Structure and Interpretation of Classical Mechanics
6.946J/8.351J/12.620J PSET 10

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1. Truncation is Dangerous!

- Show that I2 and I3 are conserved quantities.

If I2 and I3 are conversed by the evolution under HToda, then {{I2, I3}, HToda} = 0.

- Extend F to a canonical transformation, and find the transformed Hamiltonian H' and the transformed conserved quantity I3'. Check that the transformed I3' is still a conserved quantity.

{I3', HToda'} = 0, so I3' is still conserved.

```
(/
 (+ (* 1/24 (expt root-2 2) (expt root-3 2) (expt (exp y) 2) (expt (exp (* root-3 x)) 2))
    (* 1/24 (expt root-2 2) (expt root-3 2) (expt (exp y) 2) (expt (exp (* -1 root-3 x)) 2))
    (* 1/24 (expt root-2 2) (expt root-3 2) (expt (exp (* -1 y)) 4))
    (* 1/2 (expt p_y 2) (expt root-2 2) (expt root-3 2))
    (* 3/2 (expt p_x 2) (expt root-2 2))
    (* 3 (expt p_cm 2)))
 (* (expt root-2 2) (expt root-3 2)))
```

Figure 1: HToda-prime.

```
(/
 (+ (* -3/4 p_y (expt root-3 3) (expt (exp y) 2) (expt (exp (* root-3 x)) 2))
    (* 3/4 p_y (expt root-3 3) (expt (exp y) 2) (expt (exp (* -1 root-3 x)) 2))
    (* 3/4 p_x (expt root-3 2) (expt (exp y) 2) (expt (exp (* root-3 x)) 2))
    (* 3/4 p_x (expt root-3 2) (expt (exp y) 2) (expt (exp (* -1 root-3 x)) 2))
    (* -3/2 p_x (expt root-3 2) (expt (exp (* -1 y)) 4))
    (* -18 p_x (expt p_y 2) (expt root-3 2))
    (* 18 (expt p_x 3)))
 (expt root-3 3))
```

Figure 2: I3-prime.

- The transformed Hamiltonian contains a trivial center of mass degree of freedom.Write a new two degree of freedom Hamiltonian H" in x and y that eliminates the center of mass freedom. Similarly, write the conserved quantity I3" for this two degree of freedom problem. Check that the I3" is still a conserved quantity.Write out the final Hamiltonian H".

∘ H' as produced by hamiltonian code for canonical composition, has terms that can be simplified further. We do away with root-2 (or $\sqrt{2}$) and root-3 (or $\sqrt{3}$).

- Simplified:

$$H'(t; x, y, cm; p_x, p_y, p_{cm}) = \frac{1}{24}\left(e^{2y+2\sqrt{3}} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) + \frac{1}{2}\left(p_x^2 + p_y^2\right) + \frac{p_{cm}^2}{2}$$

- Since cm does not appear in the Hamiltonian, p_{cm} is conserved. Set $\frac{p_{cm}^2}{2} = k_{cm} =$ constant.

$$H''(t; x, y; p_x, p_y) = \frac{1}{24}\left(e^{2y+2\sqrt{3}x} + e^{2y-2\sqrt{3}x} + e^{-4y}\right) + \frac{1}{2}\left(p_x^2 + p_y^2\right) + k_{cm}$$

- The previous I_3' had neither cm nor p_{cm} in its terms, so I_3'' remains the same. As such, I_3'' is still a conserved quantity.

Figure 3: H".

- Expand the Hamiltonian H" to third order in x and y. Do you recognize the Hamiltonian? We know that this Hamiltonian exhibits the divided phase space, with chaotic zones and regular regions, yet the problem we started with had enough conserved quantities to be solvable.

• Find out what $\frac{1}{24}\left(e^{2y+2\sqrt{3}x}+e^{2y-2\sqrt{3}x}+e^{-4y}\right)$ equals to expanded:

$$\frac{1}{24}\left(e^{2y}\right)\left(e^{2\sqrt{3}x}+e^{-2\sqrt{3}x}\right)+\frac{1}{24}e^{-4y}=$$

$$=\frac{1}{24}\left(1+2y+2y^2+\frac{4}{3}y^3\right)\left(1+\cancel{2\sqrt{3}x}+6x^2+\cancel{4\sqrt{3}x^3}+1-\cancel{2\sqrt{3}x}+6x^2-\cancel{4\sqrt{3}x^3}\right)$$
$$+\frac{1}{24}\left(1-4y+8y^2-\frac{32}{3}y^3\right)$$

$$=\frac{1}{24}\left(1+2y+2y^2+\frac{4}{3}y^3\right)\left(2+12x^2\right)+\frac{1}{24}\left(1-4y+8y^2-\frac{32}{3}y^3\right)$$

$$=\frac{1}{8}-\frac{y^3}{3}+\frac{y^2}{2}+\frac{2y^3x^2}{3}+\frac{x^2}{2}+y^2x^2+yx^2$$

• Replace into H'':

$$H''=\underbrace{\left(\frac{1}{2}(p_x^2+p_y^2)+\frac{1}{2}(x^2+y^2)+x^2y-\frac{y^3}{3}\right)}_{\text{Hénon-Heiles Hamiltonian}}+y^2x^2+\frac{2}{3}y^3x^2+\frac{1}{8}+K_{CM}$$

Figure 4: Expanded H" contains the Henon-Heiles Hamiltonian plus some extra terms, some of which are constants.