MASSACHVSETTS INSTITVTE OF TECHNOLOGY
6.946J, 8.351J, 12.620J

Classical Mechanics: A Computational Approach

Problem Set 6—Fall 2020

Issued: 7 October 2020 Due: 16 October 2020

Note: Project 3 is due on Friday, 30 October 2020

Reading: SICM2 Chapter 3 through section 3.2

Introduction

We have been using the Lagrangian formulation of mechanics in which the equations of motion are expressed in terms of coordinates and velocities. Continuous symmetries correspond to conserved quantities. By concentrating our attention on the potentially conserved quantities, energy and momentum, we are led to the Hamiltonian formulation. We obtain Hamilton's canonical equations that give us the rates of change of coordinates and momenta in terms of partial derivatives of the Hamiltonian function, which is the energy, written as a function of coordinates and momenta. The Lagrangian formulation is equivalent to the Hamiltonian formulation: each can be obtained from the other by a Legendre transformation, if it is not singular.

Exercises

- Exercise 3.1: Deriving Hamilton's equations. SICM2 page 201
- Exercise 3.3: Computing Hamilton's equations. SICM2 page 205
- Exercise 3.4: Simple Legendre transforms, parts a and c. SICM2 page 209
- Exercise 3.5: Conservation of the Hamiltonian. SICM2 page 211
- Exercise: Hamiltonian formulation of particles in an electromagnetic field.

  Note: This problem requires loading some definitions of the vector calculus operators `Grad, Curl, Div`, and `Lap`. These are online in the `code` subdirectory of the problem set. Download the file `"code/3vector-operators.scm"` to your directory and then execute `(load "3vector-operators")`

  One way of formulating electromagnetic fields is by specifying a scalar potential (for the electric component) and a vector potential (for the magnetic component). For static fields the scalar potential ϕ is a real-valued function on the position and the vector potential A is a vector-valued function of position, here represented by a tuple of three component functions:

```scheme
(define A
  (literal-function 'A
    (-> (UP Real Real Real) (UP Real Real Real))))

(define phi
  (literal-function 'phi
    (-> (UP Real Real Real) Real)))
```

In this formulation a Lagrangian for the motion of a particle of mass m and charge q, with speed of light c is:

```scheme
(define ((L-em c m q) s)
  (let ((xyz (coordinate s))
        (xyzdot (velocity s)))
    (- (* 1/2 m (square xyzdot))
       (* q
          (- (phi xyz)
             (dot-product (/ xyzdot c)
                          (A xyz)))))))
```

1. Compute the Lagrange equations for this Lagrangian. Identify the two forces acting on the particle: the electric force and the magnetic force. Observe that the magnetic force is four terms involving spatial derivatives of the vector potential.
2. The electric field $E = -\mathrm{Grad}\phi$. The magnetic field $B = \mathrm{Curl}A$. Show, by adding the Lorentz force $q(E + (v/c) \times B)$ to the Lagrange equations, that the Lagrange equations correctly implement the Lorentz force law.
3. Now convert to a Hamiltonian formulation. We compute:

   ```scheme
   (define (Raise d)
     (up (ref d 0) (ref d 1) (ref d 2)))

   (let ((s (up 't (up 'x 'y 'z) (down 'p_x 'p_y 'p_z))))
     (let ((xyz (coordinates s))
           (pxyz (momenta s)))
       (- ((Lagrangian->Hamiltonian (L-em 'c 'm 'q)) s)
          (+ (/ (square (- (Raise pxyz)
                           (* (/ 'q 'c) (A xyz))))
                (* 2 'm))
             (* 'q (phi xyz))))))
   #| 0 |#
   ```

   Why is this zero? The kinetic energy term of the Hamiltonian is usually $p^2/2m$. But here the momentum `pxyz` must be modified by subtracting $(q/c)A$ to get this answer. Interpret this.