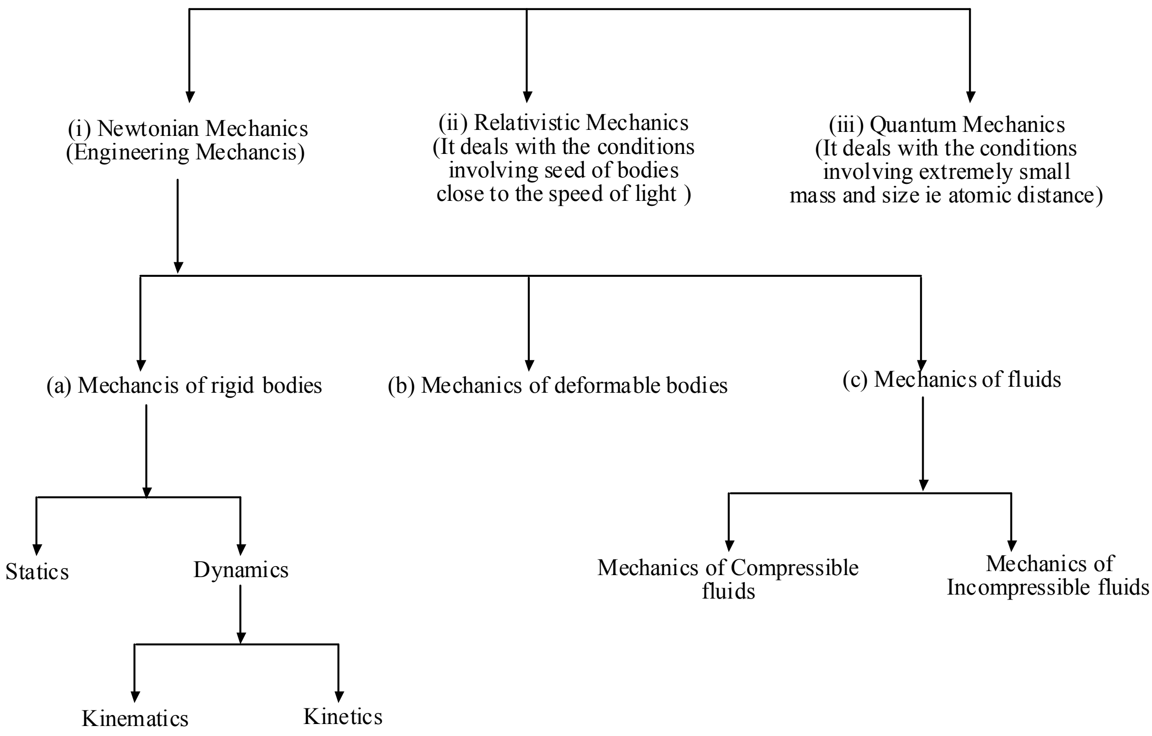
CHAPTER- 1:
INTRODUCTION TO DYNAMICS

Mechanics as the origin of Dynamics:

Mechanics is defined as that science which describe and predicts the conditions of rest or motion of bodies under the action of forces. It is the foundation of most engineering sciences. It can be divided and subdivided as below:



Dynamics:

It is which of Newtonian Mechanics which deals with the forces and their effects, while acting upon the bodies in motion. When we talk about the motion of the planets in our solar system, motion of a space craft, the acceleration of an automobile, the motion of a charged particle in an electric field, swinging of a pendulum, we are talking about Dynamics.

Kinematics:

It is that branch of Dynamics which deals with the displacement of a particles or rigid body over time with out reference to the forces that cause or change the motion. It is concerned with the position, velocity and acceleration of moving bodies as functions of time.

Kinetics:

It is that branch of Dynamics which deals with the motion of a particle or rigid body, with the reference to the forces and other factor that cause or influence the motion. For the study of motion Newton's Second Law is widely used.

Chapter:- 2

Determination of motion of particles:

- In general motion of particles (position, velocity and acceleration) is expressed in terms of function as,
  $X = f(x)$, [$x = 6t^2 + t^3$]

* But in practice the relation of motion may be defined by any other equation with function of x, v,& t .

$a = f(t)$
$a = f(x)$
$a = f(v)$ etc.

so these given relation are integrated to get the general relation of motion $x = f(t)$.

Case-I: When acceleration is given as function of time [i.e $a = f(t)$] [$a = 6t^2+t^3$]
We know,
$a = dv/dt \rightarrow dv = adt$
or, $dv = f(t)\, dt$
Now integrating both sides taking limit as time varies from 0 to t and velocity varies form v_o to v.

$$\int_{v_o}^{v} dv = \int_0^t f(t)dt$$

$$v - v_o = \int_o^t f(t)dt$$

$$v = v_o + \int_o^t f(t)dt \quad ....(i)$$

Again, velocity is given by,
$V = dx/dt \rightarrow dx = vdt$
Again integration both sides of equation similarly form time 0 to t and position x_o to x.
We get,

$$\int_{x_o}^{x} dx = \int_0^t vdt$$

$x - x_o = \int_o^t \left[v_0 + \int_0^t f(t)dt \right] dt$ Putting value of V form equation (i)

$$x = x_o + \int_0^t \left[v_o + \int_0^t f(t)dt \right] dt \quad ....(ii)$$

Thus position is obtained from equation of $a = f(t)$

\# Find the velocity and position of a particles after its 5 sec from Rest, which moves with equation of $a = 6t^2-4t$.
Solution:
Given equation $a = f(t) \rightarrow a = 6t^2 - 4t$
$x_o = 0$, $v_o = 0$ and $t = 5$.
We know,

$$V_0 = \int_o^t f(t)dt = \int_o^t f(t)dt = \int_0^5 (6t^2 - 4t)dt = \left[\frac{6t^3}{3} - \frac{4t^2}{2} \right]_0^5$$

$$v = \left[2t^3 - 2t^2 \right]_0^5 = 200\, m/s$$

Again,

$$X = x_o + \int_0^t \left[v_o + \int_0^t f(t)dt \right] dt$$

$$= 0 + \int_0^t [0 + 200] dt = \int_0^5 200 dt = [200t]_0^5$$

$= 100$ m

Therefore, x = 100m and v = 200m/s after 5 second of motion.

Case-II When the acceleration is a given function of position [i.e a = f(x) eg. x^2+4x]

We know,

a = dv/dt = dv/dx . dx/dt = v.dv/dx

or, vdv = adx

or, vdv = f(x)dx [$\because$ a = f(x)]

Now , Integrating both sides of above equation , taking limit as velocity varies from V_o to v as position p varies form x_o to x.

$$\text{i.e } \int_{v_o}^{t} vdv = \int_{x_o}^{x} f(x)dx \Rightarrow \left[\frac{v^2}{2} \right]_{v_0}^{v} = \int_{x_o}^{x} f(x)dx$$

$$\text{or, } \frac{v^2}{2} - \frac{v_0^2}{2} = \int_{x_0}^{x} f(x)dx$$

$$\because\ v = \left[v_0^2 + 2\int_{x_0}^{x} f(x)dx \right]^{\frac{1}{2}} \text{.......(1)}$$

Again We know,

V = dx/dt $\Rightarrow$ dx = vdt.

Integrating both sides with limits as time varies from 0 to t and position from x_o to x .

$$\text{i.e } \int_{x_o}^{x} dx = \int_{o}^{t} vdt \text{ ....(1)}$$

Putting value of v varies from equation (1) we get,

$$x - x_o = \int \left[v_0^2 + 2\int_{x_o}^{x} f(x)dx \right]^{\frac{1}{2}} dt$$

$$\therefore\ x = x_0 + \int_0^t \left[v_o^2 + 2\int_{x_0}^{x} f(x)dx \right]^{\frac{1}{2}} dt$$

Case III : When acceleration is a given function of velocity (i.e a =f(v) eg. a = v^2+v)

We know, a = v dv/dx $\Rightarrow$ f(v) = v dv/dx

Or , dx = v dv/f(v)

Integrating both sides taking limit as velocity varies form v_o to v and position varies from x_o to x

$$\int_{x_0}^{x} dx = \int_{v_0}^{v} v \frac{dv}{f(v)} \Rightarrow x - x_0 = \int_{v_0}^{v} v \frac{dv}{f(v)}$$

$$\therefore\ x = x_0 + \int_{v_0}^{v} v \frac{dv}{f(v)}$$

e.g The acceleration of a particle is defined as a = $-0.0125v^2$, the particle is given as initial velocity v_0, find the distance traveled before its velocity drops to half.

Solution:

Given, a = $-0.0125v^2$ i.e a = f(v) ,

Initial velocity v_o, final velocity $v_o/2$

For motion a = f(v) $x_o = 0$, x = ?

$$x = x_o + \int_{v_0}^{v} v\frac{dv}{f(v)}$$

$$x = \int_{v_0}^{\frac{v_0}{2}} \frac{v}{-0.0125v^2}dv = \frac{-1}{0.0125}\int_{v_0}^{\frac{v_0}{2}} \frac{1}{v}dv$$

$$= -\frac{1}{0.0125}\left[\ln v\right]_{v_0}^{\frac{v_0}{2}} = -\frac{1}{0.0125}\ln\left[\frac{v_0}{2v_0}\right]$$

Or, x = 24.08 m ans.

2.2 Uniform Rectilinear motion:

* Uniform motion means covering equal distance over equal intervals of time. ie velocity = constant.

We have,

V = dx/dt = v [v = constant velocity of body]

$\therefore$ dx = udt $\Rightarrow \int_{x_0}^{x} dx = \int_0^t vdt$ [Integrating both sides under limits as position varies from x_o to x and time 0 to t]

$\therefore x - x_o = vt$

$x = x_o + vt$

$\therefore$ Change in position (or displacement) is equal to uniform velocity x change in time [i.e s = vt]

2.3 Uniform Accelerated Rectilinear motion:

If constant acceleration be 'a' then,

dv/dt = a = constant $\Rightarrow$ dv = adt

Integrating both sides with limit v_0 to v and 0 to t .

We get,

$$\int_{v_0}^{v} dv = a\int_0^t dt \Rightarrow v - v_0 = at$$

$$v = v_0 + at \quad ....(1)$$

Again for position , we have

v = dx/dt .....(2) from 1 and 2.

dx = (v_o +at) dt ,

Integrating both sides over the limits

$$\int_{x_0}^{x} dx = \int_0^t (v_0 + dt)at \Rightarrow x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

Also, a = $v\frac{dv}{dx}$

Or, vdv = adx

Integrating both sides under limits

$$\int_{v_0}^{v} vdv = a\int_{x_0}^{x} dx$$

$$\frac{1}{2}\left(v^2 - v_0^2\right) = a\left(x - x_0\right)$$

$$v^2 = v_0^2 + 2a\left(x - x_0\right)$$

x_o = Initial position
x = Final position
v_0 = Initial velocity.
v = Final velocity
0 = Initial time
t = Final time

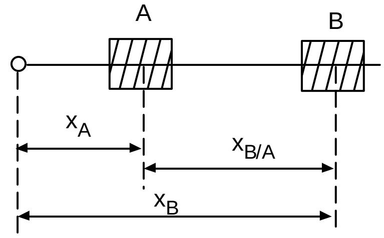
2.4 Motion of several particles:

Two or more particles moving in straight line.

Equations of motion may be written for each particles as:

(a) Relative motion of two particles
(b) Dependent motion.

(a) Relative motion of two particles:
Consider two particles A and B moving along the same straight line as follows:



$[x_{B/A} = x_B - x_A]$

Position co-ordinates of A = x_A
Position co-ordinates of B = x_B
Relative position co-ordinate of B w.r.t A = x_B-x_A = $x_{B/A}$

$\therefore$ $x_B = x_A + x_{A/B}$ ....(1) [+ $x_{B/A}$ $\rightarrow$ B is right to A in position]
[-$x_{B/A}$ $\rightarrow$ B is left to B in position]

Differentiating equation (1) w.r.t time we get,

$V_B = V_A + V_{B/A}$ ------(2) [V_A , V_B $\rightarrow$ absolute velocity of pt. A and B]
[$V_{B/A}$ $\rightarrow$ velocity of B observed from pt. A]

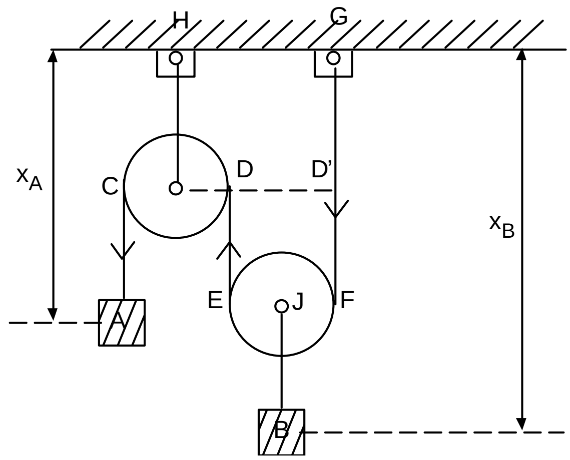
Differentiating equation (2) w.r.t time we get,
$a_B = a_A + a_{B/A}$ .... (3)

[a_B , a_A $\rightarrow$ absolute velocity of pt A and B and $a_{B/A}$ acceleration of pt B w.r.t pt A.]
(b) Dependent Motion:
When the position of a particles will dependent upon the position of another or several other particles, the motions are said to be dependent. eg :- pulley systems, Gear system etc.

Pulley system as Dependent motion:
Consider the pulleys system in which the position of Block 'B' depends upon the position of Block A as follows:

From Figure:

IH = Constant



JB = Constant
Arc CD = Constant
Arc EF = Constant

AC+DE+FG = Constant
Now,
X_A = AC +IH = AC + Constant .... (i)
X_B = FG +JB = FG + Constant ....(ii)

Multiplying equation (ii) by 2 and Adding to (ii) we get,
$X_A + 2x_B$ = AC+2FG+ Constant
= AC + FG+FG +Constant
= AC +FG+DE+D'G+ constant [Since FG = DE+D'G]
=AC+FG+DE+ Constant [Since D'G = Constant]

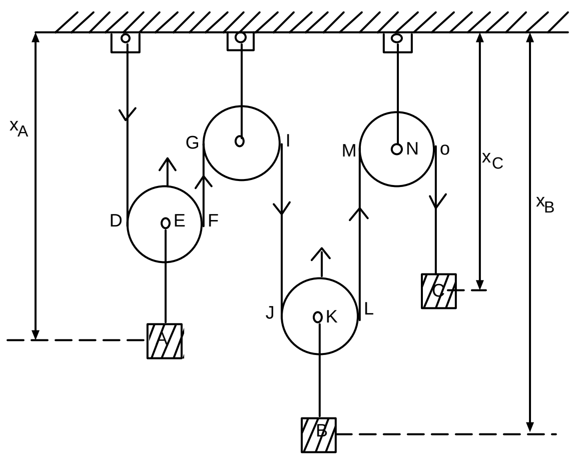
$\therefore$ x_A+2x_B = Constant ……..(iv) [AC+DE+FG = Constant]
If A block 'A' is given Δx_A motion it will produce $x_B = (-\Delta x_A/2)$ as the motion of Block B.

Differentiating equation (iv) w.r.t time , we get,
$V_A+2V_B = 0$, ………(v)
Or, $V_B = -(V_A/2)$
Similarly differentiating equation (v) w.r.t time we get,
$A_A + 2a_B = 0$ ………(vi)
Or, $a_B = -(a_A/2)$ [Negative sign denotes opposite in direction]

* In this case displacement, velocity and acceleration of one body gives the displacement, velocity and acceleration of other body. This arrangement is called 1-degree freedom .

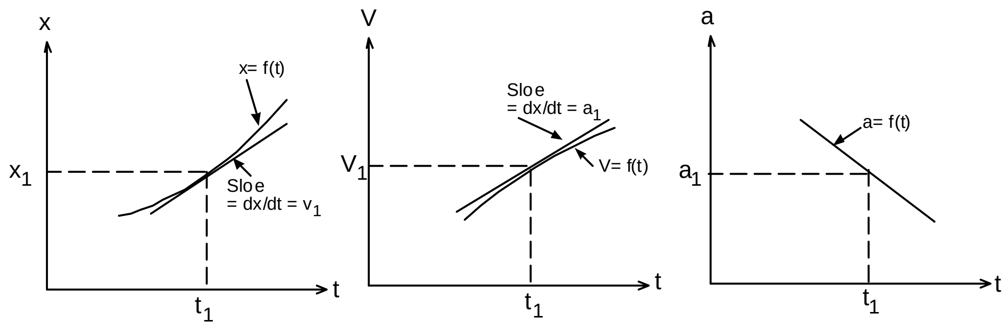
Derive the equation of motion of the given pulley system.

Solution:
$2x_A + 2x_B + x_c$ = Constant ……….(1)
$2v_A + 2v_B + v_c = 0$ …………..(2)
$2a_A + 2a_B + a_c = 0$ ………….(3)



2.6 Graphical solution of Rectilinear motion problems:
* Graphical solution are very helpful to simply and solve the problems of Dynamics.
* Using the motion graphs (i.e x –t , v-t and a-t) the missing value at any point can be obtained .
* If any on equation of motion is known all the three graphs can be obtained as follows.

If equation of displacement x = f(t) ……..(i) is known,
Then , V = dx/dt …….(ii)
a = dv/dt …….(ii)
i.e velocity is slope of x-t curve and acceleration is slope of v-t curve.



If x-t curve is given, then computing slope at each point of x-t curve corresponding v-t curve can be generated and computing slope at each point of v-t curve a-t curve can be generated.

Again,

From equation (ii)

$dx = vdt$

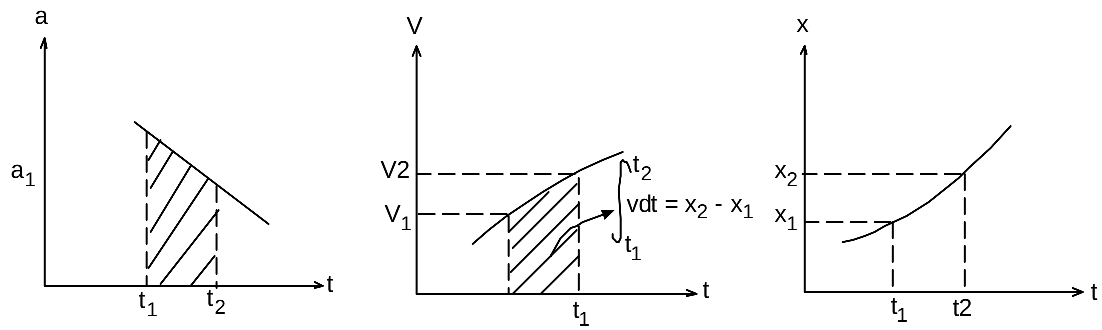
$$\int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} vdt \Rightarrow x_2 - x_1 = \int_{t_1}^{t_2} vdt \quad ....(iv)$$

And from equation (iii) ,

dv = adt

$$\int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} adt \Rightarrow v_2 - v_1 = \int_{t1}^{t_2} adt \quad ....(v)$$

- This means change in position in given by the area under curve v-t and change in velocity is given by area under the curve a-dt.



Tutorial Examples:

1) The motion of a particles is defined by the position vector $\vec{r} = 6t\hat{i} + 4t^2\hat{j} + \frac{t^3}{4}\hat{k}$ where r in meter and t in second. At the instant when t = 3 sec, find the unit position vector, velocity and acceleration.

Solution:

We have , $\vec{r} = 6t\hat{i} + 4t^2\hat{j} + \frac{t^3}{4}\hat{k}$

At time t = 3 sec. $\vec{r} = 18\hat{i} + 36\hat{j} + \frac{27}{4}\hat{k}$

$$r = |\vec{r}| = \sqrt{(18)^2 + (36)^2 + \left(\frac{27}{4}\right)^2} = 40.81m$$

Now unit position vector at t = 3 sec.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{18\hat{i} + 36\hat{j} + \frac{27}{4}\hat{k}}{40.81}$$

$$\therefore \hat{r} = \left(0.44\hat{i} + 0.88\hat{j} + 0.165\hat{k}\right) \quad Ans$$

Again,

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(6t\hat{i} + 4t^2\hat{j} + \frac{t^3}{4}\hat{k}\right)$$

$$\vec{v} = 6\hat{i} + 8t\hat{j} + \frac{3}{4}t^2\hat{k}$$

At time t = 3 sec.

$$\vec{v} = \left(6\hat{i} + 24\hat{j} + \frac{24}{4}\hat{k}\right)$$

Velocity (v) = $|\vec{v}| = \left[6^2 + 24^2 + \left(\frac{24}{7}\right)^2\right]^{\frac{1}{2}} = 25.64\, m/s$

V = 25.64 m/s

Again, acceleration $(\vec{a}) = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(6\hat{i} + 8t\hat{j} + \frac{3}{4}t^2 k\right)$

$$\vec{a} = 8\hat{j} + \frac{3}{2}t\hat{k}$$

At t = 3 sec , $\vec{a} = 8\hat{j} + 4.5\hat{k}$

Acceleration, (a) = $|\vec{a}| = \left[8^2 + (4.5)^2\right]^{\frac{1}{2}} = 9.18 \quad m/s^2 \quad Ans$

2) A ball is thrown vertically upward with a velocity of 9.15m/s. After 1s another ball is thrown with the same velocity. Find the height at which the two ball pass each other?

Solution:

Let the initial velocity of both balls

$V_{01} = v_{02} = v_o = 9.15$ m/s

h be the height at which two balls pass each other t_1 be the time elapsed by the first ball before passing second and t_2 be the time elapsed by second.

From the given condition:

$t_1 - t_2 = 1$ ....(i)

for 1st ball , $n = v_0 t_1 - \frac{1}{2}\, 9t_1^2$ ....(ii)

For 2nd ball $n = v_0 t_2 - \frac{1}{2}\, 9t_2^2$ ....(iii)

Substituting equation (iii) form equation (ii) , we get,

$$\frac{1}{2}9\left(t_1^2 - t_2^2\right) = v_0\left(t_1 - t_2\right)$$

$$\frac{1}{2}9\left(t_1 + t_2\right)\left(t_2 - t_1\right) = v_0\left(t_1 - t_2\right)$$

$$\therefore \left(t_1 + t_2\right) = \frac{2v_0}{9} = \frac{2 \times 9.15}{9.81} = 1.865$$

$$\left(t_1 + t_2\right) = 1.865 \quad ....(iv)$$

Adding equation (i) and (ii) , we get $t_1 = 1.43$ sec and $t_2 = 0.43$ sec

$$\therefore h = v_0 t_1 - \frac{1}{2}9t_1^2 = 3.05m$$

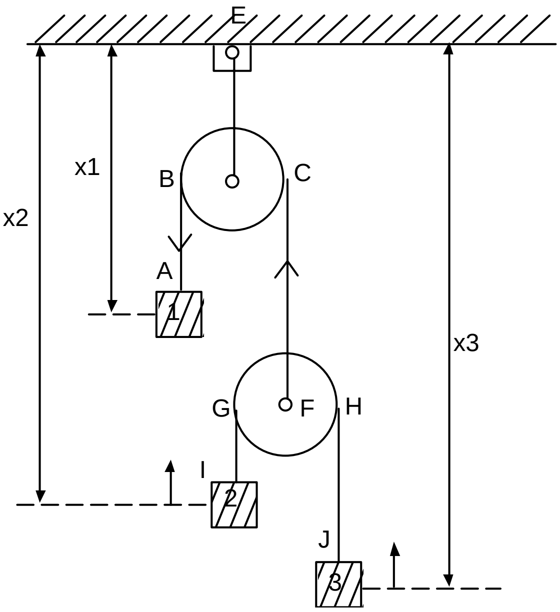
h = 3.05 m

Hence , two balls pass each other at 3.05m above the ground.

3) In the following pulley system, Block 2 has velocity 2m/s upward and its acceleration is 3m/s² downward while block 3 has velocity and acceleration 2m/s up ward and 4m/s² downward respectively. Find the velocity and acceleration of block 1.

Solution:

Given, $V_2 = 2m/s\ (\uparrow)$
$a_2 = 3m/s^2\ (\downarrow)$
$v_3 = 2m/s\ (\uparrow)$
$a_3 = 4m/s^2\ (\downarrow)$



Here ,
AB +CF = constant
GI +HJ = Constant
DE = Constant
Portion of rope around the pulley is also constant.
Now,
X_1 = AB + constant ……..(i)
X_2 = GI +CF+ constant …….(ii)
X_3 = HJ+CF+ Constant ……(iii)

Multiplying equation (i) by (2) and adding (i), (ii) and (iii) , we get
$2x_1 + x_2 + x_3 = 2AB + CF + CF + GI + HJ + Const.$
$2x_1 + x_2 + x_3 = const.$ ………..(iv)

Differentiating equation (iv) w.r.t time.
$2v_1 + v_2 + v_3 = 0$ ………(v)
$2a_1 + a_2 + a_3 = 0$ ……..(vi)

From equation 'v'

$$v_1 = -\frac{v_2 - v_3}{2} = \frac{-2-2}{2} = -2\,m/s$$

Therefore, velocity of block 1 (v_1) = 2 m/s (↓)
From equation (vi)

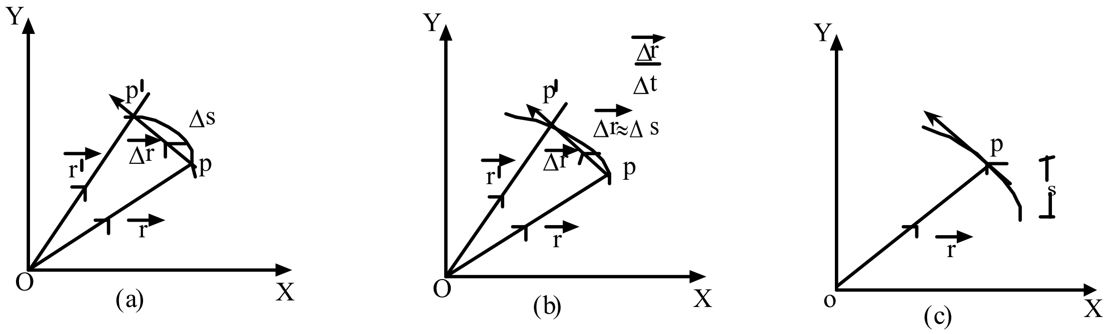
$$a_1 = \frac{-a_2 - a_3}{2} = \frac{+3+4}{2} = 3.5\,m/s^2$$

Therefore, acceleration of block 1 (a_1) = 3.5 m/s² (↑)

Chapter – 3
Curvilinear Motion of Particles

3.1 Position vector, Velocity and Acceleration:

When a particle moves along a curve path other than a straight line, it is said to be in curvilinear motion. Vector analysis is used to analyze the change in position and direction of motion of particles.′



Let, at time 't' the position vector of particle be $\vec{r}$ and at another time $(t+\Delta t)$ the particle takes a new position p' and its position be $\vec{r}$!. Then $\Delta\vec{r}$ represents the change in directoin as well as magnitude of the position vector $\vec{r}$. (fig. a) The average velocity of the particles at time interval

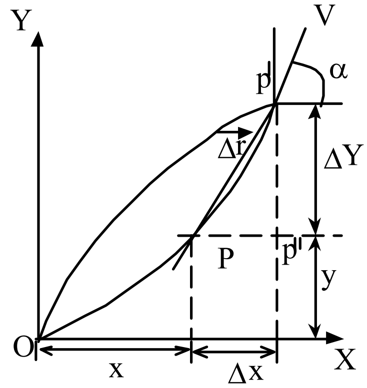
$$\Delta t = \frac{\Delta\vec{r}}{\Delta t} \quad \text{(in magnitude and direction of } \Delta\vec{r}\text{)}$$

$$\therefore \text{Instantaneous Velocity, } \vec{v} = \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

As $\Delta\vec{r}$ and Δt becomes shorter, $P \,\&\, P'$ gets closer and $\vec{v}$ is tangent to the path of the particle. (fig c)

And, As Δt decreases, length of PP' ($\Delta\vec{r}$) equals to length of arc Δs (fig b)

$$\therefore v = \lim_{\Delta t\to 0}\frac{PP'}{\Delta t} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$



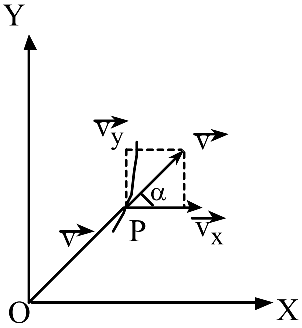
Change in position ($\Delta\vec{r}$) can be resolved into two components,

i One parallel to x-axis and ($\overrightarrow{PP''}$)

ii Other parallel to y-axis ($\overrightarrow{P''P'}$)

$$\therefore \Delta\vec{r} = \overrightarrow{PP''} + \overrightarrow{P''P'}$$

$$or, \lim_{\Delta t\to 0}\frac{\Delta\vec{r}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\overrightarrow{PP''}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\overrightarrow{P''P'}}{\Delta t}$$



$$or, \vec{v} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}\hat{i} + \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}\hat{j}$$

$$or, \vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$or, \vec{v} = v_x\hat{i} + v_y\hat{j}$$

$$or, \vec{v} = x\hat{i} + y\hat{j}$$

Then,

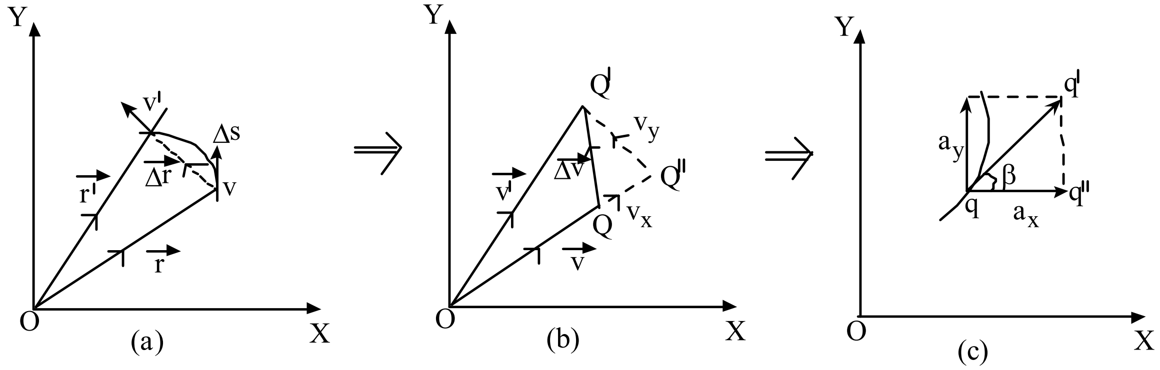
$$V = \sqrt{v_x^2 + v_y^2} \text{ [Magnitude of Velocity]}$$

$$\tan\alpha = {v_y}/{v_x}$$

$$\therefore \alpha = \tan^{-1}\frac{v_y}{v_x} \text{ [Direction of Velocity]}$$

Positive Value of $v_x \rightarrow$ Right Direction
Positive Value of $v_y \rightarrow$ Upward Direction
$v_x = \frac{dx}{dt} = x; v_y = \frac{dy}{dt} = y$

Similarly, acceleration by curvilinear motion can be computed as:



If $\vec{v}$ and $\vec{v'}$ be the velocities at time 't' 4 $(t + \Delta t)$ i.e. tangents at P and P', then the average acceleration of the particle over the timer interval is given by

$$\Delta t(\vec{a}) = \frac{\Delta\vec{v}}{\Delta t}$$

$$or, \vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

Again, Δv can be resolved into $\overrightarrow{QQ''}$ & $\overrightarrow{Q''Q'}$ parallel to x & y-axes respectively. Then,

$$\vec{v} = \overrightarrow{QQ''} + \overrightarrow{Q''Q'}$$

$$or, \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \lim_{\Delta t \to 0}\frac{\overrightarrow{QQ''}}{\Delta t} + \lim_{\Delta t \to 0}\frac{\overrightarrow{Q''Q'}}{\Delta t}$$

$$or, \vec{a} = \lim_{\Delta t \to 0}\frac{\Delta v_x}{\Delta t}\hat{i} + \lim_{\Delta t \to 0}\frac{\Delta v_y}{\Delta t}\hat{j}$$

$$or \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$or, a_x\hat{i} + a_y\hat{j}$$

$$or, \vec{a} = x\hat{i} + y\hat{j}$$

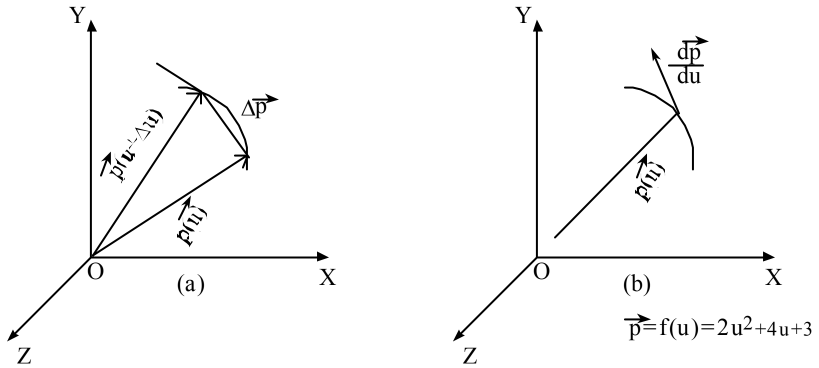
$$or, a = \sqrt{(a_x)^2 + (a_y)^2} \Rightarrow acceleration\, in\, magnitude$$

$$a_x = \frac{dv_x}{dt}$$

$$a_y = \frac{dv_y}{dt}$$

$$\tan\beta = {a_y}\big/{a_x} \Rightarrow \beta = \tan^{-1}(a_y/a_x) \Rightarrow in\, direction$$

3.2 Derivatives of a vector function:



Let, $\vec{P}(u)$ be a vector function of scalar variable u. If value of 'u' is varied, '$\vec{P}$' will trace a curve in space. Considering change of vector $\vec{P}$ corresponding to the values u 4 $(u+\Delta u)$ as shown in figure(a). Then

$$\Delta\vec{p} = \vec{p}(u+\Delta u) - \vec{p}(u)$$

$$\boxed{i.e.\ \frac{d\vec{p}}{du} = \lim_{\Delta u\to 0}\frac{\Delta\vec{p}}{\Delta u} = \lim_{\Delta u\to 0}\left\{\frac{\vec{p}(u+\Delta u)-\vec{p}(u)}{\Delta u}\right\}} \quad - (1)$$

As $\Delta u \to 0$, $\Delta\vec{p}$ becomes tangent to the curve. Thus $\frac{d\vec{p}}{du}$ is tangent to the curve as shown in figure(b).

Again,

Considering the sum of two vector functions $\vec{p}(u) \,\&\, \vec{Q}(u)$ of the same scalar variable u. Then the derivative of the vector $(\vec{P}+\vec{Q})$ is given by:

$$\frac{d}{du}(\vec{P}+\vec{Q}) = \lim_{\Delta u\to 0}\frac{\Delta(\vec{P}+\vec{Q})}{\Delta u} = \lim_{\Delta u\to 0}\left[\frac{\Delta\{\vec{P})}{\Delta u} + \frac{\Delta\{\vec{Q})}{\Delta u}\right] = \lim_{\Delta u\to 0}\frac{\Delta\vec{P}}{\Delta u} + \lim_{\Delta u\to 0}\frac{\Delta\vec{Q}}{\Delta u}$$

$$\boxed{\therefore \frac{d}{du}(\vec{P}+\vec{Q}) = \frac{d\vec{P}}{du} + \frac{d\vec{Q}}{du}} \quad - \quad (2)$$

Again, product of scalar function f(u) and pf a vector function $\vec{P}(u)$ of the same scalar variable u. Then, derivative of f$\vec{P}$ is given by:

$$\frac{df.\vec{p}}{\overrightarrow{du}} = \lim_{\Delta u\to 0}\frac{(f+\Delta f)(\vec{P}+\overrightarrow{\Delta P}) - f\vec{P}}{\Delta u}$$

$$= \lim_{\Delta u\to 0}\left[\frac{\Delta f}{\Delta u}\vec{P} + f\frac{\Delta\vec{P}}{\Delta u}\right]$$

$$\boxed{\therefore \frac{df\vec{P}}{du} = \frac{df}{du}\vec{P} + f\frac{d\vec{P}}{du}} \quad - \quad (3)$$

Similarly, scalar product and vector product of two vector functions $\vec{P}(u)\, and\, \vec{Q}(u)$ may be obtained as:

$$\boxed{\frac{d}{du}(\vec{P}.\vec{Q}) = \frac{d\vec{P}}{du}.\vec{Q} + \vec{P}.\frac{d\vec{Q}}{du}}$$ - (4) [Scalar Product]

$$\boxed{\frac{d}{du}(\vec{P}\times\vec{Q}) = \frac{d\vec{P}}{du}\times\vec{Q} + \vec{P}\times\frac{d\vec{Q}}{du}}$$ - (5) [Vector Product]

Again,

$$\boxed{\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}}$$ - (6)

where, $P_x, P_y\ \&\ P_z$ are the rectangular scalar components of vector P & $\hat{i}, \hat{j}, \hat{k}$ are the unit vector.

$$\boxed{\therefore \frac{d\vec{P}}{du} = \frac{dP_x}{du}\times\hat{i} + \frac{dP_y}{du}\times\hat{j} + \frac{dP_y}{du}\times\hat{k}}$$ - (7) [where, $\vec{P} = f(u)$]

And,

$$\boxed{\therefore \frac{d\vec{P}}{dt} = \frac{dP_x}{dt}\times\hat{i} + \frac{dP_y}{dt}\times\hat{j} + \frac{dP_y}{dt}\times\hat{k}}$$ - (8) [where, $\vec{P} = f(t)$]

3.3 Rectangular Components of Velocity and Acceleration:

When, the position of a particle is defined by at any instant by its rectangular co-ordinates x, y, z as:

$$\boxed{\vec{r} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}}$$ - (i)

Then, differentiating both sides w.r.t. time, we get,

$$\vec{v} = \frac{d\vec{r}}{dt} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$$

where,

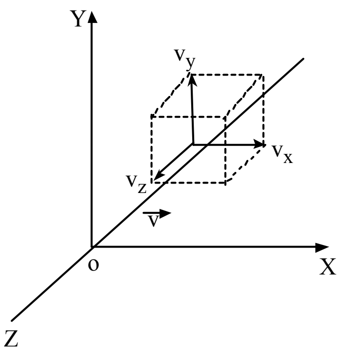
$$\dot{x} = v_x, \dot{y} = v_y\ \&\ \dot{z} = v_z$$

$$\ddot{x} = a_x, \ddot{y} = a_y\ \&\ \ddot{z} = a_z$$

So,

$$\boxed{\vec{V} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}} \quad \therefore \left|\vec{V}\right| = \sqrt{(v_x^2 + v_y^2 + v_z^2)}$$

$$\boxed{\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}} \quad \therefore \left|\vec{a}\right| = \sqrt{(a_x^2 + a_y^2 + a_z^2)}$$



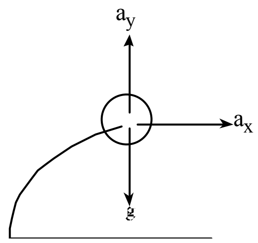
When the motion in each axis can be represented independent with each other then the use of rectangular components to describe the position, velocity and acceleration of a particle is effective i.e. motion in each axis can be considered separately.

For e.g. for projectile motion, neglecting air resistance, the components of acceleration are:

$$a_x = 0 \ and \ a_y = -g$$

$$or, v_x = 0 \ and \ v_y = -g$$

$$or, dv_x = 0 \ and \ dy = -gdt$$



On Integrating both sides under the limits,

$$\int_{v_{x_o}}^{v_x} dv_x = 0 \qquad and \qquad \int_{v_{y_o}}^{v_y} dy = -g\int_0^t dt$$

$$v_x - v_{x_o} = 0 \qquad and \qquad v_y - v_{y_o} = -gt$$

$$v_x = v_{x_o} \qquad and \qquad v_y = v_{y_o} = -gt \ - \ (v)$$

$$or, \dot{x} = v_{x_o} \qquad and \qquad \dot{y} = v_{y_o} - gt$$

$$or, \frac{dx}{dt} = v_{x_o} \qquad and \qquad \frac{dy}{dt} = v_{y_o} - gt$$

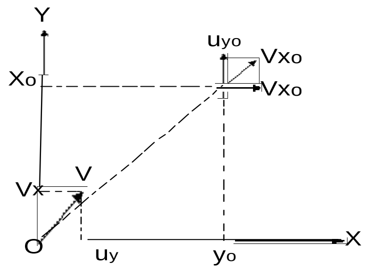
$$or, dx = v_{x_o} dt \qquad and \qquad dy = v_{y_o} dt - gtdt$$

Integrating both sides under limits considering motion starts from origin by co-ordinates i.e. x at t=0 ; x=0 ; y=0 and at $t=t_0$, $x=x_0$ and $y=y_0$

$$\int_0^{x_0} dx = v_{x_0}\int_0^t dt \qquad and \qquad \int_0^{y_o} dy = v_{y_0}\int_0^t dt - g\int_0^t tdt$$

$$x_0 = v_{x_0} t \qquad and \qquad y_0 = v_{y_0} t - \frac{1}{2}gt^2 \ - (vi)$$

Thus motion under projectile can be represented by 2-independent rectilinear motion.



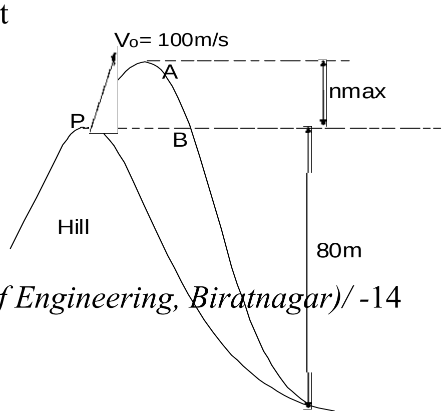
Problems:

1. A bullet is fired upward at an angle of 30° to the horizontal from point P on a hill and it strkies a target which is 80m lower than the level of projection as shown in figure. The initial velocity of the bullet is 100m/s. Calculate:

a. The maximum height to which the bullet will rise above the horizontal.

b. The actual velocity with which it will strike the target

c. The total time required for the flight of the bullet.



Solution:

$$V_0 = 100m/s$$

$$V_{x_0} = V_0 \cos 30 = 86.60m/s$$

$$V_{y_0} = V_0 \sin 30 = 50m/s$$

a) $$h_{\max} = \frac{V_0^2 \sin^2 30}{2g} = 127.42$$

$$\therefore h_{\max} = 127.42m(Ans)$$

b) Let, V_{1y} = vertical component of velocity at highest point A = 0

V_{2y} = vertical component of velocity striking target

H = vertical distance between point A and target = 127.42+80 = 207.42m

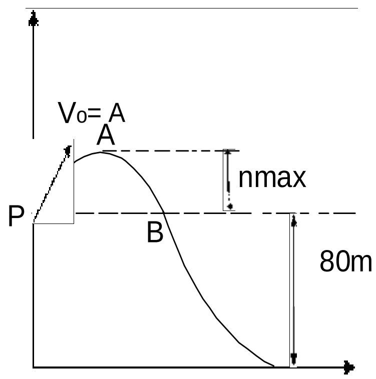
Then, $V_{2y}^2 - V_{1y}^2 = 2gH \qquad [V_{1y} = 0]$

$$or, \Rightarrow V_{2y} = 63.79m/s$$

$$V_{2x} = V_{1x} = V_{x_0} = 86.60m/s \quad [V_{x_0} = cons\tan t]$$

$$\therefore V_2 = \sqrt{V_{2x}^2 + V_{2y}^2} = 107.55m/s \text{ (Ans)}$$

& $$\theta = \tan^{-1}\frac{V_{zy}}{V_{xy}} = 36.37°$$



c) $$h = (V_y)_0 t_2 - \frac{1}{2}gt_2^2$$

$$or, -80 = 50 \times t_2 - \frac{1}{2} \times 9.8 \times t_2^2$$

$$or, 4.905t_2^2 - 50t_2 - 80 = 0$$

$Solving,$

$$xv = t_2 = 11.60\sec(Ans)$$

Total time of flight is the sum of time to reach B from A & to C from B

T=t_1+t_2

t_1=$\frac{2V_0 \sin\alpha}{g}$ & PB=Range=$\frac{V_0^2 \sin\alpha}{g}$

2) The motion of a vibrating particle is defined by the equation x=100sinπt and y=25cos2πt, where x & y are expanded in mm & t in sec.

a) Determine the velocity and acceleration when t=15

b) Show that the path of the particle is parabolic.

Solution:

a) We have,

$$x = 100\sin\pi t \Rightarrow V_x = \dot{x} = 100\pi\cos\pi t$$

$$a_x = \ddot{x} = -100\pi^2 \sin\pi t$$

Again, y=25cos2$\pi t \Rightarrow V_y = \dot{y} = -50\pi\sin 2\pi t$

$$a_y = \ddot{y} = -100\pi^2 \cos 2\pi t$$

Then, for t=2sec,

$$V_x = [100\pi\cos\pi \times 1]: V_y = [-50\pi^2 \sin\pi \times 1]$$

$$V = \left|\vec{V}\right| = \sqrt{V_x^2 + V_y^2}$$

$$= \sqrt{(100\pi\cos\pi t)^2 + (-50\sin 2\pi)^2}$$

$$V = 100\pi\, mm/s \qquad \alpha = \tan^{-1}(\frac{V_y}{V_x}) = 0$$

And, For t=1sec

$$a_x = [(-100\pi^2 \sin\pi]: a_y = -100\pi^2 \cos 2\pi \times 1$$
$$a = \left|\vec{a}\right| = \sqrt{a_x^2 + a_y^2} = 100\pi^2 mm/s^2$$
$$\beta = \tan^{-1}\left(\frac{a_y}{a_x}\right) = 270°$$

b) Since, x=100sin πt

$$\therefore \frac{x}{100} = \sin\pi t$$

$$\therefore \left(\frac{x}{100}\right)^2 = \sin^2\pi t \qquad -(i)$$

Again, $y = 25\cos 2\pi t$

$$or, \frac{y}{25} = 2\cos^2\pi t - 1$$
$$or, \left(\frac{y}{25} + 1\right) = 2\cos^2\pi t$$
$$\therefore \frac{y+25}{50} = \cos^2\pi t \qquad -(ii)$$

Adding equations (i) and (ii), we get ;

$$\frac{x^2}{10000} + \frac{y+25}{50} = \sin^2\pi t + \cos^2\pi t = 1$$
$$or, x^2 + 200y + 5000 = 10000$$
$$or, x^2 + 200y + 5000, \text{which is the equation of parabola} = [ax^2 + by = c]$$

3) The motion of a particle is given by the relation V_x=2cost & V_y=sint. It is known that initially both x & y co-ordinates are zero. Determine

a) Total acceleration at the instant of 25

b) Equation of the parabola

a) Here, V_x=2 cost & V_y =sint

Then, $a_x = \frac{dv_x}{dt} = -2\sin t$

and, $a_y = \frac{dv_y}{dt} = \cos t$

$$\therefore \vec{a} = a_x\hat{i} + a_y\hat{j}$$
$$\vec{a} = -2\sin t\,\hat{i} + \cos t\,\hat{j}$$

At t=2sec

a_x=-2sinz=-1.82

a_y=cos2=-0.42

$$\alpha = \sqrt{\overrightarrow{a_x^2} + \overrightarrow{a_y^2}} = 1.865 m/s^2$$
$$\beta = \tan^{-1}\frac{a_y}{a_x} = 193°$$

For t = 2 sec

$$or, y = -\cos t + 1 \Rightarrow (y-1)^2 = \cos^2 t \qquad -(ii)$$
$$Adding\ (i)\ and\ (ii), \qquad x^2/4 + (y-1)^2 = 1$$
$$or, x^2 + 4y^2 - 8y = 0, \quad \text{which is the required equation of parabola}$$

$\vec{a} = -2\sin 2\hat{i} + \cos 2\hat{i}$ [$where$, 2 is in radian]

$\because \vec{a} = -1.82\hat{i} - 0.42\hat{j}$

$\therefore \alpha =$ Total acceleration

$\alpha = \left|\vec{\alpha}\right| = \sqrt{(-1.82)^2 + (-0.42)^2} = 1.865 m/s^2$

$\beta = \tan^{-1}\frac{a_y}{a_x} = \tan^{-1}\left(\frac{-0.42}{-1.82}\right) = 193° (Ans)$

b) $V_x = 2\cos t \Rightarrow \frac{dx}{dt} = 2\cos t \Rightarrow \int_0^x dx = 2\int_0^t \cos dt$

$or, x = 2\sin t$

$or, \frac{x^2}{4} = \sin^2 t \; -(i)$

Again, $V_y = \frac{dy}{dx} = \sin t \Rightarrow \int_0^y dy = \int_0^t \sin t dt$

3.4 Motion Relative to a Frame in Translation:

Let A and B be the particles moving is a same plane with $\vec{r_A} \& \vec{r_B}$ be their position with respect to XY axis.

Considering New axes (X'-Y') centered at 'A' and parallel to original axes X-Y, the motion of particle 'B' can be defined with respect to motion of particle 'A' such that:

From vector triangle OAB

$\vec{r_B} = \vec{r_A} + \vec{r}_{B/A} \qquad -(i)$

and similarly,

$$\left.\begin{aligned}\vec{X_B} &= \vec{X_A} + \vec{X}_{B/A}\\ \vec{Y_B} &= \vec{Y_A} + \vec{Y}_{B/A}\end{aligned}\right\} \quad -(ii)$$

where,

- X_A, Y_A & X_B & Y_B are co-ordinates of A & B w.r.t. XY axes
- $X_{B/A}$, $Y_{B/A}$ are co-ordinates of 'B' .r.t. X'-Y' axis

Differentiating equ(i) w.r.t. time, we get:

$\vec{V_B} = \vec{V_A} + \vec{V_B} \quad -(iii)$

$\vec{r} \rightarrow$ Equation of motion in both x and y axes

$\vec{X} \rightarrow$ Equation of motion in x - axis

$\vec{Y} \rightarrow$ Equation of motion in y - axis

In scalar form:

$$\left.\begin{aligned}\dot{X}_B &= \dot{X}_A + \dot{X}_{B/A}\\ \dot{Y}_B &= \dot{Y}_B + \dot{Y}_{B/A}\end{aligned}\right\} \quad -(iv)$$

OR

$V_{(B)_x} = V_{(A)_x} + V_{(B/A)_x}$

$V_{(B)_y} = V_{(A)_y} + V_{(B/A)_y}$

where,

$\dot{X}_A, \dot{Y}_A$ are X & Y components of $\vec{V_A}$

$\dot{X}_B, \dot{Y}_B$ are X & Y componens of $\vec{V_B}$

$\dot{X}_{B/A}$ $\dot{Y}_{B/A}$ are X & Y components of $\vec{V}_{B/A}$

Again,

differentiating (iii) with respect to time, we get:

$\vec{a_B} = \vec{a_A} + \vec{a}_{B/A} \qquad -(v)$

In scalar,

$a_B = \sqrt{(a_{Bx})^2 + (a_{By})^2}$

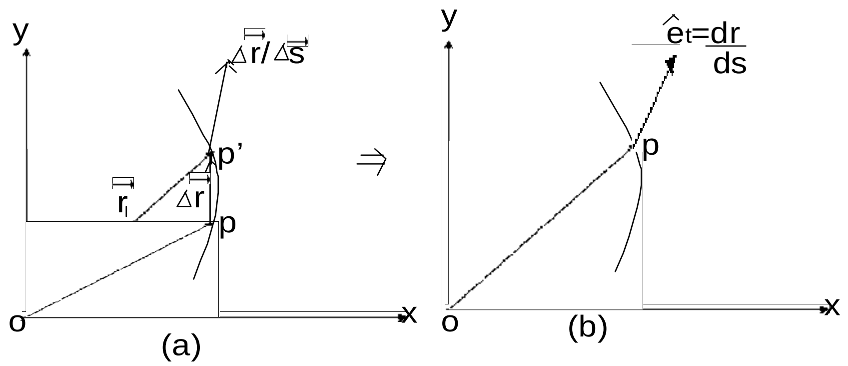
$\beta = \tan^{-1}\frac{a_{By}}{a_{Bx}}$

$\ddot{X}_B = \ddot{X}_A + \ddot{X}_{B/A}$ OR $a_{(B)x} = a_{(A)x} + a_{(B/A)x}$ - (vi)

$\ddot{Y}_B = \ddot{Y}_A + \ddot{Y}_{B/A}$ $\quad a_{(B)y} = a_{(A)y} + a_{(B/A)y}$

3.4 Tangential and Normal Components:

- The velocity of particle is vector tangent to the path of particle. But acceleration may not be tangent in curvilinear motion.
- The acceleration vector may be resolved into two components perpendicular with each other in directions
  - i First component along the tangent of path of particle (a_t)
  - ii Second component along the normal of path of particle (a_n)
- Let $\hat{e}_t$ and $\hat{e}_n$ be the unit vectors directed along the tangent and normal of the path respectively. Then, in curvilinear motion, $\hat{e}_t$ and $\hat{e}_n$ would change the direction as particle moves from one point to another.



From fig (a)

$$\overrightarrow{\Delta r} = \vec{r}' - \vec{r} \;\&\; \left[\lim_{\Delta s \to 0} \left|\overrightarrow{\Delta r}\right| = \Delta s\right]$$

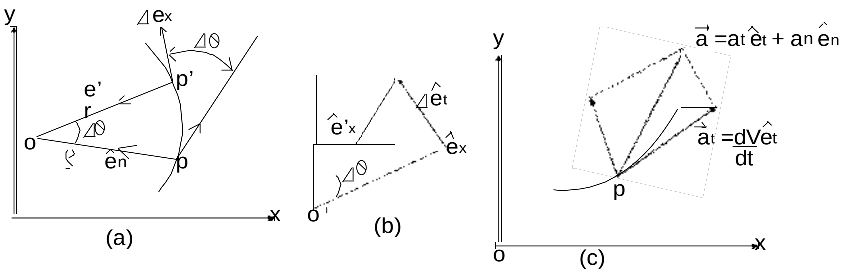
Then, $\hat{e}_t = \lim\limits_{\Delta s \to 0} \dfrac{\overrightarrow{\Delta r}}{\Delta s} = \dfrac{\overrightarrow{dr}}{ds}$

$$\therefore \hat{e}_t = \frac{\overrightarrow{dr}}{ds} \qquad - (i)$$

Again, $|\hat{e}t| = \lim\limits_{\Delta s \to 0} \dfrac{|\Delta \vec{r}|}{|\Delta s|} = \dfrac{\Delta s}{\Delta s} = 1 \qquad \left[\lim\limits_{\Delta s \to 0} |\Delta \vec{r}| = \Delta s\right]$

Therefore, $\hat{e}t = \dfrac{d\vec{r}}{ds}$ is the unit vector along the tangetn to path.

Let, ρ be the radius of curvature of the path at the point P and $\hat{e}_t$ & $\hat{e}_{t'}$ be the tangent unit vectors at P and P'. $\Delta \hat{e}_t$ be the change in unit vector while the particle moves from P to P'.



Now, from fig,

$\Delta s = PP' = \rho \Delta \theta$

$\Delta \hat{e}_t = \hat{e}_{t'} - \hat{e}_t \equiv \Delta\theta\, \hat{e}_t = \Delta\theta\, \hat{e}_n$ $\qquad \left[\text{As } \Delta s \rightarrow 0 \;\; \hat{e}_t = \hat{e}_n \rightarrow 1 \textit{ in magnitude}\right]$

$\therefore \hat{e}_n = \lim_{\Delta s \to 0} \frac{\Delta \hat{e}_t}{\Delta \theta} \Rightarrow \left[\hat{e}_n = \frac{d\hat{e}_t}{d\theta}\right]$ $\qquad - (ii)$

Similarly,

$\left[\frac{d\theta}{ds} = \frac{1}{\rho} \text{ and } \frac{d\hat{e}_t}{d\theta} = \hat{e}_n\right]$ $\qquad - \text{(iii)}$

Also,

$$V = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{\rho \Delta \theta}{\Delta t} = \rho \frac{d\theta}{dt}$$

$\left[\therefore\; V = \rho \frac{d\theta}{dt} = \rho\theta\right]$ $\quad - (iv)$

$$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}.\frac{ds}{dt} = \hat{e}_t v = v\hat{e}_t$$

$\therefore \left[\vec{V} = V\hat{e}_t\right]$ $\quad - (v)$

And,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(V\hat{e}_t) = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} = \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{d\theta}.\frac{d\theta}{ds}.\frac{ds}{dt}$$

$or,\; \vec{a} = \dot{V}\hat{e}_t + V(\hat{e}_n)(\frac{1}{\rho})$

$or,\; \left[\vec{a} = \dot{V}\hat{e}_t + \frac{V^2}{\rho}\hat{e}_n\right]$ $\qquad - (vi)$

which can be represented as in fig(c).

$\vec{a} = a_t\hat{e}_t + a_n\hat{e}_n$, where

$a_t =$ Tangential component of acceleration = $\frac{dv}{dt} = \dot{v}$

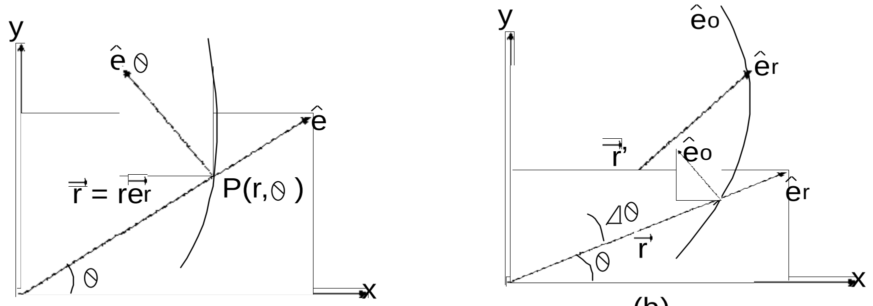
$a_n =$ Normal component of acceleration = $\frac{V^2}{\rho} = \rho\dot{\theta}^2$

Notes:

- For increasing velocity at will be in the direction of velocity and f_x decreasing velocity a_t will be in opposite to the direction of velocity.
- If the speed is constant a_t=0 but $a_n \neq 0$. [a_n=0 f_x Recti $\rho = \alpha$]
- a_n is always directed towards the centre of curvature
- For higher velocity and smaller radius higher is a_n.

3.6 Radial and Transverse Components:

- ✓ For the motion described by polar co-ordinates.
- ✓ Position of particles P is defined by the co-ordinates r & θ, where r is the length and θ is the angle in radians.
- ✓ The unit vectors in radial and transverse direction are denoted by $\hat{e}_r$ and $\hat{e}_\theta$ respectively along radius and 90° clockwise to the radius in direction. fig (a)



(a) (b)

As the particle moves from $P\,to\,P'$. The unit vectors $\overrightarrow{e_r}, \& \overrightarrow{e_\theta}$ change to $\hat{e}'_r$ & $\hat{e}'_\theta$ by $\Delta\overrightarrow{e_r}$ and $\Delta\overrightarrow{e_\theta}$ respectively.

Here,

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \frac{d\hat{e}_r}{d\theta}.\frac{d\theta}{dt} = \hat{e}_\theta\dot{\theta} \text{-----(i)} \left[\text{direction of } \frac{\mathrm{d}\hat{\mathrm{e}}_\mathrm{r}}{\mathrm{d}\theta} \text{ is in direction of } \hat{\mathrm{e}}_\theta\right]$$

$$\dot{\hat{e}}_\theta = \frac{d\hat{e}_\theta}{d\theta} = \frac{d\hat{e}_\theta}{d\theta}.\frac{d\theta}{dt} = -\hat{e}_r\dot{\theta} \text{-----(ii)} \left[\text{direction of } \frac{\mathrm{d}\hat{\mathrm{e}}_\theta}{\mathrm{d}\theta} \text{ is in direction of - } \hat{\mathrm{e}}_\mathrm{r}\right]$$

Now,

$$\vec{r} = r\hat{e}_r$$

Then, $\vec{V} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\hat{e}_r) = \dfrac{dr}{dt}\hat{e}_r + r\dfrac{d\hat{e}_r}{dt} = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$

$\therefore \left[\vec{V} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta\right]$ - (iii) $\left[\dot{\hat{\mathrm{e}}}_\mathrm{r} = \hat{e}_\theta\dot{\theta}\right]$

which can expressed as

$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta$, where

V_r = Radial component of velocity = $\dot{r}$

And, V_θ = Transverse component of velocity = r$\dot{\theta}$

Similarly,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\hat{e}_r + r\dot{\theta}\overrightarrow{e_\theta}\right)$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$= \ddot{r}\hat{e}_r + \dot{r}\dot{\theta}\hat{e}_\theta + \dot{r}\dot{\vartheta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta - r\dot{\theta}^2\hat{e}_r \quad \left[\because \dot{e}_r = \hat{e}_\theta\dot{\theta} \ \& \ \dot{\hat{e}}_\theta = -\hat{e}_r\dot{\theta}\right]$$

$$\therefore \left[\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta\right] \qquad -(v)$$

which can be represented as,

$$\vec{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta$$

where,

a_r = Radial component of acceleration = $\left(\ddot{r} - r\dot{\theta}^2\right)$

and, a_θ = Transverse component of acceleration = $\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$

In case of a particle moving along a circular path with its centre at the origin O, we have r=constant

or, $\dot{r} = 0$ & $\ddot{r} = 0$

Then,

$$\vec{v} = r\dot{\theta}\,\hat{e}_\theta \qquad - \quad (vi)$$

$$\vec{a} = -r\dot{\theta}^2\hat{e}_r + r\ddot{\theta}\hat{e}_\theta \qquad - \quad (vii)$$

Problems:

1) The motion of a particle is defined by the position vector, $\vec{r} = 3t^2\hat{i} + 4t^3\hat{j} + 5t^4\hat{k}$, where r is in m and t is in sec. At instant when t=4 sec, find the normal and tangential component of acceleration and the radius of curvature.

Solution, we have

$$\vec{r} = 3t^2\hat{i} + 4t^3\hat{j} + 5t^4\hat{k}$$

$$\therefore \vec{V} = \frac{d\vec{r}}{dt} = 6t\hat{i} + 12t^2\hat{j} + 20t^3\hat{k}$$

$$\& \ \vec{a} = \frac{d\vec{v}}{dt} = 6\hat{i} + 24t\hat{j} + 60t^2\hat{k}$$

Again,

$$V = \left|\vec{V}\right| = \left(36t^2 + 144t^4 + 400t^6\right)^{\frac{1}{2}} \qquad - \ (i)$$

$$a = \left|\vec{a}\right| = \left[36 + (24t)^2 + \left(60t^2\right)^2\right]^{\frac{1}{2}} \qquad - \ (ii)$$

Now,

At t = 4sec

V=1294.54m/s [putting t=4 in equ-(i)]

a=964.81m/s^2 [putting t=4 in equ-(ii)]

Again,

Tangential component of acceleration,

$$a_t = \frac{dv}{dt} = \frac{d}{dt}\left(36t^2 + 144t^4 + 400t^6\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\cdot\frac{1}{\left(36t^2 + 144t^4 + 400t^6\right)^{\frac{1}{2}}} \times \left(72t + 576t^3 + 2400t^5\right)$$

At time t=4 sec, a_t=963.56m/s^2 (Ans)

Now,

$$a_n = \sqrt{a^2 - a_t^2} = \sqrt{(964.81)^2 - (963.56)^2}$$

$$\therefore a_n = 49.1 m/s^2 \ (Ans)$$

Again,

$$\rho = \frac{V^2}{a_n} = \frac{(1294.54)^2}{49.1} = 34131.03m\,(Ans)$$

2. A car is traveling on a curved section of the road of radius 915m at the speed of 50km/hr. Brakes are suddenly applied causing the car to slow down to the 32 km/hr after 6 sec. Calculate the acceleration of the car immediately after the brake have been applied.

Solution: Given,

$$\rho = 915\text{m}$$
$$V_0 = 50\text{km/hr} = 13.88 \text{ m/sec}$$
$$V_1 = 32\text{km/hr} = 8.88 \text{ m/sec}$$

At the instant when the brake is applied,

$$a_n = \frac{V^2}{\rho} = \frac{(13.88)^2}{915} = 0.210 m/s^2$$

$$a_t = \frac{V_1 - V_0}{\Delta t} = -0.833 m/s^2$$

$$\vec{a} = a_n \hat{e}_n + a_t \hat{e}_t$$

$$\vec{a} = 0.210\hat{e}_n - 0.833\hat{e}_t$$

$$a = |\vec{a}| = \sqrt{(0.21)^2 + (-0.83)^2} = 0.856 m/s^2 \ (Ans)$$

$$\beta = \tan^{-1}\frac{a_n}{a_t} = \tan^{-1}\left(-\frac{0.21}{0.83}\right) = 14.2° \ (Ans)$$

3. The plane curvilinear motion of the particle is defined in polar co-ordinates by $r=t^{3/4}+3t$ and $\theta=0.5t^2$ where r is in m, θ is in radian and t is in second. At the instant when t=4 sec, determine the magnitude of velocity, acceleration and radius of curvature of the path.

Solution: We have,

$$r = \frac{t^3}{4} + 3t \Rightarrow \dot{r} = \frac{3t^2}{4} + 3 \Rightarrow \ddot{r} = 3t/2$$

Again, $\theta = 0.5t^2 \Rightarrow \dot{\theta} = t \Rightarrow \ddot{\theta} = 1$

Now, we have

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = \left(\frac{3t^2}{4} + 3\right)\hat{e}_r + \left(\frac{t^3}{4} + 3t\right)t\hat{e}_\theta \quad -(i)$$

Again, at t = 4 sec

$$\vec{v} = 15\hat{e}_r + 112\hat{e}_\theta$$

$$\therefore v = |\vec{v}| = \sqrt{(15)^2 + (112)^2} = 133 m/s^2 \qquad (Ans)$$

Again,

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$= \left\{\frac{3t}{2} - \left(\frac{t^3}{4} + 3t\right)t^2\right\}\hat{e}_r + \left\{\left(\frac{t^3}{4} + 3t\right)\times 1 + 2\left(\frac{3t^2}{4} + 3\right)t\right\}\hat{e}_\theta$$

At t = 4 sec,

$$\vec{a} = -442\hat{e}_r + 148\hat{e}_\theta$$

$$a = |\vec{a}| = \left[(-442)^2 + (148)^2\right]^{\frac{1}{2}} = 466.12 m/s^2 \ (Ans)$$

Again, from equ(i) [for ρ]

$$v = \left|\vec{v}\right| = \left[\left(\frac{3t^2}{4} + 3\right)^2 + \left(\frac{t^4}{4} + 3t^2\right)^2\right]^{\frac{1}{2}}$$

$$\because a_t = \frac{dv}{dt} = \frac{d}{dt}\left[\frac{t^6}{16} + \frac{33t^4}{16} + \frac{27t^2}{2} + 9\right]^{\frac{1}{2}}$$

$$\because a = \frac{1}{2} \times \frac{1}{\left(\frac{t^6}{16} + \frac{33t^4}{16} + \frac{27t^2}{2} + 9\right)^{\frac{1}{2}}} \times \left(\frac{3t^5}{8} + \frac{33t^3}{4} + 27t\right)$$

At t = 4 sec,

$$a_t = 16.055 m/s^2$$

$$\because a_n = \left[a^2 - (a_t)^2\right]^{\frac{1}{2}} = \left[(466.12)^2 - (16.005)^2\right]^{\frac{1}{2}}$$

$$\therefore a_n = 465.84 m/s^2$$

$$\therefore \rho = \frac{v^2}{a_n} = \frac{(113)^2}{465.84} = 27.41m$$

Hence, Radius of curvature = 27.41m (Ans)

Chapter – 4

KINETICS OF PARTICLES

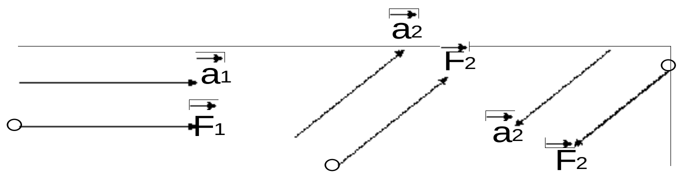
NEWTONS SECOND LAW

4.1 Newton's Second Law of Motion:

- Newton has given his understanding of motion of particles and their causes and effects in 3 laws.
- The first and third law of motion deals with the bodies at rest or moving with uniform velocity i.e. without any acceleration.
- For the bodies under the motion with acceleration the analysis of motion and forces producing it is done by the application of Newton's Second Law.

Statement of Newton's 2nd Law:

"If the resultant force acting on a particle is not zero, the particle will have an acceleration proportional to the magnitude of the resultant and in the direction of this resultant force.



If $\vec{F}_1, \vec{F}_2, \vec{F}_3$, etc be the resultant forces of different magnitude and direction acting on the particle. Each time the particle moves in the direction of the force acting on it and if $\vec{a}_1, \vec{a}_2, \vec{a}_3$, etc be the magnitude of the accelerations produced by the resultant forces. Then,

$F_1 \propto a_1, \qquad F_2 \propto a_2, \qquad F_3 \propto a_3 \qquad .... \quad etc$

$$\frac{\vec{F}_1}{\vec{a}_1} = \frac{\vec{F}_2}{\vec{a}_2} = \frac{\vec{F}_3}{\vec{a}_3} = ... = \text{constant} = \text{mass of particle (m)}$$

So, when a particle of mass 'm' is acted upon by a force $\vec{F}$ and acceleation $\vec{a}$, they must satisfy the relation,

$\vec{F} = m\vec{a} \qquad -(i) \quad$ [where direction of $\vec{F}$ & $\vec{a}$ are same]

i.e. $F_x\hat{i} + F_y\hat{j} + F_z\hat{k} = m\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)$

which is Newton's Second Law.

When a particle is subjected simultaneously to several forces equation(i) is modified as:

$\sum \vec{F} = m\vec{a} \qquad i.e. \sum\left(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}\right) = m\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)$

where, $\sum \vec{F}$ = sum of resultant of all forces acting.

$\sum F_x, \sum F_y, \sum F_z$, a_x, a_y, a_z are x, y and z component of the forces and acceleration acting on the particle respectively.

Notes:

(i) When the resultant force is zero, the acceleration of the particle is zero.

(ii) When V_0=0 and $\sum \vec{F} = 0$, Then particle would remain at rest.

(iii)When V_0=V and $\sum \vec{F} = 0$, Then particle would move with constant velocity, V along the straight line.

(iv)All the above cases defines the first law, hence the Newton's 1st Law of Motion is a particular case of Newton's 2nd Law of Motion.

4.2 Linear Momentum and Rate of Change [Impulse Momentum Theorem}:

From Newton's 2nd Law, $\vec{F} = m\vec{a}$

or, $\vec{F} = m\frac{d\vec{v}}{dt}$

$\therefore \vec{F} = \frac{d}{dt}(m\vec{v}) \quad -(i)$

Multiplying both sides by dt and integrating under the limits, we get:

$$\int_{t1}^{t2} Fdt = \int_{v1}^{v2} dm\vec{v} \Rightarrow \underbrace{I_{1-2}}_{\text{Improper Path Function}} = \overrightarrow{mv_2} - \overrightarrow{mv_1} \quad -(ii)$$

The term $\int_{t1}^{t2}\vec{F}dt$ is called the impulse ($\vec{I}$) of the force during time interval (t_2-t_1) whereas $m\vec{v}$ is the linear momentum vector of the particle.

So, equation (ii) states that

" The impulse ($\vec{I}$) over the time interval (t_2-t_1) equal the change in linear momentum of a particle during that interval." [Impulse Momentum Theorem]

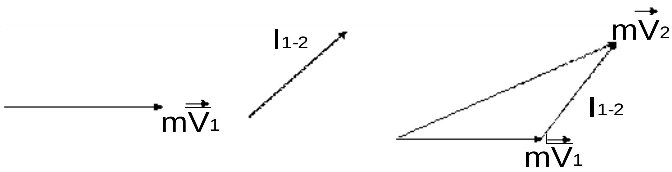
The impulse of force is known even when the force itself may not be known.

Again, from equatin(ii)

$$\overrightarrow{mv_2} = \overrightarrow{mv_1} + \vec{I}_{1-2} \quad -(iii)$$

i.e. Final momentum $(\overrightarrow{mv_2})$ of the particle may be obtained by adding vectorically its initial momentum $\overrightarrow{mv_1}$ and the impulse of the force $\vec{F}$ during the time interval considered.

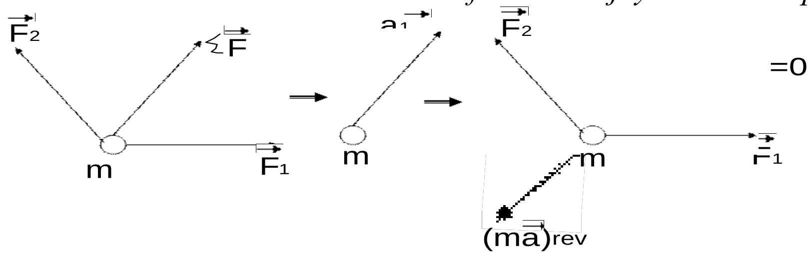
Or, showing in vector form.



When several forces act on a particle, the impulse produced by each of the forces should be considered.

i.e. $\overrightarrow{mv_1} + \sum \vec{I}_{1-2} = \overrightarrow{mv_2} \quad -(iv)$

where, $\sum \vec{I}_{1-2} = \int_{t1}^{t2}\left(\sum \vec{F}\right)dt = \int_{t1}^{t2}\left(\vec{F_1} + \vec{F_2} + \vec{F_3}.....\right)dt = \int_{t1}^{t2}\vec{F_1}dt + \int_{t1}^{t2} F_2 dt + ...$



4.3 System of Unit:

Units of measurement should be consistent and one of the standards should be followed. Generally 2 standard units are taken:

a) System de' International Unit (SI unit)
b) U.S. Customary Units (used by American Engineers)

<u>SI Units:</u>

SI stands for System de' International. SI units are the world-wide standards for the measuring system. SI units are fundamental or derived.

<u>Fundamental and Derived Units:</u>

Fundamental and Derived units are the SI units. Fundamental units are independent of any other measuring units and are the basic units for all other system whereas Derived units are the units which are expressed in terms of powers of one or more fundamental units.

Fundamental Units	**Derived Units**
Length = metre (m)	Velocity = L/T = m/s
Mass = kilogram (kg)	Acceleration = V/T = L/T^2 = m/s^2
Time = second (s)	Force = ma = ML/T^2 = kgm/s^2 (N)

SI units are the absolute system of units and results are independent upon the location of measurement.

<u>US Customary Units</u>:

This system is not absolute system of unit. They are gravitational system of units.

Base Units
length = foot(ft)
force = pound (lb)
time = second (s)

Conversion from US Customary Units to SI Units:

length : 1 ft = 0.3048 m
force : 1 lb = 4.448 N
mass : 1 slug = 14.59 kg
: 1 pound = 0.4536 kg

4.4 Equations of Motion and Dynamic Equilibrium

Considering a particle mass ‘m’ acted upon by several forces. Then from second law,

$$\sum \vec{F} = m\vec{a} \qquad -(i) \quad \Rightarrow \sum \vec{F} = m\left(a_x\hat{i} + a_y\hat{j} + a_z\hat{k}\right)$$

Using rectangular components, the equation of motions are

$$\sum \vec{F}_x = m\vec{a_x}, \sum \vec{F}_y = ma_y, \sum \vec{F_z} = ma_z \qquad -(ii)$$

(i) and (ii) gives the equation of motion of particle under the force $\vec{F}$

$$\text{or, } \sum F_x = m\ddot{x}, \sum F_y = m\ddot{y}, \sum F_z = m\ddot{z}$$

Integrating these equation as done in 3.3, the equation of motion can be obtained.
Again, the equation(i) may be expressed as

$$\sum \vec{F} - m\vec{a} = 0$$

i.e., if we add vector $-m\vec{a}$ to the resultant force in opposite direction, the system comes under the equilibrium state. This force ($-m\vec{a}$) opposite to the resultant force is called Inertial Force or Inertia Vector. This equilibrium state of a particle under the given forces and the inertia vector is said to be dynamic equilibrium.

At the dynamic equilibrium,

$$\sum F_x = 0, \ \sum F_y = 0 \ \& \ \sum F_z = 0$$

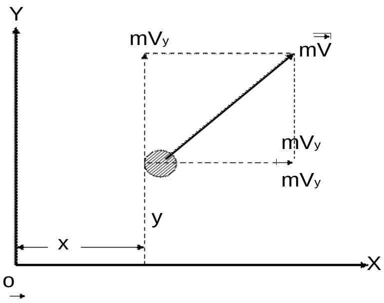
Inertia vector measure the resistance that particles offer when we try to set them in motion or when we try to change the condition of their motion.

- **Angular Momentum and Rate of Change (Angular Momentum Theorem)**

Statement

“The rate of change of angular momentum of the particle about any point at any instant is equal to the moment of the force $(\vec{F})$ acting on that particle about the same point.”

Let a particle of mass ‘m’ moving in the XY-plane and the linear momentum of the particle is equal to the vector $m\vec{v}$.



The moment about O of the vector $m\vec{v}$.(linear momentum) is called angular momentum of the particle about O at that instant and is denoted by $\vec{H_0}$

Now, mv_x and mv_y are components of $m\vec{v}$. in x & y direction.

Then, from definition,

(+↶ $H_0 = x(mv_y) - y(mv_x)$

Total momentum about a point is sum of moment about the point - Varignon's Theorem

$$\therefore H_o = m(xv_y - yv_x) \qquad -(i)$$

Differentiating equ(i) with respect to time, we get:

$$\left(\because \dot{x} = v_x \ \& \ \dot{y} = v_y ; \quad \dot{v}_x = a_x \ \& \ \dot{v}_y = a_y \right)$$

$$\dot{H}_0 = m\frac{d}{dt}\left(xv_y - yv_x\right)$$

$$= m\left(\dot{x}v_y + x\dot{v}_y - \dot{y}v_x - y\dot{v}_x\right)$$

$$= m\left(xa_y - ya_x\right)$$

$$\because \dot{H} = xma_y - yma_x$$

$$\dot{H} = xF_y - yF_x$$

$$\therefore \dot{H} = moment\ of\ \ Force\ about\, O$$

$$\therefore \left[\dot{H} = m_0\right] \quad - \quad (ii)$$

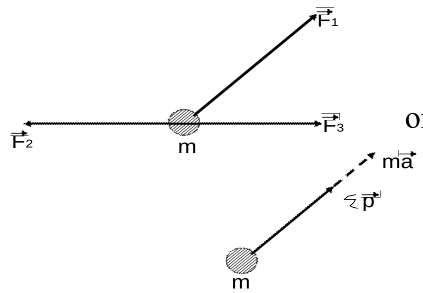
Thus, the rate of change of angular momentum of the particle about any point to any instant is equal to the moment of force $\left(\vec{F}\right)$acting on that particle about the same point.

i.5 Equation of Motion

(a) Rectilinear motion of particles:

If a particle of mass 'm' is moving in a straight line under the action of coplanar forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, etc$ Then the motion of particle can be written as

$$\sum \vec{F} = m\vec{a} \qquad -(i), \quad where \sum \vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3}$$



For Rectilinear motion, motion is only along the single co-ordinate,
i.e. $a_x = a$ & $a_y = 0$

$\therefore$ Equ^n-(i) may be written as

$$\left.\begin{aligned} \sum F_x &= ma_x \\ \sum F_y &= 0 \end{aligned}\right\} \quad \text{- (ii)}$$

These are the equation of motion for the particle moving in the straight line.

(b) Curvilinear motion of particles:

i Rectangular components
ii Tangential and Normal components
iii Radial and Transverse Components

i. Rectangular components

From Newton's second law,

$$\sum F_x = ma_x \quad ; \qquad \sum F_y = ma_y$$

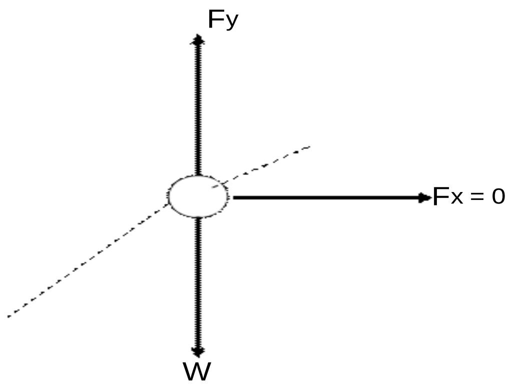
For Projectile motion, neglecting air resistance

$$\sum F_x = 0 \Rightarrow \sum F_x = ma_x = 0 \Rightarrow a_x = 0$$

$$\sum F_y = ma_y = -w = -mg$$

$$or,\ a_y = -\frac{mg}{m} = -g$$

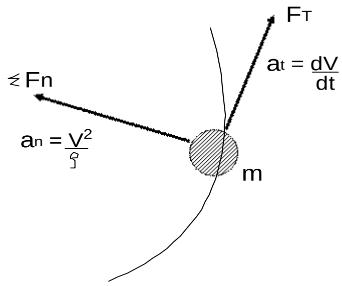
$$\left.\begin{aligned} \therefore a_x &= 0 \\ a_y &= -g \end{aligned}\right\} \quad (i)$$



These are the equations of motion.

ii. **Tangential and Normal components:**

From Newton's 2nd law,



$$\sum \overrightarrow{F_t} = m\overrightarrow{a_t} = m\frac{dv}{dt}$$

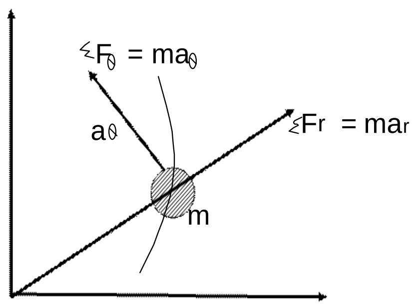
$$\sum \overrightarrow{F_n} = m\overrightarrow{a_n} = m\frac{v^2}{\rho}$$

$$\sum \overrightarrow{F} = \sum \overrightarrow{F_t} + \sum \overrightarrow{F_x}$$

These are the equations of motion.

iii. Radial and Transverse components:

From Newton's 2nd law,



$$\sum F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$\sum F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\grave{}$$

$$\sum \overrightarrow{F} = \sum \overrightarrow{F_r} + \sum \overrightarrow{F_\theta}$$

These are the equations of motion.

Note:

In case of Dynamic Equilibrium all the components of forces are balanced by Inertial Vector or Inertia force. So, for dynamic equilibrium condition, the equation of motion becomes

$$\left.\begin{aligned} \sum F_r &= ma_x = 0 \\ \sum F_y &= ma_y = 0 \end{aligned}\right\} \quad \text{- (i)}$$

$$\left.\begin{aligned} \sum F_t &= ma_t = 0 \\ \sum F_n &= ma_n = 0 \end{aligned}\right\} \quad \text{- (ii)}$$

$$\left.\begin{aligned} \sum F_r &= ma_r = 0 \\ \sum F_\theta &= ma_\theta = 0 \end{aligned}\right\} \quad \text{- (iii)}$$

i.6 Motion due to Central Force-Conservation of Angular Momentum

When the force $\overrightarrow{F}$ acting on a particle P is directed towards or away from the fixed point O, the particle is said to be moving under a central force. The fixed point 'O' is called the center of force.

As shown in the figure, particle P moves along the curve path.

O = origin of co-ordinates

Now,

F_r = Radial component of force $\overrightarrow{F}$

F_θ = Transverse component of force $\overrightarrow{F}$

For central motion $F_\theta = 0$

$$\because F\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$or, \frac{1}{r}\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right) = 0$$

$$or, \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \Rightarrow d\left(r^2\dot{\theta}\right) = 0$$

Integrating both sides we get

$r^2\ \dot{\theta}$=constant=h - (i)

Now, if elementary section are swept in time 'dt' be dA

$$\because dA = \frac{1}{2} r.rd\theta \qquad [\theta = S/R; S = 1 \ \ for\, d\theta \to 0]$$

$$or, dA = \frac{1}{2} r^2 d\theta$$

$$or, \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} \qquad [\text{Dividing both sides by dt}]$$

Here, $\frac{dA}{dt} = Rate\, of\ change\, of\ sweeping\ Area\, or\ Area\, Velocity\, (A.V.)$

$$\because A.V. = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h$$

$$\therefore h = 2\, A.V. \quad -(ii)$$

Thus, when a particle moves under the central force, the areal velocity is constant. This is also called Kepler's Law

Again, Angular momenum = momentum of linear momentum about the fixed point.

$$H_0 = mv_\theta \times r$$

Now,

$$v_\theta = r\dot{\theta}$$

$$\therefore H_0 = mr\dot{\theta} r$$

$$\therefore H_0 = mr^2\dot{\theta} \quad -\ (iii)$$

$$or, H_0 = mh \quad [\because r^2\dot{\theta} = h]$$

$$or, H_0 = 2m(A.V.) \qquad [\because h = 2\, A.V.]$$

$$\therefore\ H_0 = cons\tan t \qquad [\sin ce, A.V. = cons\tan t\, \&\, m = cons\tan t]$$

Hence, when a particle is moving under a central force, Angular momentum is always conserved.

i.7 Newton's Universal Law of Gravitation:

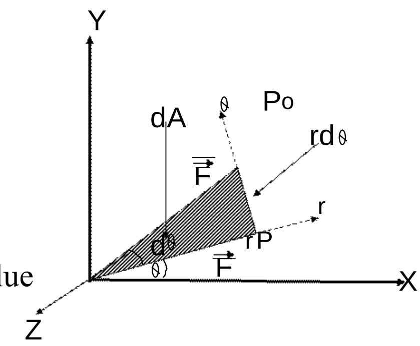
Statement:

Every particle in the universe attracts every other particle with a force, which is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

Mathematically,

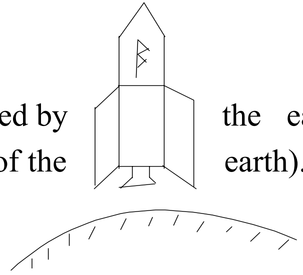
$$F \propto \frac{Mm}{d^2}$$

$$\therefore F = \frac{GMm}{d^2} \quad - \ (i) \quad \begin{bmatrix} \text{where, m and M are masses in kg} \\ \text{d is the distance in m} \end{bmatrix}$$



Also, G is the Universal Gravitation constant with its value $6.673\times10^{-11} Nm^2/kg^2$ and

F is the force of attraction between them.



For a body of mass ‘m’ located on or near the surface of earth, force exerted by the earth on a body equals to the weight of the body i.e. F = mg and d = R (radius of the earth).

$$F = mg = \frac{GMm}{R^2}$$

$$\because g = \frac{GM}{R^2} \qquad -(ii)$$

where, g is the acceleration due to gravity with its standard value 9.81 m/s^2 at the sea level.

Since, earth is not perfectly spherical so the value of R is different and hence g varies according to the variation of altitude and latitude.

i.8 Application in space mechanics:

Earth satellite and space vehicles are subjected only to the gravitational pull of the earth after crossing the atmosphere. The gravitation force acts as a central force on them and hence their motions can be predicted as follows:

From central force motion,

$$r^2\dot{\theta} = h \qquad -(i)$$

Trajectory of a particle under a central force:

Considering a particle P under central force $\vec{F}$ (i.e. directed towards center ‘O’)

Then we have

Radial component of force,

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right) = -\vec{F} \qquad -(i)$$

And, Transverse component of force

$$\sum F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \qquad -(ii)$$

From equ(ii) since m≠0

$$\therefore r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$or, \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$$

On integrating,

$$r^2\dot{\theta} = constant = h \qquad -(iii)$$

$$\therefore \dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2}$$

Again,

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}.\frac{d\theta}{dt} = \frac{h}{r^2}.\frac{dr}{d\theta} = -h\frac{d}{d\theta}\left(\frac{1}{r}\right) \qquad -(iv)$$

$$\therefore \ddot{r} = \frac{d\dot{r}}{dt} = \frac{d\dot{r}}{d\theta}.\frac{d\theta}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = \frac{h}{r^2}.\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right]$$

$$\therefore \ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) \qquad -(v)$$

Putting values of $\dot{\theta}$ & $\ddot{r}$ $in\, equ(i)\ we\ get$

$$m\left[=\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)-r\frac{h^2}{r^4}\right]=-F$$

Putting $v=\frac{1}{r}, we\ get$

$$-m\left[h^2u^2\frac{d^2u}{d\theta^2}+h^2u^2\right]=-F$$

$$\therefore \frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2} \quad -(vi)$$

This is second order differential equation, which is the trajectory followd by the particle which is moving under a central force $\vec{F}$.

Note:

- **i.** $\vec{F}$ is directed towards 'O'
- **ii.** Magnitude of $\vec{F}$ is +ve if $\vec{F}$ is actually towards O (i.e. attractive force)
- **iii.** F should be '-ve' if $\vec{F}$ is directed away from O.

The trajectory of a particle under a central force is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2} \quad -(ii)$$

Again,

$$F=\frac{GMm}{r^2}=GMmu^2 \quad -(iii)$$

where, M = mass of earth
m = mass of the space vehicle
r = distance from the centre of earth to the space vehicle, u = $\frac{1}{r}$

From equ (ii) & (iii)

$$\frac{d^2u}{d\theta^2}+u=\frac{GMmu^2}{mh^2u^2}=\frac{GM}{h^2}=\text{constant} \quad -(iv)$$

This equ^n(iv) is second order differential equation with constant co-efficient $\left(\frac{GM}{h^2}\right)$. The general solution of the differential equation is equal to the sum of the complementary i.e.

$U = U_c+U_p$

wherem,

U_c = complementy solution i.e. for tangient condition
U_p = particular solution i.e. for steady state condition

U_c = A sinθ + B cosθ

$U_p = \frac{GM}{h^2}$

Again,

U_c = A sinθ + B cosθ = C (cosθ cosθ_0 + sinθ sinθ_0) = C cos(θ-θ_0)

$$\therefore U=C\cos(\theta-\theta_0)+\frac{GM}{h^2}$$

Now,

choosing $\theta_0 = 0$ and $U = \frac{1}{r}$ $[i.e.\ \text{inital line is axis of symmetry}]$

we, get:

$$\frac{1}{r} = \frac{GM}{h^2} + c\cos\theta \qquad -(v)$$

Again, we have the equation of conic section,

$$r = \frac{l}{l + e\cos\theta}$$

$$\therefore \frac{1}{r} = \frac{1}{l} + \frac{e}{l}\cos\theta \qquad -(vi)$$

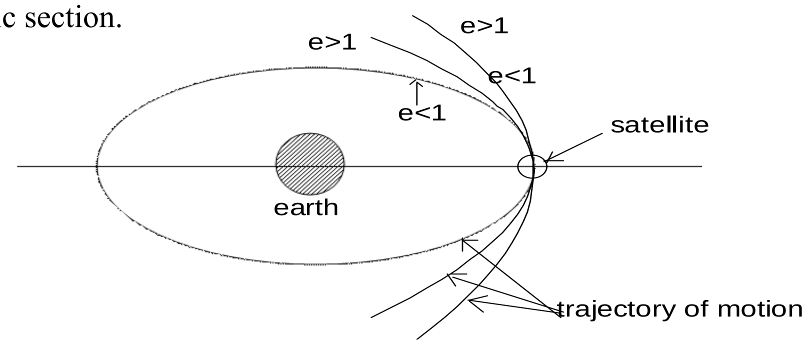
Comparing equ^n (v) & (vi), we get:

$$c = \frac{e}{l} \Rightarrow e = cl$$

Again,

$$\frac{1}{l} = \frac{GM}{h^2} \Rightarrow l = \frac{h^2}{GM}$$

$\therefore e = \frac{ch^2}{GM}$ which is eccentricity of the conic section.



So, three cases may arise:

a) If e>1 (i.e. conic is a hyperbola)

i.e. $\frac{ch^2}{GM} > 1 \qquad or, c > \frac{GM}{h^2}$

b) If e=1 (i.e. conic is a parabola)

i.e. $\frac{ch^2}{GM} = 1 \qquad or, c = \frac{GM}{h^2}$

c) If e<1 (i.e. conic is an ellipse)

i.e. $\frac{ch^2}{GM} < 1 \qquad or, c < \frac{GM}{h^2}$

Special Cases:

- When e=c=0 the length of radius vector is constant and the conic section reduces to circle.
- At the last stage of launching satellite into orbit, it has the velocity parallel to the surface of the earth and the satellite begins its free flight at the vertex ‘A’.

Let, r_0, v_0 be the radius vector and velocity at the beginning of free flight. Here, velocity reduces to transverse components only.

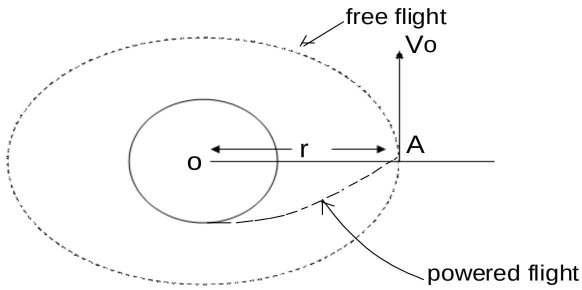
$$\therefore v_0 = v_\theta = r_0\dot{\theta}_0$$

Again,

$$h = r^2\dot{\theta} = r_0^2\dot{\theta}_0 \qquad -(vii) \qquad h = r_0^2\omega_0 = r_0 v_0$$

$$\begin{pmatrix} \dot{\theta} = \omega \\ \omega r = v \end{pmatrix}$$

Then from equ^n(v) $\left[\frac{1}{r} = \frac{GM}{h^2} + c\cos\theta\right]$



At vertex 'A', θ=0, r=r_o and v=v_o

$$\therefore c = \frac{1}{r_0} - \frac{GM}{(r_0 v_0)^2} \qquad -(viii) \qquad [\cos 0 = 1]$$

For parabolic trajectory, $c = \frac{GM}{h^2}$

$$\therefore c = \frac{GM}{r_0^2 v^2} \qquad -(ix)$$

From equ^n(viii) & (ix)

$$\frac{GM}{r_0^2 v_0^2} = \frac{1}{r_0} - \frac{GM}{r_0^2 v_0^2}$$

$$or, \frac{2GM}{r_0^2 v_0^2} = \frac{1}{r_0}$$

$$\therefore v_0 = \sqrt{\frac{2GM}{r_0}} \qquad -(x)$$

This velocity v_0 is called the escape velocity. Since, this is the minimum velocity required for the vehicle so that it does not return to its starting point.

$$\therefore V_{esc} = \sqrt{\frac{2GM}{r_0}} = \sqrt{\frac{2gR^2}{r_0}} \qquad \begin{bmatrix} \therefore GM = gR^2 \\ mg = \frac{GMm}{R^2} \end{bmatrix}$$

$$\therefore V_{esc} = \sqrt{\frac{2gR^2}{r_0}} \qquad -(xi)$$

Note:

If $V_o > V_{esc}$, trajectory will be hyperbolic
$V_o = V_{esc}$, trajectory will be parabolic
$V_o < V_{esc}$, trajectory will be elliptical

Among the elliptical orbit, if c=0 then the ellipse reduces to circle.
i.e. putting c=0 in equ^n(viii), we get

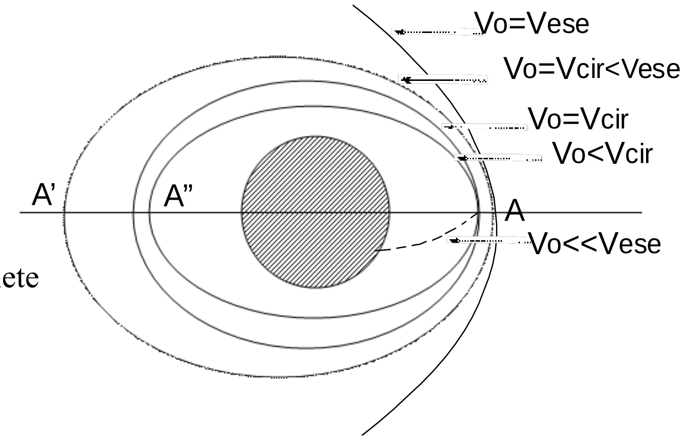
$$\frac{1}{r_0} = \frac{GM}{r_0^2 v_0^2} \Rightarrow v_{circ} = \sqrt{\frac{GM}{r_0}}$$

$$v_{circ} = \sqrt{\frac{gR^2}{r_0}} \qquad -(xii)$$

Perigee and Apogee

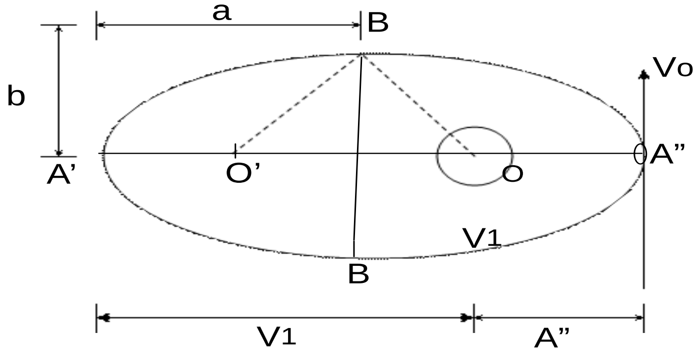
The closest point of the orbit from the earth is called perigee and the farthest point of orbit from the earth is called apogee.

(ii) For $V_{circ} < V_o < V_{esc}$, A=perigee and A'=Apogee
(iii) For $V_o < V_{circ}$, A=apogee and A''=perigee
(iv) For $V_o << V_{circ}$, the vehicle doesn't go to orbit.



Time Period (or Periodic time) of Space Vehicle

It is the time required for the satellite to complete its orbit and is denoted by τ.



$$\tau = \frac{\text{Area inside orbit (i.e. Area of ellipse)}}{\text{Area Velocity}}$$

$$\tau = \frac{\pi ab}{h/2}$$

$$or, \tau = \frac{2\pi ab}{h} \quad - (xiii)$$

where, a = semi-major axis of ellipse = $\frac{r_1 + r_0}{2}$

b = semi-minor axis of ellipse = $\sqrt{r_0 r_1}$

h = $r_0 v_0$

Tutorials:

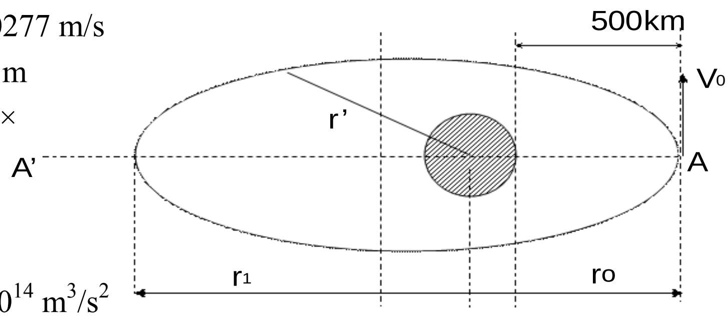
1) A satellite is launched in a direction parallel to the surface of the earth with a velocity of 37000 km/hr from an altitude of 500 km. Determine the altitude attained by it when it covers the angular distance equal to 135°. Also calculate the periodic time of the satellite. Take radius of earth, R=6370 km.

Sol^n:- Here,

Launching velocity (v_o) = 37000 km/hr = 10277 m/s

Radius of earth (R) = 6370 km = 6.37×10^6 m

Altitude of launching (h) = 500 km = 5×10^5m



Then, $r_o = 6.37 \times 10^6 + 5 \times 10^5 = 6.87 \times 10^6$ m

$h = r_o v_o = 6.87 \times 10^6 \times 10277 = 7.06 \times 10^{10}$

$GM = gR^2 = 9.81 \times (6.37 \times 10^6)^2 = 3.98 \times 10^{14}$ m^3/s^2

$$\therefore \frac{GM}{h^2} = \frac{3.98 \times 10^{14}}{(7.06 \times 10^{10})^2} = 7.98 \times 10^{-8}$$

We know, $\frac{1}{r} = \frac{GM}{h^2} + c\cos\theta \qquad - (i)$

At point 'A' θ=0 and $r = r_o = 6.87 \times 10^6$

$$\because \frac{1}{6.87 \times 10^6} = 7.98 \times 10^{-8} + c$$

$$\therefore c = 6.576 \times 10^{-8}$$

Again, at θ=135°

$$\frac{1}{r} = 7.98 \times 10^{-8} + 6.576 \times 10^{-8} \cos 135° \Rightarrow r' = 30029.44km$$

$\therefore$ Altitude gained by satellite (H) = r'-R = 23659.44 km

Again, to calculate time period:

When the satellite covers 180°, it will make

$$\frac{1}{r_1} = 7.98\times10^{-8} + 6.576\times10^{-8}$$

$$\Rightarrow r_1 = 71255.07 km$$

$$\text{Then, } a = \frac{r_o + r_1}{2} = \frac{6870 + 71225.07}{2} = 39047.54\,km$$

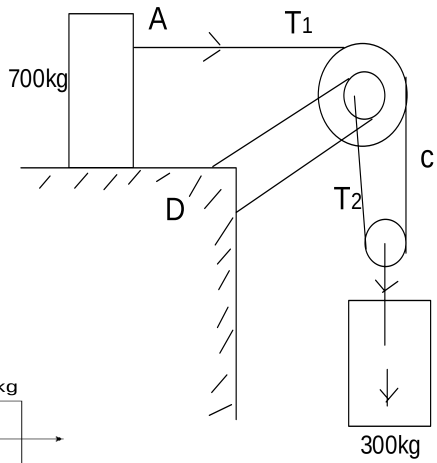
$$b = \sqrt{r_o \times r_1} = 22120.49\,km$$

$$\because \text{Time period of the satellite, } \tau = \frac{2\pi ab}{h} = \frac{2\pi\times39047.54\times10^3\times22120.49\times10^6}{7.06\times10^{10}}$$

$$\therefore \tau = 4.670647\times10^4 \text{ sec}$$

$$\tau = 21\text{hrs}\,18\min 26\sec$$

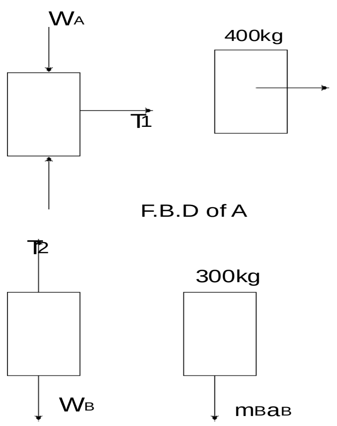
2. The two blocks shown in the figure start from rest. The horizontal plane and the pulleys are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in each cord.



Sol^n: Let, tension in the cord ACD be T_1 & cord BC be T_2. From figure, if block 'A' moves through distance S_A then block 'B' moves through $S_{A/2}$.

$$\because S_B = \frac{S_A}{2} \Rightarrow V_B = \frac{V_A}{2} \Rightarrow a_B = \frac{a_A}{2} \quad -(i)$$

Using Newton's 2nd law for Block 'A', Block 'B' and Pulley 'C'



Block 'A':

$$\sum F_x = m_A a_A$$

$$T_1 = 100 a_A \quad -(ii)$$

Block 'B':

$$\sum F_y = m_B a_B$$

$$W_B - T_2 = 300 a_B$$

$$300\times9.81 - T_2 = 300 a_A/2$$

$$\therefore T_2 - 2T_1 = 0 \quad -(iii)$$

Pulleys

Since mass of pulley is considered zero, we have:

$$\sum F_y = m_c a_c = 0$$

$$T_2 - 2T_1 = 0 \quad -(iv)$$

Putting values of T_1 & T_2 in equ^n (iv), we get:

2943-1500A-2×100a_A=0

$$\because a_A = 8.41 m/s^2$$

$$\therefore a_B = \frac{a_A}{2} = 4205 m/s^2$$

$$\because T_1 = 100a_A = 841\,N$$
$$T_2 = 2T_1 = 1682\,N$$

3. The bob of a 3 m pendulum describes an arc of a circle, in a vertical plane. If the tension is twice of the weight of the bob for the position when it is displaced through an angle of 30° from its mean position, then find the velocity and acceleration of the bob.

Sol^n:

Applying Newton's Second Law,

$$\sum F_t = ma_t$$
$$or, mg\sin 30° = ma_t$$
$$\therefore a_t = g\sin 30° = 4.9\,m/s^2$$

Again,

$$\sum F_x = ma_x$$
$$2mg - mg\cos 30 = ma_x$$
$$\because a_x = 2g - g\cos 30° = 11.12 m/s^2$$
$$\therefore a = \left[a_t^2 + a_r^2\right]^{\frac{1}{2}} = 12.15 m/s^2$$
$$\beta = \tan^{-1}\frac{a_n}{a_t} = 36.22°$$

Velocity of Bob $(v) = \sqrt{\rho a_x}$ $\quad \left[a_x = \frac{v^2}{\rho}\right]$

$$\because v = \sqrt{3\times 11.12}$$
$$v = 5.78/\sec\ \textit{perp. to the chord}$$

4. The motion of a 500 gm Block 'B' in a horizontal plane is defined by the relation r=2(1+cos2πt) and θ=2πt, where r is expressed in meters, t in seconds and θ in radians. Determine the radial and transverse component of the force exerted on the block when t=0 & t=0.75 sec.

Sol^n:

Here, m = 500 gm = 0.5 kg

r = 2(1+cos2πt) - (i)

θ = 2πt - (ii)

Differentiating with respect to time, we get

$\dot{r}$ = -4π sin2πt $\quad \dot{\theta} = 2\pi$

$\ddot{r} = -8\pi\cos 2\pi t \quad \ddot{\theta} = 0$

Now,

$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$
$$\sum F_\theta = ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

When, t = 0, r = 4, $\dot{r} = 0$ & $\ddot{r} = -8\pi^2$

$\theta = 0\ \ \dot{\theta} = 2\pi$ & $\ddot{\theta} = 0$

$\because \sum F_r = 0.5\left(-8\pi^2 - 4\times 4\pi^2\right)$

$F_r = -118.43\,N$

$\sum F_\theta = 0.5\left(4\times 0 - 2\times 0\times 2\pi\right) = 0$

$\because F_\theta = 0$

$$\left(\begin{array}{lll} \dot{r} = 4\pi & \ddot{r} = 0 & r = 2 \\ \theta = 1.5\pi & \dot{\theta} = 2\pi & \ddot{\theta} = 0 \end{array}\right)$$

Similarly for t = 0.75 sec,

$F_r = -39.5N$, $F_\theta = 79.0\,N$

Chapter – 5
Kinetics of Particle : Energy and Momentum Method

5.1 Work done by a Force:

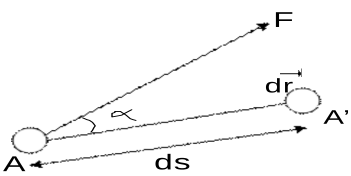
When a particle moves by the application of force $\vec{F}$ producing the displacement ds, then the work done by the force during the displacement ds is defined by:

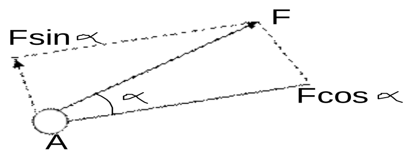
du = component of force along the direction of motion × distance travelled.

$\because du = F\cos\alpha . ds$

$du = Fds.\cos\alpha \qquad -(i)$

where, $ds = \left|d\vec{r}\right|$





$[\alpha \text{ is the angle between the force and direction of motion}]$

<u>Particular cases:</u>

(a) When $\vec{F}$ is along the direction of $d\vec{r}$, then

$du = Fds \qquad [\cos\alpha = \cos 0 = 1]$

(b) If $\vec{F}$ is perpendicular to the direction of $d\vec{r}$, then

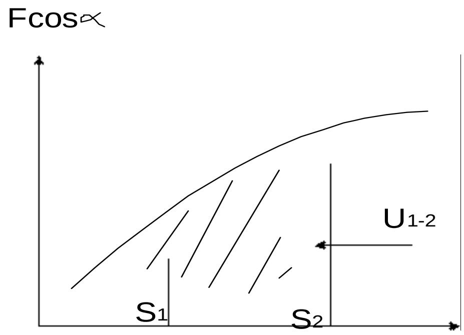
$du = 0 \qquad [\cos\alpha = \cos 90 = 0]$

(c) For finite work done from s_1 to s_2,

Integrating (i), we get:

$U_{1-2} = \int_{s_1}^{s_2} (F\cos\alpha) ds \quad -(ii)$

U_{1-2} = Area under the curve Fcosα - s

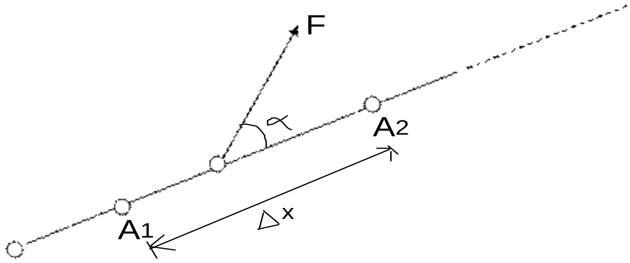


5.1.1 Work of a const Force in Rectilinear Motion

$U_{1-2} = (F\cos\alpha)\Delta x$

Δx = Displacement from A_1 to A_2

[Re*ctilinear Motion*]



5.1.2 Work of a weight (or Force of gravity)

The work du of the weight is equal to the product of weight (w) and the vertical displacement of the center of gravity G of the body.

i.e. du = -wdy

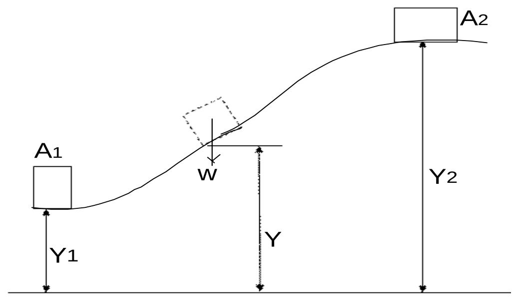
$\therefore du_{1-2} = -\int_{y_1}^{y_2} wdy$

$U_{1-2} = -w(y_2 - y_1)$

$\therefore U_{1-2} = -w\Delta y$

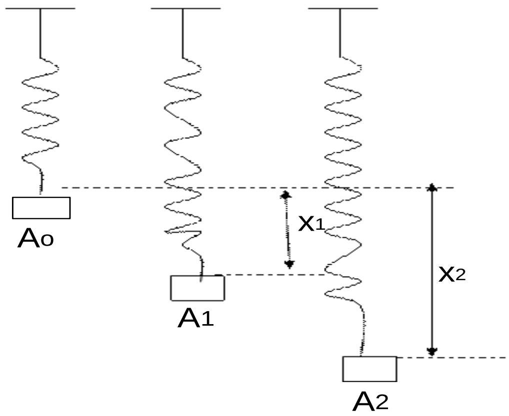
U_{1-2} is -ve when work is done on the body

U_{1-2} is +ve when work is done by the body.



5.1.3 Work of a force exerted by a spring

When a spring is deformed, the magnitude of force $\vec{F}$ exerted by it on the body is proportional to the elongation of the spring.



i.e. F = kx - (i)

where, k = spring constant

x = elongation length

Again, Elementary work

$$du = -Fdx = -k \times dx$$

$\therefore$ finite work done during elongation from x_1 to x_2

$$U_{1-2} = -\int_{x_1}^{x_2} k \times dx$$

$$U_{1-2} = -\frac{1}{2}k\left(x_2^2 - x_1^2\right)$$

work is positive, when x<x, i.e. spring is returning.

5.1.4 Work of the Gravitational Forces:

We have,

$$F = G\frac{Mm}{r^2} \qquad -(i)$$

Now, the elementary work

$$du = -Fdr = -\frac{GM}{r^2}dr \qquad -(ii) \quad [\sin ce\text{ F is directed opposite in direction of motion}]$$

So, the work for finite displacement

$$U_{1-2} = -\int_{r_1}^{r_2} \frac{GMm}{r^2}dr = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad -(iii)$$

$$\therefore \left[U_{1-2} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)wR^2\right] \qquad -(iv) \quad \left[\begin{array}{l}\because GMm = mgR^2 = wR^2 \\ Here, r > R\end{array}\right]$$

5.2 Kinetic Energy of a Particle:

For a mass 'm' acted upon by a force $\vec{F}$ and moving along the curve path, the component of force along the direction of motion is given by:

$$F_t = ma_t$$

$$or,\ F\cos\alpha = mv\frac{dv}{ds} \quad \left[\because a_t = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = v\frac{dv}{ds}\right]$$

$$or,\ F\cos\alpha ds = mvdv$$

Integrating both sides, taking limits we get

$$\int_{s_1}^{s_2} F\cos\alpha ds = m\int_{v_1}^{v_2} vdv$$

$$or,\ U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$or,\ U_{1-2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

$$or, U_{1-2} = T_2 = T_1 \qquad -(i)$$

where, T_2 and T_1 is final and initial K.E. of the particle.

Hence, the work of the force $\vec{F}$ is equal to the change of K.E. of the particle. This is also called as principle of work and energy.

5.3 Applications of Principle of work and energy:

With the help of work energy principle, solution of problems, involving force, displacement and velocity can be obtained in simple form,

e.g. Analysis of Pendulum

To determine the velocity of bob as it falls freely from A_1 to A_2,

we've

$$U_{1-2} = wL$$

Again, at KE at A_1

$$T_1 = 0 \quad [\because V_1 = 0]$$

KE at A_2

$$T_2 = \frac{1}{2}mv_2^2$$

Now, using principle of work and energy,

$$T_2 - T_1 = U_{1-2}$$

$$or,\ T_2 = U_{1-2} \Rightarrow \frac{1}{2}mv_2^2 = wL = mgL$$

$$\therefore v_2 = \sqrt{2gL} \qquad [L \text{ is the vertical height of bob from reference point}]$$

Advantages of this method:

- ❖ To find v_2 it is not necessary to find a_2
- ❖ Equation is in the form of scalar, hence it is easy to handle.
- ❖ Forces which do not work (e.g. Tension on strings), etc are eliminated.

5.4 Power and Efficiency:

Power is defined as rate of change of work.

$$P_{avg} = \frac{\Delta u}{\Delta t},\ \text{Taking limit as } \Delta t \to 0 \text{ we get}$$

$$\mathrm{P} = \frac{\mathrm{du}}{\mathrm{dt}} \qquad -(i),\ \text{Putting } du = (F\cos\alpha)ds$$

$$P = F\cos\alpha\frac{ds}{dt}$$

$$or,\ P = FV\cos\alpha \qquad -(ii) \quad \left[\frac{ds}{dt} = V\right]$$

where, V is magnitude of the velocity at the point of application of force $\vec{F}$.

$$Efficiency(\eta) = \frac{Output\ work}{Input\ work}$$

$$or,\ \eta = \frac{Power\ output}{Power\ input} < 1\ [\text{due to losses due to friction}]$$

5.5 Potential Energy:

Consider a body of weight W, which moves along a curve path from A_1 to A_2.

∵ work done due to weight during the displacement,

$$U_{1-2} = Wy_1 - Wy_2,$$

Then, work at any position,

$$U = Wy \qquad -(i)$$

The work done by gravity is independent of path and is proportional to position, work done by gravity at any position is denoted by V_g.

$$U_{1-2} = (V_g)_1 = (V_g)_2 \qquad -(ii)$$

And, $U = V_g \quad -(iii)\ where\ V_g = Wy$

If $(V_g)_2 > (V_g)_1$ then work is –ve (i.e. PE increases)

If $(V_g)_2 < (V_g)_1$ then work is +ve (i.e. PE decreases)

(a) Potential Energy of Gravitational Force:

We know that work of Gravitational Force, when the body is displayed from A_1 to A_2,

i.e. $U_{1-2} = -\left[\frac{GMm}{r_1} - \frac{GMm}{r_2}\right]$

Then, work done at any position, $U = -\frac{GMm}{r} \qquad -(i)$

Again, we know that work done, $U = V_g \qquad -(ii)$

From (i) and (ii)

$$V_g = -\frac{GMm}{r} \qquad -(iii)$$

$$\left[\begin{array}{l} F = mg = w = \frac{GMm}{R^2} \\ \therefore GMm = wR^2 \end{array}\right]$$

$$\left[V_g = -\frac{wR^2}{r}\right] \qquad -(iv)$$

r is the distance of the body from the center of the earth. For large value of r, $V_g \to 0$

(b) Potential Energy due to spring:

The work of force exerted by the spring on the body for the elongation from x_1 to x_2

$$U_{1-2} = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right) \qquad -(i)$$

$$or,\ U_{1-2} == [(V_e)_2 - (V_e)_1]$$

When, $(V_e)_1$ and $(V_e)_2$ is the PE due to elastic force, then potential energy at any elongation of spring x

$$V_e = \frac{1}{2}kx^2 \qquad -(ii)$$

- ❖ During the elongation the potential energy of the spring increases.
- ❖ The work of the force is independent of the path followed and is equal to minus charge of potential energy.

i.e. $U_{1-2} = -(V_2 - V_1) \qquad -(iii)$

The force which satisfy the equation is called as the conservation force. The gravity force and the elastic force are examples of conservative force.

5.6 Conservation of Energy:

We know, work of a force is equal to

$$U_{1-2} = -(V_2 - V_1)$$

$$\therefore U_{1-2} = -V_2 + V_1 \qquad -(i)\ [V \text{ be the PE}]$$

Again, work of a force is equal to change in KE

$$\therefore U_{1-2} = T_2 - T_1 \qquad -(ii)\ [T \text{ be the KE}]$$

From (i) and (ii)

$$V_1 - V_2 = T_2 - T_1$$

$$T_1 + V_1 = T_2 + V_2$$

$$\therefore E_1 = E_2 \qquad -(iii)$$

where, E = T+V = mechanical energy of the system. Hence, conservation of energy states that mechanical energy of the system always remains constant.

Examples of Conservation of Energy:

Analysis of Pendulum

For free fall of pendulum from A_1, Then

For position A_1,

KE, $T_1 = 0$ $\quad$ [$V_1 = 0$]

PE, $V_1 = wL$

$\therefore$ $T_1 + V_1 = wL$ $\qquad$ - (i)

At position A_2,

$V_2 = \sqrt{2gL}$

$\therefore$ KE at A_2 (T_2) = $\frac{1}{2}mv_2^2 = \frac{1}{2}\left(\frac{w}{g}\right)(2gL) = wL$

PE at A_2 $(V_2) = 0$ $\quad$ [L = 0 at A_2, datum]

$\therefore T_2 + V_2 = wL \qquad -(ii)$

For position A

KE at A, $(T_A) = \frac{1}{2}mV_A^2 = \frac{1}{2}\left(\frac{w}{g}\right)2gL\sin\theta = wL\sin\theta$

PE at A $(V_A) = w(L - L') = wL - wL\sin\theta$

$\therefore\ T_A + V_A = wL \quad - \text{(iii)}$

From equation (i), (ii) and (iii), the total mechanical energy of pendulum at any position is same and is equal to wL.

At A_1, Total energy is entirely due to PE

At A_2, Total energy is entirely due to KE

At A, Total energy is sum of PE + KE

Note:

For the system interacting with other forms of energy as electrical, frictional, etc all the forms of energy should be considered. In that case as well the total energy of system is always conserved. Hence, energy is conserved in all the cases.

5.7 Principle of Impulse and Momentum:

Considering a particle of mass 'm' acted upon by a force $\vec{F}$.

Then from Newton's 2nd Law,

$$\vec{F} = m\vec{a}$$

In x & y components,

$$F_x = ma_x \qquad \& \qquad F_y = ma_y$$

$$\therefore F_x = m\frac{dv_x}{dt} \qquad \& \qquad F_y = m\frac{dv_y}{dt}$$

Since mass 'm' is constant

$$\therefore F_x = \frac{d}{dt}(mv_x) \qquad \& \qquad F_y = \frac{d}{dt}(mv_y) \qquad -(i)$$

Vectorically, we have

$$\overleftarrow{F} = \frac{d}{dt}\left(m\vec{v}\right) \qquad -(ii)$$

This equation states, "Force $\vec{F}$ acting on the particle is equal to the rate of change of momentum $\left(m\vec{v}\right)$ of the particle.

Multiplying equation (i) by dt and integrating on both sides, we get:

$$\int_{t_1}^{t_2} F_x dt = m\int_{v_1}^{v_2} dv_x = (mv_x)_2 - (mv_x)_1$$

$$\therefore (mv_x)_1 + \int_{t_1}^{t_2} F_x dt = (mv_x)_2 \qquad -(iii)$$

Similarly,

$$(mv_y)_1 + \int_{t_1}^{t_2} F_y dt = (mv_y)_2 \qquad -(iv)$$

In vector form,

$$m\overrightarrow{v_1} + \int_{t_1}^{t_2} \vec{F}dt = mv_2 \qquad -(v) \qquad \begin{bmatrix} \overrightarrow{v_1} = \overrightarrow{v_{x_1}} + \overrightarrow{v_{y_1}} \\ \overrightarrow{v_2} = \overrightarrow{v_{x_2}} + \overrightarrow{v_{y_2}} \\ \vec{F} = \overrightarrow{F_x} + \overrightarrow{F_y} \end{bmatrix}$$

where,

$$\int_{t_1}^{t_2} \vec{F} dt = \text{Impulse of force} = \overrightarrow{I_{mp}}_{(1-2)}$$

Given, by area under the curve F-t

Hence,

$$\overrightarrow{mv_1} + \overrightarrow{I_{mp}}_{(1-2)} = \overrightarrow{mv_2} \qquad -(vi)$$

From vector diagram,

When several forces are acting on a particle,

$$\overrightarrow{mv_1} + \sum I_{mp\,(1-2)} = \overrightarrow{mv_2} \qquad -(vii)$$

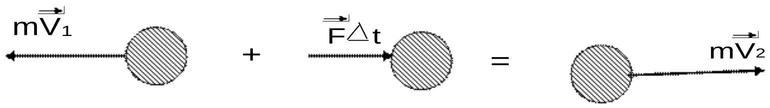
5.8 Impulsive motion and Impact

(1) Impulsive Motion:

When a very large force is acted during a very short time interval on a particle and produce a definite change in momentum, such a force is called as impulsive force and the resulting motion is called impulsive motion.

Example of Impulsive motion:

Striking the ball with a cricket bat, large force $(\vec{F})$ is applied in a small time (Δt), the resulting impulse $\vec{F}\Delta t$ is large enough to change the direction of motion of ball.



From impulse momentum principle,

$$\overrightarrow{mv_1} + \vec{F}\Delta t = \overrightarrow{mv_2} \qquad -(i)$$

Here non-impulsive forces (like weight of ball, bat, etc) are not included.

2. Impact

A collision between two bodies, which occurs in very short interval of time and during which the two bodies exert relatively large forces on each other is called an Impact.

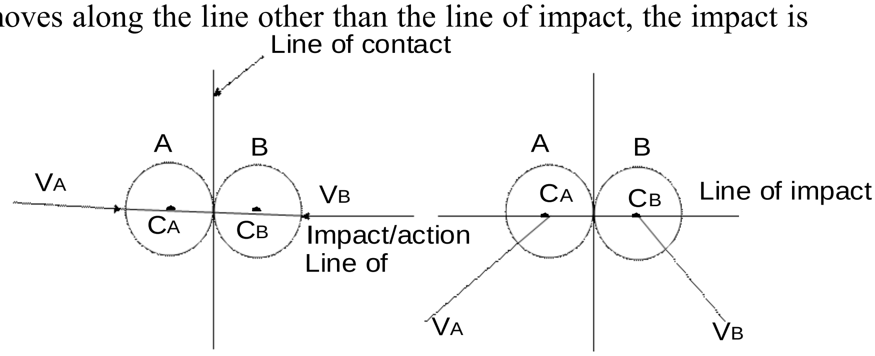
The common normal to the surfaces in contact during the impact is called the line of impact or line of action.

Types of Impact:

- If the mass centers of the two colliding bodies are located on this line of impact, the impact is central impact otherwise eccentric impact.
- If the velocities of the two particles are directed along the line of impact, it is said to be direct impact. If either or both particle moves along the line other than the line of impact, the impact is said to be an oblique impact.

Hence, four types of impact may occur. They are:

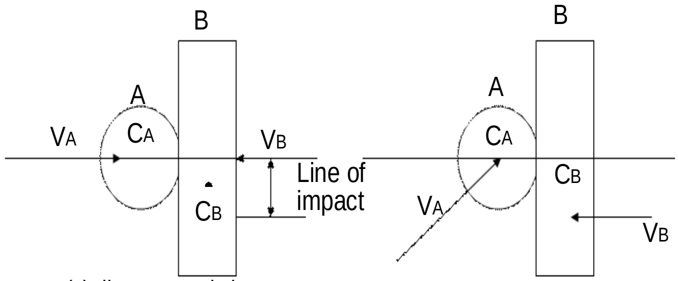
a) Direct Central Impact
b) Oblique Central Impact
c) Direct Eccentric Impact



(a) direction central impact

(b) oblique central impact

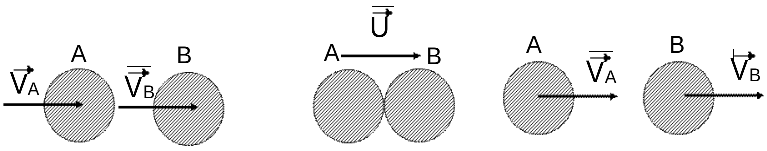
d) Oblique Eccentric Impact



(c) direct ecentric impact (d) oblique ecentric impact

5.9 Direct Central Impact:

- Two particles A and B of mass m_A and m_B are moving in a straight line with velocities v_A & v_B. If $v_A > v_B$ the particle A strikes B.
- Under the impact, they deform and at the end of period of deformation they will move with the same velocity u.
- After the impact the particles may gain their original shape or are permanently deformed, depending upon the magnitude of impact and material involved which is called restitution.
- After the impact and separation the particles move with v_A' and v_B' velocities.
- The duration of time of impact when the particles comes under the deformation and restitution during impact is called deformation period and restitution period respectively.



Considering that only impulsive forces are acting, the total momentum of the system is conserved.

i.e. $m_A\overrightarrow{v_A} + m_B\overrightarrow{v_B} = m_A\overrightarrow{v_A'} + m_B\overrightarrow{v_B'} \qquad -(i)$

In scalar form,

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B' \qquad -(ii)$$

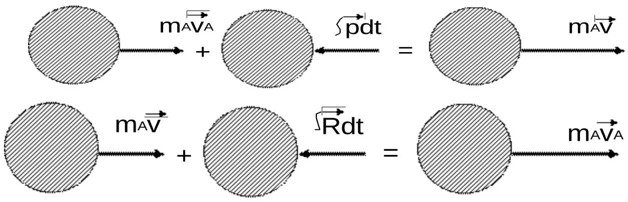
+ve value is for +ve axis and –ve value is for –ve axis.

<u>Analysis during Impact</u>

Following phenomena will occur for the particle A.

$$m_A v_A - \int pdt = m_A u \qquad -(iii)$$

$$m_A u - \int Rdt = m_A v_A' \qquad -(iv)$$



where, $\int \vec{p}dt$ and $\int \vec{R}dt$ are the impulses during the period of deformation and restitution respectively.

Then the co-efficient of restitution is defined as:

$$e = \frac{\int Rdt}{\int Pdt} \qquad -(v)$$

Value of e depends upon

- Materials of particles
- Impact velocity
- Shape & size of colliding bodies

Generally, 0<e<1

For perfectly elastic collision, e = 1

For perfectly plastic collision, e = 0

From equ(iii), $\int Pdt = m_A v_A - m_A u$

From equ(iv), $\int Rdt = m_A u - m_A v_A^{'}$

$$\because e = \frac{\int Rdt}{\int Pdt} = \frac{m_A u - m_A v_A^{'}}{m_A v_A - m_A u} = \frac{u - v_A^{'}}{v_A - u}$$

$$\therefore e = \frac{u - v_A^{'}}{v_A - u}$$

Similarly for particle B,

$$e = \frac{v_B^{'} - u}{u - v_B} \qquad -(vii)$$

Adding respectively the numerators and denominators of equ^n (vi) and (vii), we get:

$$e = \frac{u - v_A^{'} + v_B^{'} - u}{v_A - u + u - v_B} = \frac{v_B^{'} - v_A^{'}}{v_A - v_B}$$

$$\therefore v_B^{'} - v_A^{'} - e(v_A - v_B) \ -(viii)$$

i.e. Relative velocity after impact = e × Relative velocity before impact

For Perfectly Plastic Impact:

e=0, i.e. there is no period of restitution.

from equ(viii),

$$v_B^{'} = v_A^{'} = v^{'} \quad (\text{common velocity})$$

Then from equ(ii)

$$m_A v_A + m_B v_B = (m_A + m_B)v^{'}$$

$$\therefore \text{Common velocity}\left(\text{v}^{'}\right) = \frac{m_A v_A + m_B v_B}{m_A + m_B} \qquad -(ix)$$

For Perfectly Elastic Impact:

Since e=1, from equ (viii), $v_B^{'} - v_A^{'} = v_A - v_B \qquad -(x)$

from equ (ii), $m_A\left(v_A - v_A^{'}\right) = m_B\left(v_B^{'} - v_B\right) \qquad -(xi)$

from equ (x) $v_A + v_A^{'} = v_B + v_B^{'} \qquad -(xii)$

Multiplying LHS and RHS of equ(xi) & (xii) respectively and dividing both sides by 2 we get:

$$\frac{1}{2}\left[m_A\left(v_A - v_A^{'}\right)\left(v_A + v_A^{'}\right)\right] - \frac{1}{2}\left[m_B\left(v_B^{'} - v_B\right)\left(v_B^{'} + v_B\right)\right]$$

$$or, \frac{1}{2}m_A v_A^2 - \frac{1}{2}m_A v_A^{'2} = \frac{1}{2}m_B v_B^{'2} - \frac{1}{2}m_B v_B^2$$

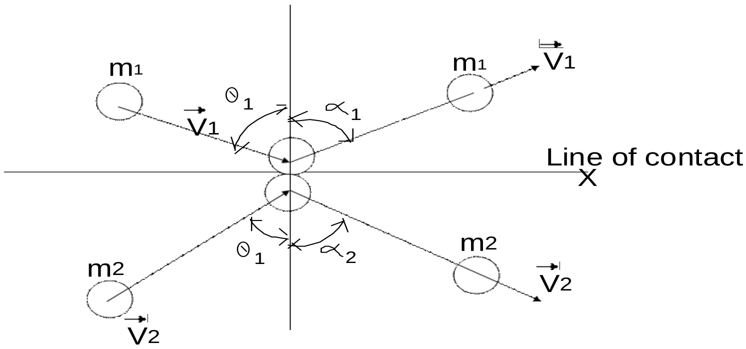
$$or, \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}m_A\left(v_A^{'2}\right) + \frac{1}{2}m_B\left(v_B^{'2}\right)$$

i.e. Initial KE of system = Final KE of system

- Hence, for perfectly elastic condition, KE of the system is conserved.
- When e≠, there is loss of KE and this lost energy is converted into heat, sound and other forms of energy.

5.10 Oblique Central Impact:

When the velocities of the two colliding bodies are not directed along the line of impact, then it is called oblique impact as shown in the figure.



Here, line of impact is along y-axis and line of contact is along x-axis. Then the following phenomena occur.

(a) x-component of the momentum of the particle 1st is conserved

i.e. $m_1 v_{1x} = m_1 v_{1x}^{'} \Rightarrow v_1 x = v_1^{'} x \quad -(i)$

(b) x-component of the momentum of 2nd particle is conserved

i.e. $m_2 v_{2x} = m_2 v_{2x}^{'} \Rightarrow v_2 x = v_{2x}^{'} \quad -(ii)$

From (a) and (b) $\left[v_1 x - v_2 x\right] = \left[v_{1x}^{'} - v_{2x}^{'}\right]$

(c) From (a) and (b), the total momentum of the particles in x-direction is also conserved

i.e. $m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}^{'} + m_2 v_{2x}^{'} \quad -(iii)$

(d) y-component of total momentum of the particle is conserved.

i.e. $-m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}^{'} - m_2 v_{2y}^{'} \quad -(iv)$

(e) y-component of relative velocity after impact is obtained by multiplying y-component of relative velocity before impact by co-efficient of restitution.

i.e. $$\left(-v_{2y}^{'} - v_{1y}^{'}\right) = e\left(-v_{1y} - v_{2y}\right)$$
$$\therefore \left(v_{2y}^{'} + v_{1y}^{'}\right) = e\left(v_{1y} + v_{2y}\right) \quad -(v)$$

The above five equation are applied for the analysis of the problems related to oblique impact.

Remember:

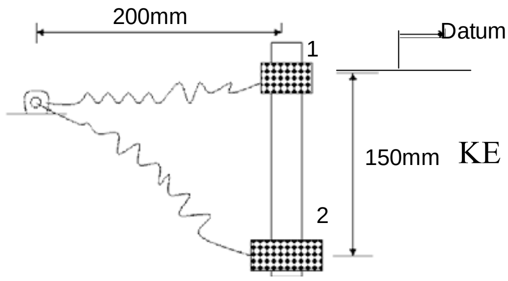
(a) Along the line of contact, momentum of each particle is conserved.

(b) Along the line of impact, the total momentum of particles is conserved.

<u>Tutorials</u>:

(a) A 10 kg collar slides without friction along a vertical rod. The spring attached to the collar has an undeformed length of 100 mm and a constant of 500 N/m. If the collar is released from rest in position 1, determine its velocity after it has moved 150mm to position2.

Solution:



Given, K = 500 N/m

Undeformed length of spring = 100mm = 0.1m

We have from conservation of energy, KE + PE at 1 = KE + PE at 2

i.e. $T_1 + V_1 = T_2 + V_2 \qquad -(i)$

$T_1 = 0 \quad [V_1 = 0]$

$V_1 = V_{e_1} + V_{g_1} = \frac{1}{2}kx_1^2$

$\therefore V_1 = \frac{1}{2} \times 500 \times (0.1)^2$

$V_1 = 2.5Nm$

x_1 = elongation of spring at 1

$x_1 = 0.2 - 0.1 = 0.1^2$

$v_{g_1} = 0 \qquad$ at datum

Again,

$Y_2 = \frac{1}{2} \times 10 \times v_2^2 = 5v_2^2$

$v_2 = v_{e_2} + v_{g_2}$

$= \frac{1}{2}kx_2^2 + wy$

$= \frac{1}{2} \times 500 \times (0.15)^2 + 10 \times 9.81(-0.15)$

$v_2 = -9.09Nm$

At point 2, the total length of spring is

$\sqrt{(0.2)^2 + (0.15)^2} = 0.25$

$\therefore x_2 = 0.25 - 0.1 = 0.15$

Putting all the values in equ(1), we get

$0 + 2.5 = 5v_2^2 - 9.09 \Rightarrow v_2 = 1.52m/s$

2) A particle having mass 0.5 kg is released from rest and strikes. The stationary particle of mass 0.4 kg as shown in the figure. Assume the impact is direct and elastic. If the horizontal surface has a dynamic co-efficient of friction $\mu = 0.3$, locate the final position of each mass from the origin of the axis.

Solution:

Applying conservation of energy at Pt. A & B

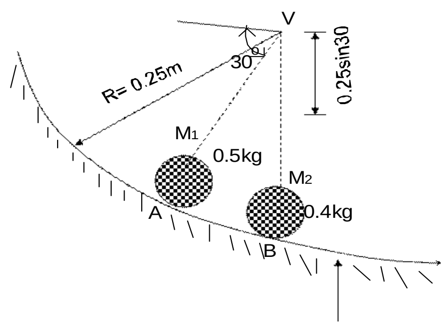
Lost of energy = work done against friction

Now for mass m_1,

KE at A (T_{A1}) = 0 $\qquad [v_A=0]$

PE at A (V_{A1}) = mgh_A = 0.5×9.81×(0.25-0.25sin30°)

(V_{A1}) = 0.613 J



KE at B (T_{B1}) = $\frac{1}{2}mv_{B1}^2 = \frac{1}{2} \times 0.25v_{B1}^2 = 0.25v_{B1}^2$

PE at B (V_{B1}) = 0 $\qquad$ [B is datum]

Now, from conservation of energy,

$$T_{A_1} + V_{A_1} = T_{B_1} + V_{B_1}$$
$$or,\ 0 + 0.613 = 0.25v_B^2 + 0 \qquad \Rightarrow v_{B_1} = 1.56m/s$$
$$\therefore \text{Velocity at } M_1 \text{ at pt. } B\left(v_{B_1}\right) = 1.56m/s$$

Now, at the point of impact

Velocity of m_1 before impact (v_1) = v_{B_1} = 1.56 m/s

Velocity of m_1 after impact $\left(v_1^{'}\right) = v_1^{'}$

Velocity of m_2 before impact (v_2) = 0

Velocity of m_2 after impact $(v_2^{'}) = v_2^{'}$

Now, for direct impact

$$m_1v_1 + m_2v_2 = m_1v_1^{'} + m_2v_2^{'}$$
$$or,\ 0.5 \times 1.56 + 0 = 0.5 \times v_1^{'} + 0.4 \times v_2^{'}$$
$$or,\ 0.51v_1^{'} + 0.4v_2 = 0.78 \qquad -(i)$$

Again, we have:

$$e = \frac{v_2^{'} - v_1^{'}}{v_1 - v_2} = 1 = \frac{v_2^{'} - v_1^{'}}{1.56 - 0}$$
$$\therefore v_2^{'} - v_1^{'} = 1.56 \qquad -(ii)$$

Solving equ(i) and (ii), we get:

$$v_1^{'} = 0.173m/s$$
$$v_2^{'} = 1.733m/s$$

Now, using work energy relation to find the distance travelled by the particle

<u>For M_1</u>:

Work done due to friction = $\Delta T + \Delta V \qquad -(iii)$

(Energy lost due to friction)

ΔT_1 = Final KE - Initial KE

$$= 0 - \frac{1}{2} \times 0.5 \times (0.173)^2 = -0.00748J$$

$\Delta V_1 = 0 \qquad [h = 0]$

$\therefore$ work done due to friction for mass m1 = $\Delta T + \Delta V = -0.00748J$

<u>For M_2</u>:

$$\Delta T_2 = 0 - \frac{1}{2} \times 0.4 \times (1.733)^2 = -0.60065J$$
$$\Delta V_2 = 0$$
$$\therefore \Delta T + \Delta V = -0.60065J$$

Now,

Work done due to friction = $-\mu mg \times$ distance travelled

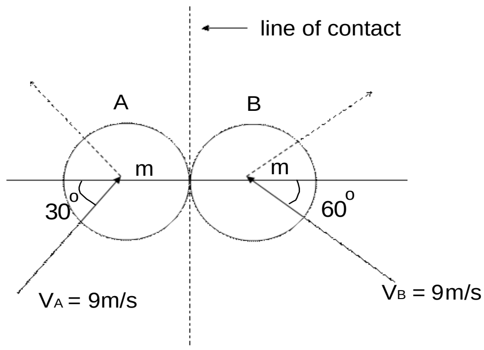
$$\therefore \text{distance travelled (x)} = -\frac{\text{work done due to friction}}{\mu mg}$$

For mass 1,

$$x_1 = \frac{-(-0.00748)}{0.3 \times 0.5 \times 9.81} = 0.00508m$$
$$x_1 = 0.00508m$$
$$x_2 = \frac{-(-0.6065)}{0.3 \times 0.4 \times 9.81}$$
$$x_2 = 0.510m$$

3. The magnitude and direction of the velocities of two identical frictionless balls before they strike each other as shown. Assuming e=0.90 determine the magnitude and direction of the velocity of each ball after the impact.

Solution:



$$V_{Ax} = V_A \cos 30° = 7.8m/s$$
$$V_{Ay} = V_A \sin 30° = 4.5m/s$$
$$V_{Bx} = -V_B \cos 60° = -6.1m/s$$
$$V_{By} = V_B \sin 60° = 10.6m/s$$

Now, in oblique impact

For motion along the line of contact,

$$V_{Ay} = V_{Ay}^{'} = 4.5m/s$$
$$And, V_{By} = V_{By}^{'} = 10.6m/s$$

For motion along the line of impact,

$$m_A V_{Ax} + m_B V_{Bx} = m_A V_{Ax}^{'} + m_B V_{Bx}^{'} \qquad [m_A = m_B]$$
$$\Rightarrow V_{Ax} + V_{Bx} = V_{Ax}^{'} + V_{Bx}^{'}$$
$$\Rightarrow 7.8 - 6.1 = V_{Ax}^{'} + V_{Bx}^{'}$$
$$\therefore V_{Ax}^{'} + V_{Bx}^{'} = 1.7 - (i)$$

Again, $e = \dfrac{V_{Bx}^{'} - V_{Ax}^{'}}{V_{Bx} - V_{Ax}}$

$$\because V_{Bx}^{'} - V_{Ax}^{'} = 0.90(7.8 - (-6.1)) = 12.5$$
$$\therefore V_{Bx}^{'} - V_{Ax}^{'} = 12.5 \qquad -(2)$$

Solving (1) and (2), we get:

$$V_{Bx}^{'} = 7.1 \qquad \& \qquad V = -5.4$$

Resultant Motion:

Adding components of velocities after impact, we get:

$$V_A^{'} = \sqrt{(V_{Ax}^{'})^2 + (V_{Ay}^{'})^2} = 7m/s$$
$$V_B^{'} = \sqrt{(V_{Ax}^{'})^2 + (V_{Ay}^{'})^2} = 12.8m/s$$

Chapter – 6
System of Particles

6.0 System of Particles:

From Newton's law, equation of motion for each particle is

$$\sum \vec{F} = m\vec{a} \Rightarrow \sum F_x = ma_x, \sum F_y = ma_y \ \& \sum F_z = ma_z \qquad -(i)$$

And for Dynamic Equilibrium,

$$\sum F_x = 0, \sum F_y = 0 \ \& \sum F_z = 0 \qquad -(ii)$$

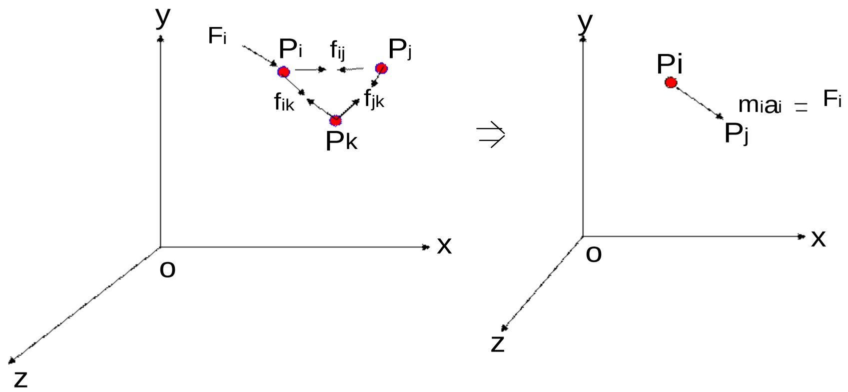
But, for the system involving a large no. of particles, and if the system as a whole is considered, then each particle of system is subjected to two types of forces:

(a) External Forces:

Forces exerted by the body outside the system and weight of particle.

(b) Internal Forces: [Vanderwaal's Molecular Attraction]

Forces exerted by the other particle of the same system.

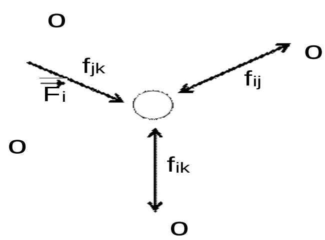


If P_1 be a particle in a system of particles P_j, P_k, etc and F_i be the resultant of external forces on i^{th} particle and F_{ij}, F_{ik} be the internal forces on i^{th} particle from j^{th} and k^{th} particle. Then,

The sum total of forces on i^{th} particle sum of external and internal forces acting the particle.

Now, from Newton's 2^{nd} law, the resultant of forces acting on the i^{th} particle is equal to $m_i a_i$. When all the particles are considered simultaneously the internal forces cancel out and the only external forces acts on the system. Hence, for the whole system

$$\left.\begin{aligned} \because \sum (F_x)_{ext} &= \sum ma_x \\ \sum (F_y)_{ext} &= \sum ma_y \\ \sum (F_z)_{ext} &= \sum ma_t \end{aligned}\right\} - \text{(iii)}$$



6.1 Newton's Law and System of Particles:

Applying Newton's 2^{nd} law for the i^{th} particle, we have:

$$m_i \frac{d\vec{v_i}}{dt} = \text{External force} + \text{Internal force}$$

$$m_i \frac{d^2}{dt^2}\vec{r_i} = \vec{F_i} + \sum_{\substack{j=1 \\ i \neq j}}^{n} \vec{F_{ij}} \quad -(i)$$

For all n particles the equation (i) can be written as:

$$\sum_{i=1}^{n} m_i \frac{d^2\overrightarrow{r_i}}{dt^2} = \sum_{i=1}^{n} \overrightarrow{F_i} + \sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n} \overrightarrow{F_{ij}} \quad - (ii)$$

For considering all particles the summation of internal forces is zero. Hence, equ(ii) modifies as :

$$\vec{F} = \sum_{i=1}^{n} m_i \frac{d^2\overrightarrow{r_i}}{dt^2} \qquad \left[\because \sum_{i=1}^{n} \overrightarrow{F_i} = \vec{F}\right]$$

$$or, \vec{F} = \frac{d^2}{dt^2}\sum_{i=1}^{n} m_i \vec{r_i} \quad - (iii)$$

If, $\vec{r_c}$ be position vector of mass centre of system of particles and M is the total mass of particles, then from principle of first moment of inertia (moment due to entire mass = sum of moments due to individual mass),

$$M\vec{r_c} = \sum M_i \vec{r_i} \quad - (iv)$$

From (iii) and (iv), we get

$$\vec{F} = \frac{d^2}{dt^2}\left(M\vec{r_c}\right)$$

$$\Rightarrow F = M\frac{d^2\vec{r_c}}{dt^2} \quad - (v)$$

6.2 Linear and Angular Momentum for a system of particles

(1) Linear Momentum for a system of particles:

For a system of particles, applying Newton's 2nd law to any jth particle, we have:

$$\vec{F} = \sum_{j=1}^{n} M_j \frac{d\overrightarrow{v_j}}{dt} \quad - (i)$$

Multiplying (i) by dt and integrating from t_1 to t_2, we get:

$$\int_{t_1}^{t_2} \vec{F}dt = \vec{I}_{ext} = \left[\sum_{j=1}^{n} M_j \overrightarrow{V_j}\right]_{t_2} - \left[\sum_{j=1}^{n} M_j V_j\right]_{t_1} \quad - (ii)$$

This shows, "The impulse of the total external force on the system of particles during a time interval equals to the sum of the changes of the linear momentum vector of the particles during the same time interval."

From the concept of mass center,

$$M\vec{r_c} = \sum_{j=1}^{n} M_j \vec{r_j} \quad - (iii)$$

Differentiating with respect to time, we get

$$\therefore M\overrightarrow{V_c} = \sum_{j=1}^{n} M_j \overrightarrow{V_j} \quad - (iv)$$

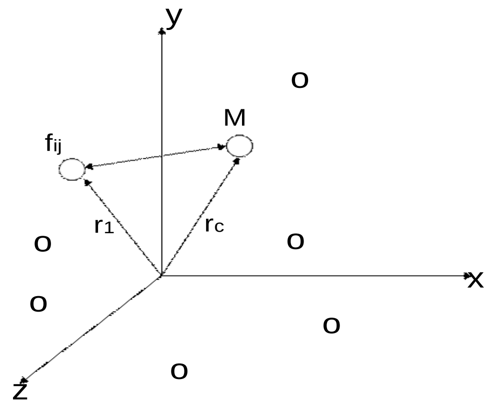
From (ii) and (iv),

$$\vec{I}_{ext} = \int_{t1}^{t2} \vec{F}dt = M(\overrightarrow{V_c})_2 - M(\overrightarrow{V_c})_1 \quad - (v)$$

Thus, the total external impulse of a system particles is equal to the change in linear momentum of the particles, moving with the mass center velocity.

2. <u>Angular Momentum For a system of particles</u>:

Angular Momentum of i^{th} particle,



$$\mathrm{H} = m_i v_i \times r_i = \frac{d}{dt}(r_i P_i)$$

$$\therefore \dot{\mathrm{H}} = m_i v_i \times r_i = \frac{d}{dt}(r_i P_i) = r_i \times m a_i$$

For system of particles the angular momentum equation for the i^{th} particle about origin 'O' is given by:

$$\vec{r_i} \times \vec{F_i} + \vec{r_i} \times \left[\sum_{j=1}^{n} \vec{F_{ij}}\right] = \frac{d}{dt}\left[\vec{r_i} \times \vec{P_i}\right] \qquad -(i)$$

where, $\vec{P_i}$ = Linear momentum of particle

For the system of n particles, equ^n(i) becomes:

$$\sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq 1}}^{n} \left(\vec{r_i} \times \vec{F_{ij}}\right) = \frac{d}{dt}\left[\sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} = \dot{\mu}_o\right]$$

Since, internal force vanishes for all particles, the moment also become zero for all particles. Hence,

$$\dot{\mu}_o = \sum_{i=1}^{n} \vec{r_i} \times \vec{F_i} = M_o \Rightarrow [M_o = \dot{\mu}_o] \; -(ii)$$

Similarly for any other fixed point 'A',

$$\overrightarrow{M_A} = \overrightarrow{\dot{\mu}_A} \qquad -(iii)$$

Thus, the total moment of external forces acting on an aggregate of particles about a fixed point 'A' in an inertial reference equals the rate of change of the angular momentum relative to the same point A and same inertial reference.

Again, considering center of mass of the aggregate of particles

For i^{th} particle, $\qquad$ [$\rho_{ci} \rightarrow$ position of i w.r.t. CG]

$$\vec{r_i} = \vec{r_c} + \overrightarrow{\rho_{ci}} \qquad -(iv)$$

Now, the angular momentum for aggregate particle about 'O' is then,

$$\overrightarrow{\mathrm{H}_o} = \sum_{i=1}^{n} \vec{r_i} \times \vec{p_i} \qquad \left[\because \vec{p_i} = linear\ \ momentum\right]$$

$$or, \vec{\mathrm{H}}_o = \sum_{i=1}^{n} \left[\left(\vec{v_c} + \overrightarrow{\rho_{ci}}\right) \times \left\{m_i\left(\vec{\dot{v}_i} + \overrightarrow{\dot{\rho}_{ci}}\right)\right\}\right] - (v) \qquad \begin{bmatrix} p_i = m_i v_i - m_i \dot{v}_i \\ \& \; \dot{r}_i = \dot{r}_c + \dot{\rho}_{ci} \end{bmatrix}$$

Carrying out the cross-product and extracting $\vec{r_c}$ from the summation we get:

$$\vec{\mathrm{H}}_o = \sum_{i=1}^{n}\left[\vec{r_c} \times m_i \vec{\dot{r}}_c + \vec{r_c} \times m_i \vec{\dot{\rho}}_{ci} + \overrightarrow{\rho_{ci}} \times m_i \vec{\dot{r}_c} + \overrightarrow{\rho_{ci}} \times m_i \overrightarrow{\dot{\rho}_{ci}}\right]$$

$$= \vec{r_c} \times \sum_{i=1}^{n}\left(m_i \vec{\dot{r}_c}\right) + \vec{r_c} \times \sum_{i=1}^{n}\left(m_i \overrightarrow{\dot{\rho}_{ci}}\right) + \sum_{i=1}^{n}\left(m_i \overrightarrow{\rho_{ci}}\right) \times \dot{r}_c + \sum_{i=1}^{n}\left(\overrightarrow{\rho_{ci}} \times m_i \overrightarrow{\dot{\rho}_{ci}}\right)$$

We know that the sum of the first moment of mass about the centroid is zero i.e.

$$\sum m_i \overrightarrow{\rho_{ci}} = 0 \text{ and hence} \sum m_i \dot{\rho}_{ci} = 0, \; Then$$

$$\overrightarrow{\mu_o} = \vec{r_c} \times M\dot{\vec{r_c}} + \sum_{i=1}^{n}\left(\overrightarrow{\rho_{ci}} \times m_i \dot{\overrightarrow{\rho_{ci}}}\right) \quad \left[\sum m_i = M\right]$$

$$\left[\overrightarrow{\mu_o} = \vec{r_c} \times M\dot{\vec{r_c}} + \overrightarrow{\mu_c}\right] \quad -(v) \qquad \left[\because \overrightarrow{\mu_c} = \overrightarrow{\rho_{cu}} \times m_i \dot{\overrightarrow{\rho_{ci}}}\right]$$

where, $\overrightarrow{\mu_c}$ be the sum of the angular momentum about the center of mass .Similarly for any point 'A', we get:

$$\therefore \left[\overrightarrow{\mu_A} = \overrightarrow{r_{AC}} \times M\overleftarrow{\dot{r}_{AC}} + \overrightarrow{\mu_c}\right] \quad -(vi)$$

where, $\overrightarrow{\dot{r}_{AC}}$ is the velocity of center of mass relative to fixed point A = $\vec{V}_{AC}$

Now, differentiating equ^n(vi) with respect to time, we get:

$$\overrightarrow{\dot{\mu}_A} = \overrightarrow{r_{AC}} \times M\overrightarrow{\dot{V}_{AC}} + \overrightarrow{\dot{\mu}_c}$$

$$or, \left[M_A = \overrightarrow{r_{AC}} \times M\overrightarrow{a_{AC}} + \overrightarrow{\dot{\mu}_c}\right] \quad -(vii) \qquad [\dot{\mu}_A = \overrightarrow{M_A}; \overrightarrow{V_{AC}} = a_{AC}]$$

6.3 Motion of mass center of a system of particle:

Center of mass for particles is not the center of mass of system.

We know that (from 6.0) for system of particles

$$\left.\begin{aligned} \sum (F_x)_{ext} &= \sum (ma_x) \\ \sum (F_y)_{ext} &= \sum (ma_y) \\ \sum (F_z)_{ext} &= \sum (ma_z) \end{aligned}\right\} \text{(i)}$$

If mass center of the system of particles is considered with co-ordinates $G(\bar{x}, \bar{y}, \bar{z})$, Then we have:

$$\left.\begin{aligned} \left(\sum m\right)\bar{x} &= \sum (mx) \\ \left(\sum m\right)\bar{y} &= \sum (my) \\ \left(\sum m\right)\bar{z} &= \sum (mz) \end{aligned}\right\} \text{(ii)}$$

(i.e. component of moment due to entire mass = component of sum of the moment due to individual mass)

Differentiating equ(ii) twice with respect to time, we get:

$$\begin{aligned} \left(\sum m\right)\ddot{\bar{x}} &= \sum (m\ddot{x}) \\ \left(\sum m\right)\ddot{\bar{y}} &= \sum (m\ddot{y}) \\ \left(\sum m\right)\ddot{\bar{z}} &= \sum (m\ddot{z}) \end{aligned} \quad \Rightarrow \quad \left.\begin{aligned} \left(\sum m\right)\bar{a}_x &= \sum (ma_x) \\ \left(\sum m\right)\bar{a}_y &= \sum (ma_y) \\ \left(\sum m\right)\bar{a}_z &= \sum (ma_z) \end{aligned}\right\} \text{ - (iii)}$$

where, $\bar{a}_x, \bar{a}_y$ & $\bar{a}_z$ are the components of acceleration $\vec{a}$ of G (i.e. center of mass) of the system,

From equ(i) & equ(iii), we have:

$$\left.\begin{aligned} \sum (F_x)_{ext} &= \left(\sum m\right)\bar{a}_x \\ \sum (F_y)_{ext} &= \left(\sum m\right)\bar{a}_y \\ \sum (F_z)_{ext} &= \left(\sum m\right)\bar{a}_z \end{aligned}\right\} \text{ - (iv)}$$

which defines the motion of center of mass of system. It shows that the center of a system of particles move as if the entire mass of the system and all the external forces were concentrated at that point G.

6.4 Conservation of Momentum:

We know that the final momentum of the particle is obtained by adding vectorically its initial momentum and the impulse of the force $\vec{F}$ during the time interval considered i.e.

$$\overrightarrow{mv_1} + \vec{I}_{mp(1-2)} = \overrightarrow{mv_2} \qquad -(i)$$

Considering system of particles,

$$\sum \overrightarrow{mv_1} + \sum \vec{I}_{mp(1-2)} = \sum \overrightarrow{mv_2} \quad -(ii)$$

since the internal forces vanish considering the all particles only impulse due to external force exists i.e.

$$\sum \overrightarrow{mv_1} + \sum \left(I_{mp(1-2)}\right)_{ext} = \sum \overrightarrow{mv_2} \qquad -(iii)$$

When the impulse of external forces is zero, then equ(iii) becomes,

$$\sum \overrightarrow{mv_1} = \sum \overrightarrow{mv_2} \qquad -(iv)$$

Thus, "when the sum of the impulses of the external forces acting on a system of particles is zero, the total momentum of the system remains constant."

Again, considering mass center $G(\bar{x}, \bar{y})$ of the system, we have,

$$\left.\begin{aligned}\left(\sum m\right)\bar{x} &= \sum (mx) \\ \left(\sum m\right)\bar{y} &= \sum (my)\end{aligned}\right\} \quad \text{- (v)}$$

Differentiating equ(v) with respect to time, we get:

$$\left.\begin{aligned}\left(\sum m\right)\overrightarrow{\bar{v}_x} &= \sum (mv_x) \\ \left(\sum m\right)\overrightarrow{\bar{v}_y} &= \sum (mv_y)\end{aligned}\right\} \Rightarrow \left(\sum m\right)\vec{\bar{v}} = \sum \left(m\vec{v}\right) \quad -(vi)$$

We have [from 6.3]

$$\vec{v} = \vec{\bar{v}} \Rightarrow \overrightarrow{v_1} = \overrightarrow{\bar{v}_1} \;\;\&\;\; \overrightarrow{v_2} = \overrightarrow{\bar{v}_2}$$

Now from equ(iv) we get:

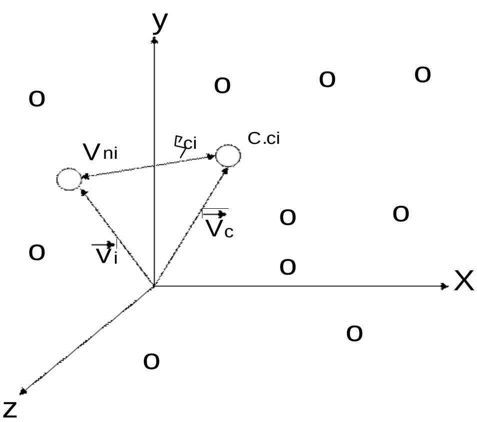
$$\overrightarrow{\bar{v}_1} = \overrightarrow{\bar{v}_2} \quad -(vii)$$

Thus, "When the sum of impulses of the external forces acting on a system of particles is zero, then mass center of the system moves with a constant velocity $\vec{v}$.

6.4 Kinetic Energy of a system of particles:

Considering a system of particles, the total K.E. relative to xyz axes of system of n particles can be obtained as

$$KE = \sum_{i=1}^{n} \frac{1}{2} m_i \overrightarrow{v_i^2} \qquad -(i)$$



Again, $\overrightarrow{r_i} = \overrightarrow{r_c} + \overrightarrow{\rho_{ci}} \quad -(ii)$

Differentiating equ(ii) with respect to time, we get,

$$\overrightarrow{v_i} = \overrightarrow{v_c} + \overrightarrow{\rho_{ci}} \quad -(iii)$$

From (i) & (iii)

$$KE = \sum_{i=1}^{n} \frac{1}{2} m_i \left(\overrightarrow{v_c} + \overrightarrow{\dot{\rho}_{ci}}\right)^2 = \frac{1}{2}\sum_{i=1}^{n} m_i \left(\overrightarrow{v_c} + \vec{\dot{\rho}}_{ci}\right)\left(\overrightarrow{v_c} + \overrightarrow{\rho_{ci}}\right)$$

$$\therefore KE = \frac{1}{2}\sum_{i=1}^{n} m_i v_c^2 + \sum_{i=1}^{n} m_i \overrightarrow{v_c}.\dot{\rho}_{ci} + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\rho}_{ci}^2 \qquad \left[\because \left(\vec{a}+\vec{b}\right)\left(\vec{a}+\vec{b}\right) = a^2 + 2\vec{a}.\vec{b} + b^2\right]$$

But,

$$\sum m_i = M \qquad and \qquad \sum_{i=1}^{n} m_i \vec{\dot{\rho}}_{ci} = 0$$

So, we get,

$$KE = \frac{1}{2} M v_c^2 + \frac{1}{2} \sum_{i=1}^{n} m_i \dot{\rho}_{ci}^2$$

Hence, KE of the system of particles = KE of the total mass moving with the mass center velocity + sum of the KE of individual particles having velocity relative to the center of mass.

6.5 Work-Energy Principle: Conservation of energy for a system of particles:

Work of the force $\vec{F}$ exerted on the particle during the displacement = change in KE

i.e. $u_{1-2} = t_2 - t_1 \quad -(i)$

For the entire system,

$$U_{1-2} = T_2 - T_1 \quad -(ii)$$

Similarly work of the conservative force is independent of path followed and is equal to minus of the change in potential energy.

i.e. $u_{1-2} = -(v_2 - v_1) \qquad -(iii)$

For the entire system,

$$U_{1-2} = -(V_2 - V_1) = V_1 - V_2 \qquad -(iv)$$

From (i) & (iv), we get

$$V_1 - V_2 = T_2 - T_1$$

$$\therefore T_1 + V_1 = T_2 + V_2 \qquad -(v)$$

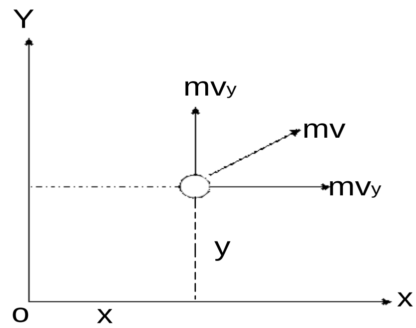
Thus for the particle moving under the conservative force, "the sum of KE and PE remains constant."

KE + PE = Total Energy (E) = constant

$\therefore$ $E_1 = E_2$ Hence, Energy is conserved.

6.7 Principle of Impulse and Momentum for a system of particles:

The angular momentum of particle about origin 'O' = moment of the first momentum of particle about the origin.



$$\because \vec{h_o} = mv_{y \times x} - mv_{x \times y} \quad (+\ )$$

Then for the total system of particle

$$\vec{H_o} = \sum m(xv_y - yv_x)$$

Differentiating with respect to time, we get,

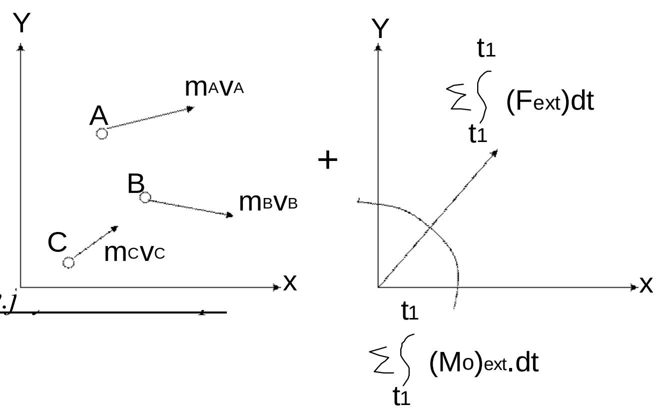
$$\dot{\vec{H_o}} = \sum \left(\vec{M_o}\right)_{ext} \qquad \left[\text{Proved in 6.22 equ}^n (iii)\right]$$

where, $\vec{M}_{o\ ext}$ is the moment of external force acting on the system.

This external force can be categorized into two types:

(a) Forces exerting at origin

These provide linear impulse



i.e. $\sum \int_{t_1}^{t_2} \left(\overrightarrow{F_{ext}}\right) dt = \sum \left(m\vec{v}\right)_2 = \sum \left(m\vec{v}\right)_1$

(b) Forces which are away from the origin

These provide angular impulse.

i.e. $\sum \int_{t_1}^{t_2} \left(F_{ext}\right) dt = \int_{t_1}^{t_2} \left(\overrightarrow{M_o}\right)_{ext} dt$

[***Note:*** - External Impulse changes the linear momentum of system

External Moment changes the angular momentum of system.

Now, from linear impulse momentum principle, we have

$$\sum \left(m\vec{v}\right)_1 + \sum \left(I_{1-2}\right)_{ext} = \sum \left(m\vec{v}\right)_2 \qquad -(i)$$

Again, from angular momentum principle

$$\left(\overrightarrow{H_o}\right)_1 + \sum \int_{t_1}^{t_2} \left(M_o\right)_{ext} dt = \left(\overrightarrow{H_o}\right)_2 \qquad -(ii)$$

Comparing these two equ^n it shows that sum of momenta (linear momentum or angular momentum) of particles at time t_1 and the impulse of the external forces are equipollent (Not actually equivalent for particles but equivalent for rigid body).

When there is no external forces,

(System momenta)$_1$ = (System momenta)$_2$

Types of system of particles:

(a) Constant System of Particles:

The system which neither gain nor lose particles during their motion are called the constant system of particles. e.g. Rigid bodies

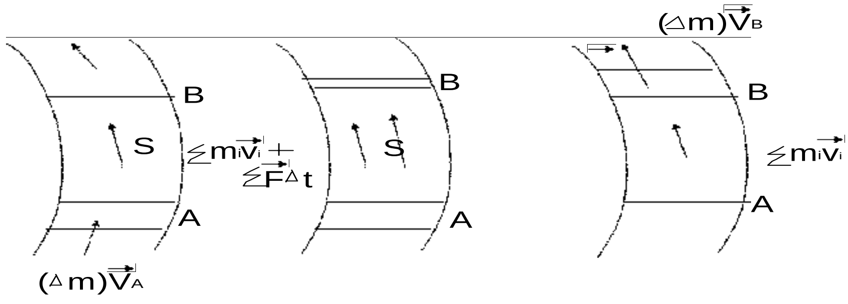
(b) Variable System of Particles:

The system which are continuously gaining or losing particles or doing both at the same time are called variable system of particles. e.g. Hydraulic turbines, Rockets, etc.

6.8 Steady Stream of Particles with variable mass

In steady stream of particles the flow of particles at any point remains constant. [All the above equ^n derived so far are for steady streams].

Let us consider a steady stream of particle(s) such as stream of water diverted by a fixed vane or a flow of air through blower.



In this system it continuously gains particles flowing in and looses an equal no. of particles flowing out, so this type of variable system of particles is reduced to auxiliary constant system of particles, which remains constant for a short time Δt

If Δm amount of mass enters the system 's', at time Δt. Then,

The momentum of the particles entering the system = $(\Delta m)\overrightarrow{v_A}$

The momentum of the particles leaving the system = $(\Delta m)\overrightarrow{v_B}$

And, the impulses of the forces exerted on s = $\sum \vec{F}.dt$ and $\sum m_i v_i$ (momentum for each particles) cancels from both sides.

Hence, we may conclude that,

The system formed by the momentum $(\Delta m)\overrightarrow{v_A}$ of the particles entering the system 's in the time Δt and impulses of the forces exerted on 's' during that time is equipollent to momentum of $(\Delta m)\overrightarrow{v_B}$ of the particles leaving 's' in the same time Δt.

i.e. $(\Delta m)\overrightarrow{v_A} + \sum \vec{F}\Delta t = (\Delta m)\overrightarrow{v_B} \quad -(i)$

Dividing both sides by Δt and taking limit, we get

$$\sum \vec{F} = \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t}\left(\overrightarrow{v_R} - \overrightarrow{v_A}\right)$$

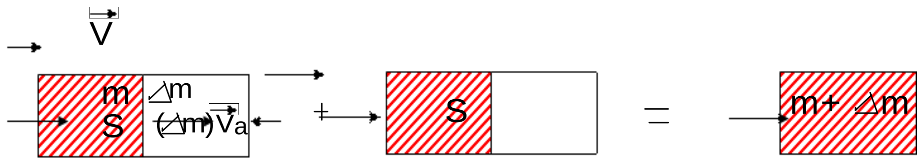
$$\therefore \sum \vec{F} = \frac{dm}{dt}\left(\overrightarrow{v_B} - \overrightarrow{v_A}\right) - (ii)$$

This equation gives the resultant of the forces exerted by the vane on the stream. This principle is applicable to the following mass as:

(a) Fluid diverted by a vane/ hydro-turbines/ properties
(b) Fluid flowing through a pipe
(c) Jet engine
(d) Fan

6.9 **System with variable mass**:

A system which gains mass by continuously absorbing particles or looses mass by continuously expelling particles is the system with variable mass. Consider at time t, the system 's' absorbs the particles of mass Δm during the time interval Δt.



Here, velocity at time t = $\vec{v}$

velocity at time $(t + \Delta t) = \vec{v} + \Delta\vec{v}$

Absolute velocity of the absorbing mass = $\overrightarrow{v_a}$

From the principle of impulse and momentum,

$$m\vec{v} + (\Delta m)\overrightarrow{v_a} + \sum \vec{F}\Delta t = (m + \Delta m)(\vec{v} + \Delta\vec{v})$$

$$or, \sum \vec{F}\Delta t = m\overrightarrow{\Delta v} + \Delta m(\vec{v} - \overrightarrow{v_a}) + \Delta m(\Delta\vec{v}) \qquad -(i)$$

Relative velocity with respect to s of the particle

$\left(\vec{u}\right) = \vec{v_a} - \vec{v}, \text{since } \vec{v_o} < \vec{v}$

$\therefore \sum \vec{F} \Delta t = m\Delta \vec{v} - (\Delta m)\vec{u} \qquad \left[\textit{neglecting} \left(\Delta m \ \Delta \vec{u}\right) \textit{due to being very very small} \right]$

$$or, \sum \vec{F} = m \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} - \lim_{\Delta t \to 0} \frac{\Delta m}{\Delta t} \vec{u}$$

$$or, \sum \vec{F} = m \frac{d\vec{v}}{dt} - \frac{\mathrm{dm}}{\mathrm{dt}} \vec{u}$$

$$or, \sum \vec{F} + \frac{dm}{dt} \vec{u} = m \frac{d\vec{v}}{dt}$$

$$\therefore \left[\vec{F} + \frac{dm}{dt} \vec{u} = m\vec{a} \right] \quad - (ii)$$

In the above equ[n] $\frac{dm}{dt}\vec{u}$ is the thrust which tends to slow down the motion of 's'. This is in case of gaining mass. If system 's' looses mass (as propulsion of rocket) the thrust generated would increase the motion of 's'.

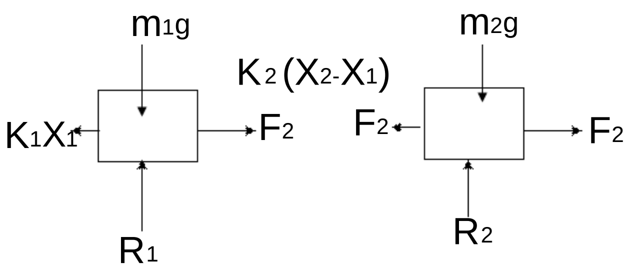
<u>Tutorials</u>:

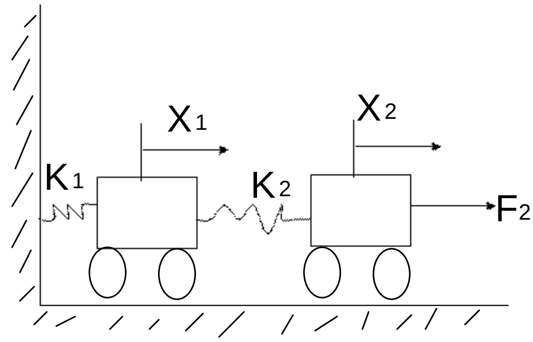
(1) Two particles shown in fig oscillate on the smooth plane in the r-direction.

(a) Write the differential equation of motion for each mass.

(b) Find equation of motion for the center of mass

(c) Write the expression for KE & PE of the system of particles.





(a) Differential equation of motion for mass m_1:

$$\sum F_x = m_1 \ddot{x}_1$$

$$\Rightarrow -k_1 x_1 + k_2 (x_2 - x_1) = m_1 \ddot{x}_1$$

Differential equation of motion for mass m_2:

$$\sum F_x = m_2 \ddot{x}_2$$

$$\Rightarrow F_2 - k_2 (x_2 - x_1) = m_2 \ddot{x}_2$$

(b) Equation of motion for the center of mass

Here the resultant internal forces due to K_2 cancles out so effect is due to external forces only.

If x_c is the c.m. of the entire system then

$$\sum F_x = \left(\sum m\right) \ddot{x}_c \qquad [\ddot{x}_c = \textit{acceleration of c.m.}]$$

$$\Rightarrow \qquad F_2 - k_1 x_1 = (m_1 + m_2) \ddot{x}_c,$$

Taking $m_1 + m_2 = M$ = Total mass

$$F_2 - k_1 x_1 = M\ddot{x}_c$$

(c) K.E./P.E. of the system of particles:

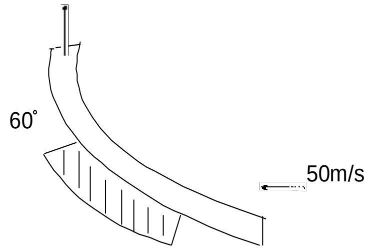
KE (T) = $\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2$

$PE(V) = V_g + V_e = O + V_e \qquad \lfloor V_g = 0,\ being\ at\ datum \rfloor$

$\therefore \qquad = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$

$\therefore (V) = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2$

(2) A nozzle discharges a stream of water of cross-sectional area A=100 mm^2 with a speed of v=50m/s, and the stream is deflected by a fixed vane as shown in the figure. The density of water is 10^3 kg/m^3. Determine the resultant force $\vec{F}$ exerted on the stream by the fixed vane.



Solution:

We've from the principle of Impulse-Momentum for particles

Here,

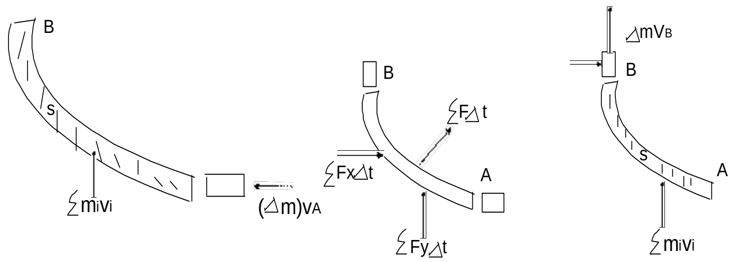
$V_A = V_B = 50$ m/s

Area, A = 100 mm^2

mass flow-rate = $\frac{dm}{dt} = \rho AV = 5\,kg/s$

Now, Cancelling $\sum m_i v_i$ from both sides, we have

Solving for x-axis,



$-(\Delta m)\overrightarrow{v_A} + F_x\Delta t = (\Delta m)v_B\cos 60°$

$\Rightarrow F_x = \frac{dm}{dt}(1 - \cos 60°)$

$\therefore F_x = 125$

Again, solving for y-axis

$F_y\Delta t = (\Delta m)v_B \sin 60°$

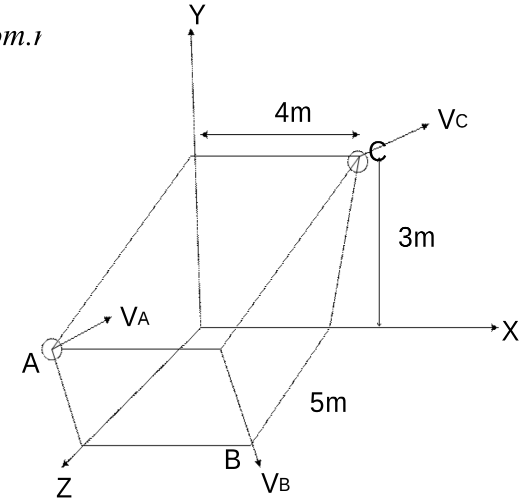
$\Rightarrow F_y = \frac{dm}{dt}v\sin 60° = 216.5$

$\therefore \vec{F} = F_x\hat{i} + F_y\hat{j} = 125\hat{i} + 216.5\hat{j}$

$\therefore F = 250N \quad \theta = 60°$

(3) A system consists of three particles A, B and C as shown in the figure and have velocity, $\overrightarrow{V_A} = (5\hat{i} - 8\hat{k})m/s,\ \overrightarrow{V_B} = (4\hat{i} + V_y\hat{j} + V_z\hat{k})m/s\ \ and\ \ \overrightarrow{V_c} = (6\hat{i} + 3\hat{j} - 2\hat{k})m/s$ respectively. If the masses of these particles are m_A = 3 kg, m_B = 1 kg and m_c = 2 kg respectively and the resultant angular momentum $\overrightarrow{\mu_o}$ of the system about origin 'O' is parallel to the z-axis, determine the value of $\overrightarrow{\mu_o}$.

Solution:
We have,
Resultant angular momentum



$$\overrightarrow{\mu_o} = \vec{r_i} \times \sum m_i \vec{v_i}$$
$$= \overrightarrow{r_A} \times m_A \overrightarrow{v_A} + r_B \times m_B \overrightarrow{v_B} + r_c \times m_c \overrightarrow{v_c}$$
$$= \left(3\hat{j} + 5\hat{k}\right) \times 3\left(5\hat{i} - 8\hat{k}\right) + \left(4\hat{i} + 5\hat{k}\right) \times 1\left(4\hat{i} + V_y\hat{j} + V_z\hat{k}\right) + \left(4\hat{i} + 3\hat{k}\right) \times 2\left(6\hat{i} + 3\hat{j} - 2\hat{k}\right)$$
$$= -\left(5V_y + 84\right)\hat{i} + \left(111 - 4V_z\right)\hat{j} + \left(4V_y - 57\right)\hat{k}$$

Since, $\overrightarrow{\mu_o}$ is parallel to the z-axis,

$$\overrightarrow{\mu_x} = 0 \quad \& \quad \overrightarrow{\mu_y} = 0$$
$$\therefore -\left(5V_y + 84\right) = 0 \qquad -(ii)$$
$$\left(111 - 4V_z\right) = 0 \qquad -(iii)$$

Solving these two equations, we get:

$$V_y = -16.8 \quad \& \quad V_z = 27.75$$

Then, $\overrightarrow{\mu_o} = \left(4V_y - 57\right)\hat{k} = -124.2\hat{k}$

$$\therefore \overrightarrow{\mu_o} = 124.2\,kgm^2/s \ \ in\ direction\ -ve\ z-axis$$

Chapter – 7
Kinematics of Rigid Bodies

7.1 Introduction:

Particle:

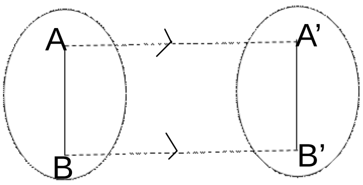
- It is a material body which is so small that its dimension can be treated as negligible in comparison to other dimensions involved.

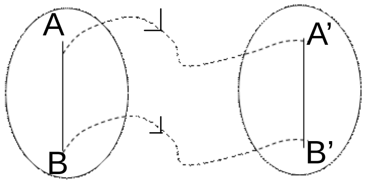
Rigid bodies:

- It is combination of two or more particles, which are connected in such a way that they do not change their relative positions due to application of external forces.
- The various points or particles in rigid bodies may have different motions but their motion are so related such that their relative position remains unchanged.
- In reality all the rigid bodies deform in application of external forces but in negligible amount.

7.2 Translation:

A motion is said to be translation if any straight line drawn on the body obeys the same direction. If all the particles move parallely along straight line, it is said to be rectilinear translation and if the path are curved, the motion is said to be curvilinear translation.





If $\Delta\overrightarrow{r_A}\, and\, \Delta\overrightarrow{r_B}$ be the displacement vectors of the particle A & B during Δt, then for translation

$$\frac{\Delta\overrightarrow{r_A}}{\Delta t} = \frac{\Delta\overrightarrow{r_B}}{\Delta t} \Rightarrow \overrightarrow{v_A} = \overrightarrow{v_B} \quad -(i)$$

Similarly,

$$\frac{\Delta\overrightarrow{v_A}}{\Delta t} = \frac{\Delta\overrightarrow{v_B}}{\Delta t} \Rightarrow \overrightarrow{a_A} = \overrightarrow{a_B} \quad -(ii)$$

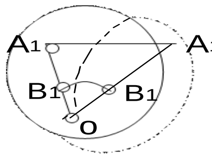
Thus, for any body in translation, all the points have the same velocity and acceleration at any given instant.

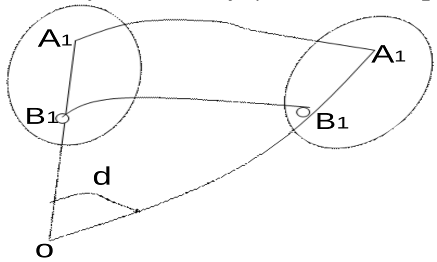
For curvilinear translation, there is change in direction and magnitude of velocity and acceleration at every instant. For rectilinear translation velocity and acceleration follow same direction during entire motion.

7.2 Rotation:

A motion is said to be rotation when the particles in the rigid bodies (slab) moves in concentric circles, with common fixed center 'O'.

If the particle in the slab moves 'dθ' at time dt, then the angular velocity is given by:





$$\omega = \frac{d\theta}{dt} \qquad -(i)$$

And, the angular acceleration is given by:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad -(ii)$$

From (i) $\quad dt = \dfrac{d\theta}{\omega}$

$$\therefore \alpha = \frac{d\omega}{\frac{d\theta}{\omega}} = \omega \frac{d\omega}{d\theta}$$

$$\therefore \alpha = \omega \frac{d\omega}{d\theta} \qquad -(iii)$$

Anti-clockwise direction is taken +ve and clockwise direction is taken –ve.

(1) For Uniform Rotation:

$\alpha = 0$, ω is constant, and $\theta = \theta_0 + \omega t$......

(2) For Uniformly Acceleration Rotation:

$$\alpha = constant \qquad \left.\begin{aligned} \omega &= \omega_o + \alpha t \\ \theta &= \theta_o + \omega_o t + \frac{\alpha t^2}{2} \\ \omega^2 &= \omega_o^2 + 2\alpha(\theta - \theta_o) \end{aligned}\right\} \text{From the equation of linear motion}$$

<u>Linear and Angular Velocity, Linear and Angular Acceleration</u>

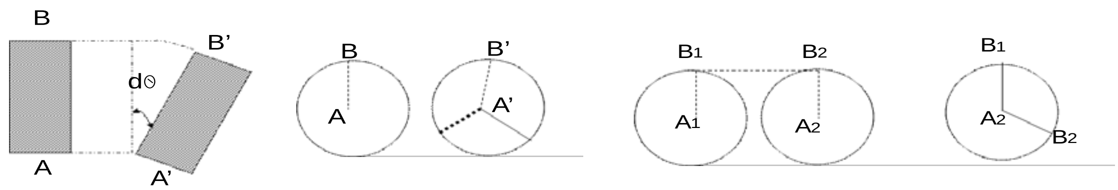
$$v = \frac{ds}{dt} \; ; \omega = \frac{d\theta}{dt} ; v = r\omega$$

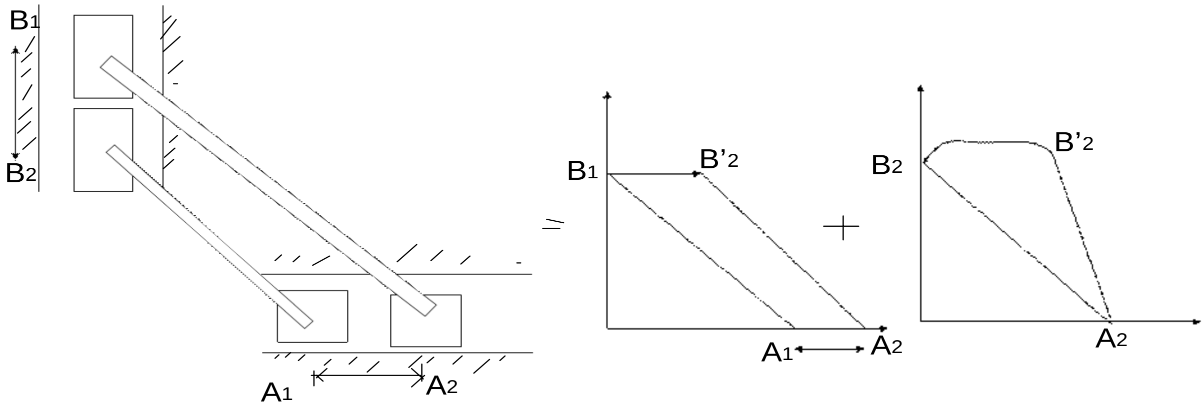
Tangential component of acceleration, $a_t = \dfrac{dv}{dt} = r\dfrac{d\omega}{dt} = r\alpha$

And Normal component of acceleration, $a_r = \dfrac{(r\omega)^2}{r} = r\omega^2$

7.3 General Plane Motion:

Any plane motion which is sum of a translational and rotational motion is called plane motion. A part of motion is translation and another part is rotational.





7.4 Absolute and Relative velocity in plane motion

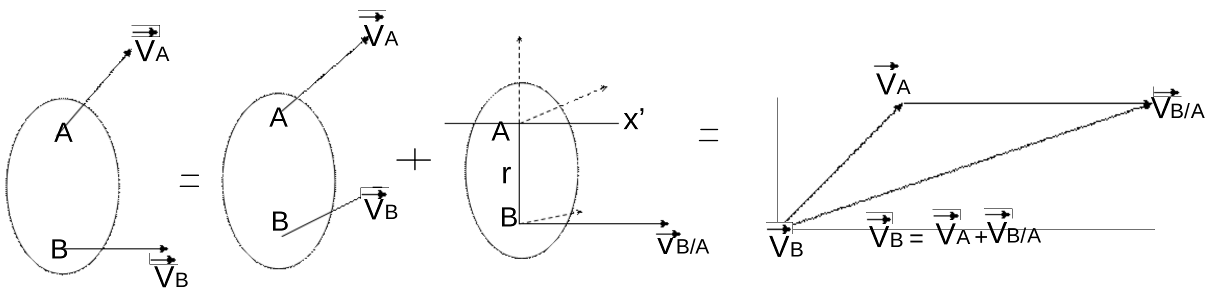
Absolute motion means motion with respect to fixed axes and relative motion means motion of one with respect to other axes in motion.

For any plane motion of rigid body, when replaced by sum of translation and rotation about A, then absolute velocity of particle B is given by:

$$\overrightarrow{V_B} = \overrightarrow{V_A} + \overrightarrow{V_{B/A}} \qquad -(i)$$

where, V_A = absolute velocity of A is translation of slab with A

$V_{B/A}$ = relative velocity 'B' with respect to 'A' i.e. rotation of slab measured with respect to 'A'



Plane motion translation with A Rotation of B about A

The plane motion of Rod can be explained as:

(a) choosing A as reference \A

Translation with 'A' & rotation about 'A'

i.e. $\overrightarrow{V_B} = \overrightarrow{V_A} + \overrightarrow{V_{B/A}}$

From figure 'd'

$$V_B = V_A \tan\theta \qquad \& \qquad V_{B/A} = V_{A/\cos\theta}$$

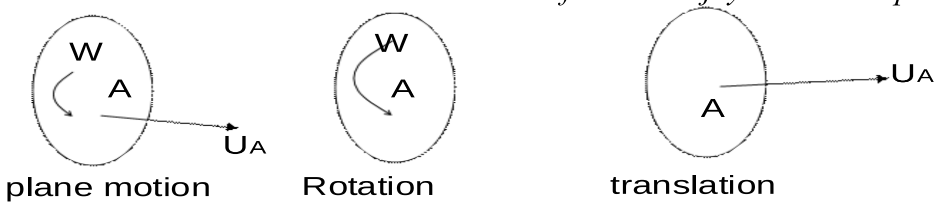
$$\omega = \frac{V_{B/A}}{l} \qquad [lv = \omega r]$$

$$\therefore \omega = \frac{V_A}{l\cos\theta} \quad -(ii) \qquad [\text{same result can be obtained by choosing 'B' as reference point}]$$

Hence, Angular velocity of Rigid body in plane motion is independent of the reference point.

7.5 Instantaneous Center of Rotation

For any general plane motion, there exist a center with reference to which the velocities of all the particles in the rigid body are same at any instant of time. This center is called as instantaneous center of rotation and the axis is called instantaneous axis of rotation.

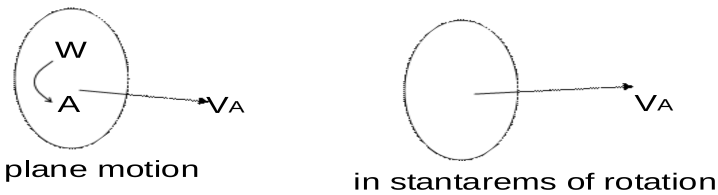


If $V_A = 0 \Rightarrow$ 'A' will be the instantaneous center of rotation

If $\omega = 0 \Rightarrow$ No instantaneous center of rotation

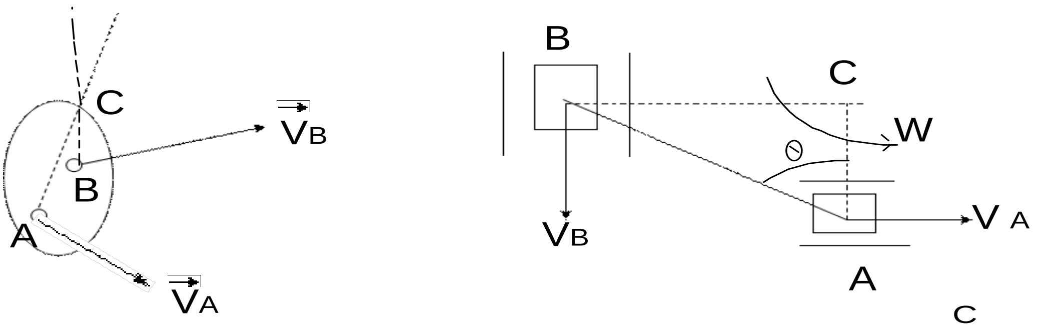
But, If $V_A \neq 0$ & $\omega \neq 0$, Then

If the slab is rotated with angular velocity $'\omega'$ about a point 'c' located on the perpendicular to V_A at distance $r = V_A/\omega$ from 'A' then all the particles will appear to rotate about 'c' with same velocity $'\omega'$.

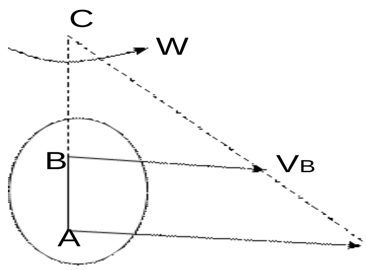


<u>Methods to locate instantaneous center of rotation</u>

(a) If direction of velocity of particles 'A' & 'B' are known but are not parallel, then 'c' is the intersection of perpendicular drawn on $\vec{V_A}$ & $\vec{V_B}$



(b) If magnitude and direction of velocity of particles 'A' & 'B' are known and are perpendicular to line AB, then 'c' is located as



(c) If magnitude $V_A = V_B$, then 'c' would be at infinite and the body would translate.

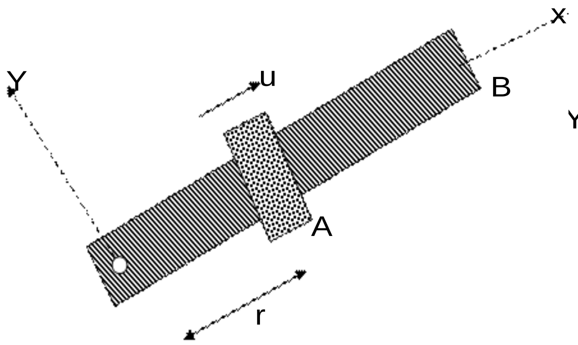
Note:

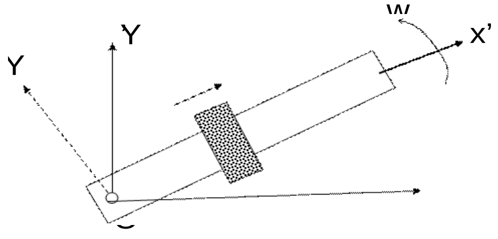
- Instantaneous center of rotation may be located either on the slab or outside.
- If 'c' is located on the slab then $V_c =$ *and* $V_B = V_{B/C}$.
- The point 'c' is different at different time interval Δt
- Acceleration of various particle on the slab cannot be determined by this method. Hence, it is used to compute only the velocity of particles at any instant of time.

7.7 Absolute and Relative Frame: Coriolis Acceleration in plane motion

Definition:

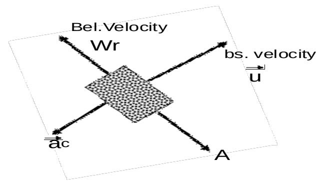
A particle with in a system of rigid body may have motion with respect to a moving frame inside the body. In such case an additional comp. of acceleration of the particle with respect to the moving frame comes into the existence, this complementary acceleration in case of moving frame is called coriolis acceleration, 'a_c' which is perpendicular to the direction of relative velocity of particle with respect to frame.





Absolute frame
Stationary frame

Relatix frame
or moving frame
Axis x'ay rotates
With avg. velocity w



Calculation:

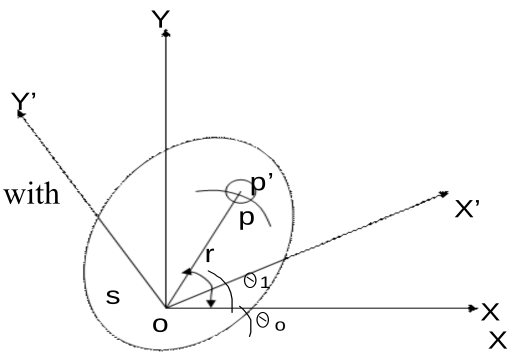
Consider the motion of particle P, which moves along a path on a slab 's'. The slab rotates along a fixed point O. Motion of P is given by r & θ with respect to fixed axis (XOY axis) and by r & θ, with respect to axis attached to slab 's' and rotating with it. It is required to determine the absolute motion of 'P' and relative motion of 'P' with respect to 'S'.

We know from radial and transverse component of velocity

$$(V_p)_r = \dot{r} \qquad \& \qquad (V_p)_\theta = r\dot{\theta} = r(\dot{\theta}_o + \dot{\theta}_1) \qquad -(i) \quad [\theta = \theta_o + \theta_1]$$

Here, p is position with respect to x-y axis

p' is position with respect to x'y' axis.



Case I:

When P is fixed on slab 's' and the slab allowed to rotate with respect to XOY, then P coincides with P'

$$\therefore V_p = V_p', \qquad r = cons\tan t \qquad \theta_1 = cons\tan t$$

$$\therefore \dot{r} = 0 \quad \& \quad \dot{\theta}_1 = 0$$

So from equation (i)

$$(V_p')_r = 0 \qquad \& \qquad (V_p')_\theta = r\dot{\theta}_o \quad -(ii)$$

Case-II:

When slab is fixed and P is allowed to move, then

$$V_p = V_{p/s} \qquad \theta_o = cons\tan t \qquad \therefore \dot{\theta}_o = 0$$

Then from equation (i)

$$(V_{p/s})_r = \dot{r} \qquad \& \qquad (V_{p/s})_\theta = r\dot{\theta}_1 \quad -(iii)$$

Now, the total velocity is given by:

from the vector components, we have:

$$V_p = V_{p'} + V_{p/s} \qquad -(iv)$$

The Radial and Transverse components of velocity

$$(V_p)_r = (V_{p'})_r + (V_{p/s})r \qquad \& \qquad (V_p)_\theta = (V_{p'})_\theta + (V_{p/s})_\theta$$ [also for acceleration]

Again for acceleration, we have:

$$(a_p)_r = \ddot{r} - r\dot{\theta}^2 = \ddot{r} - r(\dot{\theta}_o + \theta_1)^2 = \ddot{r} - r(\dot{\theta}_o^2 + 2\dot{\theta}_o\dot{\theta}_1 + \dot{\theta}_1^2) \qquad -(v)$$

$$(a_p)_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = r(\ddot{\theta}_o + \ddot{\theta}_1) + 2\dot{r}(\dot{\theta}_o + \dot{\theta}_1) \qquad -(vi)$$

Calculating the acceleration of point 'p' when the slab is rotates as 'P' also moves with respect to slab it is observed that, instead of being

$\overrightarrow{a_p} = \overrightarrow{a_{p'}} + \overrightarrow{a_{p/s}}$, we get

$\overrightarrow{a_p} = \overrightarrow{a_{p'}} + \overrightarrow{a_{p/s}} + \overrightarrow{a_c}$,

where $\overrightarrow{a_c}$ has the vector component as

$$(a_c)_r = 2\ddot{r}\dot{\theta}_o = -2\omega(V_{p/s}) \qquad -(vii)$$

and, $(a_c)_\theta = 2\omega(V_{p/s})_r \qquad -(viii)$

$\overrightarrow{(a_c)_r} \rightarrow 90°$ to the rotation of slab

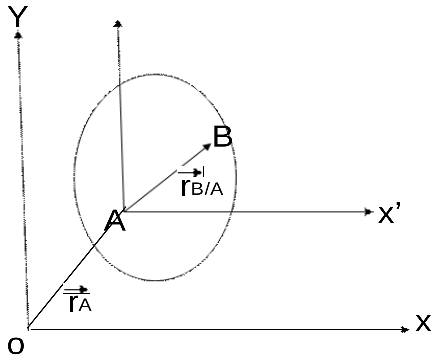
$(\overrightarrow{a_c})_\theta \rightarrow$ along the rotation of slab

The coriolis acceleration $\overrightarrow{a_c}$ is thus a vector perpendicular to the relative velocity $\vec{V}_{p/s}$ and of magnitude to $z\omega.\ V_{p/s}$ and direction of $\overrightarrow{a_c}$ is perpendicular to the vector $\vec{V}_{p/s}$.

7.10 General Motion:

Considering two particles A & B of the rigid body, then, we have

$$\overrightarrow{V_B} = \overrightarrow{V_A} + V_{B/A} \qquad -(i)$$



If the body rotates with angular velocity 'ω' about the point 'A', then

$$\overrightarrow{V_B} = \vec{\omega} \times r_{B/A} \qquad -(ii) \qquad \left[\vec{V}_{B/A} = \vec{\omega} \times r_{B/A}\right]$$

Further, we have

$$\overrightarrow{a_B} = \overrightarrow{a_A} + \vec{a}_{B/A}$$

$$or,\ \overrightarrow{a_B} = \overrightarrow{a_A} + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \qquad -(iii)$$

$$\left[\because \vec{a}_{B/A}\right] = \frac{d}{dt}(V_{B/A}) = \frac{d}{dt}(\vec{\omega} \times \vec{r}_{B/A})$$

$$= \frac{d}{dt}\vec{\omega} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{d}{dt}\vec{r}_{B/A}$$

$$= \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \vec{r}_{B/A}$$

$$\therefore \vec{a}_{B/A} = \vec{r} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{\omega} \times \vec{r}_{B/A})$$

where α is the angular acceleration of the body at instant considered. The equ (ii) and (iii) show that the most general motion of a rigid body is equivalent to the sum of a translation (in which all particles of the body have the same velocity and acceleration) and of a motion in which particle 'A' is assumed to be fixed.

i.e. The motion of any particle 'B' with respect to 'A' would be characterized by the same vectors $\vec{\omega}$ & $\vec{\alpha}$. Thus $\vec{\omega}$ & $\vec{\alpha}$ are independent of the choice of reference point but the moving frame should remain parallel to the fixed frame of reference.

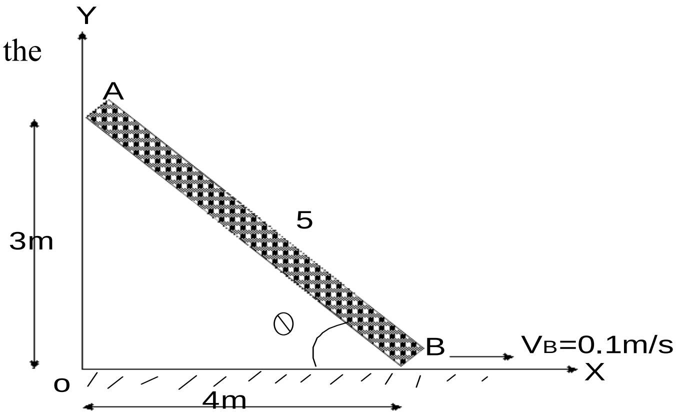
Tutorials:

1. The end 'B' of the rod AB as shown in the figure moves with constant velocity, $V_B = 0.9$ m/s ($\rightarrow$)

Determine:

(a) Angular velocity and angular acceleration of the rod

(b) Velocity and acceleration of end 'A'



Solution:

Here,

$$Y_A = 3m \;\;\&\;\; X_B = 4m$$

$$\therefore \theta = \tan^{-1}(3/4)$$

Again,

$$V_B = \dot{X}_B = 0.9 m/s \Rightarrow a_B = \ddot{x}_B = 0$$

(a) Now from ΔAOB,

$$X_B = 5\cos\theta \quad [\text{taking} + \text{ve for clockwise direction}]$$

$$\because \dot{X}_B = -5(\sin\theta)\dot{\theta} = 0.9 \Rightarrow \dot{\theta} = -0.3$$

$$\therefore \omega = \dot{\theta} = 0.3\, rad/\sec$$

Again, $a_B = \ddot{x} = -5\left[(\cos\theta)\dot{\theta}^2 + (\sin\theta)\ddot{\theta}\right] = -5\left[\frac{4}{5}\times(0.3)^2 + \frac{3}{5}\times\ddot{\theta}\right] = .0$

$$\therefore \ddot{\theta} = -0.12$$

$$\therefore \alpha = \ddot{\theta} = 0.12 rad/\sec$$

(b) We have

$$Y_A = 5\sin\theta \Rightarrow \dot{Y}_A = 5(\cos\theta)\dot{\theta} \;\&\; a_A = \ddot{Y}_A = 5\left[(\cos\theta)\ddot{\theta} - (\sin\theta)\dot{\theta}^2\right]$$

$$or,\; a_A = 5\left[\left(-\frac{3}{5}\right)\times(-0.3)^2 + \left(\frac{4}{5}\right)\times(-0.12)\right] = -0.75$$

$$or,\; V_A = Y_A = 5\times\frac{4}{5}\times -0.13 = -1.2 m/s$$

$$\therefore V_A = 1.2 m/s \qquad \& \qquad a_A = 0.75 m/s^2$$

(2) In the engine system shown, the crank AB has a constant clockwise angular velocity of 2000rpm. For the crank position indicated, determine:

(a) the angular velocity of the connecting rod BD

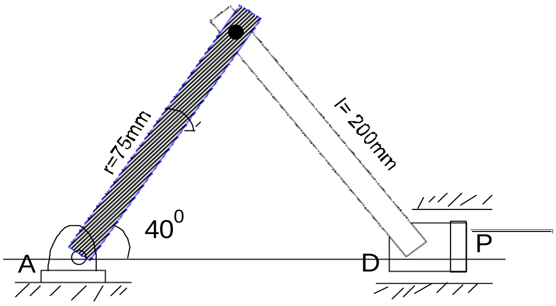
(b) the velocity of position P.

Solution:

Here, the crank AB rotates about point A,

$$\omega_{AB} = 2000rpm = \frac{2\pi N}{60} rad/\sec = \frac{2\times\pi\times 2000}{60} = 209.4 rad/\sec$$

$$V_B = (AB)\times\omega_{AB} = 209.4\times 75 = 15.68 m/s$$

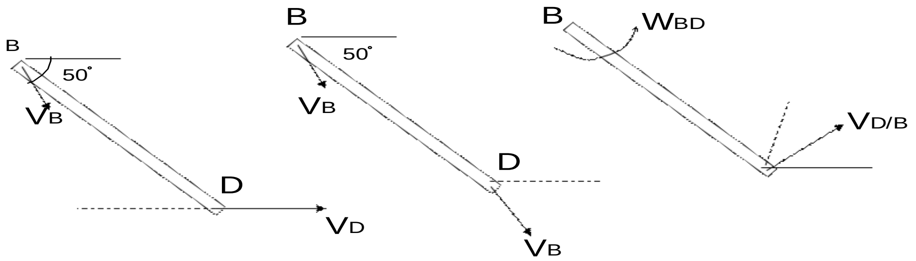


Using sine law,

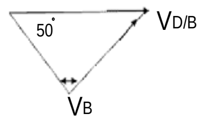
$$\frac{\sin 90°}{BD} = \frac{\sin B}{AB} \Rightarrow \frac{\sin 40°}{3} = \frac{\sin B}{8} \Rightarrow B = 13.94°$$

Now, for the motion of connection rod BD, which is a plane motion:

the velocity V_D is horizontal and V_B is as obtained above. Resolving motion of BD into a translation with B and rotation about B, we get:



Plane motion = Translation + Rotation



Now, we have

$$\overrightarrow{V_D} = \overrightarrow{V_B} + \overrightarrow{V}_{D/B}$$

Again, using sine law,

$$\frac{V_D}{\sin 53.9°} = \frac{V_{D/B}}{\sin 50°} = \frac{V_B}{\sin 76.1°} = \frac{15.68}{\sin 76.1} \qquad -(i)$$

Solving the equation separately, we have,

$$V_{D/B} = 12.37 m/s$$

$$V_D = V_p = 13.05 m/s$$

Again,

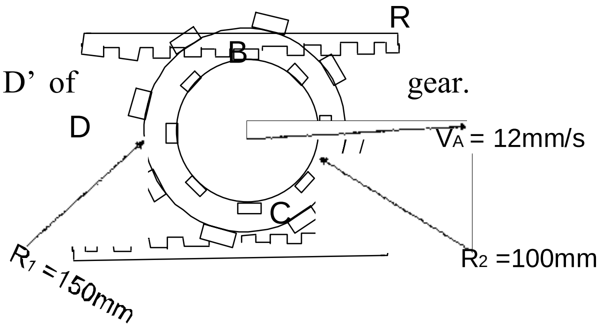
$$V_{D/B} = \omega_{BD}\times BD \Rightarrow \omega_{BD} = \frac{V_{D/B}}{BD} = \frac{12.37}{0.2} = 61.9 rad/s$$

$$\therefore \omega_{BD} = 61.9 rad/s$$

(3) A double gear rolls on the stationary lower rack. The velocity of its center 'A' is 1.2 m/s directed to the right. Determine:

(a) the angular velocity of the gear

(b) the velocities of the upper rack 'R' and of point 'D' of gear.



Solution:

(a) Angular velocity of the gear:

Distance moved by the center of gear for each revolution $= 2\pi r_1$

Then, $\dfrac{x_A}{2\pi r_1} = \dfrac{\theta}{2\pi}$ $\qquad\qquad [x_A = 2\pi r_1 \Rightarrow \theta = 2\pi]$

$$\therefore x_A = r_1\theta \qquad -(i)$$

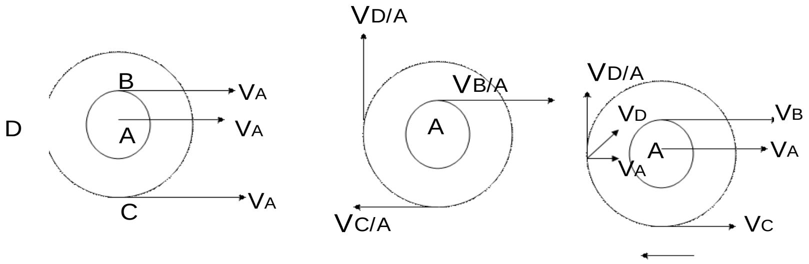
Differentiating,

$$\dot{x}_A = V_A = r_1\omega$$

$$or, 1.2 = 0.15\omega$$

$$\therefore \omega = 8 rad/\sec$$

Analyzing the plane motion:



Now, velocity of upper rack

$$\overrightarrow{V_R} = \overrightarrow{V_B} = \overrightarrow{V_A} + \vec{V}_{B/A} = 1.2 + r_2\omega = 1.2 + 0.1 \times 8 = 2m/s$$

$$\therefore \overrightarrow{V_B} = 2m/s$$

Velocity of point D, $\therefore \overrightarrow{V_D} = \overrightarrow{V_A} + V_{D/A} = 1.2 + r_1\omega = 1.2 + 0.15 \times 8$

$$\therefore \overrightarrow{V_D} = 1.69\, m/s$$

$$\phi = \tan^{-1}\frac{1.2}{1.2}$$

$$\therefore \phi = 45°$$

Chapter – 8

Plane Motion of Rigid Bodies

Forces, Moments & Acceleration

8.1 Considering a rigid body acted upon by several external forces, $\vec{F_1}, \vec{F_2}, \vec{F_3}, etc.$ Let, the rigid body be made of a large number of particles of mass $\Delta m_{ig} = (i = 1,2.....n)$ and G be the mass center of the rigid body whose motion can be considered with respect to xyz axis.

Then, from Newton's 2nd law:

$$\sum \vec{F} = m\vec{a} \qquad \text{- (i)}$$

where, m = Total mass of the body

$\vec{a}$ = Acceleration of the CG 'G'

$\sum \vec{F}$ = Sum of applied force on the body

Again from Angular-Momentum Theorem,

$$\sum \overrightarrow{M_G} = \dot{\overrightarrow{H_G}} \qquad -(ii)$$

where, $\sum \overrightarrow{M_G} =$ Sum of the moment of forces about 'G'

$\dot{\overrightarrow{H_G}} =$ Rate of change of angular momentum about G.

The equations of motion (i) and (ii) express that the system of external forces is equivalent to the system consisting of vector m$\vec{a}$ attached at G and couple of moment $\dot{\overrightarrow{H_G}}$.

8.2 Angular Momentum of Rigid body in plane motion:

Considering a rigid slab consisting of a large number of particles P_1 of mass Δm_i be in plane motion. The angular momentum $\overrightarrow{H_G}$ of the slab about its mass center G may be computed as:

$$\overrightarrow{H_G} = \sum_{i=1}^{n}\left(\vec{r_i^{'}} \times \Delta m_i v_i^{'}\right) \qquad -(i)$$

where, $r_i^{'}$ and $\Delta m_i v_i^{'}$ be the position vector and linear momentum of P_i about the 'G' with respect to x'y' frame.

Again,

$$v_i^{'} = \vec{\omega} \times r_i^{'}$$

$$\therefore \overrightarrow{H_G} = \sum_{i=1}^{n} \vec{r_i^{'}} \times (\vec{\omega} \times r_i)\Delta m_i$$

$$= \vec{\omega}\sum_{i=1}^{n} r_i^{'2}\Delta m_i$$

$$\overrightarrow{H_G} = \vec{\bar{I}}\vec{\omega} \qquad -(ii) \qquad \left[\sum_{i=1}^{n} r_i^{'2}\Delta m_i = \bar{I}\text{; motion about G perpendicualr to slab}\right]$$

Differentiating equ(ii) with respect to time, we get:

$$\overrightarrow{\dot{H}_G} = \bar{I}\vec{\dot{\omega}} = \bar{I}\vec{\alpha} \qquad -(iii)$$

$$\therefore \overrightarrow{\dot{H}_G} = \bar{I}\vec{\alpha}$$

Thus the rate of change of the angular momentum of the slab is equal to $\bar{I}\vec{\alpha}$ and in perpendicular to the direction of the slab.

8.3 Plane motion of a Rigid body : D'Alembat's Principle:

Statement:

"For a rigid body of mass 'm' moving under the action of several forces $\vec{F_1}, \vec{F_2}, \vec{F_3}, etc$, the external forces acting on the rigid body are equivalent to the effective forces of the various particles forming the body."

Considering mass center 'G' of the slab as reference point, we have:

$$\left.\begin{aligned}\sum(F_x)_{eff} &= \sum(\Delta m)a_x \\ \sum(F_y)_{eff} &= \sum(\Delta m)a_y\end{aligned}\right\} \quad \text{(i)}$$

Again, we have:

$$\left.\begin{aligned}\sum(\Delta m)a_x &= \sum(\Delta m)\bar{a}_x \\ \sum(\Delta m)a_y &= \sum(\Delta m)\bar{a}_y\end{aligned}\right\} \quad \text{(ii)}$$

where, $\bar{a}_x$ *and* $\bar{a}_y$ are components of $\bar{a}$ at 'G'.

$\sum \Delta m = \text{Total mass of the slab} = \text{M}$

Then from equations (i) and (ii), we have:

$$\left.\begin{aligned}\sum(F_x)_{eff} &= M\bar{a}_x \\ \sum(F_y))_{eff} &= M\bar{a}_y\end{aligned}\right\} \quad \text{(iii)}$$

The total acceleration $\bar{a}$ of any given particle 'P' of the slab is equal to

Total acceleration $(\vec{a})$= Linear acceleration of 'G' $(\overline{\vec{a}})$+ Angular acceleration of P w.r.t. x'y' $\vec{a}'$

$$\therefore \vec{a} = \vec{\bar{a}} + a'$$

$$\therefore \vec{a} = \vec{\bar{a}} + a_n'\hat{e}_n + a_t'\hat{e}_t \quad -(iv)$$

Thus, effective force is also resolved into two parts:

$$\sum \vec{F}_{eff} = \sum \Delta m\vec{a} = \sum \Delta m\vec{\bar{a}} + \sum \Delta m\overrightarrow{a_n}\hat{e}_n + \sum \Delta m\overrightarrow{a_t}\hat{e}_t$$

i.e. effective force = force due to translation about G + Force due to rotation about G

Considering the case of Translation:

During translation, all the particles in the body moves along 'G' such that total moment of force about the 'G' is zero [because Δma_t *and* $\Delta ma_x = 0$]

$\therefore$ For Translation,

$$\sum(M_G)_{eff} = 0$$

$$or, \sum \dot{H}_{eff} = 0$$

$$\therefore \vec{F}_{eff} = m\vec{\bar{a}} \quad -(v)$$

Considering the case of rotation:

For rotation $\Delta m a_x^{'}$ *and* $\Delta m a_t^{'}$ are associated with the slab, Now

Moment of $\Delta m a_x^{'}$ about G = 0 and $\left[a_x^{'}\ \text{is along G}\right]$

Moment of $\Delta m a_t^{'}$ about G = $r'(\Delta m)a_t^{'}$

Again, we have

$$a_t^{'} = r'\alpha,$$

Now considering all the particles of slab under rotation

$$\sum (M_G)_{eff} = \sum r'(\Delta m)r'\alpha = \alpha \sum r'^2 \Delta m = \bar{I}\alpha$$

For rotation motion,

$$\sum \overrightarrow{\mu_G} = \sum (M_G)_{eff} = \bar{I}\alpha$$

For plane motion:

For General Plane Motion, the system of effective forces can be replaced by an equivalent force coupe system consisting of m$\vec{\bar{a}}$ force at 'G' and couple of I$\vec{\alpha}$ in the directon of $\vec{\alpha}$ along 'G'

Then The D'Alembert's Principle becomes,

" The external forces acting on the body are equivalent to a force-couple system consisting of a vector m$\vec{a}$ attached to the mass center 'G' of the body and a couple $\bar{I}\vec{\alpha}$."

Then equation of motion becomes,

$$\left.\begin{aligned} \sum F_x &= M\bar{a}_x \\ \sum F_y &= M\bar{a}_y \\ \sum \overrightarrow{M}_G &= \bar{I}\vec{\alpha} \end{aligned}\right\} \qquad \text{(x)}$$

Note:

For translation, effective force reduces to m$\vec{\bar{a}}$. For rotation, effective force reduces to $\bar{I}\vec{\alpha}$ and for the plane motion effective force is the sum of the both.

8.3 Application of Rigid body motion in plane:

For a rigid body in plane motion, there exists a fundamental relation between the forces $(\vec{F})$ acting on the body, the acceleration of its mass center $(\vec{a})$ and the angular acceleration $(\vec{\alpha})$ of the body.

$$\vec{F} = m\vec{\bar{a}}_x + m\vec{\bar{a}}_y + \bar{I}\vec{\alpha}$$

This relation may be used to determine the acceleration $(\vec{a})$ and angular acceleration $(\vec{\alpha})$ produced by a given system of forces acting on a rigid body, or to determine the forces which produce a given motion of the rigid body.

The equation may be separated as:

$$\overrightarrow{F_x} = m\overrightarrow{\bar{a}_x} \quad ; \quad \overrightarrow{F_y} = m\overrightarrow{\bar{a}_y} \quad ; \quad \sum M_c = \bar{I}\vec{\alpha}$$

The equation is solved to get the unknown quantity of motion.

It gives the better understanding of problem, easy to draw free body diagrams, develop and solve equations for the 3-D and 2-D motions.

8.5 Constrained Motion in the Plane:

The plane motion, where each bodies move with definite relation with the other bodies is called constraint motion. For example: rotation of crank is related with translation of piston and oscillation of connecting rod in an engine.

In such cases, the definite relationship exists between mass center, acceleration $\vec{\bar{a}}$ and angular acceleration $\vec{\alpha}$ between the several bodies, under constrained motion.

Solution for such motion is obtained in two steps:

(i) Kinetic Analysis $\left(\vec{\bar{a}}, \vec{\alpha}, \vec{\bar{I}}, \overrightarrow{M}\right)$ of the problem

(ii) Use of D'Alembert's Principle or Dynamic Equilibrium Method to solve the unknown quantity.

When a mechanism consists of several moving parts, each moving part is considered separately and the problem is solved.

Two particular cases of constrained motion are:

(i) Non-centroidal rotation:

- Rigid body is constrained to rotate about a fixed axis which does not pass through main center.
- The mass center 'G' of the body moves along a circle of radius $\vec{r}$ centered at the point 'O'.

If $\omega = const$ $\quad$ $i.e.\ \alpha = .0$

$\therefore a_t = r\vec{\alpha} = 0,\ then\ \vec{a} = \overrightarrow{a_x}$ i.e. normal component only and the force generated thus is called centrifugal force.

Rolling Motion:

If the disk is constrained to roll without sliding the acceleration of its mass center 'G' and its angular acceleration 'α' are dependent.

Here,

$\bar{x} = r\theta \qquad -(i)$

Differentiating w.r.t time, we get

$\ddot{\bar{x}} = \bar{a} = r\bar{\alpha} \qquad -(ii)$

If the mass center 'G' does not coincide with its geometric center 'O' , the relation (ii) does not hold true.

Then, the relation becomes

$$a_o = r\alpha \ -(iii),\ where\ a_o = \text{acceleration of the geometric center}$$

Then,

$$\vec{\bar{a}} = \overrightarrow{a_G} = \overrightarrow{a_o} + \vec{a}_{G/O} = \overrightarrow{a_o} + \left(\vec{a}_{G/O}\right)_t + \left(\vec{a}_{G/O}\right)_G$$

where,

$$a_o = r\alpha,\ \left(a_{G/O}\right)_t = (OG)\alpha \quad and \quad \left(a_{G/O}\right)_n = (OG)\omega^2$$

Tutorials:

1. A cord is wrapped around a homogenous disk of radius r = 0.5 m and mass = 15 kg. If the cord is pulled upwards with a force T of magnitude 180N, determine

(a) Acceleration of center of disc

(b) Acceleration of the cord

Solution:

From (b) and (c), system of external forces = system of effective forces

$$\sum F_x = \sum (F_x)_{eff} \Rightarrow 0 = m\bar{a}_x \Rightarrow \bar{a}_x = 0$$

$$\sum F_y = \sum (F_y)_{eff} \Rightarrow T - \omega = m\bar{a}_y$$

$$\bar{a}_y = \frac{T-\omega}{m}$$

$$\bar{a}_y = \frac{180 - 15 \times 9.81}{15} = 2.19 ms^{-2}$$

$$\therefore \bar{a}_y = 2.19 ms^{-2}$$

$$\because \vec{\bar{a}} = \vec{a}_G = 2.19 ms^{-2}$$

Again,

Moment of external forces = Moment of effective turns

i.e. $\sum M_G = \sum (M_G)_{eff}$

$$-Tr = \bar{I}\alpha = \left(\frac{1}{2}mr^2\right)\alpha$$

$$\therefore \alpha = \frac{-2T}{mr} = \frac{-2 \times 180}{15 \times 0.5} = -48.0 \ rad/s^2$$

$$\therefore \alpha = 48 \, rad/s^2$$

<u>Acceleration of Cord</u>

$$\vec{a}_{cord} = \left(\overrightarrow{a_A}\right)_t = \vec{\bar{a}}_G + \left(\vec{a}_{A/G}\right)_t$$

$$= \overrightarrow{\bar{a}_G} + \alpha . AG$$

$$= 2.19 + 48 \times 0.5$$

$$\therefore \vec{a}_{cord} = 26.2 \, ms^{-2}$$

Tutorials 2 contd .....

Resolving $\vec{a}$ into two components:

$$\bar{a}_x = \bar{a}\cos 60° = 1.339\alpha$$

$$\bar{a}_y = \bar{a}\sin 60° = -0.520\alpha$$

Now, equating system of external forces to a system of equivalent forces, we get

Now,

$$\bar{I} = \frac{1}{2}ml^2 = \frac{25}{12}(1.2)^2 = 3kgm^2$$

$$\therefore \bar{I}\alpha = 3\alpha$$

$$w = mg = 25 \times 9.81 = 245N$$

$$m\bar{a}_x = 25 \times 1.339\alpha = 33.5\alpha$$

$$m\bar{a}_y = -(25)(0.52)\alpha = -13\alpha$$

Now, we have:

$$\sum M_E = \sum (M_E)_{eff} \Rightarrow w \times 0.52 = m\bar{a}_x \times 1.339 + m\bar{a}_y \times 0.52 + \bar{I}\alpha$$

$$or,\ 245 \times 0.52 = (33.5\alpha)(1.339) + (13\alpha) \times 0.52 + 3\alpha$$

$$or, \alpha = 2.33 rad/s$$

Again,

$$\sum (F_x) = \sum (F_x)_{eff}$$

$$or,\ R_B \sin 45° = m\bar{a}_x = 33.5\alpha = 33.5 \times 2.33$$

$$\therefore R_B = 110.5N$$

$$\sum F_y = \sum (F_y)_{eff}$$

$$or,\ R_A + R_B \cos 45° - 245 = (-13) \times 2.33$$

$$\therefore R_A = 136.6N(\uparrow)$$

Chapter –9
Plane Motion of Rigid Bodies
Energy and Momentum Method

9.1 <u>Principle of Work and Energy for a Rigid Body</u>

For a rigid body of mass 'm', 'P' be a particle of the body with mass 'Δm' moving with velocity 'v'. Then,

KE of the particle = ½ (Δm)v^2

If the particle moves from position P_1 to P_2, then from the principle of work and energy, the work done is calculated as:

$$\Delta U_{1-2} = \text{Change in KE} = \Delta T_2 - \Delta T_1$$
$$\therefore \Delta T_1 + \Delta U_{1-2} = \Delta T_2$$

where, ΔT_1 − KE at P_1
ΔT_2 − KE at P_2

Considering all the particles in a system, we have:

$$T_1 + U_{1-2} = T_2$$

9.2 <u>Work of Force acting on the Rigid Body</u>

The work of force $\vec{F}$ during the displacement from its position s_1 *to* s_2 is:

$$U_{1-2} = \int_{s_1}^{s_2} (F\cos\theta)ds \quad -(i)$$

where, F is magnitude of force $\vec{F}$ and θ is the angle between the force and the direction of motion.

If F and F' are two forces turning a couple, then moment of couple is:

$$\vec{M} = \vec{F}.r$$

Under the application of external forces (F & F'), the displacement of points A & B to A'B", the motion may be divided into two parts:

(a) Point A & B undergo equal displacement dr_1 to point A' & B'.
(b) Point A' remains fixed and B' moves to B" with the displacement dr_2 equal to $rd\theta$.

During the first part of motion, work due to force $\vec{F}$ = Work due to F' and cancels out due to opposite in sign.

During the second part of motion, work done is given by:

$$du = F.dr_2 = Frd\theta$$
$$\therefore du = Md\theta \quad [\because M = rd\theta]$$

$\therefore$ Total work done is $U_{1-2} = \int_{\theta_1}^{\theta_2} Md\theta$

9.3 Kinetic Energy for a system:

(A) <u>Kinetic Energy in Translation</u>:

For a rigid body in translation, all the particles have the same velocity as of CG.

i.e. $v = \bar{v}$

$\therefore$ KE of the entire body in Translation is given as:

$$T = \sum_{i=1}^{n} \frac{1}{2}(\Delta m_i)\bar{v}^2 = \frac{1}{2}\left(\sum_{i=1}^{n}(\Delta m_i)\right)\bar{v}^2$$

$$\therefore T = \frac{1}{2}M\bar{v}^2$$

(B) Kinetic Energy in Rotation:

Considering a rigid body rotating about fixed axis and point 'O', then

$$KE = \sum \frac{1}{2}(\Delta m)v^2 = \sum \frac{1}{2}(\Delta m)(r\omega)^2 = \frac{1}{2}\omega^2 \sum r^2 \Delta m$$

$$\therefore KE = \frac{1}{2}I_o\omega^2 \qquad \left[\because \sum r^2\Delta m = \text{M.O.I. of the body about the axis of rotation} = \mathrm{I_o}\right]$$

This formula is valid for any axis of rotation.

(C) Kinetic Energy in Plane Motion:

For a body in plane motion, at any given instant, the velocities of all the particles of the body are same as if the body were rotating about the instantaneous axis of rotation, and about the instantaneous center of rotation. Then the kinetic energy is

$$T = \frac{1}{2}I_c\omega^2 \qquad -(i)$$

where,

I_c = M.O.I. of the body about the instantaneous axis

ω = Angular velocity at the instant considered.

Now, from Parallel axis Theorem

$$I_C = \bar{I} + m\bar{r}^2$$

where,

$\bar{I}$ = M.O.I. of the body about a centroidal axis perpendicular to the reference plane

$\bar{\mathrm{r}}$ = Distance from 'C' to mass center 'G'

Then,

$$T = \frac{1}{2}\left(m\bar{r}^2 + \bar{I}\right)\omega^2 = \frac{1}{2}m(\bar{r}\omega)^2 + \frac{1}{2}\bar{I}\omega^2$$

$$or,\ T = \frac{1}{2}m\bar{v}^2 + \frac{1}{2}\bar{I}\omega^2 \qquad -\ (ii) \quad \left[\because \bar{r}\omega = \bar{v} = velocity\ of\ G\right]$$

So, the total KE of rigid body in plane motion is the sum of KE of body due to Translation about CG $\left(\frac{1}{2}m\bar{v}^2\right)$ and KE of body due Rotation about CG $\left(\frac{1}{2}\bar{I}\omega^2\right)$.

9.4 Conservative and Non-Conservative System:

A system is said to be conservative if the work done by the system is independent of the path followed but depends upon the initial and final position of it. Examples are KE, PE, gravity, etc.

In the non-conservative system, work done depends upon the path followed by the system. Examples are friction, elastic, etc.

The principle of conservation of energy is valid for the conservative force only. When a rigid body or a system of rigid bodies move under the action of conservative forces, then the principle of work and energy may be expressed as:

$$T_1 + V_1 = T_2 + V_2$$

i.e. The sum of the KE and PE of the system remains constant.

9.5 Work-Energy Applications:

Considering a rod AB of length 'L' and mass 'm' whose extremities are connected to blocks of negligible mass sliding along horizontal and vertical tracks as shown in figure.
If the initial velocity of the rod is zero and datum as shown is considered, then

$\because$ Initial KE (T_1) = 0

Initial PE (V_1) = 0

If the rod moves by an angle 'θ', the CG of the rod moves by $\frac{L\sin\theta}{2}$ vertically downwards from its initial position.

$$\therefore V_2 = -mg\frac{1}{2}\frac{\sin\theta}{2} = \frac{1}{2}mgL\sin\theta$$

Now, for the final position, the instantaneous center of rotation is located at 'C' and also we have that $CG = \frac{1}{2}L\omega$

Then, $\vec{V}_2 = \omega_{CG} = \omega\frac{1}{2}L = \frac{1}{2}L\omega$

$$\therefore T_2 = \frac{1}{2}m\bar{V}_2^2 + \frac{1}{2}\bar{I}\omega_2^2 = \frac{1}{2}m(L\omega)^2 + \frac{1}{2}\left(\frac{1}{12}mL^2\right)\omega^2 \qquad \left[\bar{I} = \frac{1}{12}mL^2 \ \ for\ rod\right]$$

$$T_2 = \frac{1}{2}\left(\frac{mL^2}{3}\right)\omega^2$$

Applying Principle of Conservation of Energy, we get:

$$T_1 + V_1 = T_2 + V_2$$

$$0 + 0 = \frac{1}{2}\left(\frac{mL^2}{3}\right)\omega^2 - \frac{1}{2}mgL\sin\theta$$

$$\therefore \omega = \left(\frac{3g}{L}\sin\theta\right)^{\frac{1}{2}}$$

Note: D'Alembert's Principle should be used to find the reactions at the sliders.

9.6 Impulse and Momentum for System of Rigid Bodies

The principle of Impulse and Momentum can be applied to the system of rigid bodies as well. We know that,

System Momentum-1 + System External Impulse $1 \rightarrow 2$ = System Momentum-2

IF $L = \sum_{i=1}^{n} \Delta m_i v_i$ = sum of momentum of all particles

or, $L = m\bar{v}$

And, $H_G = \sum_{i=1}^{n} r_i' \times \Delta m_i v_i$ = sum of moment of momentum of all particles

or, $H_G = \bar{I}\omega$

Then,

Then, we have :

The fig(iii) gives three equation of motion, as

$$(m\bar{v}_1)_x + \int_1^2 (Fdt)_x = (m\bar{v}_2)_x \quad -(i)$$

$$(m\bar{v}_1)_y + \int_1^2 (Fdt)_y = (m\bar{v}_2)_y \quad -(ii)$$

$$\bar{I}\omega_1 + \left(\int_1^2 Fdt\right)_y x\bar{x} + \left(\int_1^2 Fdt\right)_x \bar{y} = \bar{I}\omega_z \quad -(iii)$$

where, $\bar{x}$ & $\bar{y}$ are perpendicular distances of impulse from x and y axis

9.7 Conservation of Angular and Linear Momentum:

When no external forces acts on a rigid body, the impulses of the external forces are zero and the system of momentum at time t_1 is equal to system of momentum at time t_2.

Thus, total linear momentum is conserved in any direction and angular momentum is conserved about any point.

i.e. if $\sum F_{ext} = 0$

$m\bar{v}_1 = m\bar{v}_2$ &

$(H_o)_1 = (H_o)_2$

In some cases as when the line of action of all external forces pass through 'G' or when sum of the angular impulses of the external forces about 'G' is zero, then

9.8 Impulsive Motion and Eccentric Impact:

Remember : The definition of Impulsive Force and Impulsive motion . Eccentric impact

Considering two bodies which collide under the eccentric impact. Let V_A and V_B be the velocities before impact of the points of contact A and B. Under the impact, the two bodies will deform and at the end of period of deformation,

component of U_A = Component of U_B, along the line of impact.

[During deformation both bodies move with same velocity.]

At the end of period of restitution, V_A' *and* V_B' be the velocities of A and B as shown, Assuming the bodies are frictionless. Then, component of restitution is given by:

$$e = \frac{\int Rdt}{\int Pdt} \quad -(i) \quad \left[\begin{array}{l} where, \\ \int Pdt = \text{magnitude of impulse during the period of deformation} \\ \int Rdt = \text{magnitude of impulse after the period of deformation} \end{array}\right]$$

The relative velocities of two particles before and after impact along the line of impact is related as:

$$\left(V_B'\right)_n - \left(V_A'\right)_n = e\left[\left(V_A\right)_n - \left(V_B\right)_n\right] \quad -(ii)$$

$$or,\ e = \frac{\left(V_B'\right)_n - \left(V_A'\right)_n}{\left(V_A\right)_n - \left(V_B\right)_n} = \frac{\text{Relative velocity along the line of action after collision}}{\text{Relative velocity along the line of action before collision}}$$

Applying Impulse-Momentum Equation for the particle 'A'

Let $\bar{v}\ and\ \bar{u}$ be the velocities of mass center at the beginning and at the end of the period of deformation. $\omega\ and\ \omega_1$ be the angular velocities at the same instant.

Then along the line of impact the component of momentum and impulse becomes:

And,
$$m\bar{v}_n - \int Pdt = m\bar{u}_n \quad -(iii) \quad \Rightarrow \quad \int Pdt = m\bar{v}_n - m\bar{u}_n$$
$$\bar{I}\omega - r\int Pdt = \bar{I}\omega_1 \quad -(iv) \quad \Rightarrow \quad \int Pdt = \bar{I}\omega - \bar{I}\omega_1$$

where r is the perpendicular distance from 'G' to line of impact.

Similarly, considering for the period of restitution:

$$m\bar{v}_x - \int Rdt = m\bar{v}_x' \quad -(v) \quad \Rightarrow \quad \int Rdt = m\bar{v}_x - mv_x'$$

$$\bar{I}\omega_1 - r\int Rdt = \bar{I}\omega' \quad -(vii) \quad \Rightarrow \quad \int Rdt = \bar{I}\omega_1 - Iw'$$

where, $\bar{v}'\ and\ \bar{\omega}'$ represent the velocity of mass center and angular velocity of body after impact.

From (i), (iii) & (v), we get:

$$e = \frac{m\bar{u}_n - mv_n'}{m\bar{v}_n - m\bar{u}_n}$$

$$e = \frac{\bar{u}_n - \bar{v}_n'}{\bar{v}_n - \bar{u}_n} \quad -(vii)$$

Similarly, from equ(i), (iv) and (vi), we get:

$$e = \frac{\bar{I}\omega_1 - \bar{I}\omega'}{\bar{I}\omega - \bar{I}\omega_1}$$

$$e = \frac{\omega_1 - \omega'}{\omega - \omega_1} \quad -(viii)$$

Again,

Multiplying the numerator and denominator of equ(viii) and adding respectively to the numerator and denominator of (vii), we get:

$$e = \frac{\bar{u}_n + r\omega_1 - \left(\bar{v}_n' + r\omega'\right)}{\bar{v}_n + r\omega - \left(\bar{u}_n + r\omega_1\right)} \quad -(ix)$$

$$e = \frac{\left(u_A\right)_n - \left(v_A'\right)_n}{\left(\bar{v}_A\right)_n - \left(u_A\right)_n} \quad -(x)$$

Similarly for 2nd body,

$$e = \frac{(v'_B)_n - (u_B)_n}{(u_B)_n - (v_B)_n} \qquad -(xi)$$

Adding the numerator and denominator of (x) and (xi) respectively, we get:

$$e = \frac{(u_A)_n - (v'_A)_n + (v'_B)_n - (u_B)_n}{(v_A)_n - (u_A)_n + (u_B)_n - (v_B)_n}$$

$$e = \frac{(v'_B)_n - (v'_A)_n}{(v_A)_n - (v_B)_n} \qquad -(xii)$$

$$\therefore e = \frac{\text{Relative velocity of colliding body along the line of impact after collision}}{\text{Relative velocity of colliding body along the line of impact before collision}}$$

Tutorials:

1. Gear A has a mass of 10 kg and a radius of gyration of 200mm, while gear B has a mass of 3 kg and a radius of gyration of 80mm. The system is at rest when a couple $\overrightarrow{M}$ of magnitude 6 Nm is applied to gear B. Neglecting friction, determine:

(a) The number of revolutions executed by gear B before its angular velocity reaches 600 r/min.

(b) The tangential force which gear B exerts on gear A.

Solution:

(a) Considering the peripheral speed of gears be equal, we have

$$r_A w_A = r_B w_B$$

$$\therefore w_A = w_B \frac{r_B}{r_A} = w_B \frac{100}{250} = 0.4 w_B$$

We have, $w_B = 600 r/\min = 62.8 rad/s \left[w = \frac{\pi d N}{60} \right]$

$$w_A = 0.4 w_B = 25.1 rad/s$$

$$\bar{I}A = m_A \bar{k}_A^2 = 10 \times (0.2)^2 = 0.4 kgm^2$$

$$I_B = m_B \bar{k}_B^2 = 3 \times (0.08)^2 = 0.0192 kgm^2$$

Applying work-energy principle to the system,

Since, system is initially at rest, $\therefore T_1 = 0$

And,

$$T_2 = \frac{1}{2}\bar{I}_A w_A^2 + \frac{1}{2}\bar{I}_B w_B^2$$

$$= \frac{1}{2}(0.4)(25.1)^2 + \frac{1}{2}(0.0192)(62.8)^2$$

$$\therefore T_2 = 163.9 J$$

Let θ_B be the angular displacement of gear B,

Then,

Work $U_{1-2} = M\theta_B = 6 \times \theta_B = 6\theta_B J$

<u>By the principle of work and energy:</u>

$T_1 + U_{1-2} = T_2$

$0 + 6\theta_B = 163.9$

$\therefore \theta_B = 27.32\,rad$

$or,\ \theta_B = 4.35\,rev \qquad \left[\mathrm{Re}\,v = \dfrac{\theta_B}{2\pi}\right]$

(b) Considering the motion of gear 'A'

KE of gear A at rest T_1=0

When w=600 rpm

KE of gear A, 'T_2' = $\dfrac{1}{2}\bar{I}_A w_A^2 = \dfrac{1}{2}(0.4)(25.1)^2 = 126J$

Work:

We have,

Angular Arc covered by both gears is same i.e. $s = r_A\theta_A = r_B\theta_B \qquad [s = r\theta]$

Then, work done by gear A during the motion,

$$U_{1-2} = F(\theta_A r_A) = F(\theta_B r_B) = F(27.3)(0.1) = F(2.73)$$

Principle of Work-Energy: Applying for Gear 'A'

$$T_1 + U_{1-2} = T_2$$

$$0 + F \times 2.73 = 126J$$

$$F = 46.2N$$

2. A 2 kg sphere moving horizontally to the right with an initial velocity of 5 m/s strikes the lower end of an 8 kg rigid rod AB. The rod is suspended from a hinge at A and is initially at rest. Knowing that the co-efficient of restitution between the rod and sphere is 0.80, determine the angular velocity of the rod and the velocity of the sphere immediately after the impact.

Solution:

We have from Impulse-Momentum Theorem,

(System Momentum)$_1$ + (System External Impulse)$_{1-2}$ = (System Momentum)$_2$

Let, V_R and $V_R^{'}$ be the initial and final velocity of Rod 'CG', ω *and* ω' be the angular velocities of the rod and V_s *and* $V_s^{'}$ be the velocities of sphere.

Now,

Taking moment about A,

$$m_s V_s \times 1.2 = m_s V_s^{'} \times 1.2 + m_R v_R^{'} \times 0.6 + \bar{I}\omega' \quad -(i)$$

Here,

$$\overline{V_R^{'}} = \bar{r}\omega' = 0.6\omega'$$

$$\bar{I} = \frac{1}{12}ml^2 = \frac{1}{12} \times 8 \times (1.2)^2 = 0.96kgm^2$$

Substituting these values in equ(i) ,

$$2 \times 5 \times 1.2 = 2 \times V_s^{'} \times 1.2 + 8 \times 0.6 \times \omega' \times 0.6 + 0.96 \times \omega'$$

$$\therefore 2.4V_s^{'} + 3.84\omega' = 12 \quad -(ii)$$

Now, we have coefficient of restitution, $\varepsilon = 0.8$

$$or,\ \varepsilon = \frac{\text{Relative velocity after collision}}{\text{Relative velocity before collision}}$$

$$or,\ \varepsilon = \frac{V_R' - V_s'}{V_s - V_R} \qquad [V_s = 5m/s, \quad V_R = 0, \qquad \varepsilon = 0.8]$$

$$or,\ V_R' - V_s' = 0.8 \times 5$$

$$or,\ V_R' - V_s' = 4 \qquad -(iii)$$

When the rod rotates about A, then

$$V_R' = 0.6\omega' \qquad -(iv)$$

From (iii) and (iv),

$$1.2\omega' - v_s' = 4 \quad -(v)$$

Solving (ii) and (v), we get:

$$\omega' = 3.21\ rad/s$$

$$V_c' = 0.143m/s$$

3. A 20 gm bullet 'B' is fired with a horizontal velocity of 450 m/s into the side of a 10 kg square panel suspended from a hinge at 'A'. Knowing that the panel is initially at rest, determine:

(a) The angular velocity of the panel immediately after the bullet becomes embedded.

(b) The impulsive reaction at 'A' assuming that the bullet becomes embedded in 0.0006 sec.

Solution:

Applying Impulse-Momentum Equation:

(System Momentum)$_1$ + (System External Impulse)$_{1\text{-}2}$ = (System Momentum)$_2$

<u>Taking moments about A</u>:

$$m_B v_B \times (0.35) = m_p \bar{v}_2 (0.225) + \bar{I}_p w_2 \quad -(1)$$

<u>x-components:</u>

$$m_B v_B + A_x \Delta t = m_p \bar{v}_2 \quad -(2)$$

<u>y-components</u>:

$$0 + A_y \Delta t = 0 \quad -(3)$$

M.O.I. for square panel $\bar{I} = \frac{1}{6} mb^2 = \frac{1}{6}(10)(0.45)^2 = 0.3375 kgm^2$

Putting values in equ(i), we get:

$$\bar{v}_2 = (0.225) w_2 \qquad -(4)$$

Again, putting value of $\bar{v}_2$ in equ(i)

(0.020) (450) (0.350) = (10) (0.225w_2) (0.225) + 0.3375 w^2

w_2 = 3.73 rad/s

$$\therefore \bar{v}_2 = 0.225 \times w_2 = 0.839 m/s$$

Putting $\bar{v}_2$ *and* $\Delta t = 0.0006s$ into equ(2)

(0.020) (450)+A_x (0.0006) = 10 (0.839)

A_x = -1017N

$\therefore A_x = 1017\ (\leftarrow)$

From equ(3)

$\therefore\ A_y = 0$

Chapter – 10
Mechanical Vibrations

10.1.1 Introduction

- A mechanical vibration is the motion of a particle or a body which oscillates about a position of equilibrium. When a system is displaced from a position of stable equilibrium, it tends to return to its initial position under the action of restoring force or elastic force.
- The time interval required for the system to complete a full cycle of motion is called period of vibration (τ).
- The number of cycles per unit time is called frequency(f).
- The maximum displacement from mean position is called as amplitude of the vibration (A).

<u>Types of Vibrations</u>:

Free Vibration	:	Vibration only due to restoring force like elastic force.
Forced Vibration	:	Vibration due to external periodic force.
Undamped Vibration	:	Effect of friction may be neglected
Damped Vibration	:	Effect of friction is considered ; the vibration slowly decreases and comes to halt.

10.1.2 Undamped Free Vibration of Particles:

Let us consider a mass 'm' attached to a spring having constant K.

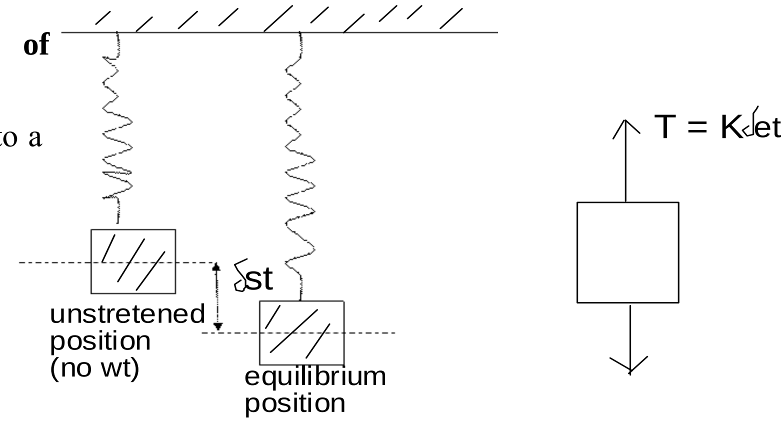
At equilibrium condition,

$$W = T = K\delta_{st} \quad -(i)$$

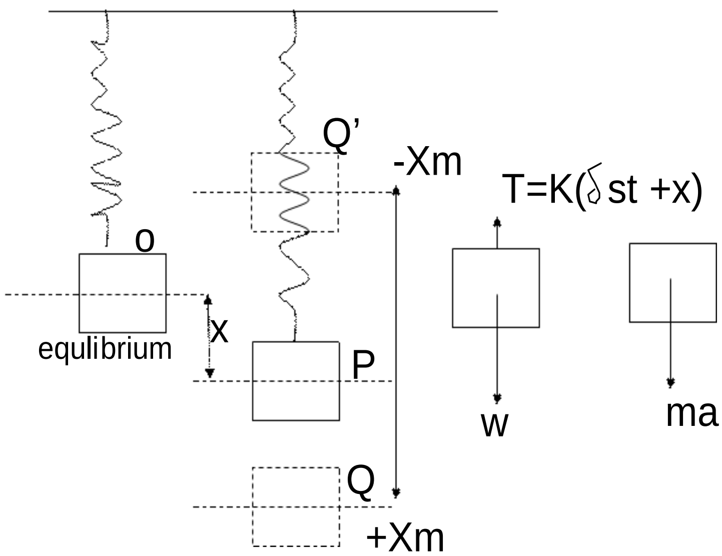
where, W = Weight of particle

T = Force exerted by spring

δ_{st} = Elongation of the spring



The particle is displaced through a distance 'x' m from its equilibrium position and released with no initial velocity. Then it will move back and forth through its equilibrium.



At any position of displacement x, the

magnitude of the resultant force ($\vec{F}$) is given by:

$$F = W - K(\delta_{st} + x)$$
$$= W - K\delta_{st} - Kx$$
$$\therefore F = -Kx \qquad -(ii) \quad [\text{since } K\delta_{st} = W]$$

The negative sign indicates that the force is towards the equilibrium position.

From (ii)

$$F + Kx = 0 \Rightarrow m\ddot{x} + kx = 0$$
$$or,\ m\frac{d^2x}{dt^2} + Kx = 0 \qquad -(iii) \quad [\text{which is linear dist. equ}^n \text{ of 2nd order}]$$
$$or,\ \frac{d^2x}{dt^2} + \left(\frac{K}{m}\right)x = 0$$
$$or,\ \frac{d^2x}{dt^2} + P^2x = 0 \qquad -(iv) \quad \left[\text{Putting } \frac{K}{m} = P^2, \text{ where } P = \sqrt{\frac{K}{m}} \text{ is circular frequency}\right]$$

The motion defined by equ(iv) is called as Simple Harmonic Motion. Particular solution of the above differential equation i.e. $x_1 = \sin pt$ & $x_2 = \cos pt$

And, the general solution :
$$x = Ax_1 + Bx_2$$
$$\therefore x = A\sin pt + B\cos pt \qquad -(iv)$$

Differentiating above equation with respect to time, we get:

$$v = \dot{x} = AP\cos pt - BP\sin pt \quad -(vi)$$

Again, differentiating w.r.t. time, we get:

$$a = \ddot{x} = -AP^2\sin pt - BP^2\cos pt \qquad -(vii)$$

The value of arbitrary constant A and B depends upon the initial conditions of motion.

At time t=0, position $x = x_o$, then from equ(v)

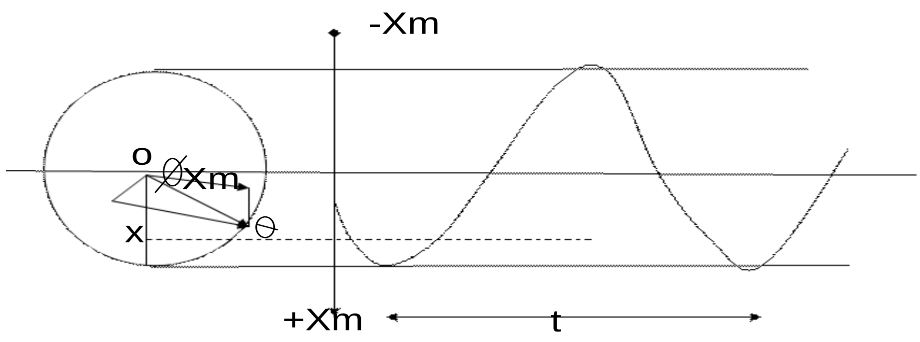
$$B = x_o \qquad -(ix)$$

Again, at time t=0, Velocity $v = v_o$ then from equ(vi)

$$A = \frac{v_o}{P} \quad -(x)$$

The vibration also can be represented in circular motion of point Q with angular velocity P and radius of rotation x m which form a simple harmonic motion.

From the equation of simple harmonic motion in circular motion, we have:



$x = x_m \sin(pt + \phi)$
$\therefore v = \dot{x} = x_m P \cos(pt + \phi)$
$a = \ddot{x} = -x_m P^2 \sin(pt + \phi)$
$Period, \tau = \dfrac{2\pi}{P}$ and
Frequency of Vibration, $f = 1/e = \dfrac{P}{2\pi}$

Again,
Maximum Velocity $(v_m) = x_m P$
Maximum Acceleration $(a_m) = x_m P^2$

10.1.3 Free Vibration of Rigid Bodies:

Analysis of vibration of rigid bodies is done as that of particle and the D'Alembert's Principle is applied to solve the unknown parameters and calculate t & p.
We have differential equations for free vibration as

$$\frac{dt^2 x}{dt^2} + P^2 x = 0 \qquad and \qquad \text{For angular displacement}$$
$$\frac{d^2\theta}{dt^2} + P^2\theta = 0 \quad -(i)$$

Let us consider a square plate of '2b' sides, which is suspended from the mid-points 'O' on one of its sides.
If the plate is tilted such that 'OG' makes angle 'θ' then we have,

$$\omega' = \dot{\theta} \quad \& \quad \alpha = \ddot{\theta}$$
$$m\bar{a}_t = mb\ddot{\theta} \qquad \left[\because a_t = r\alpha = b\ddot{\theta} \;\; \& \;\; a_n = r\omega'^2 = b\dot{\theta}^2\right]$$
$$ma_x = mb\dot{\theta}^2$$

Now, Applying D'Alembert' Principle,

$$\sum m_{ext} \text{ about } O = \sum m_{eff}\, about\, O$$
$$or, -w\sin\theta.b = ma_t b + \bar{I}\alpha$$
$$or, -wb\sin\theta = m(b\ddot{\theta})b + \bar{I}\ddot{\theta}$$

Now for Square Plate,

$$\bar{I} = \frac{2}{3}mb^2 \qquad \text{\& Putting w = mg, we get :}$$
$$-mgb\sin\theta = mb^2\ddot{\theta} + \frac{2}{3}mb^2\ddot{\theta}$$
$$or, -mgb\sin\theta = \frac{5}{3}mb^2\ddot{\theta}$$
$$or, \ddot{\theta} = -\frac{3}{5}\frac{g}{b}\sin\theta \Rightarrow \ddot{\theta} + \frac{3g}{5b}\sin\theta = 0 \qquad -(ii)$$

$$\left[\ddot{\theta} + \frac{3g}{5b}\theta = 0\right] \qquad -(iii) \quad [\sin\theta = \theta \;\; for \;\; \theta = small]$$

which is the equation of simple harmonic motion comparing with equ(i), we get:

$$P = \left(\frac{3g}{5b}\right)^{1/2}$$

Then, $\tau = \frac{2\pi}{P} = 2\pi\sqrt{\frac{5b}{3g}} \qquad -(vi)$

Comparing this equation with that of pendulum,

$$\tau_p = 2\pi\sqrt{\frac{l}{g}}$$

we get effective length, $l = \frac{5b}{3}$, This shows that the square plate will oscillate as a pendulum with effective length $\frac{5b}{3}$ with 'O' as its center.

10.2 Steady Harmonic Force Undamped Vibration

These vibrations occur when a system is subjected to a periodic force or when it is elastically connected to a support.

Consider a body of mass 'm' suspended from a spring and subjected to period force P of magnitude $= P_m sn\omega t$ undergoing damped vibration.

At any displacement x from equilibrium position, then the equation of motion will be given by:

$$+\downarrow \sum F = ma$$

$$or,\ P_m \sin\omega t + w - k(\delta_{st} + x) = m\bar{a} = m\ddot{x}$$

$$or,\ P_m \sin\omega t - w - k\delta_{st} - kx = m\ddot{x}$$

$$or,\ m\ddot{x} + kx = P_m \sin\omega t \qquad -(i) \quad [\because k\delta_{st} = w]$$

This is homogenous differential equation.

Its general solution = Particular solution + Complementary Solution

Particular solution of (i) can be obtained by trail method of

$$x = x_m \sin\omega t \quad -(ii)$$

$$\ddot{x} = -x_m\omega^2 \sin\omega t \qquad -(iii)$$

Putting values of (ii) & (iii) in (i), we get:

$$-mw^2 x_m \sin\omega t + kx_m \sin\omega t = P_m \sin\omega t$$

Solving,

$$Amplitude(X_{m)} = \frac{P_m}{k - mw^2}$$

$$\therefore X_m = \frac{P_m/k}{1 - w/p} \quad -(iv) \qquad [Putting\ k/m = P^2, \text{P is circular frequency}]$$

The general solution is given by

$$x = A\sin pt + B\cos pt + x_m \sin\omega t \qquad -(v)$$

The first two terms of equ(v) represent a free vibration of the system and the third part represents forced vibration and the frequency involved is forced frequency.

$$Magnification\ Factor = \frac{x_m}{P_m/k} = \frac{1}{1-(w/P)^2} \quad -(v)$$

<u>Tutorial</u>

1. A 50 kg block moves between vertical guides as shown. The block is pulled 40 mm down from its equilibrium position and released. Determine the period of vibration, the maximum velocity and maximum acceleration of the block.

Solution:

Here, $\delta = \delta_1 + \delta_2 = \frac{P}{k_1} + \frac{P}{k_2}$

For equivalent single spring with 'k'

$$\delta = \frac{P}{k} = \frac{P}{k_1} + \frac{P}{k_2}$$

$$\therefore \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$\therefore k = \frac{k_1 + k_2}{k_1.k_2} = \frac{4+6}{4\times 6} = \frac{10}{24}$$

$$\therefore P = \sqrt{\frac{k}{m}}$$

Now,

$$\text{Time Period } (\tau) = \frac{2\pi}{\text{P}} =$$

$$v_m = x_m P = 0.04x$$

$$a_m = x_m P^2 = 0.04x$$

2. A cylinder of mass 'm' and radius 'r' is suspended from a looped cord. One end of the cord is attached directly to a rigid support, while the other end is attached to a spring of constant 'k'. Determine the period and frequency of vibration of the cylinder.

Solution:

Taking positive sense as clockwise and measuring the displacement from the equilibrium position:

Here,

$\bar{x} = r\theta \qquad \therefore \delta = 2\bar{x} = 2r\theta$

$\alpha = \ddot{\theta} \qquad \therefore \bar{a} = r\alpha = r\ddot{\theta}(\downarrow) \quad -(i)$

Applying D'Alembert's principle

Taking Moments

$$\sum m_{ext}\, about\, 'A' = \sum m_{eff}\ about\ 'A'$$

$$mgr - T_2(2r) = m\bar{a}r + \bar{I}\alpha \qquad -(2)$$

For the unstretched condition, the tension in each cord is

$$T_o = \frac{1}{2}\omega = \frac{1}{2}mg$$

When the cylinder is rotated by 'θ' the tension T_2 becomes

$$T_2 = T_o + k\delta = mgr + k.2r\theta$$
$$\therefore T_2 = mgr + 2kr\theta \qquad -(3)$$

From (1) and (3), putting in (2), we get:

$$mgr - \left[\frac{1}{2}mg + k(2r\theta)\right](2r) = m(r\ddot{\theta})r + \frac{1}{2}mr^2\ddot{\theta} \qquad \left[\bar{I} = \frac{1}{2}mr^2\right]$$

Solving, we get:

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

The motion is S.H.M. where,

$$P^2 = \frac{8}{3}\frac{k}{m}$$
$$\therefore P = \sqrt{\frac{8}{3}\frac{k}{m}}$$
$$\tau = \frac{2\pi}{P} \qquad \therefore \tau = 2\pi\sqrt{\frac{3}{8}\frac{m}{k}}$$
$$f = \frac{P}{2\pi} \qquad \therefore f = \frac{1}{2\pi}\sqrt{\frac{8}{3}\frac{k}{m}}$$