Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J Project 3

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1. Exercise 3.14: Periodically driven pendulum

2. (A) Determination of the Inverted Equlibrium Parameter Space

Inverted equilibrium around θ=+/ − π, or the points (+/ − π, 0).

We need to determine a test whether a given set of parameters (A, ω) exhibits the inverted pendulum behavior. From the surface of section, we know that around the equilibrium point, say θ=π, the momentum p_θ is a small positive number, on the order of .1 or .01. (Unless it is zero, but it's hard to get it at exactly zero,and we're interested in nearby displacements anyway.) We can set one such value as the initial value for p_0, with initial point (π, p_0).

Any point in the range of stability should have a momentum approximately close to the value of p_0; if an angle's corresponding p_θ is significantly higher (say above 1.1p_0, to allow for rounding error) than the absolute momentum of the initial point, this indicates that the trajectory has left the stability area. (This test does not however guarantee the lack of false positives; although unlikely, there could be a chaotic section "hiding" within the small region of -1.1p_0 $\leq$ p_θ $\leq$1.1p_0.)

Now we just need to generate a large number of points around the initial point, and use the above test to sift our way through various pairs of (A, ω). Results shown in figure 1.

3. (B) Investigation of Period Doubling

Set A=.4. By trial, we find that the range 1.3$\leq$ ω $\leq$2.1 is likely to exhibit the period doubling phenomenon, for this particular value of A.

Because of the shape of the trajectory, when it comes to the momentum value around the equilibrium points, we can make a good guess by averaging the highest and lowest momenta the top half of the trajectory achieves. We have to be mindful here that we don't stray too much from the "center" of the connected twin islands here or else we end up in a sea of chaos.

Results for stable equilibrium point shown in figure 2; while those for unstable equilibrium are shown in figure 3.

4. (C) Transition to Large-scale Chaos

I approached this problem by figuring out the opposite problem: that of the presence of three large and distinct chaotic zones. A trajectory starting inside any of the zones will not leave and enter the other two, as they are usually separated by invariant curves.

Trying out some values of (A, ω), we see that separated chaotic zones occurs for low values of A and high values of ω. We also notice that the central chaotic zone also doesn't seem to range higher than p about 250 (conservatively) or so. So a very crude test, would be to let the system run for a large number of iterations, and see if the range of momentum values exceeds some given range, which, in that case, would indicate transition to large-scale chaos. Results in figure 4.

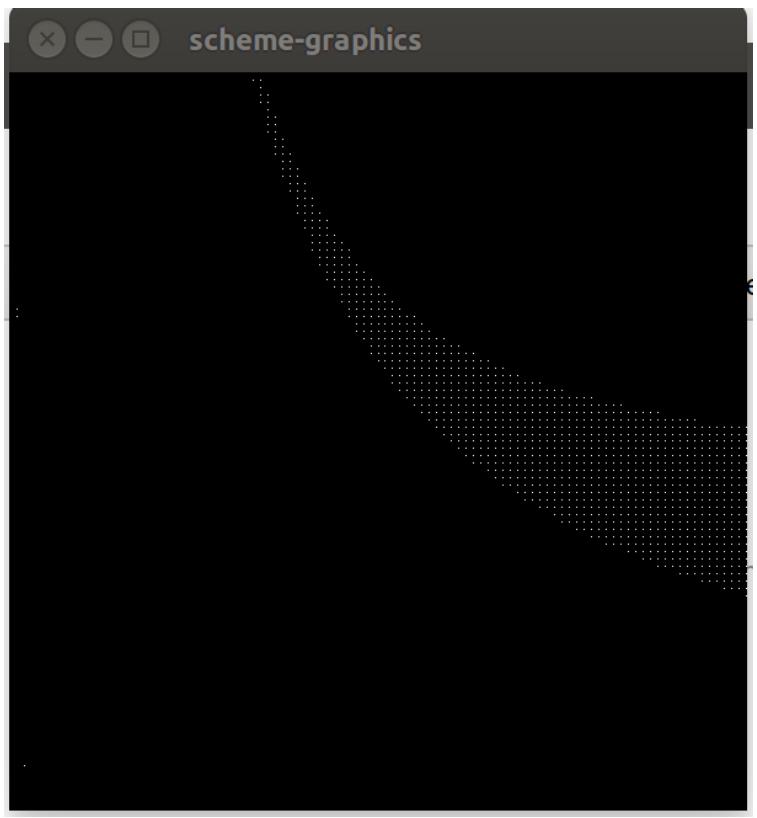

Figure 1: Region of parameter space for which an inverted equilibrium exists. The abscissa is $0 \leq \omega \leq 5$, spaced .05 apart. The ordinate is $0 \leq A \leq 1$, spaced .01 apart.

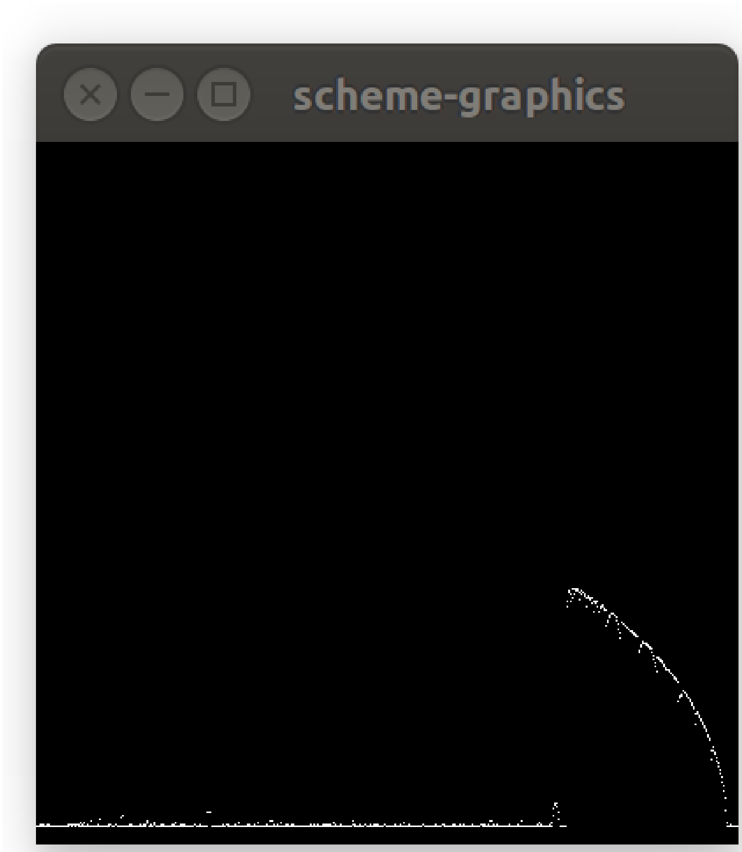

Figure 2: Plot of the momentum of the stable equilibrium $p(\omega)$ for A=.4. The ordinate ranges from 0≤p≤200, and the abscissa from 1.3≤ ω ≤2.1. This provides graphical evidence that as ω increases, the period doubled region emerges suddenly, then slowly drops back into the center. (A thing I could have done is averaging the value of the momentum for a given ω in multiple iterations, thus getting a smoother curve.)

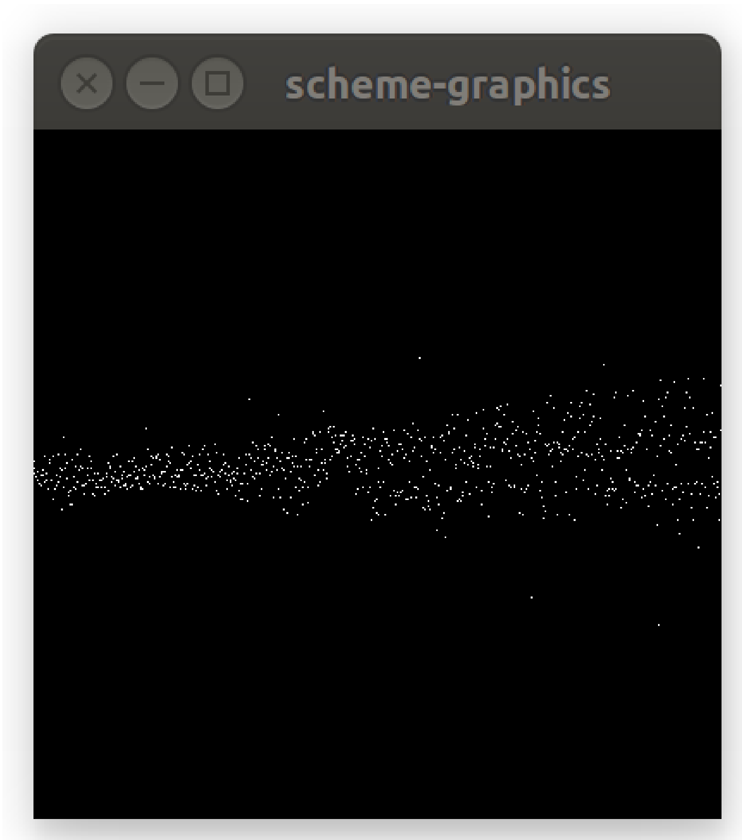

Figure 3: Plot of the momentum of the unstable equilibrium $p(\omega)$ for A=.4. The ordinate ranges from 0≤p≤200, and the abscissa from 1.3≤ ω ≤2.1. Unlike the graph of the momentum around the stable point, here no such period-doubling graph is observed. What seems to be happening is the momentum rising roughly linearly with the driving frequency.

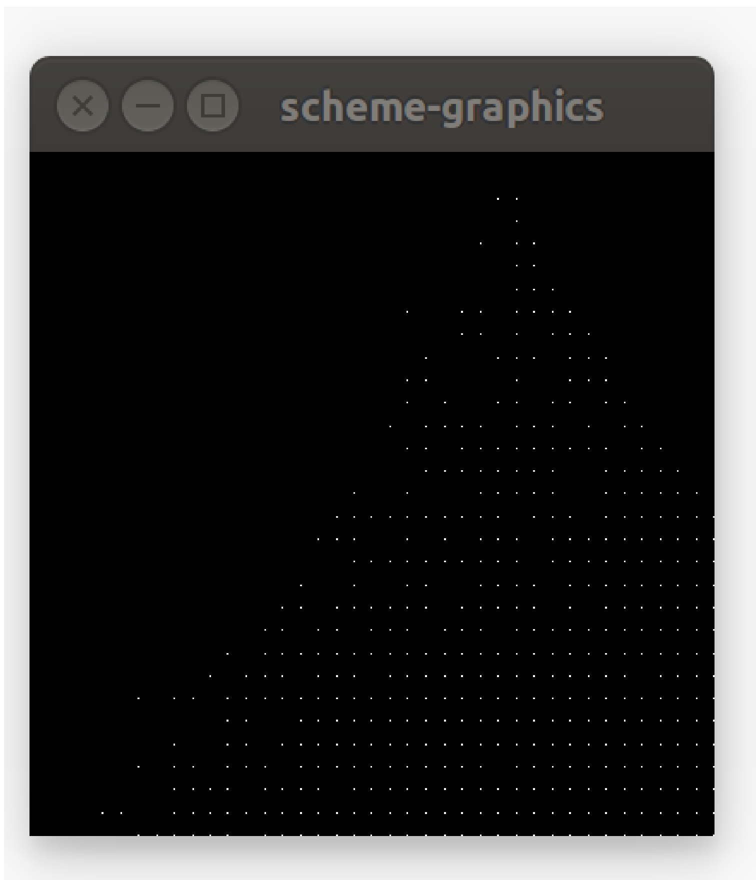

Figure 4: Region of parameter space for which the principal chaotic regions are linked: dotted. The region above the dotted plot is where the three chaotic regions are possibly separated by invariant curves. The ordinate is .1≤A≤2, and the abscissa is 4≤ ω ≤10.