Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J - PSET 1

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1. 1.1: Degrees of Freedom

Answered together with 1.2 below.

2. 1.2: Generalized Coordinates

a. Three Juggling pins

18 degrees of freedom. 3 separate pins with 6 degrees of freedom for each.
Generalized Coordinates: (x_i, y_i, z_i, θ_i, ω_i, ϕ_i), i=0,1,2
(x, y, z) the coordinates of an arbitrary atom we choose in the pin, let's say the atom at the base of the pin.
(θ, ω, ϕ) the rotation angles with respect to an axis.

b. Spherical Pendulum

2 degrees of freedom. θ from the vertical plumb line, ϕ about the radius (arbitrary pick one as axis) of the circle of possible positions this angle from the vertical would form.
Generalized Coordinates: (θ, ϕ) the two rotation angles described above.

c. Spherical Double Pendulum

4 degrees of freedom. Same as above, but the one point mass acts as the fixed point for the second point mass.
Generalized Coordinates: (θ_0, ϕ_0, θ_1, ϕ_1)

d. Point Mass on Rigid Curved Wire

1 degree of freedom; the distance from the end of the wire.
Generalized Coordinates: d (where d is the distance from a fixed point at the end of the wire)

e. Axisymmetric Top

3 degrees of freedom. (it's a bit like fixing a juggling pin's tip and now only the orientation determines its position). θ from the axis of symmetry line (y-axis), ϕ from the x-z plane, ω as rotation around the body's axis of symmetry.
Generalized Coordinates: (θ, ϕ, ω)

f. Not Axisymmetric Top

3 degrees of freedom bc symmetry does not reduce the number of degrees of freedom. We can choose a rotation axis that goes through the fixed point in this case, and the angles would work as above, even if this axis doesn't divide the body in two symmetrical halves.
Generalized Coordinates: (θ, ϕ, ω), same as described in e.

3. 1.5: Solution Process

Nothing to hand in, only observe.

4. 1.8: Implementation of δ

See ps1.scm

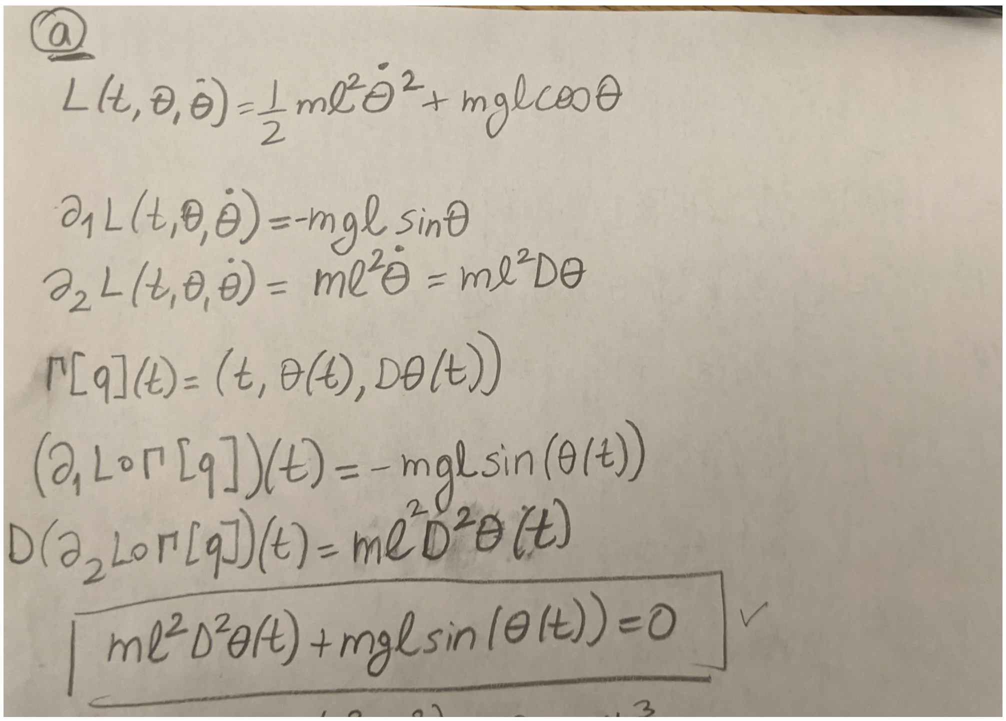
5. 1.9: Langrange's Equations

Answered along with 1.12

6. 1.12: Computing Langrange Equations

For the code see ps1.scm

First as worked out by hand, then the computed result.



(a)

$$L(t,\theta,\dot\theta) = \frac{1}{2} m l^2 \dot\theta^2 + m g l \cos\theta$$

$$\partial_1 L(t,\theta,\dot\theta) = -mgl\sin\theta$$

$$\partial_2 L(t,\theta,\dot\theta) = ml^2\dot\theta = ml^2 D\theta$$

$$\Gamma[q](t) = (t, \theta(t), D\theta(t))$$

$$(\partial_1 L \circ \Gamma[q])(t) = -mgl\sin(\theta(t))$$

$$D(\partial_2 L \circ \Gamma[q])(t) = ml^2 D^2\theta(t)$$

$$\boxed{ml^2 D^2\theta(t) + mgl\sin(\theta(t)) = 0}$$

$$glm\sin(\theta(t)) + l^2 m D^2\theta(t)$$

(b) $V(x,y) = \frac{(x^2+y^2)}{2} + x^2 y - \frac{y^3}{3}$

$$L(t; x, y; v_x, v_y) = \frac{1}{2} m (v_x^2 + v_y^2) - V(x,y)$$

$$\Gamma[q](t) = (t; x(t), y(t); Dx(t), Dy(t))$$

$$\partial_1 L(t; x, y; v_x, v_y) = [-x - 2xy, -y + y^2 - x^2]$$

$$\partial_2 L(t; x, y; v_x, v_y) = [m v_x, m v_y] \rightarrow [mDx, mDy]$$

$$(\partial_1 L \circ \Gamma[q])(t) = [-x(t) - 2x(t)y(t), -y(t) + (y(t))^2 - (x(t))^2]$$

$$D(\partial_2 L \circ \Gamma[q])(t) = [mD^2x(t), mD^2y(t)]$$

$$\boxed{\begin{aligned} mD^2x(t) + x(t) + 2x(t)y(t) &= 0 \\ mD^2y(t) + y(t) + (x(t))^2 - (y(t))^2 &= 0 \end{aligned}}$$

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$$\begin{bmatrix} mD^2x(t) + 2y(t)\,x(t) + x(t) \\ mD^2y(t) - (y(t))^2 + (x(t))^2 + y(t) \end{bmatrix}$$

(c)

$$L(t;\theta,\varphi;\alpha,\beta) = \frac{1}{2} m R^2 \left(\alpha^2 + (\beta \sin\theta)^2\right)$$

$$\Gamma[q](t) = (t;\theta(t),\varphi(t);D\theta(t),D\varphi(t))$$

$$\partial_1 L(t;\theta,\varphi;\alpha,\beta) = [mR^2\beta^2 \sin\theta\cos\theta,\ 0]$$

$$\partial_2 L(t;\theta,\varphi;\alpha,\beta) = [mR^2\alpha,\ mR^2\beta(\sin\theta)^2]$$

$$(\partial_1 L\circ\Gamma[q])(t) = [mR^2(D\varphi(t))^2\sin(\theta(t))\cos(\theta(t)),\ 0]$$

$$(\partial_2 L\circ\Gamma[q])(t) = [mR^2 D\theta(t),\ mR^2 D\varphi(t)(\sin(\theta(t)))^2]$$

$$D(\partial_2 L\circ\Gamma[q])(t) = [mR^2D^2\theta(t),\ mR^2D^2\varphi(t)(\sin\theta(t))^2 + 2mR^2 D\varphi(t)\sin(\theta(t))\cos(\theta(t))D\theta(t)]$$

$$\boxed{\begin{aligned} &mR^2D^2\theta(t) - mR^2(D\varphi(t))^2\sin(\theta(t))\cos(\theta(t)) = 0 \\ &mR^2D^2\varphi(t)(\sin(\theta(t)))^2 + 2mR^3D\varphi(t)D\theta(t)\sin(\theta(t))\cos(\theta(t)) = 0 \end{aligned}}$$

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$$\begin{bmatrix} -R^2 m \sin(\theta(t))\cos(\theta(t))(D\varphi(t))^2 + R^2 m D^2\theta(t) \\ 2.R^2 m \sin(\theta(t)) D\theta(t)\cos(\theta(t)) D\varphi(t) + R^2 m D^2\varphi(t)(\sin(\theta(t)))^2 \end{bmatrix}$$