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Classical Mechanics: A Computational Approach

Problem Set 10—Fall 2020

Issued: 4 November 2020 Due: 13 November 2020

Note: Project 4 is also due on Friday, 20 November 2020 Note: Project 5 announcement is at the end of this document.

Reading: SICM2 Chapter 4 through end; Chapter 5 through section 5.3

Introduction

Among structures in phase space, we observe that there are invariant curves that form barriers to motion. A trajectory can live on one side of an invariant curve and never appear on the other side. However, as the coupling of a nonlinear system increases, the invariant curves are destroyed. The last curves to be destroyed are the "most irrational" in that they have continued fractions that have a tail of all ones and thus converge very slowly.

We want to be able to predict the appearance and size of the features we observe in phase space from features of the Hamiltonian. This will require substantial machinery, including "Canonical Transformations" that mix coordinates and momenta but preserve Hamilton's equations. Ordinary coordinate transformations that we can use to transform Lagrangians can be extended to "point transformations" by inheriting the momenta that we get from the Lagrangian.

Exercises

- Exercise: Truncation is dangerous!

  The Toda problem is a famous problem: We imagine three particles on a circle that interact with a potential energy that is exponential in the angle between them. This problem has enough constants of the motion to reduce its solution to quadratures. Consider the Hamiltonian

  $$H_{m,k}(t; q^1, q^2, q^3; p_1, p_2, p_3) = \frac{p_1^2 + p_2^2 + p_3^2}{2m} + k(e^{q^1-q^2} + e^{q^2-q^3} + e^{q^3-q^1})$$

  with $m = 1/8$ and $k = 1/24$.

```
(define (HToda s)
   (let ((q (coordinate s)) (p (momentum s)))
     (let ((q1 (ref q 0)) (q2 (ref q 1)) (q3 (ref q 2))
           (p1 (ref p 0)) (p2 (ref p 1)) (p3 (ref p 2)))
       (+ (* 4 (+ (square p1) (square p2) (square p3)))
         (* 1/24 (+ (exp (- q1 q2))
                    (exp (- q2 q3))
                    (exp (- q3 q1))))))))
```

The fact that the coordinates all enter as differences suggests that the center of mass can be eliminated and that the total momentum is conserved. Thus

$$I_2(t; q^1, q^2, q^3; p_1, p_2, p_3) = p_1 + p_2 + p_3$$

is a conserved quantity.

```
(define (I2 s)
   (let ((p (momentum s)))
     (let ((p1 (ref p 0))
           (p2 (ref p 1))
           (p3 (ref p 2)))
       (+ p1 p2 p3))))
```

It was dicovered numerically by computing surfaces of section that there appeared to be another conserved quantity. This conserved quantity was discovered by Hénon. For our problem it is

$$I_3(t; q^1, q^2, q^3; p_1, p_2, p_3) = r_3 e^{q^1-q^2} + r_1 e^{q^2-q^3} + r_2 e^{q^3-q^1} - 3((4r_1/3)^3 + (4r_2/3)^3 + (4r_3/3)^3),$$

where $r_1 = 2p_1 - p_2 - p_3$, $r_2 = 2p_2 - p_1 - p_3$, and $r_3 = 2p_3 - p_1 - p_2$.

```
(define (I3 s)
   (let ((q (coordinate s))
         (p (momentum s)))
     (let ((q1 (ref q 0))
           (q2 (ref q 1))
           (q3 (ref q 2))
           (p1 (ref p 0))
           (p2 (ref p 1))
           (p3 (ref p 2)))
       (- (+ (* (- (* 2 p2) (+ p1 p3)) (exp (* (- q3 q1))))
             (* (- (* 2 p3) (+ p1 p2)) (exp (* (- q1 q2))))
             (* (- (* 2 p1) (+ p2 p3)) (exp (* (- q2 q3)))))
         (* 3 (+ (cube (* 4/3 (- (* 2 p2) (+ p1 p3))))
                 (cube (* 4/3 (- (* 2 p3) (+ p1 p2))))
                 (cube (* 4/3 (- (* 2 p1) (+ p2 p3))))))))))
```

– Show that I_2 and I_3 are conserved quantities.

We can make a canonical transformation to eliminate the center of mass. The coordinate part of this transformation is

$$\begin{aligned}
q^1 &= \frac{2}{3}\sqrt{2}\sqrt{3}x_{cm} + \frac{2}{3}\sqrt{3}x + 2y \\
q^2 &= \frac{2}{3}\sqrt{2}\sqrt{3}x_{cm} - \frac{4}{3}\sqrt{3}x \\
q^3 &= \frac{2}{3}\sqrt{2}\sqrt{3}x_{cm} + \frac{2}{3}\sqrt{3}x - 2y.
\end{aligned}$$

```
(define (F s)
   (let ((q (coordinate s)))
     (let ((x (ref q 0)) (y (ref q 1)) (cm (ref q 2)))
       (up (+ (* (* 2/3 'root-3 'root-2) cm)
              (* 2/3 'root-3 x)
              (* 2 y))
           (+ (* (* 2/3 'root-3 'root-2) cm)
              (* -4/3 'root-3 x))
           (+ (* (* 2/3 'root-3 'root-2) cm)
              (* 2/3 'root-3 x)
              (* -2 y))))))
```

The constants `root-2` and `root-3` are introduced to avoid using `(sqrt 2)` and `(sqrt 3)`, which would turn into inexact numbers. This is a dumb property of the algebra we use. Luckily, the constants are unimportant in the computation.

- Extend F to a canonical transformation, and find the transformed Hamiltonian H' and the transformed conserved quantity I_3'. Check that the transformed I_3' is still a conserved quantity.

- The transformed Hamiltonian contains a trivial center of mass degree of freedom. Write a new two degree of freedom Hamiltonian H'' in x and y that eliminates the center of mass freedom. Similarly, write the conserved quantity I_3'' for this two degree of freedom problem. Check that the I_3'' is still a conserved quantity. Write out the final Hamiltonian H''.

- Expand the Hamiltonian H'' to third order in x and y. Do you recognize the Hamiltonian? We know that this Hamiltonian exhibits the divided phase space, with chaotic zones and regular regions, yet the problem we started with had enough conserved quantities to be solvable.

Project 5

Project 5 will be due on 4 December 2020. There will also be problem set 11 due on that date, but it will be shorter. Project 5 is the last project this term.
Project 5 will be about symplectic integration, a numerical technique that approximates the trajectory of a system by interleaving the exact trajectories of two solvable parts.

- Exercise 6.12: Symplectic Integration SICM2 page 453