Lecture 3: Asset Price Modeling and Black-Scholes PDE

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Outline

1. Review of Lecture 2
2. Review of Some Math Knowledge
3. Asset Returns
4. Asset Price Modeling
5. Option Valuation
6. Hedging Portfolio
7. Black-Scholes PDE
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9. Summary
10. Implied Volatility

Review of Lecture 2

- Time value of money, risk-free interest rate
- short-selling of stocks
- Call-Put Parity: $C_0^T + Ke^{-rT} = P_0^T + S(0)$.
- No arbitrage Principle
- Discrete Asset Model

Taylor Series Expansion

- The Taylor series of a real-valued function $f(x)$ that is infinitely differentiable at a real number a is the power series

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + O(x^{n+1}) \end{aligned}$$

- The sum of two normally distributed independent variables $N(\mu_x, \sigma_x^2)$ and $N(\mu_y, \sigma_y^2)$ is another normal random variable

$$N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$$

.

Road Map to Option Valuation

- Our target is to be able to valuate options on some asset, e.g., European/American call/put options on HSBC.
- We will tackle this problem in the rest of lectures following these steps:
  - Let $S(t)$ denote the asset price at time t.
  - Set up some mathematical model to describe $S(t)$, so that we are able to know how $S(t)$ changes over time.
  - With the model for $S(t)$, we derive the mathematical model for options
  - Given the option payoff definition, we apply some concrete mathematical tools to calculate the option values

Asset Price Modeling

- Recall our target is to find some mathematical model to describe $S(t)$.
- Divide the time interval $[0, t]$ into L equal subintervals with $\delta t = t/L$.
- Let's write down our asset price model step by step based on our observations from the historical price data.
  - We first assume that over longer period of time the relative return is a linear function of δt:

$$\frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})} = \mu\delta t, \text{ where } \mu > 0, i = 1, \cdots, L.$$

  - On finer timescale, the stock price goes up and down randomly. So we then add a normal random variable to it:

$$\frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})} = \mu\delta t + \sigma\sqrt{\delta t}N(0, 1) \tag{1}$$

Asset Price Modeling: Discrete Asset Model

- By (1), the relative price change between t_i and t_{i+1} is *normally* distributed.
- For convenience, let's write the model in another equivalent form

$$S(t_i) = S(t_{i-1})(1 + \mu\delta t + \sigma\sqrt{\delta t} Z_i), \tag{2}$$

- therefore,

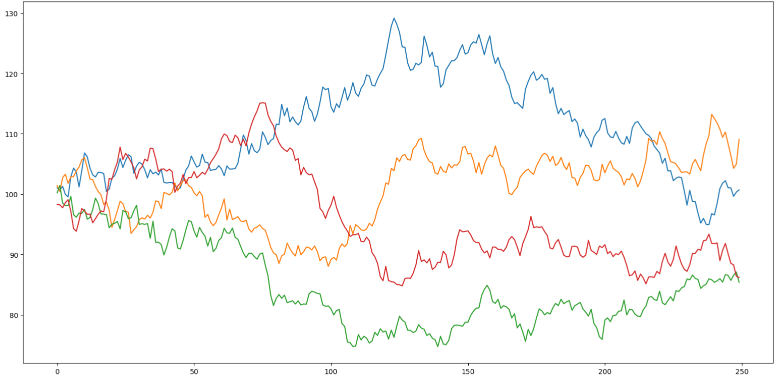
$$S(L\delta t = t) = S(0) \prod_{i=1}^{L} \left(1 + \mu\delta t + \sigma\sqrt{\delta t} Z_i\right), \tag{3}$$

where

- μ is a parameter usually called the *drift*, as it expresses how much the asset drifts upwards.
- $\sigma \geq 0$ is a parameter usually called the *volatility*, as it reflects how much the asset wobbles up and down.
- $Z_1, Z_2, \ldots$ are i. i. d $N(0, 1)$.

Simulated Asset Prices

Just to show the discrete model is a reasonable approximation to describe the asset price movements, we generate 4 possible stock price paths using parameters: $S_0 = 100$, $\mu = 0.05$, $\sigma = 20\%$. Each path consists of the simulated stock prices over 250 days.



Simulated Asset Prices

```
import numpy as np
import math
import matplotlib.pyplot as plt

mu = 0.05
sigma = 0.20
days = 250 # there are about 250 trading days in a year
deltaT = 1.0/days

s0 = 100
zArray = np.random.standard_normal(days)
factorArray = 1 + mu * deltaT + sigma * math.sqrt(deltaT) * zArray
sArray = np.cumprod(factorArray) * s0

plt.plot(sArray)
```

Continuous Asset Model

Out plan is to let $\delta t \to 0$, and hence let $L \to \infty$, to get a limiting expression for $S(t)$ from (3).

- From (3) we have

$$\ln\left(\frac{S(t)}{S(0)}\right) = \sum_{i=1}^{L} \ln\left(1 + \mu\delta t + \sigma\sqrt{\delta t} Z_i\right)$$

- We are interested in the limit $\delta t \to 0$, recall the Taylor approximation

$$\ln(1+\epsilon) \approx \epsilon - \epsilon^2/2 + \cdots, \text{ for small } \epsilon$$

  Hence,

$$\ln(1+a+b) = a + b - \frac{1}{2}a^2 - ab - \frac{1}{2}b^2 + \cdots, \text{ for small } a, b$$

- Strictly speaking, Z_i is a random variable not a real number, but what we are about to do is justifiable because $\mathbb{E}(Z_i^2)$ is finite.

Continuous Asset Model

- $$\begin{aligned}
&\ln\left(1+\mu\delta t+\sigma\sqrt{\delta t}Z_i\right) \\
&\approx \mu\delta t+\sigma\sqrt{\delta t}Z_i-\cancel{\tfrac{1}{2}\mu^2(\delta t)^2}-\cancel{\mu\sigma(\delta t)^{3/2}Z_i}-\frac{1}{2}\sigma^2\delta t Z_i^2
\end{aligned}$$

  here we ignore terms that involve the power $(\delta t)^{3/2}$ or higher.
- Now we replace the natural log terms on right-hand side with Taylor approximations

$$\ln\left(\frac{S(t)}{S(0)}\right)\approx\sum_{i=1}^{L}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right),$$

- Now let's look at the term inside the summation:

$$\begin{aligned}
\mathbb{E}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right) &= \mu\delta t-\frac{1}{2}\sigma^2\delta t, \\
\mathrm{Var}\left(\mu\delta t+\sigma\sqrt{\delta t}Z_i-\frac{1}{2}\sigma^2\delta t Z_i^2\right) &= \sigma^2\delta t+O(\delta t)^2
\end{aligned}$$

Central Limit Theorem

- Suppose we have a set of i.i.d. random variables $X_1, \ldots, X_n$ with mean μ and variance σ^2, and let

$$S_n := \sum_{i=1}^{n} X_i$$

- The central limit theorem: for large n, S_n behaves like an $\mathbf{N}(n\mu, n\sigma^2)$ random variable. More precisely, $(S_n - n\mu)/(\sigma\sqrt{n})$ is approximately $\mathbf{N}(0,1)$ in the sense that for any x we have

$$\mathbb{P}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right) \to N(x), \text{ as } n \to \infty$$

Continuous Asset Model

- Insight from the Central Limit Theorem suggests that the right hand side will behave like a normal random variable with

  mean: $L \times (\mu\delta t - \frac{1}{2}\sigma^2\delta t) = (\mu - \frac{1}{2}\sigma^2)t$,

  variance: $L \times \sigma^2\delta t = \sigma^2 t$.

- That is, approximately,

$$\sum_{i=1}^{L}\left(\mu\delta t + \sigma\sqrt{\delta t}Z_i - \frac{1}{2}\sigma^2\delta t Z_i^2\right) \approx N\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right).$$

- Both approximations in the above derivations become better as $\delta t \to 0$. Hence, the limiting expression for $S(t)$ satisfies

$$\ln\left(\frac{S(t)}{S(0)}\right) \sim N\left((\mu - \frac{1}{2}\sigma^2)t, \sigma^2 t\right),$$

  that is,

$$\ln S(t) - \ln S(0) = (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z. \qquad (4)$$

Continuous Asset Model

- Sometimes it is convenient to use another form of expression for $S(t)$:

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}Z}. \tag{5}$$

- We assume that the underlying asset follows (5).

Independent Increments

If we look at the relationships between $S(t_i)$, $S(t_j)$, and $S(t_k)$, where $0 \leq i < j < k \leq L$, the model (4) implies that

- $\ln\left(\frac{S(t_j)}{S(t_i)}\right)$ is independent of $\ln\left(\frac{S(t_k)}{S(t_j)}\right)$.
- This can be seen from (4), where we have

$$\ln\left(\frac{S(t_{i+1}}{S(t_i)}\right) = (\mu - \frac{1}{2}\sigma^2)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}Z_i$$

Z_i are i. i. d $N(0, 1)$.

Simulated Asset Prices

Similiar to the discrete model, with this continuous model, we generate 4 possible stock price paths using parameters: $S_0 = 100$, $\mu = 0.05$, $\sigma = 20\%$.

Simulated Asset Prices

```
import numpy as np
import math
import matplotlib.pyplot as plt

mu = 0.05
sigma = 0.20
days = 250 # there are about 250 trading days in a year
deltaT = 1.0/days

s0 = 100
zArray = np.random.standard_normal(days)
factorArray = (mu - 0.5 * sigma * sigma) * deltaT
              + sigma * math.sqrt(deltaT) * zArray
returnArray = math.log(s0) + np.cumsum(factorArray)
sArray = np.exp(returnArray)
plt.plot(sArray)
```

Geometric Brownian Motion

Geometric Brownian Motion

- The asset process $S(t)$ defined by (4) is referred to as *geometric Brownian motion*.
- The standard option pricing model used by major banks is based on some extension to (4), where σ is not a constant but a function of $S(t)$ and t.
- The path that the asset price $S(t)$ takes over time t satisfies:
  - it is continuous with probability 1.
  - it is not smooth(not differentiable) at any point with probability 1.
  - timescale invariance: the non-smoothness (jaggedness) looks the same over different timescales.
- The above properties can be treated rigorously, but the concepts required are beyond the scope of this course.
- Instead, the textbook uses simulation results to qualitatively verify these properties.

Option Valuation

Motivation

- We have defined European call/put options.
- We have developed a model for asset price movement.
- We will start to consider the valuation of European options.
- We will look for a function $V(S, t)$ which gives the option value for any asset price $S \geq 0$ at any time $t < T$.
- From now on, we always use T to represent the option maturity.

Intuition

- Option writer bears risk which comes from the uncertainty in the price of the underlying asset.
- Option value (premium) is paid to the writer to compensate the risk.
- The fair value (premium) of an option should be the cost for the writer to eliminate the risk.
- We first set up a dynamic portfolio which has the same risk as the option itself, and then deduce the option value from this dynamic portfolio using no-arbitrage principle.
- In finance, the action to reduce/eliminate risk is called *hedging*.

Some Assumptions

Before going into the details of hedging, let's state a list of assumptions:

- The asset price follows the geometric Brownian motion defined by Equation (4).
- The risk-free interest rate r and the asset volatility σ are known.
- There are no transaction costs.
- The asset pays no dividends during the life of the option.
- Trading of the asset can take place continuously.
- Short selling is permitted.
- We can buy or sell any units (not necessarily an integer) of the asset.

Hedging Portfolio

- With the above assumptions, the risk embedded in an option is only related to the uncertainty in the asset price.
- We set up a portfolio $\Pi(t)$ consisting of $A(S(t), t)$ units of the asset and $D(S(t), t)$ in a cash account at time t.
- Here both the units $A(S(t), t)$ and the cash amount $D(S(t), t)$ are functions of the asset price $S(t)$ and time t.
- Thus at time t the portfolio value is

$$\Pi(t) = A(S(t), t)S(t) + D(S(t), t).$$

- We want that $\Pi(t)$ has the same risk as the option at all time.
- How to determine the number of units $A(S(t), t)$?

Hedging Portfolio

- Let T be the option maturity. Split $[0, T]$ into L equally-space tiny subintervals:

$$t = t_0 < t_1 < \cdots < t_L = T, \quad \delta t = \frac{T}{L}.$$

- For simplicity, let $S_i = S(t_i)$, $A_i = A(S(t_i), t_i)$, and $D_i = D(S(t_i), t_i)$.
- Let's look at how the portfolio value and the option value change over the tiny period of time from t_i to t_{i+1}.
- Suppose the asset price change over this period of time is $\delta S_i = S(t_{i+1}) - S(t_i)$.
- At time t, the portfolio Π has A_i units of the asset, so the change δS_i produces $A_i \delta S_i$ in Π.
- Interest accrued on the cash over δt is $rD_i\delta t$, where r is the risk-free interest rate.
- Thus, the change in portfolio Π's value from t_i to t_{i+1} is

$$\delta \Pi_i = A_i \delta S_i + rD_i \delta t.$$

Ito Lemma

- Let's look at the option value change over this same period of time δt. From Taylor's expansion, we have

$$
\begin{aligned}
&\delta V_i = V(S_{i+1}, t_{i+1}) - V(S_i, t_i) = V(S_i + \delta S_i, t_i + \delta t_i) - V(S_i, t_i) \\
&\approx \frac{\partial V}{\partial t}(S_i, t_i)\delta t + \frac{\partial V}{\partial S}(S_i, t_i) \times \delta S_i + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(S_i, t_i) \times (\delta S_i)^2
\end{aligned}
$$

- From previous slides, $\delta S_i \approx S_i(\mu \delta t + \sigma \sqrt{\delta t} Z_i)$. Then

$$
\begin{aligned}
&(\delta S_i)^2 \\
&= S_i^2 \left(\sigma^2 \delta t Z_i^2 + 2(\delta t)^{\frac{3}{2}} \mu \sigma Z_i + \mu^2 (\delta t)^2 \right) \\
&= S_i^2 \left(\sigma^2 \delta t + \sigma^2 \delta t (Z_i^2 - 1) + 2(\delta t)^{\frac{3}{2}} \mu \sigma Z_i + \mu^2 (\delta t)^2 \right) \\
&\approx S_i^2 \sigma^2 \delta t
\end{aligned}
$$

Ito Lemma

- Let's look at the last three terms $\sigma^2\delta t(Z_i^2 - 1)$, $(\delta t)^{\frac{3}{2}}\mu\sigma Z_i$, and $\mu^2(\delta t)^2$
- $\sigma^2\delta t(Z_i^2 - 1)$ is a random variable. We have

$$\begin{cases} \mathbb{E}[\delta t(Z_i^2 - 1)] & = 0 \\ \mathrm{var}[\delta t(Z_i^2 - 1)] & = (\delta t)^2\mathbb{E}[(Z_i^2 - 1)^2] = (\delta t)^2(3\sigma^4 - 2\sigma^2 + 1) \end{cases}$$

$\Longrightarrow$ roughly speaking $\sigma^2\delta t(Z_i^2 - 1)$ converges to 0 faster than $O(\delta t)$

- Let's look at the term $(\delta t)^{\frac{3}{2}}\mu\sigma Z_i$.

$$\begin{cases} \mathbb{E}[(\delta t)^{\frac{3}{2}}\mu\sigma Z_i] & = 0 \\ \mathrm{var}[(\delta t)^{\frac{3}{2}}\mu\sigma Z_i] & = (\delta t)^3\mu^2\sigma^2 \end{cases}$$

$\Longrightarrow$ roughly speaking $(\delta t)^{\frac{3}{2}}\mu\sigma Z_i$ converges to 0 faster than $O(\delta t)$

Ito Lemma

- The above approximation comes from the observation that the last three terms are trivially of $o(\delta t)$ in both the mean and variance.
- Now we have the option value change from t_i to t_{i+1}

$$\delta V_i = V(S_{i+1}, t_{i+1}) - V(S_i, t_i) = V(S_i + \delta S_i, t_i + \delta t_i) - V(S_i, t_i)$$
$$= \frac{\partial V}{\partial t}(S_i, t_i)\delta t + \frac{\partial V}{\partial S}(S_i, t_i)\delta S_i + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i)\delta t$$

- Recall that the value change of portfolio Π from t_i to t_{i+1} is

$$\delta\Pi_i = A_i\delta S_i + rD_i\delta t.$$

- To compare the value changes in Π and V,

$$\delta(V-\Pi)_i = \left(\frac{\partial V}{\partial t}(S_i, t_i) - rD_i + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i)\right)\delta t$$
$$+ \left(\frac{\partial V}{\partial S}(S_i, t_i) - A_i\right)\delta S_i$$

- Recall that we want to make Π has the same risk as the option at all time.

Black-Scholes PDE

- The portfolio $\Pi - V$ should has no risk, which means $\delta(V-\Pi)_i$ should not depend on the unpredictable quantity δS_i.
- Hence, we need to set $A_i = \frac{\partial V}{\partial S}(S_i, t_i)$, and we have

$$\delta(V-\Pi)_i = \left(\frac{\partial V}{\partial t}(S_i, t_i) - rD_i + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i)\right)\delta t$$

- On the other hand, because the portfolio $\Pi - V$ should has no risk, it must grow at the risk-free rate over δt,

$$\delta(V-\Pi)_i = r(V-\Pi)_i\delta t$$

- Hence,

$$\left(\frac{\partial V}{\partial t}(S_i, t_i) - rD_i + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i)\right)\delta t = r(V-\Pi)_i\delta t$$

Black-Scholes PDE

- By simplifying the equation, we have

$$\frac{\partial V}{\partial t}(S_i, t_i) - rD_i + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i) = r(V_i - A_iS_i - D_i)$$
$$\Longrightarrow$$
$$\frac{\partial V}{\partial t}(S_i, t_i) + rS_i\frac{\partial V}{\partial S}(S_i, t_i) + \frac{1}{2}S_i^2\sigma^2\frac{\partial^2 V}{\partial S^2}(S_i, t_i) - rV_i = 0.$$

- Since there is nothing specific about t_i in the above derivation, the above equation holds for any time $t < T$.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{6}$$

- This is the famous *Black* − *Scholes* partial differential equation (PDE).

Black-Scholes PDE

Remarks

- The drift parameter μ in the asset model does not appear in the PDE (6).
- The PDE (6) must be satisfied for any option whose value depends on S and t and is paid up-front. We didn't make use of any specific information from the option type (e.g. a call option or a put option).
- The PDE (6) can have many solutions. For example, $V(S,t) = S$ is one solution. $V(S,t) = e^{rt}$ is another one.
- This idea of continuously fine-tuning the portfolio in order to reduce or remove risk is known as *dynamic hedging*.

Derivatives Valuation

Let's summarize the framework of financial derivative valuation:

- Payoffs are linked directly to the price of an "underlying" asset.
- Valuation is mostly based on replication/hedging arguments.
  - Find a portfolio that includes the underlying asset, and possibly other related derivatives, to replicate the payoff of the target derivative asset, or to hedge away the risk in the derivative payoff.
  - Since the hedged portfolio is risk free, the payoff of the portfolio can be discounted by the risk free rate.
  - Models of this type are called "no-arbitrage" models.
- Key idea: No forecasts are involved. Valuation is based on cross-sectional comparison.
- It is not about whether the underlying asset price will go up or down (given growth rate or risk forecasts), but about the relative pricing relation between the underlying and the derivatives under all possible scenarios.

Nobel Prize in Economics

A Bit of History

- This equation is named after its inventors, Fisher Black and Myron Scholes.
- Robert Merton also made significant contributions here.
- Merton and Scholes received the 1997 Nobel Prize in Economics for this work. Unfortunately, Black passed away in 1995.
- Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish Academy.

Black-Scholes PDE

- The function $V(S,t) = S$ satisfies the PDE (6). But it is definitely not the right solution for a call option. Why?
- To uniquely determine $V(S,t)$, we have to specify other conditions that involve information about the particular option.
- Consider a European call option $C(S,t)$. Let's see what we know about $C(S,t)$:
  - *Terminal condition* of the PDE (6):

$$C(S,T) = \max(S(T) - K, 0). \tag{7}$$

  - *Lower boundary condition* of (6):

$$\lim_{S \to 0} C(S,t) = 0, \text{ for any } 0 \leq t \leq T. \tag{8}$$

  - *Upper boundary condition* of (6):

$$\lim_{S \to \infty} C(S,t) = S - Ke^{-r(T-t)}, \text{ for any } 0 \leq t \leq T. \tag{9}$$

Black-Scholes PDE

- Let's look at the conditions (7)-(9) a bit closer.
- (7) is just the payoff definition of a European call option.
- When $S = 0$, from our asset model, the asset price would stay at 0 for any time t. So the option value is 0. The lower boundary condition (8) is justified.
- When $S \to \infty$, it becomes ever more likely that the option will be exercised and the magnitude of the exercise price becomes less and less important.
- Thus, as $S \to \infty$, the call option becomes a forward contract with K as the delivery price. It can be proved that the value of the option in this case is

$$S - Ke^{-r(T-t)}.$$

which is the upper boundary condition (9).

Forward Contracts

- Recall the payoff at maturity for a long forward contract position is

$$S(T) - K.$$

- Then we can prove that the value function $V(S(t), t)$ satisfies

$$V(S, t) = S(t) - Ke^{-r(T-t)}.$$

- In Assignment 1, you are asked to prove that to make $V(S, 0) = 0$, K must be equal to Se^{rT}.

Black-Scholes Formulas

- With three conditions imposed on (6), we can derive a unique solution for the call option value.
- But the derivation is beyond the scope of this course.
- Instead, we give the solution directly, and then verify that the solution $C(S,t)$ satisfies (6) and the terminal and boundary conditions.
- The solution function $C(S,t)$ is

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{10}$$

where $N(x)$ is the cumulative density function of $\mathbf{N}(0,1)$, and

$$\begin{aligned} d_1 &= \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}, & (11) \\ d_2 &= \frac{\ln(S/K) + r(T-t)}{\sigma\sqrt{T-t}} - \frac{1}{2}\sigma\sqrt{T-t}. & (12) \end{aligned}$$

Black-Scholes Formulas

- Given $C(S,t)$, we can derive the value $P(S,t)$ of a European put option using the call-put parity equation:

$$C(S,t) + Ke^{-r(T-t)} = P(S,t) + S$$

- The solution function $P(S,t)$ is

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{13}$$

- Alternatively, let's see what we know about $P(S,t)$:
  - *Terminal condition* of the PDE (6):

$$P(S,T) = \max(K - S(T), 0). \tag{14}$$

  - *Lower boundary condition* of (6):

$$\lim_{S \to 0} P(S,t) = Ke^{-r(T-t)}, \text{ for any } 0 \le t \le T. \tag{15}$$

  - *Upper boundary condition* of (6):

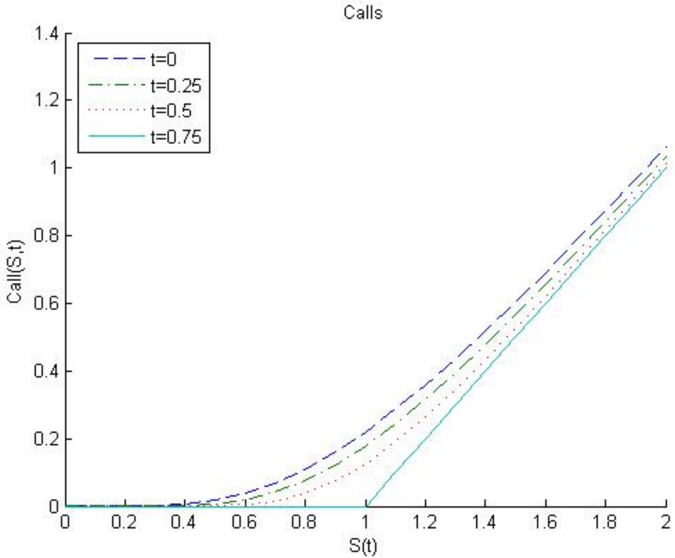
$$\lim_{S \to \infty} P(S,t) = 0, \text{ for any } 0 \le t \le T. \tag{16}$$

Black-Scholes PDE

- In summary, to valuate a financial derivative using (6), we need to specify the terminal and boundary conditions.
- Some financial insight should be utilized to choose suitable conditions for the derivative at hand.
- Most of the time, there is no closed-form solution from (6).
- It has to be solved using numerical techniques, for example, the finite difference method, the finite element method, etc.

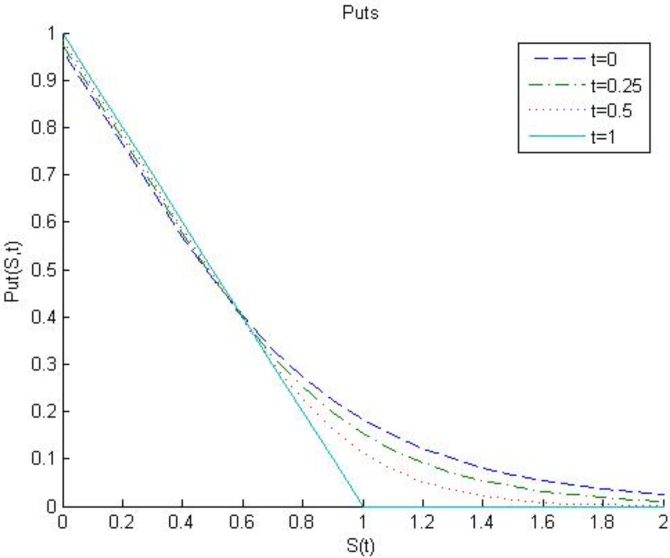
Black-Scholes Formulas

$K = 1$; $r = 0.05$; $\sigma = 0.6$; $T = 0.75$; The call option value decreases when getting closer to maturity.



Black-Scholes Formulas

The put option value can decrease or increase when getting closer to maturity, depending on the spot level.



Main Takeaway

The key points from this lecture:

- The drift term μ in the asset model (5) doesn't matter for option valuation.
- Risk can be eliminated by holding a portfolio in which the random parts of two different sub-portfolios cancel each other
- No-arbitrage principle implies that a portfolio from which risk has been eliminated must grow at the risk-free rate.
- A European option's value can be replicated by a (self-financing) portfolio consisting of dynamically trading in stock and risk-free bond.
- Get familiar with the closed-form formulas for European Call/Put options.

Practicalities of Trading Options

- So far, we have learned quite a bit about options.
- You might start thinking to get your feet wet and trade some options. But before that you have to answer the following question first.
  - With the same underlying asset, there are many different options with different strikes, different maturities, and different payoff types (Call or Put).
  - How will you decide which one to buy or sell? Intuitively, you should buy cheaper ones and sell more expensive ones.
  - You need a systematic way to decide the relative cheapness/richness among different options.
- Can we directly look at the prices of options, just similarly to what we normally do with stocks?
- Not a good idea! Why?
- The rest of this lecture gives you a new tool that could help you on this.

What is Implied Volatility

- The Black-Scholes formula gives the value of an option as a function of several inputs: S_0, K, T, r, and σ.
- Of these only one is not specified in the contract or readily observable, the volatility σ.
- What we can observe from the market are option prices.
- Since the option price is a monotonic function with respect to σ, given the option price V, there exists a unique σ when substituted into Black-Scholes formula that gives the option price V, which is called *implied volatility*.
- In practice, we calculate the implied volatilities for different strikes and different maturities, and then generate an implied volatility surface using some numerical interpolations.

Why implied volatility

- A convenient quantity to measure the cheapness or dearness of an option.
- We can look at the option premium directly for cheapness/dearness. Not a good idea:
  - Options with different strikes, different maturities, and different underlying assets are essentially different contracts. Direct comparison is meaningless.
- With implied volatilities, there are two ways of judging the cheapness or dearness of options.
- The first is simply by comparing current implied volatility with past levels of implied volatility on the same underlying asset.
- The second is by comparing current implied volatility with the historical volatility of the underlying itself (will be discussed in next lecture).

Why implied volatility

- Implied volatility is relatively more stable than stock levels.
- Complex products need the volatilities implied by the prices of simple options observed from the market.
- Later we will introduce some much more complex options, e.g., Asian options, basket options, etc.
- All of these are OTC products. Not standardized products. Their prices cannot be observed directly from the market. You have to check with various banks for the price.
- Then how do banks get the volatility information to price them?
- They use the implied volatility surface obtained from the market prices of simpler products (European/American call/put options).