MASSACHVSETTS INSTITVTE OF TECHNOLOGY
6.946J, 8.351J, 12.620J

Classical Mechanics: A Computational Approach

Problem Set 2—Fall 2020

Issued: 9 September 2020 Due: 18 September 2020

Reading: SICM2 Chapter 1 through section 1.7

Introduction

Hamilton's Principle gives us a prescription for constructing appropriate Lagrangians for systems of particles, and we have learned that the Lagrangians can be constructed for a system in any convenient coordinate system. A Lagrangian for a system of particles with rigid constraints can be constructed using a coordinate transformation that derives the redundant rectangular coordinates from irredundant generalized coordinates. This makes it easy! We find that there are many appropriate Lagrangians for any dynamical system. They differ by a "total time derivative." The Lagrange equations provide a way to compute the trajectory given an initial state of generalized coordinates and generalized velocities. We see that the trajectory of a driven pendulum is extremely sensitive to small variations of the initial state. This is chaotic behavior.

Exercises

- Exercise 1.16: Central force motion. SICM2 page 47
- Exercise 1.22: Driven pendulum. SICM2 page 59
- Exercise 1.29: Galilean invariance of kinetic energy. SICM2 page 68
- Exercise: Velocity-dependence

  Here we consider how the motion of a free particle is seen from a rotating frame of reference.

  A Lagrangian for a free particle is $L_{free}(t, x, v) = \frac{1}{2}mv \cdot v$:

```
(define ((Lfree m) s)
  (let ((t (time s))
        (q (coordinate s))
        (v (velocity s)))
    (* 1/2 m (dot-product v v))))
```

A coordinate transformation to a rotating frame of reference with angular velocity Ω is:

```
(define ((rotate Omega) s)
  (let ((t (time s))
        (q (coordinate s)))
    (let ((x (ref q 0))
          (y (ref q 1)))
      (up (- (* (cos (* Omega t)) x)
             (* (sin (* Omega t)) y))
          (+ (* (cos (* Omega t)) y)
             (* (sin (* Omega t)) x))))))
```

We can extend that coordinate transformation to a local-tuple transformation using `F->C`, to get a Lagrangian:

```
(define (Lrot m Omega)
  (compose (Lfree m)
           (F->C (rotate Omega))))
```

The Lagrange equations for this system are:

```
(((Lagrange-equations (Lrot 'm 'Omega))
  (up (literal-function 'xprime)
      (literal-function 'yprime)))
 't)
#|
(down
 (+ (* m (((expt D 2) xprime) t))
    (* -1 (expt Omega 2) m (xprime t))  ;Centrifugal force
    (* -2 Omega m ((D yprime) t)))      ;Coriolis force
 (+ (* m (((expt D 2) yprime) t))
    (* -1 (expt Omega 2) m (yprime t))  ;Centrifugal force
    (* 2 Omega m ((D xprime) t))))      ;Coriolis force
|#
```

We see that the apparent force on the particle includes centrifugal and Coriolis terms.

1. Compute the Lagrangian `(Lrot 'm 'Omega)` and say which terms result in the terms we see in the equations of motion. Notice that there are velocity-dependent terms.
2. Construct the most general (time-varying) two-dimensional potential energy function that is time-independent in the rotating coordinates.

   Make a new Lagrangian that incorporates this potential energy with the free Lagrangian. Compute the Lagrange equations and observe how the forces derived from this potential energy are rotated components of the spatial derivatives of V'.

Reminder

Project 1 (announced in Problem Set 0) will be due on Friday, 25 September 2020. So start thinking about it!

Heads up

Project 2 will be due on Friday, 9 October 2020. You will be working on the problem of the rotation of Mercury, Exercise 2.21 on page 193 of SICM2.