MAT165 ASSIGNMENT 3

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Last Date of Submission. 26 November, 2025 **before** the lecture starts.

Instructions.

- You can discuss the problems with any of your class-mates. In fact, I encourage you to talk to your friends and come up with the solutions together.
- Avoid using AI or web search to arrive at the solutions. This way would be easy, but you will learn very little.
- Write down the solutions in A4 sized sheets of paper (either blank or dotted), and staple them before submission. Use either black or blue ink for writing the solutions. **Failure to follow this will result in an immediate score of zero.**
- If you are unable to solve a problem, then mention what approach you took and how that did not work out. Sometime in mathematics, false starts can lead to promising avenues in other directions.

Questions.

(1) What is the number of ways to write the digits $1, 1, 2, 2, 3, 4$, and 5 in a line so that identical digits are not in consecutive positions?

(2) The students of a class (with more than three students) altogether speak four languages. For any three students, there is at least one language that all of them speak. There is no language that all students speak. Prove that each student speaks at least three languages. Construct an example where this happens.

(3) Using a combinatorial argument, prove that for $n > 0$, we have $2\binom{2n-1}{n} = \binom{2n}{n}$.

(4) A partition of n is a sequence of non increasing integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ such that all the λ_i's (which are called parts of the partition) sum up to n. Show that the number of partitions of n with largest part k equals the number of partitions of n with exactly k parts.

(5) Let $f(m,n)$ denote the number of lattice paths from $(0,0)$ to (m,n) with **E**, **N**, and **diagonal** steps. Then prove that

$$f(m+1,n+1) = f(m,n+1) + f(m+1,n) + f(m,n), \quad m,n \geq 0.$$

(6) Using a combinatorial argument, show that $\binom{2n}{n} < 4^n$.

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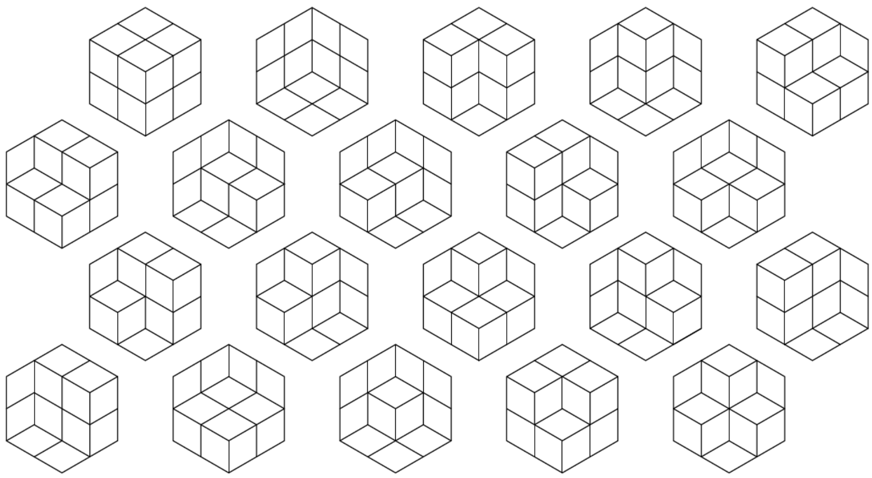
(7) For all natural numbers n, m, k, prove that

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i} \binom{n}{k-i}.$$

(8) Consider the lozenge tilings of a regular hexagon, where we tile a hexagon with equal side-lengths using the lozenges , , and . For example, all such possible tilings of a regular hexagon with side length 2 is given below.



Prove that, for a regular hexagon of side-length n, in all such tilings, the number of lozenges of the three types are always equal.