Chapter 1: Introduction to Pipe flow

1.1 Introduction

A pipe is a closed conduit of circular cross-section, which is used for carrying fluid under pressure. When the pipe is running full of liquid, the flow is under pressure.

The fluid flowing in a pipe is always subjected to resistances due to shear force between fluids particles and the boundary wall of the pipe and between the fluid particles themselves resulting from the viscosity of the fluid. The resistance to the flow of fluid is known as frictional resistance. Since certain amount of energy possessed by the flowing fluid will be consumed in overcoming frictional resistance to the flow, there will always be loss of energy in the direction of flow, which depends on the types of flow.

The flow of fluid in pipe may be of two types:

a. Laminar flow

- regular, smooth and systematic
- no intermixing of fluid particles in adjacent layers
- occurs for $Re<2000$
- low velocity
- high viscosity
- e.g. flow past tiny bodies, groundwater flow, flow of blood through vessels, rise of water in plants through their roots

b. Turbulent flow

- irregular and erratic
- violent mixing of fluid particles
- occurs for $Re>4000$
- high velocity
- low viscosity
- e.g. flow through pipes, sewers, rivers

1.2 Reynolds's experiment

The existence of two types of flow in the pipe was first demonstrated by Reynold with the help of a simple experiment.

Apparatus

- A tank containing water at constant head
- A small tank containing dye
- A horizontal glass tube with a bell mouthed entrance at one end and a regulating valve at other end.

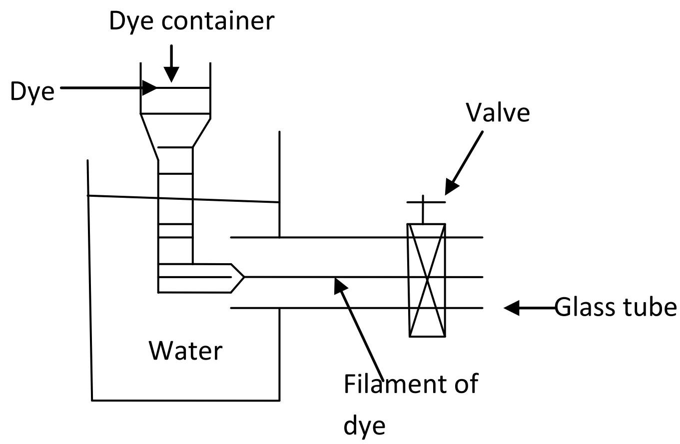


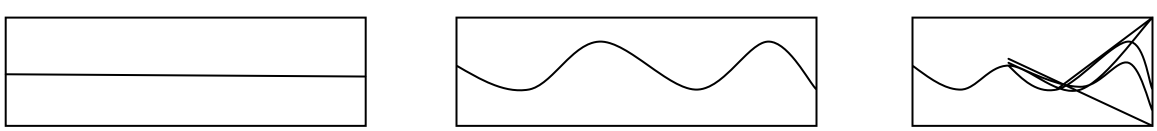
Fig 1.1: Reynolds experiment

Procedure

The water from the tank was allowed to flow through glass tube into atmosphere. The velocity of the flow was varied by the regulating valve. A liquid dye having the same specific weight as water was introduced into the flow at the bell mouthed entrance through a small tube.

Observation

1. When the velocity of the flow was low, the dye filament in the glass tube was in the form of a straight line. This was a case of laminar flow.
2. With the increase in the velocity of flow, the dye filament was no longer a straight line, but it became wavy. This shows that the flow is no longer laminar. This was a transitional state.
3. With further increase in velocity, the wavy dye-filament broke up and finally diffused in water. This was the case of a turbulent flow.



Straight dye filament   Wavy filament   Diffused filament

Fig. 1.2: Result of Reynold's experiment

Result

1. The velocity at which the flow changes from laminar to turbulent for the case of a given fluid at a given temperature and in a given pipe is a critical velocity.
2. The occurrence of a laminar and turbulent flow was governed by the relative magnitude of inertia and viscous force.

For low velocities, viscous force is predominant, whereas for higher velocities, inertial force is predominant.

$$Reynold's\ number\ (Re) = \frac{Inertia\ force}{Viscous\ force}$$
$$= \frac{mass\ x\ acceleration}{shear\ stress\ x\ Area}$$

$$= \frac{\rho x\ Vol\ x\ acceleration}{\mu \frac{\partial v}{\partial y} x Area}$$
$$= \frac{\rho x\ L^3\ x\ LT^{-2}}{\mu \frac{V}{L} x L^2} = \frac{\rho L^2 V^2}{\mu V L} = \frac{\rho V L}{\mu}$$

where ρ= density of fluid, V = mean velocity, L = Characteristic length, μ= coefficient of viscosity
In case of pipe flow, Diameter D is taken as characteristic length L.

$$R_e = \frac{\rho V D}{\mu}$$

or, $Re = \frac{VD}{\nu}$ where ν = kinematic viscosity

1.3 Steady laminar flow in a circular pipe

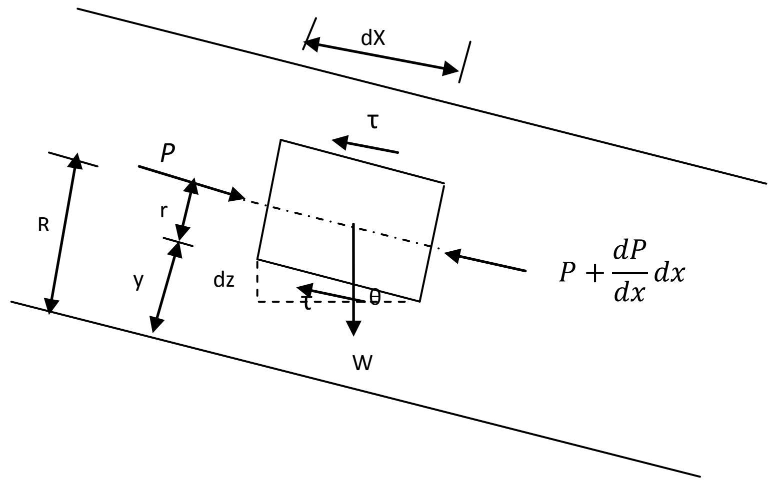


Fig. 1.3: Forces acting on fluid flowing through inclined pipe for laminar flow

Consider a horizontal pipe of radius R through which a fluid of density ρflows. Take a cylindrical element of fluid of radius 'r' and length dx. Forces acting on the fluid element are: pressure force, shear force and component of weight, W.

<u>a. Shear stress distribution</u>

Resolving forces along the direction of motion,

$$P(\pi r^2) - \left(P + \frac{dP}{dx}dx\right)\pi r^2 - \tau(2\pi r dx) + W sin\theta = 0$$

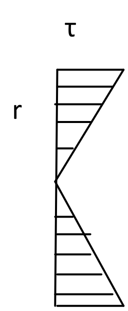
$$P(\pi r^2) - \left(P + \frac{dP}{dx}dx\right)\pi r^2 - \tau(2\pi r dx) - \gamma\pi r^2 dx\frac{dz}{dx} = 0$$

$$\tau = -\frac{d(P+\gamma z)}{dx}.\frac{r}{2} \qquad \text{(I)}$$

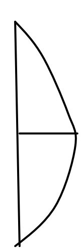
(-ve sign shows that the pressure decreases in the direction of flow.)

The shear stress varies linearly with r as $-\frac{d(P+\gamma z)}{dx}$ across section is constant.

The term $\frac{d(P+\gamma z)}{dx}$ can also be written as $\frac{\gamma\, d\left(\frac{P}{\gamma}+z\right)}{dx} = \gamma\frac{d\,h}{dx}$ where h = piezometric head



Shear stress distribution



Velocity distribution

b. Velocity distribution

Assumptions: Fluid is Newtonian and there is no slip at the boundary (zero vel. at the boundary).

From Newton's law of viscosity

$$\tau = \mu \frac{dv}{dy}$$

Distance of fluid element from pipe wall (y) is

y = R – r

On differentiation

dy = -dr

$\tau = -\mu \frac{dv}{dr}$ (II)

Equating eq. (I) and (II)

$-\mu \frac{dv}{dr} = -\frac{d(P+\gamma z)}{dx} . \frac{r}{2}$

$\frac{dv}{dr} = \frac{r}{2\mu} \frac{d(P+\gamma z)}{dx}$

Integrating w.r.t. r

$v = \frac{r^2}{4\mu} \frac{d(P+\gamma z)}{dx} + C$

At r = R (At boundary), v = 0

$0 = \frac{R^2}{4\mu} \frac{d(P+\gamma z)}{dx} + C$

$C = -\frac{R^2}{4\mu} \frac{d(P+\gamma z)}{dx}$

Substituting the value of C

$v = -\frac{1}{4\mu} \frac{d(P+\gamma z)}{dx} (R^2 - r^2)$

Here $\mu, \frac{d(P+\gamma z)}{dx}$ and R are constant, which means u varies with r. This equation is the equation of parabola. This shows that local velocity varies parabolically along diameter.

Maximum velocity

At r = 0, v = v_{max}

$v_{max} = -\frac{1}{4\mu} \frac{d(P+\gamma z)}{dx} R^2$

Local velocity in terms of maximum velocity

$v = -\frac{1}{4\mu} \frac{d(P+\gamma z)}{dx} (R^2 - r^2)$

$u = -\frac{1}{4\mu} \frac{d(P+\gamma z)}{dx} R^2 \left(1 - \frac{r^2}{R^2}\right)$

$v = v_{max} \left(1 - \frac{r^2}{R^2}\right)$

Average velocity

Average velocity (V) = $\frac{Discharge}{Cross-sectional\ area}$

Considering an elementary strip of thickness dr and area dA at a radial distance r from pipe center.

$V = \frac{\int vdA}{\pi R^2} = \frac{\int_0^R v_{max}\left(1-\frac{r^2}{R^2}\right)2\pi rdr}{\pi R^2}$

$= \frac{2v_{max}}{R^2}\left|\frac{r^2}{2} - \frac{r^4}{4R^2}\right|_0^R$

$V = \frac{v_{max}}{2}$

or, $v_{max} = 2V$

<u>The point where the local velocity is equal to the mean velocity</u>

$v = v_{max}\left(1 - \frac{r^2}{R^2}\right)$

$V = \frac{v_{max}}{2}$

For v = V

$\frac{v_{max}}{2} = v_{max}\left(1 - \frac{r^2}{R^2}\right)$

$r^2 = 0.5\ R^2$

r = 0.707R

<u>c. Pressure difference in terms of average velocity: Hagen-Poiseuille equation (Equation for head loss)</u>

Average velocity is given by

$V = \frac{v_{max}}{2} = -\frac{1}{2}.\frac{1}{4\mu}\frac{d(P+\gamma z)}{dx}R^2$

$-d(P + \gamma z) = \frac{8\mu V}{R^2}dx$

Integrating

$-\left(\int_{P_1}^{P_2} dP + \int_{Z_1}^{Z_2} \gamma z\right) = \int_{x_1}^{x_2} \frac{8\mu V}{R^2}dx$

$(P_1 - P_2) + \gamma(Z_1 - Z_2) = \frac{8\mu V}{R^2}(x_2 - x_1)$

$(P_1 - P_2) + \gamma(Z_1 - Z_2) = \frac{8\mu V}{R^2}L$

where L = Length of the pipe

In terms of diameter, D

$(P_1 - P_2) + \gamma(Z_1 - Z_2) = \frac{8\mu V}{(\frac{D}{2})^2}L$

$(P_1 - P_2) + \gamma(Z_1 - Z_2) = \frac{32\mu VL}{D^2}$

Dividing by γ

$\left(\frac{P_1}{\gamma} + Z_1\right) - \left(\frac{P_2}{\gamma} + Z_2\right) = \frac{32\mu VL}{\gamma D^2}$

As $V_1 = V_2$ for pipe of uniform diameter, $\left(\frac{P_1}{\gamma} + Z_1\right) - \left(\frac{P_2}{\gamma} + Z_2\right) = h_f$ (according to Bernoulli's equation)

where h_f = loss of head for laminar flow

$h_f = \frac{32\mu VL}{\gamma\ D^2}$ or $\frac{32\mu VL}{\rho g D^2}$

This equation is known as Hagen-Poiseuille equation. This equation was derived experimentally by Hagen and Poiseuille.

Features of Hagen-Poiseuille equation

- Head loss for laminar flows varies with the first power of velocity.
- The equation does not have any empirical roughness coefficient.
- Head loss for laminar flow depends on fluid properties and pipe geometry.

<u>Formulae for steady, laminar flow through horizontal pipe</u>

For horizontal pipe, put Z = 0 in all expressions of inclined pipe.

The equations for shear stress and velocity distribution

$\tau = -\frac{dP}{dx}.\frac{R}{2}$

$v = -\frac{1}{4\mu}\frac{dP}{dx}(R^2 - r^2)$

$v_{max} = -\frac{1}{4\mu}\frac{dP}{dx}R^2$

$v = v_{max}\left(1 - \frac{r^2}{R^2}\right)$

$v_{max} = 2V_{av}$

$(P_1 - P_2) = \frac{32\mu VL}{D^2}$

Power required to overcome the frictional resistance

Power (P_w) = force x Velocity = (pressure gradient x volume) Velocity

$= -\frac{\partial P}{\partial x}ALV = \frac{P_1 - P_2}{L}\ ALV = Q(P_1 - P_2) = Q\frac{(P_1 - P_2)}{\gamma}\gamma$

Power $= \gamma Q h_f$

1.4 Turbulent flow in pipes

In laminar flow (Re<2000), any disturbance produced is quickly damped out by the viscous resistance. At higher Re (>4000), the fluid motion is irregular, random and chaotic. There is complete mixing of fluid due to collision of fluid masses with each other. The resulting flow is known as turbulent flow. Strong eddies and vortices are formed due to the varied flow velocity of adjacent flow layers and the viscous force, rough projections of the boundary surface and sudden or sharp discontinuities in geometry. The phenomenon of turbulent motion is known as turbulence.

In turbulent flow, velocity fluctuation cause a continuous interchange of fluid masses between neighboring layers, which is accompanied by a transfer of momentum. This momentum transfer results in developing additional shear stress besides viscous shear stress. The additional shear stress is known as turbulent shear stress.

Velocity in turbulent flow

In turbulent flow, velocity does not remain constant with time. The velocity at any instant is considered to be made up of a mean value and a fluctuating component.

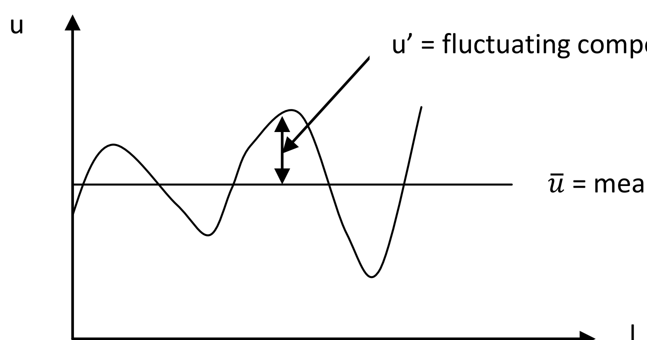


Fig. 1.4: Velocity variation in turbulent flow

Velocity at any time, $u(t) = \bar{u} + u'$

$\bar{u} = \frac{1}{T}\int_0^T u(t)dt$

$\overline{u'} = \frac{1}{T}\int_0^T u'(t)dt$

Over large time T, mean of fluctuating component = 0 and the mean value is taken for computation purpose.

Similarly in other directions, $v(t) = \bar{v} + v'$ and $w(t) = \bar{w} + w'$

Features of turbulent flow in pipes

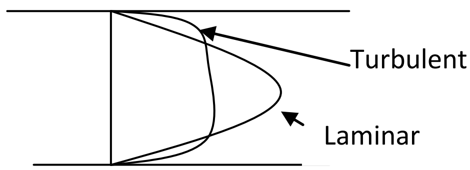


Fig. 1.5: Velocity profile

- Re>4000
- Flatter velocity (more uniform) profile due to mixing of fluid layers and exchange of momentum
- Large velocity gradient near the wall resulting in more shear
- Parabolic profile in laminar, logarithmic or power law profile in turbulent flow
- Additional shear stresses due to turbulence: Reynolds stress
- Higher frictional loss due to the formation of eddies, mixing and curving of path lines
- Experimentally derived relationship for shear stress due to complexity in mathematical analysis (Semi-empirical)

1.4.1 Shear stress in turbulent flow

a. Boussinesq theory

Similar to the Newton's law, Boussinesq developed an expression for the turbulent shear stress

$\tau_t = \eta \frac{dV}{dy} = \rho\varepsilon \frac{dV}{dy}$

τ_t =turbulent shear stress

V= average velocity at a distance y from boundary

η = Turbulent mixing coefficient (eddy viscosity)

ε = Eddy kinematic viscosity $= \frac{\eta}{\rho}$

If viscous action is also considered, the total shear stress is

$\tau = \mu \frac{dV}{dy} + \eta \frac{dV}{dy} = (\mu + \eta) \frac{dV}{dy}$

μ = function of temperature (fluid property)

η and ε = function of flow characteristics

Since η and ε cannot be predicted, Boussinesq theory is of limited use.

b. Reynold's theory

According to Reynold's theory, the turbulent shear stress between two layers of fluid at a small distance apart is given by

$\tau_t = \rho v_x v_y$

Taking average

$\overline{\tau_t} = \overline{\rho v_x v_y}$

τ =turbulent shear stress(Reynolds's shear stress)

v_x = fluctuating component of velocity in x-direction due to turbulence
v_y = fluctuating component of velocity in y-direction due to turbulence

c. Prandtl's mixing length theory

Prandtl presented a mixing length theory of turbulence, by means of which the turbulent shear stress can be expressed in terms of measurable quantities related to the average flow characteristics.

According to Prandtl, lumps of fluid particles move from one layer of some velocity to another layer of different velocity. Momentum exchange between two layers occurs due to the displacement of fluid particles. During this process, the fluid particles traveling from a layer lose its own momentum and acquire the momentum of the new layer. The distance between the two layers is the mixing length.

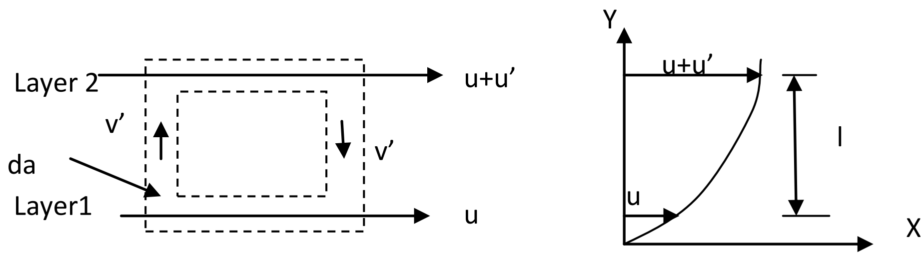


Fig. 1.6: Explanation for Prandtl's hypothesis

ρ = density of fluid
u and u+u' = velocity at a l distance apart
da = area of turbulence
v' = fluctuating velocity in y direction
Mass moving from layer 1 to layer 2 = $\rho v'da$
Total mass exchanged in two sides = $2\rho v'da$
Rate of change of momentum in X-direction (Mr) = mass u' = $2\rho v'dau'$
According to Prandtl's assumption, v' = u'
$Mr = 2\rho dau'^2$
Shear force between two layers = Mr
$\tau_t 2da = 2\rho dau'^2$
$\tau_t = \rho u'^2$
According to Prandtl, $u' = l\frac{du}{dy}$
$\tau_t = \rho l^2 \left(\frac{du}{dy}\right)^2$
This is the Prandtl's mixing length equation.

1.4.2 Velocity distribution in turbulent flow in pipes (Prandtl universal velocity distribution law)

For turbulent flow in pipes, Prandtl assumed that the mixing length, l is a linear function of distance y from the pipe wall, i.e.

$l = ky$ (a)

k = Karman universal constant = 0.4

Prandtl equation considering velocity v at a point: $\tau = \rho l^2 \left(\frac{dv}{dy}\right)^2$ (b)

From eq. (a) and (b)

$\tau = \rho (ky)^2 \left(\frac{dv}{dy}\right)^2$

$\frac{dv}{dy} = \frac{1}{ky}\sqrt{\frac{\tau}{\rho}}$

For small value of y that is very close to the boundary, the shear stress (τ) may be assumed to be constant, being approximately equal to τ_0.

$\frac{dv}{dy} = \frac{1}{ky}\sqrt{\frac{\tau_0}{\rho}}$

$\sqrt{\frac{\tau_0}{\rho}}$ has the dimensions of velocity$\left(\sqrt{\frac{ML^{-1}T^{-2}}{ML^{-3}}} = LT^{-1}\right)$, which is known as shear velocity and represented by V_*.

$\frac{dv}{dy} = \frac{V_*}{ky}$

For a given case of turbulent flow, V_* is constant. Integrating above equation

$v = \frac{V_*}{k} lny + C$

This shows that the velocity distribution in turbulent flow is logarithmic.

At y = R (radius of pipe), v= v_{max}

$C = v_{max} - \frac{V_*}{k} lnR$

Substituting C

$v = \frac{V_*}{k} lny + v_{max} - \frac{V_*}{k} lnR$

$v = v_{max} + \frac{V_*}{k}(lny - lnR)$

$v = v_{max} + \frac{V_*}{0.4} \, ln\,(y/R)$

$v = v_{max} + 2.5V_* \, ln\,(y/R)$

This equation is called Prandtl's universal velocity distribution equation. This equation is used to smooth and rough boundaries. Above equation can also be written as

$\frac{V_{max} - v}{V_*} = 2.5ln(R/y) = 5.75log_{10}(R/y)$

The difference between maximum velocity and local velocity, (v_{max}-v), is known as velocity deficit.

1.4.3 Smooth and rough boundaries

Let k is the average height of irregularities projecting from the surface of a boundary. In general, if the value of k is large for a boundary then the boundary is called rough boundary and if the value of k is less, then boundary is known as smooth boundary. However, for proper classification of smooth and rough boundaries, besides the boundary characteristics, the flow and fluid characteristics need to be considered.

For turbulent flow analysis along a boundary, the flow is divided in two portions: laminar sub layer and turbulent zone. Let δ' = thickness of laminar sub layer

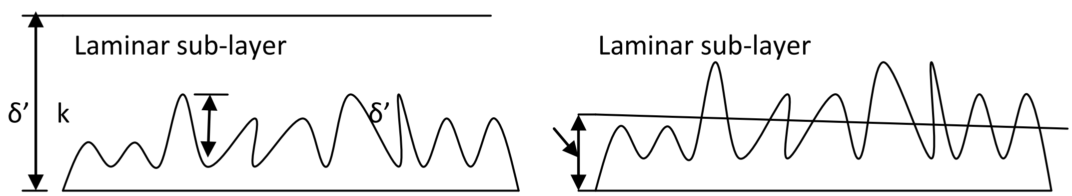
Thickness of laminar sub-layer (δ')

$$\delta' = \frac{11.6\upsilon}{V*}$$

where ν = kinematic viscosity, V_* = shear velocity

If k is much less than δ', the boundary is called hydrodynamically smooth boundary. In this case eddies cannot reach the surface irregularities. If δ' is much less than k, the boundary is called hydrodynamically rough boundary. In this case eddies come in contact with the surface irregularities and lot of energy will be lost.

Turbulent boundary layer Turbulent boundary layer



Smooth boundary Rough boundary

Fig. 1.7: Smooth and rough boundary

From Nikuradse's experiment:

$\frac{k}{\delta'}$<0.25: Smooth boundary

$\frac{k}{\delta'}$>6: rough boundary

0.25<$\frac{k}{\delta'}$<6: transition

In terms of roughness, Reynold number (Rn) = $\frac{V_* K}{\upsilon}$ where V_* = shear velocity, k = roughness height

Rn<4: smooth

Rn>70: rough

4<Rn<70: transition

1.5.5 Karman-Prandtl velocity distribution equations

Prandtl equation considering velocity v at a point: $\tau = \rho l^2 \left(\frac{dv}{dy}\right)^2$

$l = ky$ Where k = Karman universal constant = 0.4

Substituting l in Prandtl equation

$\tau = \rho(0.4y)^2 \left(\frac{dv}{dy}\right)^2$

$\frac{dv}{dy} = \frac{2.5}{y}\sqrt{\frac{\tau}{\rho}}$

For small value of y very close to the boundary, τ = τ_0 (shear stress at boundary) = const.

$\frac{dv}{dy} = \frac{2.5}{y}\sqrt{\frac{\tau_0}{\rho}}$

$\sqrt{\frac{\tau_0}{\rho}}$ = shear velocity = V_*

$\frac{dv}{dy} = \frac{2.5V_*}{y}$

Integrating above equation

$v = 2.5V_* lny + C$

For y = 0, v = 0. But in this case v = $-\infty$. So assume that at finite distance y = y', u = 0.

$C = -2.5V_* lny'$

$v = 2.5V_* ln\left(\frac{y}{y'}\right) = 5.75V_* log_{10}\left(\frac{y}{y'}\right)$ (I)

a. velocity distribution for smooth pipe in turbulent flow

From Nikuradse's experiment for smooth boundary

$y' = \frac{\delta'}{107}$ whereδ' = thickness of laminar sublayer.

$\delta' = \frac{11.6v}{V*}$

$y' = \frac{\delta'}{107} = \frac{11.6v}{107V*} == \frac{0.108v}{V*}$

Substituting y' in eq. (I) and simplifying

$v = 5.75V_* log_{10}\left(\frac{y}{\frac{0.108v}{V*}}\right)$

$\frac{v}{V_*} = 5.75log_{10}\left(\frac{V_* y}{v}\right) + 5.5$

b. velocity distribution for rough pipes in turbulent flow

From Nikuradse's experiment for rough boundary

$y' = \frac{k}{30}$ where K = roughness height

Substituting y' in eq. (I) and simplifying

$\frac{v}{V_*} = 5.75log_{10}(y/k) + 8.5$

Velocity distribution in terms of mean velocity

Average velocity for smooth pipes

$$\frac{v}{V_*} = 5.75log_{10}\left(\frac{V_* y}{\upsilon}\right) + 5.5$$

Mean velocity ($\bar{V}$)=$\frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2}$

After integration (see appendix for integration)

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{V_* R}{\upsilon}\right) + 1.75$$

Average velocity for rough pipes

$$\frac{v}{V_*} = 5.75log_{10}(y/k) + 8.5$$

$$\bar{V} = \frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2}$$

After integration (see appendix for integration)

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{R}{K}\right) + 4.75$$

Relationship between mean velocity, shear velocity and local velocity

For smooth pipes

$$\frac{v}{V_*} = 5.75log_{10}\left(\frac{V_* y}{\upsilon}\right) + 5.5 \qquad \text{(a1)}$$

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{V_* R}{\upsilon}\right) + 1.75 \qquad \text{(a2)}$$

For rough pipes

$$\frac{v}{V_*} = 5.75log_{10}(y/k) + 8.5 \qquad \text{(b1)}$$

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{R}{K}\right) + 4.75 \qquad \text{(b2)}$$

Subtracting a2 from a1 or b2 from b1

$$\frac{v-\bar{V}}{V*} = 5.75log_{10}\left(\frac{y}{R}\right) + 3.75$$

With reference to the mean velocity of flow, Karman-Prandtl expressions for the velocity distribution in both rough and smooth pipe become identical.

The point where the point velocity = mean velocity

For $v = \bar{V}$, above equation reduces to

$$0 = 5.75log_{10}\left(\frac{y}{R}\right) + 3.75$$

y= 0.223R

Relationship between maximum velocity and average velocity

At y=R, v= v_{max}

$\frac{Vmax-\bar{V}}{V*} = 3.75$

Alternative form of velocity distribution for turbulent flow

Velocity distribution in turbulent flow can also be expressed in exponential form such as

$\frac{v}{V_{max}} = \left(\frac{y}{R}\right)^{1/n} = \left(1-\frac{r}{R}\right)^{1/n}$

This is valid for both rough and smooth pipe.

Mean velocity (V)=$\frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2}$ (Considering an elementary strip of thickness dr at a radial distance r from pipe center.)

$V = \frac{\int_0^R V_{max}\left(1-\frac{r}{R}\right)^{1/n}(2\pi r)dr}{\pi R^2}$

$= \frac{2V_{max}}{R^2}\int_0^R \left(1-\frac{r}{R}\right)^{1/n} rdr$

Take $\left(1-\frac{r}{R}\right) = p$

-dr = Rdp

r = R-Rp

$V = -\frac{2V_{max}}{R^2}\int_1^0 p^{\frac{1}{n}}(R-Rp)Rdp$

$= -2V_{max}\int_1^0 \left(p^{\frac{1}{n}} - p^{\frac{1+n}{n}}\right)dp$

$= -2V_{max}\left[\frac{p^{\frac{1+n}{n}}}{\frac{1+n}{n}} - \frac{p^{\frac{1+2n}{n}}}{\frac{1+2n}{n}}\right]_1^0$

$V = \frac{2n^2}{(n+1)(2n+1)}V_{max}$

Exponent n varies from 5-10. n can be estimated from Re.

Re	n
$4x10^3$	6
10^5	7
10^6	9
$>2x10^6$	10

Empirical relationship between friction factor (f) and n

$n = \frac{1}{\sqrt{f}}$

1.5 Frictional loss in pipe flow

When a liquid is flowing through a pipe, the velocity of the liquid layer adjacent to the pipe wall is zero. The velocity of the liquid goes on increasing from the wall and thus velocity gradient and hence shear stresses are produced in the whole liquid due to viscosity. The viscous action causes loss of energy which is usually known as frictional loss. The loss of head due to friction is called major loss.

Laws of fluid friction for turbulent flow (given by Froude)

The frictional resistance in the turbulent flow is

i) proportional to the $(\text{velocity})^n$ of flow (n = 1.8 to 2)
ii) Independent of the pressure
iii) Proportional to the density of the fluid
iv) proportional to the surface area of contact
v) Dependent on the nature of the contact surface

1.5.1 Darcy-Weisbach equation: Head loss in pipe due to friction

Consider a pipe of cross-sectional area A carrying a fluid with a mean velocity V. Let P_1 and $V_1$are the pressure and velocity at section 1 and P_2 and V_2 are pressure and velocity at section 2.

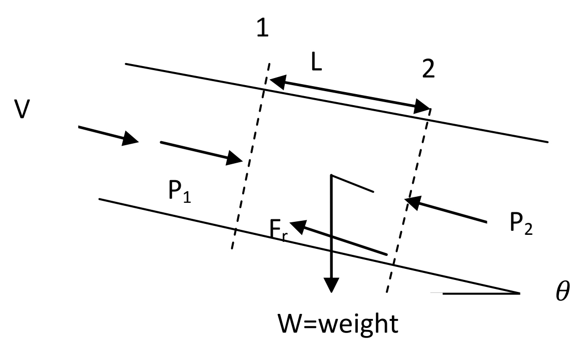


Fig. 1.8: Head loss due to friction

Applying Bernoulli's equation between section 1 and 2

$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + h_f$

V_1=V_2=V (same dia.)

h_f = loss of head due to friction

$\left(\frac{P_1}{\gamma} - \frac{P_2}{\gamma}\right) + (Z_1 - Z_2) = h_f$ (a)

Hence pressure intensity will be reduced in the direction of flow by frictional resistance.

Forces acting on fluid between section 1 and 2 area: pressure force at 1 and 2 and frictional force (F_r) and weight of fluid

Resolving all forces in horizontal direction

$P_1 A - P_2 A + W\sin\theta - F_r = 0$

$(P_1 - P_2)A + \gamma AL\frac{Z_1-Z_2}{L} = F_r$

Dividing by γA

$\left(\frac{P_1-P_2}{\gamma}\right) + (Z_1 - Z_2) = \frac{F_r}{\gamma A}$ (b)

According to Froude,

Frictional resistance (F_r) $\propto Area_{surface}V^n = f'PLV^n$ (c)

P = wetted perimeter of pipe, L = Length of pipe between section 1 and 2, and f' = coefficient

From eq. (b) and eq. (c)

$\left(\frac{P_1-P_2}{\gamma}\right) + (Z_1 - Z_2) = \frac{f'PLV^n}{\gamma A}$ (d)

From eq. (a) and eq. (d)

$h_f = \frac{f'PLV^n}{\gamma A}$

Here, $\frac{P}{A} = \frac{\pi D}{\pi D^2/4} = \frac{4}{D}$

$h_f = \frac{f'}{\gamma}\frac{4}{D}LV^n$

Taking n = 2 for turbulent flow and putting $\frac{4f'}{\gamma} = \frac{f}{2g}$

(In the equation of drag force $F = \frac{1}{2}C\rho AV^2$, $f' = \frac{1}{2}C\rho$. So $\frac{4f'}{\gamma} = 4\frac{1}{2}C\rho\frac{1}{\rho g} = \frac{4C}{2g} = \frac{f}{2g}$.)

Hence,

$h_f = \frac{fLV^2}{2gD}$

where f = Darcy's coefficient of friction or friction factor

This equation is known as Darcy-Weisbach equation, which is commonly used for computing loss of head due to friction in pipes.

If 4f is used as a friction factor, then f = Fanning's friction factor

f= func(V, D, ρ, μ, wall roughness)

in terms of discharge,

$h_f = \frac{fLV^2}{2gD} = \frac{fL}{2gD}\frac{Q^2}{A^2} = \frac{fL}{2gD}\frac{Q^2}{\left(\frac{\pi}{4}D^2\right)^2} = \frac{8fL}{\pi^2 gD^5}Q^2 = rQ^2$

Where r = $\frac{8fL}{\pi^2 gD^5}$ or $\frac{fL}{12.1D^5}$

Coefficient r is called resistance coefficient.

1.5.2 Shear stress and shear velocity for turbulent flow in terms of f

Expression of f in terms of shear stress (considering horizontal pipe)

$(P_1\text{-}P_2)\ A = F_r$

$(P_1 - P_2)\frac{\pi}{4}D^2 = \tau_0 \pi DL$

where τ_0 = shear stress at the pipe wall

$P_1 - P_2 = \frac{4\tau_0 L}{D}$ (a)

$h_f = \frac{P_1 - P_2}{\gamma} = \frac{fLV^2}{2gD}$ (b)

From a and b

$\frac{4\tau_0 L}{D} = \gamma \frac{fLV^2}{2gD}$

$\tau_0 = \frac{f\rho V^2}{8}$

Shear velocity (V_*)

$V_* = \sqrt{\frac{\tau_0}{\rho}} = V\sqrt{f/8}$

Shear stress(τ) at a certain distance r is $\tau/\tau_0 = r/R$ where τ_0 = shear stress at the boundary

In terms of V_*, shear stress at the boundary is

$\tau_0 = \rho V_*^2$

1.6 Nikuradse's experiments

Nikuradse, a German engineer, conducted a series of well- planned experiments on pipes roughened artificially by gluing uniform sand grains as closely spaced as possible on the inner side of the pipe wall. By choosing pipe of varying diameters (D) and by changing the size of sand grain (roughness height = k), he was able to vary the relative roughness 'k/D' from about 1/1014 to 1/30.

From dimensional analysis, friction factor f is given by

$f = \varphi\left(\frac{VD}{\upsilon}, \frac{k}{D}\right) = \varphi\left(Re, \frac{k}{D}\right)$

k = average height of pipe wall roughness protrusions (alternative symbol: e or ϵ)

D = Diameter of pipe

υ = Kinematic viscosity of fluid

$Re = \frac{VD}{\upsilon}$ = Reynolds number

k/D = Relative roughness

Sometimes k/D is also replaced by R/k, where R = Radius of pipe

Nikuradse's test result was plotted on logarithmic scale with friction factor as ordinate and Reynolds number as abscissa. This plot of f verse Re for on log-log scale is called a "Stanton diagram".

Salient features of Nikuradse's diagram

The experimental curve of Nikuradse showed five regions:

a. Laminar flow: Re<2000, $f = \varphi(Re)$

$f = \frac{64}{Re}$

The f versus Re curve is linear and the head loss is independent of roughness.

b. Laminar to turbulent transition: 2000<Re<4000, there is no specific relationship between f and Re

c. Turbulent flow: Re>4000, a curve for f versus Re exists for every K/D, and the horizontal lines confirm that roughness is more important than Re for determining f. friction factor is function of either Re or k/D or both, depending on whether the boundary is smooth or rough or in transition.

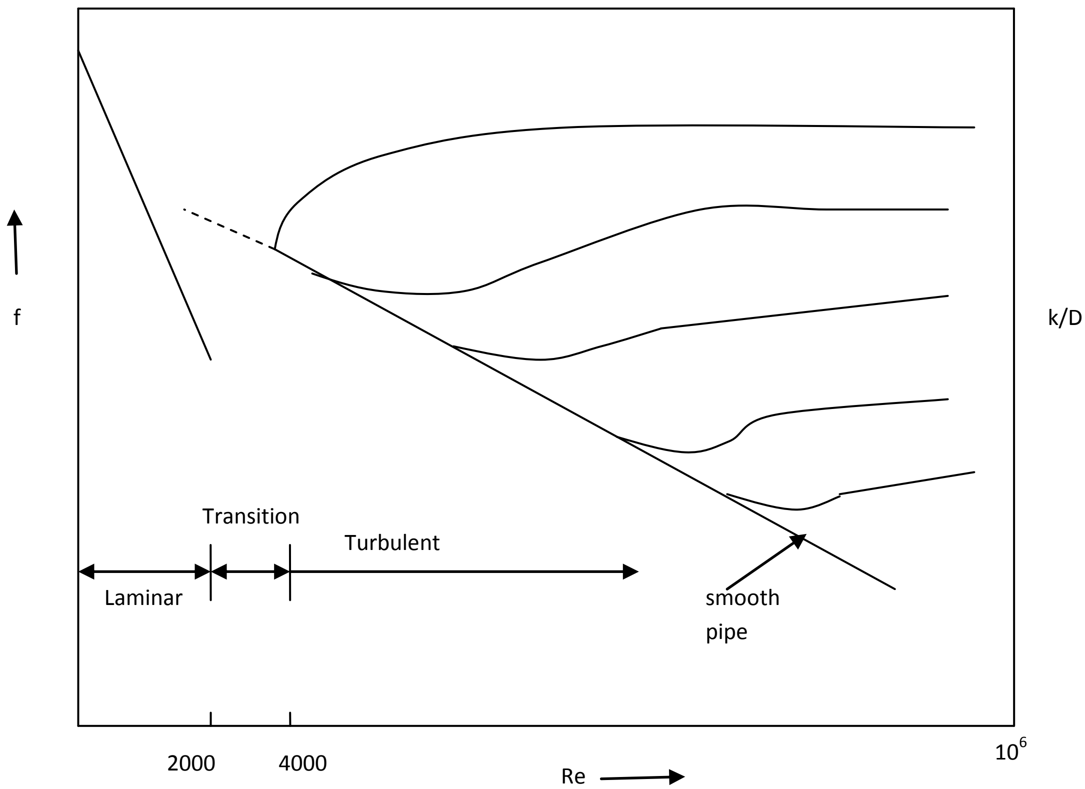


Fig. 1.9: Nikuradse's diagram

I. Turbulent flow in smooth pipes: $f = \varphi(Re)$

The pipe will act as smooth if $\frac{Re\sqrt{f}}{R/k} < 17$

For Re from 4000 to 10^5, f is given by Blasius equation.

$f = \frac{0.316}{Re^{1/4}}$

For $Re>10^5$, f is given by

$\frac{1}{\sqrt{f}} = 2.03log_{10}\left(Re\sqrt{f}\right) - 0.91$

From Nikuradse's experimental data, f for $5x10^4$ to $4x10^7$ is

$\frac{1}{\sqrt{f}} = 2.0log_{10}\left(Re\sqrt{f}\right) - 0.8$

Alternative expression of f for smooth pipes

$f = 0.0032 + \frac{0.221}{Re^{0.237}}$

II. Turbulent flow in rough pipes: $f = \varphi(R/k)$

The pipe will act as rough if $\frac{Re\sqrt{f}}{R/k} \geq 400$

For a given value of relative roughness, f is constant, shown by horizontal line.

$\frac{1}{\sqrt{f}} = 2.0log_{10}(R/k) + 1.74$

III. Intermediate roughness: $f = \varphi\left(Re, \frac{k}{D}\right)$

1.7 Colebrook-White equation

Nikuradse's experimental curves cannot be used to evaluate the friction factor for commercial pipes because the actual roughness pattern of commercial pipes is very much different from the uniform sand grain roughness used by Nikuradse. The concept of equivalent sand grain roughness (k) has been used in determining the resistance of commercial pipes. For any given K, it is possible to find a uniform sand grain diameter such that, when the pipe is coated with these sand grains, it will give the same limiting value of f as that for commercial pipe in natural form.

Based on number of experiments, Colebrook and White developed an empirical equation for both smooth and rough wall turbulence, which represents the variation of f for commercial pipes. The equation is given by

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{k/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right)$

K/D: relative roughness, Re = Reynold no.

For smooth pipe (k/D negligible). So, this equation reduces to

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{2.51}{Re\sqrt{f}}\right) = 2.0log_{10}\left(Re\sqrt{f}\right) - 0.8$

For rough pipe ($\frac{2.51}{Re\sqrt{f}}$ negligible as f is independent of f). So, this equation reduces to

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{k}{3.7D}\right) = -2.0log_{10}\left(\frac{K}{D}\right) + 1.14$

Alternative form of Colebrook-white equation:

$\frac{1}{\sqrt{f}} - 2.0log_{10}\left(\frac{R}{K}\right) = 1.74 - 2.0log_{10}\left(1 + 18.7\frac{R/K}{Re\sqrt{f}}\right)$

How to find f from Colebrook-White equation

Solve Colebrook-White equation by trial and error: Assume a value of f. Compute LHS and RHS of the equation, and check the values. Take different value of f and repeat the trials until LHS is almost equal to RHS.

1.8 Moody diagram (Moody chart)

Moody diagram is a graphical solution of Colebrook-White equation. The diagram is a plot of friction factor (f) versus Reynold number (Re) for various values of relative roughness (K/D). It is basically the Nikuradse diagram except for transition region.

Salient features of Moody's Diagram

There are four flow regimes (zones) in Moody Diagram.

I. Laminar flow: Re<2000, f is a function of Re only. (f=64/Re)

Head loss is given by Darcy-Weisbach equation

$h_f = \frac{fLV^2}{2gD}$

For laminar flow, head loss is given by Hagen-Poiseuille equation

$h_f = \frac{32\mu VL}{\rho g D^2}$

Equating both equation

$\frac{fLV^2}{2gD} = \frac{32\mu VL}{\rho g D^2}$

$f = \frac{64}{\frac{\rho DV}{\mu}}$

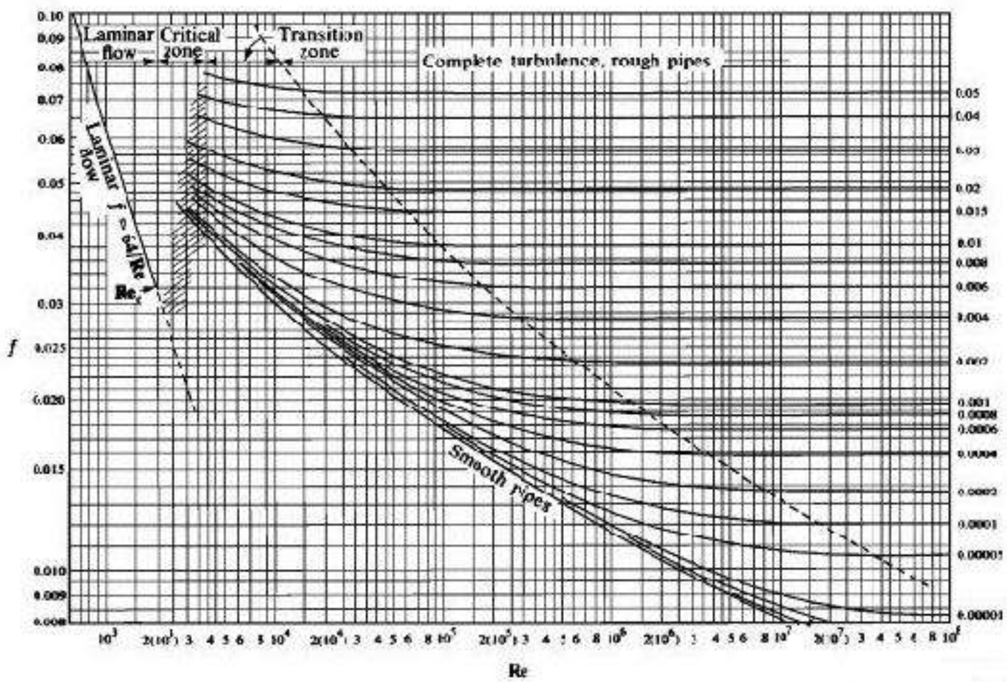
$f = \frac{64}{Re}$

This equation is an equation of straight line, which can be used for the solution of laminar flow problems.

II. Critical zone: In the critical zone, Re varies between 2000 and 4000. The flow in this zone is either laminar or turbulent (oscillatory flow that alternately exists between laminar and turbulent flow). There is no any specific relationship between f and Re in this zone.

III. Transition zone: In this zone, Re is higher than 4000. For certain ranges of Re in this zone, the roughness elements are submerged in the viscous wall layer, and the pipe has the friction factor same as that of a smooth pipe for K/D≤0.001 and for smaller Re. In this zone, f is a function of both Re and K/D.

IV. Complete turbulent zone: In this zone, average roughness height is substantially greater than the viscous wall layer thickness. As Re is high, the head loss is also higher due to the extra turbulence caused by projections. The friction factor, f is a function of K/D and is independent of Re. The horizontal lines indicate that friction factor does not change with Re. That means viscosity does not affect the head loss in this zone.



k/D

Fig. 1.10: Moody diagram

Use of Moody's diagram
i) To determine the value of f at various Re and K/D
ii) To differentiate between various stages of flow condition.

How to use Moody Chart

Mark the value of Re on X-axis. For given value of K/D, follow the K/D curve and mark the point of intersection of Re with the K/D curve. Draw a horizontal line from the point of intersection to the f-axis (primary Y-axis) and read the value of f on the f-axis. (Note: If K/D curve is not available for given value of K/D, draw a curve by following the trend of the nearby curve. For very higher values of Re, f value corresponding to given K/D can be taken without following the curve.)

<u>Roughness, K value in mm</u>

Riveted steel:	0.9-9
Concrete:	0.3-3
Cast iron:	0.25
Galvanized iron:	0.15
Commercial steel:	0.046

Determination of friction factor

a. Using Moody Chart

b. Solving Colebrook-White equation (by trial and error or writing code)

d. Using the relationship (equation) of f for different conditions (Blasius, Nikuradse, thereotical relation)

The various equations for computing f are also known as Resistance equations.

e. Swamee and Jain formula for computing f

$$f = \frac{1.325}{\left[ln\left(\frac{K}{3.7D}+\frac{5.74}{Re^{0.9}}\right)\right]^2}$$

This equation is valid for $10^{-6} \le K/D \le 10^{-2}$ and $5000 \le Re \le 10^8$.

Aging of pipe

f increase due to increase in roughness.

$$k_t = k_0 + \alpha t$$

Where k_t = roughness height after t years, K_0 = initial roughness height, t = time in years, α = coeffcient

1.9 Minor losses in pipe lines

Minor loss: Losses which occur in pipelines due to the change in velocity are called minor losses. Causes of minor losses

- Loss of head due to sudden enlargement
- Loss of head due to sudden contraction
- Loss of head at the entrance to a pipe
- Loss of head at the exit from pipe
- Loss of head due to an obstruction in a pipe
- Loss of head due to bend in the pipe
- Loss of head in various pipe fitting

For short pipes: minor loss is significant, while for long pipes, it is negligible compared to friction loss. Minor losses are negligible when they comprise only 5% or less of the head loss due to pipe friction.

a. Loss of head due to sudden enlargement

Consider a liquid flowing through a pipe, which has sudden enlargement. Consider two sections 1 and 2 before and after enlargement Let P_1, V_1 and A_1 be the pressure intensity, velocity and cross sectional area respectively at section 1 and P_2, V_2 and A_2 be the corresponding value at section 2. Due to sudden change of diameter, turbulent eddies are formed. The loss of head take place due to the formation of these eddies.

Let P_x = pressure intensity of the liquid eddies on area (A_2-A_1)

h_e = loss of head due to sudden enlargement.

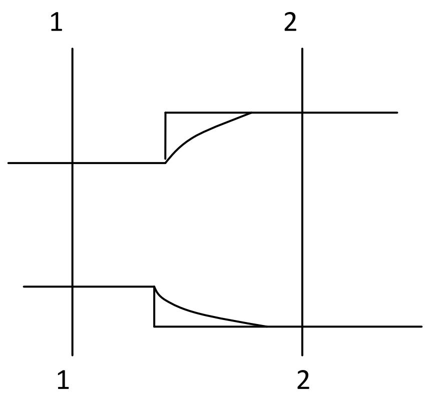


Fig. 1.11: Head loss due to sudden enlargement

Applying Bernoulli's equation between section 1 and 2

$$\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+Z_1=\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+Z_2+h_e$$

Z_1=Z_2 (horizontal)

$$h_e=\left(\frac{P_1}{\gamma}-\frac{P_2}{\gamma}\right)+\left(\frac{V_1^2}{2g}-\frac{V_2^2}{2g}\right) \quad \text{(a)}$$

According to momentum principle,

$$\sum F_x=\rho Q\,(V_2-V_1)$$

$$P_1A_1+P_x(A_2-A_1)-P_2A_2=\rho Q(V_2-V_1)$$

But experimentally it is found that P_x = P_1

$$P_1A_1+P_1(A_2-A_1)-P_2A_2=\rho Q(V_2-V_1)$$

$$(P_1-P_2)A_2=\rho Q(V_2-V_1)$$

$$(P_1-P_2)A_2=\rho A_2V_2(V_2-V_1)$$

$$\frac{(P_1-P_2)}{\rho g}=\frac{1}{g}(V_2^2-V_1V_2)$$

$$\frac{(P_1-P_2)}{\gamma}=\frac{1}{g}(V_2^2-V_1V_2) \qquad \text{(b)}$$

From eq. a and b

$$h_e=\frac{1}{g}(V_2^2-V_1V_2)+\left(\frac{V_1^2}{2g}-\frac{V_2^2}{2g}\right)$$

$$h_e=\frac{(V_1-V_2)^2}{2g}$$

$$h_e=\frac{V_1^2}{2g}\left(1-\frac{V_2}{V_1}\right)^2=\frac{V_1^2}{2g}\left(1-\frac{Q/A_2}{Q/A_1}\right)^2=\frac{V_1^2}{2g}\left(1-\frac{A_1}{A_2}\right)^2=K\,\frac{V_1^2}{2g}$$

This shows that minor loss varies as the square of velocity.

b. Loss of head due to sudden contraction

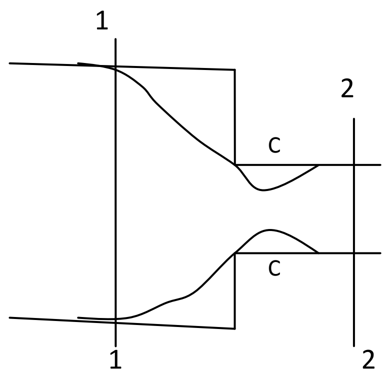


Fig. 1.12: Head loss due to sudden contraction

Consider a liquid flowing in a pipe which has a sudden contraction in area. Consider two sections 1 and 2 before and after contraction. As the liquid flows from larger pipe to smaller pipe, the area of flow goes on decreasing and becomes minimum at a section C-C, called vena-contracta. After section C-C a sudden enlargement of the area takes place.

The loss of head due to sudden contraction is actually due to sudden enlargement from vena-contracta to smaller pipe.

Let A_c and V_c be the area and velocity of flow at section C-C. Let A_2 and V_2 be the area and velocity of flow at section 2.

h_c= loss of head due to sudden contraction

h_c is actually loss of head due to sudden enlargement from section C-C to 2-2, which is given by

$$h_c = \frac{(V_C - V_2)^2}{2g} = \frac{V_2^2}{2g}\left[\frac{V_C}{V_2} - 1\right]^2 \qquad \text{(a)}$$

From continuity equation

$A_c\, V_c = A_2\, V_2$

$$\frac{V_C}{V_2} = \frac{A_2}{A_C} = \frac{1}{A_C/A_2} = \frac{1}{C_c} \qquad \text{(b)}$$

From eq. a and b

$$h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2 = \frac{KV_2^2}{2g}$$

where $K = \left[\frac{1}{C_c} - 1\right]^2$

If value of C_c is not given, K = 0.5 is taken.

$$h_c = 0.5\frac{V_2^2}{2g}$$

Value of C_C for water given

A_2/A_1	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
C_C	0.624	0.632	0.643	0.659	0.681	0.712	0.755	0.813	0.892	1.00

c. Loss of head at the entrance of a pipe

When a liquid enters a pipe from a large vessel (or tank), some loss of energy occurs at the entrance to the pipe which is known as inlet loss of energy. The loss is similar to that in the case of a sudden contraction ($h_i = \frac{KV^2}{2g}$ where v= velocity of liquid in pipe). The loss of energy actually depends on the form of the entrance.

Type of entrance	K
Sharp cornered	0.5 (common case)
Rounded	0.2
Bell mouthed	0.05

d. Loss of head at the exit of pipe

This is the loss of head due to the velocity of liquid at outlet of pipe which is dissipated either in the form of a free jet (if outlet of the pipe is free) or it is lost in the tank or reservoir (if outlet of the pipe is connected to the tank or reservoir).

Exit loss is

$h_0 = \frac{V^2}{2g}$ where V = Velocity at outlet of pipe

e. Loss of head due to bend in pipe

When there is any bend in a pipe, the velocity of flow changes, due to the separation of flow from the boundary and also formation of eddies take place. Thus the energy is lost.

$h_b = \frac{kV^2}{2g}$

h_b = loss of head due to bend

V = Velocity of flow

K = Coefficient of bend

Sharp 90^0 bend, K = 1.2

Sharp 180^0 bend, K = 2.2

f. Loss of head in various pipe fittings

$h_p = \frac{kV^2}{2g}$

h_p = loss of head in various pipe fittings such as valves, couplings

V = Velocity of flow

K = Coefficient of pipe-fitting

Fittings	K
Standard T (branch flow)	1.8
Standard T (line flow)	0.4
90^0 elbow (long radius)	0.6
90^0 elbow (short radius)	1.5
Gate valve (fully open)	0.2

g. Loss of head due to an obstruction in a pipe

Whenever there is an obstruction in a pipe, the loss of energy takes place due to the reduction of the area of the cross-section of the pipe at the place where obstruction is present. There is sudden enlargement of the area of flow beyond the obstruction due to which loss of head takes place.

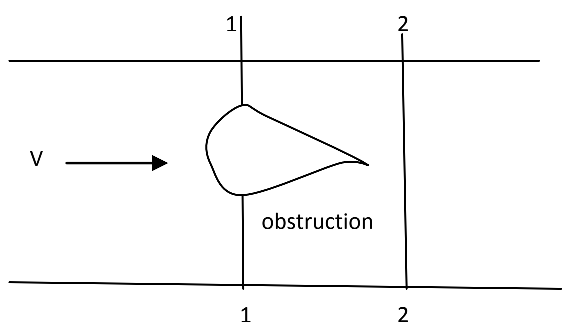


Fig. 1.13: Loss of head due to obstruction

h_o = Head loss due to obstruction , V = Velocity of liquid in pipe, A = C/s area of pipe, a = Maximum area of obstruction, V_c = velocity of liquid at vena-contracta, C_c = Coefficient of contraction

ho is the loss of head due to sudden enlargement from vena-contracta to section 2-2.

$h_o = \frac{(V_c - V)^2}{2g} = \frac{V^2}{2g}\left[\frac{V_c}{V} - 1\right]^2$ (a)

$C_c = \frac{A_c}{A-a}$

Applying continuity equation at 1-1 and 2-2,

$A_c V_c = AV$

$C_c(A-a)V_c = AV$

$\frac{V_c}{V} = \frac{A}{C_c(A-a)}$ (b)

From a and b

$$ho = \frac{V^2}{2g}\left[\frac{A}{C_c(A-a)} - 1\right]^2$$

Equivalent length of pipe representing minor loss (L_e)

$$\frac{kV^2}{2g} = \frac{fL_eV^2}{2gD}$$

$$L_e = \frac{KD}{f}$$

For more than one type of losses, K represents the sum of several losses.

This L_e can be added to L for computing head loss using Darcy- Weisbach equation.

$$h_f = \frac{f(L+L_e)V^2}{2gD}$$

1.10 Hydraulic gradient line (HGL) and Energy gradient line (EGL)

Hydraulic gradient line (HGL) is the graphical representation of sum of pressure head and datum head $\left(\frac{P}{\gamma} + Z\right)$ of a flowing fluid in a pipe with respect to some reference line.Energy gradient line (EGL) or total energy line (TEL) is the graphical representation of sum of pressure head, velocity head and datum head $\left(\frac{P}{\gamma} + \frac{V^2}{2g} + Z\right)$ of a flowing fluid in a pipe with respect to some reference line.

Features of HGL

- HGL shows the change in piezometric head along the pipeline.
- HGL may rise or fall depending on the pressure change. For pipe flow the HGL lies a distance p/ρg above the pipe centreline. Thus, the difference between pipe elevation and hydraulic grade line gives the static pressure p.
- If the HGL drops below pipe elevation this means negative gauge pressures (i.e. less than atmospheric).
- A HGL more than patm/ρg (about 10 m of water) below the pipeline is impossible.
- HGL is always below the EGL and the vertical intercept between EGL and HGL is equal to $v^2/2g$.
- For a pipe of uniform cross section the slope of the HGL is equal to the slope of EGL.
- The HGL is the height to which the liquid would rise in a piezometer tube.
- For open-channel flows, pressure is atmospheric (i.e. p = 0) at the surface; the HGL is then the height of the free surface.

Features of EGL

- EGL shows the change in total head along the pipeline.
- EGL always drops in the direction of flow due to friction.
- EGL starts at the water level in the supply reservoir.
  - small discontinuities correspond to entry loss, exit loss or other minor losses
  - steady downward slope reflects pipe friction (slope change if pipe radius changes)

- large discontinuities correspond to turbines (loss of head) or pumps (gain of head)

- The EGL represents the maximum height to which water may be delivered.

Plotting EGL and HGL

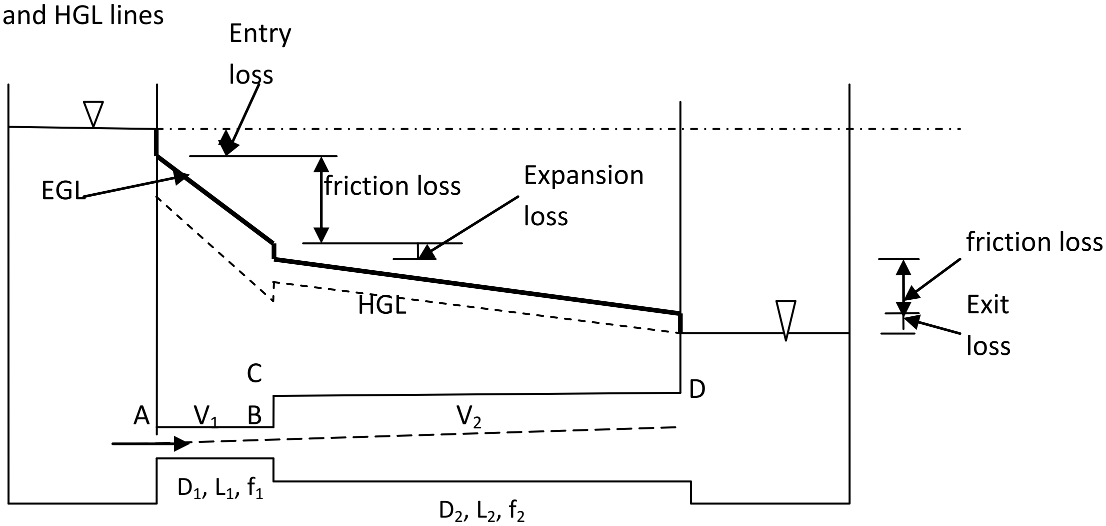
- Compute different losses.
- Find the position of EGL at starting point.
- Elevation of EGL at next point = Elevation of EGL at previous point-loss in between 2 points
- Elevation of HGL =Elevation of EGL-velocity head
  - At minor loss point, sharp fall in EGL
  - At pump (abrupt rise in EGL)

  Elevation of EGL at next point = Elevation of EGL at previous point-loss in between 2 points+head supplied by pump
  - At turbine (abrupt fall in EGL)

  Elevation of EGL at next point = Elevation of EGL at previous point-loss in between 2 points-head extracted by turbine

Example plot

EGL and HGL lines



Chapter 1: Appendix

Velocity distribution in terms of mean velocity

Average velocity for smooth pipes

$$\frac{v}{V_*} = 5.75log_{10}\left(\frac{V_* y}{v}\right) + 5.5$$

Mean velocity ($\bar{V}$)=$\frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2}$

$$\bar{V} = \frac{2V*}{R^2}\int_0^R \left[5.75log_{10}\left(\frac{V_* y}{v}\right) + 5.5\right] rdr$$

$$\frac{\bar{V}}{V*} = \frac{2}{R^2}\int_0^R \left[2.5ln\left(\frac{V_*(R-r)}{v}\right) rdr + 5.5rdr\right]$$

$$= \frac{2}{R^2}\int_0^R \left[2.5\left(ln\left(\frac{V_*}{v}\right) rdr + \ln(R-r)rdr\right) + 5.5rdr\right]$$

$= \frac{2}{R^2}\left[\frac{2.5R^2}{2} ln\left(\frac{V_*}{v}\right) + 2.5\int_0^R \ln(R-r)\, rdr + 2.75R^2\right]$ (a)

Let $I = \int_0^R \ln(R-r)\, rdr$

suppose u = ln(R-r), dv = rdr

$$du = \frac{-dr}{R-r}\ , v = \frac{r^2}{2}$$

$$I = [uv]_0^R - \int_0^R \text{vdu}$$

$$= \left[\ln(R-r)\frac{r^2}{2}\right]_0^R + \frac{1}{2}\int_0^R \frac{r^2 dr}{R-r}$$

$= \frac{R^2}{2}\ln(0) + \frac{1}{2}\int_0^R \frac{r^2 dr}{R-r}$ (b)

Let $I_1 = \int_0^R \frac{r^2 dr}{R-r}$

Suppose R-r = t, r = R-t

-dr = dt

$$I_1 = -\int_R^0 \frac{(R-t)^2 dt}{t}$$

$$= -\int_R^0 \left(\frac{R^2}{t} - 2R + t\right) dt$$

$$= -\left[R^2 lnt - 2Rt + \frac{t^2}{2}\right]_R^0$$

$=-\left(R^2\ln(0) - R^2\ln R + 2R^2 - \frac{R^2}{2}\right)$

$$I_1 = -R^2 \ln(0) + R^2 \ln R - 1.5R^2$$

Substituting the value of I_1 in b

$$I = \frac{R^2}{2}\ln(0) - \frac{R^2}{2}\ln(0) + \frac{R^2}{2}\ln R - 0.75R^2$$

$$= \frac{R^2}{2}\ln R - 0.75R^2$$

Substituting the value of I in a

$$\frac{\bar{V}}{V*} = \frac{2}{R^2}\left[\frac{2.5R^2}{2} ln\left(\frac{V_*}{v}\right) + 2.5\left(\frac{R^2}{2}\ln R - 0.75R^2\right) + 2.75R^2\right]$$

$$\frac{\bar{V}}{V_*} = 2.5ln\left(\frac{V_*R}{v}\right) + 1.75$$

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{V_*R}{v}\right) + 1.75$$

Average velocity for rough pipes

$$\frac{v}{V_*} = 5.75log_{10}(y/k) + 8.5$$

$$\bar{V} = \frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2} = \frac{2V*}{R^2}\int_0^R [5.75log_{10}(y/k) + 8.5]rdr$$

$$\bar{V} = \frac{2V*}{R^2}\int_0^R \left[2.5ln\left(\frac{R-r}{K}\right) + 8.5\right]rdr$$

$$\frac{\bar{V}}{V*} = \frac{2}{R^2}\int_0^R \left[2.5\left(ln\left(\frac{1}{K}\right)rdr + \ln(R-r)rdr\right) + 8.5rdr\right]$$

$$= \frac{2}{R^2}\left[\frac{2.5R^2}{2} ln\left(\frac{1}{K}\right) + 2.5\int_0^R \ln(R-r)\, rdr + 4.25R^2\right]$$

As computed above

$$I = \int_0^R \ln(R-r)\, rdr = \frac{R^2}{2}\ln R - 0.75R^2$$

Substituting I

$$= \frac{2}{R^2}\left[\frac{2.5R^2}{2} ln\left(\frac{1}{K}\right) + 2.5\left(\frac{R^2}{2}\ln R - 0.75R^2\right) + 4.25R^2\right]$$

$$\frac{\bar{V}}{V_*} = 2.5ln\left(\frac{R}{K}\right) + 1.75$$

$$\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{R}{K}\right) + 4.75$$

Starting from Prandtl's equation of velocity distribution for turbulent flow in pipes given by $v = v_{max} + 2.5V_* \, ln\,(y/R\,)$, show that $\frac{v_{max}}{V} = 1.33\sqrt{f} + 1$

where v = velocity at any point, v_{max} = maximum velocity, V = average velocity, y = distance from pipe wall, R = radius of pipe, and f = friction factor.

Solution:

$$v = v_{max} + 2.5V_* \, ln\,(y/R\,)$$

Average velocity V is given by

$$V = \frac{Q}{A} = \frac{\int vdA}{A} = \frac{\int_0^R v(2\pi r)dr}{\pi R^2} = \frac{2}{R^2}\int_0^R [v_{max} + 2.5V_* \, ln\,(y/R\,)]rdr$$

$$= \frac{2}{R^2}\left[\frac{R^2}{2}v_{max} + 2.5V_* \int_0^R \ln\frac{(R-r)}{R} rdr\right]$$

$$= \frac{2}{R^2}\left[\frac{R^2}{2}v_{max} + 2.5V_* \int_0^R \ln\left(\frac{1}{R}\right) rdr + 2.5V_* \int_0^R \ln(R-r)\, rdr\right]$$

$= \frac{2}{R^2}\left[\frac{R^2}{2}v_{max} + 2.5V_* \frac{R^2}{2}\ln\left(\frac{1}{R}\right) + 2.5V_* \int_0^R \ln(R-r)\, rdr\right]$ (a)

Let $I = \int_0^R \ln(R-r)\, rdr$

suppose u = ln(R-r), dv = rdr

$$du = \frac{-dr}{R-r}\,, v = \frac{r^2}{2}$$

$$I = [uv]_0^R - \int_0^R \text{vdu}$$

$$= \left[\ln(R-r)\frac{r^2}{2}\right]_0^R + \frac{1}{2}\int_0^R \frac{r^2dr}{R-r}$$

$= \frac{R^2}{2}\ln(0) + \frac{1}{2}\int_0^R \frac{r^2dr}{R-r}$ (b)

Let $I_1 = \int_0^R \frac{r^2dr}{R-r}$

Suppose R-r = t, r = R-t

-dr = dt

$$I_1 = -\int_R^0 \frac{(R-t)^2dt}{t}$$

$$= -\int_R^0 \left(\frac{R^2}{t} - 2R + t\right)dt$$

$$= -\left[R^2 lnt - 2Rt + \frac{t^2}{2}\right]_R^0$$

$=-\left(R^2\ln(0) - R^2\ln R + 2R^2 - \frac{R^2}{2}\right)$

$$I_1 = -R^2\ln(0) + R^2\ln R - 1.5R^2$$

Substituting the value of I_1 in b

$$I = \frac{R^2}{2}\ln(0) - \frac{R^2}{2}\ln(0) + \frac{R^2}{2}\ln R - 0.75R^2$$
$$= \frac{R^2}{2}\ln R - 0.75R^2$$

Substituting the value of I in a

$$V = \frac{2}{R^2}\left[\frac{R^2}{2}v_{max} + 2.5V_* \frac{R^2}{2}\ln\left(\frac{1}{R}\right) + 2.5V_*\left(\frac{R^2}{2}\ln R - 0.75R^2\right)\right]$$
$$V = v_{max} + 2.5V_* \ln\left(\frac{1}{R}R\right) - 3.75$$

$\frac{v_{max}-V}{V*} = 3.75$ (c)

We have,

$V_* = V\sqrt{f/8}$ (d)

Substituting V* in c and simplifying

$$\frac{v_{max} - V}{V\sqrt{f/8}} = 3.75$$
$$\frac{v_{max}}{V} = 1.33\sqrt{f} + 1$$

Chapter 2: Pipe flow problems and solutions

2.1 Three types of pipe flow problems and solution procedure

6 Variables: head loss (h_f), Discharge (Q), Length of pipe (L), Diameter of pipe (D), roughness (k), kinematic viscosity of fluid (υ)

L, K and υ: always known

h_f or Q or D: to compute

Use of Darcy-Weisbach equation, Continuity and Moody diagram or resistance equations (equation of f) to solve for unknown

Type 1

Given: Q, L, D, k, υ

Compute: h_f

Step:

I. Compute Re and K/D from given data.
II. For the values of Re and K/D, find f from either Moody chart or from equation (Swamee and Jain equation or Colebrook-White equation or other resistance equations).
III. Compute h_f using $h_f = \frac{fLV^2}{2gD}$.

Type 2

Given: h_f, L, D, k, υ

Compute: Q

Trial and error solution

I. Assume a suitable value of f for the known K/D by inspecting Moody diagram. (If Moody diagram is not given and resistance equation is given or known, assume a suitable value of f in the beginning.)
II. Use $h_f = \frac{fLV^2}{2gD}$ and determine V.
III. Determine $Re = \frac{VD}{\upsilon}$ and find the new value of f for this R_e and K/D using either Moody chart or equation. Use new f for next trial.
IV. Repeat steps ii) and iii) until the difference between two values of f is very small.
V. Compute Q using Q =AV.

Type 3

Given: Q, h_f, L, k, υ
Compute: D
Trial and error solution

I. Assume a suitable value of f.
II. Compute D using $h_f = \frac{fLV^2}{2gD}$. (trial value of D)

$h_f = \frac{fLV^2}{2gD} = f\frac{L}{D}\frac{Q^2}{2g[(\pi/4)D^2]^2}$

$D^5 = \left(\frac{8LQ^2}{h_f g\pi^2}\right)f = C1f$ where C1 is known quantity

III. With this trial value of D, compute value of Re.

$Re = \frac{VD}{\upsilon} = \frac{QD}{[(\pi/4)D^2]\upsilon} = \frac{4Q}{\pi\upsilon}\frac{1}{D} = \frac{C2}{D}$ where C2 is known quantity

IV. Compute k/D.
V. Using the computed value of Re and k/D, find f from either Moody chart or equation.
VI. Use new value of f for next trial.
VI. Repeat steps II to VI until the difference between two values of f is very small.

2.2 Pipes in series

If a pipeline connecting two reservoirs is made up of several pipes of different diameters and lengths, then the connection is said to be series connections. Bernoulli's energy equation and continuity equation are used to solve the problems of pipes in series.

Bernoulli's equation: Total energy at 1 = total energy at 2+ total head loss, where total head loss represents the sum of head losses in all the sections.
Continuity equation: same discharge in each section

Example

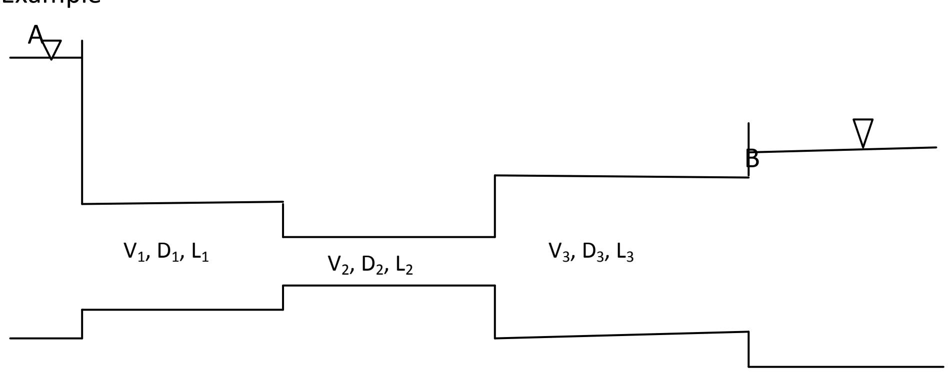


Fig. 2.1: pipes in series

Bernoulli's equation at A and B

$$\frac{P_A}{\gamma}+\frac{V_A^2}{2g}+Z_A=\frac{P_B}{\gamma}+\frac{V_B^2}{2g}+Z_B+total\ head\ loss$$

$$0+0+Z_A=0+0+Z_B+0.5\frac{V_1^2}{2g}+\frac{f_1L_1V_1^2}{2gD_1}+0.5\frac{V_2^2}{2g}+\frac{f_2L_2V_2^2}{2gD_2}+\frac{(V_2-V_3)^2}{2g}+\frac{f_3L_3V_3^2}{2gD_3}+\frac{V_3^2}{2g}$$

$$H=0.5\frac{V_1^2}{2g}+\frac{f_1L_1V_1^2}{2gD_1}+0.5\frac{V_2^2}{2g}+\frac{f_2L_2V_2^2}{2gD_2}+\frac{(V_2-V_3)^2}{2g}+\frac{f_3L_3V_3^2}{2gD_3}+\frac{V_3^2}{2g}$$

where H = difference in level between A and B

Continuity equation: $Q = Q_1 = Q_2 = Q_3$

Neglecting minor losses

$$H=\frac{f_1L_1V_1^2}{2gD_1}+\frac{f_2L_2V_2^2}{2gD_2}+\frac{f_3L_3V_3^2}{2gD_3}$$

In terms of Q

$H=r_1Q_1^2+r_2Q_2^2+r_3Q_3^2$ where $r_i=\frac{8f_iL_i}{\pi^2gD_i^5}$ or $\frac{f_iL_i}{12.1D_i^5}$

$H=Q^2\sum r_i$

Method of computation if the value of f is not given

- To determine H knowing Q, diameters and lengths: Compute Re at each section and find friction factors of all sections using Moody's diagram or resistance equations, and then compute H.

- To determine Q knowing H, diameters and lengths: First assume f value for each section, and compute Q. Find Re for each section and friction factors using Moody's diagram or resistance equations. With these new f values, compute a new value of Q. Repeat this procedure until Q does not change.

Concept of equivalent pipe for Pipes in series

When a compound pipe is replaced by a single pipe of uniform diameter having head loss and discharge equal to the loss of head and discharge of a compound pipe consisting of several pipes of different lengths and diameter, then the single pipe is called equivalent pipe. The uniform diameter of the equivalent pipe is known as equivalent diameter of the compound pipe.

Let L_1,L_2,L_3 etc are lengths and D_1,D_2,D_3 etc are the diameters respectively of the different pipes of a compound pipeline, then the total head loss in the compound pipe ,neglecting minor losses, is

$$h_L=r_1Q_1^2+r_2Q_2^2+r_3Q_3^2+\cdots$$

As $Q_1 = Q_2 = Q_3=\ldots=Q$

$$h_L=(r_1+r_2+r_3+\cdots)Q^2$$

if D and L be the diameter and length respectively of the equivalent pipe carry same discharge Q and head loss in the equivalent pipe is same that as that in the compound pipe.

Loss of head in equivalent pipe is $h_L = r\,Q^2$
Equating two head loss
$r\,Q^2 = (r_1 + r_2 + r_3 + \cdots)Q^2$
$r\ = (r_1 + r_2 + r_3 + \cdots)$
$\frac{fL}{12.D^5} = \left[\frac{f_1 L_1}{12.1 D_1^5} + \frac{f_2 L_2}{12.1 D_2^5} + \frac{f_3 L_3}{12.1 D_3^5} + \cdots\right]$
$\frac{fL}{D^5} = \left[\frac{f_1 L_1}{D_1^5} + \frac{f_2 L_2}{D_2^5} + \frac{f_3 L_3}{D_3^5} + \cdots\right]$
This equation is called Dupuit's equation, which may be used to determine the size of the equivalent pipe.
If $f = f_1 = f_2 = f_3 = \ldots = f$
$\frac{L}{D^5} = \left[\frac{L_1}{D_1^5} + \frac{L_2}{D_2^5} + \frac{L_3}{D_3^5} + \cdots\right]$

2.3 Pipes in parallel

When a main pipeline divides into two or more parallel pipes, which again join together downstream, and continue as a main line the pipes are said to be in parallels. The pipes are connected in parallels in order to increase the discharge passing through the main. Pipes in parallel problems are solved by using following principles:

Continuity: Discharge in the main line = sum of discharge in each of the parallels pipes
Head loss: Loss of head in each parallel pipe is same.

Example

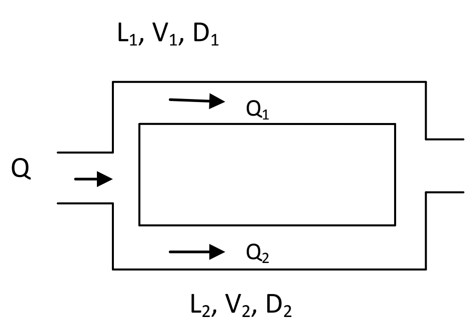


Fig. 2.2: Pipes in parallel

$Q = Q_1 + Q_2$
$h_{f1} = h_{f2}$

$$\frac{f_1 L_1 V_1^2}{2gD_1} = \frac{f_2 L_2 V_2^2}{2gD_2}$$

In terms of Q

$$r_1 Q_1^2 = r_2 Q_2^2$$

(Estimate f from Moody chart or resistance equations by trial and error approach if not given)

Equivalent pipe for pipes in parallel

We know,

$$h_f = rQ^2$$

Total discharge through the main is

$$Q = \sqrt{\frac{h_f}{r}}$$

h_f is same for all pipes in case of parallel connection.

Discharge through each pipe

$$Q_1 = \sqrt{\frac{h_f}{r_1}}, Q_2 = \sqrt{\frac{h_f}{r_2}} \ldots\ldots$$

Q= Q_1+Q_2+....

$$\sqrt{\frac{h_f}{r}} = \sqrt{\frac{h_f}{r_1}} + \sqrt{\frac{h_f}{r_2}} + \cdots$$

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} + \cdots$$

r value of equivalent single pipe replacing parallel pipes with discharge Q can be found out from this expression. In terms of f,

$$\frac{1}{\sqrt{\frac{fL}{D^5}}} = \frac{1}{\sqrt{\frac{f_1 L_1}{D_1^5}}} + \frac{1}{\sqrt{\frac{f_2 L_2}{D_2^5}}} + \cdots$$

Two types of problems in parallel pipe system

a. Determination of discharge through each pipe given head loss (h_f) in parallel pipe

As h_f is same in each branch, discharge through each branch is calculated directly from

$$h_f = r_1 Q_1^2, h_f = r_2 Q_2^2 \ldots\ldots$$

b. Determination of discharge through each pipe given discharge through the main (Q)

This problem can be solved analytically.

Continuity equation: Q= Q_1+Q_2+ Q_3.... (a)

Head loss equation: $h_{f1} = h_{f2} = h_{f3} \ldots$

$r_1 Q_1^2 = r_2 Q_2^2 = r_3 Q_3^2 \ldots$ (b)

Solving technique: In equation b, take discharge value of a particular pipe and express discharge of all other pipes in terms of discharge of that particular pipe. Then, solve for discharge.

Alternatively, this type of problem can also be solved by trial and error (assuming discharge in one pipe and checking the continuity equation) or by converting the pipe into equivalent pipe (computing equivalent r, computing h_f using equivalent r and then computing Q through each pipe using same h_f value).

2.4 Siphon

A siphon is a long bent pipe which is used to carry water from a reservoir at a higher elevation to another reservoir at a lower elevation when two reservoirs are separated by a hill or high level ground. The rising portion of the siphon is known as the inlet limb, the highest point is known as summit and the portion between the summit and the lower reservoir is known as outlet limb.

As siphon is laid over a hill, the loss of head due to friction is very large and minor losses may be neglected. The length of siphon may be taken as the length of its horizontal projection.

Uses of siphon

- To carry water from one reservoir to another reservoir separated by a hill or ridge.
- To empty a tank of water, having no outlet
- To draw a water from a canal having no outlet sluice.
- To connect two water reservoir separated by a valley (inverted siphon)
- To connect two open canals by an inverted siphon laid beneath the canal beds.

2.4.1 Starting of siphon

A siphon can be put in action either by exhausting air thus creating vacuum in it or by filling it with water. The air can be exhausted by a vacuum pump. The water in the siphon can be poured through an opening made at the summit which is only possible by closing the outlets at two ends.

While in operation the air separates itself from the flowing water and has the tendency to be collected at the bend. An air vessel can be provided to get rid of this difficulty; otherwise there will be an interruption in the flow.

2.4.2 Condition for continuous supply

Some portion of the siphon is above HGL. The vertical distance between the hydraulic gradient and the center line of the pipe represents the pressure head at any section. If the hydraulic gradient is above the center line of the pipe, the pressure is above atmospheric. However, if the hydraulic gradient is below the center line of the pipe, the pressure is below atmospheric or negative. Thus the pressure at the summit will be the least. Further as the vertical distance between the summit and the hydraulic gradient increases, the water pressure at this point reduces. Theoretically this pressure may be reduced to -10.3 m of water (if the atmospheric pressure is 10.3 m of water) or absolute vacuum, because this limit would correspond to a perfect vacuum and the flow would stop. However, in practice, if the pressure is reduced to about 2.5 m of water absolute or 7.8m of water vacuum, the dissolved air or other gases would come out of the solution and collect at the summit of the siphon in sufficient quantity to form an air-lock, which will obstruct the continuity of the flow, (or the flow will completely stop). A similar trouble may also be caused by the formation of water vapor in the region of low pressure.

For the continuous supply, following conditions should be met.

- The siphon should be laid so that no section of the pipe will be more than 7.8m above the hydraulic gradient at the section.
- In order to limit the reduction of the pressure at the summit, the length of the inlet leg of the siphon is also required to be limited. This is because of the length of inlet leg is very long, a considerable loss of head due to friction is caused, which further reduces the pressure at the summit.

2.4.3 Solving problems of siphon connecting two reservoirs

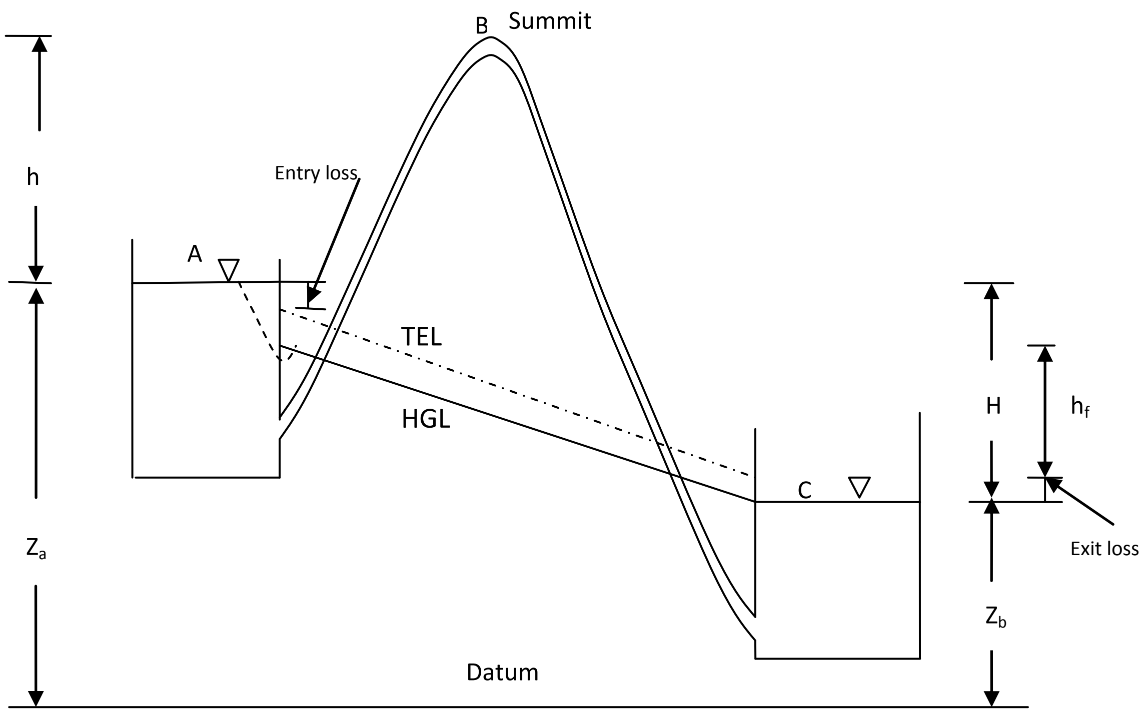


Fig. 2.3: Siphon

Two ends of pipe are submerged in water and the pipe is filled with water. As second end is below the first end, the discharge will commence. (Note: for some distance from the entrance section of the pipe, the HGL is not well defined because of the formation of vena-contracta and sudden drop in pressure head at this section.). The problems of siphon are solved by applying Bernoull's equation and continuity equation.

a. Determination of head loss between two tanks

Assuming the siphon to run full, then applying Bernoulli's equation between two points A and C, we get

$$\frac{P_A}{\gamma}+\frac{V_A^2}{2g}+Z_A=\frac{P_C}{\gamma}+\frac{V_C^2}{2g}+Z_C+Losses$$

(Working in terms of absolute pressure)

$$10.3+0+Z_A=10.3+0+Z_C+0.5\frac{V^2}{2g}+\frac{V^2}{2g}+\frac{fLV^2}{2gD}$$

V= Velocity of flow, L = Length of siphon, D = Diameter of siphon

Entry loss = $0.5V^2/2g$, exit loss = $V^2/2g$, loss due to friction (Major loss) = $\frac{fLV^2}{2gD}$

$$Z_A-Z_C=\frac{V^2}{2g}\left(1.5+\frac{fL}{D}\right)$$

$$H=\frac{V^2}{2g}\left(1.5+\frac{fL}{D}\right)$$

H= difference in levels of two reservoirs

If minor losses are neglected

$$H=\frac{fLV^2}{2gD}$$

This equation may be used to determine H if the discharge Q is known, or it may be used to determine Q if H is given.

If the value of f is not given in the problem, the following procedure is used.

- Determination of H: As Q is known, Re can be computed. Then for computed Re and given K/D, f can be determined from Moody's diagram or resistance equation.

- Determination of Q: Assume a suitable value of f. Compute Q and Re. For the computed Re and given K/D, find new f using Moody's diagram or resistance equations. With the new f value, compute a new value of Q. Repeat this procedure until Q does not change.

b. Determination of maximum height of summit

Applying Bernoulli's equation is applied between the point A and summit B. (Working in terms of absolute pressure given the absolute pressure at summit)

$$\frac{P_A}{\gamma}+\frac{V_A^2}{2g}+Z_A=\frac{P_B}{\gamma}+\frac{V_B^2}{2g}+Z_B+Losses$$

(Working in terms of absolute pressure)

$$\frac{P_A}{\gamma} + 0 + Z_A = \frac{P_B}{\gamma} + \frac{V^2}{2g} + (Z_A + h) + 0.5\frac{V^2}{2g} + \frac{flV^2}{2gD}$$

$$h = \frac{P_A}{\gamma} - \frac{P_B}{\gamma} - \frac{V^2}{2g}\left(1.5 + \frac{fl}{D}\right)$$

P_A = Atmospheric pressure = 10.3 m of water
P_B = Absolute pressure at summit
l = length of inlet leg of siphon

If absolute pressure at summit is given, above expression is used to compute h. If gauge pressure is given, then take $P_a/\gamma = 0$.

c. Determination of pressure at summit

Applying Bernoulli's equation is applied between the point A and summit B. (Working in terms of absolute pressure given the absolute pressure at summit)

$$\frac{P_A}{\gamma} + \frac{V_A^2}{2g} + Z_A = \frac{P_B}{\gamma} + \frac{V_B^2}{2g} + Z_B + Losses$$

(Working in terms of absolute pressure)

$$\frac{P_A}{\gamma} + 0 + Z_A = \frac{P_B}{\gamma} + \frac{V^2}{2g} + (Z_A + h) + 0.5\frac{V^2}{2g} + \frac{flV^2}{2gD}$$

$$\frac{P_B}{\gamma} = \frac{P_A}{\gamma} - h - \frac{V^2}{2g}\left(1.5 + \frac{fl}{D}\right)$$

(Note: If the end C is discharging to atmosphere, then take velocity at C = Velocity of flow through pipe in Bernoulli's equation.)

Chapter 3: Three reservoir problems and Pipe network

In actual practice, several pipes are interconnected forming various loops or circuit in municipal water distribution systems. There are a number of pipes connected either in series or in parallel or a combination of both. A group of interconnected pipes forming several loops or circuit in complex manner is called pipe networks. The complexity in a pipe networking is to determine the solution of pipe problem (Flow distribution) in such pipe networking.

3.1 Three reservoir problems

In a water supply system often a number of reservoirs are required to be interconnected by means of a pipe system consisting of a number of pipes namely main and branches which meet at a junction. Typical example is three reservoirs A, B and C interconnected by pipes 1, 2 and 3 which meet at a junction D.

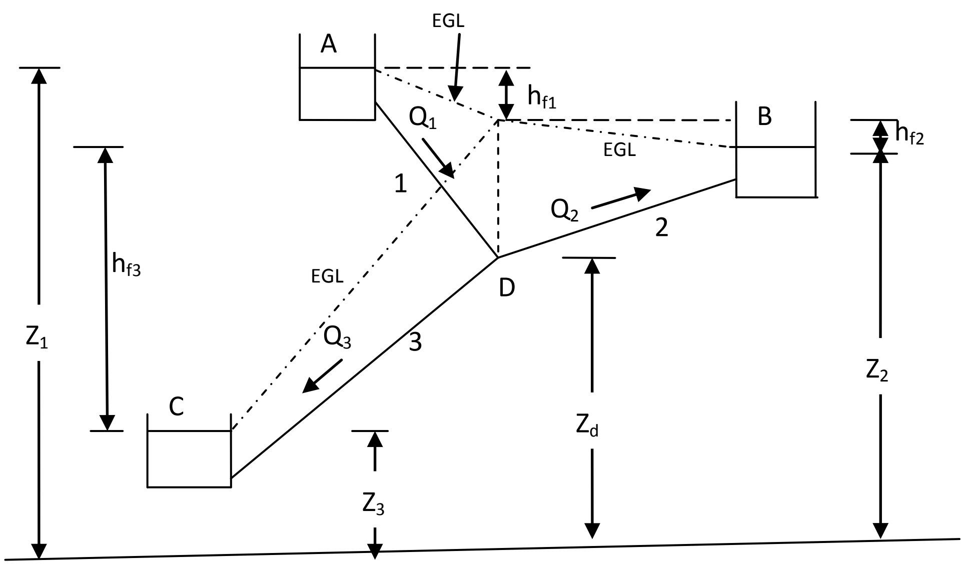


Fig. 3.1: Three reservoir system

In the problem of this type, the lengths, diameters and friction factors of the pipes are known. Further it is assumed that the flow is steady, minor losses are insignificant and the reservoirs are large enough so that their water surface elevations are constant. Three basic equations are used to solve the problems.

- Continuity equation
- Bernoulli's equation
- Darcy-Weisbach equation

Let D_1, D_2, D_3 be the diameters; L_1, L_2, L_3 be the lengths; Q_1, Q_2, Q_3 be the discharges; V_1, V_2, V_3 be the velocities of flow in pipes 1, 2 and 3 respectively. Z_1, Z_2, Z_3 are respectively the heights of the water surface in the reservoirs A,B, C; Z_D is the height of the junction D above the assumed datum; h_{f1}, h_{f2}, h_{f3} are the head loss due to friction in pipes 1,2 and 3 respectively; P_D/γ is the pressure head at the junction D.

Consider the flow of water from reservoir A to junction D and junction D to both the reservoirs B and C.
Applying Bernoulli's equation between point A and D

$$\frac{P_A}{\gamma}+\frac{V_A^2}{2g}+Z_1=\frac{P_D}{\gamma}+\frac{V_1^2}{2g}+Z_D+h_{f1}$$

Neglecting velocity heads as these values are very small in comparison to friction loss

$$0+0+Z_1=\frac{P_D}{\gamma}+0+Z_D+h_{f1}$$

$$Z_1=\left(\frac{P_D}{\gamma}+Z_D\right)+h_{f1} \qquad \text{(a)}$$

Applying Bernoulli's equation between the junction D and the reservoir B

$$\frac{P_D}{\gamma}+\frac{V_2^2}{2g}+Z_D=\frac{P_B}{\gamma}+\frac{V_B^2}{2g}+Z_2+h_{f2}$$

$$\frac{P_D}{\gamma}+0+Z_D=0+0+Z_2+h_{f2}$$

$$\left(\frac{P_D}{\gamma}+Z_D\right)=Z_2+h_{f2} \qquad \text{(b)}$$

Applying Bernoulli's equation between the junction D and the reservoir C

$$\frac{P_D}{\gamma}+\frac{V_3^2}{2g}+Z_D=\frac{P_C}{\gamma}+\frac{V_C^2}{2g}+Z_3+h_{f3}$$

$$\frac{P_D}{\gamma}+0+Z_D=0+0+Z_3+h_{f3}$$

$$\left(\frac{P_D}{\gamma}+Z_D\right)=Z_3+h_{f3} \qquad \text{(c)}$$

Applying continuity equation

$$Q_1=Q_2+Q_3$$

$$\frac{\pi}{4}D_1^2V_1=\frac{\pi}{4}D_2^2V_2+\frac{\pi}{4}D_3^2V_3 \qquad \text{(d)}$$

Let $h_D=\left(\frac{P_D}{\gamma}+Z_D\right)$

Set of equations

$Z_1=h_D+h_{f1}$ (a)

$h_D=Z_2+h_{f2}$ (b)

$h_D=Z_3+h_{f3}$ (c)

$Q_1=Q_2+Q_3$ (d)

Above four equations involve four unknowns i.e. Q_1, Q_2, Q_3 and h_D which can be calculated by solving these four equations.

If $h_D < Z_2$, the water will flow from B to D. In this case both A and B will supply water to C. In this case equations a, and c are similar to above equations and equation b and d becomes

$Z_2=h_D+h_{f2}$, $\quad Q_1+Q_2=Q_3$

If $h_D < Z_3$, the water will flow from C to D. In this case both A and C will supply water to B. In this case equations a, and b are similar to above equations and equation c and d becomes
$Z_3 = h_D + h_{f3}$, $Q_1 + Q_3 = Q_2$

3.2 Types of problems in three reservoir system and solutions

The following relationships are used to solve three-reservoir problem.

- Moody's diagram or corresponding equations for f
- Darcy-weisbach equation: $h_f = \frac{fLV^2}{2gD}$ or, $h_f = rQ^2$ where $r = \frac{fL}{12.1D^5}$
- Continuity: At the junction $\sum Q_i = 0$ i.e. the flow coming into the junction must be equal to flow going out of the junction.

Method of solving three types of problems

Type I: To determine Q_2, Q_3 and Z_3 when diameters, lengths, friction factors, Z_1, Z_2 and Q_1 are known
I. With Q_1 known, determine h_{f1}.
II. Use equation (a) to determine h_D.
III. Knowing h_D, calculate h_{f2} using equation (b) and find Q_2.
IV. Use equation (d) to determine Q_3.
V. Knowing Q_3, compute h_{f3} and use equation (c) to determine Z_3.

Type II: To determine Q_1, Q_3 and Z_2 when diameters, lengths, friction factors, Z_1, Z_3 and Q_2 are known

Method 1: Analytical method

- As Q_2 is given h_{f2} is known. Express h_{f1} in terms of Q_1 and h_{f3} in terms of Q_3.
- Eliminate h_D and reduce 4 equations to 3 equations. Now, we have three equations with three unknowns i.e. Q_1, Q_3 and Z_2. Solve these equations simultaneously to find the unknowns.

Method 2: Trial and error method

I. From equations a and c, $(h_{f1}+h_{f3}) = Z_1-Z_3$. (RHS is known)
From d, $Q_1-Q_3 = Q_2$ (RHS is known)

Trial and error method 1:
Assume trial values of Q_1 and Q_3, and compute h_{f1} and hf_3 and compare these with the known value of $(h_{f1+}\ h_{f3})$. Repeat the procedure until the computed value of the sum of $(h_{f1+}\ h_{f3})$ is equal to the known value. Knowing h_{f1} and $hf_{3,}$ compute Q_1 and Q_3.
Trial and error method 2:
Assume trial values of h_{f1} and h_{f3}, and compute Q_1 and Q_3 and compare these with the known value of (Q_1-Q_3) or (Q_3-Q_1). Repeat the procedure until two values are equal.
II. With the result of trial and error, compute h_D.
III. From given Q_2, determine h_{f2}.
IV. Using values of II and III, determine Z_2 from eq. (b).

Type III: To determine Q_1, Q_2 and Q_3 when diameters, lengths, friction factors, Z_1, Z_2 and Z_3 are known

Method 1: Analytical method

- Assign the direction of flow if not given: Assume $h_D = Z_2$, compute h_{f1} and h_{f3} and Q_1 and Q_3. If Q_1>Q_3, the direction of flow is from the junction D to the reservoir B, and the first set of equations are valid. Else, the direction of flow is from the reservoir to the junction D. In the second case, modify the equations (b) and (d).
- Eliminate h_D and reduce 4 equations to 3 equations. Now, we have three equations with three unknowns i.e. Q_1, Q_2 and Q_3. Solve these equations simultaneously to find the unknowns.

(Hint: solving method 1: Express discharge in two pipes in terms of discharge in the third pipe, e.g. express Q_1 and Q_3 in terms of Q_2, (e.g. by letting Q_3 = mQ_2), substitute Q_1 and Q_3 in continuity equation, eliminate Q and find m. Solving method 2: solve resulting equations by the by trial and error approach or by numerical method.)

Method 2: Trial and error method

Trial and error approach 1: Performing trial for h_D
Obtain the direction of flow at junction D by following similar approach as method 1.
I. Assume a trial value of piezometric head at junction D, h_D (Hint: Take average of highest and lowest reservoir elevation as starting value) compute h_{f1}, h_{f2} and h_{f3} from eqs. (a) to (c), and then determine Q_1, Q_2 and Q_3. Compute $\sum Q_i$ at junction.
III. Repeat the trials until $\sum Q_i$ at junction becomes close to zero.
Hint for reducing more trials: After performing some trials, plot h_D versus $\sum Q_i$, and fit a line. Extend the line to meet at Y- axis. The value of piezometric head for which $\sum Q_i = 0$ is the actual piezometric head at D. Knowing piezometric head at D, use eqs. (a), (b) and (c) to determine Q_1, Q_2 and Q_3.

Trial and error approach 2: Quantity balance method
In this method, correction is applied to piezometric head at D until $\sum Q_i$ at junction becomes close to zero.
Correction to be applied for piezometric head at D is $dh = \frac{2\sum Q}{\sum |Q/h_f|}$.
Calculation steps: Assume a trial value of h_D (Hint: Take average of highest and lowest reservoir elevation as starting value). Compute h_{f1}, h_{f2} and h_{f3} from eqs. (a) to (c). Compute Q_1, Q_2 and Q_3 considering flow towards junction as positive and flow away from junction as negative. Compute Q/h_f. Compute the correction dh.
h_D for next trial = h_D of previous trial+dh
Continue the trials until $\sum Q$ becomes close to zero.

3.3 Pipe Network analysis by Hardy cross method

In municipal water supply system, there are several branches, junctions and circuits, such that the system is complex. Hardy Cross method is one of the widely used methods to solve complex pipe network forming a closed-loop. This is a method of successive approximation and the approach is the head balance approach.

The basic requirements for Hardy Cross method are:
I) Continuity equation: The flow into each junction must be equal to the flow out of the junction.
II) Energy equation: The algebraic sum of the head losses around any closed circuit must be zero.
For each loop or circuit the head losses due to flow in clockwise direction is taken as +ve while the head loss in anticlockwise direction is taken as –ve.
III) Darcy-Weisbach equation: Darcy-Weisbach equation must be satisfied for flow in each pipe.

$$h_f = \frac{fLV^2}{2gD} = \frac{fL}{2gD}\frac{Q^2}{A^2} = \frac{fL}{2gD}\frac{Q^2}{\left(\frac{\pi}{4}D^2\right)^2} = \frac{8fL}{\pi^2 gD^5}Q^2 = rQ^2$$

Where r = $\frac{8fL}{\pi^2 gD^5}$ or $\frac{fL}{12.1D^5}$

Expressing h_f in the form of $h_f = rQ^n$

$r = \frac{fL}{12.1D^5}$ and n =2

$h_f = rQ^2$

Derivation for correction

For any pipe if Q_0 is the assumed discharge and Q is correct discharge, then

$Q = Q_0 + \Delta Q$

The head loss for the pipe is

$h_f = rQ^n = r(Q_0 + \Delta Q)^n$

For the complete circuit,

$\sum h_f = \sum rQ^n = \sum r(Q_0 + \Delta Q)^n$

By expanding the terms in the brackets by binomial theorem (assuming +ve sign)

$\sum rQ^n = \sum r(Q_0^n + nQ_0^{n-1}\Delta Q + \cdots)$

Assuming ΔQ to be very small compared with Q_0, higher powers of ΔQ may be neglected. Then

$$\sum h_f = \sum rQ^n = \sum r(Q_0 + \Delta Q)^n = \sum r\, Q_0^n + \sum rnQ_0^{n-1}\Delta Q$$

For the correct distribution the circuit is balanced and hence

$\sum h_f = 0$

$\sum r\, Q_0^n + \sum rnQ_0^{n-1}\Delta Q = 0$

$\Delta Q = -\frac{\sum rQ_0^n}{\sum rnQ_0^{n-1}}$

Generally in practice n = 2

$\Delta Q = -\frac{\sum rQ_0^2}{2\sum rQ_0}$

Discharge through each pipe is found by trial and error approach considering the balance of head. That means $\sum h_f = 0$ for each loop.

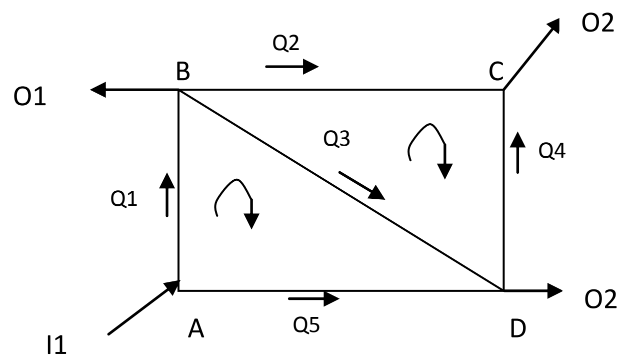


Fig. 3.2: Example of pipe network

Steps for the computation of flow by Hardy Cross Method

Given: inflow, outflow

To compute: flow through each pipe

I. Assume suitable rate of flow as well as direction of flow in each pipe, which satisfies the continuity equation at each junction.

II. Divide the pipe network into a number of closed circuits, so as to include every pipe in at least one of the circuits.

III. For each loop, compute head loss through each pipe and determine the algebraic sum of head loss as $\sum h_f = \sum rQ^2$

Consider head losses from clockwise flows as +ve and from anticlockwise flow as –ve.

Also compute $2\sum rQ$ for the circuit. This sum is calculated without considering sign (absolute value).

IV. Compute the correction (ΔQ).

$\Delta Q = -\frac{\sum rQ^2}{2\sum rQ}$

V. Compute the Corrected flow as $Q_{i-1} + \Delta Q$ where Q_{i-1} = previous value of flow

- If ΔQ is positive, add it to the flow in clockwise direction and subtract it from the flow in anti-clockwise direction.
- If ΔQ is negative, add it to the flow in anti-clockwise direction and subtract it from the flow in clockwise direction.
- For a common pipe of two loops, apply correction from both loops considering appropriate sign for each loop.

VI. Use the corrected flow for the next trial and repeat steps III to V until the correction becomes negligible.

Chapter 4: Unsteady flow through pipes (Water hammer)

Wave: Wave is temporal variation of water surface which is propagated in the fluid media.
Celerity: The relative velocity of a wave with respect to the speed of fluid is called celerity. If C is celerity, V_w is the velocity of the wave and V is the fluid speed, then C = V_w-V if both move in same direction and C = V_w+V if both move in opposite direction.

4.1 Water hammer

When the water flowing in a long pipe is suddenly brought to rest by closing the valve, the momentum of the flowing water will be destroyed and consequently a wave of high pressure will be setup. (The K.E. of flowing liquid will converted into the internal pressure energy with rise of pressure).
This wave of high pressure will be transmitted along the pipe with a velocity equal to the velocity of sound and may create a noise called knocking. This phenomenon of sudden rise in pressure in the pipe is known as water hammer or water blow.

Cause of water hammer (fast hydraulic transient)

Water hammer is caused by changes in velocity which are caused by:

- Valve operation (i.e. closure and opening of valves).
- Power failures.
- Starting or shut down of pumps (hydro- turbines).
- Fluctuation in power demand in turbines
- Rupture of the line, etc.
- Mechanical failure of the control devices like valves.

Effects of water hammer

- High-pressure fluctuations in pipelines.
- Rupture of pipe or valve if beyond safety limit
- Higher pressure requirements for the design of pipeline and penstocks, etc.

The magnitude of pressure rise depends on

- Time taken to close the valve
- Velocity of flow
- Length of pipe
- Elastic properties of the pipe materials as well as that of the flowing fluid

Time required by pressure wave to travel from the valve to tank and from tank to valve
Time taken = Distance traveled from valve to tank and back/ Velocity of pressure wave

$T = \frac{L+L}{C} = \frac{2L}{C}$

where L =length of pipe, C= velocity of pressure wave

i) The closure of valve is said to be gradual when T>2L/c

ii) The closure of valve is said to be sudden when T< 2L/c

where t= time required to close the valve, L = length of the pipe and c= velocity of the pressure wave (celerity)

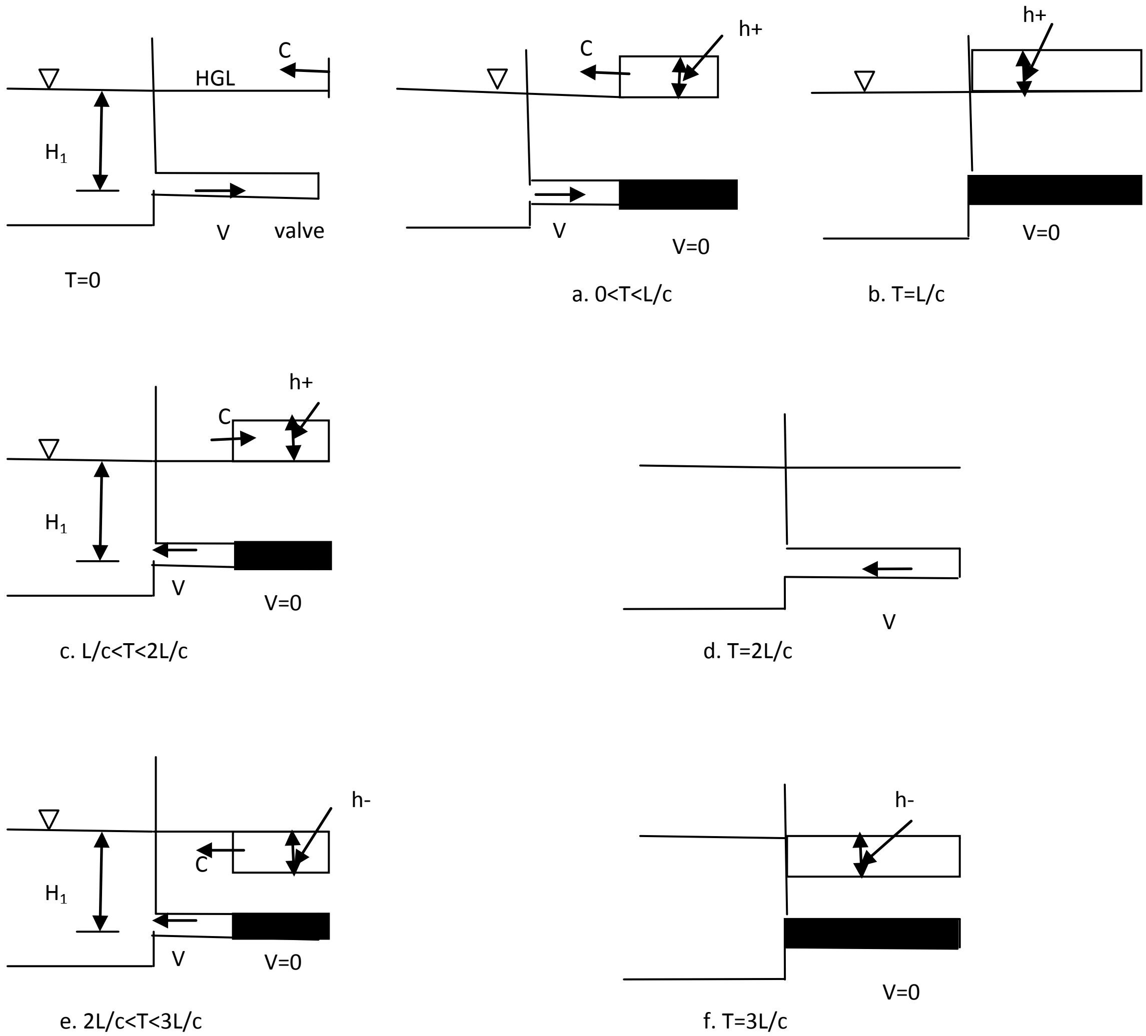
$c = \sqrt{\frac{dP}{d\rho}}$ where dp = change in pressure, $d\rho$ = change in density

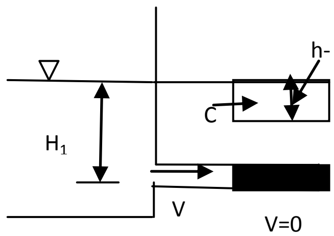
Or $c = \sqrt{\frac{K}{\rho}}$

Where K = Bulk modulus, ρ = density of fluid

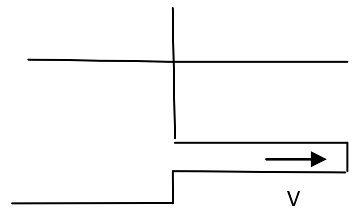
4.2 Theories of water hammer phenomenon

a. Elastic water column theory: Instantaneous closure of valve





g. 3L/c<T<4L/c



h. T=4L/c

Fig. 4.1: One cycle of wave motion in a pipe due to sudden valve closure

Descriptions of different stages

Stage 1: When the valve is closed, the layer of water close to the valve is brought to rest. The layer of water is compressed and consequently the pressure will rise by $\frac{\Delta P}{\gamma} = h^{+}$ and the pipe will get stretched. Successive layers are brought to rest in succession and the pressure rises for greater length of the pipe. The high pressure wave moves upstream with a velocity C reducing the incoming velocity to zero and reaches the reservoir at time T=L/C.

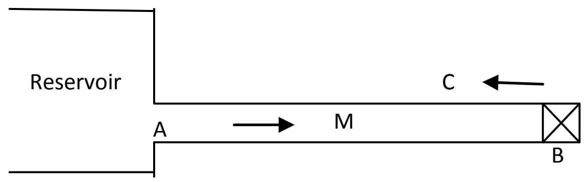
Stage 2: Since the velocity in the reservoir is approximately zero, there is no change in head. At the end of stage 1, the pressure in the pipe is equal H_1+h^+ throughout and the velocity is zero. Since the pressure in the pipe is higher than that in the reservoir, water starts to flow back to the reservoir reducing the pressure in the pipe, an expansion wave of equal magnitude travels with a velocity C towards the valve. The pressure of water falls back and the pipe walls resume their original size. Thus at time T=2L/C, the pressure in the pipe in its entire length returns to the normal or original value and water has a velocity V in the backward direction.

Stage 3: With the valve closed, water is not available to maintain backward flow. The water thus comes to rest at this end and the pressure falls by $\frac{\Delta P}{\gamma} = h^{-}$ below the normal value. This is repeated for every successive layer. The negative pressure wave travels towards the reservoir with velocity C bringing the layers of water to rest and permitting the pipe walls to contract. At T= 3L/C, the entire water body comes to rest and the pressure in the entire length of the pipe reduced to h-.

Stage 4: Since reservoir pressure is higher than that in the pipe, the water flows into the pipe attaining a velocity V towards the valve in the normal condition. At T= 4L/C, the conditions are back to what they were at the instant of the valve closure.

The stages mentioned above now repeat. The pressure is dampened with friction against pipe and elasticity of water and ultimately water comes to rest.

Pressure diagram at different points



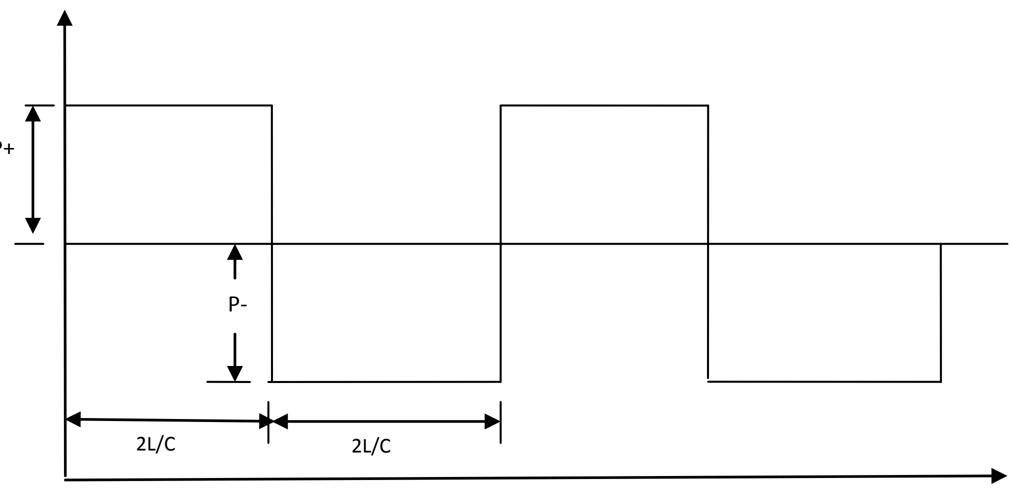
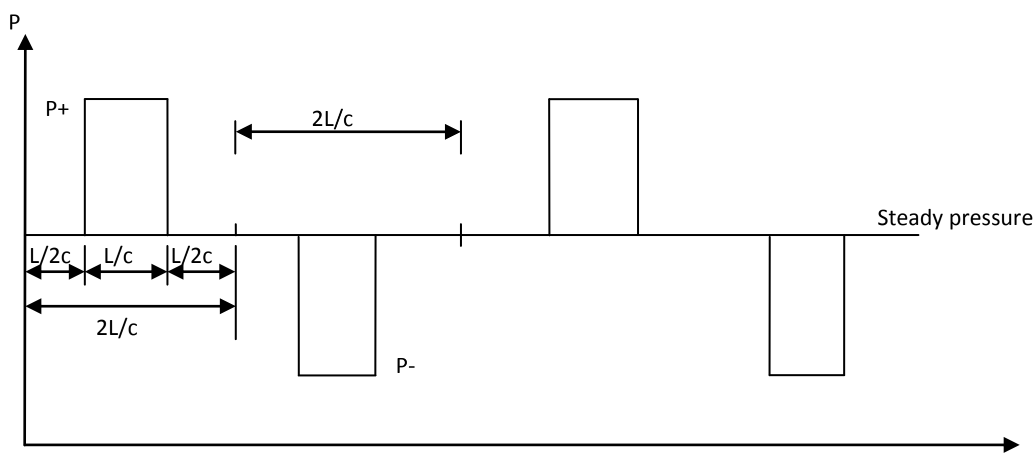
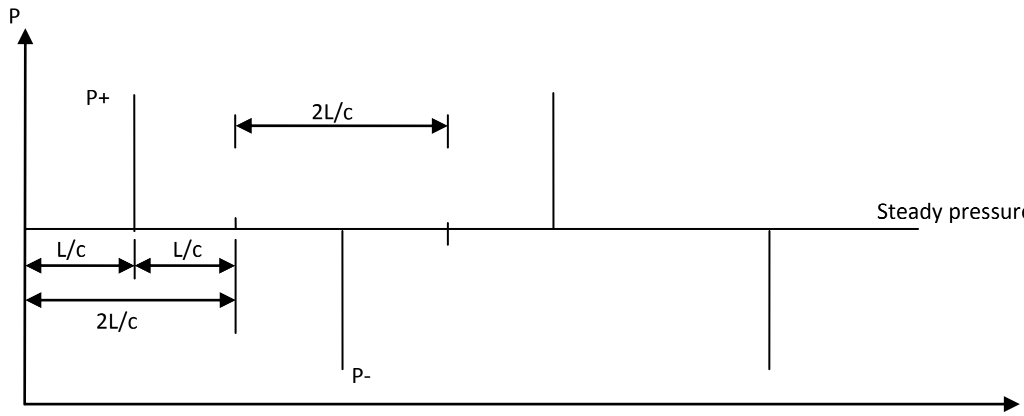


Fig. 4.2a: pressure variation at B (at the valve)



4.2b: Pressure diagram at mid point (M)



4.2c: Pressure diagram at end point (A)

b. Rigid water column theory: Sudden closure of valve

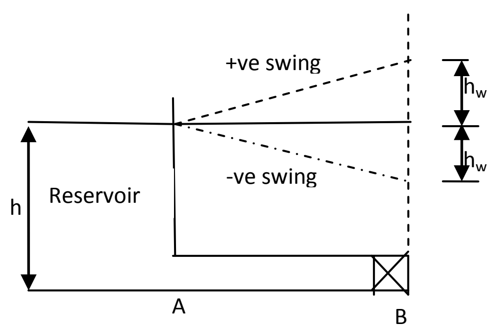


Fig. 4.3: change in HGL due to sudden valve closure

If the valve at the end is closed suddenly, the water in the pipe retards and hence there is a pressure increase. The pressure swings the normal hydraulic gradient line upwards. Since the pressure at the reservoir is atmospheric and constant, the +ve swing results in the back flow from the pipe into the reservoir. As the water flows back into the reservoir, it creates partial vacuum conditions and the pressure in the pipe swings in the –ve direction. This induces the reservoir water to flow into the pipe. But the valve being partially closed, much of water is again retarded giving rise to a +ve swing of pressure again. The water hammer head h_w is additional to the normal head h.

4.3 Basic equations of unsteady flow for pipe flow

a. Continuity equation for unsteady flow

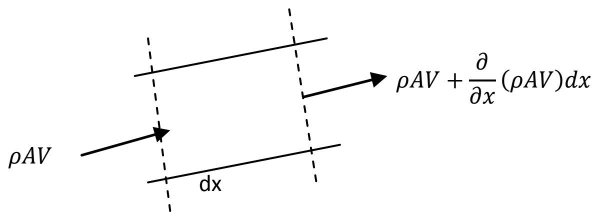


Fig. 4.4: control volume for continuity equation

Consider the fluid element as shown in the figure. A = C/s area, V = Average velocity, ρ =density of fluid
Mass of fluid entering the control volume per unit time = ρAV
Mass of fluid leaving the control volume per unit time = $\rho AV + \frac{\partial}{\partial x}(\rho AV)dx$
Net gain of mass within control volume per unit time = $\rho AV - \left[\rho AV + \frac{\partial}{\partial x}(\rho AV)dx\right] = -\frac{\partial}{\partial x}(\rho AV)dx$
Rate of increase of mass = $\frac{\partial}{\partial t}(\rho A dx)$
Net gain of mass per unit time = Rate of increase of mass

$$-\frac{\partial}{\partial x}(\rho AV)dx = \frac{\partial}{\partial x}(\rho A dx)$$
$$-\frac{\partial}{\partial x}(\rho AV)dx = \frac{\partial}{\partial t}(\rho A)dx$$
$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho AV) = 0$$
$$A\frac{\partial \rho}{\partial t} + \rho\frac{\partial A}{\partial t} + AV\frac{\partial \rho}{\partial x} + \rho\frac{\partial (AV)}{\partial x} = 0$$
$$A\frac{\partial \rho}{\partial t} + \rho\frac{\partial A}{\partial t} + AV\frac{\partial \rho}{\partial x} + \rho A\frac{\partial V}{\partial x} + \rho V\frac{\partial A}{\partial x} = 0$$

Dividing throughout by ρA

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{1}{A}\frac{\partial A}{\partial t} + \frac{V}{\rho}\frac{\partial \rho}{\partial x} + \frac{\partial V}{\partial x} + \frac{V}{A}\frac{\partial A}{\partial x} = 0$$
$$\frac{V}{A}\frac{\partial A}{\partial x} + \frac{1}{A}\frac{\partial A}{\partial t} + \frac{V}{\rho}\frac{\partial \rho}{\partial x} + \frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{\partial V}{\partial x} = 0$$

First two terms are total derivative of $\frac{1}{A}\frac{dA}{dt}$ and the next two terms are total derivative of $\frac{1}{\rho}\frac{d\rho}{dt}$

$$\frac{1}{A}\frac{dA}{dt} + \frac{1}{\rho}\frac{d\rho}{dt} + \frac{dV}{dx} = 0 \qquad \text{(a)}$$

First term: elasticity
Second term: compressibility

Consider the pipe wall with diameter D.
Rate of change of tensile force per unit length =$\frac{D}{2}\frac{dP}{dt}$
Rate of change of unit stress = $\frac{D}{2t}\frac{dP}{dt}$ where t = wall thickness
Rate of change of unit strain = $\frac{D}{2Et}\frac{dP}{dt}$ where E = Young's modulus

Rate of radial extension = $\frac{D}{2Et}\frac{dP}{dt}\frac{D}{2}$

Rate of area increase = $\frac{dA}{dt} = \frac{D}{2Et}\frac{dP}{dt}\frac{D}{2}\pi D = \frac{D}{Et}\frac{dP}{dt}\frac{\pi D^2}{4}$

Hence, $\frac{1}{A}\frac{dA}{dt} = \frac{D}{Et}\frac{dP}{dt}$

Bulk modulus (K) = $-\frac{dP}{dV/V} = \frac{dP}{d\rho/\rho}$

$\frac{d\rho}{\rho} = \frac{dP}{K}$

$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{1}{K}\frac{dP}{dt}$

Substituting values of $\frac{1}{A}\frac{dA}{dt}$ and $\frac{1}{\rho}\frac{d\rho}{dt}$ in eq. (a)

$\frac{D}{Et}\frac{dp}{dt} + \frac{1}{K}\frac{dP}{dt} + \frac{\partial V}{\partial x} = 0$

$\frac{1}{K}\frac{dp}{dt}\left(1 + \frac{DK}{Et}\right) + \frac{\partial V}{\partial x} = 0$

$c^2 = \frac{K\prime}{\rho} = \frac{K}{\rho\left(1+\frac{DK}{Et}\right)}$

$\frac{1}{K}\frac{dp}{dt}\frac{K}{c^2\rho} + \frac{\partial V}{\partial x} = 0$

Multiplying throughout by c^2

$\frac{1}{\rho}\frac{dp}{dt} + c^2\frac{\partial V}{\partial x} = 0$

$\frac{1}{\rho}\left(V\frac{\partial P}{\partial x} + \frac{\partial P}{\partial t}\right) + c^2\frac{\partial V}{\partial x} = 0$

$V\frac{\partial P}{\partial x}$ is small and can be neglected.

$\frac{1}{\rho}\frac{\partial P}{\partial t} + c^2\frac{\partial V}{\partial x} = 0$

b. Momentum equation (Euler's equation) for unsteady flow through pipe

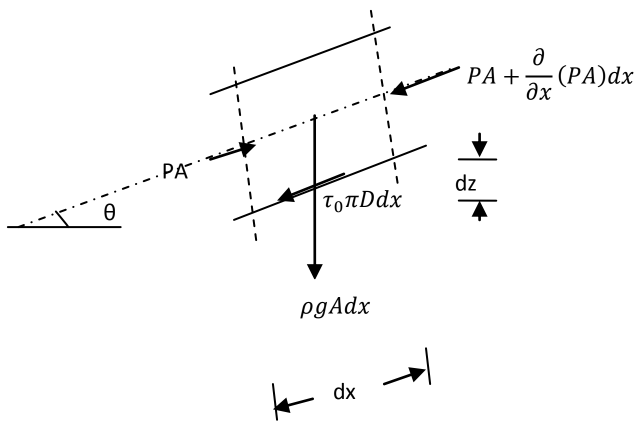


Fig. 4.5: control volume for momentum equation

Consider the fluid element as shown in the figure. Let A = C/s area, V = Average velocity, ρ =density of fluid, P = pressure on the left side

Forces acting: pressure force, gravity force, shear force

Net force = ma

$$PA-\left[PA+\frac{\partial}{\partial x}(PA)dx\right]-\rho gAdxsin\theta-\tau_0\pi Ddx=\rho Adx\frac{dV}{dt}$$

Dividing throughout by ρAdx and simplifying

$$-\frac{1}{\rho}\frac{\partial P}{\partial x}-gsin\theta-\frac{4\tau_0}{\rho D}=\frac{dV}{dt}$$

With $\tau_0=\frac{1}{8}\rho fV^2$

$$\frac{dV}{dt}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\frac{\partial z}{\partial x}+\frac{fV|V|}{2D}=0$$

$$V\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\frac{\partial z}{\partial x}+\frac{fV|V|}{2D}=0$$

In water hammer application, the term $V\frac{\partial V}{\partial x}$ is much smaller than $\frac{\partial V}{\partial t}$ and can be neglected.

$$\frac{\partial V}{\partial t}+\frac{1}{\rho}\frac{\partial P}{\partial x}+g\frac{\partial z}{\partial x}+\frac{fV|V|}{2D}=0$$

4.4 Derivation for Pressure rise due to water hammer

a. Gradual closure of valve

Consider a long pipe carrying liquid and provided with a valve, which is closed gradually.

A= area of c/s of pipe

L= Length of pipe

V= Velocity of flow of water through pipe

t= time required to close the valve

P = Intensity of pressure wave produced
Mass of water= ρAL

The valve is closed gradually in time 't' seconds and hence the water is brought from initial velocity V to zero in time t seconds.

Retardation of water = $\frac{Change\ of\ velocity}{time} = \frac{V-0}{t} = \frac{V}{t}$
Retarding force = mass x retardation = $\rho AL\frac{V}{t}$
Force due to pressure wave = P A
Equating above two
$\rho AL\frac{V}{t} = P\ A$
$P = \frac{\rho LV}{t}$
Head of pressure (H) = $\frac{P}{\rho g} = \frac{\rho LV}{\rho gt}$
$H = \frac{LV}{gt}$

b. sudden (Instantaneous) closure of valve

(considering elasticity of pipe)
Consider an elastic pipe of length L, diameter D and thickness t.
P = increase of pressure due to water hammer
E= modulus of elasticity of pipes materials
1/m = Poisson's ratio for pipe materials

When the valve is closed suddenly, wave of high pressure of intensity P will be produced in the water. Due to this high pressure, circumferential and longitudinal stresses are produced in the pipe wall.

The value of longitudinal stress $(f_l) = \frac{PD}{4t}$ and circumferential stress $(f_c) = \frac{PD}{2t}$
Strain energy stored in pipe material per unit volume = $\frac{1}{2E}\left[f_l^2 + f_c^2 - \frac{2\, f_l\, f_c}{m}\right]$

$$= \frac{1}{2E}\left[\left(\frac{PD}{4t}\right)^2 + \left(\frac{PD}{2t}\right)^2 - \frac{2x\frac{PD}{4t}x\frac{PD}{2t}}{m}\right]$$

Taking m = 4

$$= \frac{1}{2E}\left[\frac{P^2D^2}{16t^2} + \frac{P^2D^2}{4t^2} - \frac{P^2D^2}{16t^2}\right] = \frac{P^2D^2}{8Et^2}$$

Total strain energy stored in pipe material = Strain energy per unit volume x total volume

$$= \frac{P^2 D^2}{8Et^2}\, \pi D\, t\, L = \frac{P^2 DL}{2Et}\frac{\pi D^2}{4} = \frac{P^2 ADL}{2Et}$$

Loss of KE of water = $\frac{1}{2}mV^2 = \frac{1}{2}\rho ALV^2$

Gain of strain energy in water = $\frac{1}{2}$ stress x strain x volume = $\frac{1}{2}P\frac{P}{K}AL = \frac{1}{2}\frac{P^2}{K}AL$

Loss of KE of water = Gain of strain energy in water + Strain energy stored in pipe material

$$\frac{1}{2}\rho ALV^2 = \frac{1}{2}\frac{P^2}{K}AL + \frac{P^2 ADL}{2Et}$$

$$\frac{\rho V^2}{2} = \frac{1}{2}\frac{P^2}{K} + \frac{P^2 D}{2Et} = \frac{P^2}{2}\left[\frac{1}{K} + \frac{D}{Et}\right]$$

$$P = V\sqrt{\frac{\rho}{\frac{1}{K}+\frac{D}{Et}}} \qquad \text{(I)}$$

This is the equation to compute rise in pressure due to sudden closure of valve for elastic pipes.

Velocity of pressure waves in case of elastic pipe (C) = $\sqrt{\frac{1}{\rho\left(\frac{1}{K}+\frac{D}{Et}\right)}}$

0r, C = $\sqrt{\frac{K}{\rho\left(1+\frac{DK}{Et}\right)}}$

Alternatively, the above expression in terms of C can be written as

$$P = \rho VC$$

Instantaneous closure of valve for rigid pipe

For rigid pipe, $E \to \infty$. So, equation (I) becomes

$P = V\sqrt{\rho K}$

$P = V\sqrt{\frac{K\rho^2}{\rho}} = V\sqrt{K/\rho}$

$\sqrt{\frac{K}{\rho}} = C'$= Velocity of pressure waves for rigid pipe

$P = \rho VC'$

With static pressure (P_s),

Total pressure propagated = $P+P_s$

4.5 Relief Device against water hammer (Surge Tank)

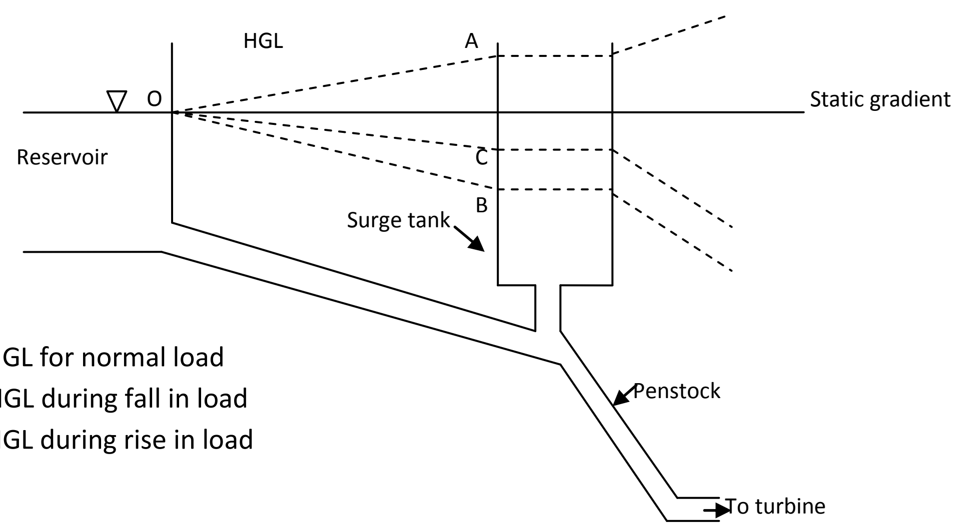
Surge

The oscillation of flow in pipelines, when compressibility effects are not important, is referred to as surge.

Surge tank

Surge tank is a vertical open tank, provided just upstream of penstock of hydroelectric plant for eliminating water hammer action. The following are the functions of Surge Tank.

- To reduce the effect of water hammering effect.(main function of surge tank is to intercept and dampen the high pressure wave and not to allow them in low pressure system i.e. power tunnel)
- To store water when the load is decreased (quick storage device)
- Temporary supply of water when the load on the turbine is suddenly increased (starting up phase).



OC: HGL for normal load
OA: HGL during fall in load
OB: HGL during rise in load

Fig. 4.6: Variation of HGL due to operation of turbine

Operation system of surge tank

Under normal operating condition, the flow through the pipeline is uniform and the pressure gradient is normal. The level of water in the surge tank is lower than that in the supply reservoir by an amount equal to frictional head loss. If the load on the turbine is suddenly decreased, the governing mechanism acts to decrease the flow accordingly. However, the rate of flow in the main pipeline cannot drop to the required quantity of supply. Water is able to flow into the surge tank. This flow causes the level in the surge tank to rise and that decelerates the flow from the supply reservoir. The usual water hammer phenomenon is limited to the short pipe length between the surge tank and the power plant.

When the power demand is suddenly increased, the valve is opened more and the surge tank supplies additional flow until the water in the supply main is accelerated. The water level in the surge tank is lowered; the difference of head along is increased thereby accelerating the flow until the flow equals that required by the turbines. In the absence of surge tank the drop in pressure at the turbines could be excessive at the time of sudden demand.

Types of surge tank

a. Simple cylindrical surge tank

The simple surge tank is of uniform cross section (cylindrical in shape) and opens to atmosphere, acting as a reservoir. The water flows into and out of the tank without appreciable loss of head. It is directly connected to the penstock and has an unrestricted opening into it and must be of adequate size so that it does not overflow.

b. Restricted orifice type

The orifice surge tank has a restricted opening at the base of tank. The restriction increases the head loss and consequently there is large acceleration and deceleration of water in the penstock. At small and rapid load changes, the orifice surge tank is not very effective in speed regulation.

c. Differential surge tank

Differential surge tank is connected at its bottom to the supply pipe through a small vertical pipe called riser. The riser is provided with a number of ports at its bottom and it reaches the height of the surge tank. When there is decrease in load on the turbine, the water rises fast in the riser and that provides a quick retarding effect. Similarly when the load increases, the water flows out first from the riser and that provides enough water for the turbine. The differential surge tank is rapid in its action and has the additional advantages of having low pressure rise and surge of limited magnitude.

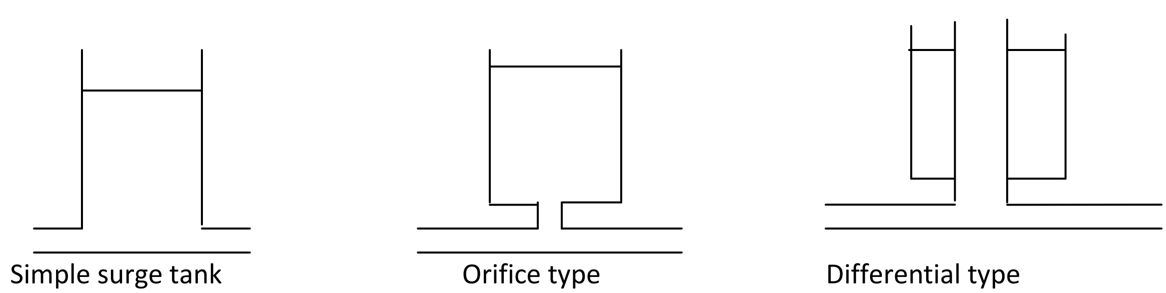


Fig. 4.7: Types of surge tank

Appendix

Pipe flow analysis

Formulae and hints for solving numerical

Continuity equation for steady flow:

Q = AV

For 2 points: $Q_1 = Q_2$

Or, $A_1V_1 = A_2V_2$

Bernoulli's (energy) equation:

For steady incompressible and frictionless flow:

Total energy head (E) = $\frac{P}{\gamma} + \frac{V^2}{2g} + Z$

For real fluid:

$E_1 = E_2$ + Loss

$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + loss$

If the pump supplies head (hp), then energy equation for two end points is

E_1+hp = E_2 + Loss

$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 + hp = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + loss$

If the turbine extracts head (ht), then energy equation for two end points is

E_1-ht = E_2 + Loss

$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 - ht = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + loss$

If h_f = head loss, slope/gradient of energy line = h_f/L

Momentum equation:

$$Net\ external\ force = Rate\ of\ chane\ of\ momentum$$
$$\sum F = \rho Q(V_2 - V_1)$$

Reynold number (Re):

$R_e = \frac{\rho VD}{\mu}$

Or, $R_e = \frac{VD}{\upsilon}$

Pipe flow

Laminar: Re<2000

Turbulent: Re>4000

Laminar flow analysis

Formulae for steady, incompressible laminar flow:

(Case of horizontal pipe)

a. Shear stress at any distance r from the center: $\tau = -\frac{\partial P}{\partial x}\frac{r}{2}$

Shear stress at pipe wall with radius R is: $\tau = -\frac{\partial P}{\partial x}.\frac{R}{2}$

b. Velocity at any distance r from the center: $u = -\frac{1}{4\mu}\frac{\partial P}{\partial x}(R^2 - r^2)$

c. Maximum velocity: $u_{max} = -\frac{1}{4\mu}\frac{\partial P}{\partial x}R^2$

d. Average velocity: $V = \frac{u_{max}}{2}$

or, $V = -\frac{1}{8\mu}\frac{\partial P}{\partial x}R^2$

0r, u_{max} = 2V

e. Local velocity in terms of maximum velocity: $u = u_{max}\left(1 - \frac{r^2}{R^2}\right)$

f. Pressure difference : $(P_1 - P_2) = \frac{32\mu VL}{D^2}$

g. Head loss in laminar flow (Hagen-Poiseuille's equation): $h_f = \frac{32\mu VL}{\rho g D^2}$

h. Power required $= \gamma Q h_f \ or \ Q(P_1 - P_2)$

i. At r = 0.707R, local velocity = mean velocity

For inclined pipe, replace $\frac{\partial P}{\partial x}$ by $\frac{\partial(P+\gamma z)}{\partial x} \ or \ \gamma\frac{\partial\left(\frac{P}{\gamma}+z\right)}{\partial x} = \gamma\frac{\partial h}{\partial x} \ where \ h = \frac{P}{\gamma} + z$ in eq. a to d.

Hints for solving numerical of laminar flow

If inclination of pipe is not mentioned, consider the case of horizontal pipe.

Computation of average velocity or other variables

- volume of flow and time given: compute Q = volume/time
- Q and A given: compute V = Q/A
- h_f given: Use $h_f = \frac{32\mu VL}{\rho g D^2}$ to compute V or other variable
- pressure drop(P_1-P_2) given: Use $P_1 - P_2 = \frac{32\mu VL}{D^2}$ to compute V or other variable
- P_1, V_1, Z_1, P_2, V_2, Z_2 known: Use Bernoulli's equation to compute h_f. Equate this h_f to $\frac{32\mu VL}{\rho g D^2}$ for laminar flow to compute average velocity V or μ.

Compute $R_e = \frac{\rho VD}{\mu} \ or \frac{VD}{\upsilon}$ and check whether Re<2000 for the condition of laminar flow is valid.

Another method of computing h_f for laminar flow: Compute Re and check whether this is less than 2000. If so, then compute f by using f = 64/Re. Then Compute h_f by using Darcy-Weisbach equation $h_f = \frac{fLV^2}{2gD}$.

Computation of power with efficiency (η) given: $Power = \frac{\gamma Q h_f}{\eta} or \ Q(P_1 - P_2)/\eta$

If static head (h_s) is given, take total head (H) = h_s+ h_f in above formula.

If τmax given: From $\tau max = -\frac{\partial P}{\partial x}.\frac{R}{2}$, compute $\frac{\partial P}{\partial x}$ and find average velocity V or μ by using $V = -\frac{1}{8\mu}\frac{\partial P}{\partial x}R^2$

<u>Criteria for smooth and rough boundary:</u>

k = Average roughness height (symbol e or ϵ is also used to represent average roughness height)
Thickness of laminar sub-layer: $\delta' = \frac{11.6\upsilon}{V*}$

Criteria 1:
$\frac{k}{\delta\prime}$<0.25: Smooth boundary
$\frac{k}{\delta\prime}$>6: rough boundary
0.25<$\frac{k}{\delta\prime}$<6: transition
Criteria2:
Roughness Reynold number (Rn) = $\frac{V_*K}{\upsilon}$ where $V_* = \sqrt{\frac{\tau_0}{\rho}}$
Rn<4: smooth
Rn>70: rough
4<Rn<70: transition

Criteria3:
a. $\frac{Re\sqrt{f}}{R/k} < 17$: smooth
b. $\frac{Re\sqrt{f}}{R/k} \geq 400$: rough
c. in between: transition

Criteria 4: Using Moody diagram
Plot the point for given Re and k/D. Check where this point lies (in the smooth pipe or rough pipe zone).

If k is known and V_* can be computed, use criteria 1 or 2
If k is not known, use criteria 3 or 4. For criteria 3, obtain f from resistance equations.

Turbulent flow analysis

a. Prandtl's equation for shear stress: $\tau = \rho l^2 \left(\frac{dV}{dy}\right)^2$

b. Prandtl's universal velocity distribution equation: $v = v_{max} + 2.5V_* \, ln\,(y/R\,)$

Shear velocity: $V_* = \sqrt{\frac{\tau_0}{\rho}}$

c. Karman- Prandtl velocity distribution equations

I. Local velocity

For smooth pipes: $\frac{v}{V_*} = 5.75log_{10}\left(\frac{V_* y}{\upsilon}\right) + 5.5$

For rough pipes: $\frac{v}{V_*} = 5.75log_{10}(y/k) + 8.5$

II. Average velocity

For smooth pipes: $\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{V_* R}{\upsilon}\right) + 1.75$

For rough pipe: $\frac{\bar{V}}{V_*} = 5.75log_{10}\left(\frac{R}{K}\right) + 4.75$

III. Relationship between local and mean velocity (common for both rough and smooth pipes): $\frac{v-\bar{V}}{V*} = 5.75log_{10}\left(\frac{y}{R}\right) + 3.75$

IV. Relationship between maximum velocity and average velocity: $\frac{Vmax-\bar{V}}{V*} = 3.75$

V. At y = 0.223R, local velocity = mean velocity

Alternative form of velocity distribution in turbulent flow (exponential or power)

(Easy to use for numerical)

$\frac{v}{V_{max}} = \left(\frac{y}{R}\right)^{1/n} = \left(1 - \frac{r}{R}\right)^{1/n}$

Average velocity V for this type of velocity distribution is

$V = \frac{2n^2}{(n+1)(2n+1)} V_{max}$

Hints for solving numerical of turbulent flow using exponential velocity profile

Exponent n varies from 5-10. Estimate n from Re using following table.

Re	$4x10^3$	10^5	10^6	$>2x10^6$
n	6	7	9	10

Knowing n, V or V_{max} can be computed by $V = \frac{2n^2}{(n+1)(2n+1)} V_{max}$

Empirical relationship between friction factor (f) and n

$n = \frac{1}{\sqrt{f}}$

If Re is not available, n = 7 can be assumed. (according to Blasius)

Knowing f, other variables can be computed, e.g.

Shear stress at the pipe wall, $\tau_0 = \frac{f\rho V^2}{8}$, $V_* = V\sqrt{f/8}$, $\delta' = \frac{11.6\upsilon}{V*}$ (In above equations, f can be obtained from Moody chart or resistance equations or f-n relationship)

Head loss due to friction and relationship of shear stress and shear velocity in turbulent flow:

a. Darcy-Weisbach equation: $h_f = \frac{fLV^2}{2gD}$

h_f = head loss due to friction (major loss)

Pressure drop $P_1 - P_2\ or \Delta P = \gamma h_f$ for horizontal pipe

b. Shear stress at the boundary in terms of f in turbulent flow: $\tau_0 = \frac{f\rho V^2}{8}$

b. Shear velocity in terms of f in turbulent flow: $V_* = V\sqrt{f/8}$

c. Shear stress (τ) at a certain distance r from center: $\tau = \tau_0\ (r/R)$

In terms of V_*, Shear stress at the boundary is

$$\tau_0 = \rho V_*^2$$

Relationship between V_{max}/V in terms of f

$\frac{v_{max}-V}{V*} = 3.75$ (I)

$V_* = V\sqrt{f/8}$ (II)

Substituting V_* in I and simplifying

$\frac{v_{max}}{V} = 1.33\sqrt{f} + 1$

Hints:

P_1-P_2, L, D, μ given: Use $P_1 - P_2 = \frac{4\tau_0 L}{D}$ to compute τ_0. Compute $V_* = \sqrt{\frac{\tau_0}{\rho}}$. If exponential velocity distribution is used, assume n (usually 7), find f using $n = \frac{1}{\sqrt{f}}$. Use $\tau_0 = \frac{f\rho V^2}{8}$ to compute V. Then compute V_{max} using $V = \frac{2n^2}{(n+1)(2n+1)} V_{max}$.

Relationship between Fiction factor (f) and Reynold's number (Re):

Equations for different cases

a. Laminar: $f = \frac{64}{Re}$ (for Re<2000)

b. Turbulent:

I. Smooth

Re 4000 to 10^5, Blasius equation: $f = \frac{0.316}{Re^{1/4}}$

For Re>10^5

$\frac{1}{\sqrt{f}} = 2.0log_{10}\left(Re\sqrt{f}\right) - 0.8$

Alternatively, for f = 4000 to $4\text{x}10^7$

$f = 0.0032 + \frac{0.221}{Re^{0.237}}$

II. Rough

$\frac{1}{\sqrt{f}} = 2.0log_{10}(R/k) + 1.74$

Or, $\frac{1}{\sqrt{f}} = 1.14 - 2.0log_{10}(K/D)$

Single equation to compute f for all conditions: Colebrook-White equation

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{k}{3.7D} + \frac{2.51}{Re\sqrt{f}}\right)$

For smooth pipe (k/D negligible). So, this equation reduces to

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{2.51}{Re\sqrt{f}}\right) = 2.0log_{10}\left(Re\sqrt{f}\right) - 0.8$

For rough pipe ($\frac{2.51}{Re\sqrt{f}}$ negligible). So, this equation reduces to

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{k}{3.7D}\right) == -2.0log_{10}\left(\frac{K}{D}\right) + 1.14$

Method to compute friction factor (f)

a. Use Moody Chart

b. Solve Colebrook-White equation by trial and error or by programming

$\frac{1}{\sqrt{f}} = -2.0log_{10}\left(\frac{k}{3.7D} + \frac{2.51}{Re\sqrt{f}}\right)$

c. Swamee and Jain formula for computing f

$f = \frac{1.325}{\left[ln\left(\frac{K}{3.7D} + \frac{5.74}{Re^{0.9}}\right)\right]^2}$

This equation is valid for $10^{-6} \leq K/D \leq 10^{-2}$ and $5000 \leq Re \leq 10^{8}$.

d. Use equations for different conditions (e.g. Nikuradse, Blasius or theoretical)

For computing range of head loss, take lower and upper limit of k value, and obtain lower and upper value of f from Moody chart or by using equations.

While using Moody chart for smooth pipe, take the curve marked smooth pipe.

For given f and Re, K/D can also be read from Moody chart. If D is known, K can be computed and type of pipe can be identified.

Minor losses in pipe flow:

a. Head loss due to sudden expansion: $h_e = \frac{(V_1 - V_2)^2}{2g}$

or, $\frac{V_1^2}{2g}\left(1 - \frac{A_1}{A_2}\right)^2 = K\frac{V_1^2}{2g}$

b. Head loss due to sudden contraction: $h_c = \frac{V_2^2}{2g}\left[\frac{1}{C_c} - 1\right]^2 = \frac{KV_2^2}{2g}$

If value of C_c is not given, take K = 0.5

c. Entry loss: $h_i = \frac{KV^2}{2g}$

Type of entrance	K
Sharp cornered	0.5
Rounded	0.2
Bell mouthed	0.05

d. Exit loss: $h_0 = \frac{V^2}{2g}$

e. Loss of head due to bend in pipe: $h_b = \frac{kV^2}{2g}$

Sharp 90^0 bend, K = 1.2

Sharp 180^0 bend, K = 2.2

f. Loss of head in various pipe fittings: $h_p = \frac{kV^2}{2g}$

Fittings	K
Standard T (branch flow)	1.8
Standard T (line flow)	0.4
90^0 elbow (long radius)	0.6
90^0 elbow (short radius)	1.5
Gate valve (fully open)	0.2

g. Loss of head due to an obstruction in a pipe: $ho = \frac{V^2}{2g}\left[\frac{A}{C_c(A-a)} - 1\right]^2$

Aging of pipe

$k_t = k_0 + \alpha t$ where K_o = initial roughness height, K_t = roughness after time t

Pipe network

a. Pipes in series:

Total Head loss = sum of head loss in all the sections

Discharge = const. in all sections

b. Pipes in parallel:

Loss of head is equal in each parallel pipe.

Discharge in the main line = Sum of discharge in each of the parallels pipes

c. Pipe network analysis by Hardy-Cross method

$$h_f = \frac{fLV^2}{2gD} = \frac{fL}{2gD}\frac{Q^2}{A^2} = \frac{fL}{2gD}\frac{Q^2}{\left(\frac{\pi}{4}D^2\right)^2} = \frac{8fL}{\pi^2 gD^5}Q^2 = \frac{fL}{12.1D^5}Q^2$$

Expressing h_f in the form of $h_f = rQ^n$

Where $r = \frac{fL}{12.1D^5}$ and n =2

.

correction to be applied

$$\Delta Q = -\frac{\sum rQ^2}{2\sum rQ}$$

Unsteady flow in pipes (Water hammer)

The closure of valve is said to be gradual when t>2L/c
The closure of valve is said to be sudden when t< 2L/c
where t= time required to close the valve, L = length of the pipe and c= velocity of the pressure wave.
Time required by pressure wave to travel from the valve to tank and back to valve (t) = 2L/c
Time period of oscillation = 4L/C

a. Pressure rise due to gradual closure of valve: $P = \frac{\rho LV}{t}$ or head = $= \frac{LV}{gt}$
b. Pressure rise due to sudden (instantaneous) closure of valve
For rigid pipe: $P = V\sqrt{\rho K}$ or, $P = \rho VC'$ where $C' = \sqrt{\frac{K}{\rho}}$ for rigid pipe and K = Bulk modulus
For elastic pipe: $P = V\sqrt{\frac{\rho}{\frac{1}{K}+\frac{D}{Et}}}$
Alternatively P for elastic pipe can be computed by using equation $P = \rho VC$ where $C = \sqrt{\frac{K}{\rho\left(1+\frac{DK}{Et}\right)}}$ for elastic pipe.

Total pressure = static pressure+water hammer pressure where static pressure = γH

Circumferential stress $= \frac{PD}{2t}$ and longitudinal stress $= \frac{PD}{4t}$
where P = total pressure (water hammer+static)

Unsteady flow equations
Continuity: $\frac{1}{\rho}\frac{\partial P}{\partial t} + c^2\frac{\partial V}{\partial x} = 0$
Momentum (Euler): $\frac{\partial V}{\partial t} + \frac{1}{\rho}\frac{\partial P}{\partial x} + g\frac{\partial z}{\partial x} + \frac{fV|V|}{2D} = 0$

Chapter 5: Basics of open channel flow

5.1 Introduction

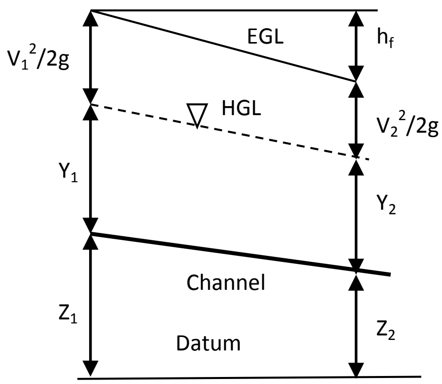
Open Channel flow/Free surface flow

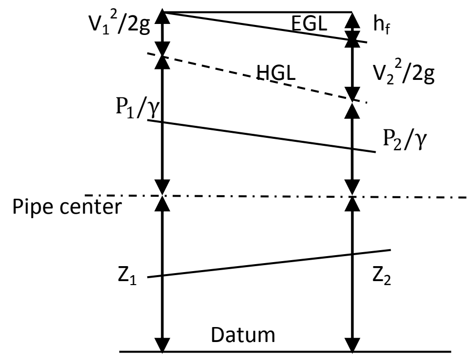
An open channel is a conduit for flow which has a free surface, which is subjected to atmospheric pressure. The free surface is actually an interface between two fluids of different density and will have constant pressure (atmospheric pressure). In case of moving fluid, the motion is caused by gravity, and the pressure distribution within the fluid is generally hydrostatic. Open channel flows are almost always turbulent and unaffected by surface tension.

Examples: flow in streams, rivers, canals, partially filled sewers

Difference between open channel and pipe flow

Aspect	**Open channel**	**Pipe flow**
Condition	Uncovered, have free surface, atmospheric pressure at free surface	Covered, no free surface
Cross-section	Any shape, e.g. rectangular, Parabolic, triangular, trapezoidal, circular, irregular	Generally circular cross section
Cause of flow	Flow due to gravity	Flow due to pressure
Surface roughness	Varies between wide limits, varies place to place	depends upon the material of the pipe
Velocity distribution	The maximum velocity at little distance below the water surface. The shape of the velocity profile dependent on the channel roughness.	The maximum velocity at the center of the flow and reducing to zero at the pipe wall velocity distribution symmetrical about the pipe axis
Piezometric head	Z+y where y = depth of flow; HGL coincides with the water surface	$Z+P/\gamma$, where p= pressure in pipe. HGL does not coincide with water surface
Surface tension	Negligible	Dominant for small diameter





Open channel flow Pipe flow

5.2 Types of flows

Basic variables: Velocity (V) or discharge (Q), flow depth (y)

a. Based on time criterion

I. Steady flow: flow propertiesat any point do not change with time (dv/dt= 0, dy/dt =0, dQ/dt =0), e.g. flow of water through a channel at constant rate
II. Unsteady flow:flow propertiesat any point do not change with time (dv/dt≠ 0, dy/dt≠ 0, dQ/dt≠ 0), e.g. flood flow

b. Based on space criterion

I. Uniform flow: flow properties do not vary with distance (dv/dx= 0, dy/dx =0) e.g. flow through a channel of constant cross-section
II. Non-uniform or varied flow: flow properties vary with distance (dv/dx ≠ 0, dy/dx ≠ 0) e.g. flow through natural channel

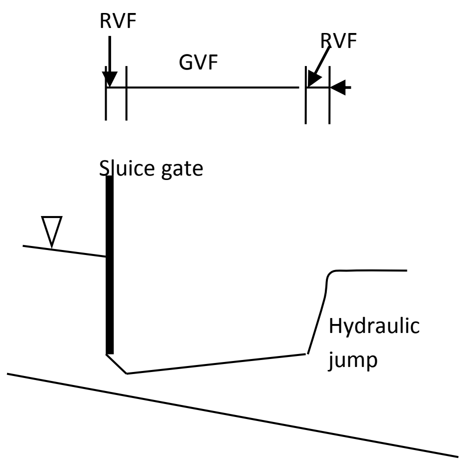
c. Based on both time and space criteria

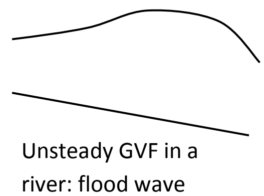
- Steady uniform flow, e.g. flow through a channel of constant cross-section at a constant rate
- Unsteady uniform flow,e.g. flow through a channel of constant cross-section at varying rate
- Steady non-uniform flow, e.g. flow behind a dam
- Unsteady non-uniform flow, e.g. flood flow

Steady uniform flow is the fundamental type of flow treated in open channel hydraulics. Unsteady uniform flow is rare. For such flow, depth of flow varies with time, but remains constant with distance, which is conceptually possible, but not possible theoretically.

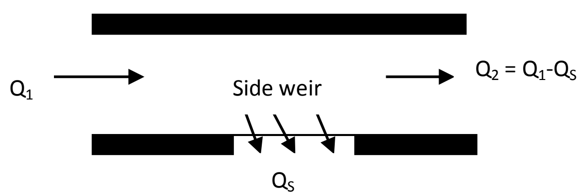
Classification of non-uniform flow

The non-uniform flow is classified as either gradually varied flow or rapidly varied flow. If the depth of flow changes rather slowly with distance, the flow is said to be gradually varied flow GVF, Examples of steady GVF:flow profile behind the dam, flow profile d/s of sluice gate from the vena-contract. Example of unsteady GVF: the passage of flood wave in a river. If the depth of flow changes rapidly over a relatively short distance, the flow is said to be rapidly varied flow, e.g. hydraulic jump, flow over a weir, flow below the sluice gate up to vena-contracta, hydraulic drop, surge (sudden change in flow that increases or decreases the depth), bore (surge of tidal origin).





Spatially varied flow (SVF): If there is addition to or withdrawal of flow from the system, the resulting varied flow is known as spatially varied flow.
Example of steady SVF: flow over a side weir
Example of unsteady SVF: flow due to rainfall, flow through gutters



Side weir (Plan)

c. Depending on the effect of viscosity relative to inertia
Reynolds number (Re) = Ratio of inertia force to viscous force
$Re = \frac{\rho VL}{\mu}$ or $\frac{VL}{\upsilon}$where ρ =density of fluid, V = mean velocity of flow, L = Characteristic length, μ = dynamic viscosity, υ = kinematic viscosity
For open channel flow, hydraulic radius (R) is taken as characteristic length.
R = cross-sectional area/wetted perimeter
I. Laminar flow: strong viscous force, very small velocity, motion of fluid in layers, occurs for Reynold no.<500 in open channel flow, e.g. flow through smooth pipe having low velocity, groundwater flow, a thin film of liquid flowing down an inclined/vertical plane

II. Turbulent flow: weak viscous force, irregular motion of fluid, occurs for Reynold no.≥ 2000in open channel flow, e.g. flow through river, high velocity flow in a conduit of large size
[For pipe flow: Re<2000: laminar and Re>4000: turbulent
For open channel, R= D/4 (D= dia. of pipe)=L. So the value is ¼ of pipe flow. But for turbulent flow, the limit is usually taken as 2000.)

d. Depending on the effect of gravity relative to inertia

Froude number (F_r) = Ratio of inertia force to gravity force
$F_r = \frac{V}{\sqrt{gL}}$ where V = mean velocity of flow, g = acceleration due to gravity, L = Characteristic length.
For open channel, L =hydraulic depth (D) where D = Cross-sectional area/width of free surface = A/T

Based on F_r, the flow is classified into three types.
I. Subcritical (tranquil): $F_r<1$, low velocity, large depth, flow controlled by downstream conditions, occurs on flat streams
II. Critical: $F_r = 1$
III. Supercritical (shooting or rapid): $F_r>1$, high velocity, small depth, flow controlled by upstream conditions, occurs on steep streams

5.3 Classification of open channel

1. On the basis of cross- sectional form of the channel
    a. Natural channel- irregular shape, developed in a natural way and not significantly improved by humans, e.g. River, streams, estuaries etc.
    b. Artificial channel- developed by human efforts, regular shape, e.g. irrigation and power canals, navigation channels, gutters and drainage ditches etc.
       Rectangular channel, trapezoidal channel, triangular channels etc
2. According to shape of channel
    a. Prismatic channel –constant cross-section and bottom slope along the length
    b. Non- prismatic channel –Varying cross-section or bottom slope along the length
3. According to type of boundary
    a. Rigid boundary channel –Not deformable boundary, e.g. lined canal, sewers and non-erodible canals
    b. Mobile boundary channel – Boundaries undergo deformation due to the continuous erosion and deposition due to the flow.

Types of artificial open channel

- Canal: long channel of mild slope
- Flume: channel of wood, metal, concrete or masonry, built above the ground surface to convey flow across depression
- Chute: steep slope channel
- Drop: Steep slope channel, in which the change in elevation is effected in a short distance

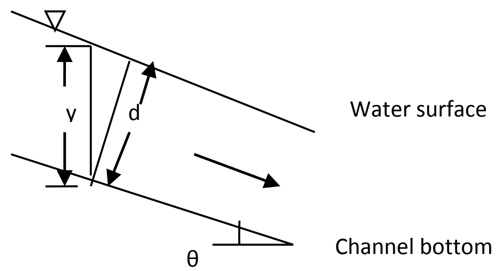
- Culvert flowing partially full: covered channel used to convey a flow under highways, railroad embankments or runways

5.4 Geometric properties of channel section

Channel section: cross-section of the channel taken normal to the direction of flow

The following are the geometric elements of basic importance.

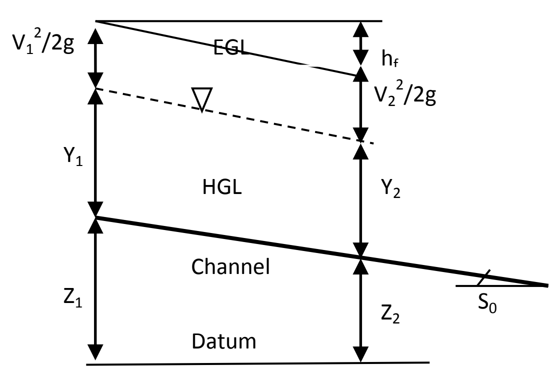
a) Depth of flow (y): Vertical distance from the lowest point of the channel section to the water surface
b) Depth of flow section (d): Depth of flow normal to the direction of flow.



c) Stage: elevation of water surface relative to datum
d) Top width (T): Width of channel section at the water surface.
e) Flow area or water area (A) : cross sectional area of the flow normal to the direction of flow.
f) Wetted perimeter (p): the length of the line which is the interface between the fluid and the channel boundary.
g) Hydraulic radius (R): the ratio of the flow area (A) to the wetted perimeter (P).
   R=A/P
h) Hydraulic depth (D): Ratio of the flow area (A) to the top width (T)
   D=A/T
i) Section factor for critical flow computation (Z): product of the flow area and the square root of the hydraulic depth.

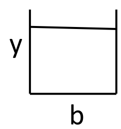
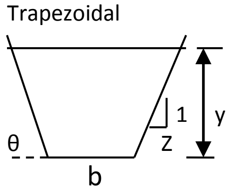
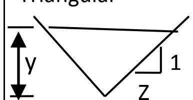
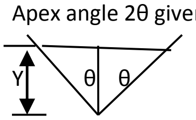
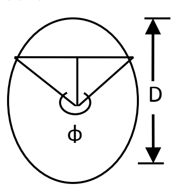
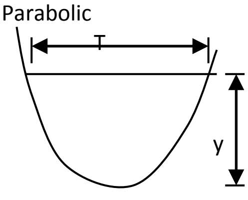
$$Z = A\sqrt{D}$$

j) Channel slope or bed slope or bottom slope or longitudinal slope (S_0) : Inclination of the channel bed
k) Water surface slope (S_w): slope made by water surface
l) Energy slope (S_f): The energy gradient line represents the sum of pressure head, velocity head and elevation head at any point along the channel. The angle of inclination of energy gradient line with the horizontal is called energy slope.

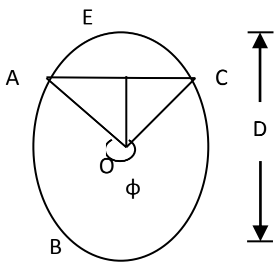


$S_f = h_f/L$, where h_f = Head loss between 1 and 2, L = Length of reach

Channel section geometric elements of channels of common shape

Channel	A	P	R	T	D
Rectangular 	by	$b+2y$	$\frac{by}{b+2y}$	b	y
Trapezoidal  $tan\theta = 1/z$ In terms of θ	$(b+Zy)y$ $\left(b+\frac{y}{tan\theta}\right)y$	$b+2y\sqrt{1+Z^2}$ $\left(b+\frac{2y}{sin\theta}\right)$	$\frac{(b+Zy)y}{b+2y\sqrt{1+Z^2}}$ $\frac{\left(b+\frac{y}{tan\theta}\right)y}{\left(b+\frac{2y}{sin\theta}\right)}$	$b+2Zy$ $\left(b+\frac{2y}{tan\theta}\right)$	$\frac{(b+Zy)y}{b+2Zy}$ $\frac{\left(b+\frac{y}{tan\theta}\right)y}{\left(b+\frac{2y}{tan\theta}\right)}$
Triangular  Apex angle 2θ given 	Zy^2 $y^2 tan\theta$	$2y\sqrt{1+Z^2}$ $\frac{2y}{cos\theta}$	$\frac{Zy}{2\sqrt{1+Z^2}}$ $\frac{ysin\theta}{2}$	$2Zy$ $2ytan\theta$	$y/2$ $y/2$
Circular 	$\frac{D^2}{8}(\varphi - Sin\varphi)$	$\phi D/2$	$\frac{D}{4}(1 - Sin\varphi/\varphi)$	$DSin\frac{\varphi}{2}$	$\frac{D}{8}\frac{(\varphi - Sin\varphi)}{Sin\frac{\varphi}{2}}$
Parabolic 	$\frac{2}{3}Ty$	$T + \frac{8}{3}\frac{y^2}{T}$	$\frac{2T^2y}{3T^2+8y^2}$	$\frac{3}{2}\frac{A}{y}$	$\frac{2}{3}y$

Circular section



Flow depth above center

Flow area (A) = Area of circle- (Area of sector AOCE-Area of triangle AOC)

$$= \frac{\pi}{4} D^2 - \left[\frac{2\pi - \varphi}{2\pi} \frac{\pi}{4} D^2 - \frac{1}{2} D \, sin\left(\frac{2\pi - \varphi}{2}\right) \frac{D}{2} cos\left(\frac{2\pi - \varphi}{2}\right)\right]$$

$$= \frac{D^2}{8} (\varphi + \sin(2\pi - \varphi))$$

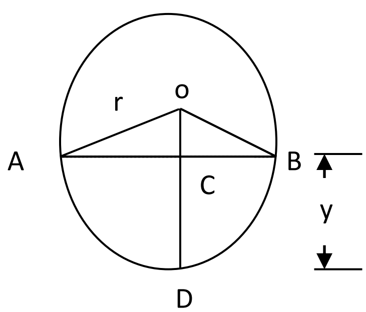
$$= \frac{D^2}{8} (\varphi - Sin\varphi)$$

Wetted perimeter (P) = $\frac{\pi D}{2\pi} \varphi = \frac{\varphi D}{2}$

(P is πD for angle 2π)

Top width (T) = AC = $D \, sin\left(\frac{2\pi - \varphi}{2}\right)$ = $D sin \frac{\varphi}{2}$

Flow depth below center



θ = angle AOC = angle COB, radius of circle = r, y = depth of flow

Wetted perimeter (p) = $\frac{2\pi r}{2\pi} 2\theta$ = 2rθ

Wetted area (A) = Area ADBA =Area of sector OADBO-Area of ΔOAB

$$= \frac{\pi r^2}{2\pi} 2\theta - \frac{1}{2} . 2r sin\theta . r cos\theta = r^2 \left(\theta - \frac{1}{2} sin 2\theta\right)$$

In terms of dia. (D)

$$= \frac{D^2}{8} (2\theta - sin 2\theta)$$

Chapter 6: Uniform flow in open channel

Conditions of uniform flow in a prismatic channel

Uniform flow in channel occurs under following conditions.

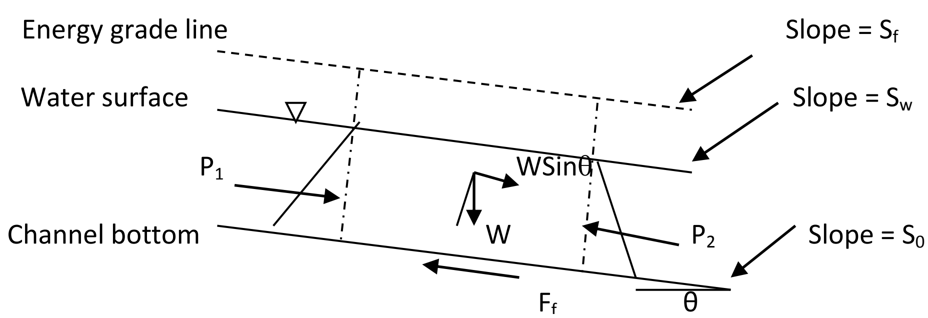
I. The depth, flow area, velocity and discharge at every section of the channel reach are constant.
II. The total energy line, water surface and the channel bottom are all parallel, or $S_f = S_w = S_o = S$
where S_f = energy line slope, S_w = water surface slope and S_o = channel bottom slope.

Although truly uniform flow seldom occurs in nature, the concept of uniform flow is central to understanding the solution of most problems in open channel hydraulics.

In general, uniform flow can occur only in very long, straight prismatic channels where a terminal velocity of flow can be achieved, i.e. the head loss due to turbulent flow is exactly balanced by the reduction in potential energy due to the uniform decrease in the elevation of the bottom of the channel.

6.1 Shear stress on the boundary

When water flows in an open channels resistance is encountered by the water as it flows downstream.



Let W = weight of water contained, L = Length of channel reach, A =cross-sectional area, θ = angle of inclination of channel bottom with the horizontal, P_1 and P_2 = pressure force at 1 and 2, V_1 and V_2 = Velocity at 1 and 2, τ_o = Boundary shear stress acting over the area of contact, γ = Specific weight of water

Applying momentum equation

$$P_1 - P_2 + W\sin\theta - F_f = \rho Q(V_2 - V_1)$$

For uniform flow, $P_1 = P_2$ and $V_1 = V_2$,and $S_o = S_w = S_f = S = \tan\theta = \sin\theta$

With these simplifications,

$$W\sin\theta = F_f$$

$\gamma ALS = \tau_0 PL$

This shows that the force of resistance is equal to the component of gravity force.

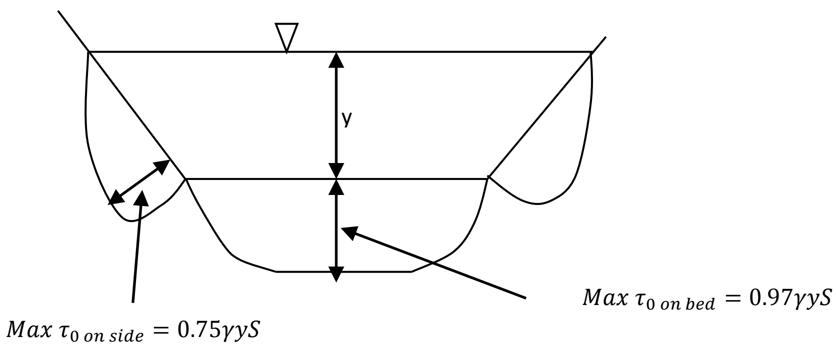
$\tau_0 = \dfrac{\gamma AS}{P}$

$\tau_0 = \gamma RS$

Where R = hydraulic radius

Shear stress distribution

- Average shear stress on the boundary, $\tau_0 = \gamma RS$
- Average shear stress on the boundary is not uniformly distributed due to the turbulence and secondary circulation.
- $\tau_0 = 0$ at the intersection of water surface with the boundary and at the corners of the boundary.
- Local maxima of τ_0 occurs on the bed and sides.



Example: Shear stress distribution for b/y = 4 and z:1 = 1.5:1

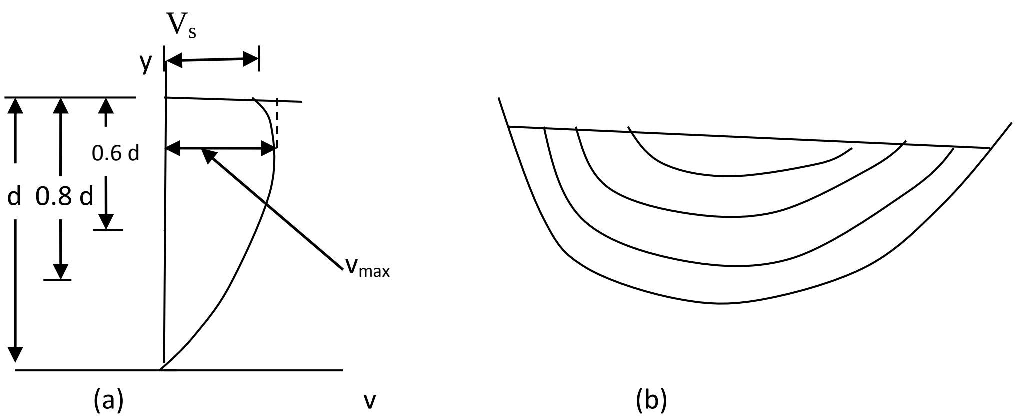
6.2 Velocity distribution in open channels

The velocity of flow at any channel section is not uniformly distributed due to the presence of a free surface and the frictional resistance along the channel boundary. The velocity is zero at the solid boundary and gradually increases with the increase in distance from the boundary. The maximum velocity occurs at some distance below the free surface. At the free surface, the velocity is less than the maximum value due to the secondary circulation and air resistance. Secondary circulation is due to the turbulence in the flow from the channel boundary to the channel center.

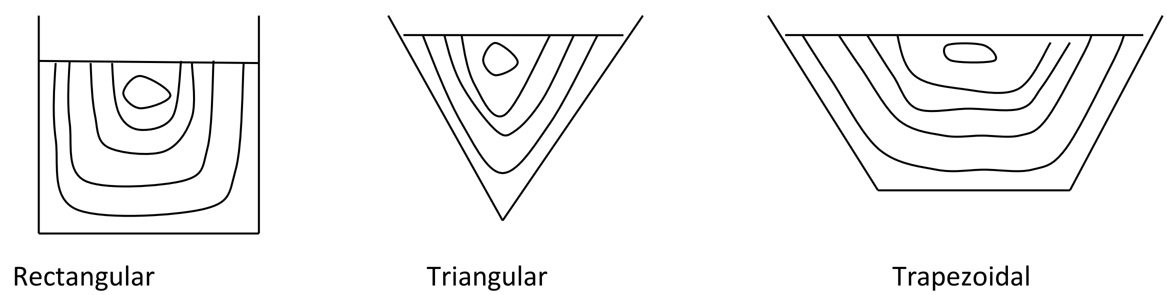
Some of the features of velocity distribution in open channel are as follows.

- As we go away from the bank to the center section, the point of maximum velocity shifts upward and at the central section, the maximum velocity occurs almost at the free surface.

- The velocity profile in vertical is logarithmic up to the maximum velocity point.
- Maximum velocity usually occurs below the free surface at a distance of 0.05 to 0.25 of the depth of flow. The higher value is applicable to section closer to the bank.
- For deep narrow channel, the location of maximum velocity point will be much lower than for a wider channel of same depth. In a broad, rapid and shallow stream or in a very smooth channel, the maximum velocity may often be found at the free surface.
- The mean velocity (V_{av}) = velocity at 0.6d where d= depth from the free surface
  or, V_{av} = (V at 0.2d + V at 0.8d)/2
- The mean velocity in the vertical = 0.85 to 0.95 times the surface velocity. The smaller value applies to the shallower streams.



Velocity distribution in open channel in (a) vertical direction, (b) across section



Velocity profiles for different cross-sections

Velocity profile for laminar flow in open channel

The velocity distribution in laminar flow in open channel is parabolic.

$v = \frac{\gamma}{\mu} S \left(Dy - \frac{y^2}{2} \right)$ where D = Depth of channel, y = flow depth, S = slope of channel

Velocity profile for turbulent flow in open channel

For turbulent flow, velocity distribution can be obtained from Prandtl's the mixing length hypothesis (as in pipe flow)

$\frac{v}{V_*} = 5.75 log_{10}\,(y/y')$ where v = velocity at any point in vertical, V_* = shear velocity, y = flow depth, y' = small depth from wall at which v is zero.
The velocity distribution in turbulent flow is logarithmic.

Simplifications (as in pipe flow)
For smooth surfaces: $\frac{v}{V_*} = 5.75 log_{10}\left(\frac{V_* y}{\upsilon}\right) + 5.5$
For rough surfaces: $\frac{v}{V_*} = 5.75 log_{10}(y/k) + 8.5$

6.3 Velocity distribution coefficients

The computation of kinetic energy at a flow section is usually based on the average velocity thereby ignoring the effect of non-uniformity of velocity distribution across the section. As a result of non-uniform distribution of velocities over a channel section, the velocity head of flow is generally greater than the value computed by using the expression $V^2/2g$, where V is the mean velocity. The true velocity is expressed as $\alpha V^2/2g$, where α is known as energy coefficient (energy correction factor) or Coriolis coefficient (first proposed by G. Coriolis).

The non-uniform distribution of velocities also affects the computation of momentum in open channel flow. The momentum of fluid passing through a channel section per unit time is expressed as $\beta\rho QV$, where β is known as momentum coefficients (momentum correction factor) or Boussinesq coefficient (first proposed by J. Boussinesq); ρ is density; Q is discharge, V is mean velocity.

Both coefficients are equal to or greater than unity. In general for straight prismatic channel α varies from 1.03 to 1.36 and β varies from 1.01 to 1.12. For straight and prismatic channel, the effect of non-uniform velocity distribution on the computed velocity head and momentum is small. Therefore, these coefficients are often assumed to be unity.

Expression for mean velocity (V)

Let v is the velocity of flowing fluid at any point through any elementary area dA.
$V = \frac{\int v dA}{A}$
where A = cross-sectional area of channel

Expression for velocity distribution coefficients

Energy coefficient (α)

Mass of fluid flowing per unit time = $\rho v dA$

Kinetic energy of fluid $=\frac{(\rho v dA)v^2}{2} = \frac{\rho}{2} v^3 dA$

Total kinetic energy possessed by the fluid across the entire cross section$=\int \frac{\rho}{2} v^3 dA$

The kinetic energy of the flowing fluid in terms of the mean velocity 'V' of flow$= \alpha \frac{(\rho VA)V^2}{2} = \alpha \frac{\rho}{2} V^3 A$

Equating above two expressions,

$$\alpha = \frac{\int v^3 dA}{V^3 A}$$

Momentum coefficient (β)

Mass of fluid flowing per unit time = $\rho v dA$
Momentum of the fluid $=(\rho v dA)v$ = $\rho v^2 dA$
Total momentum by the fluid across the entire cross section = $\int \rho v^2\, dA$
Momentum of fluid in terms of mean velocity V = $\beta \rho V^2 A$
Equating above two expressions,

$$\beta = \frac{\int v^2 dA}{V^2 A}$$

Expressions for rectangular channel of width b and water depth (y_0)

$$V = \frac{\int v dA}{A} = \frac{\int_0^{y_0} v b dy}{b y_0} = \frac{1}{y_0}\int_0^{y_0} v dy$$

$$\alpha = \frac{\int v^3 dA}{V^3 A} = \frac{\int_0^{y_0} v^3 b dy}{V^3 b y_0} = \frac{\int_0^{y_0} v^3 dy}{V^3 y_0}$$

$$\beta = \frac{\int v^2 dA}{V^2 A} = \frac{\int_0^{y_0} v^2 b dy}{V^2 b y_0} = \frac{\int_0^{y_0} v^2 dy}{V^2 y_0}$$

If v is known or expressed as an algebraic function of y, then α and β can be obtained by integration. If the relationship for v-y does not exist, the result of actual measurement or graphical method is used to determine the coefficients.

Graphical approach for given velocity distribution curve

$V = \frac{1}{y_0}\int_0^{y_0} v dy = \frac{Area\ under\ v{-}y\ curve}{y_0}$

$\alpha = \frac{\int_0^{y_0} v^3 dy}{V^3 y_0} = \frac{Area\ under\ v^3{-}y\ curve}{V^3 y_0}$

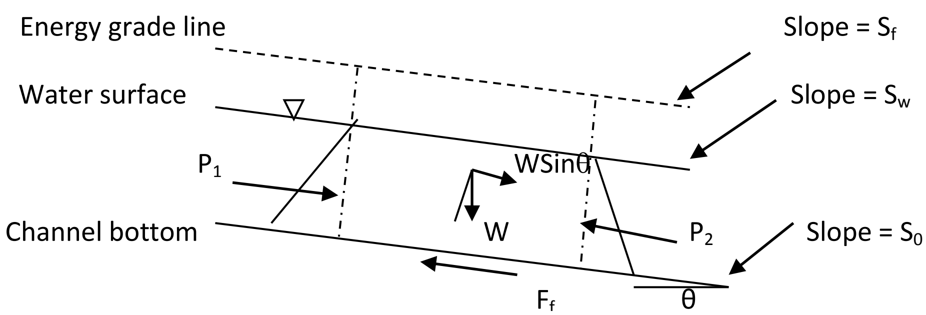
$\beta = \frac{\int_0^{y_0} v^2 dy}{V^2 y_0} = \frac{Area\ under\ v^2{-}y\ curve}{V^2 y_0}$

6.4 Computation of velocity in uniform flow: Resistance equations

a. Chezy equation

The Chezy's formula is based on the following two assumptions:

- The force resisting the flow per unit area is proportional to the square of the velocity.
- The component of gravity force causing the flow is equal to the force of resistance



Let W= weight of water contained, L = Length of channel reach, A =cross-sectional area, θ = angle of inclination of channel bottom with the horizontal, P_1 and P_2 = pressure force at 1 and 2, V_1 and V_2 = Velocity at 1 and 2, τ_o = Boundary shear stress acting over the area of contact, γ = Specific weight of water

Applying momentum equation

$$P_1 - P_2 + Wsin\theta - F_f = \rho Q(V_2 - V_1)$$

For uniform flow, $P_1 = P_2$ and $V_1 = V_2$, and $S_o = S_w = S_f = S = \tan\theta = \sin\theta$

With these simplifications

$Wsin\theta = F_f$

Resistive force $(F_f) \propto Surface\ area\ x\ V^2$

Resistive force $(F_f) = KPLV^2$ where K = coefficient, P = Wetted perimeter

$$\gamma ALS = KPLV^2$$

$$V = \sqrt{\frac{\gamma}{K}}\sqrt{RS}$$

$$V = C\sqrt{RS}$$

Where R = A/P = Hydraulic radius and C= $\sqrt{\frac{\gamma}{K}}$ = flow resistance factor known as Chezy's coefficient

This is Chezy's equation for uniform flow.

(b) Manning’s formula

The Manning’s formula for uniform flow in open channel flow is

$$V = \frac{1}{n} R^{2/3} {S_f}^{1/2}$$

where n = Manning’s roughness coefficient or rugosity Coefficient, R= Hydraulic radius, S_f = slope of the energy line, which is equal to bed slope (S) for uniform flow. For uniform flow,

$$V = \frac{1}{n} R^{2/3} S^{1/2}$$

Manning’s formula is empirical in nature. Due to its simplicity and satisfactory results it gives to practical applications, the Manning's formula has become the most widely used formula for all uniform flow computation for open channels.

(c) Darcy-Weisbach equation

For pipe flow, Darcy-Weisbach equation for head loss is

$$h_f = \frac{fLV^2}{2gD}$$

where h_f = head loss, f = friction factor, L = Length of pipe, V = Mean velocity, D = Diameter of pipe.

An open channel can be assumed to be a pipe cut into half.

Hydraulic radius (R) = $\frac{A}{P} = \frac{\pi D^2/4}{\pi D} = \frac{D}{4}$ or D = 4R

$$h_f = \frac{fLV^2}{2g.4R}$$

$$V = \sqrt{\frac{8gRh_f}{fL}}$$

$$V = \sqrt{\frac{8g}{f}} \sqrt{R\frac{h_f}{L}} = \sqrt{\frac{8g}{f}} \sqrt{RS}$$

$$V = C\sqrt{RS}$$

where $= \sqrt{\frac{8g}{f}}$, h_f/L = S = Slope of energy line = Bed slope for uniform flow

Relationship between Chezy C, Manning n and Darcy-Weisbach f

Chezy formula: $V = C\sqrt{RS}$ (a)

Manning's formula: $V = \frac{1}{n} R^{2/3} S^{1/2}$ (b)

Darcy-Weisbach formula: $V = \sqrt{\frac{8g}{f}} \sqrt{RS}$ (c)

From equations a and b

$$C\sqrt{RS} = \frac{1}{n} R^{2/3} S^{1/2}$$

$$C = \frac{1}{n} R^{1/6} \qquad \text{(d)}$$

From equations a and c, $C = \sqrt{\frac{8g}{f}}$ (e)

From equations d and e

$$\frac{1}{n} R^{1/6} = \sqrt{\frac{8g}{f}}$$

$$f = \frac{8gn^2}{R^{1/3}}$$

Equations for f

In terms of roughness, Reynold number (Rn) = $\frac{V_* K}{\upsilon}$ where V_* = shear velocity, k = roughness height and υ = kinematic viscosity

$V_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{\rho g R S}{\rho}} = \sqrt{gRS}$

Rn<4: smooth

Rn>60: rough

4<Rn<60: transition

f values

Re = Reynold number = $\frac{VD}{\upsilon} = \frac{4VR}{\upsilon}$ where D= 4R and R = Hydraulic radius

a. Laminar: $f = \frac{64}{Re}$

b. Turbulent

I. Smooth

Re 4000 to 10^5, Blasius equation: $f = \frac{0.316}{Re^{1/4}}$

For Re>10^5 : $\frac{1}{\sqrt{f}} = 2.0 log_{10}\left(Re\sqrt{f}\right) - 0.8$

II. Rough

$\frac{1}{\sqrt{f}} = -2.0 log_{10}(K/4R) + 1.14$

III. Transition

$\frac{1}{\sqrt{f}} = -2.0 log_{10}\left(\frac{k}{14.84R} + \frac{2.51}{Re\sqrt{f}}\right)$

Approximate solution of above equation by Jain

$$\frac{1}{\sqrt{f}} = 1.14 - 2.0log_{10}\left(\frac{k}{4R} + \frac{21.25}{Re^{0.9}}\right)$$

Factors affecting Manning's n

The value of n is highly variable and depends on a number of factors. These factors are

i) Surfaces roughness:
- Roughness of wetted perimeter producing retarding effect on flow
- Low n for fine grains, high n for coarse grain

ii) Vegetation:
- Vegetation is a type of roughness,
- Increases n.

iii) Channel irregularity:
- Irregularities in wetted perimeter and cross section, size and shape along the channel length
- Gradual or uniform variations will not appreciably affect the value of n, while abrupt changes increase n.

iv) Channel alignment:
- Smooth curve with large radius will give a relatively low value of n whereas sharp curve with meandering will increase n.

v) Silting and scouring:
- Silting may decrease n.
- Scouring may increase n and depends on the nature of the material deposited.

vi) Obstruction:
- Presence of logs jams, bridge piers and the like tends to increase n.

vii) Shape and size of channel
- An increase in hydraulic radius may either increase or decrease n depending on the condition of channels.

viii) Stage and discharge:
- n decreases with increase in stage and discharge
- n increases at high stages if the banks are rough and grassy.
- For flood plain, n depends on surface condition or vegetation. (usually larger than channel)

ix) Seasonal change:
- Owing to the seasonal growth of aquatic plants, grass, weeds and trees in the channel or on the banks, the value of n may increase in the growing season and diminish in the dormant season

x) Suspended materials and bed load:
- The suspended materials and the bed load whether moving or not moving would consume energy and cause head loss or increase roughness.

Determination of Manning's n

In applying the Manning's formula, the greatest difficulty lies in the determination of the roughness coefficient n. Manning's n cannot be measured directly and there is no exact method of selecting the n value.

Following approaches can be adopted for selecting n.
a. Using references, e.g. Table of Manning's by Chow
b. Determination of n by velocity or discharge measurement
c. To understand the factors that affect n and to acquire a basic knowledge of the problem and narrow the wide range of guesswork
Recognizing several primary factors affecting the roughness coefficient **Cowan** developed a procedure for estimating the value of n
$n = (n_0+n_1+n_2+n_3+n_4)m_5$
Where n_0= n value for a straight, uniform and smooth channel
n_1= value for the effect of surface irregularities
n_2= Value for the variation in shape and size of the channel
n_3= Value for obstruction
n_4= Value for vegetation and flow conditions
m_5= correction factor for meandering of channels
d. Grain size analysis
Strickler formula

$$n = \frac{d_{50}^{1/6}}{21.1}$$

where d_{50} is in meters and represents the particle size in which 50% of the bed material is finer. For mixtures of bed material with considerable coarse-grained size, the modified equation is

$$n = \frac{d_{90}^{1/6}}{26}$$

where d_{90} is in meters and represents the particle size in which 90% of the bed material is finer.

Equivalent roughness or composite roughness (n)

Equivalent roughness is the weighted average value of roughness coefficients for different parts of perimeter.
Divide the water area into N parts having wetted perimeter P_1, P_2, ……., P_N and roughness coefficients n_1, n_2, ………..,n_N.
Approaches for estimating equivalent roughness
a. Considering constant mean velocities in all sub-areas
Horton and Einstein equation

$$n = \left[\frac{\left(\sum_{i=1}^{N} P_i n_i^{3/2}\right)}{P}\right]^{2/3}$$

b. Considering total resistance force = sum of sub-area resistance forces

Pavlovskij equation

$$n = \left[\frac{\left(\sum_{i=1}^{N} P_i n_i^2\right)}{P}\right]^{1/2}$$

c. Considering total discharge = sum of sub-area discharges

Lotter equation

$$n = \frac{PR^{5/3}}{\sum_{i=1}^{N}\left(\frac{P_i R_i^{5/3}}{n_i}\right)}$$

Values of n for some cases

SN	Surface characteristics	Range of n
1) Lined channel with straight alignment		
a	Metal	0.011-0.030 (Normal: 0.012)
b	Cement	0.010-0.015
c	Wood	0.010-0.018
d	Concrete	0.011-0.027 (Normal: 0.013)
e	Masonry	0.017-0.035
f	Asphalt	0.013-0.016
g	Vegetal lining	0.03-0.5
2) Natural channels		
a	Minor streams Streams in plain Mountain streams	 0.025-0.015 0.03-0.07
b	Flood plains	0.025-0.2
c	Major streams	0.025-0.1

6.5 Terms used in the computation of uniform flow

Conveyance and section factor

Using Manning's equation

$$V = \frac{1}{n} R^{2/3} S^{1/2}$$

$$Q = AV = \frac{1}{n} AR^{2/3} S^{1/2} = K\sqrt{S}$$

where $K = \frac{1}{n} AR^{2/3}$ conveyance of the channel section, which is a measure of carrying capacity of the channel section.

$$K = \frac{Q}{\sqrt{S}}$$

The expression $AR^{2/3}$ is called the section factor for uniform flow computation.

$$AR^{2/3} = nk$$

$$AR^{2/3} = \frac{nQ}{\sqrt{S}}$$

Normal depth

For given condition of n, Q and S, there is only one possible depth for maintaining a uniform flow. This depth is called normal depth. Thus the normal depth is defined as the depth of flow at which a given discharge flows as uniform flow in a given channel.

Normal discharge

When n and s are known for at a channel section, there is only one discharge for maintaining a uniform flow through the section. This discharge is normal discharge.

Hydraulic exponent

The relationship between conveyance (K) and depth (y) is given by

$$K^2 = Cy^N$$

where C is constant and N is hydraulic exponent for computation of uniform flow.

Taking logarithm on both sides and differentiating w.r.t. y,

2lnK = lnC+Nlny

$\frac{d(lnK)}{dy} = \frac{N}{2y}$ (a)

Using Manning's equation

$K = \frac{1}{n} AR^{2/3}$

Taking logarithm on both sides and differentiating w.r.t. y,

$lnK = \ln\left(\frac{1}{n}\right) + lnA + \frac{2}{3} lnR$

$\frac{d(lnK)}{dy} = \frac{1}{A}\frac{dA}{dy} + \frac{2}{3R}\frac{d(A/P)}{dy}$

dA/dy = T

$\frac{d(lnK)}{dy} = \frac{T}{A} + \frac{2}{3R}\left[\frac{1}{P}\frac{dA}{dy} - \frac{A}{P^2}\frac{dP}{dy}\right]$

$\frac{d(lnK)}{dy} = \frac{T}{A} + \frac{2}{3A/P}\left[\frac{1}{P}T - \frac{A}{P^2}\frac{dP}{dy}\right]$

$\frac{d(lnK)}{dy} = \frac{5T}{3A} - \frac{2}{3P}\frac{dP}{dy}$ (b)

Equating a and b

$\frac{N}{2y} = \frac{5T}{3A} - \frac{2}{3P}\frac{dP}{dy}$

$N = \frac{2y}{3A}\left[5T - 2R\frac{dP}{dy}\right]$

6.6 Most (Hydraulically) efficient channel section

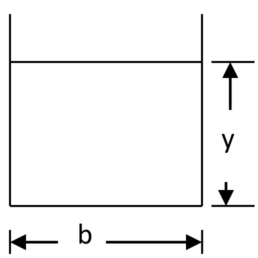
Using Manning's equation

$$Q = \frac{1}{n} A R^{2/3} S^{1/2} = \frac{A^{5/3} S^{1/2}}{n P^{2/3}}$$

A channel section is said to be most efficient when it can pass a maximum discharge for a given flow area (A), roughness coefficient (n) and slope (S). For given values of n and S, the discharge is maximum for a given area of cross-section when wetted perimeter (P) is minimum. This channel is also called the best section. Of all the various possible open channel cross sections for a given area of flow, the semi circular channel is the most efficient due to the smallest perimeter. However, there are practical limitations to using such cross-section, such as difficulty in construction, steepest stable slope etc

The most economical channel is that channel which requires minimum cost for construction. For minimum wetted perimeter, the cross-sectional area is also minimum. So, the cost of excavation and the lining is minimum for hydraulically efficient channel. However, in actual practice, the banks are higher than the full supply level and the most efficient channel does not necessarily result in minimum excavation or minimum length of lining.

a. Most efficient rectangular section



Bottom width = b and depth of flow = y

Flow area (A) = by = const.

Wetted perimeter (P) = b + 2y = $\frac{A}{y} + 2y$

For P to be minimum with A as constant, dP/dy = 0

Differentiating w.r.t. y

$$\frac{dP}{dy} = -\frac{A}{y^2} + 2 = 0$$

$$A = 2y^2$$

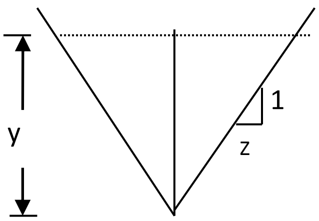
$$by = 2y^2$$

b = 2y

Hydraulic radius $(R) = \frac{A}{P} = \frac{by}{b+2y} = \frac{2y.y}{2y+2y} = \frac{y}{2}$

Thus for most efficient rectangular channel, the width should two times the depth of flow and hydraulic radius should be half the depth of flow. In other words, the depth of flow is equal to half bottom width i.e. when the channel section is half-square, best section is obtained.

b) Most efficient triangular section



Side slopes of triangular channel = z:1 and depth of flow = y

Flow area (A) = zy^2 (a)

Wetted perimeter (P) = $2y\sqrt{z^2+1}$ (b)

From (a), $y = \sqrt{\frac{A}{Z}}$

Substituting y in (b)

$$P = 2\sqrt{\frac{A}{Z}}\sqrt{z^2+1}$$

$$P^2 = \frac{4(z^2+1)}{z}A = \left(4z + \frac{4}{z}\right)A$$

For the section to be most efficient with given area, dP/dz = 0

Differentiating w.r.t. z

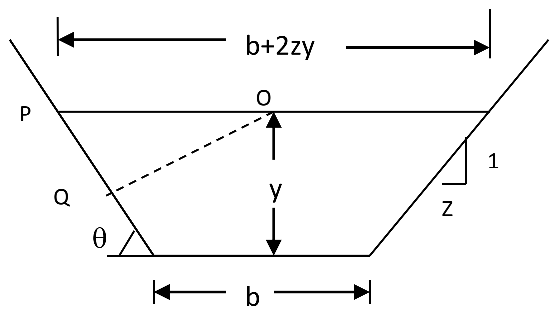
$$2P\frac{dP}{dz} = \left(4 - \frac{4}{z^2}\right)A = 0$$

Z = 1

Hence a triangular channel section will be most economical when each of its side slopes is 1:1 and sloping sides makes an angles of 45^0 with vertical. In other words, a triangular section with a central angle of 90^0 is the best.

Hydraulic radius $(R) = \frac{A}{P} = \frac{zy^2}{2y\sqrt{z^2+1}} = \frac{1.y^2}{2y\sqrt{1^2+1}} = \frac{y}{2\sqrt{2}}$

c) Most efficient trapezoidal section



b = Base width of the channel, y= Depth of flow, θ = Angle made by sides with horizontal and Side slope = z:1

Flow area (A) = (b+zy)y (a)

Wetted perimeter (P) = $b + 2y\sqrt{z^2+1}$ (b)

From a), $b = \frac{A}{y} - zy$

Substituting b in (b)

$$P = \frac{A}{y} - zy + 2y\sqrt{z^2+1}$$

I. Varying y with Keeping A and z fixed

For economical section dP/dy = 0

Differentiating w.r.t. y

$$\frac{dP}{dy} = -\frac{A}{y^2} - z + 2\sqrt{z^2+1} = 0$$

$$A = \left(2\sqrt{z^2+1} - z\right)y^2$$

$$(b+zy)y = \left(2\sqrt{z^2+1} - z\right)y^2$$

$$b + 2zy = 2\left(\sqrt{z^2+1}\right)y$$

i.e. top width (T) = twice the length of sloping sides

Hydraulic radius $(R) = \frac{A}{P} = \frac{(b+zy)y}{b+2y\sqrt{z^2+1}} = \frac{(b+zy)y}{b+b+2zy} = \frac{y}{2}$

Consider O as center and OQ as perpendicular to the side. Referring to triangle OPQ

$$OQ = OPsin\theta = \frac{T}{2}\frac{1}{\sqrt{z^2+1}} = \frac{2\left(\sqrt{z^2+1}\right)y}{2}\frac{1}{\sqrt{z^2+1}} = y$$

Thus the proportions of the hydraulically efficient trapezoidal channel section will be such that a semi-circle can be inscribed in it.

II. Varying z with Keeping A and y fixed

For economical section dP/dz = 0
Differentiating w.r.t. z

$$\frac{dP}{dz} = -y + \frac{2y}{2\left(\sqrt{z^2+1}\right)}2z = 0$$

$$2z = \left(\sqrt{z^2+1}\right)$$

$$z = \frac{1}{\sqrt{3}}$$

Or, $\theta = 60^0$
The most efficient section in this case is one half of hexagon.

III. Varying y and z keeping A fixed

For efficient section, $b + 2zy = 2\left(\sqrt{z^2+1}\right)y$
Substituting the value of side slope, $z = \frac{1}{\sqrt{3}}$ in above equation

$$b + \frac{2}{\sqrt{3}}y = 2\left(\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}\right)y = \frac{2y}{\sqrt{3}}$$

$$A = (b + zy)y = \left(\frac{2y}{\sqrt{3}} + \frac{y}{\sqrt{3}}\right)y = \sqrt{3}y^2$$

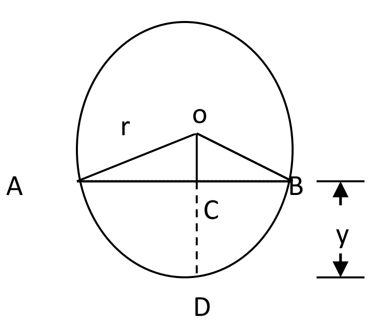
Wetted perimeter (P) = $b + 2y\sqrt{z^2+1} = \frac{2y}{\sqrt{3}} + 2y\left(\sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + 1}\right) = \frac{6y}{\sqrt{3}} = 3b$

R = A/P = y/2

Thus for a side slope of 60^0, the length of sloping side is equal to the base width of the trapezoidal section. Hence the most economical section of a trapezoidal channel will be a half hexagon.

d) Most efficient circular section

In case of circular channels running partially full, both the wetted perimeter and wetted area vary with depth of flow. The most efficient section is designed both for condition of maximum mean velocity and maximum discharge.



θ = angle AOC = angle COB, radius of circle = r, y = depth of flow

Wetted perimeter (p) = $\frac{2\pi r}{2\pi}2\theta$ = 2rθ

Wetted area (A) = Area ADBA =Area of sector OADBO-Area of ΔOAB

$$=\frac{\pi r^2}{2\pi}2\theta-\frac{1}{2}.2rsin\theta.rcos\theta=r^2\left(\theta-\frac{1}{2}sin2\theta\right)$$

I. Condition for maximum velocity

Using Chezy's formula

$V=C\sqrt{RS}=C\sqrt{(A/P)S}$

The flow velocity will have a maximum value when A/P is maximum.

$\frac{d(A/P)}{d\theta}$=0   (Where A and P both are function of θ)

$$\frac{P\frac{dA}{d\theta}-A\frac{dP}{d\theta}}{P^2}=0$$

$$P\frac{dA}{d\theta}-A\frac{dP}{d\theta}=0$$

$$\frac{dA}{d\theta}=\frac{d\left[r^2\left(\theta-\frac{1}{2}sin2\theta\right)\right]}{d\theta}=r^2(1-cos2\theta)$$

$$\frac{dP}{d\theta}=2r$$

Substituting the values of P, A, dA/dθ and dP/dθ

$$2r\theta.r^2(1-cos2\theta)-r^2\left(\theta-\frac{1}{2}sin2\theta\right).2r=0$$

$$-\theta cos2\theta+\frac{sin2\theta}{2}$$

$$tan2\theta=2\theta$$

Solving by trial and error

2θ = 257.5^0

θ = 128.75^0

Depth of flow (y) = OD-OC=r-rcosθ= r(1-cosθ)=r(1-cos128.75) = 1.62r =0.81D where D = diameter of circular channel

Thus maximum velocity occurs when the depth of flow is 0.81 times diameter of the circular channel.

Hydraulic radius $(R)=\frac{A}{P}=\frac{r^2\left(\theta-\frac{1}{2}sin2\theta\right)}{2r\theta}=\frac{r\left(\theta-\frac{1}{2}sin2\theta\right)}{2\theta}$

θ = 128.75^0 =128.75$\frac{\pi}{180}$=2.247 rad

$$R=\frac{r\left(2.247-\frac{1}{2}sin257.5^0\right)}{2x2.247}=0.611r=0.3D$$

Thus for maximum mean velocity in a channel of circular section hydraulic radius equals 0.3 times the channel diameter.

Instead of Chezy, if Manning's equation is used,

$$V = \frac{1}{n}R^{2/3}S^{1/2} = \frac{1}{n}\left(\frac{A}{P}\right)^{2/3} S^{1/2}$$

The flow velocity will have a maximum value when A/P is maximum. i.e. $\frac{d(A/P)}{d\theta}$ =0. So the derivation is same as above.

II. Condition for maximum discharge

Using chezy's equation

$$Q = AC\sqrt{RS} = AC\sqrt{\frac{A}{P}S} = C\sqrt{\frac{A^3}{P}S}$$

Maximum discharge is obtained when $\frac{A^3}{P}$ is maximum.

$$\frac{d(A^3/P)}{d\theta} = 0$$

$$\frac{Px3A^2\frac{dA}{d\theta} - A^3\frac{dP}{d\theta}}{P^2} = 0$$

$$3P\frac{dA}{d\theta} - A\frac{dP}{d\theta} = 0$$

Substituting the values of P, A, dA/dθ and dp/dθ

$$3.2r\theta.r^2(1 - cos2\theta) - r^2\left(\theta - \frac{1}{2}sin2\theta\right).2r = 0$$

Dividing by $2r^3$

$$4\theta - 6\theta cos2\theta + sin2\theta = 0$$

By trial and error, $2\theta = 308^0$

$\theta = 154^0$

Depth of flow (y)= OD-OC=r-rcosθ= r(1-cosθ)=r(1-cos154) = 1.898r =0.95D where D = diameter of circular channel

Thus maximum discharge occurs when the depth of flow is 0.95 times diameter of the circular channel.

Hydraulic radius $(R) = \frac{A}{P} = \frac{r^2\left(\theta - \frac{1}{2}sin2\theta\right)}{2r\theta} = \frac{r\left(\theta - \frac{1}{2}sin2\theta\right)}{2\theta}$

$\theta = 154^0 = 154\frac{\pi}{180}$=2.687 rad

$$R = \frac{r\left(2.687 - \frac{1}{2}sin308^0\right)}{2x2.687} = 0.573r = 0.29D$$

Thus for maximum discharge in a channel of circular section hydraulic radius equals 0.29 times the channel diameter.

Instead of Chezy, if Manning's formula is used,

$$Q = \frac{1}{n}AR^{2/3}S^{1/2} = \frac{1}{n}\frac{A^{5/3}}{P^{2/3}}S^{1/2} = \frac{1}{n}\left(\frac{A^5}{P^2}\right)^{1/3} S^{1/2}$$

For maximum discharge $\frac{A^5}{P^2}$ should be maximum.

$$\frac{d\left(\frac{A^5}{P^2}\right)}{d\theta} = 0$$

$$\frac{P^2 x 5A^4 \frac{dA}{d\theta} - A^5 x 2P \frac{dP}{d\theta}}{P^4} = 0$$

$$5P\frac{dA}{d\theta} - 2A\frac{dP}{d\theta} = 0$$

Substituting the values of P, A, dA/dθ and dp/dθ

$$5.2r\theta . r^2(1 - cos2\theta) - 2r^2\left(\theta - \frac{1}{2}sin2\theta\right).2r = 0$$

Dividing by $2r^3$

$$3\theta - 5\theta cos2\theta + sin2\theta = 0$$

By trial and error, $2\theta = 302.36^0$

θ = 151.18^0

Depth of flow (y) = OD-OC=r-rcosθ= r(1-cosθ)=r(1-cos151.18) = 1.876r =0.938D where D = diameter of circular channel

Hydraulic radius $(R) = \frac{A}{P} = \frac{r^2\left(\theta - \frac{1}{2}sin2\theta\right)}{2r\theta} = \frac{r\left(\theta - \frac{1}{2}sin2\theta\right)}{2\theta} = 0.58r$

For maximum discharge in a circular channel, y = 0.938D and R = 0.29D.

Summary of properties for the most efficient channel

I. Rectangular

Condition: B = 2y

A = By =$2y^2$

P = B+2y = 4y

R= A/P =y/2

II. Triangular

Condition: Z:1 = 1:1

A= Zy^2 = y^2

$P = \ 2y\sqrt{z^2 + 1} = 2\sqrt{2}y$

R= A/P = $y/2\sqrt{2}$ = 0.2536y

III. Trapezoidal

a) If Z:1 is given

Condition: $b + 2zy = 2\left(\sqrt{z^2 + 1}\right)y$

b) If Z:1 is not given

Condition: Z:1 = $\frac{1}{\sqrt{3}}: 1$, $b = 2y/\sqrt{3}$, P = 3b

A = (b+zy)y = $\left(\frac{2y}{\sqrt{3}} + \frac{y}{\sqrt{3}}\right) y = \sqrt{3}y^2 = 1.732y^2$
R = A/P = y/2

IV. Circular
Condition (using Chezy):
Hydraulic radius (R) = 0.3D, y = 0.81D (for maximum velocity)
Hydraulic radius (R) = 0.29D, y = 0.95D (for maximum discharge)
Condition (using Manning):
Hydraulic radius (R) = 0.3D, y = 0.81D (for maximum velocity)
Hydraulic radius (R) = 0.29D, y = 0.938D (for maximum discharge)

Semi-circular section
$A = \frac{\pi D^2}{8}$, $P = \frac{\pi D}{2}$, $R = \frac{D}{4}$, y = D

6.8 Types of flow problems in uniform flow and their solutions

The basic variables in uniform flow are: discharge (Q), mean velocity of flow (V), normal depth (y_n), roughness coefficient (n), channel slope (S), and the geometric elements. When any four of the variables are given, the remaining two unknowns can be determined by the two equations.

The following are the six types of problems and the method of solutions.
a. To find Q given y, n, S and geometric elements
Solution: use $Q = \frac{1}{n} AR^{2/3}S^{1/2}$
b. To find V given y, n, S and geometric elements
Solution: use $V = \frac{1}{n} R^{2/3}S^{1/2}$
c. To find n given Q, y, S and geometric elements
Solution: use $Q = \frac{1}{n} AR^{2/3}S^{1/2}$ to compute n.
d. To find S given Q, y, n and geometric elements
Solution: use $Q = \frac{1}{n} AR^{2/3}S^{1/2}$ to compute S.
e. To find geometric elements given Q, y, n and S
Solution: Apply Manning's equation and solve for unknown. Solve by trial and error if no direct solution exists.
f. To find normal depth (y_n) given Q, n, S and geometric elements
Solution: Solve by trial and error

Trial and error method to find normal depth (y_n)

- Compute A, P and R in terms of y_n.
- From Manning's equation

$$Q = \frac{1}{n} A R^{2/3} S^{1/2}$$

or, $AR^{2/3} = \frac{nQ}{\sqrt{S}}$

Substitute the values of variables.

- Find y by graphical or trial and error or by programming.

a. Trial and error approach (analytical approach)

Assume different values of y, compute LHS of equation. Perform computations until LHS becomes almost equal to the numerical value in the RHS of the given. The value of y for which LHS is almost equal to RHS is the normal depth.

b. Graphical approach

For different values of y, compute corresponding section factor $AR^{2/3}$. Plot a graph of $AR^{2/3}$ versus y. For the computed value of section factor obtained from the expression $\frac{nQ}{\sqrt{S}}$, find the corresponding y value from the graph. This gives the normal depth.

Trial and error approach to find geometric elements

- Similar to the normal depth approach

Wide rectangular channel

b is very large compared to y.

A = by, P = b+2y

$R = \frac{A}{P} = \frac{by}{b+2y} = \frac{by}{b\left(1+\frac{2y}{b}\right)}$

For b very large, y/b is very small, which can be neglected.

So R = y for wide rectangular channel.

Deep gorge

y is very large compared to b.

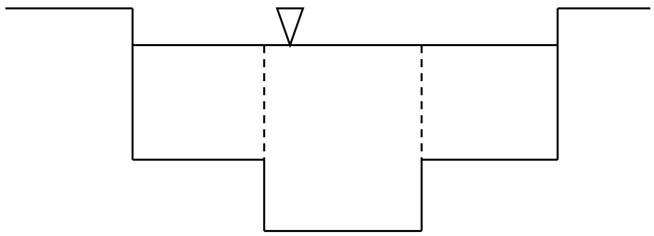
A = by, P = b+2y

$R = \frac{A}{P} = \frac{by}{b+2y} = \frac{by}{y\left(2+\frac{b}{y}\right)}$

For y very large, b/y is very small, which can be neglected.

So R = y/2 for deep gorge.

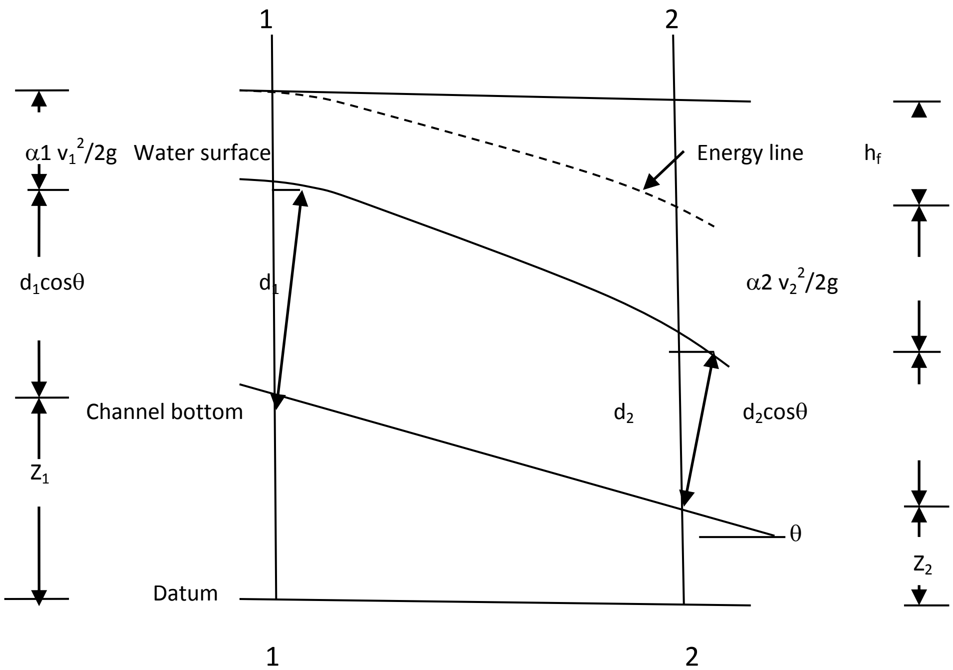
Compound channel



- Divide the channel into sub-sections and compute discharge for each sub-section. Then sum up.

Chapter 7: Energy and momentum principle in open channel

7.1 Energy principle (Bernoulli's equation) in open channel flow



The total energy of water at any point through a channel section is equal to the sum of elevation head, pressure head and velocity head.

$$H = Z + d\cos\theta + \alpha\frac{V^2}{2g}$$

H = total head, Z = Datum head, d = depth of water, θ = slope angle of channel bottom, $V^2/2g$ = velocity head, α = velocity distribution coefficient

According to the principle of conservation of energy for two sections 1 and 2,

Total energy head at section 1 = total energy head at section 2 + loss of energy between two sections

$$Z_1 + d_1\cos\theta + \alpha_1\frac{{V_1}^2}{2g} = Z_2 + d_2\cos\theta + \alpha_2\frac{{V_2}^2}{2g} + h_f$$

For a channel of small slope, $\theta \approx 0$ and $\cos\theta = 1$ and $d \approx y$

$$Z_1 + y_1 + \alpha_1\frac{{V_1}^2}{2g} = Z_2 + y_2 + \alpha_2\frac{{V_2}^2}{2g} + h_f$$

where y_1 and y_2 = water depth at 1 and 2

This equation is known as Bernoulli's equation.

When $\alpha_1 = \alpha_2 = 1$ and $h_f = 0$, the Bernoulli's equation is

$$Z_1 + y_1 + \frac{{V_1}^2}{2g} = Z_2 + y_2 + \frac{{V_2}^2}{2g}$$

7.2 Specific energy

The Specific energy in a channel section is defined as the energy per unit weight of water measured with respect to the channel bottom. Thus Specific energy E at any section is the sum of the depth of flow at that section and the velocity head. With Z = 0, the total energy is the specific energy (E), which is given by

$$E = d\cos\theta + \alpha\frac{V^2}{2g}$$

For a channel of small slope, $\theta \approx 0$, and taking $\alpha = 1$,

$$E = y + \frac{V^2}{2g}$$

This shows that the specific energy is equal to the sum of water depth and velocity head. i.e. specific energy = static energy + kinetic energy

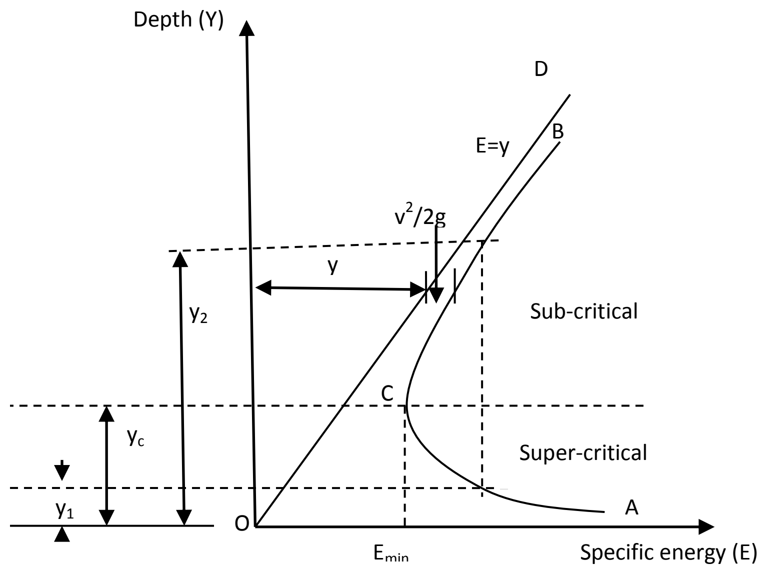
With V = Q/A, above expression becomes

$$E = y + \frac{Q^2}{2gA^2}$$

This shows that for given channel section and discharge Q, the specific energy is a function of the depth of the flow only.

Specific energy curve/diagram

Specific energy curve is the graphical plot of depth of flow versus specific energy for a given channel section and discharge.



Specific energy diagram for discharge (Q)

Features of specific energy diagram

a. At any point on the specific energy curve, the ordinate represents the depth and the abscissa represents the specific energy. , which is equal to the sum of pressure head and velocity head.
b. The specific energy curve has two limbs AC (upper) and BC (lower).
c. The plot of pressure head (E=y) is straight line, which passes through the origin O and has an angle of inclination equal to 45^0. The plot of velocity head is parabolic. Specific energy curve is obtained by summing pressure head and velocity head.
d. The specific energy curve is asymptotic to the horizontal axis for small value of y and asymptotic to the 45^0 line for high value of y.

$E = y + \frac{V^2}{2g}$, Taking V=q/y where q = discharge per unit width
$y \to 0, V = \frac{q}{y} \to \infty$. So $E \to \infty$ (asymptotic to X-axis)
$y \to \infty, V = \frac{q}{y} \to 0$. So $E \to \infty$ (asymptotic to E=y line)

e. Specific energy curve shows that at a certain point (point C in the figure), the specific energy is minimum. The condition of minimum specific energy is called critical flow condition and the corresponding depth is called critical depth (y_c). The velocity at critical depth is called critical velocity (V_c).
f. The curve shows that, for a given specific energy other than minimum specific energy, there are two possible depth of flow, e.g. low stage $y_1 < y_c$ and high stage $y_2 > y_c$. These two depths for the same specific energy are called alternative depths.
g. The zone above the critical state is the sub-critical zone and below it is supercritical zone.
$y>y_c$: subcritical ($V<V_C$, $F_r<1$)
$y=y_c$: critical ($V=V_C$, $F_r=1$)
$y<y_c$: supercritical ($V>V_C$, $F_r>1$)
h. Specific energy increases with the increase of depth for subcritical flow, whereas it increases with the decrease of depth for supercritical flow.
h. The specific energy curve shifts towards right for increase in discharge.

7.3 Criterion for critical state for given discharge

Specific energy (E) is given by

$$E = y + \frac{Q^2}{2gA^2}$$

Differentiating w.r.t. y (keeping Q constant)

$$\frac{dE}{dy} = 1 - \frac{Q^2}{gA^3}\frac{dA}{dy}$$

dE/dy = 0

dA/dy = top width (T)

(For an element of thickness dy and top width T, elementary area (dA) = Tdy)

$$1 - \frac{Q^2T}{gA^3} = 0$$

$$\frac{Q^2T}{gA^3} = 1$$

or, $\frac{Q^2}{g} = \frac{A^3}{T}$

This is the condition for minimum specific energy for constant discharge.

Other form,

$$\frac{Q^2}{A^2g} = \frac{A}{T}$$

Substituting V= Q/A and Hydraulic depth (D) = A/T

$$\frac{V^2}{g} = D$$

$$\frac{V^2}{2g} = \frac{D}{2}$$

This is the criterion for critical flow, which shows that the velocity head is equal to half the hydraulic depth. The above equation can also be written as

$$\frac{V}{\sqrt{gD}} = 1$$

Fr = 1

where $\frac{V}{\sqrt{gD}} = Fr$

This shows that Froude number (Fr) is unity for critical flow. In other words, specific energy is minimum at critical state.

7.4 Computation of critical depth

Given data: Q and geometry

- Express cross-section area and top width in terms of given geometric data and critical depth (y_c).
- Substitute values of Q, A and T in equation $\frac{Q^2}{g} = \frac{A^3}{T}$ and solve for y_c. (In case of no direct solution, solve by trial and error method).

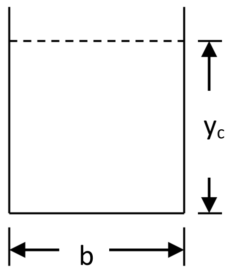
Computation of critical velocity (V_c)

$$V_c = \sqrt{gD} = \sqrt{gA/T}$$

Computation of minimum specific energy $(E_c) = y_c + \frac{V_c^2}{2g}$

Critical depth computation of some channel sections

a. Rectangular



Flow area (A) = b y_c and top width (T) = b
y_c = critical depth of flow
At critical state,

$$\frac{Q^2}{g} = \frac{A^3}{T}$$

$$\frac{Q^2}{g} = \frac{(by_c)^3}{b}$$

$$y_c = \left(\frac{Q^2}{b^2 g}\right)^{1/3}$$

or, $y_c = \left(\frac{q^2}{g}\right)^{1/3}$ where q = discharge per unit width

Critical velocity $(V_c) = \sqrt{gD} = \sqrt{gA/T}$

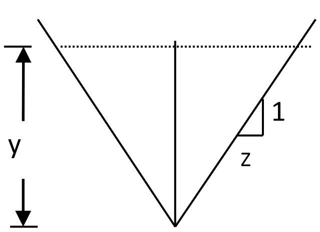
$= \sqrt{g\frac{by_c}{b}} = \sqrt{gy_c}$

The minimum specific energy E_c corresponding to critical depth and critical velocity is given by

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{gy_c}{2g}$$

$$E_c = \frac{3}{2}y_c$$

b) Triangular channel



Side slopes of triangular channel = z:1 and depth of flow = y
Flow area (A) = zy_c^2

Top width (T) = $2zy_c$
At critical state,

$$\frac{Q^2}{g} = \frac{A^3}{T}$$

$$\frac{Q^2}{g} = \frac{(zy_c^2)^3}{2zy_c}$$

$$y_c = \left(\frac{2Q^2}{gz^2}\right)^{1/5}$$

Critical velocity $(V_c) = \sqrt{gD} = \sqrt{gA/T}$

$= \sqrt{g\,\frac{zy_c^2}{2zy_c}} = \sqrt{g\,\frac{y_c}{2}}$

The minimum specific energy E_c corresponding to critical depth and critical velocity is given by

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{gy_c}{4g}$$

$$E_c = \frac{5}{4}y_c$$

c. Parabolic

$T = k\sqrt{y_c}$

$A = \frac{2}{3}Ty_c = \frac{2}{3}k\sqrt{y_c}y_c = \frac{2}{3}ky_c^{3/2}$

At critical state,

$$\frac{Q^2}{g} = \frac{A^3}{T}$$

$$\frac{Q^2}{g} = \frac{(\frac{2}{3}ky_c^{3/2})^3}{k\sqrt{y_c}}$$

$$y_c = \left(\frac{27Q^2}{8gk^2}\right)^{1/4}$$

Critical velocity $(V_c) = \sqrt{gD} = \sqrt{gA/T}$

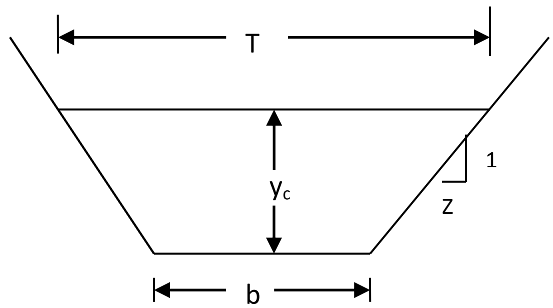
$= \sqrt{g\,\frac{\frac{2}{3}ky_c^{3/2}}{k\sqrt{y_c}}} = \sqrt{g\,\frac{2y_c}{3}}$

The minimum specific energy E_c corresponding to critical depth and critical velocity is given by

$$E_c = y_c + \frac{V_c^2}{2g} = y_c + \frac{2gy_c}{6g}$$

$$E_c = \frac{4}{3}y_c$$

d) Trapezoidal



A= $(b+zy_c)y_c$
T = $b+2zy_c$

$$\frac{Q^2}{g} = \frac{A^3}{T}$$
$$\frac{Q^2}{g} = \frac{[(b+zy_c)y_c]^3}{(b+2zy_c)}$$
$$\frac{Q^2}{g} = \frac{\left[b^2\left(1+z\frac{y_c}{b}\right)\frac{y_c}{b}\right]^3}{\left(1+2z\frac{y_c}{b}\right)\mathrm{b}}$$
$$\frac{Q^2}{gb^5} = \frac{\left[\left(1+z\frac{y_c}{b}\right)\frac{y_c}{b}\right]^3}{\left(1+2z\frac{y_c}{b}\right)}$$

No direct solution exists. Solve for y_c by trial and error.

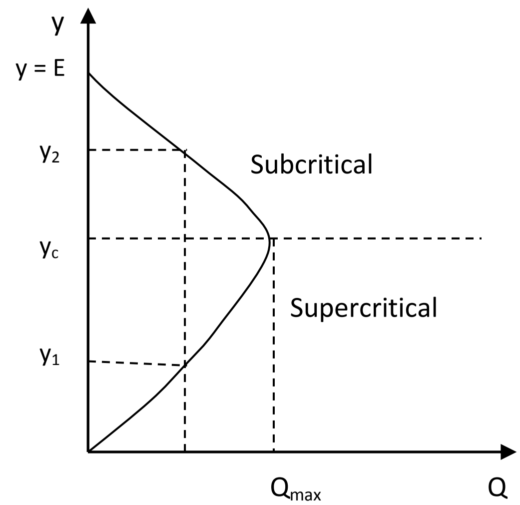
7.5 Depth-discharge diagram

Condition for maximum discharge for a given value of specific energy
Specific energy (E) is given by

$$E = y + \frac{Q^2}{2gA^2}$$
$$E - y = \frac{Q^2}{2gA^2}$$
$$Q = A\sqrt{2g}\sqrt{E-y}$$

A =f(y) for a given channel and Q=f(y) for a given specific energy.

- At y=0, A=0, then Q=0
- At y=E, the expression (E-y) =0, then Q=0
- Q is maximum at an intermediate depth within y=0 and y=E



Depth-discharge diagram

For maximum discharge, dQ/dy =0.
Differentiating w.r.t. y

$$\frac{dQ}{dy} = \sqrt{2g}\left[-\frac{A}{2\sqrt{E-y}} + \sqrt{E-y}\frac{dA}{dy}\right] = 0$$

dA/dy = Top width (T)

$$T\sqrt{E-y} = \frac{A}{2\sqrt{E-y}}$$

$$2(E-y) = \frac{A}{T}E - y = \frac{Q^2}{2gA^2}$$

From above two expressions

$$\frac{Q^2}{gA^2} = \frac{A}{T}$$

$$\frac{Q^2T}{gA^3} = 1$$

Or, $\frac{Q^2}{g} = \frac{A^3}{T}$
In terms of velocity

$$\frac{Q^2}{A^2g} = \frac{A}{T}$$

$\frac{V^2}{g} = D$ where Hydraulic depth (D) = A/T

$$\frac{V}{\sqrt{gD}} = 1$$

i.e. F_r =1

For the discharge to be maximum for a given specific energy, the flow should be critical.

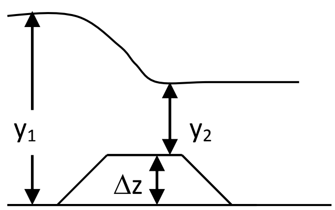
7.6 Application of specific energy concept

The concept of specific energy and critical depth are extremely useful in the analysis of flow problems, such as flow through transitions (reduction in channel width), flow over raised channel floors, flow through sluice gate opening.

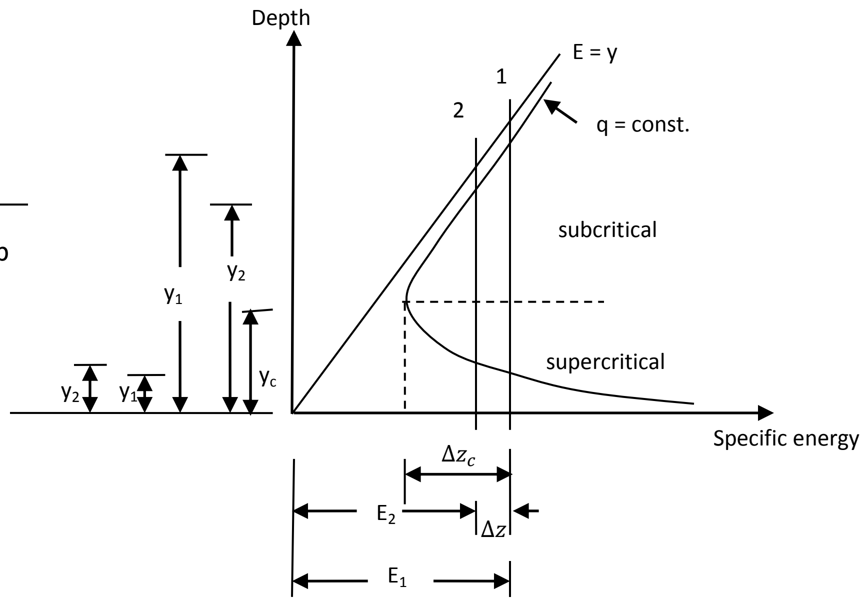
I. Flow through transitions

A channel transition refers to a certain stretch of channel in which the cross-section of the channel varies.

b. Rise in bed level of channel section (Channel with hump)



Channel transition with hump



Specific energy diagram for channel bed rise

Consider a channel of constant width (b) carrying a discharge Q with a rise in bed level (ΔZ) in a certain reach. Let ΔZ_c is the minimum height of hump under critical condition.

$q_1=q_2=q$ = discharge per unit width = constant

Specific Energy equation for hump: $E_1=E_2+\Delta Z$

Analysis of flow for hump

y_1 = flow depth at section 1, y_2 = flow depth at section 2, y_c = flow depth at section for critical condition.

First compute ΔZ_c by computing E_1 and E_2 which is equal to E_c at section 2. So, $\Delta Z_c = E_1 - E_C$

Case I: $\Delta Z < \Delta Z_c$

Flow profile:

- y_1 = constant for both subcritical and supercritical flow
- y_2 reduces for subcritical flow and increases for supercritical flow until y_2 is equal to y_c.

Given: Q, y_1, ΔZ

To find: y_2

Compute $E_1 = y_1 + \frac{Q^2}{2gA_1{}^2}$

$E_1 = E_2 + \Delta Z$

$E_2 = E_1 - \Delta Z$

$E_2 > E_c$ where E_c is minimum specific energy.

$$E_2 = y_2 + \frac{Q^2}{2gA_2{}^2}$$

Find the value of y_2 by trial and error method.

Case II: $\Delta Z = \Delta Z_c$

Flow profile:

- y_1 = constant for both subcritical and supercritical flow
- $y_2 = y_c$ for both subcritical and supercritical flow

Given: Q, y_1

To find: ΔZ_c (minimum rise or minimum size of hump under critical condition)

In this case, energy is minimum and the flow is critical at 2.

$E_2 = E_c$

$E_1 = E_2 + \Delta Z_c$

or, $\Delta Z_c = E_1 - E_2 = E_1 - E_C$

For rectangular section

$$E_c = 1.5y_c = 1.5\left(\frac{q^2}{g}\right)^{1/3}$$

$$\Delta Z_c = E_1 - E_c = y_1 + \frac{v_1{}^2}{2g} - \frac{3}{2}\left(\frac{y_1{}^2 v_1{}^2}{g}\right)^{1/3}$$

$$= y_1\left[1 + \frac{v_1{}^2}{2gy_1} - 1.5\left(\frac{v_1{}^2}{gy_1}\right)^{1/3}\right]$$

$$= y_1\left[1 + \frac{F_{r1}{}^2}{2} - 1.5F_{r1}{}^{2/3}\right]$$

where F_{r1} = Froude no. at section 1 = $\frac{v_1}{\sqrt{gy_1}}$

Case III: $\Delta Z > \Delta Z_c$

In this case, $(E_1 - \Delta Z = E_2) < E_c$ which is physically not possible. The flow cannot take place with the available specific energy (choking condition). The flow will be critical at section 2 ($E_2 = E_c$, $y_2 = y_c$) and the upstream depth will be changed.

Flow profile:

- y_2 remains constant at y_c for both subcritical and supercritical flow.
- y_1 increases for subcritical flow and decreases for supercritical flow.

Given: Q, ΔZ
To find: u/s depth (y_{1a})

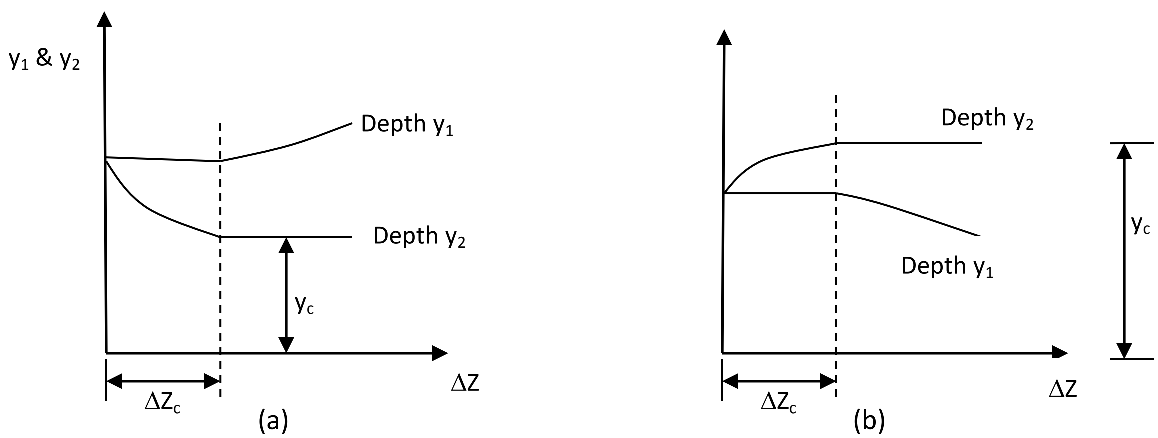
$$E_{1a} = \Delta Z + E_2$$

$$E_{1a} = \Delta Z + E_C = \Delta Z + 1.5E_C = \Delta Z + 1.5\left(\frac{q^2}{g}\right)^{1/3}$$

New u/s depth y1a is computed by

$$y_{1a} + \frac{q^2}{2gy_{1a}^2} = E_{1a}$$

Solve this equation by trial and error to get y_{1a}.

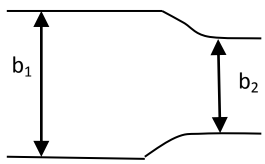


Variation of depth for (a) subcritical flow and (b) supercritical flow in case of hump

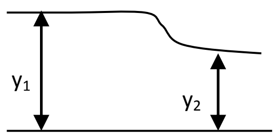
If loss if energy due to hump (h_L) is considered, then
$E_1 = E_2 + \Delta Z + h_L$

b. Reduction in channel width

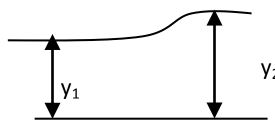
Let us consider that the width of channel is reduced from b_1 to b_2, the floor level remaining the same. Also neglect energy loss between two sections. For a constant discharge Q, the discharge per unit width in the approach channel is $q_1 = Q/b_1$ and in the throat portion is $q_2 = Q/b_2$. For subcritical approach flow, depth of flow at section 2 is less than that at section 1. For supercritical approach flow, depth of flow at section 2 is greater than that at section 1. If the width b_2 is reduced, q_2 increases and approaches q_c, i.e. the flow becomes critical at a particular width $b_2 = b_c$. The width b_c is therefore the limiting width at which maximum unit discharge q_{max} passes through the throat under critical condition. If the throat width is further reduced to b_3 less than b_c resulting in $q_3 > q_{max}$ which is obviously not possible at the given value of specific energy E_1.The energy upstream must increase under this condition to make the flow at contraction at critical state.



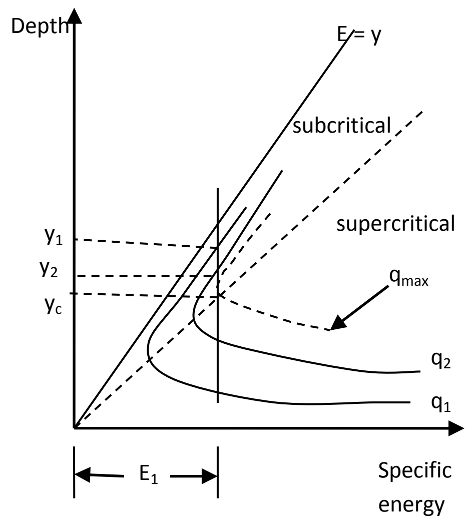
Width constriction



Subcritical flow approach



Supercritical approach flow



Specific energy diagram for width constriction

Analysis of flow for width constriction

b_1 = width at section 1, b_2 = width at section 2, b_c = width at section 2 under critical condition
y_1 = flow depth at section 1, y_2 = flow depth at section 2, y_c = flow depth at section for $b_2 = b_c$
First compute b_c by computing E_c at section 2. ($E_2 = E_c$. As $E_1 = E_2$, $E_1 = E_c$)
Case I: $b_2 > b_c$
Flow profile:

- y_1 = constant for both subcritical and supercritical flow
- y_2 reduces for subcritical flow and increases for supercritical flow until b_2 is equal to b_c

Given: Q, y_1, b_1, b_2
To find: y_2
Compute $E_1 = y_1 + \frac{Q^2}{2gA_1{}^2}$
$E_2 = E_1$
$y_2 + \frac{Q^2}{2gA_2{}^2} = E_2$
Solve by trial and error to get y_2.

Case II: $b_2 = b_c$
Flow profile:

- y_1 = constant for both subcritical and supercritical flow

- $y_2 = y_c$ for both subcritical and supercritical flow

Given: Q, y_1, b_1

To find: minimum width at contraction (b_c) at critical condition

Compute $E_1 = y_1 + \frac{Q^2}{2gA_1{}^2}$

$E_1 = E_2$

$E_2 = E_C$

Hence, $E_1 = E_C$

For rectangular channel

$$E_1 = E_c = 1.5y_c$$

$$= 1.5\left(\frac{q^2}{g}\right)^{1/3} = 1.5\left(\frac{\left(\frac{Q}{b_c}\right)^2}{g}\right)^{1/3}$$

$$E_1^3 = 3.375\frac{Q^2}{gb_c^2}$$

$$b_c = 1.84\frac{Q}{E_1^{3/2}\sqrt{g}}$$

Case 3: $b_2<b_c$

Flow profile:

- y_2 is critical with new critical depth, y_{c1} for both subcritical and supercritical flow. y_{c1} is computed by taking width = b_2. So $y_{c1}>y_c$. As specific energy at 2 (E_{c1}) rises, y_{c1} also rises.
- y_1 increases for subcritical flow and decreases for supercritical flow.

Given: Q, b_1, b_2

To compute: Depth at 2 and 1

Given discharge cannot passes through the section 2 with the available energy E_1 (choking condition). The specific energy at section 2 should be minimum and the upstream depth is changed to y_{1a}.

$E_2 = E_c$ and y_2 = critical depth (y_{c1})

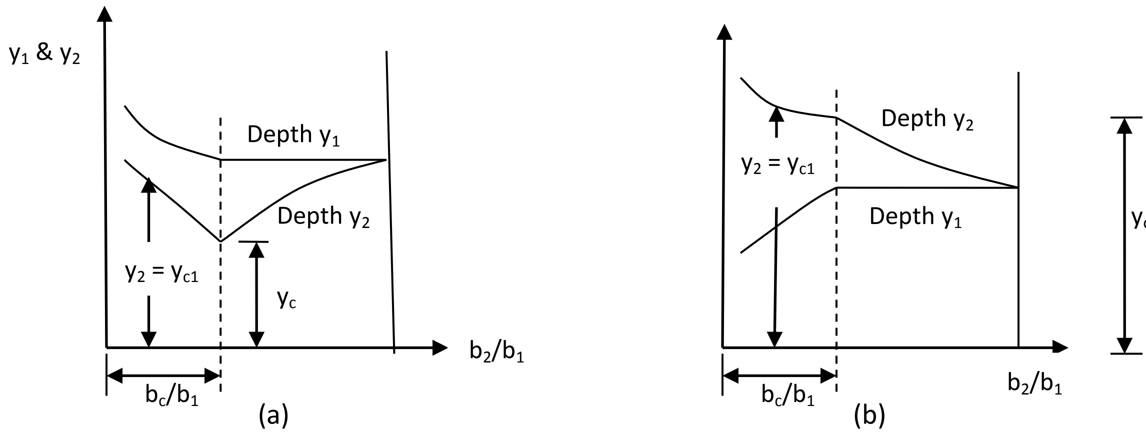
New specific energy upstream (E_{1a}) = E_2

$E_{1a} = E_c$

$$E_{1a} = E_c = 1.5y_{c1} = 1.5\left(\frac{\left(\frac{Q}{b_2}\right)^2}{g}\right)^{1/3}$$

$$E_{1a} = y_{1a} + \frac{q_1^2}{2gy_{1a}^2}$$

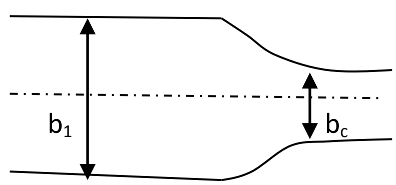
Find y_{1a} by trial and error.



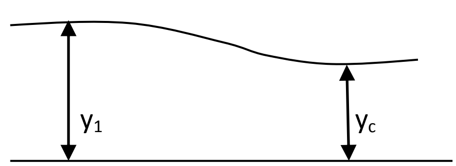
Variation of depth for (a) subcritical flow and (b) supercritical flow in case of width constriction

c) Venturiflume

It is a simple device for measuring discharge in open channel flow in which the width of contraction is equal to or less than critical width (b_c).



Venturiflume



Assumptions

- Head loss is negligible
- Flow level is horizontal
- Specific energy is constant
- Contraction is adequate to result critical flow

$$E_1 = y_1 + \frac{v_1^2}{2g}$$

$$E_c = \frac{3}{2} y_c$$

$$E_1 = E_C$$

$$y_1 + \frac{v_1^2}{2g} = \frac{3}{2} y_c$$

$$y_1 + \frac{v_1^2}{2g} = \frac{3}{2}\left(\frac{\left(\frac{Q}{b_2}\right)^2}{g}\right)^{1/3}$$

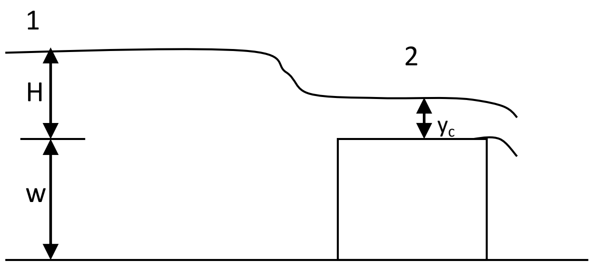
Neglecting velocity of approach $\left(\frac{v_1^2}{2g}\right)$

$$Q = 1.7 b_2 y_1^{3/2}$$

Thus discharge can be calculated by merely measuring the depth upstream of the throat.

d) Broad crested weir

Board crested weir is a simple device for measuring the discharge in open channel flow.



$E_1 = E_2 + w$

$H + w + \frac{v_1^2}{2g} = E_c + w$

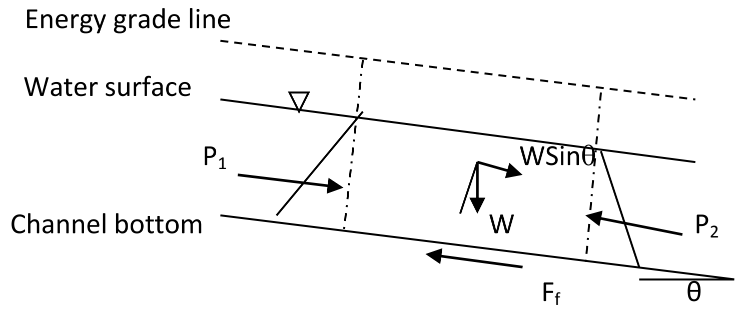
Neglecting velocity of approach

$$H = E_c$$

$$H = \frac{3}{2} y_c = \frac{3}{2}\left(\frac{(Q/B)^2}{g}\right)^{1/3}$$

$$Q = 1.704 B H^{3/2}$$

7.7 Momentum equation and specific force



Consider two sections 1 and 2 along the length of channel. Let P1 and P2 are the hydrostatic pressure force at section (1) and 2 respectively; W is the weight of water enclosed between two sections; and F_f is the resistance force due to friction. Applying the momentum principle,

Net force = Change in momentum

$$P_1 - P_2 + Wsin\theta - F_f = \rho Q(V_2 - V_1)$$

For a short reach of prismatic channel, θ = 0 and friction force, F_f can be neglected. Hence the momentum equation becomes

$$P_1 - P_2 = \frac{\gamma Q}{g}(V_2 - V_1)$$

$$\gamma A_1\overline{Z_1} - \gamma A_2\overline{Z_2} = \frac{\gamma Q}{g}\left(\frac{Q}{A_2} - \frac{Q}{A_1}\right)$$

where A_1 and A_2 are cross-sectional area, and $\overline{Z_1}$ and $\overline{Z_2}$ are the distances of centroids of the areas below the surface of flow. The above equation can be written as

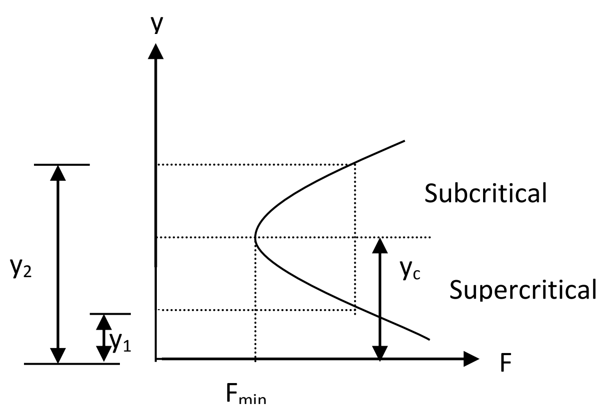
$$\frac{Q^2}{gA_1} + \overline{Z_1}A_1 = \frac{Q^2}{gA_2} + \overline{Z_2}A_2$$

The two sides of above equation are identical and may be represented by force F.

$$F = \frac{Q^2}{gA} + \bar{Z}A$$

The first term is the momentum flux per unit weight and the second term is the force per unit weight of the liquid. The sum of these two terms is known as specific force. The equation shows that the specific force is a function of depth.

Specific force curve



Specific force curve

The plot of specific force versus depth for a given channel section and discharge is called specific forces curve. The curve has two limbs. The lower limb approaches the horizontal axis asymptotically towards the right. The upper limb rises upward and extends indefinitely to the right.

For a given value of specific force, the curve has two possible depths y_1 and y_2. These depths are known as conjugate depths or sequent depths. The depth y_1 lies in the super critical region and is known as initial depth while the second depth y_2 corresponds to sub critical flow and is known as the sequent depth. These depths are also known as conjugate depths. For a particular depth (y_c), the specific force becomes minimum and the corresponding depth of flow is called critical depth.

a. Criterion for minimum specific force for a given discharge

$F = \frac{Q^2}{gA} + \bar{Z}A$ (a)

For a minimum value of specific force dF/dy = 0

Differentiating F w.r.t. y

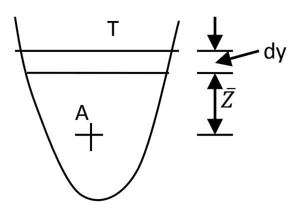
$\frac{dF}{dy} = -\frac{Q^2}{gA^2}\frac{dA}{dy} + \frac{d(\bar{Z}A)}{dy} = 0$ (b)

$d(\bar{Z}A)$ represents the change in static moment of water area about the free surface.
Moment of area about water surface = $\bar{Z}A$
For a change in depth dy,
Moment of area about water surface = Moment of A with arm $(\bar{Z} + dy)$ plus moment of elementary area = $\left[A(\bar{Z} + dy) + T\,dy\frac{dy}{2}\right]$

$$d(\bar{Z}A) = [A(\bar{Z} + dy) + T\,(dy)^2/2] - \bar{Z}A$$

Neglecting differential of higher order

$$d(\bar{Z}A) = Ady$$



With this, eq. b becomes

$$-\frac{Q^2}{gA^2}\frac{dA}{dy} + A = 0$$

Substituting dA/dy = top width (T)

$$\frac{Q^2T}{gA^3} = 1$$

Or, $\frac{Q^2}{g} = \frac{A^3}{T}$

In terms of velocity

$$\frac{Q^2}{A^2g} = \frac{A}{T}$$

$\frac{V^2}{g} = D$ where Hydraulic depth (D) = A/T

$$\frac{V}{\sqrt{gD}} = 1$$

i.e. $F_r = 1$

At critical state of flow, specific force is minimum for given discharge.

b. Criterion for maximum discharge for a given specific force

$F = \frac{Q^2}{gA} + \bar{Z}A$ (a)

Rearranging

$Q = \sqrt{g}\sqrt{A(F - \bar{Z}A)}$ (b)

For discharge to be maximum dQ/dy = 0

Differentiating Q w.r.t. y (keeping F fixed)

$\frac{dQ}{dy} = \sqrt{g}\left[\sqrt{(F - \bar{Z}A)}\frac{d(\sqrt{A})}{dy} + \sqrt{A}\frac{d\left(\sqrt{(F-\bar{Z}A)}\right)}{dy}\right] = 0$

$\sqrt{g}\left[\sqrt{(F - \bar{Z}A)}\frac{1}{2\sqrt{A}}\frac{dA}{dy} + \sqrt{A}\frac{1}{2\left(\sqrt{(F-\bar{Z}A)}\right)}x - \frac{d(\bar{Z}A)}{dy}\right] = 0$ ©

$d(\bar{Z}A)$ represents the change in static moment of water area about the free surface.

Moment of area about water surface = $\bar{Z}A$

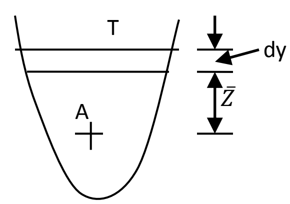
For a change in depth dy,

Moment of area about water surface = Moment of A with arm $(\bar{Z} + dy)$ plus moment of elementary area = $\left[A(\bar{Z} + dy) + T\,dy\frac{dy}{2}\right]$

$$d(\bar{Z}A) = [A(\bar{Z} + dy) + T\,(dy)^2/2] - \bar{Z}A$$

Neglecting differential of higher order

$$d(\bar{Z}A) = Ady$$



dA/dy = top width (T)

substituting dA/dy and $d(\bar{Z}A)$

$\sqrt{g}\left[\sqrt{(F - \bar{Z}A)}\frac{1}{2\sqrt{A}}T + \sqrt{A}\frac{1}{2\left(\sqrt{(F-\bar{Z}A)}\right)}x - A\right] = 0$

$\frac{\sqrt{g}}{2\left(\sqrt{A(F-\bar{Z}A)}\right)}[T(F - \bar{Z}A) - A^2] = 0$

$T(F - \bar{Z}A) - A^2 = 0$ (d)

From b,

$(F - \bar{Z}A) = \frac{Q^2}{gA}$ (e)

From d and e

$\frac{Q^2T}{gA^3} = 1$

Or, $\frac{Q^2}{g} = \frac{A^3}{T}$

In terms of velocity

$\frac{Q^2}{A^2g} = \frac{A}{T}$

$\frac{V^2}{g} = D$ where Hydraulic depth (D) = A/T

$\frac{V}{\sqrt{gD}} = 1$

i.e. $F_r = 1$

At critical state of flow, discharge is maximum for given specific force.

Summary of critical flow condition

Conditions to be fulfilled for the flow to be critical

- Specific energy is minimum for a given discharge.
- Discharge is maximum for a given specific energy.
- Specific force is minimum for a given discharge.
- Discharge is maximum for a given force.
- Froude number is unity.

Occurrence of critical flow conditions

- At a point where subcritical flow changes to supercritical flow (e.g. at a break from mild to steep slope)
- Entrance to a channel of steep slope from a reservoir
- Free outfall from a channel with a mild slope
- Change of bed level or channel width

Chapter 8: Gradually varied flow

Gradually varied flow (GVF) is a steady non-uniform flow in which the depth of flow varies gradually. In the GVF, the velocity varies along the channel and consequently the bed slope, water surface slope and energy slope will all differ from each other. The boundary friction is important in GVF.

The GVF may be caused due to one or more of the following factors:

- Change in shape and size of the cross-section
- change in slope of channel
- presence of obstruction, e.g. weir
- change in frictional forces at the boundary

Examples: flow upstream of a weir or dam, flow downstream of a sluice gate, flow in channels with break in bottom slope.

8.1 Differential Equation of GVF

Assumptions for deriving dynamic equation of GVF

- The uniform flow formula (i.e. Manning's or Chezy) is used to evaluate the energy slope of a gradually varied flow and the corresponding coefficient of roughness developed primarily for uniform flow are applicable to GVF.
- The bottom slope of a channel is very small so that the depth of flow is same whether the vertical or normal direction is used.
- The flow is steady.
- The pressure distribution over the channel section is hydrostatic.
- The channel is prismatic.
- The velocity distribution in the channel is fixed.
- The roughness coefficient is independent of the depth of flow and constant throughout the channel reach under consideration.
- The energy correction factor is unity.

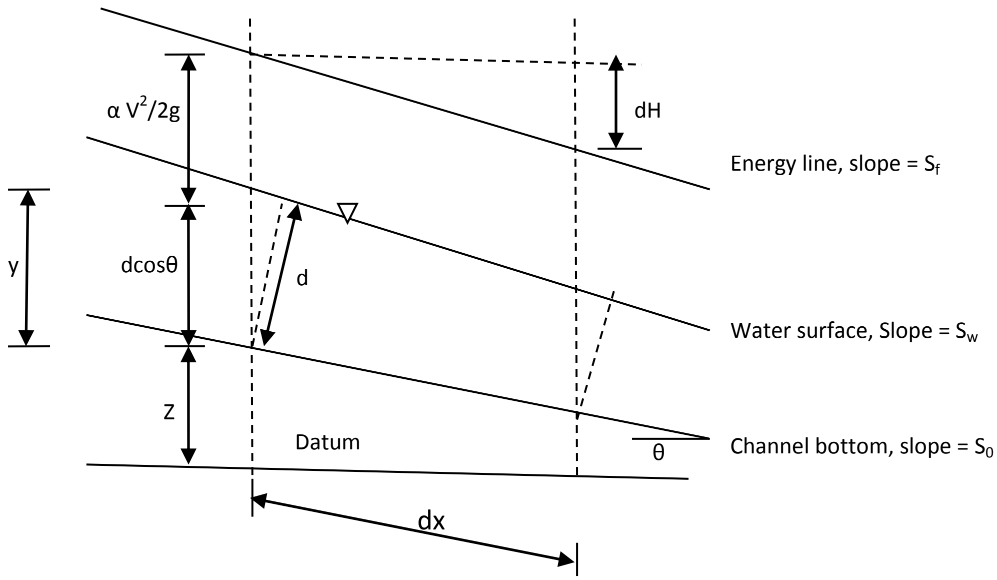
Consider the profile of GVF in the elementary length dx of an open channel. The total head above the datum at the upstream section is

$$H = z + dcos\theta + \alpha\frac{V^2}{2g}$$

where H = total head, z = vertical distance of channel bed above the datum, d = depth of flow normal to the direction of flow and V = mean velocity. For small slope, $dcos\theta \approx y$, where y = depth of flow. α =1

$$H = z + y + \frac{V^2}{2g}$$

Let S_f, S_w and S_0 are slope of energy line, slope of water surface and channel bottom slope respectively.



Taking the bottom of the channel as the x axis and the vertically upwards direction measured from the channel bottom as the y axis and differentiation w.r.t. x

$$\frac{dH}{dx} = \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\frac{V^2}{2g}\right)$$

$$= \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dx}\left(\frac{Q^2}{2gA^2}\right)$$

$$= \frac{dz}{dx} + \frac{dy}{dx} + \frac{d}{dy}\left(\frac{Q^2}{2gA^2}\right)\frac{dy}{dx}$$

$$= \frac{dz}{dx} + \frac{dy}{dx}\left[1 - \frac{Q^2}{gA^3}\frac{dA}{dy}\right]$$

$dH/dx = -S_f$, $dz/dx = -S_0$ and dA/dy =top width (T)

$$-S_f = -S_0 + \frac{dy}{dx}\left[1 - \frac{Q^2 T}{gA^3}\right]$$

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - \frac{Q^2 T}{gA^3}}$$

This equation is known as the dynamic equation of gradually varied flow. It can also be expressed in terms of Froude number (F_r).

$$\frac{Q^2T}{gA^3} = \frac{V^2}{gA/T} = \frac{V^2}{gD} = F_r^2$$

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}$$

If dy/dx =0, S_o=S_f, the water surface is parallel to channel bottom, thus representing a uniform flow. When dy/dx is +ve the water surface is rising and when dy/dx is –ve , the water surface is falling.

Modified form of GVF

Form 1: GVF equation in terms of conveyance and section factor

K = conveyance at any depth y and K_n = conveyance corresponding to normal depth y_n.

$K = \frac{Q}{\sqrt{S_f}}$ and $K_n = \frac{Q}{\sqrt{S_0}}$

$S_f = \frac{Q^2}{K^2}$ and $S_0 = \frac{Q^2}{{K_n}^2}$

$$\frac{S_f}{S_0} = \frac{{K_n}^2}{K^2}$$

Let Z= section factor at depth y and Z_c = section factor at the critical depth y_c.

$Z = \sqrt{\frac{A^3}{T}}$ and $Z_c = \sqrt{\frac{{A_c}^3}{T}}$

$Z^2 = \frac{A^3}{T}$ and $Z_c^2 = \frac{{A_c}^3}{T} = \frac{Q^2}{g}$

$$\frac{Z_c^2}{Z^2} = \frac{Q^2T}{gA^3}$$

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - \frac{Q^2T}{gA^3}} = \frac{S_0\left(1 - \frac{S_f}{S_0}\right)}{1 - \frac{Q^2T}{gA^3}}$$

Substituting the values of $\frac{S_f}{S_0}$ and $\frac{Q^2T}{gA^3}$

$$\frac{dy}{dx} = \frac{S_0(1 - K_n^2/K^2)}{1 - Z_c^2/Z^2}$$

This is another form of equation for GVF.

Form 2: Modified form of GVF equation considering wide rectangular channel

For wide rectangular channel, hydraulic radius (R) can be approximated as flow depth (y).

a. Using Manning's equation

Discharge = $\frac{1}{n}AR^{2/3}S_f^{1/2} = \frac{1}{n}byy^{2/3}S_f^{1/2} = \frac{1}{n}by^{5/3}S_f^{1/2}$

Discharge in terms of normal depth (y_n) is given by

Discharge = $\frac{1}{n}AR^{2/3}S_f^{1/2} = \frac{1}{n}by_n{y_n}^{2/3}S_f^{1/2} = \frac{1}{n}b{y_n}^{5/3}S_0^{1/2}$

$$\frac{1}{n}by^{5/3}S_f^{1/2} = \frac{1}{n}b{y_n}^{5/3}S_0^{1/2}$$

$$\frac{S_f}{S_0} = \left(\frac{y_n}{y}\right)^{10/3}$$

$$\frac{Q^2T}{gA^3} = \frac{Q^2b}{g(by)^3} = \frac{Q^2}{b^2g}.\frac{1}{y^3} = \frac{y_c^3}{y^3}$$

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - \frac{Q^2T}{gA^3}} = \frac{S_0\left(1 - \frac{S_f}{S_0}\right)}{1 - \frac{Q^2T}{gA^3}} \qquad \text{(a)}$$

Substituting the values of $\frac{S_f}{S_0}$ and $\frac{Q^2T}{gA^3}$

$$\frac{dy}{dx} = \frac{S_0\left[1 - (y_n/y)^{10/3}\right]}{1 - (y_c/y)^3}$$

b. Using Chezy's formula

$$Q = CA\sqrt{RS_f} = Cby\sqrt{yS_f} = Cby^{3/2}S_f^{1/2}$$

Similarly in terms of normal depth,

$$Q = Cby_n{}^{3/2}S_0^{1/2}\,\frac{S_f}{S_0} = \left(\frac{y_n}{y}\right)^3$$

Substituting the values of $\frac{S_f}{S_0}$ and $\frac{Q^2T}{gA^3}$ in (a)

$$\frac{dy}{dx} = \frac{S_0(1 - (y_n/y)^3)}{1 - (y_c/y)^3}$$

Slope of water surface

$\frac{dy}{dx}$ = slope of water surface with respect to channel bottom

Slope of water surface with respect to horizontal (S_w)

$S_w = S_0 - \frac{dy}{dx}$ for rising water level

$S_w = S_0 + \frac{dy}{dx}$ for falling water level

8.2 Classification of slopes

- Mild slope (M): channel bottom slope<critical slope, normal depth>critical depth
- Steep slope (S): channel bottom slope>critical slope, normal depth<critical depth
- Critical slope (C): Mild slope: channel bottom slope=critical slope, normal depth=critical depth
- Horizontal slope (H): Channel bottom slope = 0, Normal depth = ∞
- Adverse slope (A): Channel bottom slope <0 (negative), normal depth: imaginary or non-existent

Sustaining and non-sustaining slope

A sustaining slope is a channel slope that falls in the direction of flow. Hence it is always positive. It may be critical, mild (subcritical) or steep (supercritical).

A non-sustaining slope may be horizontal or adverse. A horizontal slope is a zero slope. An adverse slope is a negative slope that rises in the direction of flow.

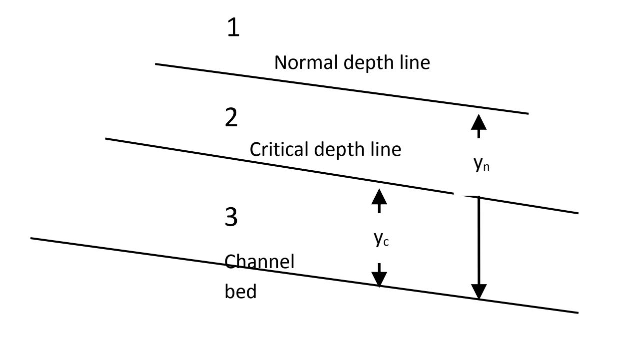
8.3 Water surface profile

For a given discharge and channel conditions, the normal depth and critical depth can be computed. A line drawn parallel to the channel bed and at height y_n is called normal depth line (NDL) and a line parallel to the channel bed and at height y_c is called the critical depth line (CDL). On the basis of these lines, the vertical space in a longitudinal section of the channel can be divided into three regions

Zone 1- space above both the critical and normal depths

Zone 2 – region lies between normal depth and critical depth

Zone 3 -region below both the normal and critical depth lines and above channel bed



The following is the classification of flow profiles for different kinds of slopes.

I. M profile (Surface profile for Mild sloped channels)

All the three zones occur with channels of mild slopes and corresponding flow profiles are designed as M1, M2 and M3 profiles.

a. M1 profile

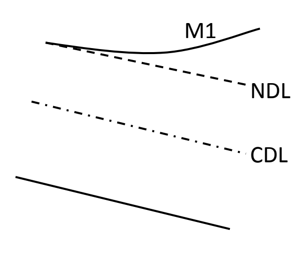
Condition: $y > y_n > y_c$

$\frac{dy}{dx} = \frac{S_0\left[1-(y_n/y)^{10/3}\right]}{1-(y_c/y)^3} = \frac{+ve}{+ve} = +ve$

$y \to y_n, \frac{dy}{dx} \to 0$: u/s end of the curve touches NDL.

$y \to \infty, \frac{dy}{dx} \to S_0$: d/s end of the curve is horizontal.

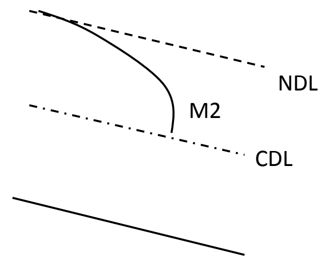
Type of profile: Backwater



Type of flow: Subcritical

Examples: flow behind dam, flow in a canal joining two reservoirs

b. M2 profile



Condition: $y_n > y > y_c$

$\frac{dy}{dx} = \frac{S_0\left[1-(y_n/y)^{10/3}\right]}{1-(y_c/y)^3} = \frac{-ve}{+ve} = -ve$

$y \to y_n, \frac{dy}{dx} \to 0$: u/s end of the curve is tangent to the NDL.

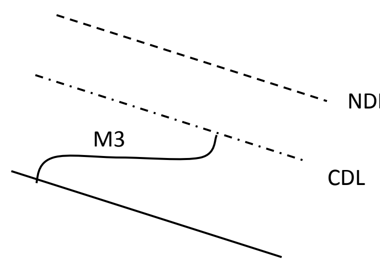
$y \to y_{c,}\ \frac{dy}{dx} \to \infty$: d/s end of the curve is normal to the CDL.

Type of profile: Drawdown

Type of flow: Subcritical

Examples: Flow over a free overfall, flow at the U/S end of a sudden enlargement in a mild channel

c. M3 profile



Condition: $y_n > y_c > y$

$\frac{dy}{dx} = \frac{S_0\left[1-(y_n/y)^{10/3}\right]}{1-(y_c/y)^3} = \frac{-ve}{-ve} = +ve$

$y \to y_c, \frac{dy}{dx} \to \infty$: d/s end of the curve is normal to the CDL.

$y \to 0, \frac{dy}{dx} \to \infty$: u/s end of the curve is normal to be bed.

Type of profile: Backwater

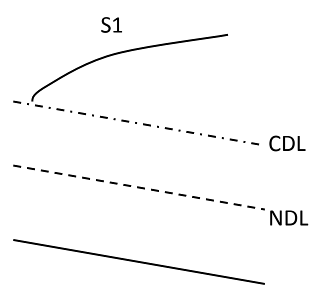
Type of flow: Supercritical

Examples: Flow downstream of a sluice gate, flow in a channel where bottom slope changes from steep to mild

II. S profile (Surface profile for Steep sloped channel)

All the three zones occur with channels of steep slopes and corresponding flow profiles are designed as S1, S2 and S3 profiles.

a) S1 profile



Condition: $y > y_c > y_n$

$\frac{dy}{dx} = \frac{S_0\left[1-(y_n/y)^{10/3}\right]}{1-(y_c/y)^3} = \frac{+ve}{+ve} = +ve$

$y \to y_c, \frac{dy}{dx} \to \infty$: u/s end of the curve is normal to the CDL.

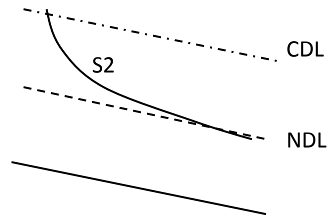
$y \to \infty, \frac{dy}{dx} \to S_0$: d/s end of the curve is horizontal.

Type of profile: Backwater

Type of flow: Subcritical

Examples: flow behind dam on a steep channel, flow behind an overflow weir

b) S2 profile



Condition: $y_c > y > y_n$

$\frac{dy}{dx} = \frac{S_0[1-(y_n/y)^{10/3}]}{1-(y_c/y)^3} = \frac{+ve}{-ve} = -ve$

$y \to y_c, \frac{dy}{dx} \to \infty$: u/s end of the curve is normal to the CDL.

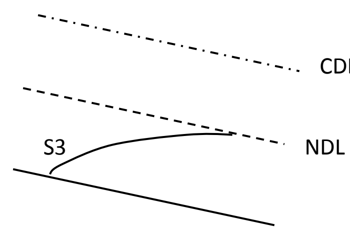
$y \to y_n, \frac{dy}{dx} \to 0$: d/s end of the curve is tangent to the NDL.

Type of profile: Drawdown

Type of flow: Supercritical

Examples: Flow from steep to steeper channel, Profile formed when there is sudden enlargement in a steep slope

c) S3 profile



Condition: $y_c > y_n > y$

$\frac{dy}{dx} = \frac{S_0[1-(y_n/y)^{10/3}]}{1-(y_c/y)^3} = \frac{-ve}{-ve} = +ve$

$y \to y_n, \frac{dy}{dx} \to 0$: d/s end of the curve is tangential to the NDL.

$y \to y_c, \frac{dy}{dx} \to \infty$: u/s end of the curve is normal to the bed.

Type of profile: Backwater

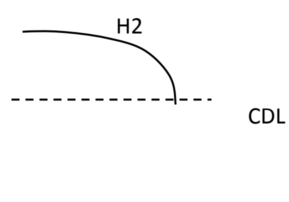
Type of flow: Supercritical

Examples: Flow down of the sluice gate, flow through channel where bottom slope change from steeper to steep

III. H profile (Surface profile in horizontal channels)

Since y_n is infinite, H1 profile does not exist. So there are two H profiles: H2 and H3 (Assume S_0 to be very small.)

a. H2 profile



Condition: $y > y_c$

$\frac{dy}{dx} = \frac{S_0 - S_f}{1-(y_c/y)^3} = \frac{-ve}{+ve} = -ve$

$y \to \infty, \frac{dy}{dx} \to S_0$: u/s end of the curve is horizontal.

$y \to y_c, \frac{dy}{dx} \to -\infty$: d/s end of the curve is normal to the CDL.

Type of profile: Drawdown

Type of flow: Subcritical

Example: Flow at free fall overran edge

b. H3 profile

Condition: $y < y_c$

$\frac{dy}{dx} = \frac{S_0 - S_f}{1-(y_c/y)^3} = \frac{-ve}{-ve} = +ve$

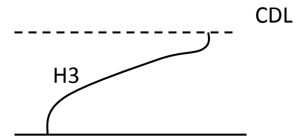
$y \to y_c, \frac{dy}{dx} \to -\infty$: d/s end of the curve is normal to CDL.

$y \to 0, \frac{dy}{dx} \to \infty$: u/s end of the curve is normal to bed.

Type of profile: Backwater

Type of flow: Supercritical

Example: Flow below a sluice gate provided in a horizontal channel



IV. C profile (Surface profile in critical-sloped channels)

As $y_n = y_c$, C2 profile does not exist. So there are two C profiles: C1 and C3.

a. C1 profile

Condition: $y > y_c = y_n$

$\frac{dy}{dx} = \frac{S_0[1-(y_n/y)^3]}{1-(y_c/y)^3} = \frac{+ve}{+ve} = +ve$

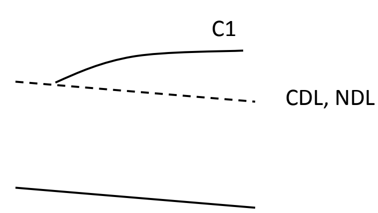
$y \to y_c, \frac{dy}{dx} \to S_0$: u/s end of the curve is horizontal.

$y \to \infty, \frac{dy}{dx} \to S_0$: d/s end of the curve is horizontal.

Type of profile: Backwater

Type of flow: Subcritical

Example: Flow behind an overflow weir, flow behind a sluice gate



b. C3 profile

Condition: $y < y_c = y_n$

$\frac{dy}{dx} = \frac{S_0[1-(y_n/y)^3]}{1-(y_c/y)^3} = \frac{-ve}{-ve} = +ve$

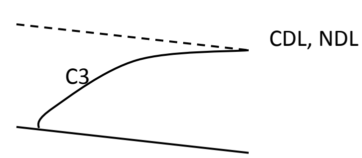
$y \to y_c, \frac{dy}{dx} \to S_0$: d/s end of the curve is horizontal.

$y \to 0, \frac{dy}{dx} \to \infty$: : u/s end of the curve is normal to the bed.

Type of profile: Backwater

Type of flow: Supercritical

Example: Flow below a sluice gate provided in a channel with critical slope



V. A profile (Surface profile is adverse-sloped channels)

Since $S_0<0$ (-ve) and y_n is imaginary, A1 profile does not exist. So there are two A profiles: A2 and A3.

a. A2 profile

Condition: $y < y_c$

$\frac{dy}{dx} = \frac{S_0 - S_f}{1-(y_c/y)^3} = \frac{S_0\left(1-S_f/S_0\right)}{1-(y_c/y)^3} = \frac{-ve}{+ve} = -ve$

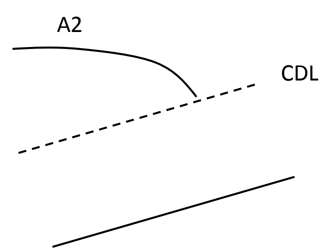
$y \to y_c, \frac{dy}{dx} \to -\infty$: d/s end of the curve is normal to CDL.

$y \to \infty, \frac{dy}{dx} \to S_0$: u/s end of the curve is horizontal.

Type of profile: Drawdown

Type of flow: Subcritical

Example: Flow profile at the D/S end of a adverse slope



b. A3 profile

Condition: $y<y_c$

$\frac{dy}{dx} = \frac{S_0 - S_f}{1-(y_c/y)^3} = \frac{S_0\left(1-S_f/S_0\right)}{1-(y_c/y)^3} = \frac{-ve}{-ve} = +ve$

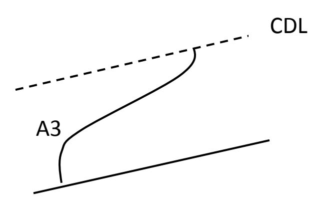
$y \to y_c, \frac{dy}{dx} \to -\infty$: d/s end of the curve is normal to CDL.

$y \to 0, \frac{dy}{dx} \to -\infty$: u/s end of the curve is normal to bed.

Type of profile: Backwater

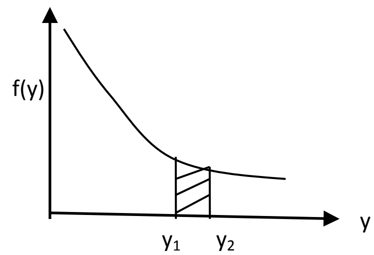
Type of flow: Supercritical

Example: Flow profile below a sluice gate provided in a adverse channel



8.4 Computation of GVF

a. Graphical integration method



This method is used to compute the distance from the given depth.

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - F_r^2}$$

x =Distance between x_1 and x_2

$$x = x_2 - x_1 = \int_{x_1}^{x_2} dx = \int_{y_1}^{y_2} \frac{dx}{dy} dy$$

$$= \int_{y_1}^{y_2} \left(\frac{1 - F_r^2}{S_0 - S_f}\right) dy = \int_{y_1}^{y_2} f(y) dy$$

Take different value of y and compute corresponding f(y), and plot f(y) vs y curve. The area enclosed between y_1 and y_2 gives the distance.

If the interval is very small, the area can be computed by multiplying the average value of f(y) with the interval dy.

b. Numerical approach

I. Direct integration method

Bresse's method

Bresse's method of integrating the varied flow equation is applicable in the case of very wide rectangular channels. In this method Chezy's formula is used for the evaluation of the effect of frictional resistance to the flow.

Using Chezy's formula

$$\frac{dy}{dx} = \frac{S_0(1 - (y_n/y)^3)}{1 - (y_c/y)^3}$$

Let $y/y_n = u$

$$\frac{dy}{dx} = \frac{S_0(1 - 1/u^3)}{1 - (y_c/y_n)^3 (1/u^3)}$$

$$S_0 dx\,(u^3 - 1) = (u^3 - (y_c/y_n)^3) dy$$

$$S_0 dx \; = ((y_c/y_n)^3 - u^3)\frac{1}{(1-u^3)}dy$$

y= uy_n
dy = $y_n du$

$$dx \; = \frac{y_n}{S_0}((y_c/y_n)^3 - u^3)\frac{1}{(1-u^3)}du$$
$$dx \; = \frac{y_n}{S_0}((y_c/y_n)^3 - 1 + 1 - u^3)\frac{1}{(1-u^3)}du$$
$$dx \; = \frac{y_n}{S_0}\left[du + ((y_c/y_n)^3 - 1)\frac{du}{1-u^3}\right]$$

Integrating

$$x \; = \frac{y_n}{S_0}\left[u - (1-(y_c/y_n)^3)\int\frac{du}{1-u^3}\right] + Constant$$

The distance x can be calculated from this equation if the value of the integral $\int\frac{du}{1-u^3}$ is known. The function $\int\frac{du}{1-u^3}$ is known as Bresse's varied flow function, which is given by

$$\int\frac{du}{1-u^3} = \varphi(u) = \frac{1}{6}ln\frac{u^2+u+1}{(u-1)^2} - \frac{1}{\sqrt{3}}tan^{-1}\frac{\sqrt{3}}{2u+1}$$

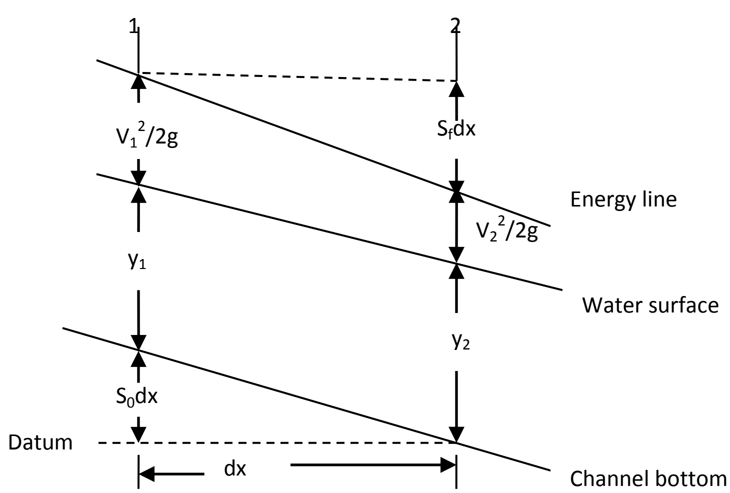
$\left(\frac{y_c}{y_n}\right)^3 = \frac{C^2 S_0}{g}$ (Using chezy)

For two sections

$$x2 - x1 \; = \frac{y_n}{S_0}\left[(u2-u1) - \left(1-\left(\frac{y_c}{y_n}\right)^3\right)\left(\varphi(u2)-\varphi(u1)\right)\right]$$

II. Direct step method

In this method, the entire length of the channel is divided into short reaches and the computation is carried out step by step from one end of the reach to the other.



Consider a channel reach of length dz. Equating the total energies at section 1 and 2

$S_0\,dx + y_1+V_1^2/2g = y_2 + V_2^2/2g + S_f\,dx$

where V_1 and V_2 are mean velocity at section 1 and 2, S_0 is bed slope and S_f is energy line slope.

$E_1 = y_1+V_1^2/2g$ and $E_2 = y_2 + V_2^2/2g$

where E_1 and E_2 are the specific energy at section 1 and 2.

From these two equations,

$S_0\,dx + E_1= E_2+ S_f\,dx$

$$dx = \frac{E_2 - E_1}{S_0 - S_f} = \frac{\Delta E}{S_0 - S_f}$$

where ΔE represent the change in specific energy between the section 1 and 2. For S_f, average friction slope is used in the computation.

$$dx = \frac{\Delta E}{S_0 - \bar{S_f}}$$

For the computation of the length x of the surface profile the various quantities that should be known, are: discharge Q, the channel shape, bottom slope S_0, Manning's n or Chezy's C and the depth of flow at one of the section (at one of the control point). The procedure for computation is as follows

Tabular form

y	A	P	R	V	E	S_f	$\bar{S_f}$	Δx	x

- For the first value of y, compute A, P, R, V, E and S_f.
- Take another value of y (increasing/decreasing from previous), and compute all variables.
- Perform computation by taking different values of y to obtain the required flow profile. (Assumed depth will be either more or less than the known depth at control section, depending on the surface profile (rising or falling)).

Formulae:

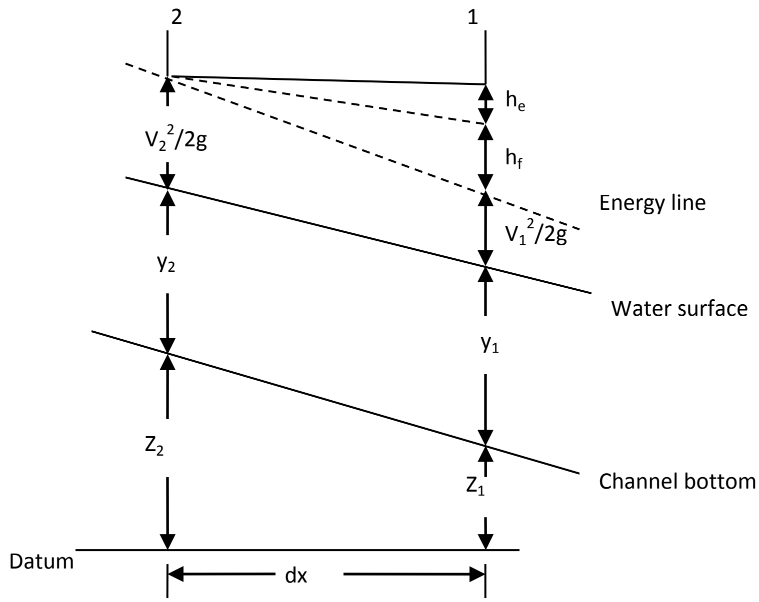
R=A/P, V= Q/A, E = y +V^2/2g, $S_f = \frac{n^2V^2}{R^{4/3}}$ using n or $S_f = \frac{V^2}{C^2R}$ using C

$\bar{S_f} = \frac{1}{2}\left[\left(S_f\right)_{i-1} + \left(S_f\right)_i\right]$

$\Delta x = \frac{E_i - E_{i-1}}{S_0 - \bar{S_f}}$, x = cumulative sum of Δx

III. Standard step method

Direct step method is suitable for prismatic channels, but there are some basic difficulties in applying it to natural channels. In natural channels, the cross-sectional shapes are likely to vary from section to section and also cross-sectional information is known only at a few locations along the channel. In such case, standard step method is applied.



Consider section 1 downstream of section 2. Proceed the calculation upstream.

Applying Bernoulli's equation at 1 and 2

$Z_1 + y_1+V_1^2/2g+ h_f + h_e = Z_2 + y_2+V_2^2/2g$

h_f = friction loss, h_e = eddy loss

$h_f = S_f\,dx$

$H_1 = Z_1 + y_1+V_1^2/2g$ = elevation of energy line at 1

$H_2 = Z_2 + y_2+V_2^2/2g$ =elevation of energy line at 2

$H_2 = H_1 + h_f + h_e$

The frictional loss, h_f is estimated by

$$h_f = \bar{S_f}dx = \frac{1}{2}(S_{f1} + S_{f2})dx$$

where $S_f = \frac{n^2V^2}{R^{4/3}}$ using n or $S_f = \frac{V^2}{C^2R}$ using C

$$h_e = C_e\left|\frac{V_1^2 - V_2^2}{2g}\right|$$

C_e = coefficient

For prismatic channel, C_e = 0.

<u>Trial and error approach</u>

Solution in tabular form (trial and error)

Stn.	Z	y	A	P	R	V	H(1)	S_f	$\bar{S_f}$	Δx	h_f	h_e	H(2)

Z= water surface elevation, y = depth of flow

- For station 1, Z is known. Compute y from the given information. (e.g. y= Z-El of dam site-S_0 dx). Then compute A, P, R, V, H(1) and S_f. H(2) = H(1) in the first step.
- For station 2, assume a value of Z or y and compute all variables. Check if H(1) is close to H(2). If not, take another value of Z and repeat the procedure until H(1) is almost equal to H(2).
- Perform similar computations for all remaining stations to obtain a flow profile.

Formulae:

R= A/P, V= Q/A, H (1) = Z + V^2/2g, $S_f = \frac{n^2 V^2}{R^{4/3}}$ using n or $S_f = \frac{V^2}{C^2 R}$ using C

$\bar{S_f} = \frac{1}{2}\left[\left(S_f\right)_{i-1} + \left(S_f\right)_i\right]$

$h_f = \Delta x \bar{S_f}$

H(2) = $(H(2))_{i-1}$ + h_f +h_e

Alternative solution for standard step

Neglecting h_e

$$H_2 = H_1 + h_f$$

$$H_2 = H_1 + \frac{1}{2}\left(S_{f1} + S_{f2}\right)(X_2 - X_1)$$

$$Z_2 + y_2 + \frac{Q^2}{2gA_2^2} - \frac{1}{2}S_{f2}(X_2 - X_1) - H_1 - \frac{1}{2}S_{f1}(X_2 - X_1) = 0$$

$$f(y_2) = Z_2 + y_2 + \frac{Q^2}{2gA_2^2} - H_1 - \frac{1}{2}\left(S_{f1} + S_{f2}\right)(X_2 - X_1)$$

In this equation, the values of variables at 1 are known. A_2 and S_{f2} are function of y_2. Hence y_2 may be determined by solving above equation. This equation may be solved for y_2 by numerical method such as Newton-Raphson method.

Chapter 9: Rapidly varied flow

9.1 Hydraulic jump

When the depth of flow changes rapidly from a low stage to a high stage, there is abrupt rise of water surface. This local phenomenon is known as hydraulic jump. The hydraulic jump occurs when a supercritical flow changes into subcritical flow. It is classified as rapidly varied flow. The flow in the hydraulic jump is accompanied by the formation of extremely turbulent rollers and hence there is a considerable dissipation of energy. The turbulent eddies break up into smaller ones as they move downstream. The energy is dissipated into heat through these small eddies. Further, air is entrained due to the breaking of number of wavelets on the surface.

The hydraulic jump occurs frequently in a canal below a regulating sluice, at the foot of a spillway, or at the place where a steep channel bottom slope suddenly turns flat.

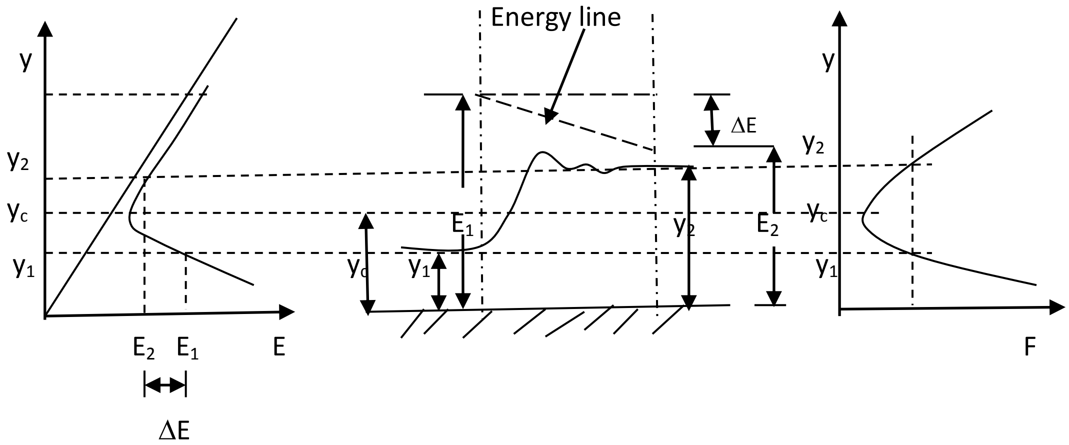
The energy equation is not applicable for the analysis of hydraulic jump because the hydraulic jump is associated with an appreciable loss of energy which is initially unknown. Therefore, momentum equation is used by considering the portion of the hydraulic jump as the control volume.

Applications of hydraulic jump

- As an energy dissipater, to dissipate the excess energy of flowing water downstream of hydraulic structure such as spillway and sluice gates
- Mixing of chemical
- Aeration of stream polluted by biodegradable waste
- Raising the water level in the channel for irrigation
- Desalination of seawater
- efficient operation of flow measurement flumes

Specific energy and specific force curves for hydraulic jump

Consider a hydraulic jump formed in a prismatic channel with horizontal floor carrying a discharge Q. The depth of flow before the jump (y_1) is called initial depth and that after the jump (y_2) is called sequent depth. The initial depth and the sequent depth are commonly known as conjugate depths. The specific force is same in case of conjugate depths, whereas the specific energy is same for alternate depths. Both the specific energy curve and the specific curve attain minimum value at the critical depth.



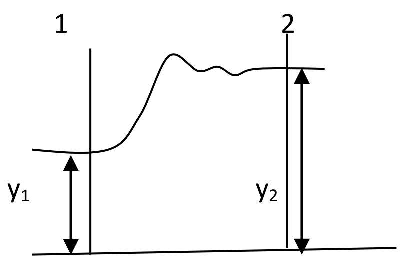
Specific energy curve Hydraulic jump Specific force curve

9.2 Hydraulic jump in a rectangular channel

Momentum equation for hydraulic jump

Assumptions

- Loss of head due to friction is negligible
- The flow is uniform and the pressure distribution is hydrostatic before and after the jump
- Channel is horizontal hence the weight component in the direction is neglected
- The momentum correction factor is unity.
- The flow is steady



Consider two sections 1 and 2 before and after the jump. Based on above assumptions, the only external forces acting on the mass of water between sections 1 and 2 are hydrostatic pressures P_1 and P_2 at sections 1 and 2 respectively. Let Q be the discharge flowing through the channel and V_1 and V_2 be the velocities at section 1 and 2 respectively. W is the weight of water enclosed between two sections; and F_f is the resistance force due to friction.

The momentum equation for the jump is given by

$$P_1 - P_2 + W sin\theta - F_f = \rho Q(V_2 - V_1)$$

For a short reach of prismatic channel, θ = 0 and friction force, F_f can be neglected.

$$P_1 - P_2 = \rho Q(V_2 - V_1)$$

a. Expression for sequent depth

Consider unit width of the channel. The discharge per unit width $q = V_1 y_1 = V_2 y_2$

$P_1 = \gamma A_1 \overline{x_1} = \gamma(1. y_1).\frac{y_1}{2} = \frac{1}{2}\gamma y_1^2$ and $P_2 = \gamma A_2 \overline{x_2} = \gamma(1. y_2).\frac{y_2}{2} = \frac{1}{2}\gamma y_2^2$

Substituting the values of P_1 and P_2

$$\frac{1}{2}\gamma y_1^2 - \frac{1}{2}\gamma y_2^2 = \rho q(V_2 - V_1)$$

From continuity equation: $q = y_1 V_1 = y_2 V_2$

$V_1 = q/y_1$, $V_2 = q/y_2$

Substituting the values of V_1 and V_2

$$\frac{1}{2}\rho g(y_1^2 - y_2^2) = \rho q\left(\frac{q}{y_2} - \frac{q}{y_1}\right)$$

$$\frac{g}{2}(y_1 + y_2)(y_1 - y_2) = q^2\left(\frac{y_1 - y_2}{y_1 y_2}\right)$$

$\frac{2q^2}{g} = y_1 y_2(y_1 + y_2)$ (I)

Dividing both sides by y_1 and simplifying

$$y_2^2 + y_1 y_2 - \frac{2q^2}{g y_1} = 0$$

Solving for y_2

$$y_2 = \frac{-y_1 \pm \sqrt{y_1^2 + 4\frac{2q^2}{g y_1}}}{2}$$

As negative root is not possible

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2q^2}{g y_1}}$$

This is the relationship between conjugate depths

Conjugate depths in terms of Froude number

Substituting $q = y_1 v_1$

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2y_1^2 {V_1}^2}{g y_1}}$$

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2\left(1 + \frac{8{V_1}^2}{g y_1}\right)}$$

$$\frac{{V_1}^2}{gy_1} = F_{r1}^2$$

$$y_2 = \frac{y_1}{2}\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)$$

Similarly

For y_1

$y_1 = -\frac{y_2}{2} + \sqrt{\left(\frac{y_2}{2}\right)^2 + \frac{2q^2}{gy_2}}$ and $y_1 = \frac{y_2}{2}\left(-1 + \sqrt{1 + 8F_{r2}^2}\right)$

The above equation is known as Belanger momentum equation.

$F_{r1}>1$

$F_{r2}<1$

The higher F_{r1}, the lower F_{r2}

Alternate derivation: Expression for sequent depth using specific force analysis

Specific force at 1 = specific force at 2

$$\frac{Q^2}{gA_1} + A_1\overline{x_1} = \frac{Q^2}{gA_2} + A_2\overline{x_2}$$

For rectangular channel of bottom width b,

$$\frac{Q^2}{gby_1} + by_1\frac{y_1}{2} = \frac{Q^2}{gby_2} + by_2\frac{y_2}{2}$$

$$\frac{b}{2}\left(\frac{2Q^2}{gb^2y_1} + y_1^2\right) = \frac{b}{2}\left(\frac{2Q^2}{gb^2y_2} + y_2^2\right)$$

$$y_2^2 - y_1^2 = \frac{2Q^2}{gb^2}\left(\frac{1}{y_1} - \frac{1}{y_2}\right)$$

$$(y_2 + y_1)(y_2 - y_1) = \frac{2q^2}{g}\left(\frac{y_2 - y_1}{y_1y_2}\right)$$

$\frac{2q^2}{g} = y_1y_2(y_1 + y_2)$ (I)

Dividing both sides by y_1 and simplifying

$$y_2^2 + y_1y_2 - \frac{2q^2}{gy_1} = 0$$

Solving for y_2

$$y_2 = \frac{-y_1 \pm \sqrt{y_1^2 + 4\frac{2q^2}{gy_1}}}{2}$$

As negative root is not possible

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2q^2}{gy_1}}$$

This is the relationship between conjugate depths

Conjugate depths in terms of Froude number

Substituting q = y_1 v_1

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 + \frac{2y_1^2 {V_1}^2}{gy_1}}$$

$$y_2 = -\frac{y_1}{2} + \sqrt{\left(\frac{y_1}{2}\right)^2 \left(1 + \frac{8{V_1}^2}{gy_1}\right)}$$

$$\frac{{V_1}^2}{gy_1} = F_{r1}^2$$

$$y_2 = \frac{y_1}{2}\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)$$

Similarly

For y_1

$y_1 = -\frac{y_2}{2} + \sqrt{\left(\frac{y_2}{2}\right)^2 + \frac{2q^2}{gy_2}}$ and $y_1 = \frac{y_2}{2}\left(-1 + \sqrt{1 + 8F_{r2}^2}\right)$

b. Relationship between downstream and upstream Froude numbers

$F_{r1} = \frac{V_1}{\sqrt{gy_1}}$ and $F_{r2} = \frac{V_2}{\sqrt{gy_2}}$

or, $V_1 = F_{r1}\sqrt{gy_1}$ and $V_2 = F_{r2}\sqrt{gy_2}$ (a)

From continuity

$V_1 y_1 = V_2 y_2$ (b)

From a and b

$$y_1\, F_{r1}\sqrt{gy_1} = y_2\, F_{r2}\sqrt{gy_2}$$

$\frac{y_1}{y_2} = \left(\frac{F_{r2}}{F_{r1}}\right)^{2/3}$ (c)

Also,

$$y_2 = \frac{y_1}{2}\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)$$

$\frac{y_1}{y_2} = \frac{2}{-1+\sqrt{1+8F_{r1}^2}}$ (d)

Equating c and d

$$\left(\frac{F_{r2}}{F_{r1}}\right)^{2/3} = \frac{2}{-1 + \sqrt{1 + 8F_{r1}^2}}$$

$$F_{r2}^2 = \frac{8F_{r1}^2}{\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)^3}$$

c. Expression for energy loss in terms of conjugate depths (Analysis using specific energy)

$$\Delta E = E_1 - E_2$$
$$= \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right)$$

From continuity $V_1 = q/y_1$, $V_2 = q/y_2$

$$= \left(y_1 + \frac{q^2}{2gy_1^2}\right) - \left(y_2 + \frac{q^2}{2gy_2^2}\right)$$
$$= \frac{q^2}{2g}\left(\frac{y_2^2 - y_1^2}{y_1^2 y_2^2}\right) - (y_2 - y_1)$$

Substituting $\frac{q^2}{g} = \frac{1}{2} y_1 y_2 (y_1 + y_2)$ from eq. (i)

$$\Delta E = \frac{1}{4} y_1 y_2 (y_1 + y_2)\left(\frac{y_2^2 - y_1^2}{y_1^2 y_2^2}\right) - (y_2 - y_1)$$

$$\Delta E = \frac{(y_2 - y_1)^3}{4 y_1 y_2}$$

d. Energy loss in terms of velocity

$$\Delta E = E_1 - E_2$$
$$= \left(y_1 + \frac{V_1^2}{2g}\right) - \left(y_2 + \frac{V_2^2}{2g}\right)$$
$$= \left(\frac{V_1^2}{2g} - \frac{V_2^2}{2g}\right) - (y_2 - y_1)$$

From continuity $y_1 = q/V_1$, $y_2 = q/V_2$

$$= \left(\frac{V_1^2 - V_2^2}{2g}\right) - \left(\frac{q}{V_2} - \frac{q}{V_1}\right)$$
$$= \left(\frac{V_1^2 - V_2^2}{2g}\right) - q\left(\frac{V_1 - V_2}{V_1 V_2}\right)$$

We have, $\frac{2q^2}{g} = y_1 y_2 (y_1 + y_2)$

$$\frac{2q^2}{g} = \frac{q^2}{V_1 V_2}\left(\frac{q}{V_1} + \frac{q}{V_2}\right)$$

$$q = \frac{2(V_1 V_2)^2}{g(V_1 + V_2)}$$

Substituting the value of q

$$\Delta E = \left(\frac{V_1^2 - V_2^2}{2g}\right) - \frac{2(V_1 V_2)^2}{g(V_1 + V_2)}\left(\frac{V_1 - V_2}{V_1 V_2}\right)$$

$$\Delta E = \frac{(V_1 - V_2)^3}{2g(V_1 + V_2)}$$

e. Other characteristics of jump

I. Relative loss= $\Delta E/E_1$ where E_1 = specific energy before jump

Relative loss in terms of Froude numbers

$$\frac{\Delta E}{E_1} = \frac{\frac{(y_2 - y_1)^3}{4y_1 y_2}}{y_1 + \frac{V_1^2}{2g}}$$

$$= \frac{\frac{y_1^3\left(\frac{y_2}{y_1} - 1\right)^3}{4y_1 y_2}}{y_1\left(1 + \frac{V_1^2}{2gy_1}\right)} = \frac{\left(\frac{y_2}{y_1} - 1\right)^3}{4\frac{y_2}{y_1}\left(1 + \frac{V_1^2}{2gy_1}\right)}$$

We have,

$y_2 = \frac{y_1}{2}\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)$ or, $\frac{y_2}{y_1} = \left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8F_{r1}^2}\right)$

$$= \frac{\left(\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8F_{r1}^2}\right) - 1\right)^3}{4\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8F_{r1}^2}\right)\left(1 + \frac{F_{r1}^2}{2}\right)} = \frac{\left(-3\sqrt{1 + 8F_{r1}^2}\right)^3}{8(2 + F_{r1}^2)\left(-1 + \sqrt{1 + 8F_{r1}^2}\right)}$$

II. Height of jump (h_j) = $y_2 - y_1$

Relative height = $\frac{h_j}{E_1}$

Relative height in terms of Froude numbers

$\frac{h_j}{E_1} = \frac{y_2 - y_1}{y_1 + \frac{V_1^2}{2g}} = \frac{y_1\left(\frac{y_2}{y_1} - 1\right)}{y_1\left(1 + \frac{V_1^2}{2gy_1}\right)}$

Substituting for y_2/y_1

$$\frac{h_j}{E_1} = \frac{\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8F_{r1}^2}\right) - 1}{\left(1 + \frac{F_{r1}^2}{2}\right)} = \frac{-3 + \sqrt{1 + 8F_{r1}^2}}{(2 + F_{r1}^2)}$$

III. Length of jump = $6(y_2 - y_1)$

IV. Efficiency of jump = E_2/E_1 where E_2 = specific energy after jump and E_1 = specific energy before jump

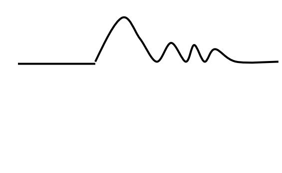
V. Power dissipated by the jump = $\gamma Q(\Delta E)$

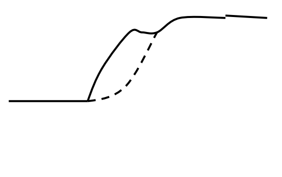
9.3 Classification of jump

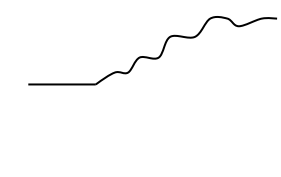
a. Based on the Froude number

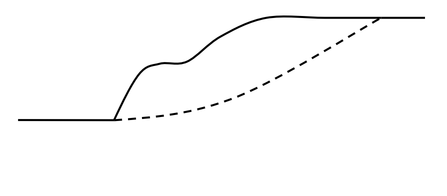
Based on the Froude's number (F_{r1}) of the supercritical flow, the jump can be classified into following five categories.

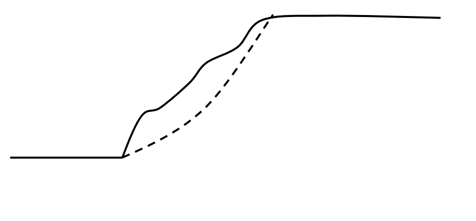
- Undular jump: $1.0<F_{r1}<1.7$, Undulating water surface with very small ripples on the surface, insignificant energy loss
- Weak jump: $1.7<F_{r1}<2.5$, A series of small roller on the surfaces of water, low energy loss.
- Oscillating jump: $2.5<F_{r1}<4.5$, Oscillation between the bed and the surface due to high velocity, moderate energy loss
- Steady jump: $4.5<F_{r1}<9.0$, Well established, significant energy loss
- Strong jump: $F_{r1}>9.0$, Rough (wavy) water surface, very efficient energy dissipation


Undular jump


Weak jump


Oscillating jump


Steady jump


Strong jump

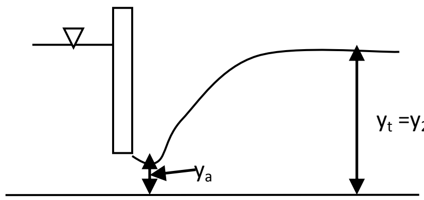
b. Based on tail water depth

The depth downstream of a hydraulic structure is called tailwater depth.
y_t = tailwater depth, y_a = Depth at the vena-contracta, y_2 = sequent depth to y_a

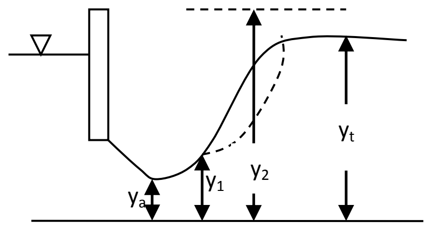
Free jump: The jump with sequent depth equal to or less than y_2 is called free jump. When $y_t = y_2$, a free jump will form at the vena-contracta.

Repelled jump: If $y_t < y_2$, the jump is repelled downstream of the vena-contracta through an M_3 curve (or may be H_3). The depth at the toe of the jump is larger than y_a. Such a jump is called a repelled jump.

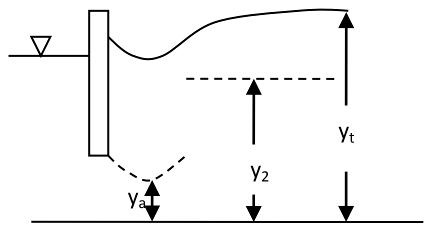
Submerged jump: If $y_t > y_2$, the jump is no longer free but gets drowned out. Such a jump is called drowned jump or submerged jump. The loss of energy in a submerged jump is smaller than that in a free jump.



Jump at vena-contracta



Repelled jump



Submerged jump

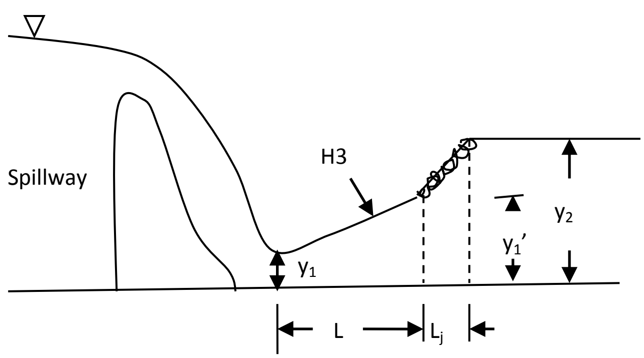
9.4 Location of jump

The formation of jump depends upon the initial depth, sequent depth and Froude number of flow. The position of jump can be fixed by introducing an obstruction or can be determined by using Belanger equation in case of naturally occurring jump.

$$y_1 = \frac{y_2}{2}\left(-1 + \sqrt{1 + 8F_{r2}^2}\right)$$

For given value of y_2, y_1 can be calculated using this equation. The length of jump can be computed by jump equation and the length of flow profile formed before or after the jump can be calculated by using any gradually varied profile computation method.

a. Jump below overfall spillway



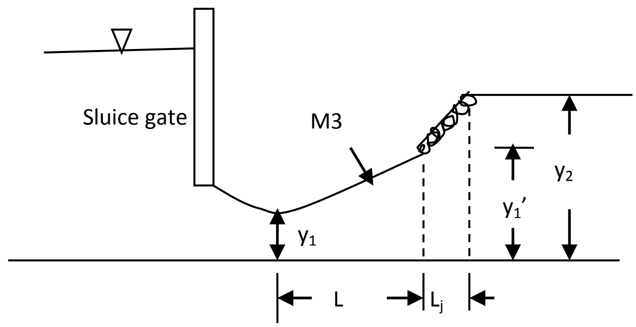
y_1 = Depth at vena-contracta (given)
y_2 = known or equal to y_n if not given
y_1' = depth sequent to y_2
Compute y_1' from Belanger equation.

$L_j = 6(y_2 - y_1)$

Compute length (L) of H3 profile by direct step or other method taking depths at two ends as y_1 and y_1'.

Location of jump = distance L from vena-contracta.

b. Jump below sluice gate in mild slope



y_1 = Depth at vena-contracta (given)

y_2 = known or equal to y_n if not given

y_1' = depth sequent to y_2

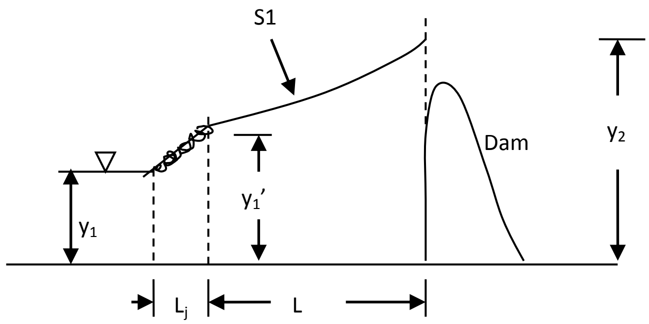
Compute y_1' from Belanger equation.

$L_j = 6(y_2 - y_1)$

Compute length (L) of M3 profile by direct step or other method taking depths at two ends as y_1 and y_1'.

Location of jump = distance L from vena-contracta.

c. Jump in steep slope with barrier, e.g. dam



y_2 = Known depth at dam section

y_1 = Normal depth

y_1' = Depth sequent to y_1

y_2 should be greater than y_1' for forming jump.

Compute y_1' from Belanger equation.

$L_j = 6(y_1' - y_1)$

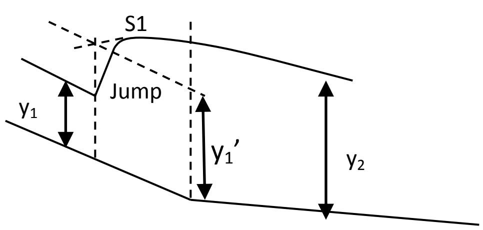
Compute length (L) of S1 profile by direct step or other method taking depths at two ends as y_1' and y_2.

Location of jump = distance $L+L_j$ from dam.

d. Jump formed due to change in slope from steep to mild/flat

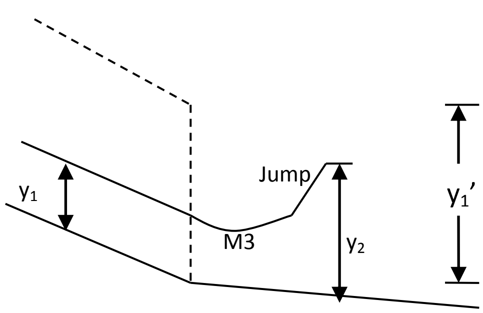
y_1 = normal depth at u/s channel (known or compute from given data)
y_2 = normal depth at d/s channel (known or compute from given data)
y_1' = Depth sequent to y_1 (Compute from Belanger equation)

(I) If $y_1' < y_2$, the jump will be formed on the u/s channel.



Compute the length (L) of S1 profile by direct step or other method taking depths at two ends as y_2 and y_1'.
location of jump = Distance L upstream from the point of change in slope.

(II) If $y_1' > y_2$, the jump will be formed on the d/s channel.



y_2' = Depth sequent to y_2 (Compute from Belanger equation)
Compute the length (L) of M3 profile by direct step or other method taking depths at two ends as y_1 and y_2'.
Location of jump = Distance L downstream from the point of change in slope.

Chapter 10: Flow in mobile boundary channel

10.1 Introduction

Sediment (Alluvium)

- loose non-cohesive material transported or deposited by the action of flowing water

Types of sediments

Bed load: sediment which moves on or near bed, movement by sliding or rolling
Suspended load: sediment which is in suspension

Types of Open channel

a. Rigid boundary channel (Non erodible boundary)

Rigid boundary channels are those channels whose bed and banks are made up of non-erodible material. This type of channel can resist erosion satisfactorily. The resistance to the water flowing in a rigid boundary channel depends only on the nature of the boundary surface and it can be determined up to fair degree of accuracy. The dimensions of the channel can be calculated by uniform flow equations, such as Manning or Chezy.

b. Mobile boundary channel (Erodible boundary channel)

Mobile boundary channels are those channels whose boundary is made up of loose soil which can be easily eroded and transported be the flowing water. The mobile boundary channel includes rivers and unlined alluvial canals. In case of mobile boundary channels, the resistance to the flowing water depends not only on the boundary surface but also on the condition of bed and the banks of the channel. Uniform flow formula is insufficient condition for designing movable boundary. This is because stability of erodible channel depends mainly on the properties of the material for the channel body.

10.2 Minimum permissible velocity approach for the design of rigid boundary channel

It represents the lowest velocity which will prevent both sedimentation and vegetation growth. In general 0.6 to 0.9 m/s will prevent both sedimentation and vegetation growth when the silt load in the flow is low. This is an important criterion for designing rigid boundary channel.

Maximum permissible velocity

It is the greatest mean velocity that will not cause erosion of the channel bed and bank. This is an important criterion for designing movable boundary channel.

Permissible velocity method for design of channels

In this method, the channel size is selected such that the mean velocity of flow for the designing discharge is less than permissible flow velocity. The permissible velocity basically depends on type of soil, the size of particles and the depth of flow and alignment of channel.

Recommended permissible velocity for 1m depths

Material	V m/s
Fine sand	0.6
Coarse sand	1.2
Earthen channel	
Sandy -silt	0.6
Silt - clay	1.1
Clay	1.8
Grass lined earthen channel	
Sandy-silt	1.8
Silt-clay	2.1
Self stone	2.4
Hard rock	6.1

Suggested side slope

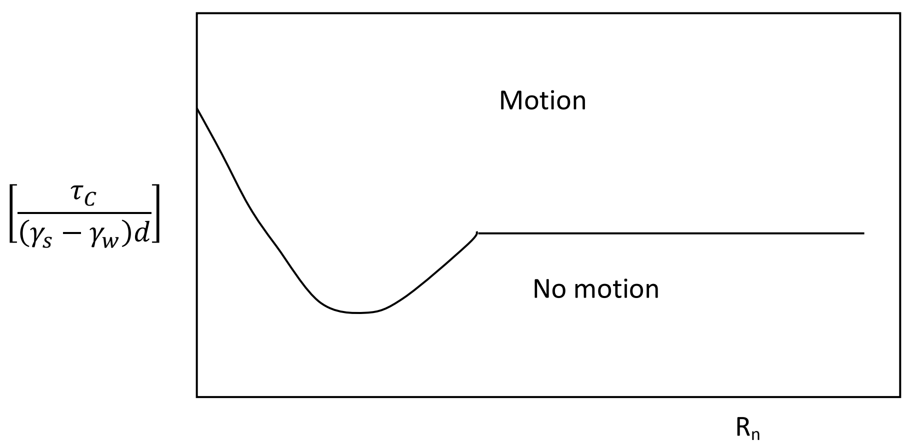
Material	Side slope H:V
Rock	Nearly vertical
Stiff clay	0.5:1 to 1:1
Loose sandy clay	2:1
Sandy loam	3:1

10.3 Incipient motion condition

At low discharge, the sediment remains stationary on the bed and clear water flows over the sediment. At low flow, depth is low and shear stress is also small. If the discharge is gradually increased, a stage will come when the shear stress begins to exceed the force opposing the movement of particles. In such stage, particles on the bed begin to move intermittently. This condition in which sediment just begins to move is called incipient motion condition or the critical condition. The bed shear stress corresponding to incipient motion is called critical shear stress or critical tractive force. When average shear stress on the bed (τ_0) is equal to or greater than the critical value (τ_c), the particles on the bed starts to move in the direction of flow. The knowledge of incipient motion condition is useful in fixing the bed slope or depth of clear water flow to reservoir to minimize silting problem in reservoir.

10.4 Shield's diagram for the studying incipient motion

According to Shield, the tractive force exerted by the flowing water on the sediment to cause motion is equal to boundary shear, τ_0. The relative magnitude of tractive force and resistance to motion of uniform sediment grain is expressed by the dimensionless ratio $\left[\frac{\tau_c}{(\gamma_s-\gamma_w)d}\right]$, which is a significant parameter in the bed load movement. In this expression, γ_s and γ_w are specific weight of sediment and water respectively and d is the grain diameter. The dimensional less ratio is termed as entrainment function or non-dimensional shear stress. Another non-dimensionless number is the Shear Reynold's number (R_n), which is given by $R_n = \frac{V*d}{\upsilon}$, where V_* = shear velocity, υ = kinematic viscosity



Shield's curve

By plotting $\left[\frac{\tau_C}{(\gamma_s-\gamma_w)d}\right]$ against $\frac{V*d}{\upsilon}$, a curve is obtained which is known as Shield's curve. This curve is used to establish the criteria for incipient sediment motion. The salient features of the curve are as follows:

- For $R_n \leq 2$, the grain is completely enclosed in laminar sub-layer and τ_C is not affected by the particle size.
- For $2 < R_n < 400$, the flow is in transition stage where both the particle size and fluid viscosity affect τ_C.
- For R_n >400, the curve becomes horizontal. At that time the value of the entrainment function becomes constant at about 0.06 (independent of Reynolds no.), thereby indicating that in turbulent flow τ_C is proportional to $(\gamma_s - \gamma_w)d$. That means τ_C is a function of particle size only. For$R_n > 400$ and the entrainment function greater than or equal to 0.06, the sediment attains incipient motion condition.
- The minimum value of the entrainment function = 0.03 at R_n = 10. That means for R_n > 10, the entrainment function should be greater than or equal to 0.03 for the sediment motion to occur.

If the median size of the particle (d) is greater than 6mm, then critical shear stress is given by

$$\tau_c = 0.06(\gamma_s - \gamma_w)d$$

Taking sp gr of sediment = 2.65,

$\tau_c = 0.06(2.65x9810 - 9810)d/1000$ = 0.905d

If d<6mm, trial and error approach has to be adopted to find critical shear stress from Shield's curve. Swamee and Mittal have expressed the relationship for Shield's curve for d<6mm as

$\tau_c = 0.155 + \frac{0.409d^2}{(1+0.177d^2)^{1/2}}$

where d is in mm and τ_c is in Pa.

Determination of size of sediments (d_c) that will not be removed from the bed (valid for d≥6mm)

$$\frac{\tau_c}{(\gamma_s - \gamma_w)d} = 0.06$$

$$\tau_c = 0.06(\gamma_s - \gamma_w)d_c$$

$$\gamma_w RS = 0.06(\gamma_s - \gamma_w)d_c$$

where R = Hydraulic radius and S = longitudinal slope

Taking sp gr of sediment = 2.65,

$$9810xRS = 0.06(2.65x9810 - 9810)d_c$$

d_c = 10RS

10.5 Alluvial channel

An alluvial channel is defined as channel which transports water as well as sediment and the sediment transported by the channel has same properties as the material of the channel boundary. Such a channel is said to be stable if the sediment inflow into a channel reach is equal to sediment outflow. Under such an equilibrium condition, the bed of the channel neither rises nor falls. Obviously, the shape, longitudinal slope and cross-sectional dimensions of such a stable channel depend on the discharge, the size of the sediment and the sediment load to be carried.

10.6 Design of mobile boundary channel

a. Minimum permissible velocity method

Design procedure

- Select Manning's n for the given material of channel boundary.
- Select side slope for given material.
- Find permissible velocity.
- From permissible velocity and S, calculate R using $V = \frac{1}{n}R^{2/3}S^{1/2}$
- For given discharge Q, find area A = Q/V. Solve the equation for A and R simultaneously and find b and y.
- Final criteria : V< Vpermissible

b . Tractive force method

Tractive force is the drag force exerted by the flowing water on the sediment particles, thereby causing their motion. This force is due to boundary shear stress, which is equal to the tractive force per unit area. The tractive force required to initiate general movement of grains is called critical tractive force. It is the function of material size and the sediment concentration.

Critical tractive force approach for the design of stable channel

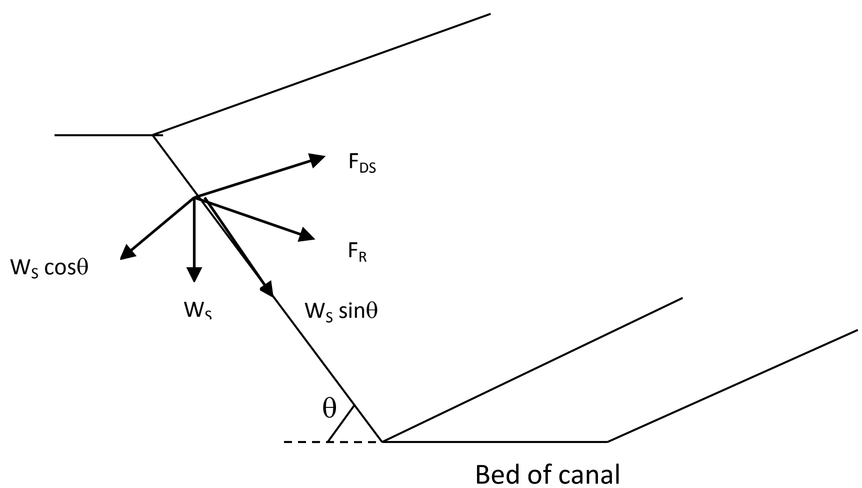
This approach attempts to restrict shear stress anywhere in the channel to a value less than the critical shear stress of bed material. If the channel carries clear water, there will be no deposition problem and hence the channel will be of stable cross-section.

Stability of particle on side slope

Consider a particle on the side of channel of inclination θ to the horizontal.

d= size of particle so that its effective area, a = $C_1 d^2$ where C_1 is coefficient

W_s = submerged weight of particle = $C_2(\gamma_s - \gamma_w)d^3$ where γ_s and γ_w are specific weight of sediment and water respectively , and C_2 is coefficient



Components of W_s are $W_s\sin\theta$ and $W_s\cos\theta$. Due to the flow, a shear stress, τ_w exists on the particle situated on the slope. The drag force on the particle due to shear is

$$F_{Ds} = \tau_w a$$

Resultant force tending to move the particle is

$$F_R = \sqrt{F_{Ds}^2 + W_s^2 sin^2\theta}$$

Stabilizing force is

$$F_s = W_s cos\theta$$

At the condition of incipient motion, $\frac{F_R}{F_S} = tan\varphi$ where φ = angle of repose of sediment particles under water (maximum angle at which the pile of sediments will be accumulated without sliding). Substituting the values of F_R and F_S

$$\frac{F_{Ds}^2 + W_s^2 sin^2\theta}{W_s^2 cos^2\theta} = tan^2\varphi$$

$$F_{Ds} = W_s cos\theta\ tan\varphi \left[1 - \frac{tan^2\theta}{tan^2\varphi}\right]^{1/2}$$

For θ = 0, the drag force on a particle situated on a horizontal bed at the time of incipient motion is obtained. Thus if τ_b = shear stress on the bed of a channel,

$$F_{Db} = \tau_b\ a = W_s\ tan\varphi$$

$$\frac{F_{Ds}}{F_{Db}} = \frac{\tau_w}{\tau_b} = cos\theta\ \left[1 - \frac{tan^2\theta}{tan^2\varphi}\right]^{1/2} = K_1$$

$K_1 = cos\theta\ \left[1 - \frac{tan^2\theta}{tan^2\varphi}\right]^{1/2}$ Or, $K_1 = \left[1 - \frac{sin^2\theta}{sin^2\varphi}\right]^{1/2}$

$$\tau_w = K_1\tau_b$$

$\tau_b = K_2\tau_c$ where τ_c = critical shear stress and K_2 = coefficient

$$\tau_w = K_1 K_2 \tau_c$$

Value of K_2
Straight channel = 0.9
Slightly curved = 0.81
Moderately curved = 0.67
Very curved = 0.54

Design of chanel using tractive force method

I. Determine angle of repose φ .
II. Establish longitudinal slope from topographical consideration. Fix side slope from practical and constructional aspects.
III. Estimate Manning's n. (e.g. by using Strickler's formula, n= $d^{1/6}$/21.1 or from references)
IV. Compute critical shear stress by using Shield's curve or empirical equations.
V. Design the non-erodible channel to withstand the maximum shear stress that may occur anywhere in the perimeter of the channel.

For trapezoidal channel of normal depth y_0 and longitudinal slope S_0, maximum shear stress on the sides $(\tau_w)_{max}$ and bed $(\tau_b)_{max}$ can be given by

$$(\tau_w)_{max} = 0.75\gamma S_0 y_0$$
$$(\tau_b)_{max} = \gamma S_0 y_0$$

For non-erodibility condition,

$$\tau_w \le (\tau_w)_{max}$$

$K_1 K_2 \tau_c \le 0.75\gamma S_0 y_0$ (a)

and

$$\tau_b \le (\tau_b)_{max}$$

$K_2 \tau_c \le \gamma S_0 y_0$ (b)

The lesser of the two values of y_0 obtained from a and b is adopted. With y_0 known, the width of channel is determined by using Manning's formula.

c. Regime theory approach

A channel in which neither silting nor scouring takes place is called regime channel or stable channel. This stable channel is said to be in a state of regime if the flow is such that silting and scouring need no special attention. The basis of designing such an ideal channel is that whatever silt has entered the channel at its canal head, it is always kept in suspension and not allowed to settle anywhere along its course. Simultaneously, velocity of water is such that it does not produce local silt by erosion of channel bed or sides.

Lacey's regime theory

According to Lacey, dimensions of bed width, depth and slope of channel attain a state of equilibrium with time which is called regime state. Lacey defined a regime channel as a channel which carries constant discharge under uniform flow in an unlimited incoherent alluvium having the same characteristics as that transported without changing bottom slope, shape or size of cross-section over a period of time. Thus in regime channel, there will be suspended load, bed load and formation of bed forms. In the initial state, depth, width and longitudinal slope may change. The continuous action of water overcomes the resistance of banks and sets up a condition such that the channel adjusts its complete section, then final regime condition is reached. After attaining regime state, the dimensions of the channel will remain constant over time.

Design procedure by Lacey's theory

Values of discharge (Q), sand size (d_{mm} in mm), side slope Z:1 (if not given assume 0.5:1) are given.

- Compute silt factor (f_s) by using eq. $f_s = 1.76\sqrt{d_{mm}}$.
- Compute longitudinal slope by using eq. $S = \frac{0.0003 f_s^{5/3}}{Q^{1/6}}$.
- Compute hydraulic radius by using eq. $R = 0.48\left(\frac{Q}{f_s}\right)^{1/3}$.
- Compute wetted perimeter by using eq. $P = 4.75\sqrt{Q}$.
- Compute cross sectional area (A) from P and R (A=PR).
- For trapezoidal section

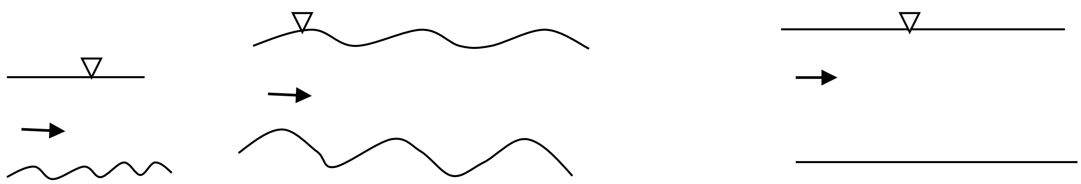
$$A = (b + Zy)y$$
$$P = b + 2y\sqrt{1 + Z^2}$$

- Substitute the values of A and P, and solve for y and b.

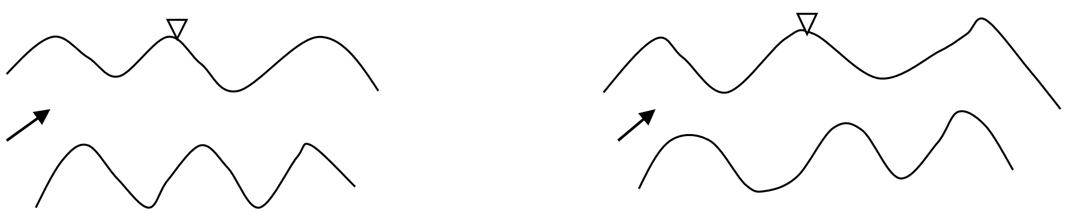
10.7 Bed forms in Alluvial Stream

When average shear stress on the bed of alluvial channel (τ_0) is greater than the critical value (τ_c), the particles on the bed starts to move in the direction of flow. Depending on the sediment size, fluid and flow condition, the bed and water surface attain different forms. The features that form on the bed under different conditions are called bed forms. The characteristics of different bed forms are known as regime of flow.

Types of bed forms (deformations)



Ripples Dunes Washed out dunes



Standing wave Antidunes

a. Plane bed with no sediment motion: When $\tau_0 < \tau_c$, the sediments on the bed do not move. The bed remains plane and the channel behaves as a rigid boundary channel.

b. Ripples and Dunes

Ripples: When τ_0 is moderately greater than τ_c, the sediment particles start moving and small unsymmetrical triangular undulations appear on the bed. These are known as ripples. The length of these undulations is less than 0.4m and the height is less than 40mm. The sediment size is normally below 0.6mm. The sediments move by sliding or rolling. Water surface is fairly smooth.

Dunes: With the increase in discharge, the ripples grow in size with flat upstream face and steep downstream face. They are larger in size than ripples and are called dunes. Water surface is not smooth and some of the particles may remain in suspension. Their length varies from 0.3m to several m and height from 30mm to several cm. The sediment is eroded on u/s side. This process results in the apparent d/s movement of the dunes at a speed much less than mean stream velocity. The ripples or dunes or both offer a relatively large resistance to flow as compared with plane bed. Both are formed when Fr<1 (subcritical condition).

c. Transition

If the discharge is increased in duned bed, dunes are washed away leaving only a small undulation. In some cases, the dunes may be completely washed out creating plane bed. A very small increase in discharge on such a bed may lead to the formation of sinusoidal waves on the bed and water surface, which are known as standing waves. These two types of bed form are designated as transition. Fr is relatively high and bed form is unstable. The transition regime of flow offers relatively low resistance to flow.

d. Antidunes: With further increase in discharge, the intensity of sediment transport increases and symmetrical bed and water surface waves appear. Although the sediment moves d/s, the crest of the bed wave move u/s. These undulations are called antidunes. The waves gradually grow steeper and then break. Antidunes occur in supercritical flow and the sediment transport rate will be very high. However, the resistance to flow is small compared to ripples and dunes.