Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J PSET 8

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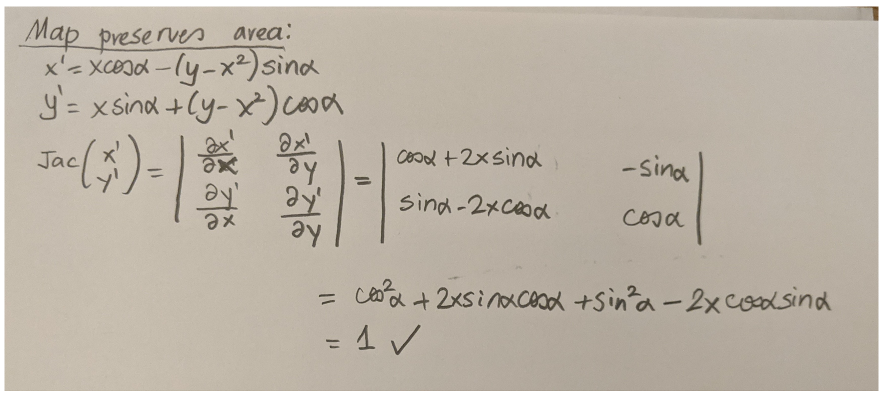
1. 3.13: Fun with Henon's quadratic map.



Map preserves area:

$x' = x\cos\alpha - (y - x^2)\sin\alpha$

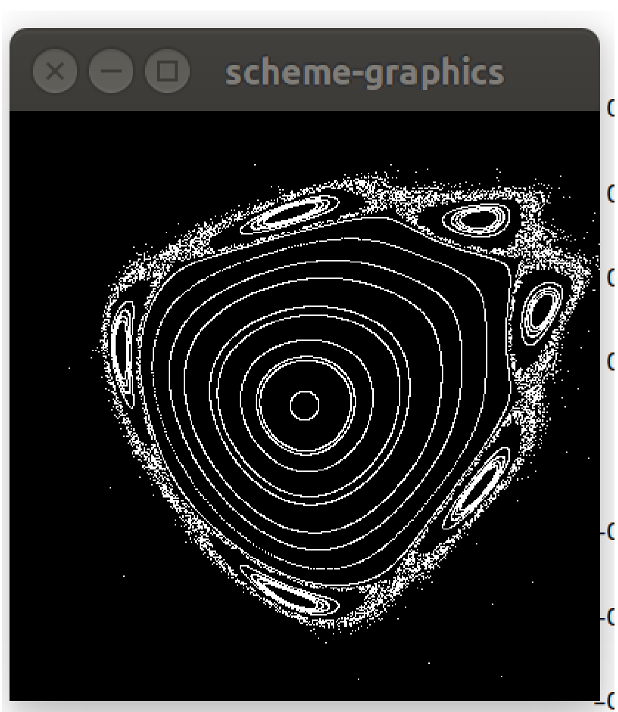
$y' = x\sin\alpha + (y - x^2)\cos\alpha$

$$\mathrm{Jac}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{vmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\alpha + 2x\sin\alpha & -\sin\alpha \\ \sin\alpha - 2x\cos\alpha & \cos\alpha \end{vmatrix}$$

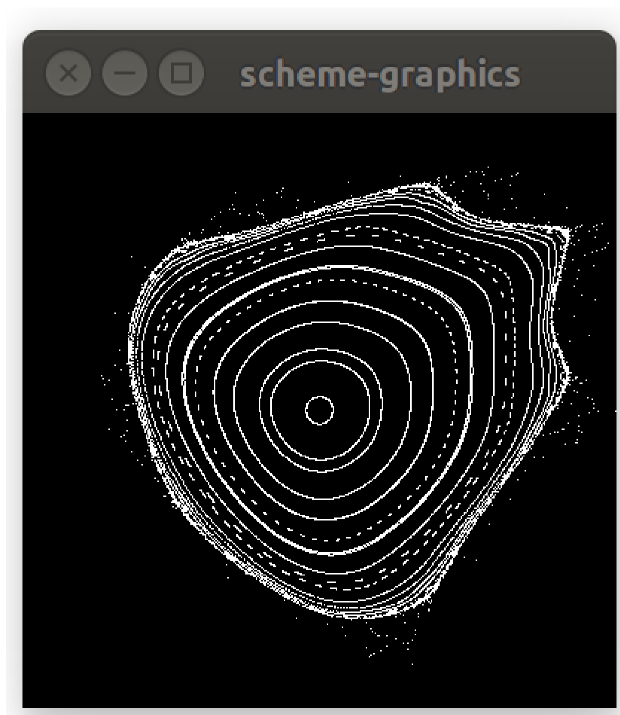
$$= \cos^2\alpha + 2x\sin\alpha\cos\alpha + \sin^2\alpha - 2x\cos\alpha\sin\alpha$$

$$= 1 \checkmark$$

Values tried for α: 1.2, 1.25, 1.29, 1.32, 1.35. As α evolves, islands of stability form and move outwards until they escape the main region of stability to disappear entirely.



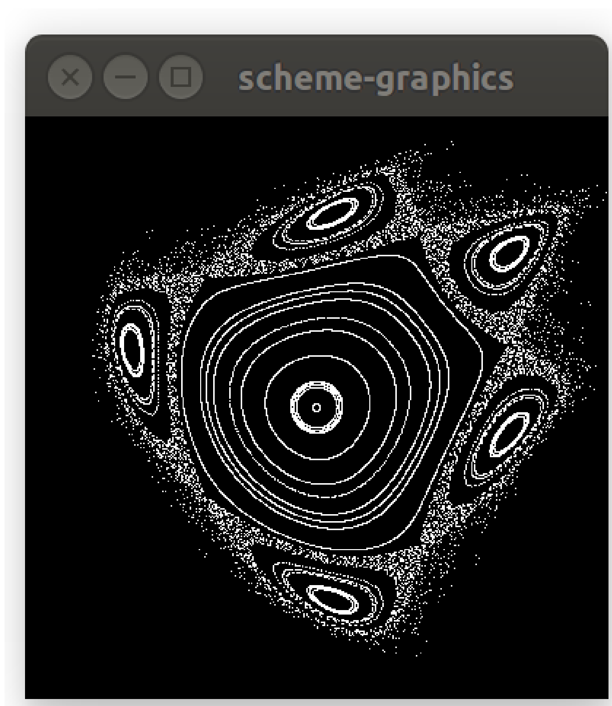
(a) α=1.2



(b) α=1.25

(a) α=1.29

(b) α=1.32



(a) α=1.35

2. 4.3: Standard map eigenvalues and eigenvectors.

$I' = (I + K\sin\theta) \bmod 2\pi$

$\theta' = (\theta + I') \bmod 2\pi$

$$DT(I,\theta) = \begin{bmatrix} 1 & K\cos\theta \\ 1 & 1+K\cos\theta \end{bmatrix}$$

$$M = DT(0,0) = \begin{bmatrix} 1 & K \\ 1 & 1+K \end{bmatrix}$$

$$\det(M-\rho I) = 0 = (1-\rho)(1+K-\rho) - K = \rho^2 - \rho(K+2) + 1$$

$$\rho = \frac{1}{2}\left(K+2 \pm \sqrt{K^2+4K}\right)$$

- $K>0 \rightarrow$ real ρs

$\sqrt{K^2+4K} > 0$, look @ largest eigenvalue only

$\frac{1}{2}\left|K+2+\sqrt{K^2+4K}\right| > \frac{1}{2}|K+2| > \frac{2}{2} = 1$

(unstable)

- $-4 < K < 0$

$K^2+4K < 0 \rightarrow$ complex ρs

- complex part affects shape only (spiral)

look @ real part $(K+2)$

$-\frac{2}{2} < \frac{1}{2}(K+2) < \frac{2}{2} \Rightarrow \left|\frac{1}{2}(K+2)\right| < 1$

(stable)

- $K < -4$

$K^2+4K > 0$, real ρs

$\frac{1}{2}\left|K+2-\sqrt{K^2+4K}\right| > \frac{1}{2}|K+2| > \frac{1}{2}|-2| = 1$

(unstable)

$$M = DT(0,\pi) = \begin{bmatrix} 1 & -K \\ 1 & 1-K \end{bmatrix}$$

$$\det(M-\rho I) = 0 = (1-\rho)(1-K-\rho) + K = \rho^2 - \rho(K-2) + 1$$

$$\rho = \frac{1}{2}\left(K-2 \pm \sqrt{K^2-4K}\right)$$

- $K>4$

$K^2-4K>0$, real ρs

$\frac{1}{2}\left|K-2+\sqrt{K^2-4K}\right| > \frac{1}{2}|K-2| > \frac{1}{2}|4-2| = 1$

(unstable)

- $0<K<4$

$K^2-4K<0$, complex ρs

$-\frac{2}{2} < \frac{1}{2}(K-2) < \frac{2}{2} \Rightarrow \frac{1}{2}|K-2| < 1$

(stable)

- $K<0$

$K^2-4K>0$, real ρs

$\frac{1}{2}\left|K-2-\sqrt{K^2-4K}\right| > \frac{1}{2}|K-2| > \frac{1}{2}|-2| = 1$

(unstable)

3. More fun with standard map.