Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J PSET 7

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Code can be found in ps7.scm

1. Exercise: An effective Lagrangian

1. If we were to simplify the terms, we'd get the effective Lagrangian of Chapter 2. Technically we get (minus L-effective), but it's the same equation with a flipped sign.

$$\frac{-\frac{1}{2}A^2C\dot{\theta}^2(\sin(\theta))^2 + AC\cdot gMR\cdot(\sin(\theta))^2\cos(\theta) + \frac{1}{2}A{p_\psi}^2(\sin(\theta))^2 + \frac{1}{2}C{p_\psi}^2(\cos(\theta))^2 - Cp_\phi p_\psi\cos(\theta) + \frac{1}{2}C{p_\phi}^2}{AC(\sin(\theta))^2}$$

```
(/
 (+ (* -1/2 (expt A 2) C (expt thetadot 2) (expt (sin theta) 2))
    (* A C gMR (expt (sin theta) 2) (cos theta))
    (* 1/2 A (expt p_psi 2) (expt (sin theta) 2))
    (* 1/2 C (expt p_psi 2) (expt (cos theta) 2))
    (* -1 C p_phi p_psi (cos theta))
    (* 1/2 C (expt p_phi 2)))
 (* A C (expt (sin theta) 2)))
```

2. As expected the first equation is Lagrange's equations for θ, which is also L-effective from part 1, followed by the Hamiltonian equations for (ϕ, ψ) and (pϕ, pψ). The derivatives of the momenta are equal to zero, hence the momenta are conserved.

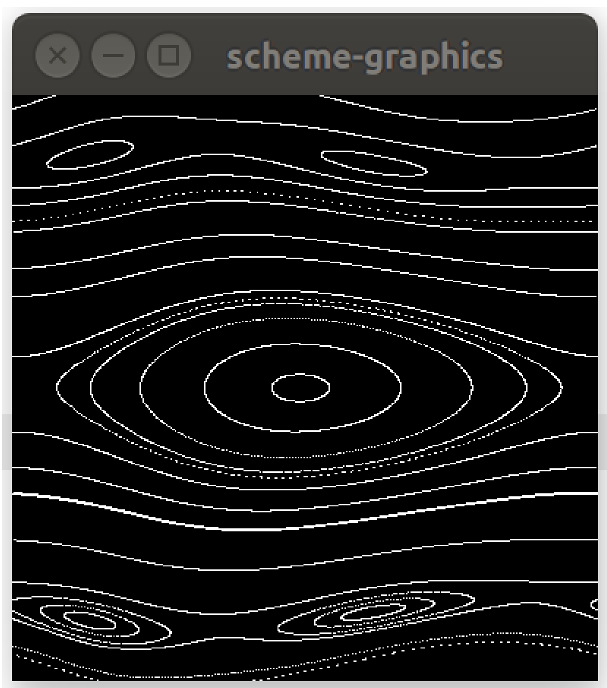
$$\begin{pmatrix} \frac{-A^2(\sin(\theta(t)))^3D^2\theta(t) + A\cdot gMR\cdot(\cos(\theta(t)))^4 - 2A\cdot gMR\cdot(\cos(\theta(t)))^2 - (\cos(\theta(t)))^2p_\psi(t)\,p_\phi(t) + \cos(\theta(t))\,(p_\psi(t))^2 + \cos(\theta(t))\,(p_\phi(t))^2 + A\cdot gMR - p_\psi(t)\,p_\phi(t)}{A(\sin(\theta(t)))^3} \\ \begin{pmatrix} 0 \\ \begin{pmatrix} \frac{A(\sin(\theta(t)))^2D\phi(t) + \cos(\theta(t))\,p_\psi(t) - p_\phi(t)}{A(\sin(\theta(t)))^2} \\ \frac{AC(\sin(\theta(t)))^2D\psi(t) - A(\sin(\theta(t)))^2p_\psi(t) - C(\cos(\theta(t)))^2p_\psi(t) + C\cos(\theta(t))\,p_\phi(t)}{AC(\sin(\theta(t)))^2} \end{pmatrix} \\ \begin{bmatrix} Dp_\phi(t) \\ Dp_\psi(t) \end{bmatrix} \end{pmatrix} \end{pmatrix}$$

```
(up
 (/
  (+ (* -1 (expt A 2) (expt (sin (theta t)) 3) (((expt D 2) theta) t))
     (* A gMR (expt (cos (theta t)) 4))
     (* -2 A gMR (expt (cos (theta t)) 2))
     (* -1 (expt (cos (theta t)) 2) (p_psi t) (p_phi t))
     (* (cos (theta t)) (expt (p_psi t) 2))
     (* (cos (theta t)) (expt (p_phi t) 2))
     (* A gMR)
     (* -1 (p_psi t) (p_phi t)))
  (* A (expt (sin (theta t)) 3)))
 (up
  0
  (up
   (/ (+ (* A (expt (sin (theta t)) 2) ((D phi) t)) (* (cos (theta t)) (p_psi t)) (* -1 (p_phi t)))
      (* A (expt (sin (theta t)) 2)))
   (/
    (+ (* A C (expt (sin (theta t)) 2) ((D psi) t))
       (* -1 A (expt (sin (theta t)) 2) (p_psi t))
       (* -1 C (expt (cos (theta t)) 2) (p_psi t))
       (* C (cos (theta t)) (p_phi t)))
    (* A C (expt (sin (theta t)) 2))))
  (down ((D p_phi) t) ((D p_psi) t))))
```

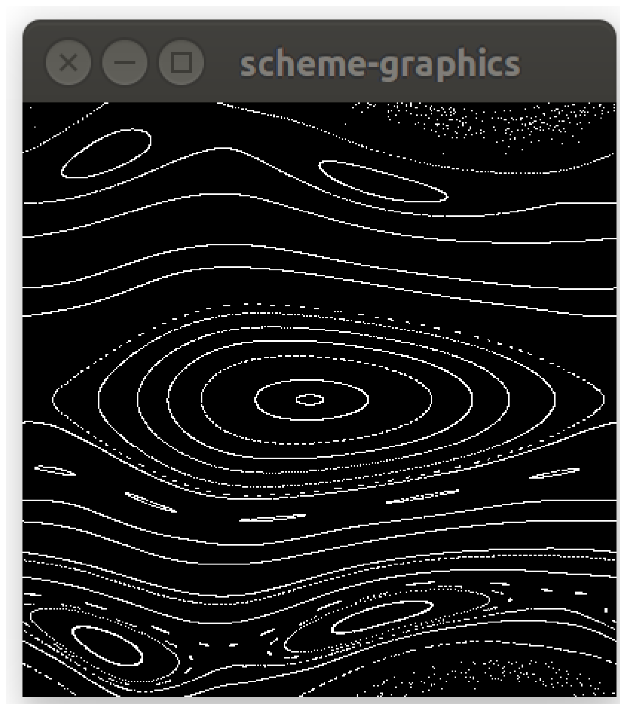
2. Exercise: Horizontally-driven pendulum

Varying the parameters varies the structure of the surfaces of section. I investigated to see if I could produce the equivalent of the period-doubled central island we saw for the vertically-driven case.

I start with A=0.05, ω=10ω_0, and slowly change the values of A, maintaining ω constant. The systems seems to be getting more chaotic over time, with smaller and smaller islands of regularity.

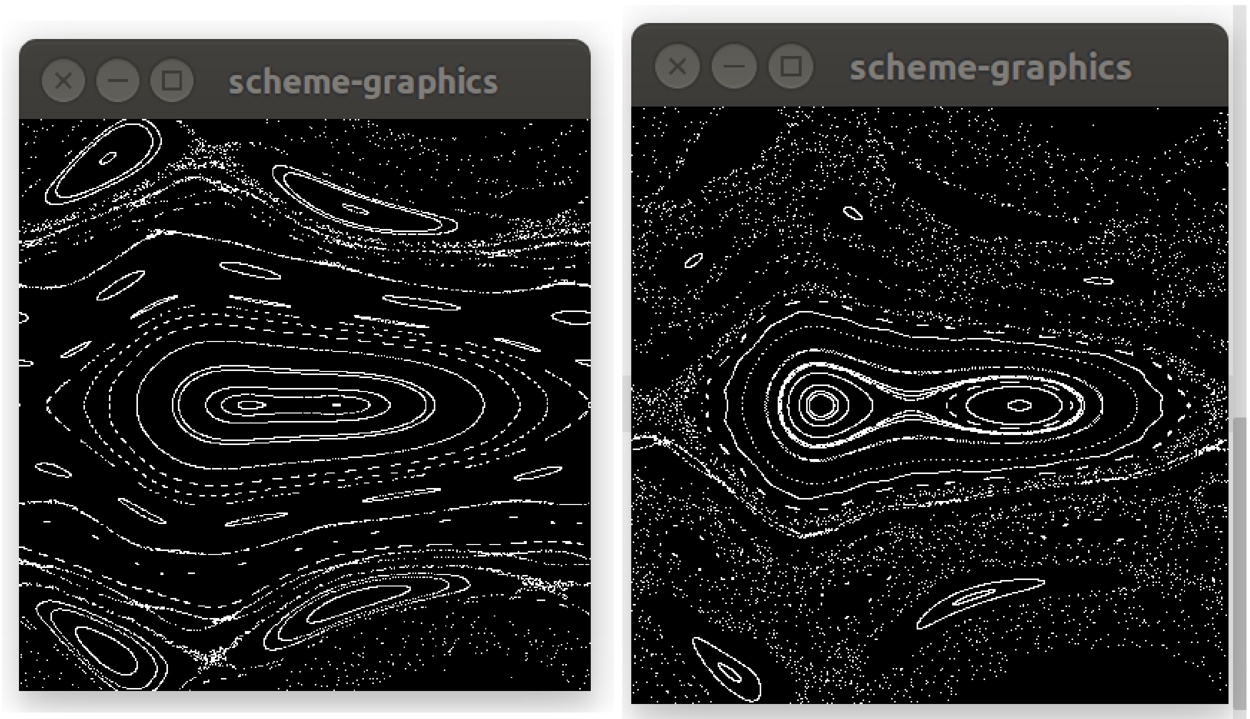


(a) A=0.05, ω=10ω_0



(b) A=0.1, ω=10ω_0

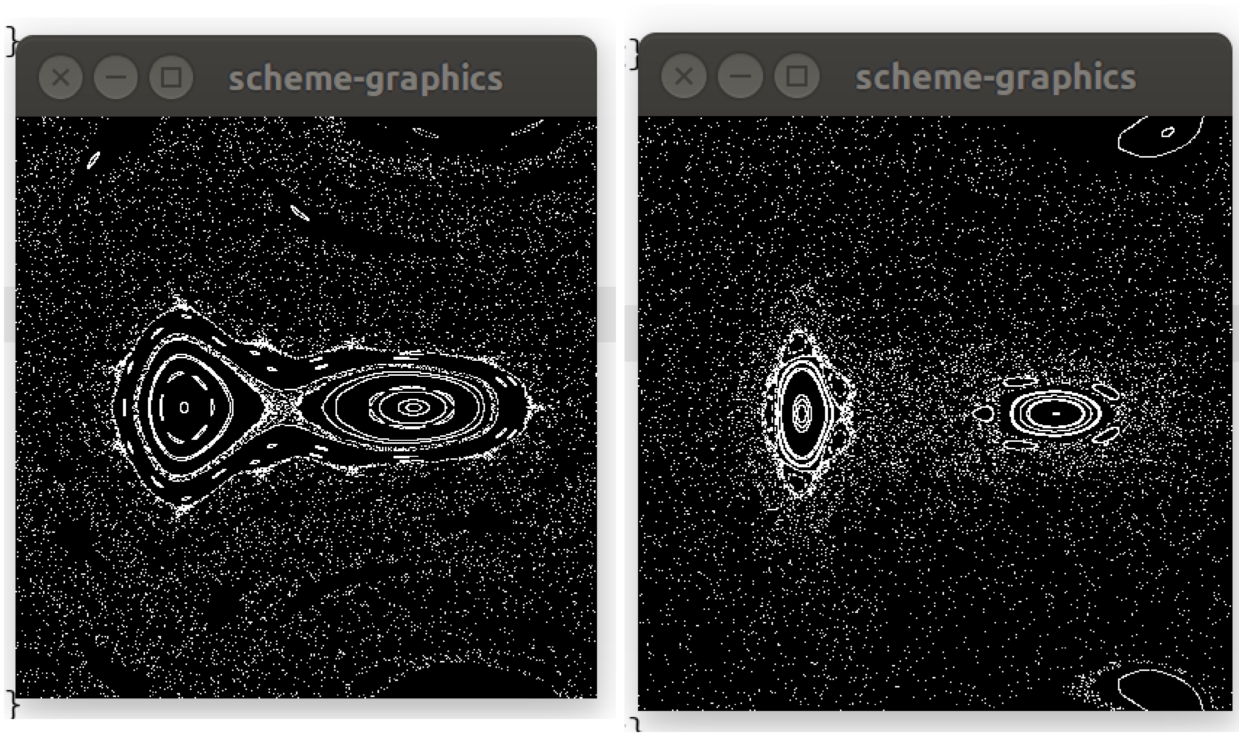
Figure 1: In the center we see the remnant of the oscillating region of the undriven pendulum, as well as the circulating trajectories towards the corners. However, in between circulating trajectories there seem to be minor islands of stability as well. As A is increased, we see the "ellipse" in the middle begin to stretch a little, more minor stability islands appear, and even chaotic regions like those in the top-right and top bottom corners.



(a) A=.15, ω=10ω_0

(b) A=0.2, ω=10ω_0

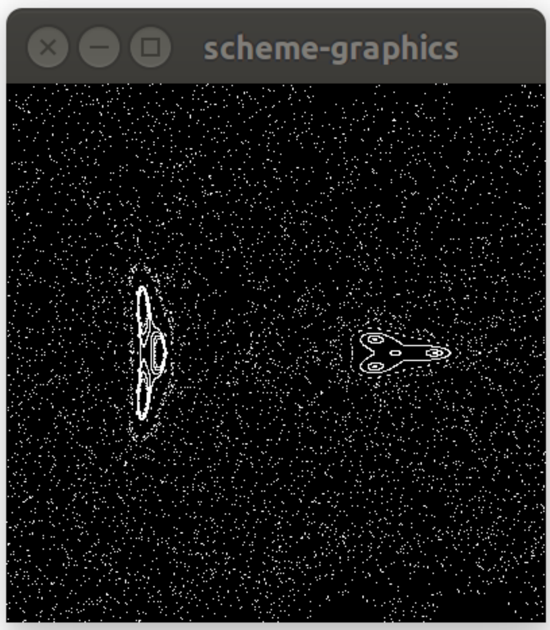
Figure 2: On the left, the ovoid is beginning to separate into two "major" stability centers. The corner chaotic regions are advancing inward. On the right, we can clearly see the bifurcation of two clear stability islands that can be alternately visited, as the support goes left and right. What's interesting here is that the islands are of different sizes, whereas in the case of the vertically-driven pendulum they had similar sizes. On both pictures, the dotted curves possibly denote very small islands of stability that show up like dots in this plot, but would look as curves if we blew up the image. Or they may as well be miniature pockets of chaos.



(a) A=0.25, ω=10ω_0

(b) A=0.3, ω=10ω_0

Figure 3: Left: This picture is even more similar to the period-doubled central island with a small chaotic region connecting in between. We can clearly see the two major island of stability have different sizes. Right: There are now two separate stability islands, with a big gap of chaos in between. These trajectories no longer go through θ=0. Another interesting thing is the small stability curves surrounding the major stability centers. On both images we see the chaotic sea gets larger, there are almost no minor islands of stability anymore.



(a) A=0.4, ω=10ω_0

Figure 4: For fun, it seems the two stability islands are now dividing into mini-islands. Their shapes are also very much deformed from what they once were. It's also interesting how one seems to be vertically squished version of the other.