)$ be the asset price now, then we have

$$C_0^T + Ke^{-rT} = P_0^T + S(0) \tag{8}$$

  This equation is called *call-put parity*.
- The relationship doesn't rely on assumptions on the movement of the underlying asset.
- We apply the "no arbitrage" rule to derive the relationship.

Call-Put Parity

Proof

- Consider two portfolios at time $t = 0$

  π_A: one call option plus Ke^{-rT} risk-free zero-coupon bond,
  π_B: one put option plus one unit of the asset.

- It is trivial to see that at time $t = 0$

$$\begin{cases} \pi_A \text{ is worth } C_0^T + Ke^{-rT}, \\ \pi_B \text{ is worth } P_0^T + S(0). \end{cases}$$

- At the expiry date T, we have

$$\begin{cases} \pi_A : \overbrace{\max(S(T) - K, 0)}^{\text{Call option}} + \overbrace{K}^{\text{bond}} = \max(S(T), K), \\ \pi_B : \underbrace{\max(K - S(T), 0)}_{\text{Put option}} + \underbrace{S(T)}_{\text{asset}} = \max(S(T), K). \end{cases}$$

Call-Put Parity

Proof

- Since the two portfolios always give the same payoff at time T, they must have the same value at time $t = 0$. Otherwise, there would be arbitrage opportunities.
- To see why, let's first assume that π_A is worth more than π_B, then we would sell π_A and buy π_B and we would pocket some money $\pi_A - \pi_B$ right away. At maturity T, since the payout we receive from π_B equals the payout we give for π_A, we do not gain or lose anything.
- We have made some money without taking any risk!
- Similar argument applies if π_B is worth more than π_A.
- Hence $C_0^T + Ke^{-rT} = P_0^T + S(0)$.

Upper/Lower bounds on Option Values

Bounds for Calls on Non-Dividend-Paying Stocks

- $C_0^T \geq \max(S(0) - Ke^{-rT}, 0)$,
- $C_0^T > 0$,
- $C_0^T \leq S(0)$.

To give an example, let's prove the first inequality. Since the call option cannot have a negative value, we only need to prove

$$C_0^T \geq S(0) - Ke^{-rT}.$$

Proof

- Consider two portfolios at time $t = 0$,

  π_A: one call option plus Ke^{-rT} risk-free zero-coupon bond,
  π_B: one unit of the asset.

Upper/Lower bounds on Option Values

Proof

- At time $t = 0$
$$\begin{cases} \pi_A \text{ is worth } C_0^T + Ke^{-rT}, \\ \pi_B \text{ is worth } S(0). \end{cases}$$
- At the expiry date T, we have
$$\begin{cases} \pi_A : \max(S(T), K), \\ \pi_B : S(T). \end{cases}$$
- The payoff for π_A is never less than the payoff for π_B. By "no arbitrage principle", at time $t = 0$, the value of π_A is no less than the value of π_B, which means
$C_0^T + Ke^{-rT} \geq S(0) \Rightarrow C_0^T \geq S(0) - Ke^{-rt}$.

Upper/Lower bounds on Option Values

Bounds for Puts on Non-Dividend-Paying Stocks

- $P_0^T \geq \max(Ke^{-rT} - S(0), 0)$,
- $P_0^T > 0$,
- $P_0^T \leq Ke^{-rT}$.
- Prove these inequalities as exercises.

European Call Value and Maturity T

- Consider two European call options with expiry dates T_1 and T_2, $T_2 > T_1$. They have the same strike price K. Then

$$C_0^{T_1} \leq C_0^{T_2}$$

.

- Formally speaking, the time-zero value of a European call option on a non-dividend-paying asset, C_0^T, is non-decreasing as a function of the expiry date T.

European Call Value and Maturity T

Proof

Let's look at what would happen at time $t = T_1$.

- At T_1, the stock price is $S(T_1)$.
- The first option matures, and the payoff from it is

$$\max(S(T_1) - K, 0).$$

- The second option has not matured yet. Let's denote its value by $C_{T_1}^{T_2}$.
- The second option can be seen as a new call option starting from T_1 and maturing at T_2 with strike K. Its time to maturity is $T_2 - T_1$.
- From the lower bound on call option,

$$\begin{aligned} C_{T_1}^{T_2} &\geq \max(S(T_1) - Ke^{-r(T_2 - T_1)}, 0) \\ &\geq \max(S(T_1) - K, 0) \end{aligned}$$

European Call Value and Maturity T

Proof (continued)

- At T_1, the second option is definitely not worth less than the first option.
- By no arbitrage principle, at $t = 0$, the second option must also be no less valuable than the first one.
- The text book has a slightly more complicated proof without using the lower bound result.
- Learn the proof yourself.

No arbitrage Principle

Let's formalize our arguments used in the above proof and put them in a theorem.

Theorem

If portfolios A and B are such that in every possible state of the market at time T, portfolio A is worth at least as much as portfolio B, then at any time $t < T$ portfolio A is worth at least as much as portfolio B.

If in addition, portfolio A is worth more than portfolio B in some states of the market at time T, then at any time $t < T$, portfolio A is worth more than portfolio B.

Road Map to Option Valuation

- Our target is to be able to valuate options on some asset, e.g., European/American call/put options on HSBC.
- We will tackle this problem in the rest of lectures following these steps:
  - Let $S(t)$ denote the asset price at time t.
  - Set up some mathematical model to describe $S(t)$, so that we are able to know how $S(t)$ changes over time.
  - With the model for $S(t)$, we derive the mathematical model for options
  - Given the option payoff definition, we apply some concrete mathematical tools to calculate the option values

Taylor Series Expansion

- The Taylor series of a real-valued function $f(x)$ that is infinitely differentiable at a real number a is the power series

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots \\
&= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + O(x^{n+1})
\end{aligned}
$$

- The sum of two normally distributed independent variables $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ is another normal random variable

$$N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

.

Efficient Market Hypothesis

Definition

The current Asset price reflects all past information. (the weak form of the *efficient market hypothesis*)

Implications

- To predict the asset price at some future time, knowing the history of the asset price gives no advantage over just knowing its current price.
- Mathematically speaking, to model the asset price movement from t to $t + \Delta t$, only the asset price at t (but not the prices at any earlier times) is needed.

Asset Returns

- When discussing the movement of an asset, it is more convenient to look at the returns.
- Given a series of asset prices, $S(t_0), S(t_1), \ldots, S(t_n)$, the returns (or the relative price changes) are defined as

$$r_i = \frac{S(t_{i+1}) - S(t_i)}{S(t_i)}, i = 0, \ldots, n-1. \tag{9}$$

- We usually look at the logarithmic returns

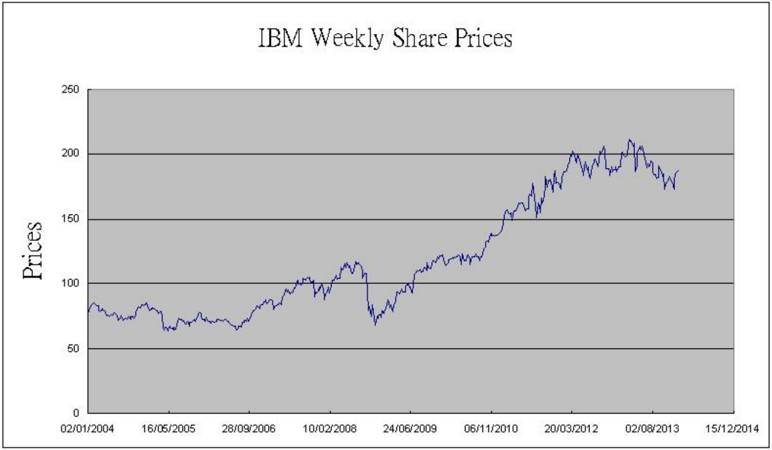
$$r_i = \ln\left(\frac{S(t_{i+1})}{S(t_i)}\right), i = 0, \ldots, n-1. \tag{10}$$

- By Taylor expansion we have

$$\ln(1+x) = x + O(x^2) \Rightarrow \ln(1+x) \approx x \text{ when } |x| \text{ is small}$$

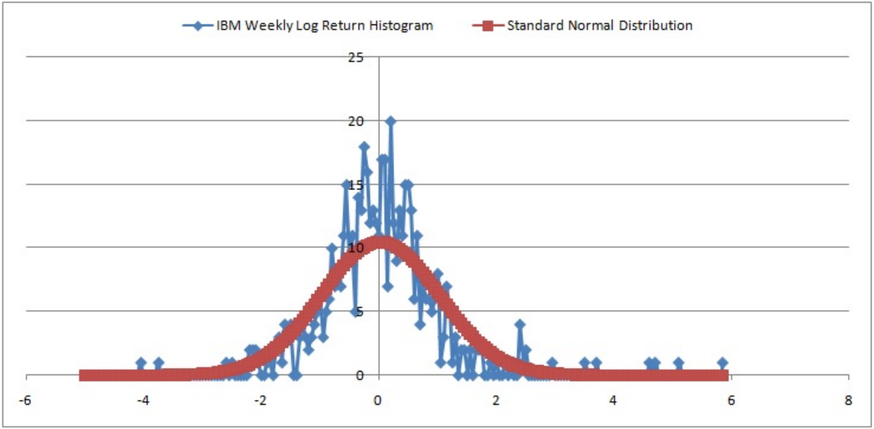
$$\ln\left(\frac{S_{i+1}}{S_i}\right) = \ln\left(1 + \frac{S_{i+1} - S_i}{S_i}\right) \approx \frac{S_{i+1} - S_i}{S_i}$$

Historical Price Data



Historical Returns

- Calculate weekly returns: $r_i^{\text{weekly}} = \ln \frac{S(t_{i+1})}{S(t_i)}$.
- Normalize the returns: $\hat{r}_i^{\text{weekly}} = \frac{r_i^{\text{weekly}} - \mu}{\sigma}$.



Asset Price Modeling

Motivation

Let's divide the time interval $[0, t]$ into N equal subinterval with $\Delta t = t/N$. We consider the relative price change from time t_i to t_{i+1}. Let's write down our asset price model step by step based on our observations from the historical price data.

- We first assume that over longer period of time the relative return is a linear function of Δt:

$$\frac{S(t_{i+1}) - S(t_i)}{S(t_i)} = \mu \Delta t, \text{ where } \mu > 0.$$

- On finer timescale, the stock price goes up and down randomly. So we then add a normal random variable to it:

$$\frac{S(t_{i+1}) - S(t_i)}{S(t_i)} = \mu \Delta t + \sigma \sqrt{\Delta t} N(0, 1) \tag{11}$$

Asset Price Modeling: Discrete Asset Model

By (11), we assume that the relative price change between t_i and t_{i+1} is normally distributed. For convenience, let's write the model in another equivalent form

$$S(t_{i+1}) = S(t_i)(1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_i), \tag{12}$$

and therefore we have

$$S(t) = S(0)\prod_{i=1}^{N}\left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}Z_i\right), \tag{13}$$

where

- μ is a parameter usually called the *drift*, as it expresses how much the asset drifts upwards.
- $\sigma \geq 0$ is a parameter usually called the *volatility*, as it reflects how much the asset wobbles up and down.
- $Z_1, Z_2, \ldots$ are i. i. d $N(0, 1)$.

Asset Price Modeling: Discrete Asset Model

Remarks

- In this model, the mean of the relative change in the stock price in time δt is $\mu \delta t$, and standard deviation of the relative change is $\sigma \sqrt{\delta t}$.
- The drift μ is the expected relative change (i.e. return) on stock per unit time (e.g. per year). Typically it is positive so that it represents that the stock price on average goes up.
- The volatility σ is the standard deviation of the relative change per unit time. It determines the strength of the random fluctuations of the relative price changes.

Continuous Asset Model

Out plan is to let $\delta t \to 0$, and hence let $L \to \infty$, to get a limiting expression for $S(t)$ from (13).

- From (13) we have

$$\ln\left(\frac{S(t)}{S(0)}\right) = \sum_{i=1}^{L} \ln\left(1 + \mu\delta t + \sigma\sqrt{\delta t} Z_i\right)$$

- We are interested in the limit $\delta t \to 0$, recall the Taylor approximation

$$\ln(1+\epsilon) \approx \epsilon - \epsilon^2/2 + \cdots\,, \text{ for small } \epsilon$$

  Hence,

$$\ln(1+a+b) = a + b - \frac{1}{2}a^2 - ab - \frac{1}{2}b^2 + \cdots\,, \text{ for small } a, b$$

- Strictly speaking, Z_i is a random variable not a real number, but what we are about to do is justifiable because $\mathbb{E}(Z_i^2)$ is finite.

Continuous Asset Model

- $$\ln\left(1+\mu\delta t+\sigma\sqrt{\delta t}Z_i\right)$$
  $$\approx \mu\delta t+\sigma\sqrt{\delta t}Z_i-\cancel{\tfrac{1}{2}\mu^2(\delta t)^2}-\cancel{\mu\sigma(\delta t)^{3/2}Z_i}-\frac{1}{2}\sigma^2\delta t Z_i^2$$
  here we ignore terms that involve the power $(\delta t)^{3/2}$ or higher.
- Now we replace the natural log terms on right-hand side with Taylor approximations
  $$\ln\left(\frac{S(t)}{S(0)}\right)\approx\sum_{i=1}^{L}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right),$$
- Now let's look at the term inside the summation:
  $$\mathbb{E}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right) = \mu\delta t-\frac{1}{2}\sigma^2\delta t,$$
  $$\mathrm{Var}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right) = \sigma^2\delta t+O(\delta t)^2$$

Continuous Asset Model

- Insight from the Central Limit Theorem suggests that the right hand side will behave like a normal random variable with

$$\begin{aligned} \text{mean: } & L \times (\mu\delta t - \tfrac{1}{2}\sigma^2\delta t) = (\mu - \tfrac{1}{2}\sigma^2)t, \\ \text{variance: } & L \times \sigma^2\delta t = \sigma^2 t. \end{aligned}$$

- That is, approximately,

$$\sum_{i=1}^{L}\left(\mu\delta t + \sigma\sqrt{\delta t}Z_i - \frac{1}{2}\sigma^2\delta t Z_i^2\right) \approx N\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right).$$

- Both approximations in the above derivations become better as $\delta t \to 0$. Hence, the limiting expression for $S(t)$ satisfies

$$\ln\left(\frac{S(t)}{S(0)}\right) \sim N\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right),$$

that is,

$$\ln S(t) - \ln S(0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z. \tag{14}$$

Continuous Asset Model

- Sometimes it is convenient to use another form of expression for $S(t)$:

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}. \tag{15}$$

- We assume that the underlying asset follows (15).

Independent Increments

If we look at the relationships between $S(t_i)$, $S(t_j)$, and $S(t_k)$, where $0 \leq i < j < k \leq L$, the model (14) implies that

- $\ln\left(\frac{S(t_j)}{S(t_i)}\right)$ is independent of $\ln\left(\frac{S(t_k)}{S(t_j)}\right)$.
- This can be seen from (14), where we have

$$\ln\left(\frac{S(t_{i+1})}{S(t_i)}\right) = (\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_i$$

Z_i are i. i. d $N(0,1)$.

Geometric Brownian Motion

Geometric Brownian Motion

- The asset process $S(t)$ defined by (14) is referred to as *geometric Brownian motion*.
- The standard option pricing model used by major banks is based on some extension to (14).
- The path that the asset price $S(t)$ takes over time t satisfies:
  - it is continuous with probability 1.
  - it is not smooth(not differentiable) at any point with probability 1.
  - timescale invariance: the non-smoothness (jaggedness) looks the same over different timescales.
- The above properties can be treated rigorously, but the concepts required are beyond the scope of this course.
- Instead, the textbook uses simulation results to qualitatively verify these properties.