~~Change~~ of Variables:

- We want something analogous to substitution for multiple integrals so that for regions where we cannot directly apply Fubini's theorem we transform them into one where we can.
- We want to transform $\iint_S f(x,y)\,dx\,dy$ over a region S in the xy-plane to $\iint_T F(x, y, v)\,du\,dv$, defined on a new region T in the uv-plane.

For single integral we had $\int_a^b f(x)dx = \int_{g(a)}^{g(b)} f(g(t))\,g'(t)\,dt$.

Here we need two fns, $x = X(u,v)$, $y = Y(u,v)$.

- We assume that the mapping from T to S is one-one.
- The fns X and Y are cont. and have cont. partial derivatives, $\frac{\partial X}{\partial u}$, $\frac{\partial X}{\partial v}$, $\frac{\partial Y}{\partial u}$ and $\frac{\partial Y}{\partial v}$.
- The Jacobian $J(u,v) = \begin{vmatrix} \partial X/\partial u & \partial Y/\partial u \\ \partial X/\partial v & \partial Y/\partial v \end{vmatrix} \neq 0$.

Then, $\iint_S f(x,y)\,dx\,dy = \iint_T f(X(u,v), Y(u,v))\,|J(u,v)|\,du\,dv$.

eg: Find the area of the region S bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the curves $xy^2 = 3$ and $xy^2 = 4$.

The reqd. area is $\iint_S dx\,dy$.

Put $u = xy$, $v = xy^2$. $\Rightarrow x = u^2/v$, $y = v/u$.

The region T is $1 \leq u \leq 2$ and $3 \leq v \leq 4$. The Jacobian $= 1/v$.

So, we get the area as $\iint_T \frac{1}{v}\,du\,dv = \int_3^4 \int_1^2 \frac{1}{v}\,du\,dv$.//

...lar Coordinates:

We use the eq^{n}s $x = X(r,\theta) = r\cos\theta$, $y = Y(r,\theta) = r\sin\theta$.

We assume $r > 0$, $\theta \in [0, 2\pi)$ or $\theta_0 \le \theta < \theta_0 + 2\pi$ for some θ_0 so that the mapping is 1-1.

$$J(u,v) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

eg: Find the vol^m of the sphere of radius a.

- The vol^m is $V = 2\iint_S \sqrt{a^2 - x^2 - y^2}\,dx\,dy$, with

$$S = \{(x,y) \mid x^2 + y^2 \le a^2\}.$$

In polar co-ordinates we get,

$$V = 2\iint_T \sqrt{a^2 - r^2}\, r\,dr\,d\theta, \text{ with } T = [0,a]\times[0,2\pi].$$

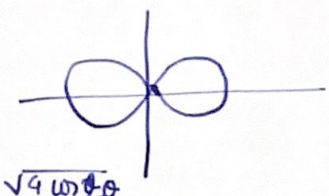
$$= 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}\, r\,dr\,d\theta = 4\pi \int_0^a r\sqrt{a^2 - r^2}\,dr$$

$$= 4\pi\left[\frac{(a^2 - r^2)^{3/2}}{-3}\right]_0^a = \frac{4\pi a^3}{3} \text{ //.}$$

Area in polar coordinates: Area of a closed and bounded region R is given by $A = \iint_R r\,dr\,d\theta$.

eg: Find the area enclosed by $r^2 = 4\cos 2\theta$.

- This is a lemniscate, so the area is 4 times the area in the 1st quadrant.



$$\text{So, } A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4 \text{ //.}$$

Ex: Evaluate $\iint_R e^{x^2+y^2}\,dy\,dx$, where R is the semi-circle bounded by $y=0$ and $y=\sqrt{1-x^2}$.

Sol^n: $x = r\cos\theta$, $y = r\sin\theta$

$$\iint_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2}\, r\,dr\,d\theta = \int_0^{\pi}\left[\frac{e^{r^2}}{2}\right]_0^1 d\theta$$

$$= \int_0^{\pi} \frac{1}{2}(e-1)\,d\theta = \pi/2\,(e-1).$$

Ex: $\displaystyle\int_0^1\int_0^{1-x} e^{\frac{x-y}{x+y}}\,dx\,dy = I$.

Let $u = x-y$, $v = x+y$. Then, $I = \iint_S e^{u/v}\,|J|\,du\,dv$,

where S is the region in the uv-plane bounded by $u=-v$, $u=v$ and $v=1$.

$$I = \frac{1}{2}\int_0^1\int_{-v}^{v} e^{u/v}\,du\,dv = \frac{1}{2}\sinh(1).$$

Ex: Evaluate $\iint_R \cos(9x^2+4y^2)\,dx\,dy$, where $R = \{(x,y) \mid 9x^2+4y^2 \le 1\}$.

Sol^n: Let $x = r/3\,\cos\theta$, $y = r/2\,\sin\theta$. Then $|J| = r/6$.

$$\iint_R \cos(9x^2+4y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 \cos(r^2)\, r/6\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \cos u\,\frac{du}{12}\,d\theta.$$
