Comp 7405 Assignment 1

Due time: 7:00 pm, 10-02-2021

1. **(No arbitrage forward price, 4 marks)** Suppose we have an non-dividend-paying stock S and the risk-free rate is r. Now, at time $t = 0$, Party A agrees to purchases one share of the stock from Party B at a specified delivery time time $t = T$ for a specified price F. This is a *forward* contract which was discussed in class. Appealing to the no arbitrage assumption, show that the delivery price F should be $S(0)e^{rT}$.

   Remarks: From this problem, you should notice that the delivery price does not depend on any fundamental economic analysis of the stock. If two stocks have the same share prices now, then their forwards for the same maturity should be the same as well. This is closely related to a fundamental concept in derivative pricing *risk neutrality*, which will be covered later.

2. **(No arbitrage conditions, 6 marks)** From the market, normally you can observe the values of quite a few options on a same asset with different strikes and different maturities. In this problem, you will see the conditions the option values have to satisfy to avoid arbitrage. Let $C_t^T(K)$ denote at time t the value of a European call option on a non-dividen-paying stock with strike K and maturity T. To avoid arbitrage, $C_t^T(K)$ has to satisfy the following conditions:

   (2.1) vertical spread condition :

   $$0 > \frac{C_t^T(K_1) - C_t^T(K_2)}{K_1 - K_2} > -e^{-r(T-t)},$$

   where r is the risk-free interest rate.

   (2.2) butterfly condition :

   $$C_t^T(K + \Delta K) - 2C_t^T(K) + C_t^T(K - \Delta K) > 0,$$

   where ΔK is a positive number and $\Delta K < K$.

   (2.3) Non-negative ratio butterfly condition :

   $$aC_t^T(K_1) - bC_t^T(K_2) + (b-a)C_t^T(K_3) > 0,$$

   where $0 \leq K_1 < K_2 < K_3$, and $\frac{a}{b} = \frac{K_3 - K_2}{K_3 - K_1}$.

(2.4) calendar spread condition : $C_t^{T_1}(, K) \leq C_t^{T_2}(K)$ where $T_1 < T_2$, when the underlying asset doesn't pay dividends between time T_1 and T_2.

We have proved (2.4) in the lecture. You are asked to prove only (2.1) and (2.2). The condition (2.3) is just for your information. Note that you cannot use (2.3) to prove (2.1) or (2.2)