Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J PSET 3

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1. 1.28:Identifying total time derivatives

1.28

$$G(t,q,v) = G_0(t,q) + G_1(t,q)\,v$$

(a) $G(t,x,v_x) = m v_x$

$G = D_t F(t,x) \rightarrow \boxed{F(t,x) = mx}$

(b) $G(t,x,v_x) = m v_x \cos t$

$G_0 = 0 \;,\; G_1 = m\cos t$

$\partial_1 G_0 = 0 \neq \partial_0 G_1 = -m\sin t$

$G \neq D_t F(t,x)$

(c) $G(t,x,v_x) = v_x \cos t - x\sin t$

$\boxed{F(t,x) = x\cos t}$

(d) $G(t,x,v_x) = v_x\cos t + x\sin t$

$G_0 = x\sin t,\; G_1 = \cos t$

$\partial_1 G_0 = \sin t \neq \partial_0 G_1 = -\sin t$

$G \neq D_t F(t,x)$

(e) $G(t;x,y;v_x,v_y) = 2(x v_x + y v_y)\cos t - (x^2+y^2)\sin t$

$\boxed{F(t,x) = (x^2+y^2)\cos t}$

(f) $G(t;x,y;v_x,v_y) = 2(x v_x + y v_y)\cos t - (x^2+y^2)\sin t + y^3 v_x + x v_y$

$G_0 = -(x^2+y^2)\sin t$

$$G_1 = \begin{bmatrix} 2x\cos t + y^3 \\ 2y\cos t + x \end{bmatrix}$$

$$\partial_1 G_1 = \begin{bmatrix} 2\cos t & 3y^2 \\ 1 & 2\cos t \end{bmatrix} \rightarrow \text{not symmetric, so}$$

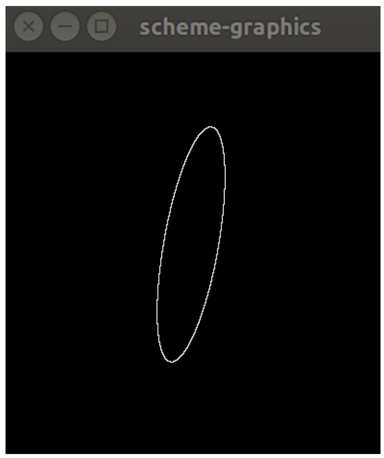
$G \neq D_t F(t,x)$

2. 1.30: Orbits in a central potential

Code can be found in ps3.scm

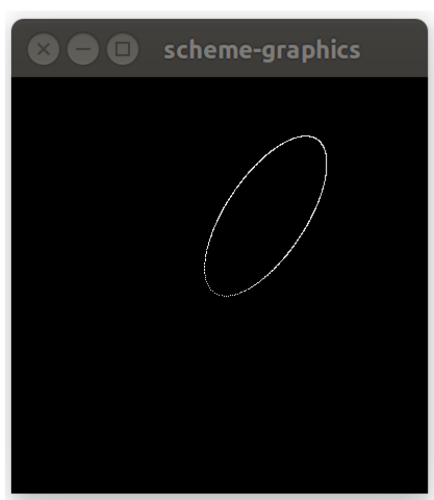
b.

```
((evolve central-potential-state-derivative 1.0 2.0 -1.0)
(up 0.0 (up 1.0 .0) (up 1.0 0.5))
(monitor plot-win)
0.01
10
1.0e-13)
```

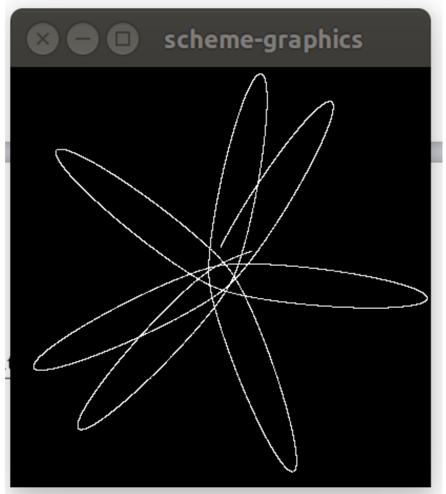


c.

```
((evolve central-potential-state-derivative 1.0 -1.0 1.0)
(up 0.0 (up 1.0 .0) (up 1.0 0.5))
(monitor plot-win)
0.01
10
1.0e-13)
```



d.
((evolve central-potential-state-derivative 1.0 1/4 -1.0)
(up 0.0 (up 1.0 .0) (up 1.0 0.5))
(monitor plot-win)
0.01
200
1.0e-13)



3. 1.36: Noether integral.

(Code can be found in ps3.scm)

The rectangular coordinates in term of a, b, θ, and ϕ are:
x $=$ asin$(\theta)\cos(\phi)$
$y = b\sin(\theta)\sin(\phi)$
$z = b\cos(\theta)$

The Lagrangian obtained is:

```
(+ (* 1/2 (expt a 2) m (expt phidot 2) (expt (sin phi) 2) (expt (sin theta) 2))
   (* -1 (expt a 2) m phidot thetadot (sin phi) (sin theta) (cos phi) (cos theta))
   (* 1/2 (expt a 2) m (expt thetadot 2) (expt (cos phi) 2) (expt (cos theta) 2))
   (* 1/2 (expt b 2) m (expt phidot 2) (expt (sin theta) 2) (expt (cos phi) 2))
   (* (expt b 2) m phidot thetadot (sin phi) (sin theta) (cos phi) (cos theta))
   (* -1/2 (expt b 2) m (expt thetadot 2) (expt (cos phi) 2) (expt (cos theta) 2))
   (* 1/2 (expt b 2) m (expt thetadot 2)))
```

$$\frac{1}{2}a^2m\dot{\phi}^2(\sin(\phi))^2(\sin(\theta))^2-a^2m\dot{\phi}\dot{\theta}\sin(\phi)\sin(\theta)\cos(\phi)\cos(\theta)+\frac{1}{2}a^2m\dot{\theta}^2(\cos(\phi))^2(\cos(\theta))^2+\frac{1}{2}b^2m\dot{\phi}^2(\sin(\theta))^2(\cos(\phi))^2+b^2m\dot{\phi}\dot{\theta}\sin(\phi)\sin(\theta)\cos(\phi)\cos(\theta)-\frac{1}{2}b^2m\dot{\theta}^2(\cos(\phi))^2(\cos(\theta))^2+\frac{1}{2}b^2m\dot{\theta}^2$$

We rotate around the x-axis with (Rx s):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(s) & -\sin(s) \\ 0 & \sin(s) & \cos(s) \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} x' \\ y'\cos(s) - z'\sin(s) \\ y'\sin(s) + z'\cos(s) \end{bmatrix}$$

such that: $\mathrm{x}^2/a^2 + y^2/b^2 + z^2/b^2 - 1 = x'^2/a^2 + y'^2/b^2 + z'^2/b^2 - 1$.

Below is the Noether integral, first in quadratic coordinates, then in angular ones.

$$my\dot{z} - m\dot{y}z$$

$$-b^2 m\dot{\phi}\sin(\theta)\cos(\phi)\cos(\theta) - b^2 m\dot{\theta}\sin(\phi)$$

4. Velocity-dependence continued

When we compute the time-independent Lagrangian in rotating coordinates, we see that it is the difference of the rotated Lagrangian we computed in pset2 with the most general potential energy we computed in part 2 of that same pset.

First the Lagrangian is shown, then the computed energy.

$$\frac{1}{2}\Omega^2 m {x_r}^2 + \frac{1}{2}\Omega^2 m {y_r}^2 + \Omega m x_r \dot{y}_r - \Omega m \dot{x}_r y_r + \frac{1}{2} m \dot{x}_r^2 + \frac{1}{2} m \dot{y}_r^2 - V_r\left(\begin{pmatrix} x_r \\ y_r \end{pmatrix}\right)$$

$$-\frac{1}{2}\Omega^2 m {x_r}^2 - \frac{1}{2}\Omega^2 m {y_r}^2 + \frac{1}{2} m \dot{x}_r^2 + \frac{1}{2} m \dot{y}_r^2 + V_r\left(\begin{pmatrix} x_r \\ y_r \end{pmatrix}\right)$$